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AN ADAPTIVE METHOD OF TEST SELECTION IN SYSTEM DEVELOPMENT

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PREFACE

Experience indicates that the major sources of uncertainty in weapon systems development can be traced to those activities which involve testing and redesign. Yet, surprisingly little of a conceptual nature has been done to improve the decisionmaking process involved in performing these activities. In this paper, an adaptive model of the development process is presented which enables decisiormakers to determine which tests, if any, should be performed at a given stage and what corrective action, if any, should be taken once the test results are known.

This study should be useful to the System Project offices, the Air Force Research and Technology Division, the Air Staff, and the directors of various test facilities. It should also be of interest to the Office of the Secretary of Defense and to defense contractors, as well as to persons concerned with commercial development.

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SUMMARY

This Memorandum considers the problem of selecting "optimal" or "best" programs for the conduct of testing and redesign activities in the development of military hardware. An adaptive model of the testing process is presented which is designed to (1) provide the project director and his staff with a means for determining the best test to perform at a given stage in the development of a system, and (2) enable the same decisionmakers to choose intelligently between the available redesign actions once the test results are known. While the model is presented in terms of relatively simple systems and tests."

The final checkout phase of a new system is considered first. The model developed by Johnson [5] for the purpose of determining the optimal trouble-shooting procedure when an operating system fails is found to be an appropriate abstraction of this phase and is adopted as a starting point. This model is then generalized to include the case in which the object tested is destroyed when the test is unsuccessful. In addition, sufficiency conditions for the optimality of the Lost procedure suggested by Johnson are given.

In Sec. III the problem of test selection in the early stages of development is superimposed on the final checkout phase. It is well known that the imperfect nsture of the information gained in early testing tends to offset the advantages of early redesign. A dynamic programming formulation is given which enables one to determine which test, if any, should be performed at a given stage and what corrective action, if any, should be taken once the test result is known. As a

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by-product, a means for determining the net benefit of a given test at a given stage is obtained. The use of the multistage decision model developed in the a section is illustrated by means of a comprehensive example.

The situation a_{i} which there is serial dependence among certain test information messages, as in the case of a multistage rocket, is examined in Sec. IV. In the last section, it is shown that Nelson's model of parallel R&D efforts [6] may be viewed as a special case of the model developed in this paper.

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I. INTRODUCTION

The aim of this study is to establish a framework and a methodology for selecting "optimal" or "best" programs for the conduct of tests in the development of military hardware. For expository purposes, the end product of the development process may be considered to be a new weapon system of some complexity, such as a missile, a fighter, or a radar system. The hardware to be developed, which will be called a <u>system</u>, will be viewed as consisting of several <u>subsystems</u> (e.g., propulsion, guidance). Each subsystem, in turn, is considered to be made up of several <u>components</u>; each component may consist of <u>parts</u>, etc. Systems whose subsystems are indivisible will be referred to as onelevel systems; systems whose subsystems are at most made up of indivisible components will be called two-level systems, etc.

Development testing is generally regarded as distinct from and preceding production or operational testing. Its purpose is to determine whether a particular new system design will work (i.e., satisfy given performance requirements, including endurance) in the environment in which it is intended to function. Development testing is concluded as soon as a prototype design is found to operate satisfactorily in this environment. Production testing in the usual sense is then initiated to determine, and perhaps to improve, the reliability of the production process instituted to manufacture the system in quantity. In the present study, we concentrate exclusively on development testing.

BASIC CHARACTERISTICS OF THE DEVELOPMENT ENVIRONMENT

Development in the area of defense takes place in an environment which, although highly difficult to cope with, can easily be described.

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Thus, there is general agreement that the chief characteristics of the environment faced by the developer are uncertainty, the necessity of making decisions based on current knowledge, and the generation of new knowledge as the development proceeds. Each of these characteristics is formally incorporated into the models constructed in this study via the approach of modern decision theory.

SETTING OF THE PROBLEM

The problem to be discussed is that faced by the project director and those of his subordinates and contractors who must decide what tests to perform, if any, at each stage in the development of a system. A closely related decision problem concerns the extent of redesign, if any, to be done once the results of a given test are known. While the project director and the test supervisors concerned must, by reference to a flexible or conditional listing of possibilities, consider the whole future of the project in deciding what to do next at a given point in the development, they cannot specify the precise sequence of steps to be taken for the whole project in advance. This is because they are in a situation which is adaptive: the best action to take at a given point is a function of the test results obtained so far.

Accordingly, the models in this paper will be developed with two purposes in mind. First, to provide the project director and his staff with a means for determining the "best" test to perform at a given stage in the development. Second, to enable the same decisionmakers to choose intelligently among available actions once the test results are known.

Consider the abstraction given in Fig. 1 of a typical (but highly

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Key

Begin development 0 Complete construction of component 1 1 Complete construction of component 2 2 3 Complete construction of component 3 4 Complete testing and redesign of component 1 Complete testing and redesign of component 2 5 Complete testing and redesign of component 3 6 Complete assembly of subsystem 1 7 Complete testing and radesign of subsystem 1 8 Complete assembly of system 9 10 Complete testing and redesign of system

Fig. 1 - PERT Network for a System Consisting of Three Components

simplified) development project. Experience indicates that the major sources of uncertainty in development can be traced to those activities that involve testing and redesign (i.e., those that end in events 4, 5, 6, 8, and 10). These activities, then, should logically be the prime targets in an attempt to provide a basis for improved decisionmaking in the development field, and will be so viewed in this paper.

OBJECTIVE FUNCTION

The objective of development is to bring to "completion" a new system or item not previously available. "Completion" may be charac艨,

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terized by a triplet of measurements: money cost x, time elapsed t, and system performance r. It will be assumed that the developer[†] has preferences over the various triplets (x, t, r) that the system could exhibit once it is "completed." These preferences will be represented by the function U(x, t, r) which is defined for all non-negative values of x, t, and r. It has the property that $U(x_1, t_1, r_1) > U(x_2, t_2, r_2)$ if and only if the triplet (x_1, t_1, r_1) is preferred to the triplet (x_2, t_2, r_2) . Presumably, U is decreasing in x, nonincreasing in t, and nondecreasing in r.

In most cases, the system performance element of the triplet would itself be a vector $r = (r_1, ..., r_n)$, the components of which would correspond to such magnitudes as speed, range, CEP, weight, reliability, etc. To simplify the exposition we assume that a given performance level \overline{r} is to be achieved and that no additional utility is derived by exceeding \overline{r} . Thus,

$$U(x, t, r') = U(x, t, \overline{r})$$

for all x, t, and all r' such that $r' \ge \overline{r}$, that is, $r'_i \ge \overline{r}_i$ for all i, and

$$U(x, t, r') = U(x, t, 0)$$

for all x, t, and all r'' such that $r'' < \overline{r}$.

Let $V(x, t) = -U(x, t, \overline{r}) + K$, where K is a constant such that V(0, 0) = 0. Referring to t as the time cost, we may call V the timeand-money cost function.

V will now be specified more precisely. The utility of money in

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The person or persons charged with making the basic decisions for a given system.

a large organization, such as the government, tends, as a result of the aggregation of many individuals' functions, to be linear. Also, a single development project usually requires only a small portion of the government's total resources, so that a nonlinear utility function would be well approximated by its tangent at some point in the relevant region. Thus we lose little in our representation of the real world and gain much by way of simplicity by assuming V to be linear in money.

Now it can easily be shown that this condition implies that V is decomposable and may be written

$$V(x, t) = x + a(t),$$

where, since V is nondecreasing in t, a is a nondecreasing function (not necessarily linear). While we would expect a to be linear for most projects (that is, the "cost" per time unit to be fixed), it is probably a sharply increasing function in the typical crash program. In this Memorandum, a(t) will be assumed to be identically zero to keep the discussion as simple as possible; in a later study we will consider both the money cost and the time cost.[†]

PROBABILITIES

The required performance level r of the system establishes a standard against which, at any point in time, the developer may subjectively derive a two-valued probability function in advance for each part, component, and subsystem. As the development proceeds, the developer

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^TThere is no reason why the inverse procedure could not be chosen. The decision to consider the money cost rather than the time cost first was completely arbitrary.

will have numerous opportunities to modify his initial probabilities by conducting tests. The conditional distributions of the outcomes (messages) of these tests will be assumed to be known. Of course, in real world situations, these distributions would also have to be determined subjectively since in development there is, by definition, no directly applicable experience to fall back on.

