FTD-MT-64-390

AD 650520

TT67-61575

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COURSE IN SPHEROIDAL GEODESY

By

G. V. Bagratuni

FOREIGN TECHNOLOGY DIVISION

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FTD-MT- [illegible]

EDITED MACHINE TRANSLATION

COURSE IN SPHEROIDAL GEODESY

By: G. V. Bagratuni

English Pages: 303

TM [illegible]

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FTD-MT- [illegible]

Date 15 Feb. 19 67

Edited By: [illegible]

G. V. Bagratuni

KURS SFEROIDICHESKOY GEODEZII

Dopushcheno
Ministerstvom vysshego i srednego
spetsial'nogo obrazovaniya RSFSR
v kachestve uchebnika dlya studentov
astronomo-geodezicheskikh spetsial'nostey

Izdatel'stvo geodezicheskoy literatury

Moskva - 1962

Pages 1-252

CIRC A[illegible]RACT WORK SHEET

(01)Acc Nr.	(65)SIS Acc Nr.	([illegible]) Country of Info	(41)Translation Nr.
TM6503073		UR	MT6400390

(42)Author	Priority / Distribution
BAGRATUNI, G. V.	II STD

(43)Source

KURS SFEROIDICHESKOY GEODEZII

(02)Ctry	(03)Ref	(04)Yr	(05)Vol	(06)Iss	(07)B. Pg	(45)E. Pg	(73)Date	(47)Subject Code
UR	0000	62	000	000	0001	0252	NONE	08

Language			
RUSS	n/a	Moscow	GEODEZIZDAT

(39)Topic Tags geodesy, astronomic geodesics, education institute, training aid

(66)Foreign Title

KURS SFEROIDICHESKOY GEODEZII

(09)English Title COURSE OF SPHEROIDAL GEODESY

(97) Header Clas	(63) Clas	(64)Rel	(60) Release Expansion
0	00	0	

ABSTRACT: This textbook covers the materials taught on spheroidal geodesy for fourth-year geodesy students at Soviet colleges and also serves as a guide for post-graduate students and practising geodetic engineers. Ellipsoidal curves are considered and the theory of the geodetic triangle on the surface of an ellipsoid is explained. The calculation of geodetic coordinates is covered and an entire chapter is devoted to the solution of long-distance geodetic problems. There are chapters on problems of representing an ellipsoid on a sphere and planes, geodetic projections of an ellipsoid on a plane, and problems on the surface of the terrestrial ellipsoid. English translation: 302 pages.

FTD FORM SEP 64 0-493d

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ë in Russian, transliterate as yë or ë. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

PREFACE

In accordance with the new educational plan for astronomic-geodetic specialty spheroidal geodesy is studied in the IV course of geodetic colleges in USSR for 7 hours a week during the entire scholastic year. Independent setting of this department of higher geodesy has as an aim, on one hand, to give future engineers the necessary knowledge for treatment of results of geodetic measurements of the spheroid and, on the other, to prepare them for study of theoretical geodesy, mathematical cartography and theory of the figure of the Earth.

Till now in USSR there was no special textbook on spheroidal geodesy. The work of professor N. A. Urmayev "Spheroidal geodesy" (1955), being a scientific treatise, contains mainly results of his research on this subject and does not embrace all problems of the course program. Second part of the fundamental labor of F. N. Krasovskiy "Guide to Higher Geodesy" (1942), which up to now was recommended as a textbook and where spheroidal geodesy for a period of 1942 is presented with sufficient fullness has significantly become obsolete in certain parts. Furthermore, the work of F. N. Krasovskiy, in the contemporary understanding can not be considered as a textbook. This scientific guide, is intended not only for students and post graduates, but also for engineers-geodesists working on large astronomic-geodetic nets, and for beginner scientists.

The offered textbook embraces all questions of the course program on spheroidal geodesy, where in many cases presentation exceeds the bounds of program requirements. Such approach should be considered as fully acceptable, since majority of the students after mastering the course wish to study the problems deeper and to become wider acquainted with the direction of the development of scientific thought in the area of spheroidal geodesy.

Author held, as a rule, to analytic method of presentation, the geometric approach is used for clarity of discourse and interpretation of complex analytic relationships. The classical mathematical apparatus is used. However, in order that cumbersome artificial transformations and reckonings would not over shadow the fundamental ideas and dependencies, non-fundamental details of derivations of certain formulas and expressions in a number of cases were omitted. Along with this an attempt is made to improve accepted till now symbolism.

Essentially the content of the textbook will indicate the following.

1. The chapter on ellipsoid curves is substantially expanded. Here for the first time in our educational literature is presented a resolution of geodetic problems with the help of normal sections and chords of ellipsoid. In connection with this the study about normal sections and chords of ellipsoid are expounded with considerable fullness. Teaching on geodesic and their application to resolution of problems of spheroidal geodesy occupies substantial place in the textbook. It is shown that application of geodesic in the resolution of geodetic problems has definite advantages as compared to application of other curves on the surface of the ellipsoid.

2. The theory of geodetic triangle on the surface of an ellipsoid is presented according to Gauss work: "Investigation of Curved Surfaces".

3. On the basis of results of investigations of the author and other scientists the chapter on calculation of geodetic coordinates is considerably expanded. For the first time the methods of nonlogarithmic calculations of geodetic coordinates and the method of chords by M. S. Molodenskiy are presented.

4. The resolution of geodetic problems for long distances is presented in a completely new fashion. Instead of one paragraph, devoted to this question in former courses, this textbook has a whole chapter with account of basic methods of resolution of direct and inverse geodetic problems for long distances.

. The problems of representation of ellipsoid on a sphere and planes are presented in a single plan on a basis of equations of Kosal - Riemann type, obtained by the author from general equations by Gauss.

. An important place is held by the description of geodetic projections of ellipsoid on a plane. In this section comparative evaluation of the most widely used geodetic projections is given.

7. In connection with the development of radar and speed of light geodetic

[illegible]

measurements a necessity arose for the resolution of a number of problems on the surface of terrestrial ellipsoid. Parallel with this work the Chair of Higher Geodesy MIIGAIK, candidate of Tech. Sciences V. A. Polevoy worked on composition of a training aid "Mathematical Treatment of Radargeodetic Measurements", which were already published.[1] Therefore to avoid parallelism in this textbook, the problems of treatment of radargeodetic measurements are not shown.

8. In order not to overload the textbook with examples of calculations, the more model and universal of them are referred to the "Practicum" of professor B. N. Rabinovich.[2] But nonmodel examples are placed in corresponding places after presentation of the theory of a given problem.

The author attempts in presentation of key basic concepts to avoid "mathematical ballast", which submerges the essence. How well he succeeded it is difficult for the author to judge. However he earnestly hopes for great help and friendly criticism from geodetic society; such help became a tradition in our Soviet activities.

Of great help to the author in preparation of the manuscript for publication was rendered by assistants of the Chair B. F. Khitrov, V. A. Romanovskiy and A. N. Solov'yev. Translation of foreign literature and a check of foreign texts and names were carried out by senior teacher of the Chair of Foreign Languages MIIGAIK G. I. Zalesskaya.

The author obtained much valuable advice and recommendations on the manuscript from Asst. Professors A. I. Vitman, A. V. Butkevich and A. A. Vizgin.

I consider it my pleasant duty to express to enumerated comrades my deep gratitude for their help in my work, especially professor P. S. Zakatov, whose very valuable remarks rendered great service to author during final editing of the manuscript.

G. V. Bagratuni

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[1] V. A. Polevoy, Mathematical Treatment of Radargeodetic Measurements. M., Geodezizdat, 1961.

[2] B. N. Rabinovich, Practicum on higher geodesy. M., Geodezizdat, 1961.

CHAPTER I

INTRODUCTION

§ 1. THE OBJECT AND PROBLEMS OF SPHEROIDAL GEODESY

Higher geodesy is a science about the figure of the Earth. The main scientific problem of the higher geodesy consists of determination of the size and shape of the Earth; this problem is resolved by means of establishment of a typical mathematical figure which would geometrically present the Earth on the whole and the study of deviations from the real form of the Earth from a fixed mathematical figure. Such figure is a rotating ellipsoid with small polar compression also called a spheroid. The term "spheroidal geodesy" is derived hence.

Spheroidal geodesy is a study of the geometry of terrestrial ellipsoid and representation of important parts of its surface on a sphere and on a plane.

All geodetic measurements are made on the physical surface of the Earth, then for strict mathematical treatment the results are projected on the surface of adopted reference-ellipsoid. The ellipsoid, oriented on the body of the Earth, in a determined way on whose surface are projected the results of geodetic measurements and on which coordinates of geodetic points, are determined, is called the reference-ellipsoid. Frequently the surface of reference-ellipsoid is called the surface of relativity. In order that the surface of the reference-ellipsoid would be disposed as nearly as possible to the surface of the Earth within the limits of a given area, it is necessary that its major semiaxis and polar compression be obtained from the results of geodetic gravimetric, and astronomical measurements, carried out in this area.

When it is spoken in higher geodesy about the surface of the Earth, visible physical surface is not implied, but a sea level surface at every point of which a plumb line coincides with the normal. This condition satisfies infinite number of sea level surfaces. In higher geodesy that surface is considered which coincides with the surface of the world ocean in a state of complete equilibrium of the water masses contained in it and, consequently, not disturbed by tides, ebbs, winds, currents and other factors. If this surface is hypothetically extended through the mainlands in such a manner that the plumb lines remain normal to it, everywhere then we will obtain a closed, continuous without folds and ridges, even surface, which is called the datum of the surface of the Earth. Geometric figure, limited by this surface, is called the geoid. Thus, terms "surface of the geoid" and "datum of the surface of the Earth" have identical meaning.

In order to present the geoid on the whole, an idea is introduced in higher geodesy about the general terrestrial ellipsoid, determined by the following characteristics:

1. The volume of the ellipsoid is equal to the volume of the geoid. 2. The center of gravity and the plane of the equator of the ellipsoid coincide with the center of gravity and the plane of the equator of the Earth. 3. The sum of the squares of deflections of the geoid from ellipsoid should be minimum in height.

The problem of determination of the size and shape of general terrestrial ellipsoid enters into natural-science problem of study of the Earth as a planet and can be rigidly solved by joint use of data of geodesy, gravimetry, astronomy, geophysics, geology and other related sciences obtained for all the surface of the Earth.

Projection of the results of the geodetic measurements on the surface of the reference ellipsoid is a complex physical and mathematical problem, which is studied in the theoretical part of the higher geodesy. In spheroidal geodesy it is assumed that the results of the geodetic measurements are rigidly projected on the surface of the reference-ellipsoid and geodetic problems are resolved as if all the measurements are performed directly on the surface of the reference-ellipsoid.

§ 2. DEVELOPMENT OF KNOWLEDGE ABOUT THE MATHEMATICAL FIGURE OF THE EARTH

Contemporary views on the figure of the Earth take their beginnings from I. Newton, who for the first time had, on a basis of the law of universal gravitation expressed as a thought that geometric figure of the Earth is the result of action of two forces, the force of terrestrial attraction and of centrifugal force. Thus, purely geometric approach to the question of determination of the figure of the earth, existing prior to Newton, was put to an end. Considering the Earth as a uniform body, in which all particles are mutually attracted, and taking ratio of centrifugal force to the force of gravity on the equator as equal to 1:289, Newton obtained a value of 1:2[illegible] for compression of the earth (1686). Besides, as he noted, this value should decrease, if the density of the masses increases toward the center.

A contemporary of Newton, Dutch scientist Kh. Gyuygens, considering attraction of the Earth not from separate particles of her mass, as follows from the law of universal gravitation, but from the center and taking this for ratio of centrifugal force to gravity on equator received the very same number as Newton had obtained for compression of the earth 1:578 (1688), that is half of its actual value.

Thus, at the end of 17th Century without any direct measurements on the Earth's surface, two extreme limits for the compression of the Earth were obtained. Meanwhile, the real compression of the Earth could only be determined from materials of direct geodetic measurements. The French Academy of Sciences, founded in 1666, undertook such measurements under the leadership of the famous astronomer G. Picard in 1669. Although the measurements of Picard were the first in this direction, before their fulfillment numerous and very important for that time inventions and instruments, such as for instance, pendular and spring timepieces telescopes provided with crosshairs microscopes, cylindrical levels, and verniers etc. were already utilized. Picard considerably improved the methods of triangulation, originally proposed by the Dutch scientist Snellius in 1615.

Results of measurements of Picard and his pupils, published in 1720 by the French Academician G. Cassini, showed that within limits of France the length of arc of a degree on a meridian decreases to the north, as if it testified not about compression of the Earth at the poles, but of prolateness.

This contradiction was brought forward in the beginning by Cassini himself and then successors as refutation of the theories of Newton and Huygens, since actual measurements were considered very precise. However it was established that the error of the measurements themselves was so great for such short distances that they wholly can cover the influence of compression of the Earth. For clarification of this and the evaluation of accuracy of measurements of Picard new measurements were required, they were undertaken by the French Academy of Sciences in 1735-1743. Two arcs were measured near the Equator, in Peru, $3^{o}7'$ long and in the north of Norway, in Lapland, 1^{o} long. Results of these measurements confirmed the correctness of the theory of

Newton and simultaneously indirectly showed that the Earth is a heterogeneous body, since compression at near Equator measurements was obtained equal to 1:314, and near the poles it was 1:214.

Thus, one of the major natural-science problems about Earth, namely the determination of its size and shape, was solved in 18th century by the results of geodetic measurements. French measurements laid foundations for degree measurements along the meridian, which began to be rapidly developed from the end 18th Century in many European states. Somewhat later, with the invention of the telegraph, degree measurements along the parallels began.

Eighteenth century is also famous for still other facts, in the history of geodesy to purely geodetic method of determination of compression of the Earth were added other methods founded on theoretical positions of celestial mechanics and other sciences. The famous A. Clerot member of the French Academy of Sciences and participant of the Laplandian degree measurement, obtained an equation in 1743, which showed that with the aid of a difference of gravity at the Equator and the Pole it is possible to calculate compression of the Earth. Delambre investigated dependency between the figure and distribution of Earth masses attracted by the Moon and the Sun. LaPlace at the end 18th Century found periodic terms in equation of the motions of the Moon, which are conditioned by the shape of the Earth and distribution of masses within it.

In the second part of the celestial mechanics LaPlace on the basis of the theory of Moon's motion and results of measurements of the force of gravity obtained a value for the compression of the Earth, approximately equal to 1:300.

LaPlace simultaneously indicated that actual mathematical figure of the Earth cannot exactly coincide with the prolate spheroid. He made this conclusion on the basis of material of triangulation, at which deviations of the plumb lines were revealed, far exceeding the errors of measurements. This served as a reason for the derivation of the well known LaPlace equation, giving difference of geodetic and astronomical azimuths.

In first half of the 19th Century several attempts were made to obtain from the triangulation material the value of a major semiaxis and compression of terrestrial ellipsoid. The most essential contribution in this was made by the greatest German astronomer and geodesist F. V. Bessel (1784-1846). In 1841 on the basis of a thorough treatment of triangulation material by a method of least squares Bessel obtained values for major semiaxis of a = 6377397, and for compression α = 1:299.15. For his derivation Bessel used the European degree measurements of the general extent of about 5?°, where the greater weight in his treatment was given the part of the triangulation, carried out under direction of the great Russian astronomer-geodesist V. Ya. Struve (1793-1864).

Due to great scientific authority of Bessel, his ellipsoid was used in geodetic work almost everywhere. Even now Bessel ellipsoid is used as a reference-ellipsoid in certain European countries. Till 1941 Bessel ellipsoid was also used as a reference-ellipsoid in USSR. Investigations of F. N. Krasovskiy (1878-1948) showed that Bessel major semiaxis for area of USSR is approximately 850 m less. However the value of compression of his ellipsoid even now is considered one of best.

Work next in importance in this area is that of a well known English geodesist A. Clarke (1828-1914), author of work "Geodesy", translated into Russian by V. V. Vitkovskiy in 1890, Clarke twice, in 1866 and 1880, developed an ellipsoid from European and Indian triangulation. He used material of degree measurements of Struve extending 25°20' along the Indian arc 21°5' long and a series of small arcs of general extent of about 75°.

Geographic location of Struve arc and Indian arc are such that due to the presence of significant latitudinal waves along these arcs, compression according to Clarke turned out to be exaggerated, while the value of the major semiaxis was close enough to contemporary values:

$$a = 6378206, \ \alpha = 1:295 \quad (1866)$$

$$a = 6378249, \ \alpha = 1:293 \quad (1880)$$

In the beginning of 20th Century several major Russian Geodesists proposed adoption as a reference-ellipsoid for Russia a semiaxis according to Clarke (6378249) and compression according to Bessel (1:299.15).

Clarke 1866 ellipsoid is used in geodetic work in the United States, Canada and Mexico, and 1880 ellipsoid is used in France, Union of South Africa, and in certain French Possessions in Africa.

After Russian geodesist F. F. Shubert (1859) to Clarke also belongs one of the derivations of triaxial terrestrial ellipsoid.

In the ninetieth years of the past century Russian geodesists professors F. A. Sludskiy (1841-1897) and A. M. Zhdanov (1858-1914), completed research on derivation of parameters of terrestrial ellipsoid from Russian triangulation and as a result obtained:

Sludskiy — a = 6377494, α = 1:297;

Zhdanov — a = 6377717, α = 1:299.

In the 20th Century research on derivation of terrestrial ellipsoid continued in Europe and in America. In 1907 a well known German geodesist F. R. Helmert (1843-1917), author of a two-volume fundamental work on higher geodesy ("Die mathematischen und physikalischen Theorien der höheren Geodäsie" Theil I und II 1880), divided the problem on derivation of parameters of terrestrial ellipsoid. He proposed to derive compression from measurements of gravity and adopting it, derived a major semiaxis from triangulation. By this method, having obtained compression of 1:298.3, Helmert determined the value of major semiaxis at a = 6378200 as a mean, obtained from material in Europe and the United States up to 1906. Helmert's achievement is in that he carried out the idea of joint use of material of geodetic and gravimetric measurements.

In 1910 American geodesist Hayford treated material of extensive astronomic-geodetic net of the United States for the purpose of derivation of terrestrial ellipsoid from American arcs. Hayford in his investigation used a theory of isostatic compensation of Earth's crust. This theory assumes that the insufficiency of density of masses in upper layers of the Earth's crust is compensated by surplus of density in lower layers to a determined depth, called the depth of isostatic compensation. According to this theory, for every section of Earth's crust it is possible to accept that the total mass in an individual vertical column, from physical surface to a certain internal surface, below which there exists a static equilibrium, is approximately constant.

With the application of the theory of isostasy Hayford obtained:

a = 6378388, α = 1:297.

Value of compression according to Hayford coincided with the value of compression obtained from data of measurements of gravity, which was then considered the most reliable. Therefore in 1924 the Geodetic Association of International Geodetic and Geophysical Union (MGGS) gave preference to Hayford derivation and adopted it as an international ellipsoid. In geodetic literature the Hayford ellipsoid is called international ellipsoid in the west. Series of geodetic tables and instructions were composed in the west using the dimensions of this ellipsoid.

Investigations of F. N. Krasovskiy and A. A. Izotov showed that there is no foundation for endorsing Hayford ellipsoid for general international value, since during his derivation he used triangulation done only in the United States. Triangulation in USSR has greater weight than in the United States.

F. N. Krasovskiy studied the problem of derivation of parameters of terrestrial ellipsoid during almost all of his scientific endeavor. However his first better founded derivation pertains to a period of 1931-1934. His work on this problem in the form of separate articles were published in the journal "Geodesist" No. , 7, 10, 11, and 12 in 1936. In his investigations F. N. Krasovskiy used material of extensive triangulation in USSR, the United States, Western Europe and India. Furthermore, he used materials of gravity measurements.

From shown material and taking into account corrections for triaxis he obtained:

$$a = 6378200, \ \alpha = 1:298.6.$$

F. N. Krasovskiy considered that it is doubtful if his derivation was erroneous in value of semiaxis more than ±100 m, and in the value of compression more than one unit in denominator.

Research on the problem of the figure of the Earth in USSR continued at TsNIIGAIK under direction of Professor A. A. Izotov and at the Institute of Theoretical Astronomy of the Academy of Sciences USSR under direction of Professor I. D. Zhongolovich and after publication of the work of F. N. Krasovskiy. A. A. Izotov in his investigations fully utilized the method of F. N. Krasovskiy with addition of new important triangulation in USSR (he included all the valuable materials, obtained up to 1940). Combined treatment carried out by him of geodetic, gravimetric and astronomical materials in Europe and the United States with introduction of isostatic reductions gave the following values for the parameters of biaxial terrestrial ellipsoid

$$a = 6378295 \pm 16 \text{ m}; \ \alpha = 1:298.4 \pm 0.4.$$

On the basis of the same materials parameters of triaxial terrestrial ellipsoid are obtained:

mean radius of equator $a = 6378245$ m,

mean polar compression $\alpha = 1:298.3$,

equatorial compression $\varepsilon = 1:30{,}000$,

longitude of the prime meridian $\lambda_0 = +15^\circ$ from Greenwich.

These conclusions, taking into account geographic disposition of utilized arcs, method of treatment and analysis of materials are at present the most founded and answer the requirements of strict mathematical treatment of extensive astronomical geodetic nets for derivation of parameters of terrestrial ellipsoid. Consequent scientific investigations in USSR and foreign countries definitely indicate that the error in major semiaxis of Krasovskiy ellipsoid does not exceed ±50-60 m, and in compression ±1 unit in denominator.

The well known Austrian geodesist K. Ledershteger (1950) taking into account the corrections in reduction of bases on the surface of the reference ellipsoid, obtained major semiaxis individually for Europe and America correspondingly 6378266 and 6378... m: comparing these results with the major semiaxis of prolate spheroid of Krasovskiy, we see their coincidence. Giving these data in the latest publication of the well known "Instructions on Higher Geodesy", by Jordon, its chief editor and co-author Professor M. Kneissl writes in introduction "Very good confirmation of the results by Ledershteger are presented by the prolate spheroid of Krasovskiy (a = 6378245 m; α = 1:298.3)".

Results of observations of motions of Soviet artificial Earth satellites also confirm this derivation with indicated degree of accuracy.

In 1960 Professor I. D. Zhongolovich obtained from the treatment of results of observations of rotation of three Soviet satellites for compression of terrestrial spheroid the value of 1:298.2, with an error in denominator of ±0.1.[1]

In 1961 American scientist Yu. Kozai using material for compression of terrestrial spheroid from American satellites obtained 1:298.31.[2]

Thus, from 4 October 1957, when USSR launched the first artificial earth satellite, a new epoch was opened in the study of the figure of the Earth, a new powerful and what is especially important an absolutely independent method of resolution of the problem was obtained.

During launching of artificial satellites and space rockets very exact calculations for determination of their orbits are required. In these calculations various geophysical, astronomical and geodetic constants are applied, in a number of them the major semiaxis and compression of terrestrial spheroid play a very large role. The "indirect tests" of these values by related sciences give valuable material for evaluation of the degree of reliability of determination of these values by geodetic ....

[1] Professor I. D. Zhongolovich. Experience in determination of certain parameters of the Earth gravitational field from results of observation of satellites 1957 β_2, 1958 α_1, 1958 δ_2. Bulletin of optical observation stations of artificial earth satellites M., 19.., No. 2(12).

[2] Y. Kozai. The Gravitational Field of the Earth Derived from the Motions of Three Satellites (The Astronomical Journal No. 12.., 1961).

By the Resolution of the Council of Ministers USSR, from 7 April 1946 the parameters of ellipsoid (major semiaxis a = 6378245 m and compression α = 1:298.3), were adopted as obligatory for geodetic work in USSR as the most responding to its areas. The ellipsoid was named F. N. Krasovskiy in honor of his great services to the Soviet geodesy. The Krasovskiy ellipsoid was also adopted for geodetic work of Socialist States. [Soviet satellites]

Results of research on derivation of Krasovskiy ellipsoid are presented in the work of A. A. Izotov "Size and Shape of the Earth by Contemporary Data" (Geodezizdat. 1953).

By now the results of geodetic, astronomical and gravimetric measurements gave correct conclusion about the figure of the Earth on the whole. However investigations in this area continue with great intensity for derivation of general terrestrial ellipsoid and study of the deviations of the figure of the Earth from correct form of rotation.

New developments in the problems of the study of the figure of the Earth the last 15-20 years is introduced by the work of M. S. Molodenskiy and his school.

It is known, that the traditional scientific problem of higher geodesy was considered to be the determination of the figure of the Listing geoid. Meanwhile, rigid determination of the figure of the geoid is impossible without additional data. For obtaining these data it is necessary to resolve physically and geometrically a complex problem: to reduce on the surface of the geoid measured gravity, deviation of the plumb line and results of geometric levelling, angles of triangulation and base lines also have to be referred to the geoid. In order to rigidly satisfy the indicated reductions, it is necessary to know the density of masses outside the geoid.

However for the treatment of geodetic measurements it is necessary to know not the geoid but a figure of physical Earth's surface, gravity and deviation of the plumb line on it, also the height of points of physical earth's surface above reference-ellipsoid. With such formulation of the problem reduction problem immediately drops off and there appears a problem of the study actual shape of the Earth's surface.

Thus, the scientific merit of M. S. Molodenskiy consists in that he introduced clarity into the problem of study of the Earth and gave a new method of resolution of the problem how on the basis of results of geodetic, astronomical and gravimetric measurements to determine the shape of the Earth's surface.

§ 3. ESSENTIAL INFORMATION ON MATHEMATICS

1. Series

Majority of problems of spheroidal geodesy are resolved by means of factorization of functions in power series according to Taylor, MacLaurin and Newton's binomial theorem.

The most essential peculiarity of the geodetic series is their rapid convergence and sign alternation. In most cases the application of series in geodesty their convergence is so evident that no proof is deducted. The convergence of alternating series is determined on the basis of the following theorem.

Alternating series

$$u_1 - u_2 + u_3 - u_4 + \dots \pm u_n \mp u_{n+1} \tag{1.1}$$

(u are positive numbers) it converges if the absolute value of its terms decrease and go to zero during infinite growth of n, while the remainder of the series does not exceed the absolute value by absolute dimension of the first of dropped terms and has the same sign.

Let series:

$$v_1 + v_2 + v_3 + \dots + v_n + v_{n+1} + \dots,$$

where:

$$v_i = |u_i|,$$

converges, then it is possible to assume that:

$$\frac{v_{i+1}}{v_i} < \varepsilon,$$

with this ε is a proper fraction.

Consequently,

$$v_{i+1} < \varepsilon v_i;\ v_{i+2} < \varepsilon v_{i+1};\ \dots\ v_{i+n} < \varepsilon v_{i+n-1} \dots$$

Therefore:

$$v_1 + v_2 + v_3 + \dots + v_n < v_1(1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \dots + \varepsilon^n),$$

but

$$1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \dots + \varepsilon^n = \frac{1}{1-\varepsilon}.$$

that is:

$$\sum_{i=1}^{i=\infty} v_i < \frac{v_1}{1-\varepsilon}.$$

Since $v_1 + v_2 + v_3 + \ldots + v_n > u_1 + u_2 + \ldots + u_n$, then series (1.1) absolutely converge.

The given theorem is applicable to all sign-alternating geodetic series.

Absolutely converging series allow any distribution of terms of the series, that is they converge unconditionally. These series can be added and multiplied, the obtained series will be absolutely convergent. Rapidly converging series are very convenient for practical application, since with them in most cases it is possible to be limited by the first terms of the series. However, it is very important in every instance to determine the order of smallness of the dropped term. The sign of the dropped term does not have value, but it is necessary by all means to evaluate and to indicate the order of smallness. In spheroidal geodesy small value of the first order is usually considered the ratio of length of arc to the mean radius of the Earth. This value corresponds also to the difference of latitudes, longitudes, and azimuths. In subsequent account of the course the order of smallness of dropped term will be designated by a symbol l_i (i = 1.2... – order of smallness).

<u>The Taylor formula.</u> Let us assume that $f(x)$ is any function of x, having derivative to n-order inclusively. We will designate, a as an approximate or measured value of x, h is the correction or error of measurement of x if $x = a + h$, then:

$$f(x) = f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots + R_n(x)$$

where $R_n(x)$ is the remainder, which is usually given form:

$$R_n(x) = \frac{(1-\theta)^{n-p}\, h^n f^{(n)}(a+\theta h)}{1.2\ldots(n-1)p}.$$

here θ is correct positive fraction, unknown exactly.
In geodetic series R_n is a rapidly converging series, therefore it is possible to accept for it approximately:

$$R_n = u_1 \frac{\varepsilon}{1-\varepsilon}. \qquad (1.2)$$

which simultaneously indicates the power of smallness of the dropped last term of series.

Convergence of the series, obtained by the Taylor formula, can be improved by means of change of initial value of the argument. In particular, during introduction of mean argument $a_m = a + \frac{h}{2}$ we have:

$$f(x) = f(a+h) = f\left[\left(a+\frac{h}{2}\right)+\frac{h}{2}\right] = f\left(a+\frac{h}{2}\right)+\frac{1}{2}hf'\left(a+\frac{h}{2}\right)+$$
$$+\frac{h^2}{8}f''\left(a+\frac{h}{2}\right)+\frac{h^3}{48}f'''\left(a+\frac{h}{2}\right)+\ldots.$$
$$f(a) = f\left[\left(a+\frac{h}{2}\right)-\frac{h}{2}\right] = f\left(a+\frac{h}{2}\right)-\frac{h}{2}f'\left(a+\frac{h}{2}\right)+$$
$$+\frac{h^2}{8}f''\left(a+\frac{h}{2}\right)-\frac{h^3}{48}f'''\left(a+\frac{h}{2}\right)+\ldots$$

Difference between these two series gives

$$f(x) = f(a+h) = f(a)+hf'\left(a+\frac{h}{2}\right)+\frac{h^3}{24}f'''\left(a+\frac{h}{2}\right)+\ldots$$

or

$$f(a+h) = f(a)+hf'(a_m)+\frac{h^3}{24}f'''(a_m)+\frac{h^5}{1920}f^{V}(a_m)+\ldots \tag{1.3}$$

From (1.3) it follows that in a series with mean argument all terms with even degrees of h disappear, and terms with odd degrees of h enter decreased by 4.16 etc. times. This principle, introduced into geodesy by Gauss, is widely used in resolution of many problems of spheroidal geodesy.

If we were to take differences of functions f(a + h) and f(a - h), we will obtain rapidly converging series in the form of:

$$f(a+h)-f(a-h) = 2hf'(a)+\frac{1}{3!}h^3f'''(a)+\ldots \tag{1.4}$$

In expressions (1.1) and (1.4) remainder can be calculated by the formula:

$$R_n = \frac{h^n}{n!}f^{(n)}\left(a+\frac{h}{n+1}\right). \tag{1.5}$$

Taylor formula can be written in the form:

$$f(x+h)-f(x) = hf'(x)+\frac{h^2}{2}f''(x)+\ldots+\frac{h^n}{n!}f^{(n)}_{(x)}+R_n.$$

The left part of this expression is increase of function $y = f(x)$, therefore:

$$dy = f'(x)\,dx = f'(x)\,h; \quad d^2y = y''dx^2 = f''(x)\,h^2$$

and in general

$$d^n y = y^{(n)} dx^n = f^{(n)}(x)\,h^n.$$

Consequently,

$$\Delta y = dy + \frac{1}{2!} d^2y + \frac{1}{3!} d^3y + \dots + \frac{1}{n!} d^n y + R_n. \qquad (1.6)$$

Formula (1.6) has great value in problems of approximation reckoning and calculations.

The MacLoren formula. In a particular case, when initial value of variable x is zero, that is, $a = 0$, and $x = h$, Taylor formula assumes the form of:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + R_n. \qquad (1.7)$$

This formula is called the MacLoren formula.

The statement about evaluation of the remainder and convergence of series is obtained by the Taylor formula, is also applicable for series obtained by the MacLoren formula. Although MacLoren formula is a particular case of Taylor formula, it is used just as frequently, as the Taylor formula.

Binomial series. Expression

$$(1 \pm u)^n = 1 \pm nu + \frac{n(n-1)}{2!} u^2 \pm \frac{n(n-1)(n-2)}{3!} u^3 + \dots \qquad (1.8)$$

has meaning and absolutely converges at any n, if $u < 1$. Expression of type (1.8) is called binomial series. In distinction from remainders of series of Taylor and MacLoren, the remainder of binomial series can be obtained by direct summation, but the limit of its convergence is not always known.

The most commonly used binomial series in spheroidal geodesy, are:

$$\left.\begin{aligned}
&1. \quad \sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 - \frac{5}{128}u^4 + \dots \\
&2. \quad \sqrt{1-u} = 1 - \frac{1}{2}u - \frac{1}{8}u^2 - \frac{1}{16}u^3 - \frac{5}{128}u^4 - \dots \\
&3. \quad \sqrt{1-u^2} = 1 - \frac{1}{2}u^2 - \frac{1}{8}u^4 - \frac{1}{16}u^6 - \frac{5}{128}u^8 - \dots
\end{aligned}\right| \qquad (1.9)$$

$$
\left.
\begin{array}{l}
4.\ \sqrt{1+u^2} = 1 + \frac{1}{2}u^2 - \frac{1}{8}u^4 + \frac{1}{16}u^6 - \frac{5}{128}u^8 + \dots \\
5.\ 1:\sqrt{1+u} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}u^3 + \frac{35}{128}u^4 - \dots \\
6.\ 1:\sqrt{1-u} = 1 + \frac{1}{2}u + \frac{3}{8}u^2 + \frac{5}{16}u^3 + \frac{35}{128}u^4 + \dots \\
7.\ 1:\sqrt{1-u^2} = 1 + \frac{1}{2}u^2 + \frac{3}{8}u^4 + \frac{5}{16}u^6 + \frac{35}{128}u^8 + \dots \\
8.\ 1:\sqrt{1+u^2} = 1 - \frac{1}{2}u^2 + \frac{3}{8}u^4 - \frac{5}{16}u^6 + \frac{35}{128}u^8 - \dots \\
9.\ 1:\sqrt{(1-u)^3} = 1 + \frac{3}{2}u + \frac{15}{8}u^2 + \frac{35}{16}u^3 + \dots \\
10.\ 1:(1-u) = 1 + u + u^2 + u^3 + u^4 + \dots \\
11.\ 1:(1+u) = 1 - u + u^2 - u^3 + u^4 - \dots \\
12.\ 1:(1-u)^2 = 1 + 2u + 3u^2 + 4u^3 + 5u^4 + \dots \\
13.\ 1:(1+u)^2 = 1 - 2u + 3u^2 - 4u^3 + 5u^4 - \dots
\end{array}
\right| \qquad \text{(1.9) contd.}
$$

Logarithmic series. Relationship of decimal and natural logarithms of positive number u is expressed by formula:

$$\lg u = \mu \ln u,$$

where μ = 0.4342945 of modulus of common logarithms.

$$\lg(1+u) = \mu\left[u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots\right], \qquad (1.10)$$

$$
\left.
\begin{array}{l}
\lg(1-u) = -\mu\left[u + \frac{u^2}{2} + \frac{u^3}{3} + \frac{u^4}{4} + \dots\right] \\
\lg u = 2\mu\left[\left(\frac{u-1}{u+1}\right) + \frac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \frac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \dots\right]
\end{array}
\right| \qquad (1.11)
$$

Inverse logarithmic series:

$$u = 1 + \left(\frac{\lg u}{\mu}\right) + \frac{1}{2!}\left(\frac{\lg u}{\mu}\right)^2 + \frac{1}{3!}\left(\frac{\lg u}{\mu}\right)^3 + \dots \qquad (1.11')$$

In all these series number u is the positive value and is less than one.

Trigonometric series. In trigonometric series angles, as a rule, are expressed in radians.

