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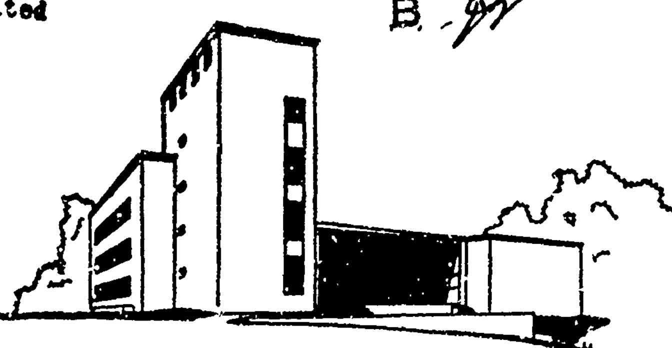

GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

William Larimer Mellon, Founder

41

Management Sciences Research Report No. 93

TRANSFER PRICING IN A DECENTRALIZED FIRM: A DECOMPOSITION ALGORITHM FOR QUADRATIC PROGRAMMING

BY

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May, 1967

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+ The author is indebted to W. W. Cooper, H. A. Simon and the referees from Management Science for their many valuable suggestions. He is, of course, alone responsible for such errors as remain.

I. INTRODUCTION

In the past ten to fifteen years considerable emphasis has been placed on "proper" methods of planning and controlling a decentralized firm. If we postulate that the firm's objective is to maximize profits, the problem becomes one of allocation of scarce resources. Decentralization is intended to combat the inefficiencies inherent in a large centrally planned and controlled organization: decision-making with time-lagged, incomplete information by the central control unit (corporate headquarters) and implementation after another costly time lag by personnel at the sub-unit (divisional) level far removed from the decision process. By making each division a profit center, thus responsible for its own planning and controlling, the informational flow is greatly reduced and decisions are more in keeping with the current state of affairs. If there were no dependencies between divisions, a simple division of the firm into smaller independent operating units would be highly advantageous.

Decentralization loses some of its efficiency, however, when divisional operations are not independent. Section II discusses some of the most obvious types of dependencies. When this situation arises, some coordinating mechanism between divisions must be introduced to insure joint profit maximization, and the obvious place for the coordinating mechanism to house itself is at corporate headquarters. Needless to say, a desirable characteristic for the coordination mechanism is that it minimize the informational flow between the divisions and corporate headquarters, hence minimizing the complexity and time dependency of the information. Economists have long recognized that demand and supply schedules in a market (simple price and quantity relations) summarize enormous amounts of information with respect to the costs of producing and benefits derived from obtaining the good in question. If a market device can be introduced, planning and control can be left to the market clearing mechanism, with each division attempting to maximize its own profits, bidding for scarce productive resources held by the firm as a whole (e.g., working capital) or produced by other divisions of the firm as intermediate products. Thus, the coordinating mechanism, operated by corporate headquarters, would be a set of markets, one for each good in question; the role of corporate headquarters would be to find the price for each market that would equate total supply to total demand. This price is called the "transfer price" for the good in question. When this set of prices has been found, an optimal allocation of scarce resources has been attained, and profits for the firm as a whole are at a maximum.

This paper develops an effective method (algorithm) for generating transfer prices under more general conditions than previous methods have allowed. The firm can be operating in imperfect markets, where the price it pays for inputs and the price at which it sells outputs depends upon the quantity it purchases or sells; it can also be selling products which compete with one another for market share or complement one another in their final usage. The algorithm is a quadratic decomposition algorithm; section II describes how the quadratic form can be used to take these imperfections into account. Following the development of the algorithm and a simple numerical example, its behavioral implications will be explored and compared with two other transfer pricing models (a linear decomposition algorithm and another quadratic decomposition algorithm) to evaluate its relative potential applicability.

II. CURRENT STATE OF THE ART

There are numerous phenomena which dictate the need for a planning and control function in a decentralized firm. The four most prevalent types are (1) demand dependence, (2) variable cost dependence, (3) corporate resource limitations, and (4) corporate policy. Examples of each are:

(1) competing goods produced by separate divisions being sold in the same imperfect market;

(2) discounts for large quantity buying when the input is used by more than one division; intermediate products;

(3) limited supply of working capital;

(4) a policy restricting the total output of the firm for one market when similar products are produced and sold in that market by more than one division (arising, e.g., from fear of anti-trust action).

Normative work in the area of decentralized decision-making has grown primarily through extensions of the basic theorem of welfare economics: under certain assumptions as to the utility function and the productive process, a competitive equilibrium can be identified with an economic optimum. The market place can be viewed as a process for solving the economic problems of coordination (equating demand and supply in all markets) and satisfying wants efficiently (i.e., in a least-cost manner in all markets) by successive approximations to the equilibrating prices. Prompted by the development of linear programming, economists began to analyze and extend this concept.[1/]

Until a 1960 paper by Danzig and Wolfe [5] on linear decomposition, the adjustment mechanisms for finding efficient prices were based upon a gradient adjustment mechanism, and convergence to an optimum takes an infinite amount of time; in the linear case, convergence is guaranteed in a finite number of iterations. Whinston [16] explored the transfer pricing problem in the general non-linear case vis-a-vis the Kuhn-Tucker conditions. He concluded that if the functions defining the problem were not separable, i.e., if externalities--at least in certain forms--existed, pure price guides would no longer give sufficient information to guide the individual decision-makers in making correct decisions even on their own accounts, much less in terms of over-all organizational goals and constraints. Utilizing the decomposable linear programming framework (where functions are separable), he was able to generate, through the use of shadow pricing, a systematic method of altering input and output prices and fixing prices for transferred goods so as to achieve joint profit maximization with decentralized decision-making. Baumol and Fabian [2] review this approach and present a numerical example.

With respect to application, it is the general concensus that most industries are oligopolistic in nature; consequently, the flat demand and supply curves required by linear programming are not truly representative. The linear decomposition model also fails to take into account any type of demand or supply dependence, which involve, at a minimum, the product of the interacting variables in their mathematical formulation.

With Wolfe's simplex method of quadratic programming [18] and Dorn's analysis of duality in quadratic programming [8], the stage was set for the extension of decomposition to quadratic programming. Quadratic decomposition remedies, to a limited extent, all of the aforementioned deficiencies of the linear decomposition algorithm.[2/] A quadratic form, as well as any linearities desired, combine to form the objective function. Hence, downward-sloping demand curves and upward-sloping (or downward-sloping) supply curves of the form

$$p_x = a + bx,$$

where p_x is the price of good x and a and b are constants, are permitted; any demand dependency which can be expressed in such a way that the total revenue (π) expression is a polynomial of degree two or less is permissible, e.g., when the amount of good y sold shifts the demand for x:

$$p_x = ay + bx + c,$$

so that

$$\pi = p_x x = (ay + bx + c)x = axy + bx^2 + cx,$$

is an acceptable expression. Similarly, supply dependencies (e.g., quantity discounts), of the form

$$p_z = a - b(cx + dy),$$

where c and d are technological coefficients relating the amount of input z used to produce a unit of x or y, are permissible since the total cost for input z, C_z, is

$$C_z = p_z z = [a - b(cx + dy)]\ (cx + dy)$$

$$= acx + ady - bc^2x^2 - 2cdxy - bd^2y^2.$$

Furthermore, all the dependencies which can be incorporated into the strictly linear decomposition algorithm are still legitimate. We, therefore, have a more complete device for generating optimal behavior in a decentralized organization through transfer pricing, a device which brings the mathematical model of the firm's operations one step closer to reality.

