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MAXIMUM-LIKELIHOOD ESTIMATION OF THE PARAMETERS OF A FOUR-PARAMETER GENERALIZED GAMMA POPULATION FROM COMPLETE AND CENSORED SAMPLES

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Maximum-Likelihood Estimation of the Parameters of a Four-Parameter Generalized Gamma Population from Complete and Censored Samples

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Consider the four-parameter generalized Gamma population with location parameter c, scale parameter a, shape/power parameter b, and power parameter p (shape parameter $d = bp$) and probability density function $f(x; c, a, b, p) = p(x - c)^{bp-1} \exp \{-[(x - c)/a]^p\}/a^{bp} \Gamma(b)$, where $a, b, p > 0$ and $x \geq c \geq 0$. The likelihood equations for parameter estimation are obtained by equating to zero the first partial derivatives, with respect to each of the four parameters, of the natural logarithm of the likelihood function for a complete or censored sample. The asymptotic variances and covariances of the maximum-likelihood estimators are found by inverting the information matrix, whose components are the limits, as the sample size $n \to \infty$, of the negatives of the expected values of the second partial derivatives of the likelihood function with respect to the parameters. The likelihood equations cannot be solved explicitly, but an iterative procedure for solving them on an electronic computer is described. The results of applying this procedure to samples from Gamma, Weibull, and half-normal populations are tabulated, as are the asymptotic variances and covariances of the maximum-likelihood estimators.

1. INTRODUCTION

Stacy [4] has studied some of the elementary properties of a three-parameter generalized Gamma population which includes, as special cases, not only the two-parameter Gamma, but also the two-parameter Weibull, the one-parameter exponential and half-normal, and other populations of interest. Parr and Webster [3] have obtained expressions for the maximum-likelihood estimators, from complete samples of size n, of the parameters of such a population and for their asymptotic variances and covariances. Stacy and Mihram [5] have reparameterized the population, generalized it further to include cases in which the power parameter p is negative, and considered estimation of parameters by the methods of moments, maximum likelihood, and minimum variance.

The author believes that the usefulness of the generalized Gamma population in the study of life distributions, which has been recognized by Parr and Webster [3], will be greatly enchanced by the addition of a fourth parameter, the location parameter c, which the above authors have assumed to be zero. In addition, he has found that it is often necessary or desirable to estimate population parameters from censored samples. In this paper, therefore, by the methods already employed by Harter and Moore [2] for the three-parameter Gamma and Weibull populations, he formulates an iterative procedure for maximum-likelihood estimation, from complete and censored samples, of the parameters of a four-parameter generalized Gamma population. Harter [1] gives the mathe-

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matical formulation and tables for the asymptotic variances and covariances of the ML estimators. A one-page excerpt from those tables is included in this paper.

2. The Four-Parameter Generalized Gamma Population

The probability density function of the random variable X having a four-parameter generalized Gamma distribution with location parameter c, scale parameter a, shape/power parameter b, and power parameter p (shape parameter

Table 1

Coefficients of 1/N Times Power of Scale Parameter A in ML Estimators, from Samples of Size N with Proportions Q1 Censored from Below and Q2 from Above, of Parameters of Four-Parameter Generalized Gamma Population with Shape/Power Parameter B and Power Parameter P

	GAMMA POPULATION WITH SHAPE PARAMETER 3		
	(B = 3.0, P = 1.0)		
	$Q1$ = 0.000, $Q2$ = 0.00		
407.99832	−795.03326	124.78596	42.27586
	1564.3831	−241.47613	−89.25062
		38.50881	12.23302
			8.52138
	$Q1$ = 0.000, $Q2$ = 0.25		
1187.3964	−2221.3827	389.30249	85.91169
	4174.9751	−725.44723	−169.21794
		128.32667	26.99814
			11.00843
	$Q1$ = 0.025, $Q2$ = 0.00		
1342.7244	−2878.2305	386.58361	308.78655
	6209.9205	−824.72781	−685.23892
		111.84878	86.72469
			85.98538
	$Q1$ = 0.025, $Q2$ = 0.25		
6949.0859	−14119.355	2138.1030	1179.6570
	28752.850	−4335.8384	−2433.3743
		659.22462	358.37341
			222.31593
	WEIBULL POPULATION WITH SHAPE PARAMETER 3		
	(B = 1.0, P = 3.0)		
	$Q1$ = 0.000, $Q2$ = 0.00		
5.30772	−14.91938	23.83340	0.99266
	56.40271	−73.18953	−8.10825
		114.80484	6.59776
			2.23764
	$Q1$ = 0.000, $Q2$ = 0.25		
15.83389	−47.06445	86.01908	2.99543
	154.56908	−263.08530	−14.22536
		483.09374	18.34561
			2.62649