Radian

$$\rho^\circ = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = 57^\circ{,}29578,$$

$$\rho' = \frac{360^\circ \cdot 60}{2\pi} = 3437'{,}74677,$$

$$\rho'' = \frac{360^\circ \cdot 3600}{2\pi} = 206264''{,}80625.$$

If an angle u is given in degrees, then in radians it is equal to:

$$u = \frac{u^{\circ}}{\rho^{\circ}} = \frac{u'}{\rho'} = \frac{u''}{\rho''}; \tag{1.12}$$

$$\left.\begin{aligned}
\sin u &= u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots = u - \frac{u^3}{6} + \frac{u^5}{120} - \ldots \\
\cos u &= 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \ldots = 1 - \frac{u^2}{2} + \frac{u^4}{24} - \ldots \\
\operatorname{tg} u &= u + \frac{u^3}{3} + \frac{2}{15}u^5 + \ldots \\
\operatorname{ctg} u &= \frac{1}{u} - \frac{u}{3} - \frac{u^3}{45} - \frac{2u^5}{945} - \ldots \\
\sec u &= 1 + \frac{u^2}{2} + \frac{5}{24}u^4 + \frac{61}{720}u^6 + \ldots \\
\operatorname{cosec} u &= \frac{1}{u} + \frac{u}{6} + \frac{7}{360}u^3 + \frac{31}{15120}u^5 + \ldots
\end{aligned}\right| \tag{1.13}$$

Inverse trigonometric series

$$x = \sin u, \qquad u = \arcsin x$$
$$x = \operatorname{tg} u, \qquad u = \operatorname{arc\,tg} x$$

$$\left.\begin{aligned}
u = \arcsin x &= x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5}{112}x^7 + \ldots \\
u = \operatorname{arc\,tg} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \\
u &= \sin u + \frac{\sin^3 u}{6} + \frac{3\sin^5 u}{40} + \frac{5\sin^7 u}{112} + \ldots \\
u &= \operatorname{tg} u - \frac{1}{3}\operatorname{tg}^3 u + \frac{1}{5}\operatorname{tg}^5 u - \frac{1}{7}\operatorname{tg}^7 u + \ldots
\end{aligned}\right|; \tag{1.14}$$

$$\left.\begin{aligned}
\lg \sin u &= \lg u\left(1 - \frac{u^2}{6} + \frac{u^4}{120} + \ldots\right) = \lg u - \mu\left(\frac{u^2}{6} + \frac{u^4}{180} + \frac{u^6}{2835} + \ldots\right) \\
\lg \operatorname{tg} u &= \lg u\left(1 + \frac{u^2}{3} + \frac{2u^4}{15} + \ldots\right) = \lg u + \mu\left(\frac{u^2}{3} + \frac{7u^4}{90} + \frac{62u^6}{2835} + \ldots\right) \\
\lg \cos u &= -\mu\left(\frac{u^2}{2} + \frac{u^4}{12} + \frac{u^6}{45} + \ldots\right) \\
\lg u &= \lg \sin u + \mu\left(\frac{\sin^2 u}{6} + \frac{11}{180}\sin^4 u + \frac{191}{5670}\sin^6 u + \ldots\right) \\
\lg u &= \lg \operatorname{tg} u - \mu\left(\frac{\operatorname{tg}^2 u}{3} - \frac{13}{90}\operatorname{tg}^4 u + \frac{251}{2835}\operatorname{tg}^6 u - \ldots\right)
\end{aligned}\right|; \tag{1.15}$$

u — positive and less than $\frac{\pi}{4}$.

In Vega and Bauschinger tables of logarithms values are given:

$$\left.\begin{aligned}
S &= \lg \frac{\sin u}{u''} \quad \text{or} \quad \lg \sin u = S + \lg u'' \\
T &= \lg \frac{\operatorname{tg} u}{u''} \quad \text{or} \quad \lg \operatorname{tg} u = T + \lg u''
\end{aligned}\right| \tag{1.16}$$

For calculation of sines and tangents of acute angles.

2. Trigonometry

Plane Trigonometry

$$\left.\begin{aligned}
\sin\alpha &= \sqrt{1-\cos^2\alpha} = \frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}} = \frac{1}{\sqrt{1+\operatorname{ctg}^2\alpha}} = \frac{1}{\operatorname{cosec}\alpha}\\
\cos\alpha &= \sqrt{1-\sin^2\alpha} = \frac{1}{\sqrt{1+\operatorname{tg}^2\alpha}} = \frac{\operatorname{ctg}\alpha}{\sqrt{1+\operatorname{ctg}^2\alpha}} = \frac{1}{\sec\alpha}\\
\operatorname{tg}\alpha &= \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha} = \frac{1}{\operatorname{ctg}\alpha}
\end{aligned}\right\}. \qquad (1.17)$$

Function of the sum and difference of angles

$$\begin{aligned}
\sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,\\
\cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,\\
\operatorname{tg}(\alpha\pm\beta) &= \frac{\operatorname{tg}\alpha\pm\operatorname{tg}\beta}{1\mp\operatorname{tg}\alpha\cdot\operatorname{tg}\beta},\\
\operatorname{ctg}(\alpha\pm\beta) &= \frac{\operatorname{ctg}\alpha\cdot\operatorname{ctg}\beta\mp 1}{\operatorname{ctg}\beta\pm\operatorname{ctg}\alpha}.
\end{aligned}$$

When $\alpha = \beta$:

$$\left.\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha\\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\
\operatorname{tg} 2\alpha &= \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}\\
\operatorname{ctg} 2\alpha &= \frac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}
\end{aligned}\right\}. \qquad (1.18)$$

When $\alpha \neq \beta$:

$$\left.\begin{aligned}
\sin\alpha+\sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}\\
\sin\alpha-\sin\beta &= 2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}\\
\cos\alpha+\cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}\\
\cos\alpha-\cos\beta &= -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\\
\frac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta} &= -\operatorname{tg}\frac{\alpha+\beta}{2}\cdot\operatorname{tg}\frac{\alpha-\beta}{2};\\
\frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta} &= \frac{\operatorname{tg}\frac{\alpha-\beta}{2}}{\operatorname{tg}\frac{\alpha+\beta}{2}}
\end{aligned}\right\}. \qquad (1.19)$$

Eyler and Moivre formulas

Eyler formula:

$$\left.\begin{aligned}
\sin u &= \frac{e^{iu}-e^{-iu}}{2i}\\
\cos u &= \frac{e^{iu}+e^{-iu}}{2}\\
\cos u \pm i\sin u &= e^{\pm iu}
\end{aligned}\right\}. \qquad (1.20)$$

or in general;

$$\cos nu \mp i\sin nu = e^{\mp inu} = e^{i\mp nu} = (\cos u \mp i\sin u)^n.$$

$e = 2.71828183$ is a base of natural logarithms, $i = \sqrt{-1}$.

Moivre Formula

$$\cos nu + i\sin nu = (\cos u + i\sin u)^n = \cos^n u - \binom{n}{1}\cos^{n-1}u \cdot i\sin u -$$
$$-\binom{n}{2}\cos^{n-2}u \cdot \sin^2 u + \ldots$$
$$\cos nu - i\sin nu = (\cos u - i\sin u)^n = \cos^n u + \binom{n}{1}\cos^{n-1}u \cdot i\sin u -$$
$$-\binom{n}{2}\cos^{n-2}u\,\sin^2 u - \ldots,$$

whence

$$\cos nu = \cos^n u - \binom{n}{2}\cos^{n-2}u \cdot \sin^2 u + \binom{n}{4}\cos^{n-4}u \cdot \sin^4 u - \ldots,$$

$$\sin nu = \binom{n}{1}\cos^{n-1}u \cdot \sin u - \binom{n}{3}\cos^{n-3}u \cdot \sin^3 u +$$
$$+\binom{n}{5}\cos^{n-5}u \cdot \sin^5 u - \ldots,$$

where

$$\binom{n}{1} = n;\quad \binom{n}{2} = \frac{n(n-1)}{2!};\quad \binom{n}{3} = \frac{n(n-1)(n-2)}{3!};$$
$$\binom{n}{i} = \frac{n(n-1)(n-2)\ldots(n-i)}{i!}.$$

For expression of even degrees of sines and cosines by cosines of multiples of arcs we have formula:

$$e^{iu} = \cos u + i\sin u = p \text{ or } p^m = \cos mu + i\sin mu,$$
$$e^{-iu} = \cos u - i\sin u = q \text{ or } q^m = \cos mu - i\sin mu,$$

whence

$$pq = 1;\ p + q = 2\cos u;\ p - q = 2i\sin u;$$
$$p^m + q^m = 2\cos mu;\quad p^m - q^m = 2i\sin mu.$$

Further

$$(2\cos u)^m = (p+q)^m = p^m + q^m + \binom{m}{1}\left(p^{m-1}q + pq^{m-1}\right) + \binom{m}{2}\left(p^{m-2}q^2 + \right.$$
$$\left. + p^2 q^{m-2}\right) + \ldots + \binom{m}{l}\left(p^{m-l}q^l + p^l q^{m-l}\right) + \ldots$$

or

$$(2\cos u)^m = 2\cos mu + \binom{m}{1} 2\cos(m-2)u + \binom{m}{2} 2\cos(m-4)u + \ldots$$
$$+ \binom{m}{l} 2\cos(m-2l)u.$$

Let m = 2n, that is, m is an even number, then:

$$\cos^{2n} u = \frac{1}{2^{2n-1}}\left\{\frac{1}{2}\binom{2n}{n} + \binom{2n}{n-1}\cos 2u + \binom{2n}{n-2}\cos 4u + \ldots\right\}. \tag{1.21}$$

Since cos (90 - u) = sin u and cos 2 (90 - u) = -cos 2u etc., then:

$$\sin^{2n} u = \frac{1}{2^{2n-1}}\left\{\frac{1}{2}\binom{2n}{n} - \binom{2n}{n-1}\cos 2u + \binom{2n}{n-2}\cos 4u - \ldots\right\}. \tag{1.22}$$

For odd degrees of sines and cosines

$$\sin^{2n+1} u = \frac{1}{2^{2n}}\left\{\binom{2n+1}{n}\sin u - \binom{2n+1}{n-1}\sin 3u + \ldots\right\}; \tag{1.23}$$

$$\cos^{2n+1} u = \frac{1}{2^{2n}}\left\{\binom{2n+1}{n}\cos u + \binom{2n+1}{n-1}\cos 3u + \ldots\right\}. \tag{1.24}$$

From these general formulas it follows:

$$\left.\begin{aligned}
\sin^2 u &= \frac{1}{2} - \frac{1}{2}\cos 2u \\
\sin^3 u &= \frac{3}{4}\sin u - \frac{1}{4}\sin 3u \\
\sin^4 u &= \frac{3}{8} - \frac{1}{2}\cos 2u + \frac{1}{8}\cos 4u \\
\sin^5 u &= \frac{5}{8}\sin u - \frac{5}{16}\sin 3u + \frac{1}{16}\sin 5u \\
\sin^6 u &= \frac{5}{16} - \frac{15}{32}\cos 2u + \frac{3}{16}\cos 4u - \frac{1}{32}\cos 6u
\end{aligned}\right\}; \quad (1.2$$

$$
\left.\begin{aligned}
\cos^2 u &= \frac{1}{2} + \frac{1}{2}\cos 2u \\
\cos^3 u &= \frac{3}{4}\cos u + \frac{1}{4}\cos 3u \\
\cos^4 u &= \frac{3}{8} + \frac{1}{2}\cos 2u + \frac{1}{8}\cos 4u \\
\cos^5 u &= \frac{5}{8}\cos u + \frac{5}{16}\cos 3u + \frac{1}{16}\cos 5u \\
\cos^6 u &= \frac{5}{16} + \frac{15}{32}\cos 2u + \frac{3}{16}\cos 4u + \frac{1}{32}\cos 6u
\end{aligned}\right| \qquad (1.26)
$$

Application of Taylor series to trigonometric functions:

$$
\left.\begin{aligned}
\sin(u+h) &= \sin u + h\cos u - \frac{h^2}{2}\sin u - \frac{h^3}{6}\cos u + \frac{h^4}{24}\sin u + \dots \\
\sin(u-h) &= \sin u - h\cos u - \frac{h^2}{2}\sin u + \frac{h^3}{6}\cos u + \frac{h^4}{24}\sin u - \dots \\
\cos(u+h) &= \cos u - h\sin u - \frac{h^2}{2}\cos u + \frac{h^3}{6}\sin u + \frac{h^4}{24}\cos u - \dots \\
\cos(u-h) &= \cos u + h\sin u - \frac{h^2}{2}\cos u - \frac{h^3}{6}\sin u + \frac{h^4}{24}\cos u + \dots \\
\mathrm{tg}(u+h) &= \mathrm{tg}\, u + \frac{h}{\cos^2 u} + \frac{h^2\sin u}{\cos^3 u} + \frac{h^3}{3}\,\frac{\cos^2 u + 3\sin^2 u}{\cos^4 u} + \dots \\
\mathrm{tg}(u-h) &= \mathrm{tg}\, u - \frac{h}{\cos^2 u} + \frac{h^2\sin u}{\cos^3 u} - \frac{h^3}{3}\,\frac{\cos^2 u + 3\sin^2 u}{\cos^4 u} + \dots
\end{aligned}\right| \qquad (1.27)
$$

In higher geodesy designation tg u = t is frequently used

$$
\left.\begin{aligned}
\cos(u+h) &= \cos u\left(1 - ht - \frac{h^2}{2} + \frac{h^3}{6}t + \frac{h^4}{24} - \dots\right) \\
\mathrm{tg}(u+h) &= \mathrm{tg}\, u + h(1+t^2) + h^2 t(1+t^2) + \frac{h^3}{3}(1 + 4t^2 + 3t^4) + \dots \\
\sin(u+h) &= \sin u\left(1 + \frac{h}{t} - \frac{h^2}{2} - \frac{h^3}{6t} + \frac{h^4}{24} + \frac{h^5}{120t} + \dots\right)
\end{aligned}\right| \qquad (1.28)
$$

For exponential functions:

$$
\left.\begin{aligned}
e^u &= 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \frac{u^5}{120} + \dots \\
a^x &= 1 + x\ln a + \frac{x^2(\ln a)^2}{2!} + \frac{x^3(\ln a)^3}{3!} + \frac{x^4(\ln a)^4}{4!} + \dots
\end{aligned}\right\} . \qquad (1.29)
$$

3. Spherical Trigonometry

Resolution of right-angle spherical triangle

Let us designate vertexes of triangle – A, B, C, angles – α, β, γ, and sides – , b, c (Fig. 1).

Formulas

$$\alpha = 90^\circ \left\{ \begin{array}{l} \cos a = \cos b \cos c; \\ \cos a = \operatorname{ctg}\beta \operatorname{ctg}\gamma; \\ \sin\beta = \dfrac{\sin b}{\sin a}; \quad \sin\gamma = \dfrac{\sin c}{\sin a}; \\ \cos\beta = \dfrac{\operatorname{tg} c}{\operatorname{tg} a}; \quad \cos\gamma = \dfrac{\operatorname{tg} b}{\operatorname{tg} a}; \\ \operatorname{tg}\beta = \dfrac{\operatorname{tg} b}{\sin c}; \quad \operatorname{tg}\gamma = \dfrac{\operatorname{tg} c}{\sin b}; \\ \cos b = \dfrac{\cos\beta}{\sin\gamma}; \quad \cos c = \dfrac{\cos\gamma}{\sin\beta}. \end{array} \right.$$

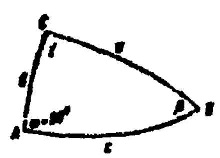


Fig. 1.

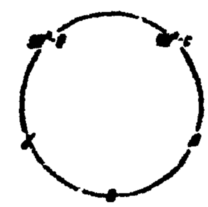


Fig. 2.

Right-angle spherical triangle can be resolved by two rules, if the arms are replaced by their supplements to 90° and elements of a triangle are disposed circularly, as is shown in Fig. 2:

First rule: the cosine of separate element is equal to the product of sines of adjacent elements. For instance:

$$\cos a = \sin(90^\circ - b)\sin(90^\circ - c) = \cos b \cdot \cos c.$$

Second rule: cosine of the mean element is equal to the product of cotangents of extreme elements, for instance

$$\cos a = \operatorname{ctg}\beta \cdot \operatorname{ctg}\gamma.$$

General Formulas for Resolution of Spherical Triangle

1. Formulas of cosines of the sides:

$$\left. \begin{array}{l} \cos a = \cos b \cos c + \sin b \sin c \cos\alpha \\ \cos b = \cos a \cos c + \sin a \sin c \cos\beta \\ \cos c = \cos a \cos b + \sin a \sin b \cos\gamma \end{array} \right\} \qquad (1.$$

2. Formula of sines

$$\frac{\sin\alpha}{\sin a}=\frac{\sin\beta}{\sin b}=\frac{\sin\gamma}{\sin c}. \qquad (1.31)$$

3. Formulas of cotangents of elements:

$$\left.\begin{aligned}
\operatorname{ctg} a\sin b &= \cos b\cos\gamma+\sin\gamma\operatorname{ctg}\alpha\\
\operatorname{ctg} b\sin c &= \cos c\cos\alpha+\sin\alpha\operatorname{ctg}\beta\\
\operatorname{ctg} c\sin a &= \cos a\cos\beta+\sin\beta\operatorname{ctg}\gamma\\
\operatorname{ctg} a\sin c &= \cos c\cos\beta+\sin\beta\operatorname{ctg}\alpha\\
\operatorname{ctg} b\sin a &= \cos a\cos\gamma+\sin\gamma\operatorname{ctg}\beta\\
\operatorname{ctg} c\sin b &= \cos b\cos\alpha+\sin\alpha\operatorname{ctg}\gamma
\end{aligned}\right\}. \qquad (1.32)$$

3'. Formulas of five elements:

$$\left.\begin{aligned}
\sin a\cos\beta &= \cos b\sin c-\sin b\cos c\cos\alpha\\
\sin b\cos\gamma &= \cos c\sin a-\sin c\cos a\cos\beta\\
\sin c\cos\alpha &= \cos a\sin b-\sin a\cos b\cos\gamma\\
\sin a\cos\gamma &= \cos c\sin b-\sin c\cos b\cos\alpha\\
\sin b\cos\alpha &= \cos a\sin c-\sin a\cos c\cos\beta\\
\sin c\cos\beta &= \cos b\sin a-\sin b\cos a\cos\gamma
\end{aligned}\right\}. \qquad (1.32')$$

4. Formulas of cosines of angles:

$$\left.\begin{aligned}
\cos\alpha &= -\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos a\\
\cos\beta &= -\cos\gamma\cos\alpha+\sin\gamma\sin\alpha\cos b\\
\cos\gamma &= -\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos c
\end{aligned}\right\}. \qquad (1.33)$$

5. Gauss – Delambre formulas

$$\left.\begin{aligned}
\sin\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2} &= \sin\frac{b+c}{2}\sin\frac{a}{2}\\
\sin\frac{\alpha}{2}\cdot\sin\frac{\beta-\gamma}{2} &= \sin\frac{b-c}{2}\cos\frac{a}{2}\\
\cos\frac{\alpha}{2}\cos\frac{\beta+\gamma}{2} &= \cos\frac{b+c}{2}\sin\frac{a}{2}\\
\cos\frac{\alpha}{2}\sin\frac{\beta+\gamma}{2} &= \cos\frac{b-c}{2}\cdot\cos\frac{a}{2}
\end{aligned}\right\}. \qquad (1.34)$$

6. Napier's analogies:

$$\left.\begin{aligned}
&\operatorname{tg}\frac{b+c}{2}=\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}}\operatorname{tg}\frac{a}{2}; \quad \operatorname{tg}\frac{b-c}{2}=\frac{\sin\frac{\beta-\gamma}{2}}{\sin\frac{\beta+\gamma}{2}}\operatorname{tg}\frac{a}{2}\\
&\operatorname{tg}\frac{\beta+\gamma}{2}=\frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}\operatorname{ctg}\frac{\alpha}{2}; \quad \operatorname{tg}\frac{\beta-\gamma}{2}=\frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}\operatorname{ctg}\frac{\alpha}{2}
\end{aligned}\right\}. \qquad (1.35)$$

7. Formula of tangents:

$$\frac{\operatorname{tg}\frac{b+c}{2}}{\operatorname{tg}\frac{b-c}{2}}=\frac{\operatorname{tg}\frac{\beta+\gamma}{2}}{\operatorname{tg}\frac{\beta-\gamma}{2}}. \qquad (1.36)$$

Spherical excess of a spherical triangle.

$$\varepsilon=\alpha+\beta+\gamma-180^\circ,$$

$$\varepsilon=\frac{F}{R^2}\cdot\rho'',$$

F — area of a triangle,

R — radius of a sphere,

ρ'' — number of seconds in radian,

$$\operatorname{tg}\frac{\varepsilon}{4}=\sqrt{\operatorname{tg}\frac{p}{2}\operatorname{tg}\frac{p-a}{2}\operatorname{tg}\frac{p-b}{2}\operatorname{tg}\frac{p-c}{2}},$$

where $p=\frac{a+b+c}{2}$;

$$\operatorname{tg}\frac{\varepsilon}{2}=\frac{\operatorname{tg}\frac{b}{2}\operatorname{tg}\frac{c}{2}\sin\alpha}{1+\operatorname{tg}\frac{b}{2}\operatorname{tg}\frac{c}{2}\cos\alpha}=\frac{k\sin\alpha}{1+k\cos\alpha}, \qquad (1.37)$$

$$k=\operatorname{tg}\frac{b}{2}\cdot\operatorname{tg}\frac{c}{2}.$$

Spherical excess of right-angle spherical triangle when $\alpha = 90^0$:

$$\operatorname{tg}\frac{\varepsilon}{2}=\operatorname{tg}\frac{b}{2}\cdot\operatorname{tg}\frac{c}{2}. \qquad (1.38)$$

4. Differential Geometry

Plane curves. Equation of a curve:

in implicit form $F(x, y) = 0$,

in evident form $y = f(x)$,

in parametric form $x = x(u)$, $y = y(u)$, u is a parameter.

The last form of assignment of curve is more frequently used in spheroidal geodesy.

Depending upon the form of assignment of curve differential of it's arc it is expressed:

1. $ds=\sqrt{1+y'^2}\,dx$ or $y=f(x)$;
2. $ds=\sqrt{x'^2+y'^2}\,du$ or $x=x(u)$; $y=y(u)$.

Curvature of a plane curve Γ in a given point P is called the limit of ratio of the angle of contiguity $\Delta\alpha$ (angle between positive directions of tangents at points P_1 and P_2 – Fig. 3) and length of arc $\overset{\smile}{P_1P_2}$, when $\overset{\smile}{P_1P_2} \to 0$

$$K = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}.$$

Fig. 3.

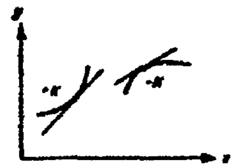


Fig. 4.

Radius of curvature R at a given point P is called value, inverse to curvature, that is:

$$R = \frac{1}{K}.$$

Curvature K – is positive, if the curve at a given point of its concavity is turned to axis x (Fig. 4).

In grid coordinates:

$$\left.\begin{aligned} K &= \pm \frac{y''}{(1+y'^2)^{3/2}} \\ R &= \pm \frac{(1+y'^2)^{3/2}}{y''} \end{aligned}\right\} \qquad (1.39)$$

<u>Space curves.</u> Equation of space curve in parametric form:

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

or

$$x = x(s), \quad y = y(s), \quad z = z(s),$$

where s – length of arc of curve.

Differential of arc of a space curve

$$ds = \sqrt{x'^2 + y'^2 + z'^2}\, du.$$

At each point P of the space curve are determined three straight lines and three planes, mutually intersecting at P at right angles (Fig. 5).

<u>Straight lines.</u> Tangent is a limiting position of a secant (Fig. 5). Principal

normal is the intersection of normal and osculating planes. Binormal is a straight line, perpendicular to osculating plane.

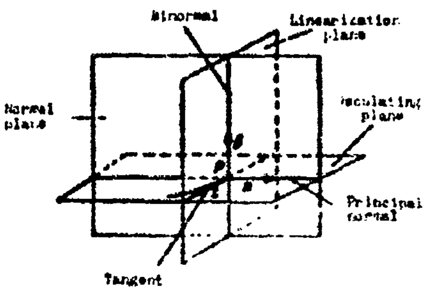


Fig. 5.

Planes. Normal are perpendicular to a tangent. Osculating, a limiting position of a plane passing through three close points of a curve $P_1 P_2$ and P_3, when $P_1 \to P_2$ and $P_3 \to P_2$ (Fig. 6). Linearizing – containing tangent and a binormal.

These three straight lines together with the planes connecting them form an accompanying trihedron of a space curve.

If for the axis of coordinates the tangent is taken, the principal normal and binormal with origin of coordinates at point P_1 of space curve, then coordinates of the other point P_2 of the curve will be:

Fig. 6.

$$\left.\begin{aligned} x &= s - \frac{s^3}{6R^2} + \ldots \\ y &= \frac{s^2}{2R} - \frac{s^3}{6R^2}\frac{dR}{ds} + \ldots \\ z &= -\frac{s^3}{6RT} + \ldots \end{aligned}\right\}, \qquad (1.40)$$

where:

s is the length of arc of the curve between points P_1 and P_2,

R – radius of curvature of space curve,

T – torsion.

The curvature of space curve at a given point is called numerical characteristic of deflection of the curve from straight line in an area of a given point of the curve, it is calculated by the formula:

$$K = \sqrt{x''^2 + y''^2 + z''^2}, \qquad R = \frac{1}{K}.$$

Torsion of space curve at a given point is called numerical characteristic of deflection of a space curve from plane curve in an area of a given point. In problems of spheroidal geodesy the curvature and torsion of space curve are rarely used.

In formulas (1,40) the values of R, T and $\frac{dR}{ds}$ are taken where s [illegible].

Surface. Equation of surface is given in the following forms:

$F(x, y, z) = 0$ is nonevident

$z = f(x, y)$ is evident

$x = x(u, v)$; $y = y(u, v)$; $z = z(u, v)$ are parametric.

Differential of arc or first quadratic form is:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{1.41}$$

where

$$\left.\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ F &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned}\right| \tag{1.41}$$

In spheroidal geodesy orthogonal system of curvilinear parametric coordinates is used which form on the surface the graticule

$$ds^2 = E\,du^2 + G\,dv^2. \tag{1.42}$$

Designating $\sqrt{\frac{E}{G}}\,du = dt$, we obtain:

$$ds^2 = G\,(dt^2 + dv^2). \tag{1.43}$$

Curvilinear coordinates (t, v) are called isometric. The isometric system of coordinates is characterized by the fact that they form on the surface a grid of squares with sides $\sqrt{G}\,dt$ and $\sqrt{G}\,dv$. Where $dt = dv$ regular squares are obtained, but they are not equal to each other, since G is a function of coordinates of a given point.

Through each point of surface it is possible to pass an infinite number of planes, passing through the normal to surface at a given point. These planes are called normals. Plane curves, obtained as traces of intersection of these planes with a surface, are called normal sections. From normal sections two main mutually perpendicular sections have essential values, one with the greatest curvature $\frac{1}{R_2}$ and the other with the least $\frac{1}{R_1}$, then the curvature of any normal section can be expressed through curvature of main sections by the Eyler formula

$$\frac{1}{R} = \frac{\cos^2 A}{R_1} + \frac{\sin^2 A}{R_2}. \tag{1.44}$$

Where A = azimuth of a given normal section.

Besides the curvature of a normal section, in spheroidal geodesy Gauss curvature is used:

$$K = \frac{1}{R_1 R_2} \tag{1.45}$$

and mean curvature:

$$K_{cp} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{1.46}$$

In certain problems the following formula is used:

$$K_0 = \frac{1}{\sqrt{R_1 R_2}}.$$

where

$\sqrt{R_1 R_2}$ is called mean radius of curvature.

The geodesic. Through each point of the surface P(u, v) it is possible to pass a line in a given direction which will be the shortest between two points. Such line is called a geodesic. The material point will move on the surface along a geodesic if external forces are absent hampering its movement. Elastic thread, stretched along the surface, takes form of a geodesic.

For spheroidal geodesy the following determination of geodesic is more essential. Geodesic on a surface is a type of a curve, whose principal normal at a given point coincides with the normal to the surface.

Let us take the initial point of the geodesic P_1 for origin of coordinates plane xoy coinciding with tangent plane at point P, then coordinates of point P_2, of geodesic will be equal:

$$\left.\begin{aligned} x &= s\cos A - \frac{1}{6R_1R}\cos A s^3 + \ldots \\ y &= s\sin A - \frac{1}{6R_2R}\sin A s^3 + \ldots \end{aligned}\right\}$$

where

s -- arc of geodesic between points P_1 and P_2,

A — azimuth of geodesic at initial point,

R_1 – meridian radius of curvature,

R_2 – radius of curvature of first vertical,

R – radius of curvature of normal section at azimuth A.

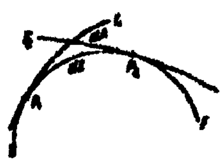


Fig. 7.

Geodesics on the surface play a role to a certain degree of straight lines on a plane, therefore many positions of differential geometry on a plane can be generalized for surfaces with substitution of straight lines by geodesic. One of such generalizations is the understanding of geodesic curvature on a surface. In solution of certain problems of spheroidal geodesy it is very expedient to start from consideration of geodesic curvature.

Geodesic curvature of surface curve is called ratio of angle of contiguity dA to the element of arc ds (Fig. 7).

$$K_g = \frac{dA}{ds}. \tag{1.47}$$

In curving of the surface the geodesic curvature is not changed. If all three lines $P_1\Gamma_1$, $P_2\Gamma_2$ and OF were geodesics, then they would have merged and the geodesic curvature would be equal to zero. In other words, geodetic curvature of geodesics is always equal to zero.

If normal sections and geodesic (Fig. 7) are projected on a tangent plane, through point P_1, then geodesics will be straight lines on this plane, the elements dA and ds will be distorted by small values of the highest order, consequently their ratio will remain constant, therefore the so-called tangential curvature is equal to the geodesic.

Projection of curve P_1P_2 to a tangent plane will have curvature of a plane curve. Consequently, if we designate an angle between tangent plane and a surface at point P_1 and osculating plane of element ds through ϑ is designated, then the geodesic curvature will be equal to the usual curvature, multiplied by the cosine of this angle:

$$K_g = K \cos\vartheta. \tag{1.48}$$

Normal section in initial point has geodesic curvature, equal to zero, since at this point the angle $\vartheta = 90^\circ$; in remaining points of normal section $\vartheta > 90^\circ$; with removal from initial point its geodesic curvature is correspondingly increased.

CHAPTER II

TERRESTRIAL SPHEROID

§ 4. ELEMENTS OF MERIDIAN ELLIPSE

Geometric solid, obtained by rotation of ellipse around its polar axis, is called prolate spheroid. Prolate spheroids with small polar compression are also called spheroids. Basic elements of a spheroid, determining its geometric figure, are the semiaxis: major, or equatorial and minor, or polar (Fig. 8). Let us designate:

a – major semiaxis of terrestrial spheroid,

b – minor semiaxis of terrestrial spheroid.

For terrestrial spheroid $a > b$. In solution of many problems of geodesy it is necessary to use different values, obtained through a and b, such as, for instance, three compressions:[1]

$$\alpha = \frac{a-b}{a}; \quad \alpha' = \frac{a-b}{b}; \quad \alpha'' = \frac{a-b}{a+b} = n \tag{2.1}$$

and three eccentricities,[1] whose squares are expressed thus:

$$e^2 = \frac{a^2-b^2}{a^2}; \; e'^2 = \frac{a^2-b^2}{b^2}; \; e''^2 = \frac{a^2-b^2}{a^2+b^2}; \tag{2.2}$$

these values are connected by relationships:

---

[1]Terms "third compression" and "third eccentricity" while not conventional, are used by certain authors (see A. P. Yushchenko "Cartography", 1941, p. 9) who call "third compression" and "third eccentricity" $2n$ and $2e''^2$ respectively.

$$
\left.\begin{aligned}
b &= a\sqrt{1-e^2} = a(1-\alpha) = a\,\frac{1-n}{1+n}\\
e^2 &= 2\alpha-\alpha^2 = \frac{4n}{(1+n)^2} = \frac{e'^2}{1+e'^2} = \frac{2e''^2}{1+e''^2}\\
e'^2 &= \frac{e^2}{1-e^2} = \frac{2\alpha-\alpha^2}{(1-\alpha)^2} = \frac{4n}{(1-n)^2} = \frac{2e''^2}{1-e''^2}\\
\alpha &= \frac{2n}{1+n} = \frac{\alpha'}{1+\alpha'} = 1-\sqrt{1-e^2}\\
n &= \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}
\end{aligned}\right\} \qquad (2.3)
$$

Values e^2, e'^2, e''^2 are expressed by a following symmetric series through n

$$
\left.\begin{aligned}
e^2 &= 4n - 8n^2 + 12n^3 - 16n^4 + \ldots\\
e'^2 &= 4n + 8n^2 + 12n^3 + 16n^4 + \ldots\\
e''^2 &= 2n - 2n^2 + 2n^3 - 2n^4 + \ldots
\end{aligned}\right\} \qquad (2.4)
$$

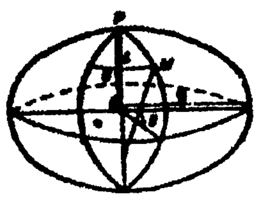


Fig. 8.

Value $\frac{a^2}{b} = c$ is radius of curvature at spheroid poles or polar radius of the spheroid.

In approximate calculations with an error of α^2 it is assumed $e^2 = 2\alpha$, or in a numerical expression $e^2 \approx 1:150$.

In the USSR geodetic work and that of socialist countries the Krasovskiy ellipsoid was adopted, in the west the greatest use is made of Bessel and Hayford ellipsoids.

The parameters of Krasovskiy ellipsoid:

$a = 6\,378\,245{,}00000$ м	$\lg a = 6.8047011973$
$b = 6\,356\,863{,}01877$ м	$\lg b = 6.8032428531$
$c = 6\,399\,698{,}90178$ м	$\lg c = 6.8061595414$
$\alpha = 0{,}003352329869$	$\lg \alpha = 7.5253467466_{-10}$
$n = 0{,}001678979181$	$\lg n = 7.2250453066_{-10}$
$e^2 = 0{,}006693421623$	$\lg e^2 = 7.8256481823_{-10}$
$e'^2 = 0{,}006738525415$	$\lg e'^2 = 7.8285648706_{-10}$

Parameters of Bessel ellipsoid:

$a = 6377397{,}15500$ м	$\lg a = 6.8046434637$
$b = 6356078{,}96325$ м	$\lg b = 6.8031892839$
$c = 6398786{,}84939$ м	$\lg c = 6.8060976435$
$\alpha = 0{,}003342773182$	$\lg \alpha = 7.5241069093_{-10}$
$n = 0{,}001674184801$	$\lg n = 7.2238033949_{-10}$
$e^2 = 0{,}006674372231$	$\lg e^2 = 7.8244104237_{-10}$
$e'^2 = 0{,}006719218798$	$\lg e'^2 = 7.8273187833_{-10}$

Parameters of Hayford ellipsoid:

$$a = 6378388000\ м \qquad \lg a = 6.8047109340$$
$$b = 6356911{,}94613\ м \qquad \lg b = 6.8032461938$$
$$c = 6399936{,}60810\ м \qquad \lg c = 6.8061756723$$
$$\alpha = 0{,}0033670033670 \qquad \lg \alpha = 7.5272435507_{-10}$$
$$n = 0{,}001686340641 \qquad \lg n = 7.2269453067_{-10}$$
$$e^2 = 0{,}006722670022 \qquad \lg e^2 = 7.8275417947_{-10}$$
$$e'^2 = 0{,}006768170197 \qquad \lg e'^2 = 7.8304712712_{-10}$$

§ 5. MERIDIAN ELLIPSE AND CONNECTED WITH IT SYSTEM OF COORDINATES

Geometric locus of points on the surface of prolate spheroid, having identical longitudes, is called meridian. Plane, passing through meridian and axis of rotation, is called meridional plane. If a plane of any meridian is taken as initial for counting longitudes, then such meridian is called prime. For counting longitudes from the initial a plane of meridian, is taken which passes through Greenwich astronomical observatory (near London).

Geodetic longitude of a point is called dihedral angle between planes of prime meridian and a meridian, passing through a given point (Fig. 8). Longitudes are counted from the prime meridian to the east and west and correspondingly are called eastern and western: they are distinguished either by corresponding letter designations, for instance L_e — eastern longitude, L_w — western longitude, or signs. In USSR minus signs are added to eastern longitudes.

Position of a point on meridian with a known longitude is fully determined, if geodetic latitude B is given as an acute angle between the equator plane and normal to surface at a given point (Fig. 8). Latitudes can be northern or southern.

Latitude and longitude fully determine the position of a point on the surface of an ellipsoid and are called geodetic coordinates. The system of geodetic coordinates on surface of a spheroid is the more natural and convenient for all surface of the terrestrial spheroid, therefore it is used both in theoretical investigations, and solution of practical problems of higher geodesy.

System of geodetic coordinates also has wide application in cartography. Conventional designation of geodetic coordinates is:

B — geodetic latitude.

L — geodetic longitude.

In certain cases, when meridian plane is given by longitude, it is convenient in theoretical problems to apply grid coordinates (x, y), referred to a plane of a given meridian (Fig. 9).

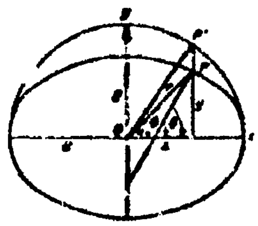


Fig. 9.

Equation of ellipse with origin of coordinates in a center is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{2.5}$$

This equation is satisfied by substitution

$$\left.\begin{aligned} x &= a\cos u \\ y &= b\sin u \end{aligned}\right\}. \tag{2.6}$$

where u — is called reduced latitude.

The reduced latitude is obtained by means of geometric construction in the following manner.

Describing from center of an ellipse a circumference by radius, equal to major semiaxis a, extend the ordinate of a given point y to intersection with circumference and connect by a straight line the obtained point with the center of ellipse. The angle between this line and the plane of equator will be the reduced latitude. The reduced latitude is also called parametric latitude. Application of a reduced latitude instead of geodetic has distinct advantages in certain theoretical problems.