We do not consider here the question of how to derive subjective probability distributions. Numerous descriptions of how this may be accomplished, on the basis of postulates presupposing certain consistencies in behavior, are available in the literature.[†]

Neither do we argue the merits of using subjective probabilities in relation to alternative inputs. We merely call attention to two relevant observations. As McKean [3] points out, a decisionmaker, in taking a position on an uncertain issue, <u>implicitly quantifies</u> considerations which he refuses to quantify explicitly. Going one step further, it follows as a theorem that a decisionmaker who observes a certain measure of consistency in deciding under uncertainty in fact <u>imputes</u> a probability distribution over the possible outcomes, regardless of what criterion is used. This distribution is such that if it is used to solve the decision problem under risk, it will give the same solution as was obtained under uncertainty with the given criterion. As a result, if one is committed to this (highly reasonable) measure of consistency, it would seem that one might as well convert the decision problem to one under risk by searching for the necessary probability distribution(s).

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[†]See, for example, the accounts of Marschak [1] and Timson [2]. [‡]See Ref. 4, pp. 287-294.

II. THE FINAL CHECKOUT PROBLEM WHEN TEST RESULTS ARE TWO-VALUED

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This section deals with the problem of minimizing the m ney cost of a particular system once the final test stage has been reached.[†] It wil¹ be assumed that in testing a system, subsystem, component, or part the only information gained is whether it works or doesn't work. In other words, the test information is two-valued.

ONE-COMPONENT SYSTEMS

To begin with, consider the case where a one-level (one-component) system consisting of M parts has reached the final test stage. The problem is how to conduct the final test and any redesign that may be required so as to maximize total utility. In this case, this is equivalent to minimizing the expected cost of the final test since all other costs have already been incurred.

Since the required performance level is given, the outcome of the final test is either "success" or "failure." The following assumptions will now be made:

<u>Assumption 1</u>. The final test of a component (subsystem, system) will fail if and only if one or more of its parts (components, subsystems) is not working.

<u>Assumption 2</u>. The failure of one part of a component is independent of the failure of any other part or combination of parts.

The test procedure given in Fig. 2 will be arbitrarily selected for this model. (Conditions under which this procedure is optimal are

⁷The final test stage is considered first because in an adaptive situation the optimal decision rules, as will become evident shortly, are obtained by working backwards in time.



Fig. 2 - Flow Chart of Test Procedure for Final Checkout of One-component System

given later.) Having so limited the problem, the task becomes to select the sequence in which the parts should be tested in order to minimize expected cost. This problem was first treated by Johnson [5]; it will be briefly reviewed here in the setting of the current study.

The following definitions will be used:

 $E_i = cost of testing component i$

M_i = number of parts in component i

 R_{ij} = cost of removing jth part from ith component

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$$B_{ij} = R_{ij} + C_{ij} + R'_{ij}$$

$$q_{ij} = 1 - P_{ij}$$

$$\pi_{ij} = \Pi_{k=j}^{N} P_{ik}$$

We now make another assumption:

<u>Assumption 3</u>. In each test, $p_{ij} > 0$ for all i and j.

initial component test when sequence S_i is used

This assumption marely expresses the common-sense rule that we will not engage in any dest unless the <u>a priori</u> probability that all parts involved will work is positive.

Given an arbitrary sequence S_i , we obtain, assuming that the system is not destroyed if the final test fails,

$$C_{i}(S_{i}) = \pi_{i1}0 + (1 - \pi_{i1})B_{i1} + q_{i1}(C_{i1} + E_{i})$$

+ (1 - $\pi_{i2})B_{i2} + q_{i2}(C_{i2} + E_{i}) + \cdots$
+ (1 - $\pi_{iM_{i}})B_{iM_{i}} + q_{iM_{i}}(C_{iM_{i}} + E_{i}) - q_{iM_{i}}E_{iM_{i}}$

or

(1)
$$C_{i}(S_{i}) = \sum_{j=1}^{n_{i}} [B_{ij} + q_{ij}(C_{ij} + E_{i})] - \sum_{j=1}^{n_{i}} \pi_{ij}B_{ij} - q_{iM_{i}}E_{iM_{i}}.$$

After the j - lth part has been checked out, the probability that the component test will fail again is $(1 - \pi_{ij})$. The last term of (1) indicates that we need not test the last part if the procedure takes us that far because then we would know that the part does not work and would immediately begin to redesign it.

For any given $\overline{p_i} = (p_{i1}, \ldots, p_{iM_i})$, it is clear that with reference to the terms in brackets, $C_i(S_i)$ is independent of the particular test sequence used. However, the values of the last two terms are clearly affected by the way the tests of the parts are sequenced.

Let us temporarily fix the final part in the test sequence and consider sequence S'_i , which is the same as S_i except that parts j and j + 1 < M_i have been interchanged. This gives

$$C_{i}(S_{i}') - C_{i}(S_{i}) = (1 - \pi_{ij})B_{ij} + q_{ij}(C_{ij} + E_{i}) + (1 - \pi_{i,j+1})B_{i,j+1} + q_{i,j+1}(C_{i,j+1} + E_{i}) - (1 - \pi_{ij})B_{i,j+1} - q_{i,j+1}(C_{i,j+1} + E_{i}) - (1 - P_{ij}P_{i,j+2}, ..., P_{iM_{i}})B_{ij} - q_{ij}(C_{ij} + E_{i}) - B_{ij}(\pi_{ij} - P_{ij}P_{i,j+2}, ..., P_{iM_{i}}) - B_{i,j+1}(\pi_{i,j+1} - \pi_{ij}) = -\pi_{i,j+2}P_{ij}B_{ij}(P_{i,j+1} - 1) - \pi_{i,j+1}B_{i,j+1}(1 - P_{ij}).$$

Simplifying further,

(2)
$$C_{i}(S'_{i}) - C_{i}(S_{i}) = -\pi_{i,j+2}(-p_{ij}B_{ij}q_{i,j+1} + p_{i,j+1}B_{i,j+1}q_{ij}).$$

Clearly, this expression is negative if

$$-p_{ij}B_{ij}q_{i,j+1} + p_{i,j+1}B_{i,j+1}q_{ij} > 0;$$

that is, if

(3)
$$\frac{P_{ij}B_{ij}}{q_{ij}} < \frac{P_{i,j+1}B_{i,j+1}}{q_{i,j+1}}.$$

As a result, if (3) holds, sequence S'_i leads to a lower expected final checkout cost than S_i .

Thus, since (3) is a transitive relation over the values of j, we can, by successively interchanging adjacent parts until (3) holds for all $j < M_i - 1$, obtain the optimal sequence for any given final part. Formally, the rule may be stated as

(4) j precedes k if
$$\frac{P_{ij}B_{ij}}{q_{ij}} < \frac{P_{ik}B_{ik}}{q_{ik}}$$

We must now consider the final part in the sequence. Let $S_{iM_{i}}$ be the sequence obtained by applying (4) to all parts $j = 1, \ldots, M_{i}$. Let S_{ik} be $S_{iM_{i}}$ with the kth part transferred to the end. Thus the problem is to determine m_{i} such that

(5)
$$G_{im_i k}^{min G_{ik'}}$$

where

$$S_{ik} = C_{i}(S_{ik}) - C_{i}(S_{iM_{i}})$$

= $(1 - \pi_{i,k+1}P_{ik})B_{i,k+1} + (1 - \pi_{i,k+2}P_{ik})B_{i,k+2} + \cdots$
+ $(1 - \pi_{iM_{i}}P_{ik})B_{iM_{i}} + (1 - P_{ik})B_{ik} - q_{ik}E_{ik}$
- $[(1 - \pi_{ik})B_{ik} + (1 - \pi_{i,k+1})B_{i,k+1} + \cdots$
+ $(1 - \pi_{iM_{i}})B_{iM_{i}} - q_{iM_{i}}E_{iM_{i}}]$

$$= -p_{ik}(i_{ik} + \pi_{i,k+1}B_{i,k+1} + \dots + \pi_{iM_{i}}B_{iM_{i}}) + \sum_{j=k}^{M_{i}} \pi_{ij}B_{ij}$$

- $q_{ik}E_{ik} + q_{iM_{i}}E_{iM_{i}}$
= $(1 - p_{ik}) \sum_{j=k+1}^{M_{i}} \pi_{ij}B_{ij} - p_{ik}B_{ik} + \pi_{ik}B_{ik} - q_{ik}E_{ik} + q_{iM_{i}}E_{iM_{i}}$

Simplifying,

(6)
$$G_{ik} = q_{ik} \sum_{j=k+1}^{M_i} \pi_{ij} B_{ij} + B_{ik} (\pi_{ik} - P_{ik}) - q_{ik} E_{ik} + q_{iM_i} E_{iM_i}.$$

Thus, G_{ik} can easily be generated, providing the desired minimizing index m_{i} .