At least two decomposition algorithms have been developed to date. One was derived by Whinston [17] and the other appears in the next section of this paper. While their objectives are identical, their methods, and consequent implications, are different; these will be explored in section V. It is worth noting at this point, however, the unique concept in control demonstrated by the algorithm discussed here. The standard description of the methodology behind all decomposition algorithms is that the intent of the central planner is to (1) erase the monopoly-monopsony conditions inherent in the transfer of goods for which no external markets exist,[3/] (2) proceed in an orderly fashion through price manipulation and response to extract from each subordinate decision-maker his feasible production region, and (3) to choose the optimal production points within these regions and, when possible, find the demand and supply curves for all products that will allow each subordinate decision-maker to act independently in maximizing his own profits, and thereby maximize joint profits.

An alternative description exists, however, which is more in keeping with traditional economics. Above we spoke of finding "efficient" transfer prices, prices under which all activities in the decentralized subdivisions will be carried on in an efficient and coordinated manner. As noted, Whinston has shown if the functions are separable, such prices exist, and if the functions are linear, it is a task of finite length to find them. When separability is not present, a pure pricing mechanism breaks down--something more is needed. In quadratic programming, we look at one type of non-separability, a relatively simple type of non-separability in the objective function. In this case we find that there exist "efficient" functions, not efficient prices. Here the functions are linear, e.g., demand curves of the form $a - bx$. When a firm is operating in imperfect markets, a supply curve does not exist in the usual sense for any given output; e.g., a monopolist does not produce a quantity simply in reference to a price: he reacts rather to the whole demand curve he faces.[4/] Here we find a set of pure prices for scarce corporate resources and a set of specific taxes and bounties. If no dependencies were present in the objective function, the latter set would be null, and we would have an "efficient" pricing mechanism. With dependencies, however, the latter set is not the null set, and what we are in effect doing is shifting demand and supply curves, finding "efficient" linear functions. Thus, the subordinated decision-makers can make independent decisions while joint profits are maximized because all externalities have been taken into account in the information with which they make their decisions. The algorithm which follows may be interpreted in this fashion.

III. A DECOMPOSITION ALGORITHM FOR QUADRATIC PROGRAMMING

Let us consider a firm with two divisions:[5/] the first division's input-output vector is denoted as X, the second division's input-output vector is denoted as Y; the input-output vector for the firm is denoted as $Z = (X:Y)$. The firm's problem is to find the Z which maximizes profits (π); it can be written

$$\max \pi(X,Y) = P'X + Q'Y + Z'\emptyset Z$$

subject to:

$$
\begin{aligned}
CZ &\le R \\
f_i(X) &\le s_i, \quad i=1,\dots,a \\
g_i(Y) &\le t_i, \quad i=1,\dots,b \\
X,Y &\ge 0
\end{aligned}
$$

where P and X are m-component column vectors,
Q and Y are n-component column vectors,
$Z = (X:Y)$, a $(m+n)$-component column vector,
$\emptyset$ is a $(m+n) \times (m+n)$ symmetric, negative-definite matrix,[6/]
C is a $k \times (m+n)$ matrix,
R is a k-component column vector
s_i and t_i are constants, and
f_i and g_i are convex functions of X and Y, respectively.[7/]

We can partition $\emptyset$ and C into four and two submatrices, respectively, with dimensions corresponding to X and Y:

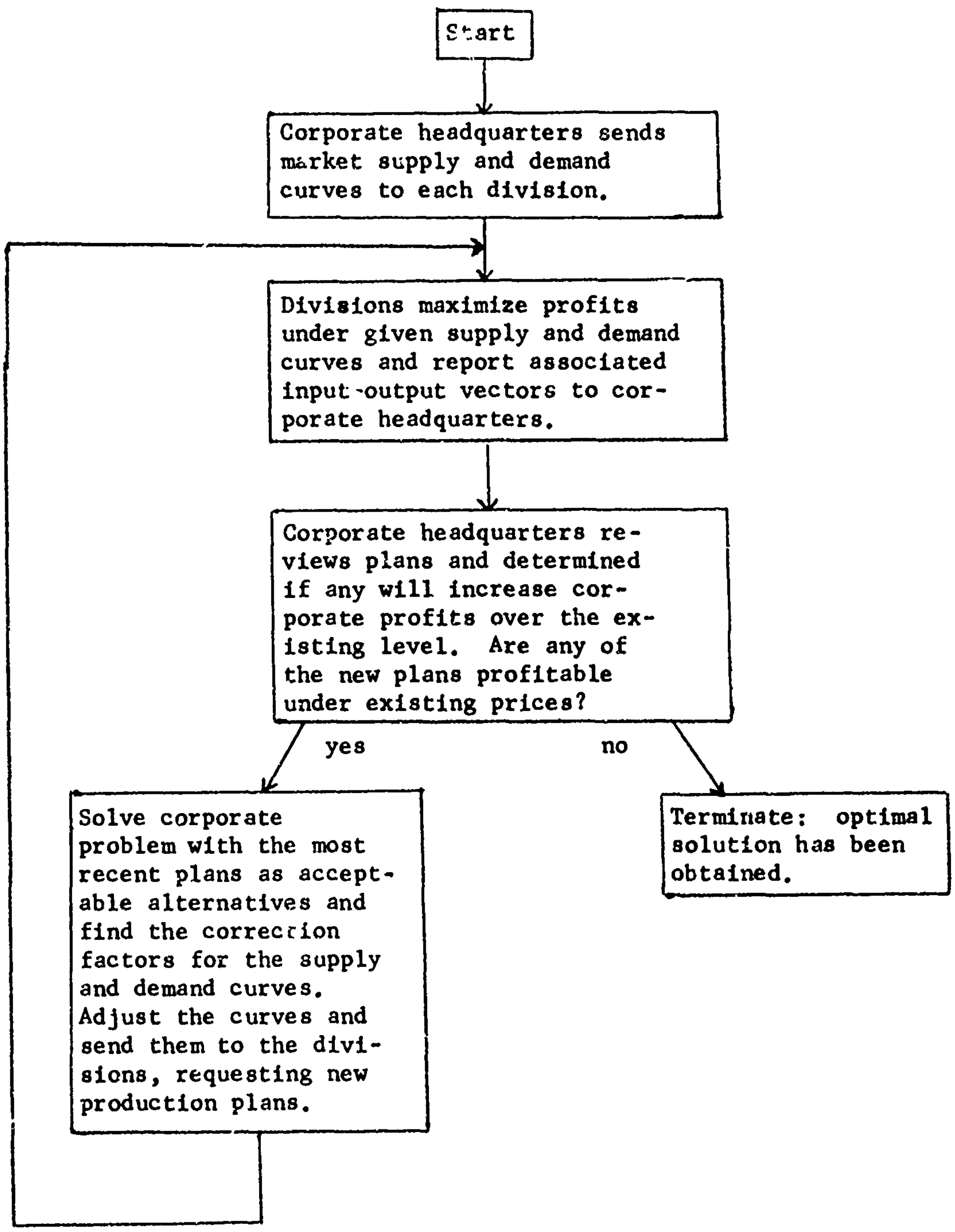
$$
\emptyset = \begin{pmatrix} \underset{(m\times n)}{\emptyset_1} & \vdots & \underset{(m\times n)}{\emptyset_3} \\ \cdots & & \cdots \\ \underset{(n\times m)}{\emptyset_3} & \vdots & \underset{(n\times n)}{\emptyset_2} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \underset{(k\times m)}{C_1} & \vdots & \underset{(k\times n)}{C_2} \end{pmatrix}
$$