TABLE 1 *Continued*

	$Q1 = 0.025$, $Q2 = 0.00$		
5.78565	−21.63456	28.30335	3.17546
	218.92256	−167.51331	−65.59402
		171.18363	39.41193
			22.75602
	$Q1 = 0.025$, $Q2 = 0.25$		
28.58330	−134.47492	178.24989	24.69657
	779.59296	−905.12068	−173.08989
		1157.5[illegible]38	180.69986
			43.51342
	HALF-NORMAL POPULATION ($B = 0.5$, $P = 2.0$)		
	$Q1 = 0.025$, $Q2 = 0.00$		
8.22773	−7.93012	19.29805	0.33543
	8.18039	−19.65633	−0.38075
		51.19087	0.73146
			0.04609
	$Q1 = 0.025$, $Q2 = 0.25$		
25.22315	−30.24875	88.96886	0.80371
	37.67493	−111.99979	−0.99136
		340.70355	2.63114
			0.05909

$d = bp$) is given by

$$f(x; c, a, b, p) = p(x - c)^{bp-1} \exp\{-[(x - c)/a]^p\}/a^{bp}\Gamma(b),$$

$$a, b, p > 0, \quad x \geq c \geq 0. \tag{2.1}$$

From a mathematical standpoint there is no reason why c cannot be negative, and Stacy and Mihram [5] have introduced a simple modification which allows p to be negative, but since negative values of either c or p are not of much interest, at least from the point of view of life distributions, we assume that c and p are non-negative. The corresponding cumulative distribution function is given by

$$F(x; c, a, b, p) = \Gamma_{[(x-c)/a]^p}(b)/\Gamma(b). \tag{2.2}$$

The fact that the cumulative distribution function of this population is an incomplete Gamma-function ratio, as is that of the Gamma population, suggests the name generalized Gamma population, though it is also a generalization of the three-parameter Weibull population and of other populations as well. Specifically, one may mention the following populations as special cases: three-parameter Gamma ($p = 1$); three-parameter Weibull ($b = 1$); two-parameter exponential ($b = p = 1$); and two-parameter half-normal ($b = 1/2$, $p = 2$). If, in addition, one sets the location parameter c equal to zero in any one of these populations, the result is the same population with the number of parameters decreased by one.

3. Asymptotic Variances and Covariances of ML Estimators

The asymptotic variance-covariance matrix for the maximum-likelihood estimators $\hat{a}$, $\hat{b}$, $\hat{p}$, and $\hat{c}$ is given by $n^{-1}[v_{ij}]$, where $[v_{ij}] = [v^{ij}]^{-1}$ and the v^{ij}

are given in a report by Harter [1]. The computation of the elements v^{ij} of the information matrix (multiplied by $1/n$) and the inversion of this matrix to obtain the coefficients of $1/n$ in the variance-covariance matrix were performed on the IBM 7094 computer for various values of the parameters b and p and the censoring proportions q_1 (from below) and q_2 (from above). Computation is quite straightforward when the shape parameter $d = bp$ is greater than 2, but when the shape parameter is less than or equal to 2, one encounters quantities which become infinite when $q_1 = 0$ and take the indeterminate form $\infty - \infty$ when $q_1 > 0$. In the latter case, one may use alternate forms which are finite and can be evaluated by numerical integration. Estimation is non-regular and hence the asymptotic variances and covariances of the estimators have not been found when $q_1 = 0$ and the shape parameter is less than or equal to 2. With this exception, the coefficients of $(1/n)$ times a power of the scale parameter a in the asymptotic variances and covariances were computed for $q_1 = 0.000$ (0.005) 0.025 and $q_2 = 0.00$ (0.25) 0.75 for the following cases: $b = 1, p = 3$ (Weibull with shape parameter 3); $b = 3, p = 1$ (Gamma with shape parameter 3): $b = 1, p = 2$ (Weibull with shape parameter 2); $b = 2, p = 1$ (Gamma with shape parameter 2); $b = p = 1$ (exponential); and $b = 0.5, p = 2$ (half-normal). Representative results, accurate to within a unit in the last place given, are shown in Table 1, arranged in the form