Let K_i denote the set of indices k' for which $p_{ik'} = 1$. Then by (4), K_i will be at the end of S_i , unless $m_i \notin K_i$ as determined by (5), in which case K_i will appear immediately preceding the final part in S_i . In the first case, $1 - \pi_{ik'} = 1 - p_{ik'} = 0$ for all $k' \notin K_i$, which implies by (1) that the testing would never reach the parts in K_i ; letting $\overline{S_i} = S_i - K_i$, we obtain $C_i(\overline{S_i}) = C_i(S_i)$. Thus, in this case, it is immaterial whether we exclude K_i from the test sequence.

In the second case, $1 - \pi_{ik} = 1 - \pi_{im} = 1 - p_{im}$ so that

$$c_i(\overline{s}_i) = c_i(s_i) - (1 - p_{im_i}) \sum_{k' \in K_i} B_{ik'}$$

or

(7)
$$C_i(\overline{S}_i) < C_i(S_i)$$

ing, we must therefore explicitly exclude them in defining the test sequence. This gives

Theorem 1. To ensure that parts known to be working are not tested for any component 1, it is necessary to limit the domain of S_{j} to those indices j for which $p_{ij} < 1$, $j = 1, ..., M_{j}$.

The main result may now be stated as

<u>Theorem 2.</u> For any component i, number the parts for which $p_{ij} < 1$ in ascending order of $p_{ij}B_{ij}/q_{ij}$, $j = 1, ..., M_i$; then transfer the mth element of S_i to the final position M'_i where m_i is given by $G_{im_i} = \min_k G_{ik}$, k = 1, ..., M', and G_{ik} is given by

(8)
$$G_{ik} = q_{ik} \sum_{j=k+1}^{M'_i} \pi_{ij} B_{ij} + B_{ik}(\pi_{ik} - P_{ik}) - q_{ik} E_{ik} + q_{iM'_i} E_{iM'_i}$$

Then this is the optimal sequence.

Let us call the optimizing sequence S_{i}^{*} . Then

$$C_{i}(S_{i}^{*}) = \min_{\substack{S_{i} \in S_{i}}} C_{i}(S_{i}),$$

where S_i is the set of all possible sequences for component i. Clearly S_i^* is a function of \overline{p}_i , the probability vector of the component, so that we should write $S_i^*(\overline{p}_i)$. It will be convenient to define $C_i^*(\overline{p}_i) \equiv C_i(S_i^*(\overline{p}_i))$; thus $C_i^*(\overline{p}_i)$ is the optimal expected checkout cost when the given test procedure is used and \overline{p}_i holds.

MULTI-COMPONENT SYSTEMS

We now consider two-level systems and find that the problem remains exactly the same, as shown by Johnson [5]. The test procedure is depicted in Fig. 3. We use the convention of testing a given component even after the last part has been redesigned, although this is not strictly necessary.

Let i denote the ith component of the system, using an arbitrary numbering scheme, where i = 1, ..., K.

We now need the following definitions:

E = cost of performing system test

 $R_i = cost$ of removing ith component from system

 $R'_{i} = \text{cost of returning working component i into system}$ $p_{i} = \prod_{j=1}^{M_{i}} p_{j} = \pi_{i1} = \underline{a \text{ priori}}$ probability that ith component works

S = sequence in which components are tested

C(S) = total expected cost of final checkout, excluding initial test

of system, when the components are tested in the sequence S

$$B_{i} = R_{i} + E_{i} + R_{i}^{\prime}$$
$$q_{i} = 1 - p_{i}$$
$$\pi_{i} = \Pi_{j=i}^{K} P_{j}$$

We now obtain

$$C(S) = 0\pi_{1} + (1 - \pi_{1})B_{1} + q_{1}\left(\frac{C_{1}^{*}(\overline{p}_{1})}{q_{1}} + E\right) + (1 - \pi_{2})B_{2} + q_{2}\left(\frac{C_{2}^{*}(\overline{p}_{2})}{q_{2}} + E\right)$$

+ ... + (1 - $\pi_{K})B_{K} + q_{K}\left(\frac{C_{K}^{*}(\overline{p}_{K})}{q_{K}} + E\right) - q_{K}E_{K},$

or

(9)
$$C(S) = \sum_{i=1}^{K} (B_i + C_i^{\dagger}(\overline{p}_i) + q_i E) - \sum_{i=1}^{K} \pi_i B_i - q_K E_{K^*}$$

Since this is analogous to (1), we obtain, by the proof of Theorems 1 and 2, Theorems 3 and 4.



Fig. 3 – Flow Chart of Test Procedure for Final Checkout of Multi-component System

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Theorem 3. To ensure that components known to be working are not tested, it is necessary to limit the domain of S to ' ose indices i for which $p_i < 1$, i = 1, ..., K.

Theorem 4. Number the components for which $p_i < 1$ in ascending order of $p_i B_i / q_i$, i = 1, ..., K; then transfer the mth element of S to the final position K' where m is given by

$$G_{m} = \min G_{k}, \quad k = 1, \dots, K'$$

and

$$G_{k} = q_{k} \sum_{i=k+1}^{K'} \pi_{i} B_{i} + B_{k} (\pi_{k} - P_{k}) - P_{k} E_{k} + P_{K'} E_{K'}$$

in analogy with (8). Then this is the optimal sequence S^* .

Let $\overline{p} = (\overline{p}_1, \ldots, \overline{p}_K)$ and let $C^*(\overline{p}) \equiv C(S^*(\overline{p}))$. Then the minimum expected final checkout cost, including the initial system test, becomes

(10)
$$f_{j}(\overline{p}) \equiv E + C^{*}(\overline{p}).$$

Clearly for $\overline{p} = I$, where I is the unit vector, $C^{*}(I) = 0$. Also, $C^{*}(\overline{p})$ appears to be continuous for $\overline{0} < \overline{p} < I$ (where $\overline{0}$ is the zero vector) with discontinuities at $p_{ij} = 1$ for all i and j, and strictly decreasing in p_{ij} for all i, j.

It is readily seen that the preceding results generalize so as to apply to systems of any number of levels.

LOSS OF SYSTEM

So far, we have assumed that failure in the final test of a: item does not result in loss of the item. We now consider the situation

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where this is the case; let the cost of a duplicate system be L and the cost of a duplicate of component i be L_i . Then

$$C(S) = \pi_1^0 + (1 - \pi_1)(L_1 + E_1 + R_1') + C_1^*(\overline{p}_1) + q_1(E + L)$$

+ $(1 - \pi_2)(L_2 + E_2 + R_2') + C_2^*(\overline{p}_2) + q_2(E + L) + \cdots$
+ $(1 - \pi_{K'})(L_{K'} + E_{K'} + R_{K'}') + C_{K'}^*(\overline{p}_{K'})$
+ $q_{K'}(E + L) - p_{K'}E_{K'}',$

or

(10a)
$$C(S) = \sum_{i=1}^{K'} (L_{i} + E_{i} + R_{i}' + C_{i}^{*}(\bar{p}_{i}) + q_{i}(E + L))$$
$$- \sum_{i=1}^{K'} \pi_{i}(L_{i} + \Sigma_{i} + R_{i}') - q_{K'}E_{K'}.$$

This is analogous to (9) with R_i replaced by L_i , and E replaced by E + L. Thus Theorems 2 and 4 also hold when the object tested is destroyed in the final checkout as long as R_i is replaced by L_i ; that is, $B_i = L_i + E_i + R_i$.

THE TEST PROCEDURE: SUFFICIENCY CONDITIONS FOR OPTIMALITY

So far, we have taken the test procedures shown in Figs. 2 and 3 as given. Although chosen arbitrarily, as pointed out earlier, the procedure appears rather reasonable. We now give the conditions under which it is optimal. While these conditions will be stated for one-level systems only, they essentially "generalize" when multilevel systems are involved.

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We first prove the following:

Lemma. Let

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(11)
$$\frac{p_i B_i}{1 - p_i} \le \frac{p_{i+1} B_{i+1}}{1 - p_{i+1}}, \quad i = 1, \dots, K' - 1,$$

<u>where</u> E, $B_i > 0$, $0 < p_i < 1$, i = 1, ..., K'. Then

(12)
$$p_1 \left(B_1 \prod_{\substack{i=1 \ i \neq j}}^{K'} p_i + E \right) - E \ge 0, \quad j = 1, ..., K'$$

implies that

(13)
$$p_k \left(B_k \prod_{\substack{i=1 \ i \neq j}}^{K'} p_i + E \right) - E \ge 0, \quad j = 1, ..., K',$$

 $k = 2, ..., K'.$

<u>Proof.</u> For any $k = 2, \ldots, K'$, (11) gives

(14)
$$p_1 B_1 \leq \frac{p_k B_k (1 - p_1)}{1 - p_k}$$

The inequality (12) is then preserved if the right-hand side of (14) is substituted for p_1B_1 in (12), giving

$$\frac{P_{k}B_{k}(1 - P_{1})}{1 - P_{k}} \prod_{\substack{i=1\\i\neq j}}^{K'} P_{i} + P_{1}E - E \ge 0, \qquad j = 1, \ldots, K',$$

which reduces to (13) upon factoring out 1 - p_1 .