Position of a point on meridional ellipse can be determined also by an angle, formed by radius-vector OP with equatorial plane (Fig. 9). This angle is called geocentric latitude. Geocentric latitude is used more frequently in astronomy and cartography, and in the theory of the figure of the Earth. Geocentric latitude is designated by Φ.

From Fig. 9 it follows that:

$$\left.\begin{aligned} x &= \bar{r}\cos\Phi \\ y &= \bar{r}\sin\Phi \end{aligned}\right\} \tag{2.7}$$

or

$$\mathrm{tg}\,\Phi=\frac{y}{x}.$$

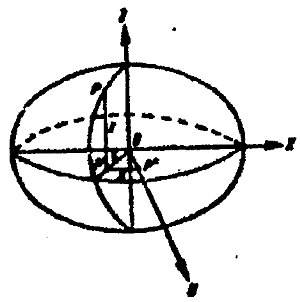


Fig. 10.

Position of point on the surface of a prolate spheroid can be determined by right-angle space coordinates with a beginning in the center of a spheroid (Fig. 10). Here the axis OZ is disposed along the axis of rotation of spheroid, and axis OX and OY in a plane of its equator. This system of coordinates are used in theoretical investigations and resolution of geodetic problems with application of chords of the ellipsoid. Equation of ellipsoid in these coordinates is in the form:

$$\frac{X^2}{a^2}+\frac{Y^2}{a^2}+\frac{Z^2}{b^2}=1.$$

This equation is satisfied by substitution:

$$\left.\begin{aligned} X &= a\cos u\cos L, \\ Y &= a\cos u\sin L \\ Z &= b\sin u. \end{aligned}\right\} \qquad (2.8)$$

since a cos u = x and b sin u = y, then:

$$\left.\begin{aligned} X &= x\cos L, \\ Y &= x\sin L \\ Z &= y \end{aligned}\right\}. \qquad (2.9)$$

Formulas (2.9) give ties between coordinates (X, Y, Z); (u, L) and (x, y).

§ 6. CONNECTION BETWEEN GEODETIC, GEOCENTRIC AND REDUCED LATITUDE

From elementary triangle $P_1P_1''P_2$ (Fig. 11) we have:

$$\operatorname{tg} B = -\frac{dx}{dy}$$

and from (2.6):

$$dx = -a\sin u\,du,$$
$$dy = b\cos u\,du.$$

Consequently:

$$\operatorname{tg} B = \frac{a\sin u}{b\cos u} = \frac{a}{b}\operatorname{tg} u,$$

since:

$$b = a\sqrt{1-e^2}.$$

then:

$$\operatorname{tg} u = \sqrt{1-e^2}\,\operatorname{tg} B. \qquad (2.10)$$

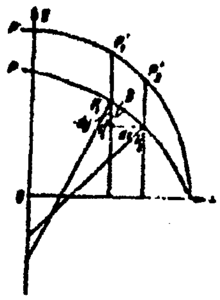


Fig. 11.

Taking into account that

$$\sin u = \frac{\operatorname{tg} u}{\sqrt{1+\operatorname{tg}^2 u}}; \quad \cos u = \frac{1}{\sqrt{1+\operatorname{tg}^2 u}},$$

and replacing the value u of geodetic latitude from (2.10), we obtain:

$$\left.\begin{aligned} \sin u &= \frac{\sqrt{1-e^2}\,\sin B}{\sqrt{1-e^2\sin^2 B}} \\ \cos u &= \frac{\cos B}{\sqrt{1-e^2\sin^2 B}} \end{aligned}\right\}. \qquad (2.11)$$

Having performed analogous transformations for sine and cosine of geodetic latitude, we obtain:

$$\left.\begin{aligned} \sin B &= \frac{\sin u}{\sqrt{1-e^2\cos^2 u}} \\ \cos B &= \frac{\sqrt{1-e^2}\,\cos u}{\sqrt{1-e^2\cos^2 u}} \end{aligned}\right\}. \qquad (2.11')$$

From expressions (2.6) and (2.11) it follows:

$$\left.\begin{aligned} x &= \frac{a\cos B}{\sqrt{1-e^2\sin^2 B}} \\ y &= \frac{a(1-e^2)\sin B}{\sqrt{1-e^2\sin^2 B}} \end{aligned}\right\}. \qquad (2.12)$$

We introduce designation:

$$\left.\begin{aligned} W &= \sqrt{1-e^2\sin^2 B} \\ w &= \sqrt{1-e'^2\cos^2 u} \end{aligned}\right\}, \tag{2.13}$$

where

$$Ww = \sqrt{1-e^2}\,.$$

Then

$$\left.\begin{aligned} \sin u &= \frac{\sqrt{1-e^2}\sin B}{W} \\ \cos u &= \frac{\cos B}{W} \\ \sin B &= \frac{\sin u}{w} \\ \cos B &= \frac{\sqrt{1-e^2}\cos u}{w} \end{aligned}\right\}. \tag{2.14}$$

$$\left.\begin{aligned} x &= \frac{a\cos B}{W} \\ y &= \frac{a(1-e^2)\sin B}{W} \end{aligned}\right\}. \tag{2.15}$$

W – is called first basic function of geodetic latitude, and w is a function of reduced latitude. These designations are conventional.

From comparison of formulas (2.9) and (2.15) it follows:

$$\left.\begin{aligned} X &= \frac{a\cos B\cos L}{W} \\ Y &= \frac{a\cos B\sin L}{W} \\ Z &= \frac{a(1-e^2)\sin B}{W} \end{aligned}\right\}. \tag{2.16}$$

For finding connection between geodetic and geocentric latitude let us consider formulas (2.7) and (2.15).

We have

$$\operatorname{tg}\Phi = (1-e^2)\operatorname{tg}B. \tag{2.17}$$

Closed expressions (2.10) and (2.17) are applied in rigid reckoning, in certain cases it is necessary to know the approximate values of differences ($\beta - \alpha$) and ($\beta - \Phi$).

Let us assume that:

$$\frac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{\operatorname{tg}\alpha + \operatorname{tg}\beta} = \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} = k.$$

We designate: $\alpha - \beta = \gamma$, then $\alpha + \beta = 2\alpha - \gamma$ and

$$\sin\gamma = k \sin(2\alpha - \gamma).$$

Using Eyler formulas (1.20), we find

$$e^{i\gamma} - e^{-i\gamma} = k(e^{2i\alpha - i\gamma} - e^{-2i\alpha + i\gamma}),$$

where $i = \sqrt{-1}$, e is a base of natural functions.

Multiply right and left part of this expression by $e^{i\gamma}$ and we will have:

$$e^{2i\gamma}(1 + ke^{-2i\alpha}) = 1 + ke^{2i\alpha}$$

or

$$2i\gamma = \ln(1 + ke^{2i\alpha}) - \ln(1 + ke^{-2i\alpha}).$$

For the right side of this formula logarithmic series can be applied (1.10):

$$\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots;$$

since $ke^{2i\alpha} < 1$, then:

$$\gamma = \alpha - \beta = k\,\frac{e^{2i\alpha} - e^{-2i\alpha}}{2i} - \frac{k^2}{2}\,\frac{e^{4i\alpha} - e^{-4i\alpha}}{2i} + \frac{k^3}{3}\,\frac{e^{6i\alpha} - e^{-6i\alpha}}{2i} - \dots$$

or

$$\alpha - \beta = k\sin 2\alpha - \frac{k^2}{2}\sin 4\alpha + \frac{k^3}{3}\sin 6\alpha - \dots$$

Applying this general formula to our case, we obtain:

$$\frac{\operatorname{tg} B-\operatorname{tg} u}{\operatorname{tg} B+\operatorname{tg} u}=k=\frac{\sin(B-u)}{\sin(B+u)}=\frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}=n,$$
$$\frac{\operatorname{tg} B-\operatorname{tg} \Phi}{\operatorname{tg} B+\operatorname{tg} \Phi}=k=\frac{\sin(B-\Phi)}{\sin(B+\Phi)}=\frac{e^2}{2-e^2}=e''^2.$$

Thus, for difference (B - u) we have k = n, and for difference (B - Φ) correspondingly $k = e''^2$, therefore:

$$B-u=n\sin 2B-\frac{n^2}{2}\sin 4B+\frac{n^3}{3}\sin 6B-\ldots \tag{2.18}$$

$$B-\Phi=e''^2\sin 2B-\frac{e''^4}{2}\sin 4B+\frac{e''^6}{3}\sin 6B-\ldots \tag{2.19}$$

For Krasovskiy ellipsoid these differences in seconds will be:

$$(B-u)''=346'',3143\sin 2B-0'',2907\sin 4B+0'',0003\sin 6B-\ldots,$$
$$(B-\Phi)''=692'',6267\sin 2B-1'',1629\sin 4B+0'',0026\sin 6B-\ldots$$

Differences (B - u) and (B - Φ), as can be seen from (2.18) and (2.19), attain maximum when $B = 45^\circ$, where

$$(B-u)_{max}\approx 5',9,\quad (B-\Phi)_{max}\approx 11',5.$$

From (2.18) and (2.19) for the most approximate calculations it follows that:

$$\left.\begin{aligned}(B-u)''&=\frac{e^2\rho''}{4}\sin 2B-\ldots\\(B-\Phi)''&=\frac{e^2\rho''}{2}\sin 2B-\ldots\end{aligned}\right\}. \tag{2.20}$$

Sometimes it is expedient to express geodetic latitude by auxiliary angle, according to the following formula:

$$e \sin B = \sin \psi. \tag{2.21}$$

With introduction of an angle ψ recording of first function of geodetic latitude W, is simplified thus, for example:

$$W = \cos \psi. \tag{2.21'}$$

Geometric meaning of angle ψ is shown in Fig. 12, where F_1 and F_2 are focuses of meridian ellipse, $\overline{F}n$ is a normal at point $\overline{F}$, and B is geodetic latitude of point F.

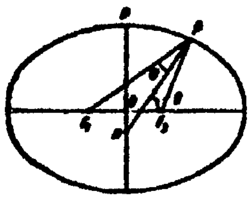


Fig. 12.

§ 7. MAIN RADII OF CURVATURE AT A GIVEN POINT OF A SPHEROID

Through normal of every point on the surface of a spheroid it is possible to pass a great number of normal planes, perpendicular to tangential plane at a given point. Trace of a normal plane on a surface is a plane curve, called <u>normal section</u>. Curvature of various normal sections of a spheroid at a given point is unequal, they have their own extremum, and <u>minimum</u> and <u>maximum</u> values. Sections with extremum curvature are called <u>principal normal sections</u>. Consequently, one of the main sections has maximum curvature and minimum radius of curvature, and another — minimum curvature and maximum radius.

Curvature of any normal section is determined by a well known Eyler formula (1.44)

$$\frac{1}{R_A} = \frac{\cos^2 A}{R_1} + \frac{\sin^2 A}{R_2}.$$

where A — azimuth of given normal section, and R_1 and R_2 = radii of curvature of principal normal sections. Where $A = 0^{\circ}$ we have $R_{0} = \frac{1}{R_1}$ and when $A = 90^{\circ}$ correspondingly $\frac{1}{R_{90^{\circ}}} = \frac{1}{R_2}$. Thus, on terrestrial spheroid one of principal normal sections coincides with meridional section, and another with section of the first vertical. In spheroidal geodesy following designations are taken for radii of curvature of principal normal sections: M is radius of curvature of meridional section; N is radius of curvature of a section of first vertical; M and N are applied in many theoretical and practical calculations as functions of latitude B of a given point. In Fig. 1

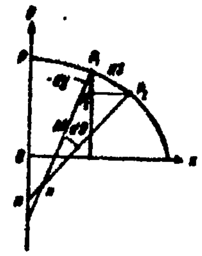


Fig. 13.

$\breve{P_1 P_2}$ = ds is an elementary arc of meridian; K is center of curvature of meridian section; M is radius of curvature of meridional section at current point from elementary triangle $P_1' P_1 P_2$

$$ds = M\,dB = \sqrt{dx^2 + dy^2} = dy\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{dy}{\sqrt{1 + \mathrm{tg}^2 B}} = \frac{dy}{\cos B}$$

$$dy = M\,dB \cos B$$

or

$$M = \frac{1}{\cos B} \cdot \frac{dy}{dB}. \tag{2.22}$$

From (2.15)

$$\frac{dy}{dB} = a(1 - e^2) \cdot \left\{+\frac{\cos B}{W} - \frac{\sin B}{W^2} \cdot \frac{dW}{dB}\right\}.$$

but

$$\frac{dW}{dB} = -\frac{e^2 \cos B \sin B}{W}$$

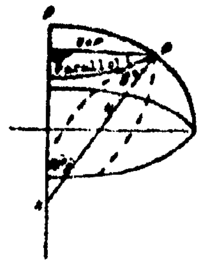
Therefore

$$\frac{dy}{dB} = \frac{a(1 - e^2)\cos B}{W^3}.$$

Consequently,

$$M = \frac{a(1 - e^2)}{W^3} = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 B)^{3/2}}. \tag{2.23}$$

Plane of parallel, perpendicular to meridional plane, is slanted to a plane of the first vertical, where angle of inclination is equal to geodetic latitude of a given point (Fig. 14). Parallel and section of first vertical at a given point have common tangent. By well known theorem of Menier the radius of curvature of slanted section is equal to the product of radius of main section (in this case first



vertical) by cosine of the angle of inclination, that is:

$$x = r = N \cos B.$$

Or, taking into account (2.15),

Fig. 14.

$$N = \frac{x}{\cos B} = \frac{a \cos B}{\cos B\, W} = \frac{a}{W} = \frac{a}{\sqrt{1 - e^2 \sin^2 B}}. \qquad (2.24)$$

Let us consider M and N as extreme values of B.

1. Where B = 0

$$M = a(1 - e^2),$$
$$N = a.$$

Consequently, M and N are minimum at points on equator.

2. Where $B = 90^{\circ}$:

$$M = \frac{a}{\sqrt{1 - e^2}} = c,$$
$$N = \frac{a}{\sqrt{1 - e^2}} = c, \text{ i.e.,}$$

That is: M and N are maximum at spheroid poles.

Formulas (2.23) and (2.24) assume more symmetric form, if in them e^2 is expressed by e'^2 in the formula:

$$e^2 = \frac{e'^2}{1 + e'^2}.$$

Then:

$$W = \sqrt{1 - \frac{e'^2}{1 + e'^2} \sin^2 B} = \frac{\sqrt{1 + e'^2 \cos^2 B}}{\sqrt{1 + e'^2}}.$$
$$1 - e^2 = \frac{1}{1 + e'^2}.$$

but:

$$c=\frac{a}{\sqrt{1-e^2}}=a\sqrt{1+e'^2}.$$

Let us designate:

$$V=\sqrt{1+e'^2\cos^2 B};\quad e'^2\cos^2 B=\eta^2.$$

Consequently,

$$V=W\sqrt{1+e'^2}=\frac{W}{\sqrt{1-e^2}}. \qquad (2.25)$$

Replacing W by V by the formula (2.25) for radii of principal normal sections, we obtain:

$$\left.\begin{aligned} M&=\frac{c}{V^3}\\ N&=\frac{c}{V}\end{aligned}\right\}. \qquad (2.26)$$

From (2.26) it follows that:

$$\frac{N}{M}=V^2=1+\eta^2=1+e'^2\cos^2 B.$$

Right side of this equality is a value essentially positive and larger than a unit, therefore at any point of spheroid $N > M$. The greater value of V^2 is on the equator and is equal to 1.00674 (Krasovskiy ellipsoid). Hence it is easy to conclude that meridional section at a given point of a spheroid has maximum curvature and minimum radius; while a section of the first vertical has minimum curvature and maximum radius. The relation $\frac{N}{M}$ at each point renders a presentation of deflection of the curvature of a spheroid from the curvature of a sphere.

In geodetic calculations M and N are used in the form of expressions $\frac{M}{\rho''}$, $\frac{N}{\rho''}$ or $\frac{\rho''}{M}$, $\frac{\rho''}{N}$, where the last ones are applied more frequently and for them special designations are taken:

$$\left.\begin{aligned} \frac{\rho''}{M}&=(1)\\ \frac{\rho''}{N}&=(2)\end{aligned}\right\}. \qquad (2.27)$$

where $\rho'' = 206265''$ is a number of seconds in a radian, values (1) and (2) constitute angles, under which arcs of meridian and first vertical 1 m in length are seen from the centers of curvature of these curves. Geometrically these values express correspondingly curvature of meridian and first vertical in seconds per unit of length. The values of M and N are expressed in meters. Expressions (1) and (2) are called first and second geodetic values. These values are used with indices, for example, $(1)_1$, $(1)_2$, $(1)_m$, signifying that they are referred to first, second and average latitudes.

In "Tables For Calculation of Geodetic Coordinates" (Geodezizdat, 1953)[1] logarithms for values (1) and (2) are given with eight decimal places for every minute of latitude from 8° to 30°.

In "Tables For Logarithmic Calculation of Gauss-Kruger Coordinates for Latitude from 30° to 80°" (Geodezizdat, 1948), F. N. Krasovskiy and A. A. Izotov[2] give lg $\frac{M}{\rho''}$ with seven and lg $\frac{N}{\rho''}$ with eight decimal places for each minute of latitude.

Value (1) or M are used for calculation of differences of latitudes of geodetic points and lengths of arcs of meridians; (2) or N, for calculation of lengths of arcs of parallels and differences of longitudes and azimuths of geodetic points.

With very approximate calculations, assuming $M = N = 6 \cdot 10^6$ m and $\rho'' = 2 \cdot 10^5$, we take:

$$(1) = (2) \approx \frac{1}{30}.$$

$\frac{1}{M}$, $\frac{1}{N}$ or in general $\frac{1}{R_A}$ give curvature of corresponding normal sections at given point of a spheroid. However frequently a need arises to know the curvature of a surface at a given point. For that in higher geodesy and in higher mathematics, an idea is introduced about full or Gauss curvature, equal to:

$$K = \frac{1}{MN} = \frac{1}{R^2};$$

R – average radius of curvature, it is defined as an average geometric form from main radii of curvature at a given point, that is:

[1]Subsequently these tables will be called – "Geodetic tables".

[2]Subsequently will be called: "Krasovskiy and Izotov Tables".

$$R = \sqrt{MN} = \frac{a\sqrt{1-e^2}}{W^2} = \frac{b}{W^2} = \frac{c}{V^2}, \qquad K = \frac{1}{MN} = \frac{1}{R^2} = \frac{W^4}{a^2(1-e^2)} = \frac{W^4}{b^2} = \frac{V^4}{c^2}. \qquad (2.28)$$

Average radius of curvature is used in the image of parts of a surface of a spheroid on a sphere or on a plane, during calculations of areas and spherical excesses of figures on the surface of a spheroid. In Geodetic tables for the indicated purpose are given:

$$\lg R, \lg \frac{\rho''}{2R^2} \text{ and } \lg \frac{1}{R^2}.$$

Radius of curvature of any normal section can be obtained from the Eyler formula:

$$R_A = \frac{MN}{N\cos^2 A + M\sin^2 A} = \frac{N}{\sin^2 A + V^2\cos^2 A} = \frac{N}{1 + \eta^2\cos^2 A}. \qquad (2.29)$$

With error in values of the order of η^4 from formula (2.29):

$$R_A = N(1 - \eta^2\cos^2 A + \ldots).$$

In resolution of certain problems it is sometimes necessary to consider the Earth as a sphere. If this is done for very approximate calculations, the radius of a sphere R_0 is taken as equal to 6370 km. Such a sphere is usually taken in cartography, its surface is equal to the surface of an ellipsoid. For Krasovskiy ellipsoid the radius of such a sphere is R = 6371.116 km. In other cases it is expedient to take $R_c = \frac{a + a + b}{3} = 6370784.3$ m (Krasovskiy ellipsoid).

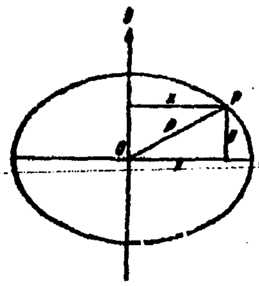
Radius of a parallel. Locus of points on the surface of a prolate spheroid, having the same latitude, are called a parallel. Terrestrial parallels are circumferences whose radii are equal to the length of a section of a perpendicular, dropped from a given point on the axis of a rotation of an ellipsoid. By this determination the radius of a parallel is equal to abscissa in system of grid coordinates in a plane of a given meridian. Usually the radius of a parallel is designated by r, consequently:

$$r = x = \frac{a\cos B}{W} = N\cos B = \frac{c\cos B}{V}. \qquad (2.30)$$

In geodetic calculations r is rarely used instead an expression $\frac{r}{\rho''} = \frac{N}{\rho''} \cos B$, is used equal to the length of an arc of parallel, corresponding to the difference of longitudes for one second. Value $\frac{r}{\rho''}$ is designated by b_1 shown in "Tables for Nonlogarithmic Calculation of Gauss-Kruger Coordinates" (Geodezizdat, 1959).[1]

Distance from the center of an ellipsoid to a given point is designated by ρ (Fig. 15) and will be called radius-vector.

From Fig. 15:



$$\rho = \sqrt{x^2 + y^2}. \tag{2.31}$$

Fig. 15.

or, taking the value of x and y from formulas (2.15), we obtain:

$$\rho = \frac{a}{W}\sqrt{\cos^2 B + (1-e^2)^2 \sin^2 B} = \frac{a}{W}\sqrt{1 - e^2(2-e^2)\sin^2 B}$$

but:

$$\frac{1}{W} = 1 + \frac{e^2}{2}\sin^2 B + \frac{3}{8}e^4 \sin^4 B + \dots \tag{I}$$

$$\sqrt{1 - e^2(2-e^2)\sin^2 B} = 1 - \frac{e^2(2-e^2)}{2}\sin^2 B - \frac{e^4(2-e^2)^2}{8}\sin^4 B - \dots \tag{II}$$

Multiplying formula (I) on (II) and retaining terms to e^4, we find:

$$\rho = a\left(1 - \frac{e^2}{2}\sin^2 B + \frac{e^4}{2}\sin^2 B - \frac{5}{8}e^4 \sin^4 B\right) + \dots \tag{2.32}$$

Radius-vector is rarely used in spheroitic geodesy. This value is used in resolution of certain problems of theory of the figure of the Earth.

We will clarify the geometric meaning of functions of geodetic latitude W and V.

Through a point P of meridional ellipse draw tangent PT and extend it to the crossing with an axis x (Fig. 16). From the center of an ellipsoid drop to tangent PT a perpendicular and designate the length of perpendicular OT' = $\overline{\rho}$. Obviously, the

[1]Subsequently these tables will be called: "D. A. Larin Tables".

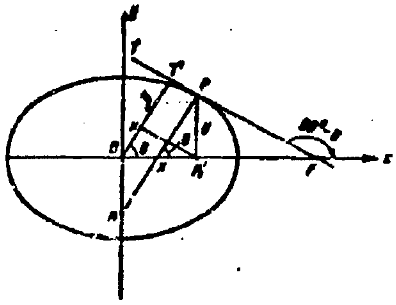


Fig. 16.

angle between the perpendicular $\bar{\rho}$ and the axis x will be a geodetic latitude. Let us consider the projection of a broken line $OP_1'PT'$ on perpendicular $\bar{\rho}$, we have

$$\bar{\rho} = x\cos B + y\sin B.$$

Obtaining the value of x and y from formula (2.15), we have:

$$\bar{\rho} = \frac{a\cos^2 B}{W} + \frac{a(1-e^2)\sin^2 B}{W} = \frac{a}{W}(1 - e^2\sin^2 B),$$

or:

$$\bar{\rho} = a(1 - e^2\sin^2 B)^{\frac{1}{2}} = aW, \qquad (2.33)$$

but:

$$aW = bV,$$

consequently:

$$\bar{\rho} = bV, \qquad (2.33')$$

Thus,

$$\left.\begin{aligned} W &= \frac{\bar{\rho}}{a} = \frac{a}{N} \\ V &= \frac{\bar{\rho}}{b} = \frac{c}{N} \end{aligned}\right\} \qquad (2.34)$$

Formulas (2.34) give geometric presentation of the functions of W and V; they are correspondingly the essence of relation of the length of perpendicular $\bar{\rho}$ to the major and minor semiaxes of an ellipsoid.

§ 8. TRANSFORMATION OF W AND V IN POWER SERIES

Functions of W and V appear in many theoretical and practical problems of spheroitic geodesy. For calculation of W and V and connected with them values it is expedient to present their series by increasing powers of e^2 and e'^2.

We have:

$$W^2 = 1 - e^2 \sin^2 B,$$
$$V^2 = 1 + e'^2 \cos^2 B,$$

or:

$$\lg W^2 = \mu \ln (1 - e^2 \sin^2 B),$$
$$\lg V^2 = \mu \ln (1 + e'^2 \cos^2 B).$$

Applying to these expressions the logarithmic series (1.10) and (1.11):

$$\ln(1 \mp u) = \mp u - \frac{u^2}{2} \mp \frac{u^3}{3} - \frac{u^4}{4} \mp \ldots,$$

we obtain:

$$\lg W^2 = -\mu\left[e^2 \sin^2 B + \frac{e^4}{2} \sin^4 B + \frac{e^6}{3} \sin^6 B + \frac{e^8}{4} \sin^8 B + \ldots\right],$$
$$\lg V^2 = \mu\left[e'^2 \cos^2 B - \frac{e'^4}{2} \cos^4 B + \frac{e'^6}{3} \cos^6 B - \frac{e'^8}{4} \cos^8 B + \ldots\right].$$

For calculations it is convenient to use even series of sines and cosines and to substitute by cosines of even arcs by the formulas in (1.25) and (1.26), then:

$$\left.\begin{aligned} \sin^2 B &= \frac{1}{2} - \frac{1}{2}\cos 2B \\ \sin^4 B &= \frac{3}{8} - \frac{1}{2}\cos 2B + \frac{1}{8}\cos 4B \\ \sin^6 B &= \frac{5}{16} - \frac{15}{32}\cos 2B + \frac{3}{16}\cos 4B - \frac{1}{32}\cos 6B \end{aligned}\right\}; \qquad (2.35)$$

$$\left.\begin{aligned} \cos^2 B &= \frac{1}{2} + \frac{1}{2}\cos 2B \\ \cos^4 B &= \frac{3}{8} + \frac{1}{2}\cos 2B + \frac{1}{8}\cos 4B \\ \cos^6 B &= \frac{5}{16} + \frac{15}{32}\cos 2B + \frac{3}{16}\cos 4B + \frac{1}{32}\cos 6B \end{aligned}\right\}. \qquad (2.36)$$

With substitution of $\sin^i B$ and cos B ($i = 2, 4, 6 \ldots$) by cosines of multiples of arcs we obtain:

$$\lg W^2 = \mu\left\{-\left(\frac{1}{2}e^2 + \frac{3}{16}e^4 + \frac{5}{48}e^6 + \frac{35}{512}e^8 + \ldots\right) + \right.$$
$$+\left(\frac{1}{2}e^2 + \frac{1}{4}e^4 + \frac{5}{32}e^6 + \frac{7}{64}e^8 + \ldots\right)\cos 2B -$$
$$-\left(\frac{1}{16}e^4 + \frac{1}{16}e^6 + \frac{7}{128}e^8 + \ldots\right)\cos 4B +$$
$$+\left(\frac{1}{96}e^6 + \frac{1}{64}e^8 + \ldots\right)\cos 6B -$$
$$\left.-\left(\frac{1}{512}e^8 + \ldots\right)\cos 8B.\right\}$$

$$\lg V^2 = \mu\left\{\left(\frac{1}{2}e'^2 - \frac{3}{16}e'^4 + \frac{5}{48}e'^6 - \frac{35}{512}e'^8 + \ldots\right) + \right.$$
$$+\left(\frac{1}{2}e'^2 - \frac{1}{4}e'^4 + \frac{5}{32}e'^6 - \frac{7}{64}e'^8 + \ldots\right)\cos 2B -$$
$$-\left(\frac{1}{16}e'^4 - \frac{1}{16}e'^6 + \frac{7}{128}e'^8 - \ldots\right)\cos 4B +$$
$$+\left(\frac{1}{96}e'^6 - \frac{1}{64}e'^8 + \ldots\right)\cos 6B -$$
$$\left.-\left(\frac{1}{512}e'^8 + \ldots\right)\cos 8B.\right\}$$

For Krasovskiy ellipsoid:

$$\begin{aligned}\lg V = {} & 0{,}0007297842112 + \\ & + 0{,}0007291713934 \cos 2B - \\ & - 0{,}0000006121318 \cos 4B + \\ & + 0{,}0000000006852 \cos 6B - \\ & - 0{,}0000000000009 \cos 8B.\end{aligned}$$

But since $W = V\sqrt{1-e^2}$, then $\lg W = \lg V + \lg\sqrt{1-e^2} = \lg V + 9.9985416558$. Logarithms of values V are given in Geodetic tables by argument of latitude for every minute with ten decimal places. With the help of tables of values lg V it is possible to compose any tables for calculation of radii of curvature and other functions of latitude. Values of $\frac{1}{W}$ are given in 93rd issue of the Works of TsNIIGAiK for 10' of latitude with eight decimal places.

§ 9. LENGTHS OF ARCS OF MERIDIAN AND PARALLEL

Elementary arc of meridian ds (Fig. 17) is equal to:

$$ds = M\,dB = \frac{a(1-e^2)\,dB}{W^3}$$

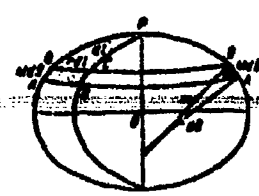


Fig. 17.

or:

$$s = a(1-e^2)\int_0^B \frac{dB}{W^3} = a(1-e^2)\int_0^B \frac{dB}{(1-e^2\sin^2 B)^{3/2}} = c\int_0^B \frac{dB}{V^3} = c\int_0^B \frac{dB}{(1+e'^2\cos^2 B)^{3/2}}. \qquad (2.37)$$

These integrals in elementary functions are not taken in a closed form, therefore it is necessary to transform W^{-3} and V^{-3} to binomial series and then to integrate term by term with a given degree of accuracy.

We have:

$$W^{-3}=(1-e^2\sin^2 B)^{-3/2}=1+\frac{3}{2}e^2\sin^2 B+\frac{15}{8}e^4\sin^4 B+$$
$$+\frac{35}{16}e^6\sin^6 B+\frac{315}{128}e^8\sin^8 B+\ldots,$$
$$V^{-3}=(1+e'^2\cos^2 B)^{-3/2}=$$
$$=1-\frac{3}{2}e'^2\cos^2 B+\frac{15}{8}e'^4\cos^4 B-\frac{35}{16}e'^6\cos^6 B+\frac{315}{128}e'^8\cos^8 B-\ldots$$

Substituting in these expressions of $\sin^i B$ and $\cos^i B$ for cos B (i = 2, 4, 6, 8 ...) by formulas (1.25) and (1.26), we obtain

$$\left.\begin{aligned}W^{-3}&=A-B\cos 2B+C\cos 4B-D\cos 6B+E\cos 8B-\ldots\\ V^{-3}&=A^*-B^*\cos 2B+C^*\cos 4B-D^*\cos 6B+E^*\cos 8B-\ldots\end{aligned}\right\}\qquad(2.38)$$

where:

$$\begin{aligned}A&=1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\ldots\\ B&=\frac{3}{4}e^2+\frac{15}{16}e^4+\frac{525}{512}e^6+\frac{2205}{2048}e^8+\ldots\\ C&=\frac{15}{64}e^4+\frac{105}{256}e^6+\frac{2205}{4096}e^8+\ldots\\ D&=\frac{35}{512}e^6+\frac{315}{2048}e^8+\ldots\\ E&=\frac{315}{16384}e^8+\ldots\end{aligned}$$

For Krasovskiy ellipsoid:

$$\begin{aligned}A&=1{,}0050517739,\\ B&=0{,}00506237764,\\ C&=0{,}00001062451,\\ D&=0{,}00000002081,\\ E&=0{,}00000000004.\end{aligned}$$

$$\begin{aligned}A^*&=1-\frac{3}{4}e'^2+\frac{45}{64}e'^4-\frac{175}{256}e'^6+\frac{11025}{16384}e'^8-\ldots\\ B^*&=\frac{3}{4}e'^2-\frac{15}{16}e'^4+\frac{525}{512}e'^6-\frac{2205}{2048}e'^8+\ldots\\ C^*&=\frac{15}{64}e'^4-\frac{105}{256}e'^6+\frac{2205}{4096}e'^8-\ldots\\ D^*&=\frac{35}{512}e'^6-\frac{315}{2048}e'^8+\ldots\\ E^*&=\frac{315}{16384}e'^8-\ldots\end{aligned}$$

For Krasovskiy ellipsoid:

$$
\begin{aligned}
A^* &= 1{,}00168250882,\\
B^* &= 0{,}00168180230,\\
C^* &= 0{,}00000070593,\\
D^* &= 0{,}00000000059,\\
E^* &= 0{,}00000000000.
\end{aligned}
$$

Taking values of W^{-3} and V^{-3} from (2.38) in (2.37) and taking into account that $\int \cos 2B\,dB = \frac{1}{2}\sin 2B$, $\int \cos 4B\,dB = \frac{1}{4}\sin 4B$ etc, we obtain:

$$
s = a(1-e^2)\left\{\frac{AB^\circ}{\rho^\circ} - \frac{B}{2}\sin 2B + \frac{C}{4}\sin 4B - \frac{D}{6}\sin 6B + \frac{E}{8}\sin 8B - \ldots\right\}. \qquad (2.39)
$$

$$
s = c\left\{\frac{A^*B^\circ}{\rho^\circ} - \frac{B^*}{2}\sin 2B + \frac{C^*}{4}\sin 4B - \frac{D^*}{6}\sin 6B + \frac{E^*}{8}\sin 8B - \ldots\right\}. \qquad (2.40)
$$

Taking $B = \frac{\pi}{2}$, from these formulas, we obtain length of a quarter of meridian $Q = a(1-e^2)A\frac{\pi}{2}$. For Krasovskiy ellipsoid:

$$
Q = 10002137{,}498 \text{ м.}
$$

After substitution of values of constants A, B, ..., A^*, B^*, ... in (2.39) and (2.40) we obtain

$$
\begin{aligned}
s = {} & 6367558{,}495874600\,\frac{B''}{\rho''}\\
& - 16036{,}4802690885 \sin 2B\\
& + 16{,}8280667831 \sin 4B\\
& - 0{,}0219752790 \sin 6B\\
& + 0{,}00003112433 \sin 8B
\end{aligned} \qquad (2.41)
$$

This expression is used in composition of tables of arcs of meridian. Lengths of arcs of meridian for every minute of latitude from 30° to 80° with accuracy of one millimeter are given in: "Tables for Logarithmic Calculation of Gauss-Kruger Coordinates" (1946) F. N. Krasovskiy and A. A. Izotov and "Tables of D. A. Larin".

In these tables are given values of the arc of meridian from equator to a parallel with a given latitude, which are designated for X. In Table 1 values of X are given for latitudes 52°-52°10′ from D. A. Larin Tables.

Table 1

Latitude	X	Δ	ΔB"	Corrections
52°00'	5763444,761	3090,812	0	0
1	765299,254	821	8	1
2	767153,756	830	21	2
3	769008,250	839	36	3
4	770862,756	848	50	4
5	5772717,267	3090,856	60	
6	774571,784	865		
7	776426,305	874		
8	778280,832	883		
9	780135,365	891		
52°10'	5781989,902	3090,900		

Note: Here Δ is a change of X by 100" for a given latitude where Δ is interpolated for an average from the given and tabular latitudes, and the corrections to Δ are taken from right column of table for ΔB".

Example. Latitude $B = 52^{\circ}05'23''$, 6257, are given to find X.

Tabular latitude $B_0 = 52^{\circ}05'$, then $X_0 = 5772717.267$

Tabular increase for B_0 is equal to $\Delta_0 = 3090.856$

Correction for $\Delta B'' = 24''$ is equal to +2

Corrected increase equal to $\Delta_m = 3090.858$

Correction for $\Delta X = \Delta_m \Delta B'' \cdot 10^{-2}$ is equal to $\Delta X = 730.237$

Required value $X = 5773447.504$

If it is necessary to determine the arc of meridian s between parallels with latitude B_1 and B_2, then, after finding X_1 and X_2 by B_1 and B_2, their difference is taken, that is: $s = X_2 - X_1$.

Expression for the length of arc of meridian for short distances, on the order of length of side or link of 1st order triangulation, can be obtained by means of application of Taylor formula with introduction of average argument.