The vextor P and the matrix $\emptyset_1$, together form that portion of the objective function involving only the inputs and outputs of the first division: $P'X + X'\emptyset_1 X$. The vector Q and the matrix $\emptyset_2$ do the same for the second division. The matrix $\emptyset_3$ contains the profit function interdependencies between the input-output vectors of both divisions: $2X'\emptyset_3 Y$. There are k corporate constraints involving both X and Y in a linear fashion: $C_1X + C_2Y \le R$, i.e., in producing (utilizing) one unit of x_i, an amount c_{ij} of scarce resource r_j is utilized (produced). The constraint set $f_i(X) \le s_i$, $i=1,\dots,a$ defines the production possibility set for activities X at the division one level of operations. The set $g_i(Y) \leq t_i$, $i=1,\ldots,b$ performs the same function for division two. If C and the off-diagonal blocks of $\emptyset$ are null matrices, then there are no externalities or profit function dependencies; if each division knows, or is informed of, the appropriate diagonal matrices, they can proceed to maximize their own profits, and the plans they derive will be optimal for the firm.

But let us assume that C and the off-diagonal blocks of $\emptyset$ are not null matrices, so that dependencies exist, and proceed. In order for the algorithm to operate, corporate headquarters must have knowledge of P, Q, $\emptyset$ and C. We can reflect this condition as a matter of company organization by supposing that marketing research (the function that derives the demand curves that the firm faces and the demand dependencies that exist between products) and purchasing (the function that determines the supply curves the firm faces) are both housed at corporate headquarters. In addition, the relations, C, that exist between the usage (or production) of the scarce corporate resources, R, and the values of the input-output vectors, X and Y, must be known by corporate headquarters.[8/] The first division has knowledge of the f_i set of relations and their corresponding limits, s_i; the second division has knowledge of the sets g_i and t_i.

Briefly, th algorithm consists of iterating over solutions proposed by divisions in response to demand and supply curves continually being manipulated by corporate headquarters in such a fashion as to proceed in an orderly manner toward the set of demand and supply curves that will lead each division, in their attempts to maximize their own profit, to the optimal X and Y for the firm as a whole. Figure 1 describes the algorithm in flow-chart form.

FIGURE 1


Start
Corporate headquarters sends market supply and demand curves to each division.
Divisions maximize profits under given supply and demand curves and report associated input-output vectors to corporate headquarters.
Corporate headquarters reviews plans and determined if any will increase corporate profits over the existing level. Are any of the new plans profitable under existing prices?
yes
no
Solve corporate problem with the most recent plans as acceptable alternatives and find the correction factors for the supply and demand curves. Adjust the curves and send them to the divisions, requesting new production plans.
Terminate: optimal solution has been obtained.

We initialize the algorithm by assuming the null production plans $X_o = 0$, $Y_o = 0$. Since no scarce corporate resources are being utilized and the current value of the interaction terms in the objective function is zero, corporate headquarters sends the market demand and supply curves $P + \emptyset_1 X$ to division one, and $Q + \emptyset_2 Y$ to division two. They respond by solving their respective profit-maximization problems:

$$\max \Pi_1(X) = (P + \emptyset_1 X)'X$$
$$\text{s.t.} \qquad f_i(X) \leq s_i, \quad i = 1,2,\ldots,a; \quad X \geq 0$$

and

$$\max \Pi_2(Y) = (Q + \emptyset_2 Y)'Y$$
$$\text{s.t.} \qquad g_i(Y) \leq t_i, \quad i = 1,2,\ldots,b; \quad Y \geq 0 .$$

Let us designate X_1 and Y_1 as the first responses of the underlying divisions to the demand and supply curves presented to them by corporate headquarters. By assumption, these X_1 and Y_1 values are optimal divisional responses to these initially presented curves. They are then presented as givens or known vectors of constants to corporate headquarters. The latter then solves the following problem:

$$(1) \qquad \max \Pi(U,V) = 0u_o + \sum_{i=1}^{d} \left[P'X_i + \sum_{w=1}^{d} (X_i'\emptyset_1 X_w) u_w \right] u_i + 0v_o + \sum_{h=1}^{e} \left[Q'Y_h + \sum_{\ell=1}^{e} (Y_h'\emptyset_2 Y_\ell) v_\ell \right] v_h + 2 \sum_{i=1}^{d} \sum_{h=1}^{e} u_i (X_i'\emptyset_3 Y_h) v_h$$

$$(2) \qquad \text{s.t.} \qquad \sum_{i=o}^{d} (C_1 X_i) u_i + \sum_{h=o}^{e} (C_2 Y_h) v_h \leq R$$

$$(3) \qquad \sum_{i=o}^{d} u_i \qquad\qquad = 1$$

$$(4) \qquad \sum_{h=o}^{e} v_h \qquad\qquad = 1$$

$$u_i \geq 0, \quad i = 0,1,\ldots,d$$

$$v_h \geq 0, \quad h = 0,1,\ldots,e$$

where $\emptyset_1$, $\emptyset_2$, $\emptyset_3$, C_1 and C_2 are defined above, and d and e are the number of accepted production plans by divisions one and two, respectively. In the initial iteration d and e both equal one.

The solution to this problem is a set of u's and v's which can be interpreted as weights each of the production plans X_i and Y_h receive in the optimal corporate solution as of the iteration in question. To put it another way, each division has come forth with a set of production plans, X_i, $i=1,\ldots,d$ and Y_h, $h=1,\ldots,e$; the corporate solution finds the convex combination of each set which maximizes corporate profits. The first two summations of the corporate problem are related to the profits of each plan without any interdependencies taken into account. The last summation of the problem adjusts the corporate profits for the demand and supply interdependencies. The first set of constraints relate the interdependency of divisional operations with respect to scarce corporate resources.