$n \operatorname{Var}(\hat{a})/a^2$	$n \operatorname{Cov}(\hat{a}, \hat{b})/a$	$n \operatorname{Cov}(\hat{a}, \hat{p})/a$	$n \operatorname{Cov}(\hat{a}, \hat{c})/a^2$
	$n \operatorname{Var}(\hat{b})$	$n \operatorname{Cov}(\hat{b}, \hat{p})$	$n \operatorname{Cov}(\hat{b}, \hat{c})/a$
		$n \operatorname{Var}(\hat{p})$	$n \operatorname{Cov}(\hat{p}, \hat{c})/a$
			$n \operatorname{Var}(\hat{c})/a^2$

4. Iterative Procedure for Obtaining ML Estimates

The maximum-likelihood estimates of the parameters are the solutions of the likelihood equations obtained by equating to zero the first partial derivatives of the likelihood function with respect to the parameters, which are given in a report by Harter [1]. Since these equations do not have explicit solutions, it is necessary to resort to iterative solution on an electronic computer. Three iterative procedures were tried, singly and in various combinations—the rule of false position, the Newton–Raphson method, and the gradient method. The procedure found to give best results was a hybrid one, in which the rule of false position was used, for the first 120 iterations, to estimate the parameters, one at a time, in the cyclic order a, b, p, and c, omitting any assumed to be known. Assuming that the first m order statistics of a sample of size n ($m \leq n$) are known, one starts by setting $r = 0$ (no censoring from below). One then chooses initial estimates for the unknown parameters. At each step, one determines the value (if any) of the parameter then being estimated which satisfies the appropriate likelihood equation, in which the latest estimates (or known values) of the other three parameters have been substituted. Positive values $\hat{a}$, $\hat{b}$, and $\hat{p}$ can always be found in this way. In estimating c, however, one may find that no value of c in the permissible interval $0 \leq c \leq x_1$ satisfies the

TABLE 2

Estimates of Parameters from First M Order Statistics of Samples of Size 40

	M	$\hat{A}$	$\hat{A} \mid B$	$\hat{A} \mid P$	$\hat{A} \mid C$	$\hat{A} \mid B, P$	$\hat{A} \mid B, C$	$\hat{A} \mid P, C$	$\hat{A} \mid B, P, C$	$\hat{B}$	$\hat{B} \mid A$	$\hat{B} \mid P$	$\hat{B} \mid C$
G3	30	240.9	67.1	53.0	248.1	59.4	44.6	75.3	53.6	0.509	3.553	3.432	0.444
	40	221.6	70.0	45.6	233.5	55.7	48.8	60.8	51.8	0.672	3.675	3.844	0.566
W3	30	112.1	82.3	101.0	53.7	99.5	95.8	100.6	95.7	0.295	0.568	0.557	2.295
	40	101.0	83.5	99.6	79.2	101.6	95.5	101.7	97.0	0.541	0.560	0.569	1.412
HN	30	7.29	30.3	36.0	33.6	36.2	37.0	37.8	36.4	2.106	0.234	0.504	0.610
	40	30.7	35.7	35.4	36.9	35.7	35.8	36.3	35.8	0.648	0.246	0.510	0.470