Let us take points P_1 and P_2 with latitude B_1 and B_2.
We designate them:

$$\Delta B = B_2 - B_1,$$
$$B_m = \frac{1}{2}(B_2 + B_1).$$

whence

$$B_1 = B_m - \frac{\Delta B}{2},$$
$$B_2 = B_m + \frac{\Delta B}{2}.$$

$$X_1 = X(B_1) = X\left(B_m - \frac{\Delta B}{2}\right) = X(B_m) - \frac{\Delta B}{2}\left(\frac{dX}{dB}\right)_m + \frac{\Delta B^2}{8}\left(\frac{d^2X}{dB^2}\right)_m -$$
$$- \frac{\Delta B^3}{48}\left(\frac{d^3X}{dB^3}\right)_m + \ldots,$$
$$X_2 = X(B_2) = X\left(B_m + \frac{\Delta B}{2}\right) = X(B_m) + \frac{\Delta B}{2}\left(\frac{dX}{dB}\right)_m +$$
$$+ \frac{\Delta B^2}{8}\left(\frac{d^2X}{dB^2}\right)_m + \frac{\Delta B^3}{48}\left(\frac{d^3X}{dB^3}\right)_m + \ldots$$

Designating difference of these arcs for s, we obtain

$$s = X_2 - X_1 = \left(\frac{dX}{dB}\right)_m \Delta B + \left(\frac{d^3X}{dB^3}\right)_m \frac{\Delta B^3}{24} + \ldots \qquad (2.42)$$

Here dX = ds, therefore

$$\left(\frac{dX}{dB}\right)_m = \left(\frac{ds}{dB}\right)_m = M_m = \frac{c}{V_m^3},$$
$$\left(\frac{d^2X}{dB^2}\right)_m = -\frac{3c}{V_m^4}\left(\frac{dV}{dB}\right)_m = \frac{3M_m \eta_m^2 t_m}{V_m^2},$$
$$\left(\frac{d^3X}{dB^3}\right)_m = \frac{3M_m \eta_m^2}{V_m^4}(1 - t_m^2 + \eta_m^2 + 4\eta_m^2 t_m^2).$$

where $t_m = \mathrm{tg}\, B_m$. Sign $_m$ indicates that the functions are calculated for average latitude. Consequently,

$$s = M_m \frac{(B_2 - B_1)''}{\rho''} + \frac{M_m \eta_m^2}{8V_m^4}(1 - t_m^2 + \eta_m^2 + 4\eta_m^2 t_m^2) \cdot \frac{(B_2 - B_1)''^3}{\rho''^3} \ldots \qquad (2.43)$$

or:

$$s = \frac{(B_2 - B_1)}{(1)_m} + k_m \Delta B''^3, \qquad (2.44)$$

where:

$$k_m = \frac{M_m \eta_m^2}{8V_m^4}(1 - t_m^2 + \eta_m^2 + 4\eta_m^2 t_m^2).$$

k_m is a small value, which can be taken from Table 2.

Table 2.

B_m	h, мм	B_m	h, мм	B_m	h, мм
0°	28,1	45	0,2	53	− 7,6
10°	26,4	46	−0,7	54	− 8,6
20°	21,7	47	−1,7	55	− 9,5
25	18,3	48	−2,7	60	−14,0
30	14,3	49	−3,7	65	−18,2
35	9,9	50	−4,7	70	−21,7
40°	5,1	51	−5,7	80	−26,8
45	0,2	52	−6,6	90	−28,6

Formula (2.44) can be applied with sufficient accuracy for difference of latitudes not more than 6°-7°. In correction member ΔB is expressed in degrees.

Example. Given: $B_1 = 55^{\circ}27'48''.245$, $B_2 = 59^{\circ}57'48''.245$. Find s by the formula (2.44)

$$\Delta B^{\circ} = 4^{\circ}30',$$
$$\Delta B'' = 16200'',$$
$$\lg \Delta B'' = 4,20951501,$$
$$\lg (1)''_m = 8,50951687,7$$
$$\lg s_0 = 5,69999813,3$$
$$s_0 = 501185,078$$
$$k\Delta B^{\circ 3} = -1,087$$
$$s = 501183,991 \text{ м}$$
$$s = 501183,983 \text{ м}$$ (D. A. Larin Tables).

From Table 2 it follows that for distances of the order of a side of triangulation (that is, 25-30 km or 15' arc) the maximum value of correction $k\Delta B^{\circ 3}$ will be at latitude 90°, where

$$k\Delta B^{\circ 3} = \frac{28,6}{64} \approx 0,5 \text{ мм}.$$

For distances less than 45 km it is possible to use correction member from formula (2.44), that is to take:

$$s = \frac{(B_2 - B_1)''}{(1)_m}, \qquad (2.45)$$

or

$$s = \Delta_m (B_2 - B_1)'' \cdot 10^{-2}, \qquad (2.45')$$

Δ_m is taken from D. A. Larin Tables for average latitude.

Example. Find s by formula (2.45')

$$B_1 = 55°27'48'',245, \quad B_2 = 55°42'50'',257,$$
$$B_m = 55°35'19'',251,$$
$$(B_2 - B_1) = \Delta B'' = 902'',012$$
$$\Delta_m = 3092,671$$
$$s = \frac{\Delta_m \Delta B''}{100} = 27896,264 \text{ м.}$$

In certain cases it is required on a given length of arc of meridian and latitude of one of its terminal points to find a difference of latitudes:

$$\Delta B = \frac{s}{\Delta_m} \cdot 100. \qquad (2.46)$$

By this formula the calculation is made by a method of approximations, since Δ_m is a function of mean latitude. In first approximation Δ is taken at a known latitude on one of terminal points of arc, after obtaining the approximate average latitude, calculate succeedingly the following approximations to coincidence of results of calculations of the last two approximations within limits of given accuracy. As a rule, second approximation gives the desired value with an accuracy of up to $0''.001$. For obtaining accuracy up to $0''.0001$ it is necessary to carry out three approximations.

Let us solve inverse problem according to data of the preceding example.

Given: $s = 27896.264$ m, $B_1 = 55°27'48''.245$. Find B_2

I approximation:

$$s = 27896,264 \text{ м}$$
$$\Delta_1 = 3092,605$$
$$\Delta B_1'' = 902,031$$
$$B_1 = 55°27'48'',245$$
$$\frac{\Delta B}{2} = 7'31'',016$$
$$B_m' = 55°35'19'',261$$

II approximation:

$$\Delta_m' = 3092'',671$$
$$\Delta B = 902'',012\,(15'02'',012)$$
$$B_1 = 55°27'48'',245$$
$$\Delta B = 15'02'',012$$
$$B_2 = 55°42'50'',257.$$

Arc of parallel. Terrestrial parallels, as was already is established, are the circumference of radii $N \cos B = r$. Central angle is the difference of longitudes of terminal points of arc. Designating the length of arc of parallel by s', and the difference of longitudes l, we obtain

$$s' = \frac{N \cos B l''}{\rho''} \qquad (2.47)$$

But by previous:

$$\frac{N\cos B}{\rho''} = b_1,$$

therefore:

$$s' = b_1 l''. \tag{2.48}$$

Formula (2.48) is used for calculation of arcs of parallels with the aid of D. A. Larin tables, where b_1 is given for every minute of latitude.

Example. Given: $B = 55°27'48''.245$

$l = 1°30'45''.457$

$l'' = 5445.457$

$b_1 = 175709.793$ (from tables, p. 63)

$s' = 95685.898$

Inverse problem, that is, finding differences of longitudes, is resolved by the formula:

$$l'' = \frac{s'}{b_1}.$$

Examples of calculations of arcs of meridian and parallel and differences of latitude and longitudes are given on p. 252-257 "Practicum on Higher Geodesy" by B. N. Rabinovich, second edition, 1961.[1]

§ 10. CALCULATION OF AREAS ON THE SURFACE OF A TERRESTRIAL SPHEROID

Knowledge of an area of all the surface of terrestrial spheroid can be necessary in examining of certain theoretical problems. In practice a typical case is the calculation of an area of parts of a surface of the ellipsoid, limited by meridians and parallels and presenting an area of surveying trapezoids or map sheets of one or another scale. Mathematically the calculation of surface areas of terrestrial ellipsoid is based on calculation of integral described below:

Let us take on the ellipsoid (Fig. 17) an elementary trapezoid dT with sides AB and BC or AD.

AB an elementary arc of meridian is equal to MdB; BC or AD are elementary arcs

[1]Subsequently, the shown work of B. N. Rabinovich will be named simply "Practicum on Higher Geodesy".

of parallel, equal to:

$$r dl = N \cos B dl.$$

Consequently,

$$dT = M dB r dl = MN \cos B dB dl.$$

Taking integral from this expression on longitude, which changes from 0 to 2π, we will find an area of spheroidal zone and, designating it by z, we obtain:

$$z = 2\pi \int_{B_1}^{B_2} MN \cos B \, dB$$

or:

$$z = 2\pi b^2 \int_{B_1}^{B_2} \frac{\cos B \, dB}{(1 - e^2 \sin^2 B)^2}.$$

But from (2.21)

$$e \sin B = \sin \psi,$$
$$e \cos B dB = \cos \psi d\psi.$$

Consequently:

$$z = \frac{2\pi b^2}{e} \int \frac{d\psi}{\cos^3 \psi}.$$

where b – minor semiaxis of a spheroid.

Last integral is tabular and is equal to:

$$\frac{1}{e} \int \frac{d\psi}{\cos^3 \psi} = \frac{1}{e} \frac{\sin \psi}{2\cos^2 \psi} + \frac{1}{4e} \ln \frac{1 + \sin \psi}{1 - \sin \psi}$$

Or, considering that $e \sin B = \sin \psi$, we obtain:

$$z = \pi b^2 \left(\frac{\sin B}{1 - e^2 \sin^2 B} + \frac{1}{2e} \ln \frac{1 + e \sin B}{1 - e \sin B} \right). \quad (2.49)$$

From (2.49) it follows, that an area of spheroidal trapezoid is expressed in a closed form in elementary functions, whereas the length of elliptic arc does not possess this property. However formula (2.49) is less convenient for calculations than the one obtained by means of transformation $(1 - e^2 \sin^2 B)^{-2}$ into binomial series.

We have:

$$(1 - e^2 \sin^2 B)^{-2} = 1 + 2e^2 \sin^2 B + 3e^4 \sin^4 B + 4e^6 \sin^6 B + \dots,$$

Therefore:

$$z = 2\pi b^2 \int_{B_1}^{B_2} (1 + 2e^2 \sin^2 B + 3e^4 \sin^4 B + 4e^6 \sin^6 B + \dots) \cos B\, dB.$$

Applying general formula of integration

$$\int \sin^n B \cos B\, dB = \frac{1}{n+1} \sin^{n+1} B,$$

we obtain:

$$z = 2\pi b^2 \Big|_{B_1}^{B_2} \left(\sin B + \frac{2}{3} e^2 \sin^3 B + \frac{3}{5} e^4 \sin^5 B + \frac{4}{7} e^6 \sin^7 B + \dots\right). \tag{2.50}$$

Placing in (2.50) $B_1 = 0$, $B_2 = \frac{\pi}{2}$, we obtain half of all the surface of the spheroid. Consequently, the area of all surface of the spheroid will be equal to:

$$\Pi = 4\pi b^2 \left\{1 + \frac{2}{3} e^2 + \frac{3}{5} e^4 + \frac{4}{7} e^6 + \frac{5}{9} e^8 + \dots\right\}. \tag{2.51}$$

For Krasovskiy spheroid

$$\Pi = 510083035{,}4\ \text{км}^2.$$

From (2.51) it follows that the radius of a sphere, is equivalent to the terrestrial spheroid,

$$R^* = \sqrt{\frac{\Pi}{4\pi}} = b\left(1 + \frac{e^2}{3} + \frac{11e^4}{45} + \frac{103e^6}{945} + \dots\right) \tag{2.52}$$

For Krasovskiy ellipsoid $R^* = 6371116$ meters.

Radius of a sphere, equal by volume to an ellipsoid, is derived equal to $R_1^* = \sqrt[3]{a^2b}$ (for Krasovskiy ellipsoid $R_1^* = 6371110$ m).

However actual area of a physical surface of the Earth is not calculated by these formulas, but by means of direct measurements of areas on topographic maps.

Calculation of considerable parts of the surface of the Earth or territories of countries constitutes one of the principal scientific problems of cartometry.

For convenience of computing areas of surveying trapezoids of sheets of topographic maps it is expedient to use formula (2.50), to transform substituting sines of odd powers by sines of odd arcs.

In accordance with formulas (1.25) we have:

$$\sin^3 B = \frac{3}{4}\sin B - \frac{1}{4}\sin 3B,$$

$$\sin^5 B = \frac{5}{8}\sin B - \frac{5}{16}\sin 3B + \frac{1}{16}\sin 5B,$$

$$\sin^7 B = \frac{35}{64}\sin B - \frac{21}{64}\sin 3B + \frac{7}{64}\sin 5B - \frac{1}{64}\sin 7B.$$

Substituting these expressions in (2.50) and replacing the differences of sines by products of sines of semidifference by cosine of half sum by the formula:

$$\sin B_2 - \sin B_1 = 2\sin\frac{B_2 - B_1}{2}\cos\frac{B_2 + B_1}{2},$$

we obtain:

$$z = 4\pi b^2\Big\{A'\sin\frac{B_2 - B_1}{2}\cos B_m - B'\sin\frac{3}{2}(B_2 - B_1)\cos 3B_m + C'\sin\frac{5}{2}\times(B_2 - B_1)\cos 5B_m - D'\sin\frac{7}{2}(B_2 - B_1)\cos 7B_m + E'\sin\frac{9}{2}(B_2 - B_1)\cos 9B_m\Big\}, \tag{2.53}$$

where

$$A' = 1 + \frac{1}{2}e^2 + \frac{3}{8}e^4 + \frac{5}{16}e^6 + \frac{35}{128}e^8 + \ldots = 1{,}0033636057,$$

$$B' = \frac{1}{6}e^2 + \frac{3}{16}e^4 + \frac{3}{16}e^6 + \frac{35}{192}e^8 + \ldots = 0{,}0011240272,$$

$$C' = \frac{3}{80}e^4 + \frac{1}{16}e^6 + \frac{5}{64}e^8 + \ldots = 0{,}0000016989,$$

$$D' = \frac{1}{112}e^6 + \frac{5}{256}e^8 + \ldots = 0{,}0000000027,$$

$$E' = \frac{5}{2304}e^8 + \ldots = 0{,}0000000000.$$

Maps of a scale of 1:1,000,000 served as a basis of listing of topographic maps the dimensions of trapezoid frames on a scale of 1:1,000,000 are equal to $B_2 - B_1 = 4^\circ$, $L_2 - L_1 = 6^\circ$. Area of such trapezoid is calculated by the formula:

$$P_{1:1\,000\,000} = \frac{\pi b^2}{15}(A'\sin 2^\circ\cos B_m - B'\sin 6^\circ\cos 3B_m + C'\sin 10^\circ\cos 5B_m + D'\sin 14^\circ\cos 7B_m). \qquad (2.54)$$

For map of scale of 1:100000, where:

$$B_2 - B_1 = 20', \quad L_2 - L_1 = 30'$$

$$P_{1:100\,000} = \frac{\pi b^2}{180}(A'\sin 10'\cos B_m - B'\sin 30'\cos 3B_m + C'\sin 50' \times \cos 5B_m). \qquad (2.55)$$

$\pi b^2 = 124061094.3\ km^2$, $\lg \pi b^2 = 8.09363561$ (Krasovskiy ellipsoid)

In addition to an area of trapezoid, in practice it is necessary to also calculate linear dimensions of its frame on a map scale. Frames of trapezoid are sections of meridians of arc and parallels, therefore, in accordance with designations in Fig. 18:

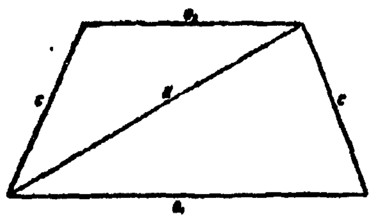


Fig. 18.

$$(a_1)_{cm} = \frac{N_1\cos B_1}{\rho'' m}\cdot 100 l'' = \frac{100}{m} b_1 l'',$$
$$(a_2)_{cm} = \frac{N_2\cos B_2}{\rho'' m}\cdot 100 l'' = \frac{100}{m} b_1' l'',$$
$$(c)_{cm} = \frac{M(B_2 - B_1)''}{\rho'' m}\cdot 100 = \frac{\Delta m (B_2 - B_1)''}{m},$$

where m — denominator of a scale, b_1 is taken from the tables of D. A. Larin for corresponding latitude and Δ_m — by mean average latitude.

Allignment of sag of a frame of topographic trapezoid is calculated by the formula:

$$h = N_m \frac{l^2}{16\rho''}\sin 2B_m.$$

In "Tables of Gauss-Kruger Coordinates," composed under direction of A. M. Virovts, for different scales of topographic maps are given a_1, a_2, c, d, and P, whence and values of these magnitudes are taken.

Numerical examples of calculation of frames and area of trapezoid on maps of 1:1[illegible] scale are given on p. 2[illegible]7-2[illegible] of Practicum on Higher Geodesy.

CHAPTER III

INVESTIGATION OF CURVES ON TERRESTRIAL SPHEROID

I. Normal Sections

§ 11. MUTUAL NORMAL SECTIONS AND AN ANGLE BETWEEN THEM

Let us present the following geometric construction on the surface of a terrestrial spheroid. Assume that the geodetic theodolite is set at a point P_1 (Fig. 19) so that its vertical axis coincides with the normal at this point and the telescope of the theodolite is directed at point P_2. Plane, passing through normal P_1n_1 and point P_2, will be a normal plane at point P_1, and its trace on the surface, a curve $a(P_1P_2)$, called the normal section:

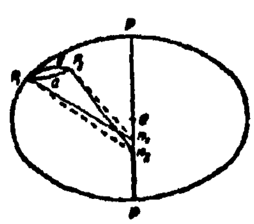


Fig. 19.

Moving with the theodolite to point P_2 and satisfying the same construction as at point P_1, we obtain normal section b. Curves a and b are called mutual normal sections, where curve a is called straight normal section at point P_1 and b an inverse, and at point P_2 by straight section will be b and inverse will be a.

We will prove that mutual normal sections on an ellipsoid in general cases do not coincide.

From triangle $P_1P_1'n_1$ (Fig. 20) we have:

$$n_1P'_1 = N_1 \sin B_1,$$
$$On_1 = n_1P'_1 - y_1 = N_1 \sin B_1 - N_1(1 - e^2)\sin B_1 = e^2 N_1 \sin B_1.$$

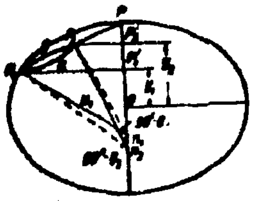


Fig. 20.

From triangle $P_2P'_2n_2$

$$n_2P'_2 = N_2 \sin B_2,$$
$$On_2 = n_2P'_2 - y_2 = N_2 \sin B_2 - N_2(1 - e^2)\sin B_2 = e^2N_2 \sin B_2.$$

Let us assume that $B_2 > B_1$, then:

$$On_2 > On_1.$$

Consequently, normals at points, not lying on one parallel, cross axis of rotation of a spheroid at various points. In general plane $n_1P_1P_2$, normal at point P_1, does not coincide with plane $n_2P_2P_1$, normal at point P_2. This means that between two points on a spheroid two normal sections pass. If point P_1 lies south of point P_2, then mutual normal sections (curves a and b) are disposed as is shown in Fig. 20, that is, curve b north of curve a.

At each triangulation point angles are measured between straight normal sections. Therefore, if on site there is a triangle, whose vertexes of angles were measured then, due to duality of normal sections, the figure obtained from measurements will have six sides, as shown in Fig. 21, where point P_2 is located further north of points P_1 and P_3, and point P_3 is further north than point P_1. Measured angles of each point are outlined by an arc.

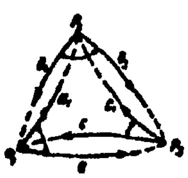


Fig. 21.

We will define the angle between mutual normal sections. Let us assume that on a spheroid two points P_1 and P_2 (Fig. 22) are given. We will pass the normal planes through these points as described above and designate segment $\overline{n_1n_2}$ by d, then:

$$d_2 - d_1 = d = On_2 - On_1 = e^2(N_2 \sin B_2 - N_1 \sin B_1) = e^2N_2\left(\sin B_2 - \frac{N_1}{N_2}\sin B_1\right).$$

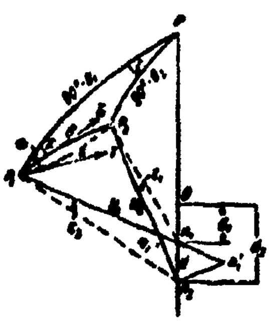


Fig. 22.

We find small angles ε_1 and ε_2, under which segment d is seen from points P_2 and P_1. From n_2 we will drop a perpendicular on continuation of a normal of point P_1. From triangle $n_2n_1n'_1$:

$$n_1n'_1 = d \sin B_1, \quad n'_1n_2 = d \cos B_1.$$

From right-angle triangle $P_1 n_2 n_1'$

$$\operatorname{tg} \varepsilon_1 = \frac{d \cos B_1}{N_1 + d \sin B_1}.$$

From right-angle triangle $n_1 n_1'' n_2$

$$n_1'' n_1 = d \sin B_2, \quad n_2 n_1'' = d \cos B_2.$$

From right-angle triangle $P_2 n_1 n_1''$

$$\operatorname{tg} \varepsilon_2 = \frac{d \cos B_2}{N_2 - d \sin B_2}.$$

Values of d_1, ε_1 and ε_2 for sides of 1st order triangulation are small values of the second order, therefore within them it is possible to substitute N_1 and N_2 by N_m a radius of curvature of first vertical for mean latitude B_m, also:

$$\begin{aligned}
\sin B_1 &= \sin\left(B_m - \frac{\Delta B}{2}\right) = \sin B_m - \frac{\Delta B}{2} \cos B_m + \ldots, \\
\sin B_2 &= \sin\left(B_m + \frac{\Delta B}{2}\right) = \sin B_m + \frac{\Delta B}{2} \cos B_m + \ldots, \\
\cos B_1 &= \cos\left(B_m - \frac{\Delta B}{2}\right) = \cos B_m + \frac{\Delta B}{2} \sin B_m + \ldots, \\
\cos B_2 &= \cos\left(B_m + \frac{\Delta B}{2}\right) = \cos B_m - \frac{\Delta B}{2} \sin B_m + \ldots
\end{aligned}$$

Dropping from formulas for d, ε_1, and ε_2 small values of order e^4, we obtain:

$$d = N_m e^2 \Delta B \cos B_m. \tag{3.1}$$

$$\varepsilon'' = \varepsilon_2'' = \varepsilon_1'' = \rho'' \frac{d}{N_m} \cos B_m = e^2 \Delta B \cos^2 B_m. \tag{3.2}$$

Difference of latitudes of points of 1st order triangulation does not exceed $20'$-$30'$. In radian measure this will be approximately $\frac{1}{120}$, therefore where $B_m = 60^\circ$:

$$\begin{aligned}
d &= \frac{64 \cdot 10^5}{150 \cdot 120 \cdot 2} \approx 180 \text{ м}, \\
\varepsilon'' &= \frac{2 \cdot 10^5 \cdot 180}{64 \cdot 10^5 \cdot 2} \approx 3''.
\end{aligned}$$

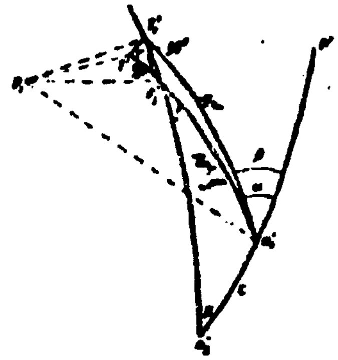


Fig. 23.

Thus, due to smallness of d it is possible to consider the lengths of normal sections b and a coinciding and to take them for the length of arc of circumference of radius N_m. Let us designate the central angle at n_1 by $\sigma = \frac{s}{N_m}$, then the angle between chord P_1P_2 and normal n_1P_1 will be equal $90^\circ - \frac{\sigma}{2}$. The angle between mutual normal sections b and a will be designated by Δ and its expression will be found by means of the following construction (Fig. 23).

From point P_1, as a center, we will describe an auxiliary sphere of arbitrary radius. On this sphere to directions, emanating from P_1, determined points will correspond. Let us assume that to directions P_1n_2, P_1n_1, P_1P_2, P_1T_1, and P_1T (Fig. 22) correspond points n_2', n_1', P_2', T' and T_1' on an auxiliary sphere. Connecting these points by arcs of great circles, we note that the great circle $n_2'n_1'P'$ depicts meridian of point P_1.

Azimuth of a straight normal section α is represented by a spherical angle $P_2'n_1'P'$, arc $P_2'n_1'$ it corresponds to angle $90^\circ - \frac{\sigma}{2}$, arc $n_1'n_2'$ to angle ε and, finally angle f at vertex P_2' is an angle between mutual normal planes. Tangent P_1T' lies in a plane of straight normal section $n_1P_1P_2$ and is perpendicular to normal n_1P_1, therefore the angle at T' in a spherical triangle $P_2'T'T_1'$ is a straight line, arc $T'n_1' = 90^\circ$, and $P_2'T' = \frac{\sigma}{2}$. Tangent P_1T_1' lies in a plane of inverse normal section and forms a right-angle with normal P_1n_1, therefore arc $n_1'T_1' = 90^\circ$. Thus, the angle between tangents P_1T' and P_1T_1' or arc $T'T_1'$ is the unknown angle Δ between mutual normal sections a and b.

From right-angle spherical triangle $P_2'T'T_1'$ we have:

$$\cos\left(90^\circ - \frac{1}{2}\sigma\right) = \mathrm{ctg}\, f\, \mathrm{ctg}\,(90^\circ - \Delta)$$

or

$$\mathrm{tg}\,\Delta = \sin\frac{\sigma}{2}\,\mathrm{tg}\, f.$$

From spherical triangle $n_2'P_2'n_1'$ by theorem of sines:

$$\sin f = \frac{\sin\varepsilon \sin\beta}{\sin\left(90^\circ - \frac{\sigma}{2}\right)} = \frac{\sin\varepsilon \sin\beta}{\cos\frac{\sigma}{2}}.$$

As was already established, Δ, ε are small values of the second order, and σ are of first order. Therefore sines and tangents of these small values substituted by angles in radians and with errors higher than the second order are shown thus:

$$\left.\begin{aligned} \Delta &= \frac{1}{2}\sigma f + \ldots \\ f &= \varepsilon \sin\beta + \ldots \end{aligned}\right\}, \qquad (3.3)$$

or

$$\Delta = \frac{1}{2}\sigma\varepsilon \sin\beta + \ldots \qquad (3.3')$$

Accuracy of formulas (3.3) and (3.3') will not be lowered, if β is substituted by α, since the difference $(\alpha - \beta)$ is small value of third order. Substituting in (3.3) and (3.3') the value ε from (3.2), we obtain

$$f = e^2 \Delta B \cos^2 B_m \sin\alpha,$$
$$\Delta = \frac{e^2}{2}\sigma \Delta B \cos^2 B_m \sin\alpha.$$

Since $\sigma = \frac{s}{N_m}$, $\Delta B = \frac{s\cos\alpha}{M_m}$ (with accuracy up to small values of third order), then:

$$\begin{aligned} f'' &= \rho'' e^2 s \cos^2 B_m \cos\alpha \sin\alpha = \frac{\rho'' e^2 s}{2M_m} \cos^2 B_m \sin 2\alpha, \\ \Delta'' &= \rho'' \frac{e^2 s^2}{2M_m N_m} \cos^2 B_m \cos\alpha \sin\alpha = \frac{\rho'' e^2 s^2}{4M_m N_m} \cos^2 B_m \sin 2\alpha. \end{aligned} \qquad (3.4)$$

In last expression with accuracy of $e^4\sigma^2$ it is possible to accept $M_m = N_m$. Consequently,

$$\Delta'' = \rho'' \frac{e^2 s^2}{4N_m^2} \cos^2 B_m \sin 2\alpha + \ldots \qquad (3.5)$$

From formulas (3.4) and (3.5) it follows that the values f and Δ revert to zero twice: when $\alpha = 0$ and when $\alpha = 90^\circ$. In other words, mutual normal sections coincide, if the points lie on one meridian or on one parallel. This conclusion is justifiable with the degree of accuracy, that are derived from formula (3.4) and (3.5), i.e., with accuracy up to small values $e^2\sigma^3$.

Besides an angle between the normal sections, we will consider their linear divergence, which, obviously, will be maximum for median points of arcs a and b. For determination of this value we will execute the following construction.

From the middle of the chord P_1P_2 we will restore a perpendicular and continue it to intersection with a surface of the spheroid. From point n_1 (Fig. 24), as a center, we will describe an arc of circumference P_1DP_2 radius N_m and determine curve pointer of sag h. It is known that:

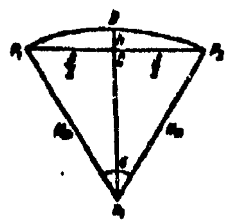


Fig. 24.

$$h = \frac{s}{2} \operatorname{tg} \frac{\sigma}{4}.$$

Being limited by smallness σ in the first term of factorization $\operatorname{tg}\frac{\sigma}{4}$ in series and taking into account that $\sigma = \frac{s}{N_m}$, it is possible to state:

$$h = \frac{s^2}{8N_m}. \qquad (3.6)$$

Now from point C let us restore perpendiculars to normal sections a and b. The angle between perpendiculars is equal to the angle between mutual normal planes f (Fig. 25). By length the perpendiculars are very close among themselves and with high degree of accuracy are equal to pointer sag curve h. Elementary arc q, i.e., linear divergence of mutual normal sections, can be determined as an arc of circumference of radius h with central angle f, i.e.,

Fig. 25.

$$q = hf.$$

Substituting value f and h from (3.4) and (3.6), we obtain

$$q = \frac{e^2 s^3}{8N_m^2} \cos^2 B_m \sin a \cos a = \frac{e^2 s^3}{16N_m^2} \cos^2 B_m \sin 2a, \qquad (3.7)$$

Formulas (3.5) and (3.7) are useful by accuracy for lengths of the order of a side of 1st order triangulation. Table 3 presents numerical values of magnitudes Δ and q.

Table 3

Extreme azimuth	Latitude	s, km	Δ''	q, mm
45°	52°	30	0,003	0,1
45	52	100	0,032	3,6
45	52	150	0,057	13,0

Values Δ and q show that for typical lengths of the sides of a triangle of 1st order triangulation in USSR, whose dimensions are 20-25 km, with duality of normal sections should not be considered. For distances of 20-25 km they can be considered merging. However for distances more than 30 km in transmission of azimuths it is necessary to introduce corresponding corrections.

In order to avoid the duality of normal sections in general, the geometric figures on the surface of a spheroid can be formed either by chords of normal sections, or geodetic lines. But for consideration of these questions it is first necessary to investigate the most intrinsic properties of geodetic lines normal sections and their chords on the surface of a spheroid.

Various attempts in the past and now have been made to develop a theory of spheroidal geodesy on the basis of application of normal sections have not succeeded. The matter is that with identical degrees of accuracy the formulas obtained with application of the geodetic line are simpler than the analogous formulas constructed by means of normal sections.

Recently certain scientists proposed to leave out the geodetic line from spheroidal geodesy and to replace it by chords of an ellipsoid. Although this leads in certain cases to closed expressions instead of infinite series, nonetheless the chord does not possess the generalization of geodetic line for solution of all problems of spheroidal geodesy. Application of geodetic line in the tightest form ties spheroidal geodesy with higher mathematics, on whose achievements its development is based to a significant degree. However in particular problems it may become expedient to use normal sections or chords of an ellipsoid as auxiliary values. Therefore basic problems, necessary for the use of normal sections and chords of a ellipsoid are expounded below.

§ 12. AZIMUTH AND CHORD OF A NORMAL SECTION

Two points are given on a spheroid: P_1 and P_2 (Fig. 26). Let us assume that plane XY coincides with the meridian plane of point P_1, i.e., $Y_1 = 0$. Consequently, for space coordinates of points P_1 and P_2 we have corresponding expressions:

$$\left.\begin{aligned} X_1 &= N_1 \cos B_1 = r_1 \\ Y_1 &= 0 \\ Z_1 &= N_1 \frac{b^2}{a^2} \sin B_1 = r_1 \frac{b^2}{a^2} \operatorname{tg} B_1 \end{aligned}\right\}; \quad \left.\begin{aligned} X_2 &= N_2 \cos B_2 \cos l = r_2 \cos l \\ Y_2 &= N_2 \cos B_2 \sin l = r_2 \sin l \\ Z_2 &= N_2 \frac{b^2}{a^2} \sin B_2 = r_2 \frac{b^2}{a^2} \operatorname{tg} B_2 \end{aligned}\right\}.$$

l – difference of geodetic longitudes of points P_1 and P_2.

We introduce a new system of grid coordinates (ξ, η, ζ) with origin at point P_1. Tangent plane at point P_1 is taken for plane $\xi\eta$; axis ξ directed along the tangent to meridian of point P_1, axis η – perpendicular to axis ξ and in parallel to axis Y; axis Z coincides with the normal of point P_1. From Fig. 26 it follows that the angle of rotation of systems of coordinates will be latitude B_1 of the point P_1.

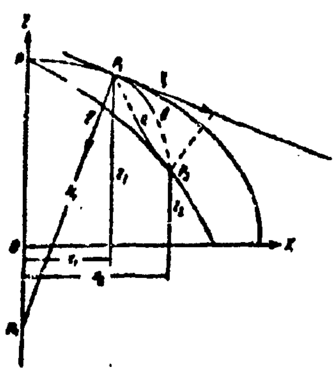


Fig. 26.

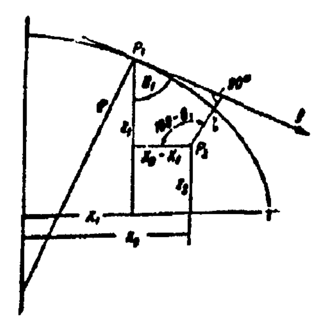


Fig. 27.

For obtaining connection between systems of coordinates (X, Y, Z) and (ξ, η, ζ) we will design Fig. 26 on a meridian plane of point P_1, then we will obtain Fig. 27, from which:

$$\left.\begin{aligned} \xi &= (X_2 - X_1) \sin B_1 - (Z_2 - Z_1) \cos B_1 \\ \eta &= Y_2 \\ \zeta &= -(X_2 - X_1) \cos B_1 - (Z_2 - Z_1) \sin B_1 \end{aligned}\right\}. \tag{3.8}$$

Let us make a normal section from P_1 to P_2; the plane of this section will intersect plane $\xi\eta$ by a straight line

$$\eta = \xi \operatorname{tg} \alpha,$$

where α is an azimuth of straight normal section from P_1 to P_2.

From (3.8) it follows that:

$$\operatorname{tg} \alpha = \frac{Y_2}{(X_2 - X_1)\sin B_1 - (Z_2 - Z_1)\cos B_1}.$$

or

$$\operatorname{tg} \alpha = \frac{N_2 \cos B_2 \sin l}{(N_2 \cos B_2 \cos l - N_1 \cos B_1)\sin B_1 - \frac{b^2}{a^2}(N_2 \sin B_2 - N_1 \sin B_1)\cos B_1}.$$

Let us introduce here a radius of parallel r = N cos B, then we obtain:

$$\operatorname{tg} \alpha_1 = \frac{\sin l}{\left(\cos l - \frac{r_1}{r_2}\right)\sin B_1 - (1 - e^2)\left(\operatorname{tg} B_2 - \frac{r_1}{r_2}\operatorname{tg} B_1\right)\cos B_1}. \qquad (3.9)$$

For inverse normal section by means of transposition of indices, contained in the formula of values, we obtain:

$$\operatorname{tg} \alpha_2 = -\frac{\sin l}{\left(\cos l - \frac{r_2}{r_1}\right)\sin B_2 - (1 - e^2)\left(\operatorname{tg} B_1 - \frac{r_2}{r_1}\operatorname{tg} B_2\right)\cos B_2}. \qquad (3.9')$$

Let us designate the chord of reciprocal normal sections by $\bar{s}$, then we obtain:

$$\bar{s}^2 = (X_2 - X_1)^2 + Y_2^2 + (Z_2 - Z_1)^2.$$

or with replacement of grid coordinates by geodetic coordinates:

$$\bar{s}^2 = (N_2 \cos B_2 \cos l - N_1 \cos B_1)^2 + N_2^2 \cos^2 B_2 \sin^2 l + \frac{b^4}{a^4}(N_2 \sin B_2 - \\ - N_1 \sin B_1)^2,$$

or:

$$\bar{s}^2 = r_2^2 \left\{ \sin^2 l + \left(\cos l - \frac{r_1}{r_2}\right)^2 + (1 - e^2)^2 \left(\operatorname{tg} B_2 - \frac{r_1}{r_2}\operatorname{tg} B_1\right)^2 \right\}. \qquad (3.10)$$

Closed expressions (3.9), (3.9'), and (3.10) can be used for calculation of azimuths of normal sections of ellipsoid chords with the help of computers, where the value r should be chosen from D. A. Larin Tables, in which $b_1 = \frac{r_{II}}{\rho}$ are given with sufficient number of decimal points.