In addition to the optimal set of u's and v's, a by-product of the quadratic programming solution is a set of dual variables, or "shadow prices," one for each of the k+2 constraints.[9/] Let us denote the first k of them as λ_i, $i=1,\ldots,k$. The λ's can be interpreted as the marginal (or revenues) associated with the use (or production) of the scarce resources;

thus, at any particular iteration, they can be used as an internal market price for the scarce resource with which they correspond. Charging or rewarding each division the internal market prices for their usage or production of scarce corporate resources is, in effect, adjusting the external demand and supply curves for X and Y. In addition, since this is a marginal analysis, each division should be fully rewarded or charged for the current values of the dependencies it produces or imposes; these also take the form of adjustments to the supply or demand curves. Hence, at each iteration, the external market demand and supply curves for X and Y are to be adjusted as follows:

$$\text{new set of curves} = \begin{cases} P + \phi_1 X - C_1'\lambda + 2\phi_3\hat{Y}, & \text{for division 1,} \\ Q + \phi_2 Y - C_2\lambda + 2\phi_3'\hat{X}, & \text{for division 2,} \end{cases}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$. The first two terms of a new set of curves is the original market set, e.g., $P + \phi_1 X$, without the dependencies. The third term, e.g., $C_1'\lambda$, is a vector of dimension m for the first division, pricing the k scarce corporate resources at λ per unit and charging or rewarding the first division for their usage or production the next time it sets X. The last term is the current tax or bounty vector for producing or using an additional unit of each component of X or Y, given the most current and, consequently, best feasible solution at this stage, $\hat{Y}$ and $\hat{X}$. This can be attained by applying the most recent set of weights, u and v, to the X_i's and Y_h's previously generated by the divisions. It should be noted that strictly internal markets are treated as if they were perfect. Hence, all monopolistic and monopsonistic power originally inherent between divisions is eradicated. This is demonstrated in the example given in the following section.

Once the supply and demand curves for X and Y have been adjusted for the current opportunity costs of the scarce resources and for the current marginal values of the dependencies they impose or benefit by, these revised curves are sent to the divisional headquarters and another set of production plans is requested. When these plans are received by corporate headquarters, it must be determined if any of them will improve the overall profitability of operations. Let us denote these plans by X_{d+1} and Y_{e+1} and examine the accept/reject criteria for one of them, say X_{d+1}. If X_{d+1} were combined with $\hat{Y}$, the profits of such a combination would be

$$\pi(X_{d+1},\hat{Y}) = P'X_{d+1} + Q'\hat{Y} + X'_{d+1}\emptyset_1 X_{d+1} + \hat{Y}'\emptyset_2\hat{Y} + 2\,X'_{d+1}\,\emptyset_3\,\hat{Y}$$

if it were feasible. Viewing λ_i as change in profits associated with a unit change in scarce resource i ($\frac{\partial\pi}{\partial r_i} = \lambda_i$), the change in X from $\hat{X}$ to X_{d+1} implies what amounts to an increment or decrement in R by the amount $\lambda' C_1(X_{d+1} - \hat{X})$.10/ Thus, if $\lambda > 0$, an implicit change in Y from $\hat{Y}$ is anticipated, and its contribution or cost to profits is $-\lambda' C_1(X_{d+1} - \hat{X})$. Since

$$\pi_1(X_{d+1}) = P'X_{d+1} + X'_{d+1}\emptyset_1 X_{d+1} - 2X'_{d+1}\,\emptyset_3\hat{Y} - \lambda' C_1 X_{d+1},$$

it will be profitable to consider X_{d+1} if

$$\pi_1(X_{d+1}, \hat{Y}) - \lambda' C_1(X_{d+1} - \hat{X}) > \pi(\hat{X}, \hat{Y}), \text{ or if}$$

(6) $\pi_1(X_{d+1}) > D_X$

where $D_X = \pi(\hat{X},\hat{Y}) - Q'\hat{Y} - \hat{Y}'\,\emptyset_2\,\hat{Y}' - \lambda' C_1\hat{X}$

Thus, the accept/reject decision for the proposed plan X_{d+1} can be formulated:

> if (6) holds for division 1, plan X_{d+1} the plan is acceptable; if not, reject. The next plan offered by division 1 will be denoted as d+2 if the current plan is accepted; it will be denoted as d+1 if the current plan is rejected.

The decision criterion for Y_i is analogous: accept Y_{e+1} if $\pi_2(Y_{e+1}) > D_Y$

where $D_Y = \pi(\hat{X},\hat{Y}) - P'\hat{X} - \hat{X}'\phi_1\hat{X} - \lambda' C_2\hat{Y}$.

Rejection of both plans via equation (6) and its counterpart will terminate the algorithm.

With one exception the most recent demand and supply curves upon termination will lead each division independently to a joint profit optimum. The one exception is in the case of goods sold in a perfect market. Under such a condition the producing division will react to a pure price greater than zero by producing as much as is possible of that good, subject to the opportunity costs of the scarce resources under its control; i.e., the response will always be a boundary point. At the optimum the basic economic fact that marginal cost equals marginal revenue prevails. While marginal revenue is the market price plus the marginal value of the externalities, marginal cost is the opportunity costs of the scarce corporate resources, the λ's, plus the opportunity costs of producing other products demanded. If the final solution is an interior point, the algorithm will lead to a case where price, which is marginal revenue for the perfect competitor, equal marginal cost, so that the net profitability is zero. Here, then, the division must be ordered by corporate headquarters to produce the optimal amount of the product. While this may appear to be an appreciable drawback, it can be reduced by observing that with a net profit of zero, the division is indifferent with respect to the amount produced; by directing them to produce as much as is possible, given the amount of scarce corporate resources they can use, for the case of the perfectly competitive good an optimum will be obtained. When the good in question is a transferred good, treated as though its market was perfect, a directive to produce as much as can be sold to the consuming division will yield the same net effect. These ideas are illustrated in the example which is found in the next section.

It should be noted that a different, informational scheme than the one described above can be used with the algorithm to achieve an optimum. While it was most easy to have corporate headquarters known P, Q, $\emptyset$, C and R, with the divisions knowing the f_i's, s_i's, g_i's and t_i's, it may be more feasible to conceive of the divisions having knowledge of P, Q, $\emptyset_1$ and $\emptyset_2$ rather than corporate headquarters. In such a case division 1 would be required, at each iteration, to send not only its most recent X_i, but also all the terms necessary for the first line of equation (1) and the first summation of equation (2); division 2 must reply with its most recent Y_i and the necessary terms for the second line of equation (1) and the second summation of equation (2). Corporate headquarters would then be in a position to solve its problem, since it is assumed to have knowledge of $\emptyset_3$. Once this problem is solved for the currently best u's and v's (and hence, $\hat{X}$ and $\hat{Y}$) and the shadow prices, λ, are obtained, the informational flows to the divisions could simply be the vector of shadow prices and the tax or bounty vectors, $2\emptyset_3\hat{Y}$ for division 1 and $2\emptyset_3'\hat{X}$ for division 2. This scheme reallocates the source of information, but does not change the solution method of the algorithm.11/

IV. A NUMERICAL EXAMPLE

Consider the following problem:

$$\max \pi(X,Y) = 0x_1 + (4 - x_2)x_2 + (3 - 0.5y_1)y_1 + (0.5x_2 + 3 - 2y_2)y_2$$

$$\begin{array}{lrrrrl}
\text{s.t.} & x_1 & + 2x_2 & + y_1 & + 2y_2 & \leq 10 \\
 & -x_1 & & + 2y_1 & & \leq 0 \\
 & x_1 & + x_2 & & & \leq 10 \\
 & x_1 & & & & \leq 6 \\
 & & x_2 & & & \leq 6 \\
 & & & y_1 & + y_2 & \leq 5.5 \\
 & & & y_1 & & \leq 4 \\
 & & & & y_2 & \leq 4 \\
 & & & & x_1, x_2, y_1, y_2 & \geq 0
\end{array}$$