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Theorem 5. For the test procedure in Fig. 2 to be optimal, it is sufficient that (12)
$$P_1\left(B_1\prod_{\substack{i=1\\i\neq j}}^{K'}P_i+E\right) - E \ge 0, \quad j=1, ..., K',$$

where the subscript i denotes the ith part and the parts have been numbered in ascending order of $p_i B_i / (1 - p_i)$.

<u>Proof</u>. We first show that the procedure in Fig. 2 is optimal given that a system test is performed first, and then show that a system test should indeed be performed as the first step.

(1) Given that the initial system test fails, an alternative test procedure must necessarily encompass the testing and repairing of two or more parts before a new system test is made. Let Q(P) be the total checkout cost when parts s, ..., s' are grouped together somewhere in the test sequence followed by parts s' + 1, ..., K' (separably or in groups). Similarly, let Q(P') be the total checkout cost when k $\in \{s, ..., s'\}$ is "broken out" of the group; that is, <u>after</u> the other parts in the group (but before parts s' + 1, ..., K') have been tested, a system test is performed to determine whether further testing is needed. We then obtain

(15)
$$Q(P) - Q(P') = \cdots + (1 - p_{g}, \dots, p_{K'})(B_{g} + \cdots + B_{g'})$$

+ $(1 - p_{g})C_{g} + \cdots + (1 - p_{g'})C_{g'}$
+ $(1 - p_{g}, \dots, p_{g'})E + \cdots$

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$$- \left[\cdots + (1 - p_{s}, \dots, p_{K'}) (B_{s} + \dots + B_{k-1} + B_{k+1} + \dots + B_{s'}) + (1 - p_{s})C_{s} + \cdots + (1 - p_{s'})C_{s'} + (1 - p_{s}, \dots, p_{k-1}p_{k+1}, \dots, p_{s'})E_{s'} + (1 - p_{k}p_{s'+1}, \dots, p_{K'})B_{k} + (1 - p_{k})E_{s'} \right]$$

$$= (1 - p_{s}, \dots, p_{k-1}p_{k+1}, \dots, p_{s'})$$

$$\cdot \left[p_{k} \left(B_{k} \prod_{i=s'+1}^{K'} p_{i} + E \right) - E_{s} \right].$$

Since the first factor of (15) is always positive and the second factor is non-negative whenever (12) is by the lemma, it follows that part k should be "broken ou⁺" whenever (12) holds, and, since k may be any part, that every part in every group should be "broken out." Consequently, the test procedure in Fig. 2 is optimal given that a system test is performed as the first step.

(2) Let $C(P_i)$ be the total final checkout cost when the test procedure in Fig. 2 is used, except that <u>no system test</u> is performed until <u>after</u> the ith part test. We then obtain

(16)
$$C(P_1) - C(P_0) = B_1 + (1 - P_1)C_1 + E + \cdots$$

 $- [E + (1 - P_1, \dots, P_K)B_1 + (1 - P_1)C_1 + (1 - P_1)C_1 + (1 - P_1)E + \cdots]$
 $= P_1(P_2, \dots, P_K)B_1 + E - E,$

which is non-negative whenever (12) is. Similarly,

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(17)
$$C(P_i) - C(P_{i-1}) = P_i(P_{i+1}, \dots, P_K, B_i + E) - E,$$

 $i = 2, \dots, K',$

which, by the lemma, is non-negative whenever (12) is. Consequently, the theorem holds.

III. MULTISTAGE TESTING

Since the minimum expected cost of the final checkout, $C^*(\vec{p})$, is a function of the <u>a priori</u> probabilities that the various parts are working, we are now in a position to assess the value of performing tests of parts, components, and subsystems at earlier stages in the development. Before constructing an analytical model to deal with this problem, several important aspects of its setting must be clarified.

We have already pointed out that each element of p expresses the <u>a priori</u> probability that a particular part will perform its job <u>in the</u> <u>final system</u>. Thus at any one point in time, the same part may have an entirely different <u>a priori</u> probability of working in one system as opposed to another.

In effect Assumption 1 states (perhaps somewhat idealistically) that the tests used for the final checkout provide perfect information. Letting z denote the state of the system and y the message of the test as to what that state is, perfect information implies the following conditional distribution P(y/z):

	P(y/z) Message		
State of System	Working ^y l	Not Working ^y 2	$\Sigma_{y} P(y/z)$
Working (z ₁)	1	0	1
Not working (z ₂)	Ú	1	1

When a test is performed at an earlier stage in the development, we would not expect to obtain a conclusive answer as to whether the

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part, component, or subsystem tested will work in the final system. Thus we might expect to have a situation like this:

	P(y Mes	/z) sage	
State of Part	У1	У ₂	$\Sigma_{y} P(y/z)$
z ₁	1	0	1
^z 2	1/4	3/4	1

In this case, $P(y_1/z_1) = 1$ and $P(y_2/z_1) = 0$. This implies that if the part is working the test will bear this out, and in most real situations this will probably be true. However, if we wish, for example, to take into account the possibility that the test equipment itself may be faulty without our discovering it, we obviously have a situation in which $P(y_2/z_1) > 0$.

In the example, $P(y_2/z_2) = 3/4$; that is, if the part is not working, the probability that the test will discover this fact is 3/4. $P(y_2/z_2)$ may be considered as a measure of the "power" of the test. In general, the earlier the stage in which the test is made, the smaller we would expect its "power" to be.

For a test to have any value, its results must provide learning, that is, for a modification of the <u>a priori</u> probabilities involved. The mechanism by which this modification is accomplished is provided by Bayes' Theorem. Letting P(z) denote the <u>a priori</u> probability of state z, we obtain by this theorem

$$P(z/y) = \frac{P(y/z)P(z)}{\sum_{z} P(y/z)P(z)}.$$

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In the above example, letting $P(z_1) = 1/2$,

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$$P(z_1/y_1) = \frac{P(y_1/z_1)P(z_1)}{P(y_1/z_1)P(z_1) + P(y_1/z_2)P(z_2)} = \frac{1 + 1/2}{1 + 1/2 + 1/4 + 1/2} = 4/5,$$

$$P(z_2/y_1) = \frac{1/4 + 1/2}{1 + 1/2 + 1/4 + 1/2} = 1/5,$$

$$P(z_1/y_2) = \frac{P(y_2/z_1)P(z_1)}{P(y_2/z_1)P(z_1) + P(y_2/z_2)P(z_2)} = 0,$$

$$P(z_2/y_2) = 1 - P(z_1/y_2) = 1.$$

Thus, on the basis of this test, we would revise the probability that the part is working from 1/2 to 4/5 if message y_1 is received, and to 0 if y_2 is obtained. Note that (if $P(y/z) = x_y$, say, for all z, the <u>a posteriori</u> probabilities would coincide with the <u>a priori</u> probabilities. Therefore, for learning to take place, the test must generate messages that are in some way correlated with the relevant states of the part tested.

When y_2 is received, it will frequently seem logical to take some corrective action in the form of rework or redesign, since this would normally be cheaper than to do so when the part has already become part of a larger unit. However, the same action appears rather illogical when message y_1 is received in the test. On the other hand, the imperfect nature of the information gained in early testing tends to offset the advantages of early redesign. Thus, we have a "dual" problem: what test to perform on the given part and what corrective action to take once the result is known.

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PRE-CHECKOUT TESTING OF PARTS

Let us make the following definitions:

- \overline{p} = a priori probability vector of system
- p' = a posteriori probability vector of system
 - y = a possible test result
 - Y = the set of possible test results of an available test
- Q = the set of available tests (includes null test, Y_n)
- a = a corrective action
- A = the set of corrective actions (includes null action, a_0)
- T(Y) = cost of performing test Y

 $c_{\gamma}(y, a) = expected cost of taking corrective action a when message y$ is received and test Y is used

y'(y, u) = transformed message y' after action a has been taken following receipt of message y

Let us now discuss some properties of c_y and y'. Clearly, $c_y(y, a_0) = 0$ for all y, and $y'(y, a_0) = y$ for all y since "no action" costs nothing and is incapable of performing a transformation of a message. Returning to our example in which y_1 is the message that the part works and y_2 that it does not, let a_1 be the action consisting of redesigning the part until it passes the test (message y_1 is obtained). We then obtain $y'(y_2, a_1) = y_1$. In general, the effect of taking a corrective action other than a_0 is to transform a "bad" message into a "good" one while at the same time incurring a cost.