Formula for chord $\bar{s}$, according to Molodenskiy can be shown in following form:

$$\bar{s}^2 = 4N_1N_2\sin^2\frac{\psi}{2} - \frac{a^2-b^2}{a^2}(N_2\sin B_2 - N_1\sin B_1)^2 + (N_2 - N_1)^2, \tag{3.10'}$$

where:

$$\sin^2\frac{\psi}{2} = \sin^2\frac{(B_2-B_1)}{2} + \cos B_1\cos B_2\sin^2\frac{l}{2};$$

formula (3.10') is less convenient for calculations.

§ 13. LENGTH OF ARC OF NORMAL SECTION

Points P_1 and P_2 are given on a spheroid with geodetic and grid coordinates. Let us designate angle between chord $\bar{s}$ and tangent T by ϑ (Fig. 28), and the azimuth of straight section as α.

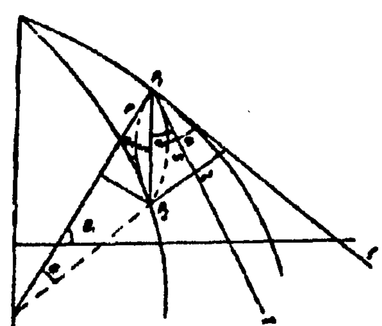


Fig. 28.

Let us define coordinates ξ, η and ζ of point P_2. Projecting chord $\bar{s}$ on tangent T and normal N, we obtain sections $\bar{s}\cos\vartheta$ and $\bar{s}\sin\vartheta$. From Fig. 28 it follows that:

$$\xi = \bar{s}\cos\vartheta\cos\alpha,$$
$$\eta = \bar{s}\cos\vartheta\sin\alpha,$$
$$\zeta = \bar{s}\sin\vartheta.$$

Taking into account (3.8), we find

$$\bar{s}\cos\vartheta\cos\alpha = (X_2 - X_1)\sin B_1 - (Z_2 - Z_1)\cos B_1,$$
$$\bar{s}\cos\vartheta\sin\alpha = Y_2,$$
$$\bar{s}\sin\vartheta = -(X_2 - X_1)\cos B_1 -$$
$$-(Z_2 - Z_1)\sin B_1$$

or, replacing values X, Y, Z by geodetic coordinates B and l, we obtain

$$1.\quad \frac{\bar{s}}{a}\cos\vartheta\cos\alpha = \frac{\cos B_2\cos l\sin B_1}{W_2} - \frac{e^2\sin B_1\cos B_1}{W_1} - \frac{(1-e^2)\sin B_2\cos B_1}{W_2} \tag{3.11}$$

$$2.\quad \frac{\bar{s}}{a}\cos\vartheta\sin z = \frac{\cos B_2 \sin l}{W_2}$$

$$3.\quad -\frac{\bar{s}}{a}\sin\vartheta = \frac{\cos B_2 \cos l \cos B}{W_2} + \frac{(1-e^2)\sin B_2 \sin B_1}{W_2} - W_1 \qquad (3.11)\ \text{cont}$$

If $\bar{s}$, B and α are given then these three equations fully and simply determine unknowns B_2, l and ϑ. Excluding from these equations B_2 and l, we obtain expression for ϑ. For that, the first of formulas (3.11) is multiplied by cos B_1, the third by sin B_1, and then conversely. If we subtract the third from the first and add them, we obtain correspondingly:

$$\left.\begin{aligned} &1.\quad \frac{\sin B_2}{W_2} = \frac{\sin B_1}{W_1} - \frac{\bar{s}}{a(1-e^2)}(\cos\vartheta\cos z_1\cos B_1 + \sin\vartheta\sin B_1).\\ &2.\quad \frac{\cos B_2\cos l}{W_2} = \frac{\cos B_1}{W_1} + \frac{\bar{s}}{a}(\cos\vartheta\cos z_1\sin B_1 - \sin\vartheta\cos B_1).\end{aligned}\right\} \qquad (3.12)$$

Second terms of right side of formulas (3.12) have definite geometric value. Let us introduce a horizontal system of coordinates, i.e., the zenithal distance z and azimuth α of chord $\bar{s}$. We designate directional cosines $\bar{s}$ in a system (X, Y, Z) by $n_1 = \cos\alpha$, $n_2 = \cos\beta$, $n_3 = \cos\gamma$. On a sphere of unit radius, which subsequently we will call Molodenskiy sphere, since it was first introduced by him, point P_1 designates geodetic zenith of point P_1 (Fig. 29); points X, Y, and Z correspond to directions of the axes of coordinates, and s to direction of chord from point P_1 to P_2.

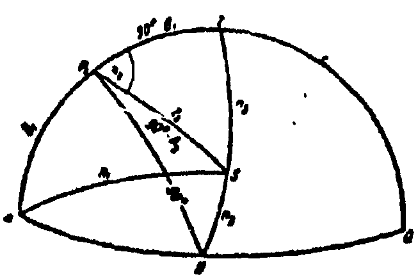


Fig. 29.

On a sphere of arc sx, sy, and sz are equal correspondingly to the cosines of directional chord $\bar{s}$.

From spherical triangle P_1xs (Fig. 29)

$$n_1 = \cos B_1\cos z_0 - \sin B_1\sin z_0\cos z_1.$$

From triangles P_1zs

$$n_3 = \sin B_1\cos z_0 + \cos B_1\sin z_0\cos z_1.$$

Consequently, angle $\vartheta = 90^\circ - z_0$ (let us call ϑ geodetic height in horizontal system of coordinates).

Reverting to equations (3.11) and (3.12) we accomplish the following actions on them: raise to a square the second from (3.12) and add to the square of second equation from (3.11), the obtained sum is multiplied $(1 - e^2)$ and is added to the square of the first from (3.12), multiplied by $(1 - e^2)^2$, then:

$$\frac{\sin\vartheta\,(1-e^2)\,2\bar{s}}{a W_1} = \frac{\bar{s}^2}{a^2}\,(1-e^2+e^2 n_3^2) = \frac{\bar{s}^2}{a^2}\,(1-e^2)\left(1+\frac{e^2 n_3^2}{1-e^2}\right).$$

But $\frac{e^2}{1-e^2} = e'^2$, therefore:

$$\sin\vartheta = \frac{\bar{s} W_1}{2a}\,(1+e'^2 n_3^2) = \frac{\bar{s}}{2N_1}\,(1+e'^2 n_3^2).$$

Instead of N_1 introduce radius of curvature of straight normal section from P_1 to P_2 by the formula:

$$\frac{1}{N_1} = \frac{1}{\rho\,(1+\eta_1^2\cos^2 a_1)}$$

$$\sin\vartheta = \frac{\bar{s}}{2\rho}\left\{\frac{1+e'^2 n_3^2}{1+\eta_1^2\cos^2 a_1}\right\} = \frac{\bar{s}}{2\rho}\left\{1+\frac{e'^2\,(n_3^2-\cos^2 B_1\cos^2 a}{1+\eta_1^2\cos^2 a_1}\right\}.$$

$$n_3^2-\cos^2 B_1\cos^2 a_1 = \sin^2\vartheta\,(\sin^2 B_1-\cos^2 B_1\cos^2 a_1)+$$
$$+\,2\sin\vartheta\cos\vartheta\sin B_1\cos B_1\cos a_1.$$

Let us designate:

$$e'^2\left(\frac{\sin^2 B_1-\cos^2 B_1\cos^2 a_1}{1+\eta_1^2\cos a_1}\right) = \mu_2,$$
$$\frac{e'^2\sin 2B_1\cos a_1}{1+\eta_1^2\cos^2 a_1} = \mu_1;$$

Therefore:

$$\sin\vartheta = \frac{\bar{s}}{2\rho}\left\{1+\mu_1\sin\vartheta+\mu_2\sin^2\vartheta-\frac{1}{2}\,\mu_1\sin^3\vartheta+\dots\right\}. \qquad (3.13)$$

Formula (3.13) has a high degree of accuracy, since it retains the values $\frac{\bar{s}}{2\rho}e'^2\vartheta^5$. Without decreasing this accuracy and taking in first approximation $\sin\vartheta = \frac{\bar{s}}{2\rho}$, we obtain:

$$\sin\vartheta = \left(\frac{\bar{s}}{2\rho}\right)+\mu_1\left(\frac{\bar{s}}{2\rho}\right)^2+\mu_2\left(\frac{\bar{s}}{2\rho}\right)^3-\frac{\mu_1}{2}\left(\frac{\bar{s}}{2\rho}\right)^4+l_1.$$

Passing from sin ϑ to angle ϑ, we obtain:

$$\vartheta = \left(\frac{\bar{s}}{2\rho}\right) + \frac{1}{6}\left(\frac{\bar{s}}{2\rho}\right)^3 + \frac{3}{40}\left(\frac{\bar{s}}{2\rho}\right)^5 + \mu_1\left(\frac{\bar{s}}{2\rho}\right)^2 + \mu_2\left(\frac{\bar{s}}{2\rho}\right)^3 + l_7. \qquad (3.14)$$

For determination of the length of arc of normal section from P_1 on P_2 we introduce polar coordinates. As radius-vector we take chord $\bar{s}$, and for polar angle – ϑ. The square of an element of arc in these coordinates will be:

$$ds^2 = d\bar{s}^2 + \bar{s}^2 d\vartheta^2. \qquad (3.15)$$

From (3.14)

$$\bar{s}\,d\vartheta = \left\{\left(\frac{\bar{s}}{2\rho}\right) + \frac{1}{2}\left(\frac{\bar{s}}{2\rho}\right)^3 + \frac{3}{8}\left(\frac{\bar{s}}{2\rho}\right)^5 + 2\mu_1\left(\frac{\bar{s}}{2\rho}\right)^2 + 3\mu_2\left(\frac{\bar{s}}{2\rho}\right)^3 + l_1\right\} d\bar{s},$$

or, squaring and substituting in (3.15), we obtain:

$$ds^2 = \left\{1 + \left(\frac{\bar{s}}{2\rho}\right)^2 + \left(\frac{\bar{s}}{2\rho}\right)^4 + \left(\frac{\bar{s}}{2\rho}\right)^6 + 4\mu_1\left(\frac{\bar{s}}{2\rho}\right)^3 + 6\mu_2\left(\frac{\bar{s}}{2\rho}\right)^4 + l_1\right\} d\bar{s}^2$$

or:

$$ds = \left\{1 + \frac{1}{2}\left(\frac{\bar{s}}{2\rho}\right)^2 + \frac{3}{8}\left(\frac{\bar{s}}{2\rho}\right)^4 + \frac{5}{16}\left(\frac{\bar{s}}{2\rho}\right)^6 + 2\mu_1\left(\frac{\bar{s}}{2\rho}\right)^3 + 3\mu_2\left(\frac{\bar{s}}{2\rho}\right)^4 + \right.$$
$$\left. + l_1\right\} d\bar{s}.$$

Integral of this equation within limits of $\bar{s} = 0$ and $\bar{s} = \bar{s}$ gives us the length of arc of normal section $P_1\breve{}P_2$.

We have:

$$s = \bar{s}\left\{1 + \frac{1}{6}\left(\frac{\bar{s}}{2\rho}\right)^2 + \frac{3}{40}\left(\frac{\bar{s}}{2\rho}\right)^4 + \frac{5}{112}\left(\frac{\bar{s}}{2\rho}\right)^6 + \frac{\mu_1}{2}\left(\frac{\bar{s}}{2\rho}\right)^3 + \right.$$
$$\left. + \frac{3}{5}\mu_2\left(\frac{\bar{s}}{2\rho}\right)^4 + l_1\right\}. \qquad (3.16)$$

It follows from this that for obtaining the length of arc of normal section by given geodetic coordinates of its terminals it is necessary to calculate by the formulas (3.9) and (3.10) first of all the azimuth and the chord of this section.

Formula (3.16) has high degree of accuracy and can be used for considerable distances between the points. In practical calculations it is expedient to have small tables for selection of μ_1 and μ_2 by arguments B_1 and A_1. Calculations by the formula (3.16) is convenient for use with computers.

If one were to allow that all our preceding reasonings pertain to point P_2, i.e., to section from point P_2 to point P_1, then the length of arc of inverse section will be expressed:

$$s' = \bar{s}\left\{1 + \frac{1}{6}\left(\frac{\bar{s}}{2\rho'}\right)^2 + \frac{3}{40}\left(\frac{\bar{s}}{2\rho'}\right)^4 + \frac{5}{112}\left(\frac{\bar{s}}{2\rho'}\right)^6 + \right.$$
$$\left. + \frac{\eta_2}{3}\left(\frac{\bar{s}}{2\rho'}\right)^3 + \frac{3\eta_2^2}{5}\left(\frac{\bar{s}}{2\rho'}\right)^4 + l_7\right\}. \qquad (3.16')$$
$$\rho' = \rho + (B_2 - B_1)\frac{d\rho}{dB} + \dots$$

Since $(B_2 - B_1)\frac{d\rho}{dB}$ is a small value of the order e^2k, then the difference $s - s'$ will be on the order of e^4k^6, i.e., a value, practically imperceptible during the most exact calculations. In other words, this difference can be discounted, the more so, because with the presence of coordinates of two points instead of $\frac{1}{\rho}$ for terminals it is possible to take $\frac{1}{\rho_m}$, i.e., to refer this value to point with a mean latitude.

For short distances, on the order of 100 km, the expression (3.16) is essentially simplified, if it is required, that s be determined with accuracy of up to 1 cm:

$$s = \bar{s}\left\{1 + \frac{1}{6}\left(\frac{\bar{s}}{2\rho}\right)^2 + l_4\right\}. \qquad (3.17)$$

The biggest term $3/40\left(\frac{\bar{s}}{2\rho}\right)^4$ is dropped where $\bar{s}$ = 200 km is less than 3 mm. If however s on the order of the length of a side of 1st order triangulation, then it is possible to substitute in the formula (3.17) $\frac{1}{\rho}$ by $\frac{1}{a}$, then:

$$s = \bar{s}\left\{1 + \frac{1}{6}\left(\frac{\bar{s}}{2a}\right)^2 + l_4\right\}. \qquad (3.18)$$

Error from replacement of value ρ by a in (3.18) will be less than 1 mm.

In joint application of formulas (3.9), (3.9'), (3.10), and (3.16) it is possible to resolve the so-called inverse geodetic problem, i.e., according to given geodetic coordinates of two points to find distance between them, and also the forward and back azimuths. Only in this case azimuths, calculated by the formulas (3.9) and

(3.), will pertain to the a chord of the ellipsoid between given points (Fig. 30). It is necessary to keep in mind that if the lengths of arcs of normal sections n and n can be considered practically equal, then it is necessary to consider the difference in their azimuths.

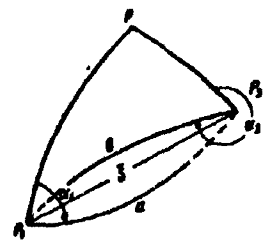


Fig. 30.

Thus, spherical triangle P_1PP_2, in which sides P_1P and PP_2 are arcs of meridian, different lengths and angles are obtained depending upon which of the azimuths of two normal sections is taken as basic: besides only the angle at the pole (l — difference of longitudes) remains constant. If however none of the sides of a triangle coincide with meridian, then two of its angles and all sides obtain different values depending upon the azimuth of the normal section, taken as initial. In the last case the inconveniences connected with the application of normal sections as basic lines, are more fully revealed, connecting geodetic points on the surface of spheroid.

Plane of meridians of points P_1 and P_2 with normal plane $P_1n_1P_2$ or $P_2n_2P_1$ form a trihedral angle with vertexes at n_1 and n_2. Let us visualize a sphere with arbitrary radius, described from point n_1. On the surface of this sphere trihedron with ribs n_1P_1, n_1P_2, and n_1P (Fig. 31) will correspond to triangle $P_1'P'P_2'$, in which the initial is the azimuth of straight normal section at the vertex P_1'. For resolution of this triangle let us find connection between B_2 and B_2'.

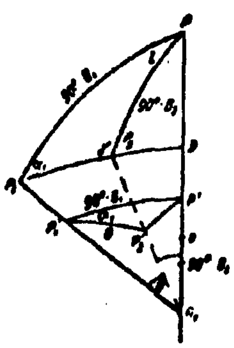


Fig. 31.

From Fig. 31 we have:

$$n_1o = e^2 N_1 \sin B_1,$$

$$oD = y_2 = \frac{a(1-e^2)\sin B_2}{W_2},$$

$$P_2D = x_2 = \frac{a\cos B_2}{W_2}.$$

From triangle n_1P_2D:

$$\mathrm{tg}\, B_2' = \frac{oD + on_1}{P_2D} = \frac{e^2\frac{\sin B_1}{W_1} + \frac{(1-e^2)\sin B_2}{W_2}}{\frac{\cos B_2}{W_2}}.$$

$$\mathrm{tg}\, B_2 = \left(1 - e^2 + e^2\frac{W_2}{W_1}\frac{\sin B_1}{\sin B_2}\right). \qquad (3.19)$$

$$\mathrm{tg}\, B_2' = (1-e^2)\,\mathrm{tg}\, B_2\left(1 + e'^2\frac{W_2}{W_1}\frac{\sin B_1}{\sin B_2}\right).$$

But $\frac{\sin B_1}{W_1}$ = sin u, therefore formula (3.19) can be written thus:

$$\operatorname{tg} B_2' = (1 - e^2) \operatorname{tg} B_2 \left(1 + e'^2 \frac{\sin u_1}{\sin u_2}\right). \qquad (3.20)$$

If geodetic coordinates of points P_1 and P_2, are given then the triangle $P_1'P'P_2'$ (Fig. 32) can be solved by the following formulas:

$$\left.\begin{aligned} \sin\theta \sin\alpha_1 &= \cos B_2' \sin l \\ \sin\theta\cos\alpha_1 &= -\cos B_1 \sin B_2' - \sin B_1 \cos B_2' \cos l \\ \sin\theta \sin\gamma &= \cos B_1 \sin l \\ \sin\theta\cos\gamma &= \sin B_1 \cos B_2' - \cos B_1 \sin B_2' \cos l \\ \cos\theta &= \sin B_1 \sin B_2' + \cos B_1 \cos B_2' \cos l \end{aligned}\right| \qquad (3.21)$$

Besides it should be underlined that angle γ is not the azimuth of normal section from point P_2 to point P_1, since in substitution of B_1 by B_2 in (3.21) we, obviously, will not obtain α_2 instead of α_1.

Let us assume that line $P_1P_2 = s$ (Fig. 31) is extended till it will not be intersected by any meridians at right-angle. Designate the latitude of this point by B_0, where it will be maximum throughout the extent of line P_1P_2 and its continuation. On auxiliary sphere we obtain a right-angle triangle $P_1'P'P_0$ (see Fig. 33), from which it follows that:

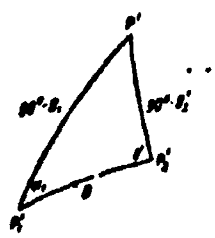


Fig. 32.

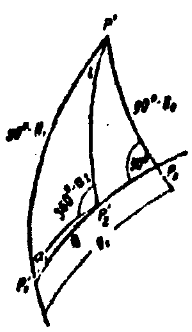


Fig. 33.

$$\cos B_1 \sin\alpha_1 = \cos B_0, \qquad (3.22)$$

$$\left.\begin{aligned} \sin B_0 \sin\theta_1 &= \cos B_1 \cos\alpha_1 \\ \sin B_0 \cos\theta_1 &= \sin B_1 \end{aligned}\right\}. \qquad (3.23)$$

Arcs θ and θ_1 are plane curves and lie in a plane of straight normal section. **If as a basic angle of triangle $P_2'P'P_1'$ $360^0 - \alpha_2$ is taken (Fig. 33), then by performing the same constructions, we obtain other values in substitution for θ and θ_1.**

§ 14. DETERMINATION OF GEODESIC AND ITS LOCATION RELATIVE TO MUTUAL NORMAL SECTIONS

The shortest lines on any mathematical surface are called geodesics. Straight on a plane, great circles on a sphere, helixes on cylinder etc. are geodesics since they are the shortest distances on these surfaces.

Two points on an arbitrary surface can be connected by a multitude of curves, possessing different geometric and analytic properties. If, at any of the given points on a surface tangent plane is established and on it all curves passing through these points, are constructed then only the geodesic will be a straight line, and all the others will be depicted by curves. Geodesic is a surface curve, having at each point a double curvature. Therefore it does not lie in one plane. For the study of plane properties of such curves an idea is introduced on an osculating plane, appearing as a limiting position of a plane, passing in three infinitely close points of a curve.

Principal normal of geodesic at each of its points coincides with normal at surface at a given point and lies in the osculating plane. This property of geodesic allows its construction analytically.

Let us assume that the aligned geodetic theodolite is set on a point P_1 so that its vertical axis coincides with normal at the surface of spheroid at this point.

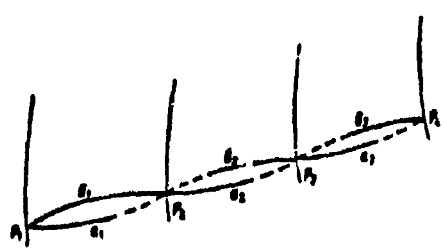


Fig. 34.

We select on spheroid a point P_2, close to point P_1, and direct the telescope of the theodolite to a point P_2. The trace of a sighting plane on the surface is curve a_1 (Fig. 34), as it is known, will be straight normal section. We move the theodolite to point P_2 and after setting it in a horizontal position, with locked plate we sight the telescope at point P_1; we obtain inverse normal section, and curve b; then, detaching alidade, will turn the telescope 180° and sight it on nearby point P_3. The sighting plane will describe a curve of straight normal section from point P_2 to point P_3, i.e., line a_2. Moving the theodolite consecutively from point P_2 to point P_3, and from point P_3 to point P_4 etc., and carrying out at each point analogous actions, we obtain construction, schematically depicted in Fig. 34.

Let us assume that points P_i ($i = 1, 2, ..., n$) are located at very minute distances one from another, and that these distances can become as small as desired, then, connecting points P_i of the curve, we obtain a geodesic between points P_1 and P_n. This ensues from our construction and determination of geodesic. Actually, by construction every close three points of P_i lie in one plane, which contains a normal to the surface at median point. In other words, passing through every point P_i the planes are osculating planes, perpendicular to the surface, and the curve, connecting these points, is a geodesic.

Let us note that for the construction of a geodesic on site between given points it is necessary to know direction of its first element or an angle between straight normal section at an initial point and the first element of geodesic.

Location of geodesic relative to mutual normal sections in general is shown in (Fig. 35 and 36), where dotted lines designate continuation of arcs of normal sections. On the whole geodesic is always closer disposed to straight normal section along all given points. If azimuths of geodesic are close to 0° or 90°, the location of geodesic with respect to normal sections is somewhat different, but these cases should be studied at a given azimuth.

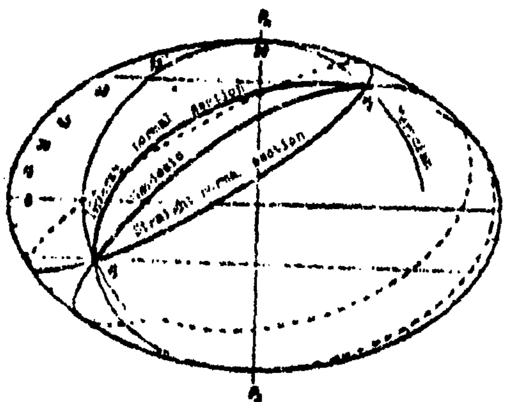


Fig. 35.

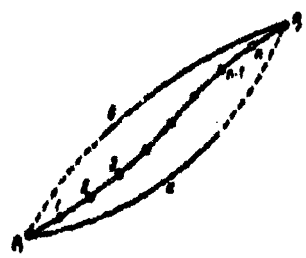


Fig. 36.

§ 15. FUNDAMENTAL EQUATION OF A GEODESIC

We derive fundamental equation of a geodesic from the fact that it is the shortest distance between points on a spheroid.

Let us take geodesic AB. We take along this line an elementary arc ds (Fig. 37) and construct it on a meridian and a parallel; we obtain sections of meridian AC = MdB and parallel DC = rdl. From elementary right-angle triangle ACD we have:

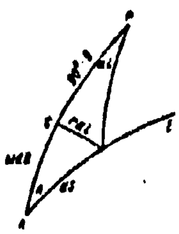


Fig. [illegible].

$$MdB = ds\cos A \text{ and } rdl = ds\sin A,$$

whence:

$$ds^2 = M^2dB^2 + r^2dl^2, \qquad (3,28)$$

$$\operatorname{tg} A = \frac{rdl}{MdB}. \qquad (3,[illegible])$$

$$ds = \sqrt{M^2dB^2 + r^2dl^2} = \sqrt{M^2\left(\frac{dB}{dl}\right)^2 + r^2}\,dl.$$

Let us designate:

$$\frac{dB}{dl} = q, \quad U = \sqrt{M^2q^2 + r^2} = U(B, q).$$

Then:

$$ds = Udl$$

or:

$$s = \int Udl.$$

Since $\int Udl$ expresses the length of arc of a geodesic, then it should have the least value. This is possible at determined dependency between B and l. Let us assume that this dependency is given by analytic function $B = B(l)$. Consequently, at any other dependency to the same l $B + b$, will correspond where b is a function of l, which becomes zero for terminal points of arcs A and B.

Thus, we have:

$$s' = \int U'dl,$$

where:

$$U' = U\left(B + b, q + \frac{db}{dl}\right).$$

According to Taylor:

$$U' = U + b\frac{\partial U}{\partial B} + \frac{\partial b}{\partial l}\frac{\partial U}{\partial q} + \ldots.$$

b — arbitrarily small value. Terms of highest order in Taylor line are omitted, since they are vanishingly minute as compared to first terms, and in further calculations cannot play a part.

We have:

$$s' = s + \int \frac{\partial U}{\partial B} b\,dl + \int \frac{\partial U}{\partial q} dl + \ldots$$

$s' > s$, since $(s' - s)$ is a value essentially positive at any b. In order that s' can be a geodesic, it is necessary and sufficient to:

$$\int \frac{\partial U}{\partial B} b\,dl + \int \frac{\partial U}{\partial q} dl = 0.$$

Let us prointegrate the second term of equation by parts, taking:

$$dJ = db \text{ and } \frac{\partial U}{\partial q} = v,$$

then:

$$\int \frac{\partial U}{\partial B} b\,dl + b\,\frac{\partial U}{\partial q} - \int b\,d\left(\frac{\partial U}{\partial q}\right) = 0$$

or:

$$\int\left[\frac{\partial U}{\partial B} dl - d\left(\frac{\partial U}{\partial q}\right)\right] b + \Big|_{A}^{B} b\,\frac{\partial U}{\partial q} = 0.$$

By condition b equals zero for points A and B, therefore the last term is identically a zero. We have:

$$\int\left[\frac{\partial U}{\partial B} dl - d\left(\frac{\partial U}{\partial q}\right)\right] b = 0.$$

In a space between points A and B $b \neq 0$, consequently:

$$\frac{\partial U}{\partial B} dl - d\left(\frac{\partial U}{\partial q}\right) = 0,$$

But:

$$\frac{dl}{d\theta} = q.$$

therefore:

$$\frac{\partial U}{q} - d\left(\frac{\partial U}{\partial q}\right) = 0 \quad \text{or} \quad \partial U - q\,d\left(\frac{\partial U}{\partial q}\right) = 0.$$

integral of this equation will give:

$$U - q\frac{\partial U}{\partial q} = \text{const},$$

but:

$$\frac{\partial U}{\partial q} = \frac{M^2 q}{U}.$$

then:

$$\frac{U^2 - M^2 q^2}{U} = \text{const}$$

or:

$$\frac{M^2q^2 + r^2 - M^2q^2}{\sqrt{M^2q^2 + r^2}} = \frac{r^2}{\sqrt{M^2q^2 + r^2}} = \frac{r}{\sqrt{1 + \frac{M^2q^2}{r^2}}} = \text{const}.$$

From (3.25)

$$\sqrt{1 + \frac{M^2q^2}{r^2}} = \sqrt{1 + \operatorname{ctg}^2 A} = \frac{1}{\sin A}.$$

Finally

$$r \sin A = \text{const}. \tag{3.26}$$

Equation (3.26) is called the basic equation of the geodesic and reads: the product of the radius of the parallel by the sine of azimuth at each point of the geodesic on a surface of a prolate spheroid is a constant value. If a series of points are taken on a geodesic, then the equation (3.26) can be stated in a more general form, thus:

$$r_1 \sin A_1 = r_2 \sin A'_2 = r_3 \sin A'_3 = \ldots \qquad (3.27)$$

Usually in higher geodesy the finite arcs of geodesics between two given points are studied. Namely for such cases an equation (3.26) can be given, for very interesting geometric interpretation (Fig. 38). We have:

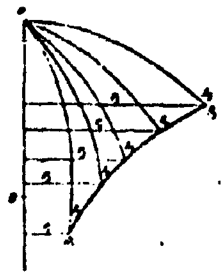


Fig. 38.

$$r_1 \sin A_1 = r_2 \sin A'_2, \qquad (3.28)$$

or:

$$\frac{r_1}{\sin A'_2} = \frac{r_2}{\sin A_1}.$$

This known relationship of a plane triangle is a theorem of sines.

Introducing the third side and the angle, opposite it, we obtain plane triangle $P'_1 P' P'_2$ (Fig. 39).

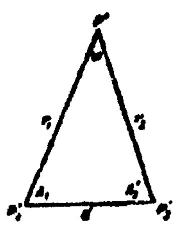


Fig. 39.

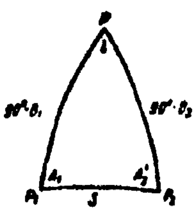


Fig. 40.

Angles of this triangle will compare with angles of spheroidal polar triangle $P_1 P P_2$ (Fig. 40).

From triangle $P'_1 P' P'_2$ and spheroidal triangle $P_1 P P_2$ we have:

$$A_1 + A'_2 + \omega = 180°,$$
$$A_1 + A'_2 + l = 180° + \varepsilon,$$

where ε – spheroidal excess of triangle $P_1 P P_2$.

Consequently:

$$\omega = l - \varepsilon. \qquad (3.29)$$

For value ε there are no closed expressions. But for approximate calculations

It is possible to consider spheroidal triangle [illegible] as spherical and we will obtain the approximate value of ε by the formula:

$$\operatorname{tg}\frac{\varepsilon}{2}=\frac{\operatorname{tg}\frac{1}{2}(90-B_1)\operatorname{tg}\frac{1}{2}(90-B_2)\sin l}{1+\operatorname{tg}\frac{1}{2}(90-B_1)\operatorname{tg}\frac{1}{2}(90-B_2)\cos l}$$

or, designating tg 1/2 $(90 - B_1)$ tg 1/2 $(90 - B_2) = k$, we obtain:

$$\operatorname{tg}\frac{\varepsilon}{2}=\frac{k\sin l}{1+k\cos l}. \qquad (3.30)$$

Further

$$\left.\begin{aligned}\frac{A_2'+A_1}{2}&=90^\circ-\frac{\omega}{2}\\ \operatorname{tg}\frac{A_2'-A_1}{2}&=\frac{r_2-r_1}{r_2+r_1}\operatorname{ctg}\frac{\omega}{2}\end{aligned}\right\}. \qquad (3.31)$$

$$\left.\begin{aligned}A_2'&=\frac{1}{2}(A_2'+A_1)+\frac{1}{2}(A_2'-A_1)\\ A_1&=\frac{1}{2}(A_2'+A_1)-\frac{1}{2}(A_2'-A_1)\end{aligned}\right|. \qquad (3.32)$$

In determination of the limit of application of the approximate method of calculation of geodetic azimuths, it is taken into consideration that the difference of spheroidal and spherical excesses of triangles with equal sides, as it will be proven in the following chapter, is the small value of third order. Therefore the shown method can be applied where it is required to know the azimuths within an accuracy of up to 1-3".

Other application of formula (3.26) consists in that during the resolution of direct and inverse geodetic problems it is possible to control the calculation of unknown values:

$$r_1\sin A_1=-r_2\sin A_2$$

where A_2 is a back azimuth of a geodesic, equal $180 + A_2'$.

An Example of Calculation of Approximate Geodetic Azimuths by the Formulas (3.29)-(3.32) is Given Below

Order of steps	Formulas	1	2	Notes
1	B_1	52°30'17"	68 58'0"	
2	B_2	54 42 51	37 45 0	
3	$^1/_2(90 - B_1)$	18 41 51	10 31 0	
4	$^1/_2(90 - B_2)$	17 38 31	26 730	
6	$tg\,^1/_2(90 - B_1)$	0,339105	0,185640	
7	$tg\,^1/_2(90 - B_2)$	0,318011	0,496436	
8	$\sin l$	0,123601	0,414429	
9	$\cos l$	0,992342	−0,910082	
10	k	0,107915	0,091045	
11	$k \sin l$	0,013342	0,037731	
12	$1 + k \cos l$	1,107117	0,917142	
13	$tg\,^1/_2\varepsilon$	0,012021	0,041140	
14	ε	1°22'13"	4 42'12"	
15	l	7 6 0	155 31'0"	
15'	[illegible]	5 41 17	150°48'18"	
16	[illegible]	2 51 38	75°24'9"	
17	$^1/_2(A_2' + A_1)$	87 08 22	14°35'51"	
25	$^1/_2(A_2' - A_1)$	27 35 24	−5°34'31"	
26	A_1	59 32 58	9°01'17"	
27	A_2'	114 43 46	201025	
28	$A_2 = 360° - A_2'$	245°16'14"	3395935	
18	r_1	18,862	11,131	From tables of E. A. Larin, argument B_1
19	r_2	17,902	24,481	From tables of E. A. Larin, argument
20	$r_1 - r_2$	0,960	−13,350	
21	$r_1 + r_2$	36,764	35,612	
22	$\frac{r_1 - r_2}{r_1 + r_2}$	0,026112	0,37487	
23	$ctg\,^1/_2\omega$	20,0120	0,260431	
24	$tg\,^1/_2(A_2' - A_1)$	0,522563	−0,097630	

equation $r \sin A = c$ is obtained as a product of two values, by whose arbitrary change the product should remain constant along a given geodesic. For meridian, where $A = 0$ we obtain $c = 0$. Consequently, terrestrial meridians are geodesics. On equator $r = a$, $A = 90^{\circ}$, i.e., at any point on equator $c = a$; consequently, terrestrial equator is also a geodesic.

Terrestrial parallels are not geodesics. This is obvious, since even on a sphere the arc of a parallel between two points is not the shortest distance.

Let us consider a general case, when a geodesic takes its beginning from a point with latitude B with azimuth between 0° and 90° (Fig. 41). Let us trace the process of a change of equation $r \sin A = c$.

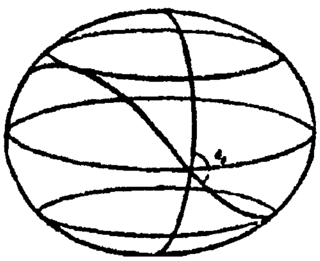


Fig. 41.

By the measure of receding from initial point along the geodesic latitudes and azimuths at all points are increased until the azimuth will not attain 90°, latitude its maximum (B_0), and r its minimum $r_0 = c$. At this point the geodesic will be tangent to parallel with latitude B_0 and will turn toward south; at its subsequent points the latitude will decrease, and the azimuth will increase, becoming more than 90°. Such change will occur prior to intersection of geodesic with equator, where r will attain a maximum (major semiaxis a). and A will obtain certain value of A_0. In southern hemisphere — the passage of a geodesic will be analogous. Attaining a point with maximum negative latitude ($-B_0$) and touching its parallel, it will turn to equator and will intersect it at a point, which does not coincide with opposite point of initial intersection of the equator by the geodetic.

Consequently, geodesic on a surface of a spheroid will describe an infinite number of turns during its continuous extension, starting at any point along the azimuth from 0° to 90°. The picture of a run of a geodesic on a spheroid will not be changed, if its first element will be required for azimuth greater than 90°.

Application of the fundamental equation of geodesic to solution of practical and theoretical problems will become more general, if the equation (3.26) is transformed while bearing in mind that:

$$r = N\cos B = a\cos u,$$

or:

$$N\cos B\sin A = a\cos u\sin A = \text{const.} \tag{3.33}$$

For finite sections of geodesic, when coordinates of its terminals and azimuths at these points are given:

$$a\cos u_1\sin A_1 = a\cos u_2\sin A_2'$$

or

$$\cos u_1\sin A_1 = \cos u_2\sin A'_2. \tag{3.34}$$

Equation (3.34) can be rewritten still thus:

$$\frac{\sin(90^\circ - u_1)}{\sin A_2'} = \frac{\sin(90^\circ - u_2)}{\sin A_1}. \tag{3.35}$$

Equation (3.35) presents a theorem of sines for spherical triangle with sides $90 - u_1$, $90 - u_2$ and opposite angles $180 - A_2'$ and A_1. Let us introduce the third

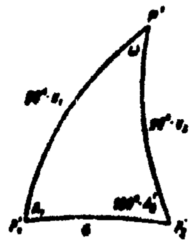


Fig. 43.

side of this triangle and its opposite angle. Let us designate this side by σ, and the angle ω. Then the shown polar spherical triangle will have a form depicted in Fig. 43.