This is a simple decomposable quadratic programming problem. The first constraint requires that 10 units of scarce corporate resource be allocated between four outputs, two produced by the first division and two produced by the second division. The first constraint relates to a good which can be produced by the first division and can be sold only to the second division. It arose from the following consideration: two units of x_1 are required to produce one unit of y_1; if we constrain supply to be equal to, or greater than, demand, the original constraint reads $x_1 \geq 2y_1$, and simple algebraic manipulation yields the constraint above. Constraints three through five limit the production of x_1 and x_2 to a convex set bounded by a production possibility curve which is piece-wise linear. Constraints six through eight do the same for the second division. The good x_1 is a transfer good, having an external market price of zero; goods x_2, y_1, and y_2 are sold in imperfect markets, and the intercepts of the demand curves for these products have been adjusted to take into account the cost per unit of variable inputs purchased in perfect markets (such as labor and raw materials); in addition, good y_2 benefits from the sale of x_2, a complementary good which shifts the demand curve for y_2 to the right. Note that the producer of x_1 will always face a flat demand curve (since all adjustments change only the level of a curve without an intercept term); thus, the potential monopolistic/monopsonistic condition between division one and division two with respect to the good x_1 is forced out of existence.

According to the notation used in the previous section of this paper, the components of the objective function are

$$P = \begin{bmatrix} 0 \\ 4 \end{bmatrix} , \quad Q = \begin{bmatrix} 3 \\ 3 \end{bmatrix} , \quad \text{and } \emptyset = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0.25 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0.25 & 0 & -2 \end{bmatrix} .$$

Partitioning $\emptyset$ along the dimensions of X and Y, we have

$$\emptyset_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \emptyset_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}$$

for the intradivisional interactions in the profit function for goods X and Y, respectively, and

$$\emptyset_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}$$

for the interdivisional interactions in the profit function for goods X and Y. The matrix of technological requirements of scarce corporate resources associated with the activity levels of X and Y is

$$C = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 2 & 0 \end{bmatrix},$$

and partitioning it along the dimensions of X and Y we have

$$C_1 = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}.$$

In this example the divisional constraints are linear, so we can write them in matrix notation. Let A be the technological requirements matrix for the first division's scarce resources, S, in producing X, and we have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 10 \\ 6 \\ 6 \end{bmatrix};$$

in similar fashion for the second division's constraint set we have

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 5.5 \\ 4 \\ 4 \end{bmatrix}.$$

We must first show that $Z'\emptyset Z$ is negative, semi-definite; i.e., that $Z'\emptyset Z \leq 0$ for every Z, and there exists some $Z \neq 0$ for which $Z'\emptyset Z = 0$, where $Z = (x_1, x_2, y_1, y_2)$ in this example. For every Z it is true that

$$0x_1^2 + 3x_2^2 + 2y_1^2 + 7y_2^2 + (y_2 - x_2)^2 \geq 0 .$$

Completing the square and dividing both sides of the inequality by (-4) yields

$$-0x_1^2 - x_2^2 - 0.5y_1^2 - 2y_2^2 + 0.5x_2y_2 \leq 0,$$

which is $Z'\emptyset Z$ for the above problem. Furthermore, for $Z = (k, 0, 0, 0)$, $k \neq 0$, the above expression equals zero.

Solving the problem directly leads to the following answer:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.3453 \\ 1.7442 \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.6726 \\ 0.7468 \end{bmatrix} \text{ and } \pi = 9.3299.$$

Now let us proceed to solve the problem according to the algorithm. The null solutions are assumed to exist at no profit by corporate headquarters.

Headquarters proceeds to inform division one of the demand curves under the null solution: 0 for good x_1 and $(4 - x_2)$ for good x_2. Similarly, they inform the second division of the demand curves for goods y_1 and y_2, which are also the original market curves under the null solution: $(3 - 0.5y_1)$ for good y_1 and $(3 - 2y_2)$ for good y_2, since under the null plan the interaction term disappears. Production plans, vectors X and Y, are requested from each division.

First Iteration

(a) Division #1 Problem:

$$\begin{aligned} \max \pi_1(X) &= 0x_1 + (4 - x_2)x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 10 \\ & x_1 \le 6 \\ & x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\text{solution:} \quad X_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \pi_1(X) = 4.0$$

(b) Division #2 Problem:

$$\begin{aligned} \max \pi_2(Y) &= (3 - 0.5y_1)y_1 + (3 - 2y_2)y_2 \\ \text{s.t.} \quad & y_1 + y_2 \le 5.5 \\ & y_1 \le 4.0 \\ & y_2 \le 4.0 \\ & y_1, y_2 \ge 0 \end{aligned}$$

$$\text{solution:} \quad Y_1 = \begin{bmatrix} 3.00 \\ 0.75 \end{bmatrix}, \quad \pi_2(Y) = 5.675$$

(c) Corporate Headquarters' Operations:

Upon receipt of X_1 and Y_1 from the two divisions, accept/reject decisions must first be made. Should X_1 be accepted as a plan which will contribute to the over-all profitability of the firm? Employing equation (6) yields

$$X_1: \text{ Is } \pi_1(X_1) \geq D_X \text{ ?}$$

$$\text{Is } \quad 4.0 \geq 0 \text{ ?} \quad \text{Yes --- accept.}$$

Should Y_1 be considered?

$$Y_1: \text{ Is } \pi_2(Y_1) \geq D_Y \text{ ?}$$

$$\text{Is } \quad 5.625 \geq 0 \text{ ?} \quad \text{Yes --- accept.}$$

Employing equations (1) through (4) yields the corporate problem:

$$\max \pi(U,V) = 0u_o + \left\{ [0 \ \ 4] \begin{bmatrix} 0 \\ 2 \end{bmatrix} + [0 \ \ 2] \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u_1 \right\} u_1$$

$$+ 0v_o + \left\{ [3 \ \ 3] \begin{bmatrix} 3 \\ 0.75 \end{bmatrix} + [3 \ \ 0.75] \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 0.75 \end{bmatrix} v_1 \right\} v_1$$

$$+ 2u_1 \ [0 \ \ 2] \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 3 \\ 0.75 \end{bmatrix} v_1$$

$$\text{s.t.} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_o + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_o + \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0.75 \end{bmatrix} v_1 \leq \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$u_o + u_1 = 1$$

$$v_o + v_1 = 1$$

$$u_o, u_1, v_o, v_1 \geq 0$$

$$\text{solution: } U = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \pi(U,V) = 4.0,$$

$$\lambda_1 = 0, \quad \text{and} \quad \lambda_2 = 2 \ .$$

Thus, at the end of the first iteration, since none of the transferred good is produced, the optimal production plan is to produce two units of good x_2. At this level of production not all ten units of the scarce corporate resource are utilized, so the current opportunity cost of an additional unit is zero. On the other hand, the marginal value of an additional unit of x_1 is 2.[12] Noting that the current optimal values of X and Y are

$$\hat{X} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \hat{Y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

new demand curves are calculated by corporate headquarters according to equation (5):

$$X:\quad P + \emptyset_1 X - C_1'\lambda + 2\emptyset_3\hat{Y}$$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 - x_2 \end{bmatrix}$$

$$Y:\quad Q + \emptyset_2 Y - C_2'\lambda + 2\emptyset_3\hat{X}$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 - 0.5y_1 \\ 4 - 2y_2 \end{bmatrix}$$

Corporate headquarters informs each division of these curves and requests a production plan from each.