We can now state the problem in which we have at some point an opportunity to test and redesign one part before the final checkout stage is reached:

(18)
$$\min_{\mathbf{Y} \in \mathcal{Q}_{\mathbf{Y}}} \left\{ T(\mathbf{Y}) + \sum_{\mathbf{y}} P(\mathbf{y}) \min_{\mathbf{a} \in \mathbf{A}} \left[c_{\mathbf{Y}}(\mathbf{y}, \mathbf{a}) + C^{*}(\overline{\mathbf{p}}'(\overline{\mathbf{p}}, \mathbf{y}'(\mathbf{y}, \mathbf{a}))) \right] \right\}.$$

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As stated earlier, the problem is two-fold. First, for a given test and test result we must find the least-cost, or "best," action a. Then, weighting this cost by the probability of receiving the given message and summing over all messages, it remains, after repeating this procedure for all tests Y, to pick the test with the least expected cost. The "best" test may, as stated earlier, be the null test.

The function $\alpha_{Y,\overline{p}}^{*}$, among all possible $\alpha_{Y,\overline{p}}^{-}$, which associates with each possible y \in Y the action a \in A that minimizes the bracketed expression of (18) is the decisionmaker's best decision rule for that test and <u>a priori</u> probability vector \overline{p} . Letting Y^{*} be the best test for the part that can be tested prior to the final checkout, the minimum expected test and redesign cost of the system is

(19)
$$T(Y^{*}) + \sum_{y \in Y^{*}} P(y) [c_{Y^{*}}(y, \alpha_{Y, p}^{*}(y)) + c^{*}(\overline{p}'(\overline{p}, y'(y, \alpha_{Y, p}^{*}(y))))].$$

Since the minimum expected test and rework cost of the system when the part is not tested prior to the final test is $C^*(\overline{p})$, the value of the information gained by conducting test Y is clearly

(20)
$$C^{*}(\overline{p}) - T(Y) - \sum_{y \in Y} P(y) [c_{Y}(y, \alpha_{Y,\overline{p}}^{*}(y)) + C^{*}(\overline{p}'(\overline{p}, y'(y, \alpha_{Y,\overline{p}}^{*}(y))))];$$

that is, the amount by which the minimum total expected cost of testing is reduced when test Y is employed.

PRE-CHECKOUT TESTING OF COMPONENTS

Just as opportunities exist to test parts before they are acsembled into components, opportunities are usually available to test components before they are assembled into subsystems, to test subsystems before

assembly, etc. Here we need to draw on the preceding discussion of pre-checkout testing of parts as well as the discussion in Sec. II.

Let p_i be the probability that the ith component works before the test (of the component) and p'_i be the <u>a posteriori</u> probability. By Assumption 2,

$$P_{i} = \prod_{j=1}^{M_{i}} P_{ij}.$$

Once the result of the test is known, p'_{ij} can now be obtained, as before, by Bayes' Theorem. Again by Assumption 2,

$$p_{i}' = \prod_{j=1}^{M_{i}} p_{ij}'.$$

In taking corrective action in response to a message y which relates to a component, we again assume that the procedure in Fig. 3 is used. Thus the cost of taking corrective action becomes a function of \overline{p}_i as well as of the message y and the action a. The minimum expected such cost, $c'_Y(y, a, \overline{p}_i)$, would be derived by the method of Sec. II.

The value of the information gained by conducting pre-checkout test Y of component i (when other components and parts are not so tested) is then, in analogy with (20),

(21)
$$C^{*}(\overline{p}) - T(Y) - \sum_{y \in Y} P(y)[c_{Y}^{*}(y, \alpha_{Y,\overline{p}}^{*}(y), \overline{p}) + C^{*}(\overline{p}'(\overline{p}, y'(y, \alpha_{Y,\overline{p}}^{*}(y)))].$$

The minimum expected test and rework cost of the system now becomes

(22)
$$\min_{Y \in Q_j} \left\{ T(Y) + \sum_{y \in Y} P(y) \min_{a \in A} \left[c_Y^*(y, a, \overline{p}) + c^*(\overline{p}'(\overline{p}, y'(y, a))) \right] \right\}.$$

THE CONPLETE PROBLEM

Having formulated the test and rework problem for two levels, we are now in a position to state the entire development problem in sequential form. Consider Fig. 4 in which the test stages involved in the development of a two-level system are indicated. Any activity, or the smallest set of activities, which can be finished before all remaining activities must begin, constitutes a stage. Thus, a stage is the smallest set of activities during which information that can be used for subsequent stages can be obtained.

It is now clear that at any stage the choice of a test is often a choice of a combination of tests; we therefore use the word "test" to refer to a particular combination of tests (see, for example, stage 1 in Fig. 4).

We now make the following definitions:

N = number of stages at which tests can be performed

- $f_N(p)$ = minimum expected cost of developing system when tests exist at N stages
 - Y_N = the set of possible test results of a test available at Nth stage from end
 - u_{N} = the set of tests available at Nth stage from end (includes null test)

 $A_N \approx$ the set of redesign actions available at Nth stage from end T(Y_N) \approx cost of performing test Y_N

$$c_{Y_N}^{(y, a, p)} = minimum expected cost of taking corrective action a whentest YN is used, test result y is received, and the a prioriprobability vector is $\overline{p}$$$

K = cost of producing and assembling parts and components exclusive of testing and redesign activities

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Fig. 4 - Test Stages of a System Consisting of Three Components

Extending our two-stage model to N stages, we obtain for the minimum expected cost of developing the system, by the principle of optimality [6, p. 83],

(23)
$$f_{N}(\overline{p}) = K + \min_{\substack{Y_{N} \in \langle \mathcal{Y}_{N} \rangle \\ Y_{N} \in \langle \mathcal{Y}_{N} \rangle}} \left\{ T(Y_{N}) + \sum_{y \in Y_{N}} P(y) \min_{a \in A_{N}} [c_{Y_{N}}^{\star}(y, a, \overline{p}) + t_{N-1}(\overline{p}'(\overline{p}, y'(y, a)))] \right\},$$

where $f_1(\overline{p}) = C^*(\overline{p})$.

At any stage, this equation enables the developer to choose the best test for that stage. (Remember that the best test may, of course, be no test.) Since the best decision at any stage always depends on past actions and test results, the model is an adaptive one.

AN EXAMPLE

At this point, we provide an example to illustrate the use of the preceding model. Let the task be to develop, at minimum cost, a new missile system (system X), which is to meet a given performance level. For purposes of illustration, we assume that system X consists of only two subsystems (parts)--a guidance subsystem and a propulsion subsystem. The PERT chart governing its development is given in Fig. 5, where the two possible test stages are indicated.



Key

- 0 Begin development
- 1 Complete construction of guidance subsystem
- 2 Complete construction of propulsion subsystem
- 3 Complete testing and redesign of guidance subsystem
- 4 Complete testing and redesign of propulsion subsystem
- 5 Complete assembly of total system
- 6 Complete testing and redesign of total system

Fig. 5 - System X

The relevant information concerning the final (second) test stage is summarized in Table 1. Tables 2 through 9 and the text following contain the necessary information required for decisionmaking with respect to test stage 1.

COSTS RELATING TO FINAL TEST STAGE

Missile System X:

E = 1 Expected cost of performing final test firing of one missile.

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L = 7 Expected cost of replacing a system used up in a final test firing.

Guidance Subsystem (Subsystem 1):

- E₁ = 10 Expected cost of tests required to be performed on guidance subsystem to ascertain whether it meets the required performance level (includes the production costs of guidance and auxiliary subsystems used up in these tests), given that the final system test failed.
- L = 4 Expected cost of (replacing) guidance subsystem (used up in a final system test firing).
- R₁['] = 1 Expected cost of integrating guidance subsystem with propulsion subsystem to form total system.
- C₁ = 8 Expected cost of redesigning and testing guidance subsystem to pass the guidance subsystem test, given that this test was not passed the first time (includes the cost of replacing guidance and auxiliary subsystems used up in this phase).

Propulsion Subsystem (Subsystem 2):

Expected cost of tests required to be performed on pro- pulsion subsystem to ascertain whether it meets the required performance level (includes the production costs of propulsion and auxiliary subsystems used up in these tests), given that the final system test failed.