For determination of all elements of this triangle following formulas of spherical trigonometry will serve

$$\left.\begin{aligned} &1.\ \sin\sigma\sin A_1 = \cos u_2 \sin\omega \\ &2.\ \sin\sigma\cos A_1 = \cos u_1 \sin u_2 - \sin u_1 \cos u_2 \cos\omega \\ &3.\ \sin\sigma\sin A_2' = \cos u_1 \sin\omega \\ &4.\ \sin\sigma\cos A_2' = -\sin u_1 \cos u_2 + \cos u_1 \sin u_2 \cos\omega \\ &5.\ \cos\sigma \quad = \sin u_1 \sin u_2 + \cos u_1 \cos u_2 \cos\omega \end{aligned}\right\} . \qquad (3.36)$$

Equation (3.34) and corresponding to geometric figure, can be represented in another form, i.e.;

$$N_1 \cos B_1 \sin A_1 = N_2 \cos B_2 \sin A_2'$$

or

$$\cos B_1 \frac{\sin A_1}{V_1} = \cos B_2 \frac{\sin A_2'}{V_2}. \qquad (3.37)$$

Designating:

$$\frac{\sin A_1}{V_1} = \sin A_1^{*};\ \frac{\sin A_2'}{V_2} = \sin A_2^{*}, \qquad (3.38)$$

we obtain:

$$\cos B_1 \sin A_1^{*} = \cos B_2 \sin A_2^{*}$$

or

$$\frac{\sin(90 - B_1)}{\sin A_2^{*}} = \frac{\sin(90 - B_2)}{\sin A_1^{*}}. \qquad (3.38')$$

Spherical triangle $P_1'P''P_2'$ (Fig. 43) corresponds to equation (3.38').

Thus, we see that depending upon the form of recording of fundamental equation of geodesic it can be interpreted by different spherical triangles.

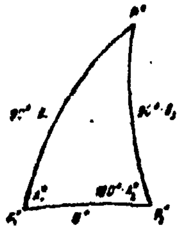


Fig. 43.

Selection of these triangles should be in accordance with the problem, that is to be resolved. However all these resolutions can differ by form, but essentially they are invariants of one and the same solution, which can be obtained with the help of an equation (3.34) and a corresponding to spherical triangle $P_1'P'P_2'$ (see Fig. 43).

For determination of geometric value of constant of c let us assume that arc $\breve{P_1P_2}$ (Fig. 44) continues to north and to south to equator. We designate azimuth of geodesic at point on the equator by A_0 the latitude of point P_0, and where the geodesic intersects a meridian at a right angle, by B_0.

We have:

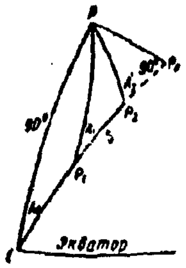


Fig. 44.

$$a\sin A_0 = N_0\cos B_0 = \frac{a\cos B_0}{W_0};$$

but:

$$\frac{\cos B_0}{W_0} = \cos u_0 \tag{3.39}$$

therefore:

$$\sin A_0 = \cos u_0 \tag{3.40}$$

or:

$$A_0 = 90 - u_0.$$

Thus, the constant c is equal to cosine of a given latitude of that point, where continuation of spherical arc σ intersects a meridian at a right-angle on an auxiliary sphere. Obviously, such intersection is possible only once, otherwise the equations (3.26) and (3.34) cannot have single value solution.

Let us deduce the differential equations of a geodesic:

From elementary right-angle triangle 1-2-3 (Fig. 45)

Fig. 45.

$$MdB = ds\cos A,$$
$$rdL = ds\sin A,$$

or:

$$\frac{dB}{ds}=\frac{\cos A}{M};\quad \frac{dL}{ds}=\frac{\sin A}{r}.$$

From (5.2[illegible]) obtain by differentiation:

$$dr\sin A+r\cos A\,dA=0,$$

But:

$$dr=-M\sin B\,dB,$$

therefore; by substitution $dB=\frac{ds\cos A}{M}$, we obtain:

$$\frac{dA}{ds}=\frac{\sin A\sin B}{r}.$$

Thus:

$$\left.\begin{aligned}\frac{dB}{ds}&=B'=\frac{\cos A}{M}=\frac{V^3\cdot\cos A}{c}\\ \frac{dL}{ds}&=L'=\frac{\sin A}{r}=\frac{\sin A}{N}\sec B\\ \frac{dA}{ds}&=A'=\frac{\sin A\cdot\sin B}{r}=\frac{\sin A\,\mathrm{tg}\,B}{N}\end{aligned}\right\}. \qquad (5.40a)$$

Obtained equations of geodesic (5.40a) constitute differential equations of a first order.

First two of them are suitable for any line on a surface, the third, obtained from fundamental equation of geodesic, is only for geodesics. Indicated equations are derivative latitudes, longitudes, and azimuths for distance s. Consequently, integrating these equations, we can obtain the difference of latitudes, longitudes, and azimuths of two points, located on the surface of a spheroid.

Passing from differentials to finite increments and designating them by ΔB, ΔL and ΔA with accuracy up to small values of third order, we have:

$$\left.\begin{aligned}\Delta B&=\frac{s\cos A}{M}+l_3=\frac{s\cos AV^3}{c}+l_3\\ \Delta L&=\frac{s\sin A}{r}+l_3=\frac{s\sin AV\sec B}{c}+l_3,\\ \Delta A&=\frac{s\sin A\sin B}{r}+l_3=\frac{s\sin A\,\mathrm{tg}\,BV}{c}+l_3\end{aligned}\right\}. \qquad (5.40b)$$

or in seconds:

$$\left.\begin{aligned}\Delta B'' &= (1)s\cos A + l_3\\ \Delta L'' &= (2)s\sin A\sec B + l_3\\ \Delta A'' &= (2)s\sin A\,\mathrm{tg}\,B + l_3\end{aligned}\right\}. \qquad (3.40c)$$

Formulas (3.40c) are frequently applied in approximate calculations. If it is accepted that: s = 30 km, A = 45° and

$$(1) \approx (2) \approx \frac{1}{30},\ \text{that}$$

$$\left.\begin{aligned}\Delta B'' &= \frac{30000\sqrt{2}}{30\cdot 2} \approx 700''\\ \Delta L'' &= \frac{30000\sqrt{2}}{30\cdot 2}\sec B \approx 700''\sec B\\ \Delta A'' &= \frac{30000\sqrt{2}}{30\cdot 2}\,\mathrm{tg}\,B = 700''\,\mathrm{tg}\,B\end{aligned}\right\}. \qquad ([illegible])$$

Thus we obtain approximate numerical values of differences of latitudes, longitudes and azimuths for adjacent points of 1st order triangulation.

§ 10. GEODETIC POLAR COORDINATES

One of the applications of the geodesics in spheroidal geodesy consists in that by its means it is possible to create a system of coordinates on a surface of a spheroid by which a position of points is determined by the length of geodesic and an angle, measured from a given initial direction. In the particular case, if this direction coincides with a meridian, then the second coordinate an angle, will be the azimuth of the geodesic. Such system of coordinates on a spheroid is analogous to polar system of coordinates on a plane, and is called geodesic polar coordinates.

On the basis of theory of geodesic polar coordinates lies a theorem.

If, on a surface from certain initial point a bundle of geodesics of equal length is drawn, then the curve, connecting their terminals, is orthogonal to each of them.

Let us assume that from point O two geodesics are drawn to length s, distant one from another by an angle dA. We will prove that arc P_1P_2 is perpendicular at points P_1P_2 to geodesics OP_1 and OP_2 (Fig. 46). We will prove this theorem from the opposite. Let us assume that angles at points P_1 and P_2 differ from polars by small value ε, whereupon by the law of continuity one of them is greater and the other less than the polar. Let us assume that at point P_2 the angle will be 90 + ε, and P_1 = = 90° − ε. We take on the line OP_1 point P_1' and connect it with P_2 by a line, composing with P_1P_2 a right-angle (Fig. 46), then from elementary right-angle triangle $P_1'P_1P_2$ we have:

$$P_1'P_2 = P_1'P_1\cos\varepsilon.$$

Further:

$$OP_1' + P_1'P_2 = OP_1' + P_1'P_1\cos\varepsilon = OP_1 - P_1'P_1 + P_1'P_1\cos\varepsilon = OP_1 - P_1P_1'(1 -$$

$$-\cos\varepsilon) = OP_2 - 2P_1'P_1\sin^2\frac{\varepsilon}{2}$$

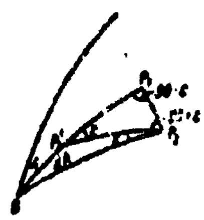


Fig. 46.

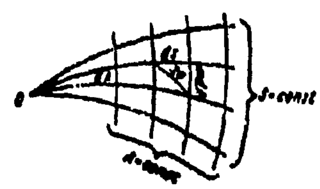


Fig. 47.

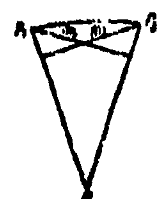


Fig. 48.

Since value $\sin^2\frac{\varepsilon}{2}$ is essentially positive, then, consequently,

$$OP_2 > OP_1' + P_1'P_2.$$

this cannot be, since by condition $OP_2 = OP_1' + P_1'P_2$.
QED.

In system of polar geodesic coordinates of line a = const are called <u>geodesic circumferences</u>. Element of geodesic circumference is equal to mdA (Fig. 47). Value m is called a reduced length of geodesic line. Lineal element of the surface in polar coordinates, as follows from Fig. 47, has the form of:

$$ds^2 = da^2 + m^2dA^2. \qquad (4.41)$$

In order to clarify the geometric meaning of the reduced length of geodesic, let us consider a specific case. Let us take the origin of coordinates at point of terrestrial pole, then, marking off along the meridians equal a and connecting their terminals, obtain geodesic circumference, which will coincide with terrestrial parallel. The reduced length of geodesic in this case will be the radius of a parallel. With small values s between initial and finite points, m — the length of a perpendicular, dropped from the initial point on to a normal, and passed into a finite point. Thus, m — the function of polar geodetic coordinates: the arc of geodesic s and the azimuth A. Between two points on a surface, regardless of which of them is taken as the initial point, m always has one value, i.e., to each geodesic there corresponds a specific m (Fig. 48).

The reduced length of geodesic is connected with Gauss curvature by differential equation, whose simplified derivation is shown below.

Let us take two points P_1 and P_2 on a spheroid at such a distance s, that it would be possible to disregard the difference of Gauss curvature $\frac{1}{R^2} = \frac{1}{MN} = K$ in them. We will draw in the area of these points a spherical surface with radius R and take on it an arc of the great circle, equal to s. We will designate by α (Fig. 49), the central angle, corresponding to arc s.

We have:

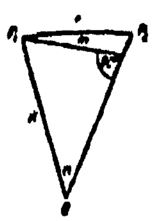


Fig. 49.

$$s = R\alpha,$$
$$m = R\sin\alpha,$$
$$ds = R\,d\alpha,$$
$$\frac{d\alpha}{ds} = \frac{1}{R},$$
$$dm = R\cos\alpha\,d\alpha,$$
$$\frac{dm}{ds} = \cos\alpha,$$
$$\frac{d^2m}{ds^2} = -\sin\alpha\,\frac{d\alpha}{ds}$$

or:

$$\frac{d^2m}{ds^2} = -\frac{m}{R^2},$$
$$\frac{d^2m}{ds^2} + mK = 0. \qquad (3.42)$$

Expression (3.42) is an ordinary second order differential equation, whose integration will give an m, if K is known, or it will give K, if m is given. In derivation of formula (3.42) strict analytic proof was not everywhere applied but the equation (3.42), if s is considered a geodesic, and K a Gauss curvature at a given point, is suitable for any surface.

Integration of equation (3.42) will be executed, while keeping in mind that for infinitesimal value of s value, m = s and, consequently, for s = 0 and m = 0, $\left(\frac{dm}{ds}\right)_0 =$ = 1. We will show m as the Maclaurin line of ascending powers of s, then:

$$m = m(s) = m_0 + sm_0' + \frac{s^2}{2} m_0'' + \frac{s^3}{6} m_0''' + \frac{s^4}{24} m_0^{IV} + \frac{s^5}{120} m_0^{V} + \ldots, \tag{3.43}$$

where:

$$m_0^i = \left(\frac{d^i m}{ds^i}\right)_0, \quad (i = 1, 2, 3, \ldots)$$

From (3.42)

$$\begin{gathered} m'' = -mK \\ m''' = -m'K - mK', \\ m^{IV} = -m''K - 2m'K' - mK'', \\ m^{V} = -m'''K - 3m''K' - 3m'K'' - mK''' \end{gathered}$$

Where s = 0:

$$\left.\begin{aligned} m_0'' &= 0 \\ m_0''' &= -K_0 \\ m_0^{IV} &= -2K_0' \\ m_0^{V} &= -m_0'''K_0 - 3K_0'' \end{aligned}\right\}, \tag{3.44}$$

$$K = \frac{V^4}{c^2},$$

$$K' = \frac{dK}{ds} = \frac{dK}{dB} \cdot \frac{dB}{ds} = \frac{4}{c^2} V^3 \frac{dV}{dB} \cdot \frac{dB}{ds}.$$

but:

$$\frac{dV}{dB} = -\frac{\eta^2 \operatorname{tg} B}{V}, \quad \frac{dB}{ds} = \frac{V^3}{c} \cos A,$$

therefore:

$$K' = -\frac{4\eta^2 V^4 \operatorname{tg} B \cos A}{c^3}. \tag{3.45}$$

From (3.45) it follows, that K' is a small value of first order, and that K'' is smaller by absolute value than K', therefore in further calculations we will take K'' = 0, which will lead to an error in final formula for m by small value carried to seventh place. Substituting the values of derivatives m_0', m_0'' m_0''', m_0^{IV} in (3.43) and considering that m_0 = 0, we find:

$$m = s - \frac{s^3}{6R^2} + \frac{s^4\eta^2 \operatorname{tg} B \cos A}{3R^3 V} + \frac{s^5}{120R^4} + l_1. \qquad (3.46)$$

In (3.46) R, B and A pertain to a point, which is taken for the initial. Applying (3.46) to spherical surface, where $\eta = 0$, we obtain

$$m_c = s - \frac{s^3}{6R^2} + \frac{s^5}{120R^4} - \ldots = R \sin\frac{s}{R}. \qquad (3.47)$$

One of the important applications of the reduced length of geodesic to problems of spheroidal geodesy consists in the proof of a theorem that the spheroidal triangles with sides, not exceeding 200-250 km, with an error of third order in small values can be solved as spherical.

Let us assume that two points P_1 and P_2 with their polar coordinates (s, A) and (s, A + ΔA) are given on a spheroid. Join them by an arc of geodesic circumference mΔA (Fig. 50). On a sphere of radius R take point O', from which with azimuths A and A + ΔA from line O'M' we draw arcs of great circles, equal to s. Join obtained points P_1' and P_2' by an arc of geodesic circumference (Fig. 51).

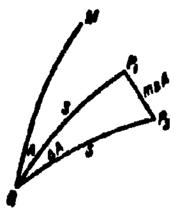


Fig. 50.

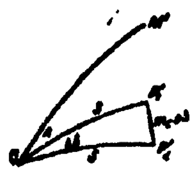


Fig. 51.

Consequently, the difference of arcs $P_1 P_2$ and $P_1' P_2'$ will be $\Delta A(m - m_c)$.

Relative error of lengths considering the value m and m_c by the formulas (3.46) and (3.47) will be:

$$\delta = \frac{(m - m_c)}{m} = \frac{s^3\eta^2 \operatorname{tg} B \cos A}{3R^3 V}.$$

Or, dropping terms with η^4,

$$\delta = \frac{s^3\eta^2 \operatorname{tg} B \cos A}{3R^3} = \frac{e'^2}{6}\left(\frac{s}{R}\right)^3 \sin 2B \cos A.$$

Value δ attains maximum where B = 45° and A = 0°, i.e.:

$$\delta_{max} = \frac{e'^2}{6}\left(\frac{s}{R}\right)^3.$$

In calculation of lengths of sides of triangles in 1st order triangulation we retain eighth decimal place.

Consequently, it is necessary that:

$$\delta_{max} < 1 \cdot 10^{-8}$$

or:

$$\frac{e'^2}{6}\left(\frac{s}{R}\right)^3 \leqslant 1 \cdot 10^{-8}.$$

Resolving this inequality where $e'^2 = \frac{1}{150}$, R = 6400 km, we find that:

$$s < 133 \text{ км.}$$

From this it follows that part of the spheroidal surface, bounded by geodesic circumference of 130-140 km radius, can be substituted by spherical radius R. With this R — mean radius of curvature of origin of the coordinates. Within the limits of this area the spheroidal triangles, the greater of which will be the inscribed equilateral triangle with sides 230-240 km, can be resolved as spherical with shown degree of accuracy. This very important derivation is used in the resolution of small spheroidal triangles.

The square of lineal element of surface in polar coordinates has the form, shown in formula (3.41).

This equation is satisfied by substitution (Fig. 52)

$$\left.\begin{aligned} ds &= d\sigma\cos\theta \\ m\,dA &= d\sigma\sin\theta \end{aligned}\right\}; \qquad (3.48)$$

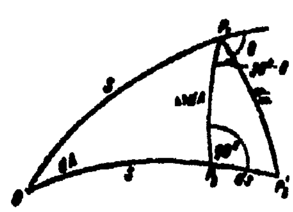


Fig. 52.

let us consider ds and dA arbitrary increases, therefore they can be taken as constants. Differentiating formula (3.48), we obtain:

$$dm\,dA = d\sigma\cos\theta\,d\theta$$

or:

$$\frac{dm\,d\sigma}{m}\sin\theta = ds\,d\theta.$$

Hence:

$$\frac{d\theta}{d\sigma} = \frac{1}{m}\frac{dm}{ds}\sin\theta. \qquad (3.49)$$

Let us assume that in a particular case $\theta = 90^{\circ}$, σ is a geodesic, designate it by p, then from (3.49) we obtain:

$$\frac{d\theta}{dp} = \frac{1}{m}\frac{dm}{ds}. \qquad (3.50)$$

This important equation is frequently used in resolution of various problems.

§ 17. RIGHT-ANGLE SPHEROIDAL COORDINATES

Let us take point O on a spheroid as initial and pass a geodesic OP_n through it. From point P construct a geodesic perpendicular to line OP_n at point P'. Designate section OP' by p, and section PP' by q (Fig. 53).

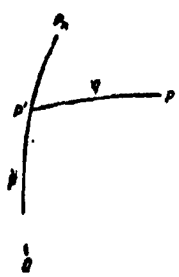


Fig. 53.

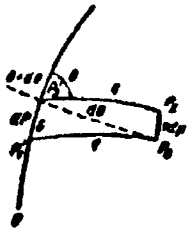
If the direction of the line OP_n on the surface of a spheroid is given, then sections OP' = p and $P'P$ = q fully determine the position of point P on the surface. In a particular case for simplifying problems to be resolved line OP_n is taken for any meridian, called axial. p and q are called right-angle spheroidal coordinates. They resemble cartesian coordinates in a plane. As on plane, p — abscissa, and q — ordinate in a system (p, q).

Introduction of spheroidal coordinates is based on a theorem: If on a given surface there are any geodetic lines from whose separate points emerge at right-angles on the same side an infinite number of geodetic lines of equal length, then the curve, connecting their other ends, intersects each of them at a right-angle.

The proof of this theorem is similar to that of a theorem for geodesic circumference. Shown in the theorem orthogonal trajectory q = const is called geodetic parallel. Geodetic parallel cannot be a geodesic.

Let us assume that on surfaces two close points (Fig. 54) with coordinates (p, q) and (p + dp, q) are given.

Let us pass a geodetic parallel through points P_1 and P_2 and designate its section P_1P_2 = ndp, where n — function of coordinates p and q. Connect points P_1 and P_2' by geodesic s. Inasmuch as points P_1 and P_2 are close together, the elementary arc of geodetic parallel ndp can be considered an elementary arc of geodetic circumference of radius s, i.e.:



$$n\,dp = m\,d\theta. \qquad (3.51)$$

Comparing equations (3.50) and (3.51) we obtain:

Fig. [illegible]

$$n = \frac{dm}{ds}. \qquad (3.52)$$

Equation (3.52) is obtained for orthogonal geodetic. In our case such line is q, consequently:

$$n = \frac{dm}{dq}. \qquad (3.53)$$

We have:

$$\frac{dn}{dq} = \frac{d^2m}{dq^2}. \qquad (I)$$

$$\frac{d^2n}{dq^2} = \frac{d^3m}{dq^3}. \qquad (II)$$

Let us differentiate equation (3.42), by preliminary substitution of s by q, then:

$$\frac{d^3m}{dq^3} + \frac{dm}{dq} \cdot K = 0.$$

Or, taking into account expressions (I) and (II), we find:

$$\frac{d^2n}{dq^2} + nK = 0. \qquad (3.54)$$

Comparing equations (3.42) and (3.54), we arrive at a conclusion that they are completely symmetric with respect to Gauss curvature. Only equation (3.42) is suitable for any geodesic while (3.54) is applicable only for ordinates in a system of right-angle spheroidal coordinates.

Geometric meaning of the value n is clearest when we study right-angle coordinates p and q for spherical surface.

Substituting s by q in (3.47) where q_c is an ordinate in a system of spherical coordinates, we obtain:

$$m_c = R \sin \frac{q_c}{R}.$$

Differentiating this formula by q_c, we obtain:

$$n_c = \cos \frac{q_c}{R}. \qquad (3.55)$$

In Fig. 55 a system of coordinates (p, q) on a sphere is depicted. Line P_1C is a geodetic parallel and Q is a pole of axial meridian. Length of geodetic parallel decreases proportionally $n_c = \cos \frac{q_c}{R}$. Consequently, n_c, can generally be called the coefficient of convergence of ordinates. All ordinates, perpendicular to axial meridian on a sphere, cross at one point are called their pole, but on a spheroid the ordinates do not cross at one point, therefore they do not have a common pole.

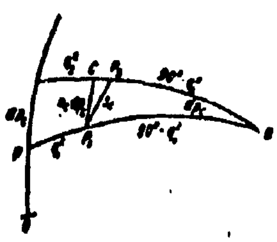


Fig. 55.

In accordance with formula (3.46) and with substitution of s by q, we obtain after differentiation:

$$n = 1 - \frac{q^2}{2R^2} + \frac{4q^3\eta^2 \operatorname{tg} B \cos A}{3R^3V} + \frac{q^4}{24R^4} + l_5. \qquad (3.56)$$

Here R, η and V pertain to point P_1, and A to azimuth of the line P_1P_1' (Fig. 53). We designate:

$$-\frac{1}{2R^2} = f, \quad \frac{4}{3}\eta^2 \frac{\operatorname{tg} B \cos A}{3R^3V} = g, \quad \frac{1}{24R^4} = h.$$

Then:

$$n = 1 + fq^2 + gq^3 + hq^4 + l_5. \qquad (3.57)$$

Coefficients f, g and h are functions of latitude and azimuth at point P_1 or abscissas of point P_1.

For various applications and practical calculations it is expedient to convert expression for n in such a manner that coefficients f, g and h become functions of latitude and azimuth of a geodesic s at the origin of coordinates. Considering that they are certain functions of p – abscissa, we will apply Maclaurin line and present them by series:

$$\left.\begin{aligned} f &= f^0 + pf' + p^2f'' + p^3f''' + \dots \\ g &= g^0 + pg' + p^2g'' + p^3g''' + \dots \\ h &= h^0 + ph' + p^2h'' + p^3h''' + \dots \end{aligned}\right\}. \qquad (3.58)$$

In series (3.58)

$$f^i = \frac{1}{i!}\frac{\partial^i f}{\partial p^i}, \quad g^i = \frac{1}{i!}\frac{\partial^i g}{\partial p^i} \text{ and } h^i = \frac{1}{i!}\frac{\partial^i h}{\partial p^i}. \ (i = 1, 2, 3, \dots)$$

Substituting (3.58) for (3.57) and retaining values of fourth order with respect to p and q, we obtain:

$$\left.\begin{aligned} n = 1 &+ f^0q^2 + f'pq^2 + \dots \\ &+ g^0q^3 + g'pq^3 + \dots \\ &+ h^0q^4 \end{aligned}\right\}. \qquad (3.59)$$

From (3.59) where $q = 0$ it follows that:

$$\left.\begin{aligned} &n_0 = 1, \\ &\left(\frac{\partial n}{\partial q}\right)_0 = 0 \end{aligned}\right\}. \qquad (3.60)$$

Coefficients f^0, f', f'', g^0, g', h^0 are the essence of the function of origin of spheroidal coordinates.

For Gauss curvature we obtain from (3.54):

$$K = -\frac{1}{n}\frac{d^2n}{dq^2}.$$

Or, taking into account (3.57):

$$K = -\frac{2f + 6gq + 12hq^2 + \dots}{1 + fq^2 + \dots} = -2(f + 3gq + (6h - f^2)q^2 - \dots).$$

We substitute f, g and h by f^0, f', f'' g^0, g', h^0 etc. according to (3.58), then:

$$K = -2f^0 - 2f'p - 6g^0q - 2f''p^2 - 6g'pq - (12h^0 - 2f^{0\,2})q^2.$$

Let us consider a case, where K is a linear function of p and q, i.e.:

$$K = -2f^0 - 2f'p - 6g^0q. \qquad (3.61)$$

Consequently, in this case it is necessary to set:

$$f' = 0, \quad g' = 0, \quad 12h^{0} = 2f^{02}.$$

With these values of coefficients the formula (3.69) will take the following form:

$$n = 1 + f^{0}q^{2} + f'pq^{2} + g^{0}q^{3} + \frac{1}{6}f^{02}q^{4} + l_{5}.$$

With the same degree of accuracy:

$$\frac{1}{n} = 1 - f^{0}q^{2} - f'pq^{2} - g^{0}q^{3} + \frac{5}{6}f^{02}q^{4} + l_{5}. \qquad (3.70)$$

The values of coefficients f^{0}, f', g^{0}, determined by means connected with Gauss curvature, will be given in the following chapter, devoted to solution of spheroidal triangles. These designations were first introduced by Gauss in "General Investigations of Curves of Surfaces,"[1] therefore subsequently we will call them Gauss coefficients.

§ 18. DIFFERENCES OF AZIMUTHS AND LENGTHS OF ARCS OF GEODESICS AND NORMAL SECTION

For formation of geodesic triangles on a surface of a spheroid it is necessary to change over from normal sections to geodesics. With this goal it is necessary to introduce corrections into the measured directions. Deductions of the formula for indicated corrections will be made with the utilization of an understanding about geodetic curvature of a normal section.

Normal section is a plane section, at each of its points a binormal is perpendicular to the normal plane. The same perpendicular will constitute to inverse normal section with an inverse normal plane an angle, equal to $90^{0} - f$, where f is an angle between mutual normal planes, equal, in accordance with (3.4) to:

$$f = \frac{e\eta_{m}^{2}\cos A \cdot \sin A}{M_{m}} + l_{5}.$$

[1]K. F. Gauss. Selected Geodesic Compositions, Vol. II, Geodezizdat, 1958.

Geodetic degree of curvature on a surface, as follows from (1.48), is equal to:

$$\frac{1}{R_g} = \frac{1}{R}\cos\vartheta.$$

where $\frac{1}{R}$ is the usual degree of curvature and ϑ is an angle between normal to surface and binormal of a curve.

In our case:

$$\vartheta = 90^\circ - f, \tag{3,63}$$

consequently:

$$\frac{1}{R_g} = \frac{1}{R}\sin f; \tag{3,64}$$

sin f or f is a small value of the second order, therefore in reference to normal section we can take $\frac{1}{R}$ as equal to $\frac{1}{M} \cong \frac{1}{N}$.

Then:

$$\frac{1}{R_g} = \frac{\sin f}{N} = \frac{f}{N}. \tag{3,65}$$

In a general case the geodesic is disposed as is shown in Fig. 56 with respect to mutual normal sections.

Let us construct on a tangent plane of points P_1 of lines, passing through this point (Fig. 56). In this projection the geodesic will be depicted as straight line and normal sections a and b as curves (Fig. 57).

Fig. 56.

Fig. 57.

Let us take P_1 as origin of grid coordinates; direct axis x along s, i.e., in this case x = s, and axis y is perpendicular to s, then b will be an angle between geodesic and normal section.

Geodetic curvature at any point of a normal section is equal to:

$$\frac{1}{R_g} = \frac{-y''}{(1+y'^2)^{3/2}} = \frac{f}{N} = \frac{s\eta^2\cos A\sin A}{N^2}. \tag{3,66}$$

In adopted system of coordinates:

$$y' = \mathrm{tg}\,\delta, \qquad (3.67)$$

but δ is a small value of third order, therefore we can with great accuracy in denominator (3.66) take $y'^2 = 0$.

Then:

$$-y'' = \frac{s\eta^2 \cos A \sin A}{N^3}. \qquad (3.68)$$

Thus, we obtained second order differential equation. Considering in equation (3.68) the latitude and azimuth at the origin of coordinates as constants, we integrate them.

We have:

$$-y' = \frac{s^2\eta^2 \sin A \cos A}{2N^3} + c_1,$$
$$-y = \frac{s^3\eta^2 \sin A \cos A}{6N^3} + c_1 s + c_2.$$

Let us determine the value of constants c_1 and c_2. At point P_1 we have $x = 0$, $y = 0$, and $y' = \mathrm{tg}\,\delta$. Consequently:

$$\left.\begin{aligned} c_1 &= -\mathrm{tg}\,\delta = -y' \\ c_2 &= 0 \end{aligned}\right\}.$$

At point P_2 we have $y = 0$, $x = s$ and then:

$$\mathrm{tg}\,\delta = \frac{s^2\eta^2 \sin A \cos A}{6N^2}$$

or:

$$\delta'' = \rho'' \frac{s^2\eta^2 \sin A \cos A}{6N^2} = \rho'' \frac{s^2\eta^2 \sin 2A}{12N^2}, \qquad (3.69)$$

Where S	B	A	δ
120 km	45°	45°	0",020
100	45	45	0 ,014
60	45	45	0 ,006
30	45	45	0 ,001

Thus, the magnitude of correction in direction of transition from normal sections to geodesics in usual by dimensions triangles of triangulation is less than 0",001;

for single transmissions this correction can be ignored. But in consecutive calculation of azimuths of the sides in triangulation along the links, disregard of this correction can lead to a systematic error in azimuth of the side of last triangle of a link on an order of $0.''1$.

Angles and azimuths of geodesics after adjustment of 1st order triangulation are calculated to $0.''001$. Therefore correction δ should be considered in mathematical treatment of results of angle measurements in state 1st order triangulation. In 2nd triangulation this correction is disregarded.

Let us find the difference in lengths of arcs of geodesics and normal section. We will express the element of an arc of normal section in polar geodetic coordinates:

$$d\sigma^2 = ds^2 + m^2 dA^2. \tag{3.70}$$

Here $d\sigma$ is an element of arc of normal section, ds — an element of the arc of geodesic, and A an angle between these arcs, i.e., $A = \delta$.

From expression (3.70):

$$d\sigma^2 = ds^2\left[1 + m^2\left(\frac{dA}{ds}\right)^2\right]$$

or:

$$d\sigma = ds\left[1 + \frac{m^2}{2}\left(\frac{dA}{ds}\right)^2 + \dots\right].$$

But from (3.69)

$$\left(\frac{d\delta}{ds}\right)^2 = \left(\frac{dA}{ds}\right)^2 = \frac{e^4\eta^4\sin^2 2A}{36N^4}.$$

Consequently,

$$d\sigma = ds\left(1 + \frac{m^2 s^4\eta^4\sin^2 2A}{72N^4}\right). \tag{3.71}$$

But, in accordance with formula (3.46)

$$m = s - \frac{s^3}{6R^2} + \dots$$

With an error of a magnitude higher than eighth order in expression (4.71) it is possible to take m = s, then:

$$d\sigma = ds\left(1 + \frac{s^4\eta^4 \sin^2 2A}{72N^4}\right).$$

Integral of this differential equation will be:

$$\sigma - s = \frac{s^5\eta^4 \sin^2 2A_1}{360N_1^4}. \qquad (4.72)$$

Where s = 1000 km, B = 0 and A = 45° we have:

$$\sigma - s = 0{,}075 \text{ мм}.$$

Thus, at any distances on a terrestrial spheroid, possible in practice in geodesic work, it is possible not to consider the difference of lengths of arcs of geodesic and normal section.

§ 10. CORRECTION FOR A HEIGHT OF AN OBSERVED POINT

Let us assume that points a and b are projections of points A and B on the surface of the spheroid (Fig. 58), n_1 and n_2 are points of intersection of normals of points A and B with axis of rotation and H the height of point B. Direction measured from A to B lies in a plane $ab'n_1$, while we have to obtain an angle between directions ab_0 and ab. Consequently, the measured direction A_1 must be corrected by value γ.

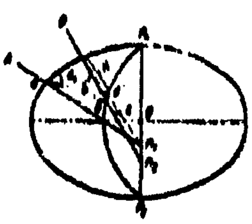


Fig. 58.

In triangle abb' angles at vertexes b and b' can be taken as equal to $360^\circ - A_2$ (A_2 is a back azimuth), and lengths ab and ab' are equal to s, then we obtain:

$$\gamma = -\frac{bb' \sin A_2}{s}.$$

In view of the smallness of bb' we can consider that this is an arc of circumference of radius H; consequently:

$$bb' = \varepsilon H.$$

But ε from expression (3.2) with replacement $\Delta B = \frac{s \cos A_1}{M_1}$ is equal to:

$$\varepsilon = \eta_1^2 \frac{s}{M_1} \cos A_1$$

or:

$$\gamma = -\frac{s^2 H \sin A_1 \cos A_1}{M_1}.$$

Since γ is a small magnitude of the second order, it is possible to take:

$$\sin A_2 \approx -\sin A_1,$$

therefore:

$$\gamma'' = \rho'' \frac{\eta_1^2 H \sin 2A_1}{2M_1},$$

or, taking $\frac{\rho''}{M_1} = (1)_1$,

$$\left.\begin{aligned} \gamma &= (1)_1 \eta_1^2 \frac{H}{2} \sin 2A_1 \\ (A_1 - A_1')'' = \gamma'' &= (1)_1 \eta_1^2 \frac{H}{2} \sin 2A_1 \end{aligned}\right\} \qquad (4.74)$$

It should be emphasized that γ in the main member does not depend on s.

When $B = 45^\circ$, $A_1 = 45^\circ$, $(1) = \frac{1}{30}$

we have:

For $H = 1000$ m we have $\gamma'' = 0''.05$,

$H = 200$ m we have $\gamma'' = 0''.01$.

We require that $\gamma \leq 0''.001$. Let us assume that B and A_1 have these same values, then:

$$H = \frac{0''.001 \cdot 30 \cdot 2}{2} \approx 0.03 \text{ км} \approx 30 \text{ м}.$$

Thus, correction γ should be considered where $H > 30$ m. Besides it should be considered that H is the height of sighting target above the reference ellipsoid.

Numerical examples of calculations of corrections for height of observed point and transition from azimuths of normal sections to azimuths of geodesics are given in "Practicum on Higher Geodesy" on p. 269-270.

CHAPTER IV

RESOLUTION OF SPHERICAL AND SPHEROIDAL TRIANGLES

§ 20. RESOLUTION OF SMALL SPHERICAL TRIANGLES BY LEGENDRE THECREM

In planning a scheme for 1st order triangulation it is necessary to deal with triangles of comparatively small dimensions. History of trigonometric work lists only several triangles, having sides more than 100-150 km in length. The longest side of a geodetic quadrangle, measured by French geodesists for connection between triangulations of Spain and Algeria in 1879, was nearly 270 km long. Triangles of namely this quadrangle serve as an example of resolution cf large triangles on Earth's surface.

Presently radargeodetic means make it possible to measure distances on the order of 400-500 km; however for layout of high-precision geodetic nets with indicated sides these means have not been used as yet. Nevertheless, considering the prospects of increase in the accuracy of radargeodetic measurements, this chapter will consider methods and obtained exact formulas, both for resolution of small dimension triangles (25-30 km), laid out according to contemporary scheme of triangulation, and for triangles of large dimensions, up to 400-500 km.

Proceeding from theorem in § 17, spheroidal triangles with sides of 230-240 km, with errors of third order values can be substituted by spherical triangles with similar sides, laid out on a sphere of radius, equal to mean radius of curvature of the center of gravity of spherical triangle. Consequently, all triangles of contemporary 1st order triangulation can be resolved as spherical, i.e., without considering their spheroidness.

The outgoing sides of triangulation are measured directly or are obtained through basic nets in meters. Therefore sides of triangulation should also be obtained in these units. In selection of methods of resolution of triangles this condition is initial. There are several such methods. Theorem of Legendre, is the most frequently used, it is so formulated that a small spherical triangle can be solved as a plane one, if every angle is decreased by one third of its spherical excess.