Second Iteration

(a) Division #1 Problem:

$$\max \pi_1(X) = 2x_1 + (4 - x_2)x_2$$

$$\begin{aligned} \text{s.t.}\quad & x_1 + + x_2 \leq 10 \\ & x_1 \leq 6 \\ & x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

solution: $X_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $\pi_1(X) = 16$

(b) Division #2 Problem:

$$\max \pi_2(Y) = (-1 - 0.5y_1)y_1 + (4 - 2y_2)y_2$$

$$\text{s.t.}\quad \begin{aligned} y_1 \quad\quad + y_2 &\leq 5.5 \\ y_1 \quad\quad\quad &\leq 4.0 \\ y_2 &\leq 4.0 \\ y_1, y_2 &\geq 0 \end{aligned}$$

solution: $Y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\pi_2(Y_2) = 2$

(c) Corporate Headquarters' Operations:

Upon receipt of X_2 and Y_2, the accept/reject decisions are made:

X_2: Is $\pi_1(X_2) \geq D_X$?

Is 16 $\geq$ 4 ? Yes --- accept.

Y_2: Is $\pi_2(Y_2) \geq D_Y$?

Is 2 $\geq$ 0 ? Yes --- accept.

Accepting both proposals and employing equations (1) through (6) yields the new corporate problem:

$$\begin{aligned} \max\ \pi(U,V) = {} & 0u_o + 8u_1 - 4u_1^2 - 8u_1u_2 - 4u_2^2 \\ & + 0v_o + 11.25v_1 - 5.625v_1^2 - 3v_1v_2 + 3v_2^2 \\ & - 2v_2^2 + 0.75u_1v_1 + u_1v_2 + 0.75u_2v_1 + u_2v_2 \end{aligned}$$

$$\text{s.t.}\quad \begin{aligned} 0u_o + 4u_1 + 10u_2 + 0v_o + 4.5v_1 + 2v_2 &\leq 10 \\ 0u_o + 0u_1 - 6u_2 + 0v_o + 6v_1 - 0v_2 &\leq 0 \\ u_o + u_1 + u_2 &= 1 \\ v_o + v_1 + v_2 &= 1 \\ u_o, u_1, u_2, v_o, v_1, v_2 &\geq 0 \end{aligned}$$

solution: $U = \begin{bmatrix} 0.127877 \\ 0.314578 \\ 0.557545 \end{bmatrix}$, $V = \begin{bmatrix} 0.113811 \\ 0.557545 \\ 0.328645 \end{bmatrix}$,

$$\lambda = \begin{bmatrix} 0.442455 \\ 0.442455 \end{bmatrix} \quad , \quad \Pi(U,V) = 9.3299 \; .$$

At the end of the second iteration, profits for the corporation are at a level of 9.3299 under a new weighted average of the production plans from divisions #1 and #2. Under this newest plan, the marginal value of the scarce corporate resource and an additional unit of x_1 are both 0.442455. The currently optimal X and Y are

$$\hat{X} = \begin{bmatrix} 3.3453 \\ 1.7442 \end{bmatrix} \quad \text{and} \quad \hat{Y} = \begin{bmatrix} 1.6726 \\ 0.7468 \end{bmatrix} .$$

Using the newest values of λ, Y and Y in equation (5), a new set of demand curves are derived:

$$X: \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0.442455 \\ 0.442455 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1.6726 \\ 0.7468 \end{bmatrix}$$

or

$$\begin{bmatrix} 0x_1 \\ 3.488492 - x_2 \end{bmatrix}$$

$$Y: \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0.442455 \\ 0.442455 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 3.3453 \\ 1.7442 \end{bmatrix}$$

or

$$\begin{bmatrix} 1.672635 - 0.5y_1 \\ 2.989213 - 2y_2 \end{bmatrix}$$

Corporate headquarters informs each division of their respective demand curves and requests production plans in response to them.

<u>Third Iteration</u>

(a) Division #1 Problem:

$$\max \Pi_1(X) = 0x_1 + (3.488492 - x_2)x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 \quad + x_2 &\le 10 \\ x_1 \quad &\le 6 \\ x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

solution: $X_3 = \begin{bmatrix} 0 \\ 1.74425 \end{bmatrix}$, $\pi_1(X) = 3.0424$

(b) Division #2 Problem:

$$\max \pi_2(Y) = (1.672635 - 0.5y_1)y_1 + (2.987213 - 2y_2)y_2$$

$$\text{s.t.} \quad \begin{aligned} y_1 \quad + y_2 &\le 5.5 \\ y_1 \quad &\le 4.0 \\ y_2 &\le 4.0 \\ y_1, y_2 &\ge 0 \end{aligned}$$

solution: $Y_3 = \begin{bmatrix} 1.672635 \\ 0.746803 \end{bmatrix}$, $\pi_2(Y) = 2.5159$

(c) Corporate Headquarters' Operations:

Upon receipt of X_3 and Y_3, the accept/reject decisions are made:

X_3: Is $\pi_1(X_3) \ge D_X$?

Is $3.0424 \ge 3.0424$? No --- reject.

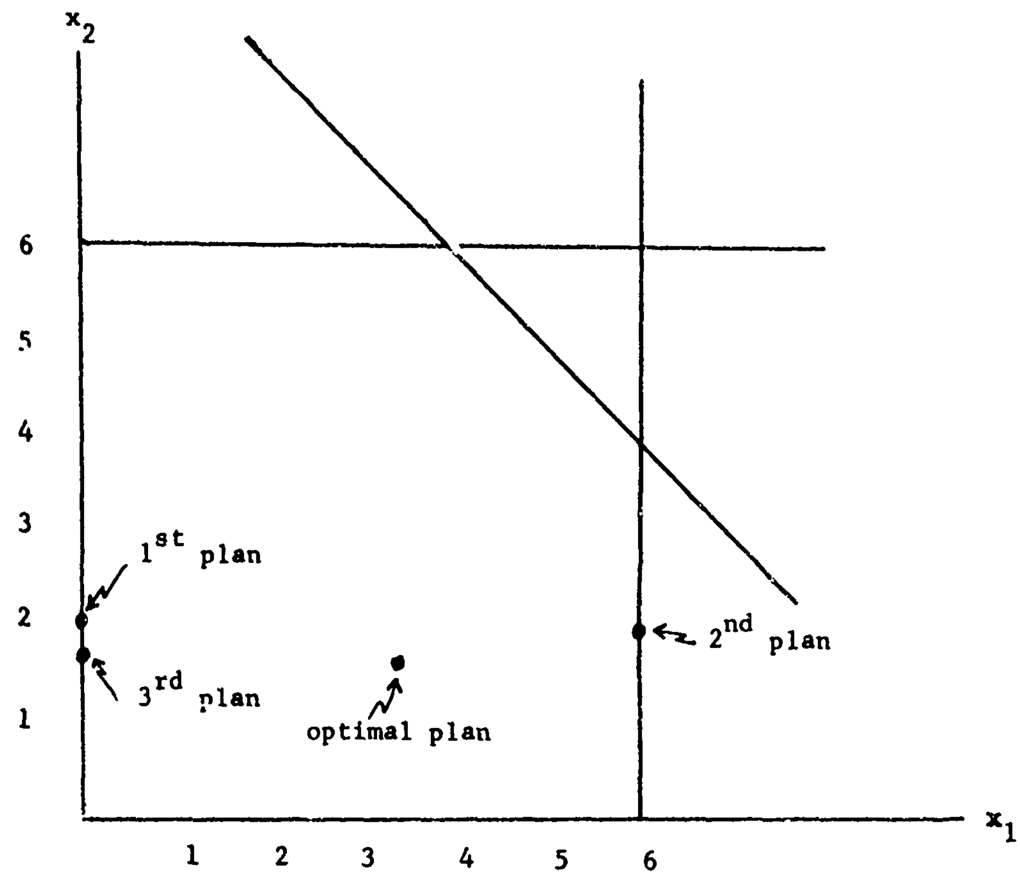
Y_3: Is $\pi_2(Y_3) \ge D_Y$?