- L₂ = 2 Expected cost of (replacing) propulsion subsystem (used up in a final system test firing).
- $R_2' = R_1' = 1$
- C₂ = 6 Expected cost of redesigning and testing propulsion subsystem to pass the propulsion subsystem test, <u>given</u> that this test was not passed the first time (includes the cost of replacing propulsion and auxiliary subsystems used up in this phase).

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	Configuration									
Designation ^A	No Test	Centrifuge Test of Guidance Subsystem	Sled Test of Guidance Subsystem	Static Firing Test of Propulsion Subsystem	Cost T(Y)					
YO	x				0					
Y ₁		x			.1					
¥2			x		2					
Y ₃				x	1					
^Y 1,2		x	x		2.1					
^Y 1,3		x		х	1.1					
¥2,3			X	х	3					
^Y 1,2,3		x	x	x	3.1					

TESTS AVAILABLE AT TEST STAGE 1

^aFor simplicity, in the case of $Y_{1,2}$, we assume that Y_1 and Y_2 provide independent messages.

Table 3

PROPERTIES OF TEST Y

(Centrifuge Test of Guidance Subsystem)

	Probability of Receiving Message y Given State $z (P(y/z))$					
State of Subsystem with respect to System X Requirements	Test Passed at High Speed (y ₁)	Test Passed at Medium Speed Only (y ₂)	Test Failed (y ₃)			
Working (z ₁)	.8	.2	о			
Not working (z_2)	.2	.4	.4			

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PROPERTIES OF TEST Y2

(Sled Test of Guidance Subsystem)

	Probability of Receiving Message y Given State z (P(y/z))					
with respect to System X Requirements	Test Passed (y ₁)	Test Failed (y ₂)				
Working (z ₁)	.9	.1				
Not working (22)	.3	.7				

Table 5

PROPERTIES OF TEST Y₃ (Static Firing Test of Propulsion Subsystem)

	Probability of Receiving Message f Given State z (P(y/z))					
with respect to System X Requirements	Test Passed (y ₁)	Test Failed (y ₂)				
Working (z ₁)	.95	.05				
Not working (z ₂)	.45	.55				

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Table 6

PROPERTIES OF TEST Y_{1,2} (Centrifuge Test and Sled Test of Guidance Subsystem)

**************************************	Probability of Receiving Message y Given State z (P(y/z))							
State of Subsystem with respect to System X Requirements	Centr: y ₁ Sled: y ₁ (y _{1,1})	Centr: y ₁ Sled: y ₂ (y _{1,2})	Centr: y ₂ Sled: y ₁ (y _{2,1})	Centr: y ₂ Sled: y ₂ (y _{2,2})	Centr: y ₃ Sled: y ₁ (y _{3,1})	Centr: y ₃ Sled: y ₂ (y _{3,2})		
Working (z ₁)	.72	.08	.18	.02	0	0		
Not working (z ₂)	.06	. 14	.12	. 28	. 12	. 28		

Table 7

COST OF AVAILABLE REDESIGN ACTIONS AS A FUNCTION OF MESSAGES RECEIVED:

TESTS Y1, Y2, AND Y3

<u> </u>	Cost of Redesign Action (c _y (y, a))							
	Т	est Y ₁		Test Y ₂		Test Y ₃		
	Message			Message		Message		
Action Taken on Conclusion of Test	У1	у ₂	у ₃	у ₁	^y 2	у ₁	^y 2	
a ₀ None	0	0	0	0	0	0	0	
a ₁ Redesign to obtain y ₁		4	8		5		6	
a Redesign to obtain y 2			3					

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COST OF AVAILABLE REDESIGN ACTIONS AS A FUNCTION OF MESSAGES RECEIVED: TEST Y1,2

		Cost of Redesign Action $(c_{Y}(y), d_{Y})$, a))	
		Message Received						
Action	Taken on Conclusion of Test	У _{1,1}	^y 1,2	^y 2,1	^y 2,2	y _{3,1}	y _{3,2}	
a 0	None	0	0	0	0	0	0	
^a 1,1	Redesign to obtain y _{1,1}		5	4	7	8	9	
^a 2,1	Redesign to obtain y _{2,1}					3	6	

ND Y _{1,2,3}
Y _{2,3} , A
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^a0, ^a1,1,1, ^a1,1,2, ^a2,2,1 ^a0, ^a1,1,1, ^a1,1,2, ^a1,2,1 ^a0, ^a1,1,1, ^a1,1,2, ^a2,1,1 **Availabl**€ Actions Test Y_{1,2,5} ^a0, ^a1,1,1 ^ao, ^a1,1,1 ^a0, ^a1,1,1 ao, a1,1,1 ഹ y1,2,1 ^y2,1,1 y2,1,2 y1,1,2 y2,2,2 y1,2,2 y1,1,1 ^y2,2,1 Message a₀, a₁,1, a₁,2, a₂,1 Available Actions Test Y2,3 ³0, ³1,1 a₀, a_{1,1} **8**0 Message y1,1 ^y2,1 y_{1,2} ^y2,2 ^a0, ^a1,1, ^a1,2, ^a2,1 ^a0, ^a1,1, ^a1,2, ^a2,1 Available ^a0, ^a1,1, ^a2,1 Actions Test Y_{1,3} ^a2,2,^a3,1 a₀, a_{1,1} ^a0, ^a1,1 a 0 Message y2,1 ^y2,2 ^y3,1 y1,1 y1,2 y3,2

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Y3,1,2 | ^a0, ^a1,1,1, ^a1,1,2, ^a2,1,1'

^a2,1,2^{, a}3,1,1

^a0, ^a1,1,1, ^a2,1,1

y3,1,1

^a0^{, a}1,1,1^{, a}1,1,2^{, a}2,1,1[,]

^y3,2,2

a2,1,2' a3,2,1

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^a0, ^a1,1,1, ^a2,1,1

^y3,2,1

from Tables 7 and 8 by simple addition since Y_1 and Y_3 , Y_2 and Y_3 , and $Y_{1,2}$ and Y_3 apply to different subsystems. Thus, for example,

$$c_{Y_{1,2,3}}(y_{3,2,2}, a_{2,1,1}) = c_{Y_{1,2}}(y_{3,2}, a_{2,1}) + c_{Y_{3}}(y_{2}, a_{1})$$

= 6 + 6 = 12.

From $B_i = E_i + L_i + R'_i$, we obtain that $B_1 = 15$ and that $B_2 = 15$. Applying Theorem 2 to (1), we find by reference to (10) that the minimum expected final checkout cost of system X at the beginning of the final test stage, $f_1(p_1, p_2)$, is

(24)
$$f_1(p_1, p_2) = 61 - 16p_1 - 14p_2 - 15p_1p_2$$

+ min {-3p_2 - 12, -5p_1 - 10}.

For example, when p_1 (the probability as of the beginning of the final test stage that the guidance subsystem will work) is .5, and p_2 (the corresponding probability for the propulsion subsystem) is .5, we obtain

$$f_1(.5, .5) = 28.75, \quad s^* = (1, 2),$$

and when $p_1 = .9$ and $p_2 = .7$,

$$f_1(.9, .7) = 12.85, \quad S^* = (2, 1).$$

Thus, in case the first test firing of the missile fails, the optimal sequence in the first case is to perfect the guidance subsystem before attempting another firing of the full system and, if necessary, proceed

to redesign the propulsion subsystem. In the second case, the reverse sequence should be used.

The total minimum expected cost of testing and redesign for each of the possible tests that may be used in stage 1, as of the beginning of that stage (assuming an optimal test sequence in stage 2), is shown in Table 10 for selected probability vectors \overline{p} . When $\overline{p} = (.5, .5)$, we note that test $Y_{1,2,3}$ gives the lowest expected cost: $f_2(.5, .5) = 20.63$. Analogously, we note that $f_2(.5, .95) = 11.06$ ($Y_{1,2}$ optimal), $f_2(.9, .7) =$ 10.42 ($Y_{1,3}$ optimal), and $f_2(.95, .95) = 3.61$ (Y_1 optimal). It should be noted from Table 10 that the null test (Y_2) is the worst test when $\overline{p} = (.5, .5)$, while it is the second best test when $\overline{p} = (.95, .95)$.

The best corrective actions $\alpha_{Y_1\overline{p}}^{*}(y)$ for the best tests in Table 10 are given in Table 11. Results for tests Y_1 and $Y_{1,3}$ illustrate that it is not always optimal to redesign so as to pass a given test on all counts when the outcome of the first test trial is unfavorable.