Let us take a given spherical triangle ABC and a corresponding plane triangle $A_1B_1C_1$ (Fig. 59).

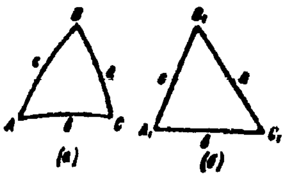


Fig. 59.

We have:

$$\sin\left(\frac{A-A_1}{2}\right)=\sin\frac{A}{2}\cos\frac{A_1}{2}-\cos\frac{A}{2}\sin\frac{A_1}{2},$$

but:

$$\sin\frac{A}{2}=\sqrt{\frac{\sin(p-b)\sin(p-c)}{\sin b\sin c}};\quad \cos\frac{A}{2}=\sqrt{\frac{\sin p\sin(p-a)}{\sin b\sin c}};$$

$$\sin\frac{A_1}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}},\quad \cos\frac{A_1}{2}=\sqrt{\frac{p(p-a)}{bc}},$$

where: $\frac{a+b+c}{2}=p$.

Consequently:

$$\sin\frac{A-A_1}{2}=\sqrt{\frac{\sin(p-b)\sin(p-c)\,p(p-a)}{bc\sin b\sin c}}-\sqrt{\frac{\sin p\sin(p-a)(p-b)(p-c)}{bc\sin b\sin c}}$$

Supplementing subradical expressions to full area of a plane triangle $A_1B_1C_1$; i.e., $\Delta=\sqrt{p(p-b)(p-a)(p-c)}$ and carrying it as a common factor, we obtain

$$\sin\frac{A-A_1}{2}=\frac{\Delta}{bc\sqrt{\frac{\sin b}{b}\cdot\frac{\sin c}{c}}}\left\{\sqrt{\frac{\sin(p-b)\sin(p-c)}{(p-b)(p-c)}}-\sqrt{\frac{\sin p\sin(p-c)}{p(p-c)}}\right\};$$

under roots a following expression is obtained:

$$\sqrt{\frac{\sin x}{x}}=\left(1-\frac{x^2}{6}+\frac{x^4}{120}+\ldots\right)^{1/2}=1-\frac{x^2}{12}+\frac{x^4}{1440}+\ldots$$

Retaining small values to fourth order inclusively and taking additional designations:

$$2p_1=b+c-a,\quad 2p_2=a+c-b,\quad 2p_3=a+b-c,$$

we obtain

$$\sin\frac{A-A_1}{2}=\frac{\Delta}{bc}\left\{\frac{(p^2-p_2^2)+(p_1^2-p_3^2)}{12}+\frac{p_2^2p_3^2-p^2p_1^2}{144}-\right.$$
$$\left.-\frac{(p^4-p_2^4)+(p_1^4-p_3^4)}{1440}\right\}\left(1+\frac{b^2+c^2}{12}\right).$$

But

$$1)\ p^2-p_2^2+p_1^2-p_3^2=(p-p_2)(p+p_2)+(p_1-p_3)(p_1+p_3)=$$
$$=b(a+c)+b(c-a)=2bc,$$
$$2)\ p_2^2p_3^2-p^2p_1^2=(p_2p_3-pp_1)(p_2p_3-pp_1)=$$
$$=\frac{1}{8}[a^2-b^2-c^2][(b+c)^2-(b-c)^2]=\frac{bc}{2}(a^2-b^2-c^2),$$
$$3)\ (p^4-p_2^4)+(p_1^4-p_3^4)=(p^2-p_2^2)(p^2+p_2^2)+(p_1^2-p_3^2)(p_1^2+p_3^2)=$$
$$=bc(3a^2+b^2+c^2).$$

Consequently,

$$\sin\frac{A-A_1}{2}=\frac{\Delta}{6}\left\{1+\frac{a^2-b^2-c^2}{48}-\frac{3a^2+b^2+c^2}{240}\right\}\left(1+\frac{b^2+c^2}{12}\right),$$

or

$$\sin\frac{A-A_1}{2}=\frac{\Delta}{6}\left\{1+\frac{b^2+c^2}{12}+\frac{a^2-b^2-c^2}{48}-\frac{3a^2+b^2+c^2}{240}\right\}=$$
$$=\frac{\Delta}{6}\left(1+\frac{a^2+7b^2+7c^2}{120}\right). \tag{4.1}$$

From (4.1) it follows that sin $(A - A_1)$ is the small value of the second order, i.e., the difference in angles of spherical and plane triangles for corresponding sides is a small value of the second order, therefore with accuracy up to small values of sixth order:

$$\sin\frac{A-A_1}{2}=\frac{A-A_1}{2}+l_6.$$

Expressing $(A - A_1)$ in seconds, sides of triangle in parts of radius of a sphere R and considering that for $(B - B_1)$ and $(C - C_1)$ we have to obtain symmetric (4.1) expressions by means of corresponding transposition of letters a, b, c, we find:

$$\left.\begin{aligned}(A-A_1)''&=\frac{\Delta}{3R^2}\rho''\left(1+\frac{a^2+7b^2+7c^2}{120R^2}\right)+l_6;\\(B-B_1)''&=\frac{\Delta}{3R^2}\rho''\left(1+\frac{7a^2+b^2+7c^2}{120R^2}\right)+l_6;\\(C-C_1)''&=\frac{\Delta}{3R^2}\rho''\left(1+\frac{7a^2+7b^2+c^2}{120R^2}\right)+l_6.\end{aligned}\right\} \tag{4.2}$$

or:

$$A+B+C-(A_1+B_1+C_1)=\varepsilon''=\frac{\Delta}{R^2}\rho''\left(1+\frac{a^2+b^2+c^2}{24R^2}\right)+l_6. \tag{4.3}$$

From (4.3

$$\frac{\Delta}{R^2}\rho'' = \varepsilon''\left(1 - \frac{a^2+b^2+c^2}{24R^2}\right) + l_4. \qquad (4.4)$$

Substituting (4.4) for (4.2) and adopting designation:

$$\frac{a^2+b^2+c^2}{3} = m^2, \qquad (4.5)$$

we finally obtain

$$\left.\begin{aligned} A_1 &= A - \frac{\varepsilon''}{3} - \frac{\varepsilon''}{60R^2}(m^2 - a^2) + l_4 \\ B_1 &= B - \frac{\varepsilon''}{3} - \frac{\varepsilon''}{60R^2}(m^2 - b^2) + l_4 \\ C_1 &= C - \frac{\varepsilon''}{3} - \frac{\varepsilon''}{60R^2}(m^2 - c^2) + l_4 \end{aligned}\right\}, \qquad (4.6)$$

$$\varepsilon'' = \frac{\Delta}{R^2}\rho''\left(1 + \frac{m^2}{8R^2}\right) + l_4. \qquad (4.6')$$

If in (4.6) the terms of fourth order are dropped, then the obtained expressions will take the following form:

$$\left.\begin{aligned} A_1 &= A - \frac{\varepsilon''}{3} \\ B_1 &= B - \frac{\varepsilon''}{3} \\ C_1 &= C - \frac{\varepsilon''}{3} \end{aligned}\right\}, \qquad (4.7)$$

this will be proof of the above mentioned Legendre theorem. Dropped terms are called correction spherical terms of Legendre theorem.

Spherical excess will be equal to:

$$\varepsilon'' = \frac{\Delta}{2R^2}\rho''. \qquad (4.8)$$

Since Δ is an area of a plane triangle, then:

$$\varepsilon'' = \rho''\frac{ab}{2R^2}\sin C_1 = \rho''\frac{ac}{2R^2}\sin B_1 = \rho''\frac{cb}{2R^2}\sin A_1. \qquad (4.9)$$

From (4.5) and (4.6) it follows that in equilateral triangles spherical terms become zero, but in isosceles triangles they become maximum.

Let us assume b = c, and investigate the obtained expression of spherical correction.

From isosceles plane triangle:

$$\begin{aligned} &1)\ b = c = \frac{a}{2\cos C_1}, \\ &2)\ \varepsilon'' = \frac{ab}{2R^2}\rho''\sin C_1 = \rho''\frac{a^2}{4R^2}\operatorname{tg} C_1. \end{aligned} \qquad (4.10)$$

Therefore:

$$\mu = \frac{\varepsilon''}{60R^4}(m^4 - a^4) = \rho'' \cdot \frac{a^4}{1440R^4}(\mathrm{tg}^4 C_1 - 2\,\mathrm{tg}\, C_1).$$

Hence:

$$\frac{\partial \mu}{\partial C_1} = \frac{3a^4 \rho''}{1440R^4}\left(\frac{\mathrm{tg}^2 C_1}{\cos^2 C_1} - \frac{1}{\cos^2 C_1}\right) = 0$$

or

$$\left(\frac{\mathrm{tg}^2 C_1}{\cos^2 C_1} - \frac{1}{\cos^2 C_1}\right) = \frac{\mathrm{tg}^2 C_1 - 1}{\cos^2 C_1} = 0.$$

But $\cos^2 C_1 \neq 1$, consequently, $\mathrm{tg}^2 C_1 - 1 = 0$, whence:

$$\mathrm{tg}\, C_1 = 1, \text{ a } C_1 = B_1 = 45^\circ, \ A_1 = 90^\circ \text{ and}$$

$$\mu_{max} = \frac{a^4}{720R^4}\rho''.$$

Let us assume that, as required in 1st order triangulation $\mu_{max} \leq 0.''001$, then:

$$\left(\frac{a}{R}\right)^4 = \frac{720}{10^3 \cdot 2 \cdot 10^5} < \frac{360}{10^8}$$

or:

$$\frac{a}{R} < \frac{4,3}{10^2}; \quad a = \frac{4,3R}{10^2} = \frac{4,3 \cdot 6000}{100} \approx 264 \text{ км}.$$

From (4.10) it follows that where $C_1 = 45^\circ$,

$$b = c \approx 185 \text{ км}.$$

Thus, the reduced calculation shows that if the largest side of a spherical triangle does not exceed 200-250 km, then such triangle can be resolved by the Legendre theorem without correction terms. Angles A_1, B_1 and C_1 are called reduced plane angles.

If the sides of a triangle exceed the shown limits, spherical corrections of Legendre theorem should be used, but then such triangles, measured on Earth's surface, can not be considered spherical; they should be solved as spheroidal (§ 23).

Spherical excesses of triangles are calculated in the triangles of 1st order triangulation with accuracy of up to $0.''001$, therefore the error in determination of ε must not exceed 4-5 units of fourth decimal place. Proceeding from this requirement, we establish, at what values of ε it is necessary to calculate them by the formula (4.6').

Second term of right side (4.6') has the form:

$$\Delta\varepsilon = \varepsilon'' \frac{m^2}{8R^2}. \tag{4.11}$$

Let us assume that:

$$\varepsilon'' \frac{m^2}{8R^2} < 0'',0004 \tag{4.11'}$$

(m will have maximum value in equilateral triangle), a = b = c = m = 100 km, then from inequality (4.11') it follows:

$$\varepsilon'' < 17''.5.$$

Thus, if $\varepsilon'' > 18''$, then for calculation of ε formula (4.9'), and where $\varepsilon \leq 18''$ use formula (4.9).

In calculation of ε'', as a rule, exact values of reduced angles are unknown, therefore it is necessary to substitute them by spherical. This substitution can lead to an error:

$$d\varepsilon'' = \varepsilon'' \operatorname{ctg} A_1 dA,$$

where:

$$dA = \frac{\varepsilon}{3}.$$

therefore:

$$d\varepsilon'' = \varepsilon''^2 \frac{\operatorname{ctg} A_1}{3\rho''}.$$

Under condition, $d\varepsilon'' \leq 0''.004$ and $A_1 = 60°$

$$\varepsilon'' < 18''.$$

In other words, preceding derivation about the accuracy of calculation of ε'' was confirmed, where $\varepsilon \leq 18''$ it should be calculated according to (4.9), with substitution of reduced plane angles A_1, B_1 and C_1 by corresponding spherical angles. If however $\varepsilon > 18''$, then ε is obtained: by taking the sum of the measured angles of a triangle less 180°, subtract a third part of this difference from angle A or B, or C and by it calculate ε by the formula (4.9').

Let us note that prior to calculation of ε by the formulas (4.9') or (4.9) its value with an error of $\pm 0''.05$ is known from preliminary calculations for determination of discrepancy of a triangle.

For triangles of 1st order triangulation ε'' can have a maximum value of 5'' with 50 km sides; with 60 km sides, 8'' and with 100-110 km sides, 20''-27''.

If triangles in triangulation are formed by ellipsoid chords, then, as was shown by M. S. Molodenskiy, their resolution can be satisfied by the formulas, analogous to formulas of Legendre theorem.

Designating chords by $\overline{a}$, $\overline{b}$ and $\overline{c}$ and assuming that $\overline{a}$ and spherical angles A, B and C for $\overline{b}$ and $\overline{c}$, are known we obtain:

$$\lg \bar{b} = \lg \bar{a}\, \frac{\sin\left(B - \frac{1}{4}\varepsilon\right)}{\sin\left(A - \frac{1}{4}\varepsilon\right)},$$

$$\lg \bar{c} = \lg \bar{a}\, \frac{\sin\left(C - \frac{1}{4}\varepsilon\right)}{\sin\left(A - \frac{1}{4}\varepsilon\right)}.$$

§ 24. RESOLUTION OF SMALL SPHERICAL TRIANGLES BY THE ADDITAMENT METHOD

For spherical triangle ABC (Fig. [illegible]), whose sides are expressed in parts of radius, we have by theorem of sines

$$\sin\frac{b}{R} = \sin\frac{a}{R}\,\frac{\sin B}{\sin A}. \qquad (4.12)$$

Fig. [illegible]

Here a — known side, b is to be determined.

For 1st order triangulation $\frac{b}{R}^{\circ} < 1^{\circ}$, therefore from sines of small angles let us turn to angles by a known formula:

$$\lg\sin x = \lg x - \frac{\mu x^2}{6} - \frac{\mu x^4}{180} - \dots \qquad (4.13)$$

(μ — modulus of natural functions).

From (4.12) and (4.13) we have:

$$\left.\begin{aligned} \lg\sin\frac{b}{R} &= \lg\frac{b}{R} - \frac{\mu\, 10^6 b^2}{6R^2} - \frac{\mu\, 10^8 b^4}{180R^4} - \dots \\ \lg\sin\frac{a}{R} &= \lg\frac{a}{R} - \frac{\mu\, 10^6 a^2}{6R^2} - \frac{\mu\, 10^8 a^4}{180R^4} - \dots \end{aligned}\right\} \qquad (4.14)$$

Where a = b = 200 km term $\frac{\mu\, 10^8 b^4}{180R^4} < 0.6\cdot 10^{-8}$. Therefore for usual sides in triangulation this term in (4.14) can be disregarded. Let us designate:

$$A_s = \frac{\mu\, 10^6 s^2}{6R^2}, \qquad (4.15)$$

s is a general symbol for designation of the side in triangulation.

Consequently, from (4.12), (4.14) and (4.15) we have

$$\lg b = \lg a + \lg\frac{\sin B}{\sin A} - A_a + A_b; \qquad (4.16)$$

A_a, A_b are called additaments, whence the name "Additament Method".

In comparison with the Legendre theorem in additament method the logarithms of sides of a triangle are changed, and the solution of a triangle is made by the following actions:

1. From the logarithm of the initial side take away its additament and obtain a reduced logarithm of the side.

2. Resolve the triangle with initial reduced side as a plane and obtain reduced

logarithms of other sides of the triangle.

4. To reduced logarithms of each side add the additament and obtain logarithms of spheroidal sides.

Using geodetic tables, additaments can be calculated in the following manner: for each degree of latitude logarithms of value (5) = $\frac{1}{3R^2}$, consequently:

$$A_s = \frac{1}{2}(5)s^2$$

or:

$$\lg b = \lg a + \lg\frac{\sin B}{\sin A} - \frac{1}{2}(5)a^2 + \frac{1}{2}(5)b^2. \qquad (4,15)$$

From (4,15) it follows that for calculation of additaments it is necessary to know Gauss curvature $\frac{1}{R^2}$ for every triangle. However $\frac{1}{R^2}$ changes so slowly with a change of latitude that, knowing $\frac{1}{R^2}$ for one triangle, it is possible to calculate by it additaments for a whole link of triangulation. Indeed in geodetic tables (p. [illegible]) in latitudinal band [illegible]° or on an extent of [illegible] km change lg $\frac{1}{R^2}$ constitutes three units of fourth decimal place. Therefore in (4,15) $\frac{1}{R^2}$ can be accepted as constant for significant latitudinal bands. By argument lg s it is possible to compose a table for selection of additaments. In Table 5 are given values of additaments for mean latitude of [illegible]°. This latitude approximately coincides with the mean latitude of the territory of the USSR.

Table 5

lgs	Additaments A in units of 8 place logarithms										A cp	lgs	lgs	A_s
	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09				
4.0	18	18	19	20	21	22	23	25	26	27	1	4.0	3.0	
4.1	28	30	31	31	34	35	37	39	41	42	3	4.1	3.1	
4.2	45	47	49	51	54	56	59	62	65	66	3	4.2	3.2	0
4.3	71	74	78	81	85	90	93	98	103	107	5	4.3	3.3	1
4.4	112	117	123	130	135	141	148	155	162	170	8	4.4	3.4	1
4.5	178	186	195	204	214	224	234	245	257	269	13	4.5	3.5	2
4.6	282	295	310	324	339	355	372	389	406	427	20	4.6	3.6	3
4.7	447	468	491	513	537	562	589	617	646	676	32	4.7	3.7	4
4.8	708	741	776	813	851	891	936	977	1023	1071	51	4.8	3.8	7

§ 22. RESOLUTION OF RIGHT-ANGLE SPHEROIDAL TRIANGLES (RELATION BETWEEN POLAR GEODETIC AND SPHEROIDAL COORDINATES)

The position of point on a surface of a spheroid can be determined either by polar geodetic coordinates (s, A), or spheroidal (p, q). Consequently, any point of a spheroid, determined by given values of polar coordinates, will correspond determined values of spheroidal coordinates, i.e.,

$$s = s(p, q),$$
$$\alpha = \alpha(p, q).$$

[illegible]:

$$\left.\begin{aligned} ds &= \frac{\partial s}{\partial p}dp + \frac{\partial s}{\partial q}dq \\ d\alpha &= \frac{\partial \alpha}{\partial p}dp + \frac{\partial \alpha}{\partial q}dq \end{aligned}\right| \qquad (4.17)$$

Let us determine the geometric value of partial derivatives, entered in (4.17).

Let us assume that point P (Fig. 61) is given in polar coordinates (s, α) and spheroidal (p, q). The origin of coordinates for systems (s, α) and (p and q) is at point O, the axis of abscissas in a system (p, q) coincides with the meridian and coordinates p and q increase to north and east.

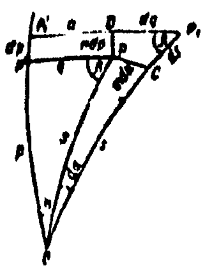


Fig. 61.

Let us consider the other point, determined by the coordinates:

$$(p + dp,\ q + dq) \text{ and } (s + ds,\ \alpha + d\alpha).$$

Let us designate an angle where vertex P in a triangle $OP'P$ by β. In elementary quadrangle $CPDP_1$ the angle where vertex P is equal to $(180^\circ - \beta)$, angles where vertexes D and C are polar and angle at P_1 is equal to β.

Let us take projection of broken line P_1DPC at CP_1 and projection CP_1DP of PC. We have:

$$\left.\begin{aligned} ds &= dq\cos\beta + ndp\sin\beta \\ md\alpha &= dq\sin\beta - ndp\cos\beta \end{aligned}\right\}. \qquad (4.18)$$

Multiplying the second equation from the systems (4.17) by an m and equating right and left parts (4.17) and (4.18), we obtain:

$$\left.\begin{aligned} &1.\ \frac{\partial s}{\partial p} = n\sin\beta, && 3.\ \frac{\partial \alpha}{\partial p} = -\frac{n}{m}\cos\beta, \\ &2.\ \frac{\partial s}{\partial q} = \cos\beta, && 4.\ \frac{\partial \alpha}{\partial q} = \frac{1}{m}\sin\beta. \end{aligned}\right|. \qquad (4.19)$$

If origin of coordinates is taken at point P (Fig. 62) and allowing that point O shifted to O_1, then angles β and α and spheroidal coordinates p and q will exchange roles. Therefore analogously with expressions (4.19) we obtain:

$$\left.\begin{aligned} \frac{\partial s}{\partial p} &= \cos\alpha \\ \frac{\partial s}{\partial q} &= n\sin\alpha \end{aligned}\right|. \qquad (4.20)$$

From (4.20) by means of identity transformation:

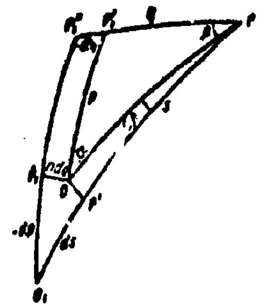


Fig. 62.

$$\frac{\partial s^2}{\partial p} = 2s\frac{\partial s}{\partial p}; \qquad \frac{\partial s^2}{\partial q} = 2s\frac{\partial s}{\partial q}$$

Consequently;

$$\left.\begin{array}{ll} 1.\ \frac{1}{n}\frac{\partial s^2}{\partial p} = 2s\sin\beta, & 3.\ \frac{\partial s^2}{\partial p} = 2s\cos\alpha \\ 2.\ \frac{\partial s^2}{\partial q} = 2s\cos\beta, & 4.\ \frac{1}{n}\frac{\partial s^2}{\partial q} = 2s\sin\alpha \end{array}\right\}. \qquad (4.21)$$

or:

$$\left(\frac{1}{n}\frac{\partial s^2}{\partial p}\right)^2 + \left(\frac{\partial s^2}{\partial q}\right)^2 = 4s^2. \qquad (4.22)$$

$$\left.\begin{array}{l} 1.\ s\cos\alpha = \frac{1}{2}\frac{\partial s^2}{\partial p}, \\ 2.\ s\sin\alpha = \frac{1}{2n}\frac{\partial s^2}{\partial q}, \\ 3.\ s\sin\beta = \frac{1}{2n}\frac{\partial s^2}{\partial p}, \\ 4.\ s\cos\beta = \frac{1}{2}\frac{\partial s^2}{\partial q} \end{array}\right\}. \qquad (4.23)$$

Differential equations (4.22) and (4.23) together provide solution of right-angle spheroidal triangle $OP'P$ (Fig. 61). They were first obtained by analytical method by Gauss in his "General Investigations of Curves of Surfaces". General integration of these equations is a very difficult problem, but for geodetic purposes this integration can be made by means of factorization into series by a method of indefinite coefficients. For integration we note important properties of a function:

$$s = s(p, q).$$

1. Where $p = 0$ we will have $s^2 = q^2$ and where $q = 0$ is correspondingly $s^2 = p^2$, consequently, in a series, presenting s^2 by p and q, with the exception of p^2 and q^2, there can be no terms, depending either only on p or only on q.

2. Since series for s^2 start with $p^2 + q^2$, they cannot contain terms in the form of kpq^i or jp^iq ($i = 1, 2, 3 \ldots$).

From these two properties of the function $s(p, q)$ it follows that series for s^2 should be symmetric with respect to p and q and can contain the following combinations of various degrees of p and q:

$$\left.\begin{array}{l} 1.\ p^2, q^2 \\ 2.\ p^2q^2 \\ 3.\ p^4q^2, p^2q^4 \\ 4.\ p^6q^2, p^4q^4, p^2q^6 \text{ etc.} \end{array}\right\}. \qquad (4.24)$$

Considering expression (4.24) and introducing indefinite coefficient

a_i ($i = 1, 2, 3, \ldots$), we have:

$$s^2 = p^2 + q^2 + a_1 p^2 q^2 + a_2 p^3 q^2 + a_3 p^2 q^3 + a_4 p^4 q^2 + a_5 p^2 q^4 + a_6 p^3 q^3 + \ldots \qquad (4.25)$$

Series (4.25) constitute solution of differential equation (4.22) in implicit form. In order to obtain it in explicit form, it is necessary to determine the value of indefinite coefficients.

From (4.25):

$$\frac{\partial s^2}{\partial p} = 2p + 2a_1 p q^2 + 3a_2 p^2 q^2 + 2a_3 p q^3 + 4a_4 p^3 q^2 + 2a_5 p q^4 + 3a_6 p^2 q^3. \qquad (I)$$

$$\frac{\partial s^2}{\partial q} = 2q + 2a_1 p^2 q + 2c_2 p^3 q + 3a_3 p^2 q^2 + 2a_4 p^4 q + 4a_5 p^2 q^3 + 3a_6 p^3 q^2. \qquad (II)$$

From (3.62)

$$\frac{1}{n^2} = 1 - 2f^0 q^2 - 2f' p q^2 - 2g^0 q^3 + \frac{8}{3} f^{02} q^4. \qquad (III)$$

Raise expression (I) and (II) to a square, multiply the square of (I) by (III) and add them, we then have:

$$\begin{aligned} s^2 = \frac{1}{4}\left\{\left(\frac{1}{n}\frac{\partial s^2}{\partial p}\right)^2 + \left(\frac{\partial s^2}{\partial q}\right)^2\right\} &= p^2 + q^2 + p^2 q^2 (4a_1 - 2f^0) + \\ &+ p^3 q^2 (5a_2 - 2f') + p^2 q^3 (5a_3 - 2g^0) + p^4 q^2 (6a_4 + a_1^2) + \\ &+ p^2 q^4 \left(6a_5 + a_1^2 - 4a_1 f^0 + \frac{8}{3} f^{02}\right) + 6a_6 p^3 q^3. \end{aligned} \qquad (4.26)$$

Comparing (4.25) with (4.26), for determination of indefinite coefficients a_1, a_2, a_3, a_4, a_5 and a_6, we obtain the following system of equalities:

1. $a_1 = 4a_1 - 2f^0$	or	$a_1 = \frac{2}{3} f^0$
2. $a_2 = 5a_2 - 2f'$		$a_2 = \frac{1}{2} f'$
3. $a_3 = 5a_3 - 2g^0$		$a_3 = \frac{1}{2} g^0$
4. $a_4 = 6a_4 + a_1^2$		$a_4 = -\frac{4}{45} f^{02}$
5. $a_5 = 6a_5 + a_1^2 - 4a_1 f^0 + \frac{8}{3} f^{02}$		$a_5 = -\frac{4}{45} f^{02}$
6. $a_6 = 6a_6$		$a_6 = 0$

With these values of coefficients a_i series (4.25) will take the form:

$$\begin{aligned} s^2 = p^2 + q^2 + \frac{2}{3} f^0 p^2 q^2 + \frac{1}{2} f' p^3 q^2 + \frac{1}{2} g^0 p^2 q^3 - \frac{4}{45} f^{02} p^4 q^2 - \\ - \frac{4}{45} f^{02} p^2 q^4 + l_8. \end{aligned} \qquad (4.27)$$

i.e., basic relationship between right-angle spheroidal coordinates (p, q) and polar geodetic coordinates (s, α) is obtained.

We have:

$$1.\ \frac{\partial s^2}{\partial p} = 2p + \frac{4}{3} f^0 p q^2 + \frac{3}{2} f' p^2 q^2 + g^0 p q^3 - \frac{16}{45} f^{02} p^3 q^2 - \frac{8}{45} f^{02} p q^4 + l_7. \qquad (V)$$

$$2\frac{\partial s^2}{\partial q} = 2q + \frac{4}{3} f^0 p^2 q + f' p^3 q + \frac{3}{2} g^0 p^2 q^2 - \frac{8}{45} f_0^2 p^4 q - \frac{16}{45} f^{0\,2} p^2 q^3 + l_7. \qquad \text{(VI)}$$

Further from (3.62)

$$\frac{1}{2m} = \frac{1}{2} - \frac{1}{2} f^0 q^2 - \frac{1}{2} f' p q^2 - \frac{1}{2} g^0 q^3 + \frac{5}{12} f^{0\,2} q^4; \qquad \text{(VII)}$$

Substituting expressions (V), (VI) and (VII) in (4.23) and retaining terms to seventh order with respect to p and q, will obtain:

$$\left.\begin{aligned} s\sin\beta &= p - \frac{1}{3} f^0 p q^2 - \frac{1}{4} f' p^2 q^2 - \frac{1}{2} g^0 p q^3 - \frac{8}{45} f^{0\,2} p^3 q^2 + \\ &\quad + \frac{7}{90} f^{0\,2} p q^4 + l_7 \\ s\cos\beta &= q + \frac{2}{3} f^0 p^2 q + \frac{1}{2} f' p^3 q + \frac{3}{4} g^0 p^2 q^2 - \frac{4}{45} f^{0\,2} p^4 q - \\ &\quad - \frac{8}{45} f^{0\,2} p^2 q^3 + l_7 \end{aligned}\right\}; \qquad (4.28)$$

$$\left.\begin{aligned} s\sin\alpha &= q - \frac{1}{3} f^0 p^2 q - \frac{1}{6} f' p^3 q - \frac{1}{4} g^0 p^2 q^2 + \frac{7}{90} f^{0\,2} p^4 q - \\ &\quad - \frac{8}{45} f^{0\,2} p^2 q^3 + l_7 \\ s\cos\alpha &= p + \frac{3}{2} f^0 p q^2 + \frac{3}{4} f' p^2 q^2 + \frac{1}{2} g^0 p q^3 - \frac{8}{45} f^{0\,2} p^3 q^2 - \\ &\quad - \frac{4}{45} f^{0\,2} p q^4 + l_7 \end{aligned}\right\}. \qquad (4.29)$$

Thus are obtained series (4.27), (4.28) and (4.29) in conjunction they give solution of right-angle spheroidal triangle $OP'P$ and simultaneously present resolution of differential equations (4.22) and (4.23). If p and q are given then series (4.27), (4.28), (4.29) determine s, β and α. By means of conversion of these series a formula can be obtained for calculation of p and q, if s and α, or s and β are given. But before inversion of series, let us find the value of coefficients f^0, f' and g^0.

From (3.61)

$$K = -2f^0 - 2f'p - 6g^0 q, \qquad (4.30)$$

where K is Gauss curvature.

Let us consider this equation for vertexes of right-angle triangle $OP'P$ (see Fig. 61).

For points:

$$\left.\begin{array}{llll} P \text{ we have } & p = 0 & q = 0 & K_0 = -2f^0 \\ P' \gg & p = p & q = 0 & K_{p0} = -2f^0 - 2f'p \\ P \gg & p = p & q = q & K_p = -2f^0 - 2f'p - 6g^0 q \end{array}\right\}. \qquad (4.31)$$

Thus, if Gauss curvature of vertexes of right-angle triangle $OP'P$ (Fig. 63), from (4.31) is given coefficients f^0, f', g^0 are determined and conversely, if

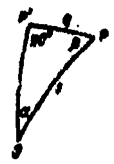


Fig. 63.

coefficients are given, then curvature is determined.

From (4.31) it follows that:

$$K_0 + 2K_{90} + K_\rho = -(8f^0 + 6f'p + 6g^0 q). \tag{4.32}$$

With the help of expression (4.32) series (4.27), (4.28) and (4.29) can be given form:

$$\left.\begin{aligned}
&1.\ s^2 = p^2 + q^2 - \frac{K_0 + 2K_{90} + K_\rho}{12} p^2q^2 - \frac{K_0^2}{45}(p^4q^2 + p^2q^4) \\
&2.\ s\sin\beta = p + \frac{K_0 + K_{90} + 2K_\rho}{24} pq^2 - \frac{K_0^2}{360}(16p^3q^2 - 7pq^4) \\
&3.\ s\cos\beta = q - \frac{2K_0 + 3K_{90} + 3K_\rho}{24} p^2q - \frac{K_0^2}{45}(p^4q + 2p^2q^3) \\
&4.\ s\sin\alpha = q + \frac{K_0 + K_{90} + 2K_\rho}{24} p^2q - \frac{K_0^2}{360}(16p^2q^3 - 7p^4q) \\
&5.\ s\cos\alpha = p - \frac{2K_0 + 3K_{90} + 3K_\rho}{24} pq^2 - \frac{K_0^2}{45}(pq^4 + 2p^3q^2).
\end{aligned}\right| \tag{4.33}$$

All these expressions are mutually controlled, since:

$$(s\sin\beta)^2 + (s\cos\beta)^2 = s^2 \quad \text{or} \quad (s\sin\alpha)^2 + (s\cos\alpha)^2 = s^2$$

From (4.33) by means of conversion of series following expressions for spheroidal coordinates p and q are obtained:

$$\left.\begin{aligned}
&1.\ p = s\sin\beta - s^3\sin\beta\cos^2\beta\,\frac{K_0 + K_{90} + 2K_\rho}{24} + \frac{K_0^2}{120}(pq^4 - 8p^3q^2) \\
&2.\ q = s\cos\beta + s^3\sin^2\beta\cos\beta\,\frac{2K_0 + 3K_{90} + 3K_\rho}{24} + \frac{K_0^2}{15}(2p^4q - p^2q^3) \\
&3.\ p = s\cos\alpha + s^3\sin^2\alpha\cos\alpha\,\frac{3K_0 + 3K_{90} + 2K_\rho}{24} + \frac{K_0^2}{15}(2pq^4 - p^3q^2) \\
&4.\ q = s\sin\alpha - s^3\sin\alpha\cos^2\alpha\,\frac{2K_0 + K_{90} + K_\rho}{24} + \frac{K_0^2}{120}(p^4q - 8p^2q^3)
\end{aligned}\right| \tag{4.34}$$

Formulas (4.34) have great application, since from the geodetic measurements polar coordinates, distance and azimuth are obtained, and in resolution of geodetic problems right-angle spheroidal coordinates are utilized.

For complete solution of right-angle spheroidal triangle OP'P it is still necessary to obtain a formula for its spheroidal excess.

We have:

$$\varepsilon = (\beta + \alpha + 90) - 180^\circ = -[90^\circ - (\beta + \alpha)],$$
$$\sin\varepsilon = \sin\alpha\sin\beta - \cos\alpha\cos\beta$$

or:

$$s^2\sin\varepsilon = s\sin\alpha\, s\sin\beta - s\cos\alpha\, s\cos\beta.$$

With the aid of formulas (4.31), omitting details of conversions we obtain:

$$s^2 \sin\varepsilon = \frac{K_0+K_{90}+K_\varphi}{6} pq(p^2+q^2) + \frac{15}{360}K_0^2 pq(p^2-q^2)^2.$$

Converting from sine of acute angle to an angle in radians, and substituting s^2 by spheroidal coordinates, with accuracy up to small values of sixth order we obtain:

$$\varepsilon = \frac{K_0+K_{90}+K_\varphi}{6}pq + \frac{K_0^2}{24}pq(p^2+q^2) + l_6. \qquad (4.35)$$

by formula (4.35) ε is obtained in a radian measure.

§ 23. RESOLUTION OF GENERAL SPHEROIDAL TRIANGLE

Let us call an arbitrary shaped triangle general. Then let us consider resolution of a general spheroidal triangle, obtained from right-angle triangle by following the construction.

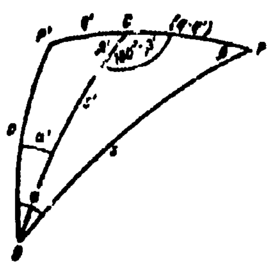


Fig. 64.

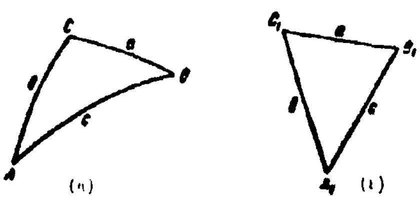


Fig. 65.

On geodesic $P'P$ (Fig. 64) take arbitrary point C with ordinate q' and join it with the origin of coordinate O by geodesic $s' = OC$. Triangle OCP, formed by geodesics OC, OP and CP, is a general spheroidal triangle. For convenience of study let us introduce new designations. Assume that the vertexes of the triangle are designated by A, C and B, and the opposite sides by a, c and b (Fig. 65a).

Elements of the new triangle ACB in former designations will be:

$$A = \alpha - \alpha', \qquad B = \beta, \qquad C = 180° - \beta',$$
$$a = q - q', \qquad b = s', \qquad c = s.$$

Substituting in formulas (4.33) the values, pertaining to vertex P, which pertain to vertex C, we obtain:

$$\left.\begin{aligned} s'^2 &= p^2 + q'^2 - \frac{K_0+2K_{90}+K_\varphi}{12}p^2q'^2 - \frac{K_0^2}{45}(p^4q'^2+p^2q'^4) \\ s'\cos\alpha' &= b\cos\alpha' = p - \frac{2K_0+3K_{90}+3K_\varphi}{24}pq'^2 - \frac{K_0^2}{45}(pq'^4+2p^3q'^2) \\ s'\sin\alpha' &= b\sin\alpha' = q' + \frac{K_0+K_{90}+2K_\varphi}{24}p^2q' - \frac{K_0^2}{360}(16p^2q'^3-7p^4q') \end{aligned}\right\}. \qquad (4.36)$$

Spheroidal excess of triangle ABC is equal to:

$$\varepsilon_1 = \varepsilon - \varepsilon', \qquad (4.37)$$

where ε — spheroidal excess of triangle $OP'P$,

ε' – spheroidal excess of triangle $OP'C$.