Is $2.5159 \ge 2.5159$? No --- reject.

Corporate headquarters terminates at this point; the optimum has been attained.

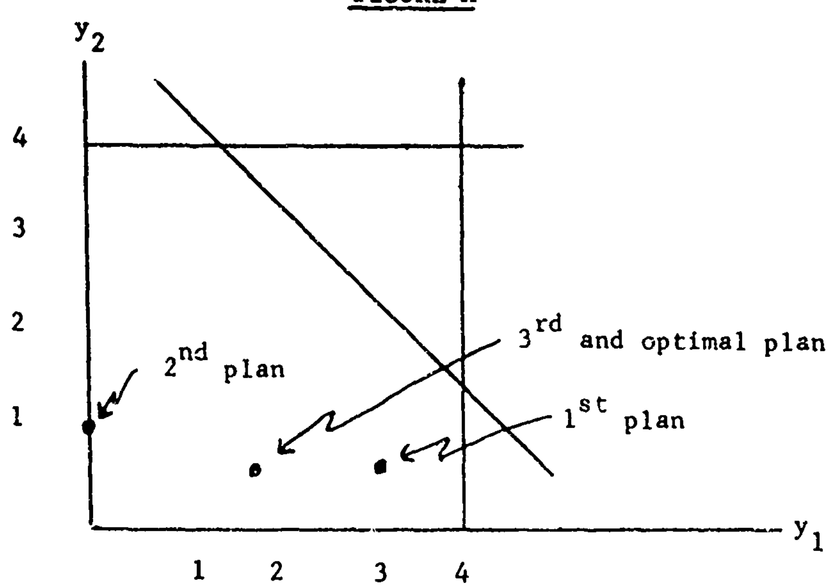
The solutions X_3 and Y_3 are optimal with the exception that in X_3, $x_1 = 0$; i.e., headquarters has derived the optimal demand curves for the goods x_2, y_1, and y_2 (or optimal transfer prices for the ten units of scarce corporate resource and x_1, and the optimal bounty per unit for goods x_2 and y_2 required to compensate for the externality between these two goods). Thus they could inform the divisions that the last demand curves conveyed are to be taken as final and the divisions should produce accordingly, with the first division producing, in addition, as much x_1 as the second division demands; it should be willing to do so since such production involves no loss (selling x_1 at a price of 0.442455 to division #2, but requiring one unit of the scarce corporate resource at price 0.442455 for each x_1 produced) and utilizes excess capacity.

Figures A and B sketch out the convergence process for divisions #1 and #2, respectively.



First Division

FIGURE A



Second Division

FIGURE B

V. SOME GENERAL COMMENTS ON FEASIBILITY.

A few comments are in order with respect to the feasibility of application of the algorithm to a real-time situation and how it compares in such a situation to the Danzig-Wolfe linear decomposition algorithm and Whinston's quadratic decomposition algorithm. These comments can be divided into two broad categories:

A. Methodology:

1. In a real-time situation the algorithm can be conceptualized as an on-going process, never terminating. Changes in technology, market conditions, and resource limitations or policies, continually change the underlying problem; thus, divisional plans generated more than a few iterations ago would not be the same as they would had they been generated under existing conditions. But this does not constitute an insurmountable drawback, for once the transfer price is within the range of the optimal transfer price under existing conditions, near optimal solutions to the existing problem will be generated by the corporate problem. These can be used as temporary production plans while another iteration is performed. The corporate problem can always be updated; the only fear is that production plans generated more than a few iterations prior are no longer feasible. Hence, they should simply be dropped as variables from the corporate problem. If the aforementioned changes are not extreme at any point, the algorithm should still track the optimal transfer prices

and, therefore, the optimal solution quite well.

2. When the algorithm is first applied, if a "guesstimate" of the transfer prices is made by corporate headquarters at the first iteration, the number of iterations required to come within reasonable bounds of the optimum is reduced if the guesstimates are anywhere "in the ballpark."

3. In its strictest sense the algorithm requires corporate headquarters to consider only one divisional production plan at a time. There seems, however, to be no loss, and perhaps even some gain, in reducing the number of iterations required for convergence, by considering as many new divisional production plans as possible (only one per division, of course) when entering new variables into the corporate problem.

4. This algorithm, unlike Whinston's, does not conflict with a profit-center accounting scheme. Each division still makes profits; although they pay the economically appropriate prices (opportunity costs) for scarce goods, they still receive the full benefit of their non-competitive power in the market and the externalities they generate.[13/] Corporate headquarters also makes a "profit" by selling the scarce resources to the divisions for not the highest price that it can (since they are in a monopolist position with respect to the divisions), but rather at a price which will equate their supply to the divisional demands. While it is true that the sum of divisional and headquarter profits does not equal the profits for the firm as a whole under this scheme, the

discrepancy is due entirely to the multiple counting of the externality benefits; e.g., in the example of the previous section, at the optimum $\pi_1 = 3.0424$, $\pi_2 = 2.5159$, and corporate headquarters sold ten units of scarce resource at 0.442455 per unit, yielding a profit of 4.4246, and the sum of these three profits is 9.9829. On the other hand, $\pi = 9.3299$, so that the difference is 0.6530, which is previsely the value of the externality, $(0.5)\hat{x}_2 \hat{y}_2 = (0.5)(1.74425)(0.746803) = 0.6530$, since both divisions are receiving this amount fo the externality. Hence, aside from the value of externalities, divisional profits can be looked upon as the profit contribution of each division to the profits of the firm as a whole. Thus, separate books might be kept for internal management accounting; how the externality is to be viewed for the firm as a whole is individual preference.