Let us illustrate briefly how $f_2(.9, .7) = 10.42$ was computed for test $Y_{1.3}$.

<u>Step 1.</u> Bayes' Theorem gives, by direct application of the conditional probabilities in Tables 3 and 5,

$$\overline{p}'(y_{1,1}, \overline{p}) = \left(\frac{.8p_1}{.8p_1 + .2(1 - p_1)}, \frac{.95p_2}{.95p_2 + .45(1 - p_2)}\right),$$

so that $\overline{p}'(y_{1,1}, (.9, .7)) = (.97, .83)$. Inserting this value in (24) we obtain $f_1(.97, .83) = 6.93$. In the same way, we obtain for the

TOTAL MINIMUM EXPECTED COST OF TESTING AND REDESIGN OF SYSTEM X, AS OF THE BEGINNING OF THE FIRST TEST STAGE, AS A FUNCTION OF THE TEST USED AT THE FIRST STAGE FOR SELECTED PROBABILITY VECTORS

<u>A Priori</u> Probability Vector		Test Used at First Stage							
P1	^р 2	Yo	Y ₁	Y ₂	Y ₃	¥1,2	¥1,3	¥2,3	¥1,2,3
.5	.5	28.95	24.10	26.62	27.14	24.32	22.18	34.59	20,63
.5	.95	17.72	11.55	14.16	19.79	11.06	12.18	15.73	11.62
.9	.7	12.85	11.93	13.76	11.47	13.61	10.42	12.27	12.26
.95	.95	4.07	3.61	5.78	4.59	5.37	4.10	6.32	5.74

Table 11

BEST CORRECTIVE ACTION (a) AS A FUNCTION OF THE MESSAGE RECEIVED (y) FOR SELECTED FIRST-STAGE TESTS AND PROBABILITY VECTORS

Test Y_1 ($\overline{p} = (.95, .95)$)		$\frac{\text{Test } Y_{1,2}}{(\vec{p} = (.5, .95))}$		Test $Y_{1,3}$ ($\overline{p} = (.9, .7)$)		Test Y _{1,2,3} $(\bar{p} = (.5, .5))$			
у ₁	a _0	y _{1,1}	a 0	y _{1,1}	a 0	^y 1,1,1	a _0	^y 2,2,1	^a 1,1,1
y ₂	a _0	y _{1,2}	a _{1,1}	y _{1,2}	^a 1,1	^y 1,1,2	^a 1,1,1	^y 2,2,2	a 1,1,1
y ₃	^a 2	^y 2,1	^a 1,1	y _{2,1}	^a 1,1	^y 1,2,1	^a 1,1,1	y _{3,1,1}	a1,1,1
		^y 2,2	^a 1,1	^y 2,2	^a 1,1	^y 1,2,2	^a 1,1,1	^y 3,1,2	a1,1,1
		^y 3,1	^a 1,1	y _{3,1}	^a 2,1	^y 2,1,1	^a 1,1,1	^y 3,2,1	a1,1,1
		^y 3,2	a 1,1	^y 3,2	a 2,1	^y 2,1,2	^a 1,1,1	^y 3,2,2	^a 1,1,1

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other possible messages:

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$$\overline{p}'(y_{1,2}, (.9, .7)) = (.97, .18) \qquad f_1(.97, .18) = 25.49$$

$$\overline{p}'(y_{2,1}, (.9, .7)) = (.82, .83) \qquad f_1(.82, .83) = 11.56$$

$$\overline{p}'(y_{2,2}, (.9, .7)) = (.82, .18) \qquad f_1(.82, .18) = 29.05$$

$$\overline{p}'(y_{3,1}, (.9, .7)) = (0, .83) \qquad f_1(0, .83) = \mathbf{n}^{\dagger}$$

$$\overline{p}'(y_{3,2}, (.9, .7)) = (0, .18) \qquad f_1(0, .18) = \mathbf{n}^{\dagger}$$

Step 2. We must next compute

$$F_{Y_{1,3}}(y, \overline{p}) \equiv \min_{a \in A_{Y,y}} \{c_{Y_{1,3}}(y, a, \overline{p}) + f_1(\overline{p}'(y'(a, \overline{p})))\}.$$

In the case of message $y_{2,2}$, for example, we note from Table 9 that

$$A_{Y_{1,5},Y_{2,2}} = \{a_0, a_{1,1}, a_{1,2}, a_{2,1}\}.$$

From Table 7 we obtain

$$c_{Y_{1,3}}(y_{2,2}, a_{0}, \overline{p}) = 0,$$

$$c_{Y_{1,3}}(y_{2,2}, a_{1,1}, \overline{p}) = 4 + 6 = 10,$$

$$c_{Y_{1,3}}(y_{2,2}, a_{1,2}, \overline{p}) = 4,$$

$$c_{Y_{1,3}}(y_{2,2}, a_{2,1}, \overline{p}) = 6.$$

When some element of \overline{p}' is 0, $f_N(\overline{p}')$ is automatically set equal to infinity in order to rule out the null action. See Assumption 3.

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Thus,

$$F_{Y_{1,3}}(y_{2,2}, (.9, .7))$$

= min {0 + 29.05, 10 + 6.93, 4 + 25.49, 6 + 11.56} = 16.72

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Consequently, action $a_{1,1}$ is optimal when message $y_{2,2}$ is related. From the formula $P(y) = \sum_{z} P(y/z)P(z)$, we obtain, by reference to Tables 3 and 5 $P(y_{2,2}) = (.2 \times .05)(.9 \times .7) + (.2 \times .55)(.9 \times .3) + (.4 \times .05)(.1 \times .7) + (.4 \times .55)(.1 \times .3) = .04$. Repeating the procedure for each message, we obtain:

	$F_{Y_{1,3}}(y, (.9, .7))$	$\alpha_{Y_{1,3}}^{*}, (.9, .7)^{(y)}$	P(y)
y _{1,1}	6.93	a ₀	. 59
^y 1,2	12.93	^a 1,1	. 15
^y 2,1	10.93	^a 1,1	. 18
^y 2,2	16.93	^a 1,1	.04
У _{3,1}	14.56	^a 2,1	.03
^y 3,2	20,56	^a 2,1	.01

Note that the first and third columns appear in Table 11.

<u>Step 3</u>. Since $T(Y_{1,3}) = 1.1$ (Table 2), we obtain from the preceding summary that

$$T(Y_{1,3}) + \sum_{y \in Y_{1,3}} P(y)F_{Y_{1,3}}(y, (.9, .7)) = 1.1 + 9.32 = 10.42.$$

Since this is the minimum value given by

$$\min_{\mathbf{Y} \in Q_j} \left\{ T(\mathbf{Y}) + \sum_{\mathbf{y} \in \mathbf{Y}} P(\mathbf{y}) F_{\mathbf{Y}}(\mathbf{y}, (.9, .7)) \right\},$$

f₂(.9, .7) = 10.42.

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IV. THE FINAL CHECKOUT PROBLEM WHEN TEST RESULTS ARE MULTIVALUED

In the preceding model, we limited the discussion to the case in which the final checkout information was two-valued. This restriction, however, has not been present with respect to the pre-checkout tests of parts, since the model is the same whether y is such that it can take only two values or whether it may assume several values.

Multivalued test results correspond to the situation in which one learns more about the object tested than merely whether it passes or fails the test. Thus, in putting a system through a final test, one may often ascertain whether specific subsystems work even though the system itself fails the test.