	M	$\hat{B} \mid A, P$	$\hat{B} \mid A, C$	$\hat{B} \mid P, C$	$\hat{B} \mid A, P, C$	$\hat{P}$	$\hat{P} \mid A$	$\hat{P} \mid B$	$\hat{P} \mid C$	$\hat{P} \mid A, B$	$\hat{P} \mid A, C$	$\hat{P} \mid B, C$	$\hat{P} \mid A, B, C$
G3	30	3.599	2.802	2.206	3.039	3.074	0.985	1.094	3.340	0.911	0.877	0.851	0.911
	40	3.549	2.900	2.555	3.007	2.524	1.036	1.171	2.716	0.964	0.954	0.954	0.968
W3	30	0.567	0.910	0.885	0.894	4.923	2.094	2.143	1.585	2.844	2.901	2.719	2.686
	40	0.565	0.910	0.869	0.894	3.091	3.033	2.330	2.214	2.844	2.880	2.715	2.775
HN	30	0.401	0.224	0.475	0.394	0.730	3.467	1.982	1.630	1.671	3.557	1.908	1.639
	40	0.391	0.241	0.486	0.385	1.689	3.282	2.008	2.049	1.799	3.275	1.964	1.765

	M	$\hat{C}$	$\hat{C} \mid A$	$\hat{C} \mid B$	$\hat{C} \mid P$	$\hat{C} \mid A, B$	$\hat{C} \mid A, P$	$\hat{C} \mid B, P$	$\hat{C} \mid A, B, P$	$\hat{D}$**	$\hat{D} \mid A$	$\hat{D} \mid C$	$\hat{D} \mid A, C$
G3	30	18.01	0.00	0.00	0.00	11.83	0.00	4.92	13.23	1.566	3.500	1.182	2.457
	40	16.07	0.00	0.00	0.00	12.42	0.00	8.19	13.09	1.696	3.808	1.536	2.765
W3	30	38.54	36.86	33.12	37.02	16.88	36.86	16.31	15.94	1.452	1.700	3.638	2.641
	40	36.99	36.82	30.80	36.74	16.69	36.78	14.86	16.18	1.673	1.698	3.126	2.622
HN	30	20.11	20.11	20.11	20.11	20.11	20.11	20.11	20.11	1.538	0.811	0.995	0.795
	40	20.11	20.11	20.11	20.11	20.11	20.11	20.11	20.11	1.094	0.806	0.963	0.791

G3—Gamma Population—$A = 50,\ B = 3,\ P = 1,\ C = 20\ (D = 3)$.
W3—Weibull Population—$A = 100,\ B = 1,\ P = 3,\ C = 20\ (D = 3)$.
HN—Half-Normal Population—$A = 50,\ B = 0.5,\ P = 2,\ C = 20\ (D = 1)$.

** The estimate $\hat{D} = \hat{B}\hat{P}$ is the product of the maximum likelihood estimates $\hat{B}$ and $\hat{P}$.

likelihood equation obtained by equating to zero the partial derivative with respect to c. In such cases, the likelihood function in that interval is either monotone decreasing, so that $\hat{c} = 0$, or monotone increasing, so that $\hat{c} = x_1$. The latter situation occurs when $\hat{b}\hat{p} \leq 1$, since then the partial derivative with respect to c, for $r = 0$, contains only positive terms. Once that has occurred, it is impossible to continue iteration with $r = 0$, since some of the terms in the likelihood equations become infinite, so it is necessary to censor the smallest observation x_1 and any others equal to it (r observations in all). Subsequently, x_1 plays no role in the estimation procedure except as an upper bound on $\hat{c}$. Iteration continues until the results of successive steps agree to within some assigned tolerance. If, however, the tolerance has not been met by the time 120 iterations have been performed, the procedure is altered. The Newton-Raphson method is used, starting with the 121st iteration, to estimate the three parameters a, b, and p simultaneously. This is alternated with estimation, by the rule of false position, of the parameter c, which, because it is restricted to the closed interval $[0, x_1]$, does not lend itself to estimation by the Newton-Raphson method, which might yield an estimate outside this interval. The altered procedure is continued until the tolerance has been met or until the total number of iterations reaches 1100, at which point the attempt to estimate the parameters is abandoned. This particular procedure is recommended because the gradient method is the most slowly converging of the three, while the Newton-Raphson method converges most rapidly if the estimates are already quite good, but behaves erratically if they are not, as is likely to be the case at the outset.