B. Behavioral

1. Both the linear decomposition algorithm and Whinston's quadratic decomposition algorithm manipulate pure prices rather than demand and supply curves in the process of arriving at an optimal solution. Hence, the greatest behavioral hindrance for both these algorithms lies in the fact that the best solution to date (or optimal solution, if attained) might require production inside the boundary defined by the divisional constraint sets, i. e., an interior point of the

production possibility set -- a point which cannot be reached by pure price manipulation alone -- a point which is obviously "non-optimal" for the local decision-making managers. On the other hand, under the condition that all inputs and outputs are bought and sold in imperfect markets, the quadratic decomposition algorithm discussed here will supply final demand and supply curves which lead each division independently to the appropriate price setting and production schedule for joint profit-maximization; the optimal transfer prices and credits for demand and supply externalities, when given to the divisional decision-makers, will adjust the original demand and supply curves in such a way that the optimal solution can be attained without corporate headquarters ever having to order a division to a point interior to its production possibility curve. When goods are purchased or sold in perfect markets (transferred goods are treated as such), production plans will always be boundary points since the response to any pure price is a boundary point. In this case, the amount of these goods to be bought and sold must be specified by the central planners if the optimal solution point is an interior point.

With this algorithm there is, consequently, no behavioral "block" in the sense of obvious non-optimal decisions. Given the final pricing structure, each manager will produce the optimal output of his own, profit-maximizing, accord; since he is indifferent about outputs with a net profitability of zero, he will be willing to produce as much as he can sell to utilize capacity; the example in the previous section illustrates this.

2. As noted in point 4 above, at each iteration full benefit of the current value of any externality is given to each producer associated with that externality. It is believed that there exists no alternative method of distributing the externality values which is rationally more appealing. While such a method will lead to an accounting problem as described above, the behavioral gain through profit-center accounting, with corporate headquarters absorbing the gain or loss due to multiple counting of externality values, may far exceed the imperfect accounting mechanism.

3. While it is true that once the optimum X and Y is known, one can easily find sets (there exists an infinite number of them) of demand and supply curves that will lead each division to these points, if one were to use an algorithm like Whinston's to generate the optimum X and Y, i.e., find them through pure price manipulation, some behavioral problems might arise when the optimum is reached, for then a different pricing mechanism would be introduced.

4. As Whinston [16] has pointed out, care must be taken if algorithms proceeding along the lines of the one described in this paper are utilized. There exists a definite opportunity for "gaming" the system, for divisional decision-makers to

deliberately misspecify their output vectors in hopes of gaining control of more scarce corporate resources and, consequently, increasing their own profits at the expense of over-all profitability. Perhaps some auditing procedure directed at decision-making could serve to minimize the possibility of such conduct arising.

5. It should finally be noted that while the profit-center concept introduced above has desirable accounting and behavioral characteristics, decisions to (dis)invest further in any division should not be made according to the profits attributable to them under the optimal transfer prices, but rather by incremental cash flow analysis.

FOOTNOTES

1. See, e.g., [1], [2], [3], [6], [7], [12], [13], [14]. Gordon [9] puts puts some of these references into proper perspective.

2. This statement is not meant to imply that there are no other deficiencies. The most obvious would seem to be that the interdivisional constraints must still be linear. The quadratic model, however, introduces no deficiencies absent in the linear model, except, perhaps, computational difficulties. No comparisons can be made with respect to speed of convergence or sensitivity since computational experience is limited for both algorithms.

3. See Charnes and Cooper [3, p. 291] or any standard economic price theory text, such as Cohen and Cyert [4,pp. 274-75].

4. Cohen and Cyert [4, p. 190].

5. The algorithm which follows is, in many respects, similar to the one described by Whinston [16] and Baumol and Fabian [2]. It is described for the two-division case for simplicity; extension to the n-division case is not conceptually different.

6. If $Z'\emptyset Z$ is negative semi-definite, it may be perturbed as in [3, p. 687] into negative definite form without influencing the numerical answer. While $\emptyset$ is required to be negative definite at least in principle, it has also been observed that the method generally works, even without perturbation, when $Z'\emptyset Z$ is semi-definite. Consequently, the usual procedure is to try it without first attempting to perturb the matrix. See the numerical problem solved in the next section.

7. The relations f_i and g_i are required to be convex in order to assure convergence. Note, however, that they need not be linear. Hence, this algorithm is applicable to problems which are not strictly defined as quadratic programming problems.

8. A word of explanation is required with respect to the set of relations $CZ \leq R$. Few of the corporate scarce resources are fixed, i.e., independent of the values of X and Y. X and Y usually generate and use resources, e.g., working capital or intermediate products, thus adding to or subtracting from the stock at the beginning of the period that planning covers. Thus, some of the elements of C would be negative, others positive.

9. Hadley [10] describes the most widely-used technique for solving quadratic programming problems and has a clear discussion of the dual problem and its associated dual variables. The IBM Share Library has an efficient code for this method, RS-QPF4, which has all the necessary output.

10. See Hass [11] for the proof of this statement and a formal convergence proof.

11. I am indebted to Bart McGuire from the University of California, Berkeley, for bringing this to my attention. This interpretation then has much of the same type of informational flows as linear decomposition, in which corporate headquarters knows only the net profits of the various proposed production plans and the amount of scarce resource each plan uses, but not the product mix of any plan. The product mix is required here due to the externalities, $\emptyset_3$. If this matrix is null, then no mix is required.

12. Shadow prices in quadratic programming are valid only at the margin due to the non-linearities involved. To show that, at the margin, the value of an additional unit is 2, suppose we had an ϵ amount of x_1 available; then we could let $v_1 = \epsilon/6$. The objective function of the corporate problem can be rewritten as

$$\max \pi = 0u_o + 8u_1 - 4u_1^2 + 0v_o + 11.25v_1 - 5.625v_1^2 + 0.75u_1v_1.$$

with $u_1 = 1$ and $v_1 + \epsilon/6$, ignoring the higher order terms in ϵ, $\pi = 4 + 2\epsilon$. We can also convert this to the original problem: with currently optimal $X' = (0 \quad 2)$ and $Y' = (0.5\epsilon \quad 0.125\epsilon)$, we can substitute these values into the objective function. Ignoring the higher order terms in ϵ, the value of the function is $4 + 2\epsilon$.

13. Shubik [15] approaches the profit-center problem from a different direction (game theory), but has very similar economic rationale.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Graduate School of Industrial Administration Carnegie Institute of Technology	2a REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP Not applicable

3 REPORT TITLE

TRANSFER PRICING IN A DECENTRALIZED FIRM: A DECOMPOSITION ALGORITHM FOR QUADRATIC PROGRAMMING

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

Hass, Jerome E.

6 REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May, 1967	39	18

8a CONTRACT OR GRANT NO. NONR 760(24) b PROJECT NO. NR 047-048 c d.	9a. ORIGINATOR'S REPORT NUMBER(S) Management Sciences Research Report No. 93 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) none

10 AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D. C. 20360

13. ABSTRACT

A discussion of the importance of transfer pricing and a review of the relevant literature are followed by a decomposition algorithm for quadratic programming, using the problem of transfer pricing with externalities as a vehicle for its development. A numerical example is given and comments are made of the applicability and behavioral implications of adoption in a real-time context.

DD FORM 1 JAN 64 1473

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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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