MULTIPLE CHOICE OF TESTS

Consider a system of K components in which, through test configuration Y, one obtains k + 1 read-outs, where $k \le K'$. These correspond to k + 1 messages, one representing the system as a whole, and the others k different components. Let the indices of these components form the set K₁. Thus, we may write $y = (y_0, y_1, y_2, ..., y_k)$, where each y_i is two-valued since the final checkout always gives perfect information. Thus 2^{k+1} different messages may be distinguished when k < K', and 2^k when k = K'. Let S be such that the components about which specific information is received are at the beginning of S. Letting E_y be the cost of the final test when test combination Y is used, the cost of the final checkout (not including the initial system test), $C_y(S)$, becomes when this sequence is used:

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$$C_{Y}(S) = \pi_{1}0 + (1 - p_{1})(R_{1} + R_{1}') + C_{1}^{*} + \dots + (1 - p_{k})(R_{k} + R_{k}')$$
$$+ C_{k}^{*} + (1 - p_{1}, \dots, p_{k})E_{y} + (1 - \pi_{k+1})B_{k+1} + C_{k+1}^{*}$$
$$+ q_{k+1}E_{y} + \dots + (1 - \pi_{K'})B_{K'} + C_{K'}^{*} + q_{K'}E_{y} - q_{K'}E_{YK'}$$

where the last term is zero when k = K'. Simplifying,

(25)
$$C_{\mathbf{Y}}(\mathbf{S}) = \sum_{\mathbf{k} \in K_{1}} \left(q_{\mathbf{k}}(\mathbf{R}_{\mathbf{k}} + \mathbf{R}_{\mathbf{k}}') + \mathbf{C}_{\mathbf{k}}^{*} \right) + (1 - p_{1}, \dots, p_{\mathbf{k}}) \mathbf{E}_{\mathbf{Y}}$$
$$+ \sum_{\mathbf{k} \notin K_{1}} \left(\mathbf{B}_{\mathbf{k}} + \mathbf{C}_{\mathbf{k}}^{*} + q_{\mathbf{k}} \mathbf{E}_{\mathbf{Y}} \right)$$
$$- \sum_{\mathbf{k} \notin K_{n}} \pi_{\mathbf{k}} \mathbf{B}_{\mathbf{k}} - \begin{cases} q_{\mathbf{k}}' \mathbf{E}_{\mathbf{Y}\mathbf{k}}' & \text{when } \mathbf{K}_{1} \neq \varphi, \\ 0 & \text{when } \mathbf{K}_{1} = \varphi. \end{cases}$$

After test E_{Y} is performed, we first redesign those components which the test shows are not working. With the remaining components, we proceed as in Fig. 3 until y_0 indicates that the system is working. It is clearly optimal to redesign the components about which specific information is received before reworking any others. To see this, let component j $\in K_1$ be transferred to the position following i $\notin K_1$. This has the sole effect of changing $(1 - \pi_m)B_m$ to $(1 - p_j\pi_m)B_m$, m = k + 1, ..., i, and $(1 - p_1, \ldots, p_k)E_Y$ to $(1 - p_1, \ldots, p_{j-1}P_{j+1}, \ldots, p_k)E_Y$ $+ (1 - p_j)E_Y$, all of which increase C_Y . This gives Theorem 6.

Theorem 6. Let each final test be such that it provides information as to whether the following work: (1) The system, and (2) each component $k \in K_1$. Then, after the initial system test, the optimal sequence is to redesign, as necessary, all components $k \in K_1$, in any order. At this point, proceed as in Theorem 4 for all components $k \notin K_1$.

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Thus the minimum expected cost of the final checkout becomes, by (25) and (10),

(26)
$$f_1(\bar{p}) = \min_{\substack{Y_1 \in q_{j_1} \\ Y_1 \in q_{j_1}}} \{E_{Y_1} + C_{Y_1}^*(\bar{p})\}.$$

SERIAL DEPENDENCE AMONG MESSAGES

We now introduce a further complication with respect to the final checkout of a system. The situation to be discussed is one in which multivalued messages are received due to the fact that subsystems function serially. That is, a particular subsystem may be designed to go into operation only after other subsystems have begun, or even completed, their functions. Thus, if the "lead" subsystem fails, the test clearly cannot provide information with respect to a "successor" subsystem. We refer to "no message" as the null message.

An example of this kind of system is given by a multistage missile. Unless the first-stage rocket operates, we will not have an opportunity to learn whether the second or third stages, for example, work.

<u>Definition 1</u>. A subsystem belongs to the first stage of a system operation^{\dagger} if its activation is not dependent on whether some other subsystem works.

This definition is readily extended by induction to

<u>Definition 2</u>. A subsystem belongs to the nth stage of a system operation, n = 2, 3, ..., if its activation is dependent on whether a subsystem in the n - 1th stage works and the subsystem does not already belong to a previous stage.

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⁷The various stages of system operation should not be confused with the stages (phases) of testing. Unfortunately, the word "stage" is the most appropriate in both instances.

Let L be the total number of operational stages obtained by this procedure and let K'_j be the set of subsystem indices belonging to the jth operational stage, j = 1, ..., L, for which $p_i < 1$. Clearly, $L \le K'$, and $\bigcup_{j=1}^{L} K'_j = \{1, ..., K'\}$; that is, each subsystem is identified with one and only one operational stage. In each stage, denote by $K_j \subseteq K'_j$ the set of subsystems about which test Y provides specific information. Now let the test sequence S be as follows: Let all subsystems of a stage be adjacent to each other, with the subsystems about which specific information is obtained by the test in front of the others. Then position the operational stages in ascending order. We then obtain

(27)
$$C_{\mathbf{Y}}(\mathbf{S}) = \sum_{j=1}^{\mathbf{L}} \left[\sum_{i \in K_{j}} \left(q_{i}(\mathbf{R}_{i} + \mathbf{R}_{i}') + C_{i}^{*} \right) + (1 - \prod_{i \in K_{j}} \mathbf{P}_{i}) \mathbf{E}_{\mathbf{Y}} + \sum_{i \in K_{j}' - K_{j}} \left(\mathbf{B}_{i} + C_{i}^{*} + q_{i}\mathbf{E}_{\mathbf{Y}} - \mathbf{R}_{ij}\mathbf{B}_{i} \right) - q_{\mathbf{s}_{j}}\mathbf{E}_{\mathbf{Y}\mathbf{s}_{j}} \right],$$

where $R_{ij} = \prod_{k=1}^{j} p_k$, and s_j is the last subsystem in the jth stage. Within each stage, the situation is clearly completely analogous to the one-stage case so that Theorem 6 applies directly.

Now suppose we change S by moving subsystem j $\epsilon K'_m$ to the position behind subsystem i $\epsilon K'_n$, where m < n. Our test and redesign operations would then clearly be trapped in the mth operational stage if the transferred subsystem does not work, since in that case we would not be able to obtain any information about subsequent stages. Since $q_j > 0$ for all j involved by Theorem 3, there is a positive probability that this will happen. Thus we obtain

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Theorem 7. Given a system that operates in L stages and a final test that provides information as to whether the system and each subsystem k $\in K_j$, j = 1, ..., L, work, the optimal test sequence, S^* , is obtained as follows: (1) Group the subsystems by stage and order the stages by their natural number; (2) within each stage, order the subsystems as per Theorem 6.

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V. PARALLEL DEVELOPMENT AS A SPECIAL CASE

An interesting attempt to model the phenomenon of parallel R&D efforts has been made by Nelson [7]. We now demonstrate how that model may be viewed as a special case of the model developed in this study.

Nelson considers the problem of developing a system to meet certain specifications at minimum cost in time and money. He distinguishes between two stages: design and development completion. In the design stage, opportunities for learning exist that may be beneficially utilized in the second stage. These opportunities are in the form of a "built-in" test which gives two-valued perfect information at the end of the first stage for a given design.

The following set of strategies is considered by Nelson: The developer has the opportunity to run independent, but merit-wise indistinguishable, projects in the first stage, each costing the same amount of money. (As before, we ignore time considerations.) Then, at the end of the first stage, he is to pick one project to run to completion. We denote a success in the first-stage test of a <u>design</u> by y_1 and failure by y_2 . For any number of designs i, there is clearly no benefit in obtaining more than one y_1 since only one design is to be jursued further. Thus there are only two macromessages that are relevant for a given i; let y_{11} be the set of messages containing at least one y_1 among the i designs tested and let y_{12} be the message that $y = y_2$ for all i. Let us also make the following definitions:

M = average cost of a design incurred in the first stage, including testing

p = probability of obtaining message y₂ for a design

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- M₁ = total test and redesign cost in second stage when y₁₁ is obtained
- $M_2 = total test and redesign cost in second stage when <math>y_{12}$ is obtained
 - K = total development cost in second stage exclusive of testing
 and redesign

When i designs are pursued in the first stage, the total expected development cost of the system becomes

(28)
$$M + (1 - 1)M + K + (1 - p^{i})M_{1} + p^{i}M_{2}$$

Thus, since M and K are fixed, the minimum expected cost of testing and redesign is

(29)
$$\min_{i \in I} \{(i - 1)M + (1 - p^{i})M_{1} + p^{i}M_{2}\},\$$

where I is the set of positive integers.

Comparing (29) with (22), we find that

$$q_{j} = I,$$

$$Y_{i} = i,$$

$$T(Y_{i}) = (i - 1)M,$$

$$P(y_{i1}) = 1 - p^{i},$$

$$P(y_{i2}) = p^{i},$$

$$c_{Y}^{*}(y, a, \overline{p}) = 0,$$

$$y' = y_{ij},$$

$$c_{(\overline{p}'(\overline{p}, y_{ij})) = M_{i},$$

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so that the preceding model of parallel development may be viewed as a special case of the testing problem considered in this study.

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