5. Numerical Examples

As illustrations, consider the simulated life tests, each on forty components, summarized in Table 2. We shall suppose that the "data" represent observed failure times (in hours). Actually, they were obtained by approporiate transformations of uniform, exponential, or normal random numbers. For each set of data, the iterative estimation procedure described in Section 4 was carried out for $m = 10(10)40$ in the following cases: (1) all four parameters unknown; (2) any three parameters unknown; (3) any two parameters unknown; and (4) any one parameter unknown. The resulting estimates for $m = 30, 40$ are shown in Table 2. The number of iterations required tends to be large when one is estimating b and p simultaneously, especially from censored samples, apparently because of the fact that there is a high negative correlation between $\hat{b}$ and $\hat{p}$, so that their product $\breve{d}$, an estimate (not ML) of the shape parameter d, tends to be more stable than either $\hat{b}$ or $\hat{p}$.

The iterative estimation procedure was programmed in FORTRAN and run on the IBM 7094 computer. Machine time tends to be somewhat excessive, averaging about a minute per hundred iterations in cases in which three or four parameters are being estimated.

6. Concluding Remarks

Asymptotic variances and covariances of the estimators of the remaining parameters when one or more of the parameters are known have been calculated for various parameter values and censoring proportions. This was accomplished by inverting all square submatrices of the information matrix. When the location parameter c is known, estimation is regular even when the shape parameter d is less than or equal to 2, so it was possible to compute asymptotic variances and covariances of the estimators of the other parameters for the cases in which $q_1 = 0$, $d \leq 2$. Because of space limitations, the results are not included in this paper.

Just how applicable the asymptotic variances and covariances are to estimates from samples of size as small as 40 is an open question. Conceptually, this question might be settled by a Monte Carlo study, but from a practical standpoint any such study large enough to be conclusive would be ruled out by the excessive machine time required. In any case, the estimates given by the iterative procedure described in Section 4, when the location parameter c is unknown, differ in two important respects from those for which asymptotic variances and covariances have been calculated, which assume that at least one observation is censored from below whenever the shape parameter d is less than or equal to 2 and that negative values of the estimate $\hat{c}$ of the location parameter are permitted. Violation of either of these conditions vitiates the property of asymptotic multivariate normality and changes the asymptotic variances and covariances. Nevertheless, the author believes that, when it converges, the iterative procedure described in Section 4, which violates both of these conditions, results in more realistic estimates. Moreover, the restriction of $\hat{c}$ to be non-negative results obviously in a reduction (which may be substantial when d is large and n and c/a are small) in the variance of $\hat{c}$, and probably, because of the high correlation between the estimators, in a reduction in the other variances and covariances. A comparison of the discrepancies of the estimates given in Table 2 for the cases in which $d = 3$ from the true values of the parameters with those which one might expect if the asymptotic formulas held tends to confirm that such reductions do occur.

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13. ABSTRACT

Consider the four-parameter generalized Gamma population with location parameter c, scale parameter a, shape/power parameter b, and power parameter p (shape parameter d = bp) and probability density function $f(x; c, a, b, p) = p(x - c)^{bp-1} \exp\{-[(x - c)/a]^p\}/a^{bp}\Gamma(b)$, where $a, b, p > 0$ and $x \geq c \geq 0$. The likelihood equations for parameter estimation are obtained by equating to zero the first partial derivatives, with respect to each of the four parameters, of the natural logarithm of the likelihood function for a complete or censored sample. The asymptotic variances and covariances of the maximum-likelihood estimators are found by inverting the information matrix, whose components are the limits, as the sample size $n \to \infty$, of the negatives of the expected values of the second partial derivatives of the likelihood function with respect to the parameters. The likelihood equations cannot be solved explicitly, but an iterative procedure for solving them on an electronic computer is described. The results of applying this procedure to samples from Gamma, Weibull, and half-normal populations are tabulated, as are the asymptotic variances and covariances of the maximum-likelihood estimators.

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