From (4.35) it follows that:

$$\varepsilon_1 = \frac{K_0 + K_{90} + K_b}{6} pq + \frac{K_0^2}{24} pq(p^2 + q^2) - \frac{K_0 + K_{90} + K_c}{6} pq' - \frac{K_0^2}{24} pq'(p^2 + q'^2). \tag{4.38}$$

Applying formula (4.31) to points P', C and P (Fig. 65), lying on a line of ordinates, where $p = 0$.

for P' we have $K_{90} = -2f$,

for C >> $K_c = -2f - 2gq'$,

for B >> $K_b = -2f - 2gq$,

hence:

$$\frac{K_b - K_c}{K_c - K_{90}} = \frac{q - q'}{q'};$$

or:

$$K_{90}(q - q') = qK_c - q'K_b. \tag{4.39}$$

Dropping in (4.38) terms of fourth order of smallness and substituting K_{90} by formula (4.39), we obtain:

$$\varepsilon_1 = \frac{p(q - q')}{2}\left(\frac{K_a + K_c + K_b}{3}\right) + l_4. \tag{4.40}$$

Here K_a, K_c, K_b are Gauss curvature of vertexes of the triangle ACB.

Value $\frac{p(q - q')}{2}$ are of a triangle $A_1B_1C_1$ (Fig. 65b); designating it by Δ, we have:

$$\varepsilon_1 = \Delta\left(\frac{K_a + K_c + K_b}{3}\right) + l_4. \tag{4.41}$$

Let us consider plane triangle $A_1B_1C_1$ with sides of a spheroidal triangle ABC (Fig. 65b). We find the difference $(A - A_1)$, $(B - B_1)$ and $(C - C_1)$.

For plane triangle $A_1B_1C_1$:

$$2bc\cos A_1 = b^2 + c^2 - a^2$$

or, substituting values b^2, c^2 and a^2, expressed by the spheroidal coordinates by formulas (4.33) and (4.36), we obtain:

$$2bc\cos A_1 = 2p^2 - 2qq' - \frac{p^2K_a(q^2 + q'^2)}{12} - \frac{p^2K_{90}(q^2 + q'^2)}{6} - \frac{p^2K_bq^2}{12} - \frac{p^2K_cq'^2}{12}. \tag{I}$$

Further, in accordance with Fig. 60,

$$\cos A = \cos(\alpha - \alpha') = \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha',$$

or, multiplying this equation by 2cb according to (4.33) and (4.36), we obtain:

$$2bc\cos A = 2p^2 - 2qq' - \frac{p^2K_a}{12}(3q^2 + 3q'^2 + 4qq') - \frac{p^2K_{90}}{12}(3q^2 + 3q'^2 + 2qq') - \frac{p^2K_c}{12}(2q^2 + qq') - \frac{p^2K_b}{12}(2q'^2 + qq'). \tag{II}$$

Difference (II) and (I) gives:

$$2bc(\cos A - \cos A_1) = -\frac{p^2(q-q')}{12}\{2K_a(q-q') + K_{90}(q-q') + K_cq + K_bq'\}.$$

Substituting $K_{90}(q - q') = qK_c - qK_b$ and considering that $\frac{p(q - q')}{2} = \Delta$, we have:

$$2bc(\cos A - \cos A_1) = -\frac{\Delta^2}{3}(2K_a + K_b + K_c).$$

But:

$$\cos A - \cos A_1 = -2\sin\frac{A - A_1}{2}\sin\frac{A + A_1}{2};$$

$(A - A_1)$ is a small value of the second order, therefore with an accuracy of up to small values of sixth order it is possible to accept:

$$\cos A - \cos A_1 = -(A - A_1)\sin A_1.$$

Consequently,

$$2bc(\cos A - \cos A_1) = -2(A - A_1)bc\sin A_1,$$

but:

$$2bc\sin A_1 = 4\Delta.$$

Therefore.

$$A - A_1 = \frac{\Delta}{12}(2K_a + K_b + K_c) + l_4. \tag{4.42}$$

or:

$$A - A_1 = \frac{\Delta}{12}(K_a + K_b + K_c) + \frac{\Delta}{12}K_a. \tag{4.42'}$$

First term of this expression is symmetric with respect to vertexes of a triangle and, consequently, should be general for $(B - B_1)$ and $(C - C_1)$; the second term pertains only to that difference, for which the formula was derived.

Therefore:

$$\left.\begin{aligned} A - A_1 &= \frac{\Delta}{12}(K_a + K_b + K_c) + \frac{\Delta}{12}K_a = \frac{\Delta}{12}(2K_a + K_b + K_c) + l_4 \\ B - B_1 &= \frac{\Delta}{12}(K_a + K_b + K_c) + \frac{\Delta}{12}K_b = \frac{\Delta}{12}(K_a + 2K_b + K_c) + l_4 \\ C - C_1 &= \frac{\Delta}{12}(K_a + K_b + K_c) + \frac{\Delta}{12}K_c = \frac{\Delta}{12}(K_a + K_b + 2K_c) + l_4 \end{aligned}\right\}. \tag{4.43}$$

sum $$A + B + C - (A_1 + B_1 + C_1) = \varepsilon_1 = \frac{\Delta}{3}(K_a + K_b + K_c) + l_4. \tag{4.44}$$

Consequently, formula (4.41) is again obtained.

Formulas (4.43) show, how the solution of spheroidal triangle can lead to solution of a plane triangle, having the same sides a, b, c. From (4.43) it follows that into spheroidal angles A, B and C must be introduced unequal reductions, so that their sines will become proportional to opposite sides. For terrestrial ellipsoid of differences of these reductions are the small values of fourth order, therefore for normal sides in triangulation these differences can be disregarded.

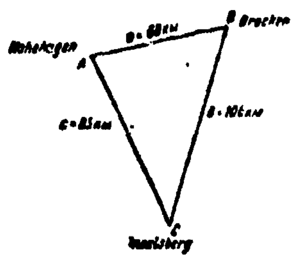


Fig. 66.

For the largest triangle of Honnover triangulation by Gauss the Hohehagen, Brocken and Inselsberg (Fig. 66), where the largest side is equal to 106 km, and spheroidal excess is nearly 4."9, the difference in reductions, according to data shown below is less than 0."0003.

Vertexes	Differences
Hohehagen . .	$A - A_1 = 4''{,}95113$,
Brocken	$B - B_1 = 4\ {,}95104$,
Inselsberg . . .	$C - C_1 = 4\ {,}95131$.

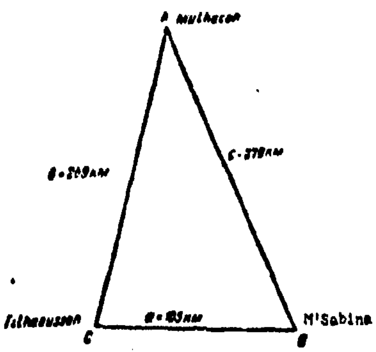


Fig. 67.

For large Algerian triangle Mulhacen – M'Sabina – Filhaoussen (Fig. 67) with the longest side Mulhacen – M'Sabina, was equal to 270 km, the difference in reductions are less than 0."001, for instance:

Mulhacen	$A - A_1 = 23''{,}5866$,
M'Sabina	$B - B_1 = 23\ {,}5866$,
Filhaoussen	$C - C_1 = 23\ {,}5875$.

Formula (4.43) and reduced numerical characteristics of differences of spheroidal reductions again confirm the basic deduction that spheroidal triangles, whose sides do not exceed 200-250 km, can be solved as spherical.

For surface of a sphere $K_a = K_b = K_c = K = \frac{1}{R^2}$. R -- radius of a sphere; from (4.43) it follows that:

$$\left.\begin{aligned} A - A_1 &= \frac{\Delta}{3R^2} \\ B - B_1 &= \frac{\Delta}{3R^2} \\ C - C_1 &= \frac{\Delta}{3R^2} \end{aligned}\right\}, \qquad (4.44')$$

i.e., we arrived at the Legendre theorem.

In the contemporary geodetic practice by radar methods (Shoran, Hiran) geodetic nets are laid out with sides of 400-500 km. It is true, the accuracy in such nets is now such that the geometric figures formed within them can be accepted as spherical with proper selection of a radius. However in time the accuracy of measurements in these nets will, probably, become so high that a necessity will arise for calculation of spheroidity of geometric figures. Besides, in adjustment of astronomic-geodetic nets figures are formed with large sides and in composition of condi'·onal equations of azimuths and coordinates a necessity arises for calculation of spheroidal corrections.

Certain geodesists here and abroad propose to form from dense nets with comparatively small sides nets of large triangles with sides 250-300 km. In resolution of such triangles spheroidal corrections should also be considered.

Formulas (4.43) are derived with an accuracy up to small values of the second order. In solution of large triangles, mentioned above, in formulas, terms of fourth order should be considered. This can easily be done, since for obtaining terms of the highest order, spheroidal triangles can be considered spherical with a very high degree of accuracy. Then to formulas (4.43) should be added the spherical terms of fourth order of Legendre theorem from (4.6), i.e.:

$$\frac{\varepsilon''}{60R_m^2}(m^2-a^2),\ \frac{\varepsilon''}{60R_m^2}(m^2-b^2)\ \text{и}\ \frac{\varepsilon''}{60R_m^2}(m^2-c^2),$$ where

$$\frac{1}{R_m^2}=\frac{1}{3}\left(\frac{1}{R_a^2}+\frac{1}{R_b^2}+\frac{1}{R_c^2}\right).$$

Then we have:

$$\left.\begin{aligned}(A-A_1)'' &= \frac{\Delta\rho''}{12}(2K_a+K_b+K_c)+\frac{\varepsilon''}{60R_m^2}(m^2-a^2)\\ (B-B_1)'' &= \frac{\Delta\rho''}{12}(K_a+2K_b+K_c)+\frac{\varepsilon''}{60R_m^2}(m^2-b^2),\\ (C-C_1)'' &= \frac{\Delta\rho''}{12}(K_a+K_b+2K_c)+\frac{\varepsilon''}{60R_m^2}(m^2-c^2)\end{aligned}\right\}. \tag{4.45}$$

Sum: $\varepsilon=\frac{\Delta}{3}\rho''(K_a+K_b+K_c).$

Let us design $K_m=\frac{K_a+K_b+K_c}{3}$, (4.46)

then:

$$\varepsilon=\Delta K_m\rho''$$

or:

$$\Delta=\frac{\varepsilon}{K_m\rho''}. \tag{4.47}$$

Substituting (4.46) and (4.47) in (4.45), we obtain:

$$
\left.\begin{aligned}
(A-A_1)'' &= \frac{\varepsilon''}{3} + \frac{\varepsilon''}{12}\frac{(K_a-K_m)}{K_m} + \frac{\varepsilon''}{60}K_m(m^2-a^2) \\
(B-B_1)'' &= \frac{\varepsilon''}{3} + \frac{\varepsilon''}{12}\left(\frac{K_b-K_m}{K_m}\right) + \frac{\varepsilon''}{60}K_m(m^2-b^2) \\
(C-C_1)'' &= \frac{\varepsilon''}{3} + \frac{\varepsilon''}{12}\left(\frac{K_c-K_m}{K_m}\right) + \frac{\varepsilon''}{60}K_m(m^2-c^2)
\end{aligned}\right\}. \qquad (4.48)
$$

In (4.48), as earlier, K_a, K_b and K_c are Gauss curvatures of vertexes of a spheroidal triangle.

Formulas (4.48) are applicable to any spheroidal triangles, whose elements can be directly measured by geodetic methods on the Earth's surface. Erroneousness of these formulas for triangles with sides 400-500 km is less than 0."001.

Examples of the solution of spherical triangles by Legendre theorem, and by additament method, also solution of large spheroidal triangles are given in § 71 and 72 (p. 270-276) of Practicum on the Higher Geodesy.

CHAPTER V

CALCULATION OF GEODETIC COORDINATES

§ 24. GENERAL CONSIDERATIONS AND DETERMINATIONS

In the contemporary practice of geodesic work in the USSR geodetic coordinates of vertexes of triangles are calculated only in 1st order triangulation. Triangulation of other orders, polygonometry and special geodetic nets are calculated on Gauss-Kruger projection by obtaining grid conformal coordinates of points. Calculation of coordinates of geodetic points is the last stage of treatment of results of geodetic measurements and is carried out with particular thoroughness and mathematical strictness. Later these coordinates are published in special lists and for very prolonged periods serve as a basis for scientific investigations, state cartographic work, research and for engineering construction, also for long distance ways of communication, exploration of the Earth's bowels etc.

Like any engineering construction, geodetic construction must possess great "reserve of durability". This position presents definite requirements for the treatment of results of geodetic measurements and; during calculation of geodetic coordinates a condition is set so that the error of state calculations is to be 10 times less than the errors of nonstate field measurements.

In adjustment of 1st order triangulation relative error of a side is equal approximately to $\frac{\Delta s}{s} = \frac{1}{300,000}$. Maximum length of side of 1st order triangulation by contemporary scheme of construction should not be larger than 25-30 km. Hence if $s = 25$ km then, $\Delta s = 0.083$ m. Thus, position of a third vertex of a triangle in triangulation relative to any two other vertexes is determined by results of field measurements with mean error not greater than 8-0 centimeters.

In order to determine the influence of this error on the differences of latitudes and longitudes, let us use formula (3.40c).

We have:

$$\left.\begin{aligned}\Delta b'' &= (1)\,\Delta s \cos A\\ \Delta l'' &= (2)\,\Delta s \sin A \sec B\end{aligned}\right\}. \tag{5.1}$$

Here Δb, Δl are errors in differences of latitudes and longitudes. Taking $\Delta s =$ $= 0.03$ m, $(1) = (2) = \frac{1}{30}$, we obtain:

$$\Delta b'' = 0'',003 \cos A,$$
$$\Delta l'' = 0'',003 \sin A \sec B.$$

With extreme azimuth:

$$\Delta b'' = 0'',003,$$
$$\Delta l'' = 0'',003 \sec B.$$

Consequently, error of calculations of coordinates have to be 10 times less than 0".003. Coordinates are calculated by means of consecutive algebraic summation of their differences which leads to accumulation of additional errors of rounding; considering, this, the differences of latitudes and longitudes of points of 1st order triangulation are calculated with an accuracy of up to 0".0001; in this case the error in difference of latitudes of two points, 200-300 km, one from another will be no larger than 0".003.

Direction azimuths in triangulation are obtained by angles, derived from link adjustments to a thousandth fraction of a second; consequently, geodetic azimuths have to be calculated with accuracy of up to 0".001.

Where the length of a side in triangulation is 30 km and extreme azimuth for differences of latitudes and longitudes is by formula (5.1), we have:

$$\Delta b'' = 1000'',$$
$$\Delta l'' = 1000'' \sec B.$$

In order to obtain these values with accuracy of up to 0".0001, obviously, it is necessary in calculations to retain the eighth decimal place. In logarithmic calculations for determination of accuracy it is essential to use known relationships:

$$d(\lg N) = \mu \frac{\Delta N}{N}$$

(μ = 0.4343 modulus of common logarithms). In our case ΔN = 0.0001, N = 1000.

Consequently,

$$d(\lg \Delta b) = \frac{0.43 \cdot 0'',0001}{1000} \approx 4{,}3 \cdot 10^{-8}.$$

In extreme case this value decreases three times, i.e., the differences of latitudes and longitudes should be calculated with the aid of eight place logarithm tables.

Formulas for calculation of differences of latitudes, longitudes and azimuths are obtained in the form of a series by ascending degrees of $\frac{s}{R}$, which are considered small values of first order. The remaining values in determination of their degree of smallness are compared with $\frac{s}{R}$. Differences of latitudes, longitudes and azimuths at maximum value, are equal to $\frac{s}{R}$, expressed in arc measure, therefore b, l and t are also small values of a first order. Where s = 30 km, $\frac{s}{R} = \frac{1}{200}$. But $e^2 - \frac{1}{150}$. The values e^2 and $\frac{s}{R}$ are small and of a 1st order. Calculations with eight place tables ensure a 1×10^{-8} fraction of a given value. If $\frac{s}{R} \approx \frac{1}{200}$, then for guarantee of indicated accuracy it is necessary to retain $\left(\frac{s}{R}\right)^4 = \frac{1}{16 \cdot 10^8}$, i.e., in derivation of formulas for calculation of differences of latitudes, longitudes and azimuths it is necessary, as a rule, to retain small values of fourth order within them.

A direct geodetic problem is where geodetic coordinates of first point, and the distance, and azimuth of direction from first point to second are given, to calculate coordinates of the second point and the back azimuth. This problem can be solved by different ways. It is possible to set as target to determine the unknown values directly, i.e., by the direct method, applied for long distances between points. For short distances, such as the sides of 1st order triangulation it is more expedient to calculate at first the differences of latitudes, longitudes and azimuths of the table determined and initial points, and then by simple summation to obtain the unknown values, i.e., latitude, longitude and azimuth for the second point. Such approach is called the indirect method. Advantage of indirect method for short distances is that the difference of geodetic coordinates in calculations with eight place tables are obtained with the same accuracy, as with ten-place calculations of the direct method. Although classification of methods of solutions of the direct geodetic problem is conditional, nonetheless this terminology is conventional and is convenient for explanation of geometric approach to solution of the problem on hand. With indirect methods several methods of solution of direct geodetic problem are distinguished. The most important of them are:

a) factorization of differences of latitudes, longitudes and azimuths by ascend-powers of length of arc of geodetic s, where this factorization can be satisfied by initial (where s = 0) and mean arguments;

b) the method of auxiliary point, when from geodetic polar coordinates (s, A) a conversion is made to angle spheroidal coordinates (p, q), and from them to geodetic (B, L);

c) one of the possible ways of solution of the problem consists in that in conformity with a determined law certain part of the surface of the ellipsoid is depicted on a sphere, i.e., a transition is accomplished from spheroidal elements of the problem to corresponding spherical; the problem is resolved on a sphere, then converted from geodetic coordinates on a sphere to geodesic coordinates on a spheroid. Very frequently in solution of an indicated problem the sphere is used as an auxiliary surface for mathematical transformations, but the final formulas for calculations are obtained on a spheroid;

d) somewhat isolated is the method of solution of the problem with the help of chords of an ellipsoid. In this method formulas for finding unknown values are obtained in a closed form and can be applied to chords of any length.

Shown above are only the basic principles, on which methods of resolution of direct geodetic problem, are based, besides there exists a multitude of different approaches and methods of application of these principles. Inasmuch as resolution of this problem is one of the mass forms of geodetic work, then the requirement of simplicity and convenience of calculations is very essential.

Calculation of geodetic coordinates, essentially, is a comparatively simple geometric problem; however in this area are many mathematical investigations and scientific work. The geodesists and mathematicians of USSR and abroad are working on the problem of the more expedient resolution of geodetic problems. The main tendency of investigation in this area currently consists of composing formulas and tables, convenient for calculations with application of calculating machines, which are gradually winning permanent positions in all areas of computer technology. All formulas and tables till now were calculated for logarithmic calculations still widely used at present. Therefore, along with the new proposals for use of nonlogarithmic methods of calculations, further below will be presented logarithmic formulas for calculation of geodetic coordinates used at present.

§ 25. FACTORIZATION OF DIFFERENCES OF LATITUDES, LONGITUDES AND AZIMUTHS BY ASCENDING POWERS OF s

Let us assume that polar coordinates of point P_2 (s, A_1) are given. It is required, knowing geodetic coordinates of point P_1, to find difference of latitudes, longitudes and azimuths of points P_2 and P_1 (Fig. 68). Imagine that we move along

Fig. 68.

the arc s; to every point on this arc will correspond a definite geodetic coordinate and azimuth of direction of the movement. Such functional dependency can be expressed by parametric equations:

$$B = B(s),$$
$$L = L(s),$$
$$A = A(s).$$

Allowing that these functions have all the derivatives with respect to s, and designating them correspondingly by B^i, L^i and A^i, we have Maclaurin series for these functions.

$$\left.\begin{aligned} B &= B_1 + \sum_{i=0}^{i=n} \frac{s^i}{i!} B_1^i \\ L &= L_1 + \sum_{i=0}^{i=n} \frac{s^i}{i!} L_1^i \\ A &= A_1 + \sum_{i=0}^{i=n} \frac{s^i}{i!} A_1^i \end{aligned}\right\}. \qquad (5.2)$$

where $B_1^i = \left(\frac{d^iB}{ds^i}\right)_1$, $L_1^i = \left(\frac{d^iL}{ds^i}\right)_1$, $A_1^i = \left(\frac{d^iA}{ds^i}\right)_1 (i = 1, 2, 3 \ldots n)$, sign "1" means that this value or derivative is calculated where s = 0 or $B = B_1$ and $A = A_1$. First derivatives B', L' and A' are obtained in Chapter III and in accordance with (3.40a):

$$\left.\begin{aligned} B' &= \frac{V^3 \cdot \cos A}{c} \\ L' &= \frac{V \cdot \sin A}{c} \sec B \\ A' &= \frac{V \cdot \sin A}{c} \operatorname{tg} B \end{aligned}\right\} \qquad (5.2')$$

From (5.2') it follows that B_1', L_1', A_1' are explicit functions of latitudes and azimuth and implicit functions of s, therefore derivatives of the higher order are found by rules of differentiation of implicit function.

General recording for derivatives of higher order:

$$\left.\begin{aligned} B^i &= \frac{\partial}{\partial B}(B^{i-1})\frac{dB}{ds} + \frac{\partial}{\partial A}(B^{i-1})\frac{dA}{ds} \\ L^i &= \frac{\partial}{\partial B}(L^{i-1})\frac{dB}{ds} + \frac{\partial}{\partial A}(L^{i-1})\frac{dA}{ds} \\ A^i &= \frac{\partial}{\partial B}(A^{i-1})\frac{dB}{ds} + \frac{\partial}{\partial A}(A^{i-1})\frac{dA}{ds} \end{aligned}\right\}. \qquad (5.3)$$

Further we have:

$$V = \sqrt{1+\eta^2} = \sqrt{1+e'^2\cos^2 B},$$
$$\frac{dV}{dB} = -\frac{e'^2 \sin B \cos B}{V} = -\frac{\eta^2 t}{V}, \quad t = \operatorname{tg} B,$$
$$\frac{dV}{ds} = \frac{dV}{dB} \cdot \frac{dB}{ds} = -\frac{\eta^2 V^2 t}{c} \cos A.$$

From formulas (5.2')

$$B'' = \frac{3V^2 dV}{c \cdot ds}\cos A - \frac{V^2}{c}\sin A\frac{dA}{ds},$$

$$L'' = \frac{\sec B \sin A}{c}\cdot\frac{dV}{ds} + \frac{V}{c}\,t\sec B\cdot\sin A\,\frac{dB}{ds} + \frac{V}{c}\sec B\cos A\,\frac{dA}{ds},$$

$$A'' = \frac{\sin A}{c}\,t\,\frac{dV}{ds} + \frac{V}{c}(1+t^2)\sin A\,\frac{dB}{ds} + \frac{V}{c}\cos A\cdot t\,\frac{dA}{ds}.$$

Substituting values $\frac{dV}{ds}$, $\frac{dB}{ds}$, $\frac{dA}{ds}$ and satisfying the reductions we obtain:

$$\left.\begin{aligned} B'' &= -\frac{V^4}{c^2}(\sin^2 A\cdot t + 3\cos^2 A\,\eta_1^2 t)\\ L'' &= \frac{2}{c^2}V^2\sec B\sin A\cos A\cdot t\\ A'' &= \frac{V^2}{c^2}\cdot\sin A\cos A(1+2t^2+\eta_1^2)\end{aligned}\right\}. \qquad (5.4)$$

$$B''' = -\frac{4V^3}{c^2}(\sin^2 At + 3\cos^2 A\,\eta_1^2 t)\frac{dV}{ds} - \frac{V^4}{c^2}[\sin^2 A(1+t^2) +$$

$$+ 3\cos^2 A\,\eta_1^2(1-t^2)]\frac{dB}{ds} - \frac{V^4}{c^2}(2\sin A\cos At - 6\cos A\sin A\,\eta_1^2 t)\frac{dA}{ds},$$

$$L''' = \frac{4V}{c^2}\sec B\sin A\cos At\frac{dV}{ds} + \frac{2V^2}{c^2}[\sec B\sin A\cos A\times$$

$$\times(1+2t^2)]\frac{dB}{ds} + \frac{2V^2}{c^2}[\sec B(\cos^2 A - \sin^2 A)t]\frac{dA}{ds},$$

$$A''' = \frac{2V}{c^2}\sin A\cos A(1+t^2+\eta_1^2)\frac{dV}{ds} + \frac{V^2}{c^2}(\cos^2 A - \sin^2 A)\times$$

$$\times(1+2t^2+\eta_1^2)\frac{dA}{ds} + \frac{2V^2}{c^2}\sin A\cos At\,[2+2t^2-\eta_1^2]\frac{dB}{ds}.$$

Substituting values $\frac{dV}{ds}$, $\frac{dB}{ds}$ and $\frac{dA}{ds}$, we obtain:

$$\left.\begin{aligned} B''' &= -\frac{V^6}{c^3}\cos A\,\{\sin^2 A(1+3t^2+\eta_1^2-9\eta_1^2t^2) + \cos^2 A(3\eta_1^2 -\\ &\quad -3\eta_1^2t^2+3\eta_1^4-15\eta_1^4t^2)\}\\ L''' &= \frac{2V^4}{c^3}\sec B\,\{\sin A\cos^2 A(1+3t^2+\eta_1^2) - \sin^3 At^2\}\\ A''' &= \frac{V^4}{c^3}\{\sin A\cos^2 At(5+6t^2+\eta_1^2-4\eta_1^4) - \sin^3 At\times\\ &\quad \times(1+2t^2+\eta_1^2)\}\end{aligned}\right\}. \qquad (5.5)$$

Omitting details of calculations of derivatives of the higher degrees, we reduce them in final form to:

$$\left.\begin{aligned} B^{IV} &= \frac{V^6}{c^4}\,t\sin^4 A(1+3t^2+\eta_1^2-9\eta_1^2t^2) - \frac{2V^6t}{c^4}\sin^2 A\cos^2 A\times\\ &\quad\times(4+6t^2-13\eta_1^2-9\eta_1^2t^2-17\eta_1^4+45\eta_1^4t^2) + \frac{V^6}{c^4}\times\\ &\quad\times t\,\eta_1^2\cos^4 A(12+69\eta_1^2-45\eta_1^2t^2+57\eta_1^4-105\eta_1^4t^2)\\ L^{IV} &= \frac{8V^6}{c^4}\sin A\cos^3 A\sec Bt(2+3t^2+\eta_1^2-\eta_1^4) - \frac{8V^6}{c^4}\times\\ &\quad\times\sin^3 A\cos At(1+3t^2+\eta_1^2)\sec B\\ A^{IV} &= \frac{V^6}{c^4}\sin A\cos^3 A(5+28t^2+24t^4+6\eta_1^2+8\eta_1^2t^2-3\eta_1^4+\\ &\quad+4\eta_1^4t^2-4\eta_1^6+24\eta_1^6t^2) -\\ &\quad-\frac{V^6}{c^4}\sin^3 A\cos A(1+20t^2+24t^4+2\eta_1^2+8\eta_1^2t^2+\eta_1^4-12\eta_1^4t^2)\end{aligned}\right\}. \qquad (5.6)$$

As it was proven in preceding paragraph, to guarantee the required accuracy of calculations of differences of latitudes, longitudes and azimuths it is necessary to retain small values to fourth order inclusively in the formulas. However for references we will give fifth derivatives in spherical presentation, i.e., we will take them as $\eta = 0$.

We have:

$$\left.\begin{aligned}
B^{\mathrm{V}} &= \frac{V^5}{c^5}\sin^4 A\cos A(1 + 30t^2 + 45t^4) - \frac{2V^5}{c^5}\sin^2 A\cos^3 A(4 + \\
&\quad + 30t^2 + 30t^4) + l_5\eta^2 \\
L^{\mathrm{V}} &= \frac{8V^5}{c^5}\sin A\cos^4 A\sec B(2 + 15t^2 + 15t^4) - \frac{8V^5}{c^5}\times \\
&\quad \times \sin^3 A\cos^2 A\sec B(1 + 20t + 30t^4) + \frac{8V^5}{c^5}\sec B\sin^5 At^2(1 + \\
&\quad + 3t^2) + l_5\eta_i^2 \\
A^{\mathrm{V}} &= \frac{V^5}{c^5}\sin A\cos^4 At(61 + 180t^2 + 120t^4) - \frac{V^5}{c^5}\sin^3 A\cos^2 At\times \\
&\quad \times(58 + 280t^2 + 240t^4) + \frac{V^5}{c^5}\sin^5 At(1 + 20t^2 + 24t^4) + l_5\eta_i^2
\end{aligned}\right\}. \tag{5.7}$$

We designate:

$$\left.\begin{aligned}
s\sin A &= v \\
s\cos A &= u
\end{aligned}\right\}; \tag{5.8}$$

$$b_1 = \frac{V^2}{c}\rho''; \quad b_2 = -\frac{V^2 t}{2c^2}\rho''; \quad b_3 = -\frac{3}{2}\frac{V^4}{c^2}\eta_i^2 t\rho'';$$

$$b_4 = -\frac{V^4\rho''}{6c^3}(1 + 3t^2 + \eta_i^2 - 9\eta_i^2 t^2); \quad b_5 = -\frac{V^4}{2c^3}\eta_i^2(1 - t^2)\rho'';$$

$$b_6 = \frac{V^4}{24c^4}\rho'' t(1 + 3t^2 + \eta_i^2 - 9\eta_i^2 t^2);$$

$$b_7 = -\frac{V^4}{12c^4}t\rho''(4 + 6t^2 - 13\eta_i^2 - 9\eta_i^2 t^2);$$

$$b_8 = \frac{V^4}{2c^4}\eta_i^2 t\rho''.$$

$$b_9 = \frac{V^5}{120c^5}(1 + 30t^2 + 45t^4)\rho'';$$

$$b_{10} = -\frac{V^5}{30c^5}\rho''(2 + 15t^2 + 15t^4).$$

$$l_1 = \frac{V}{c\cos B}\rho''; \quad l_2 = \frac{V^3}{c^2\cos B}\rho'' t; \quad l_3 = -\frac{V^3 t^2\rho''}{3c^3\cos B};$$

$$l_4 = \frac{V^3\rho''}{3c^3\cos B}(1 + 3t^2 + \eta_i^2); \quad l_5 = -\frac{V^4 t\rho''}{3c^4\cos B}(1 + 3t^2 + \eta_i^2);$$

$$l_6 = \frac{V^4 t\rho''}{3c^4\cos B}(2 + 3t^2 + \eta_i^2); \quad l_7 = \frac{V^5 t^2\rho''}{15c^5\cos B}(1 + 3t^2);$$

$$l_8 = \frac{V^5\rho''}{15c^5\cos B}(2 + 15t^2 + 15t^4); \quad l_9 = -\frac{V^5\rho''}{15c^5\cos B}(1 + 20t^2 + 30t^4);$$

$$a_1 = \frac{Vt}{c}\rho''; \quad a_2 = \frac{V^2}{2c^2}\rho''(1 + 2t^2 + \eta_i^2); \quad a_3 = -\frac{V^3}{6c^3}t\rho''(1 + 2t^2 + \eta_i^2);$$

$$a_4 = \frac{V^3 t\rho''}{6c^3}(5 + 6t^2 + \eta_i^2 - 4\eta_i^4); \quad a_5 = -\frac{V^4\rho''}{24c^4}(1 + 20t^2 + 24t^4 +$$

$$+ 2\eta_i^2 + 8\eta_i^2 t^2);$$

$$a_6 = \frac{V^4\rho''}{24c^4}(5 + 28t^2 + 24t^4 + 6\eta_i^2 + 8\eta_i^2 t^2); \quad a_7 = \frac{V^5 t\rho''}{120c^5}(1 + 20t^2 + 24t^4);$$

$$a_8 = -\frac{V^5 t}{120c^5}(58 + 280t^2 + 240t^4); \quad a_9 = \frac{V^5 t}{120c^5}(61 + 180t^2 + 120t^4).$$

With these designations differences of latitudes, longitudes and azimuths will take the form:

$$
\left.
\begin{aligned}
&1.\ b = B_2 - B_1 = b_1 u + b_2 v^2 + b_3 u^2 + b_4 v^2 u + b_5 u^3 + b_6 v^4 + \\
&\qquad + b_7 v^2 u^2 + b_8 u^4 + b_9 v^4 u + b_{10} v^2 u^3 + l_6 \eta_1^2 \\
&2.\ l = L_2 - L_1 = l_1 v + l_2 vu + l_3 v^3 + l_4 vu^2 + l_5 v^3 u + l_6 vu^3 + \\
&\qquad + l_7 v^5 + l_8 vu^4 + l_9 v^3 u^2 + l_6 \eta_1^2 \\
&3.\ t = A_2' - A_1 = a_1 v + a_2 vu + a_3 v^3 + a_4 vu^2 + a_5 v^3 u + a_6 vu^3 + \\
&\qquad + a_7 v^5 + a_8 v^3 u^2 + a_9 vu^4 + l_6 \eta_1^2 \\
&\qquad\qquad A_2 = A_2' \pm 180^\circ.
\end{aligned}
\right\} \qquad (5.9)
$$

Formulas (5.9) are final. Coefficients $b_1, b_2, \ldots, b_{10}$; $l_1, l_2, \ldots, l_9$; $a_1, a_2, \ldots, a_9$ are the functions of a latitude of a given point and can be tabulated.

The author investigated these formulas in 1955-1957 both with respect to their accuracy, and with respect to the composition of tables for coefficients a_i, b_i and l_i. Results of investigations were published in an article "About Nonlogarithmic Calculations of Geodetic Coordinates of Points of First Order Triangulation in USSR."[1]

At present the tables of coefficients a_i, b_i and l_i are available. The Bulgarian Academy of Sciences, in 1957, issued tables of Academician V. K. Khristov, "Tables for Geodetic Conversion with the Aid of Arithmometer of Geographic Coordinates on Krasovskiy Ellipsoid for Latitudes 0°-70° for Each Minute."

Academician V. K. Khristov confirms in his formulas small values of fourth order inclusively in reference to u and v. Khristov formulas have the form of:

$$
\begin{aligned}
B_2 &= B_1 + b_{10}u + b_{20}u^2 + b_{02}v^2 + b_{30}u^3 + b_{12}uv^2 + b_{22}u^2v^2 + b_{04}v^4, \\
L_2 &= L_1 + l_{01}v + l_{11}uv + l_{21}u^2v + l_{03}v^3 + l_{31}u^3v + l_{13}uv^3, \\
A_2 &= (A_1 \pm 180^\circ) + a_{01}v + a_{11}uv + a_{21}u^2v + a_{03}v^3 + a_{31}u^3v + a_{13}uv^3,
\end{aligned}
$$

here $u = 10^{-5}\, s \cos A_1$, $v = 10^{-5}\, s \sin A_1$.

Formulas and tables of academician V. K. Khristov are fully suitable for calculation of geodetic coordinates of 1st order triangulation points according to contemporary construction scheme in the USSR.

Formulas (5.9) can be applied for calculations where distances are 130-150 km. For this it is necessary to supplement V. K. Khristov tables with coefficients where u and v go to fifth order inclusively.

Formulas (5.9), as basic mathematical relationships, are also used for obtaining other formulas for resolution of geodetic problems and derivation of the so-called **differential formulas.**

---

[1]Works of MIIGAIK. Pub. 29. M., Geodezizdat, 1957, p. 27-32.

Example of Calculation of Differences of Latitudes, Longitudes and Azimuths by the Formulas in (5.9) are:

$B_1 = 42°19'53'',2714$	$b_1 = 3240'',8597$	$l_1 = 4367'',8505$	$a_1 = 2041'',379$
$L_1 = 25\ 04\ 55\ ,3915$	$b_2 = -23,1077$	$l_2 = 62,2862$	$a_2 = 67,312$
$A_1 = 279\ 07\ 50\ ,447$	$b_3 = -0,2553$	$l_3 = -0,2361$	$a_3 = -0,320$
$S = 82618,157_m$	$b_4 = -0,4568$	$l_4 = 1,2463$	$a_4 = 1,199$
$\cos A_1 = 0,158\ 68867$	$b_5 = -0,0042$	$l_5 = -0,0178$	$a_5 = -0,018$
$\sin A_1 = -0,987\ 32898$	$b_6 = 0,0016$	$l_6 = 0,0229$	$a_6 = 0,023$
$u = 0,083\ 4305$	$b_7 = -0,0044$		
$v = -0,51951431$			
$u^2 = 0,0069719$	$b_1u = 270'',6547$	$l_1v = -2269'',15045$	$a_1v = -1528'',0885$
$uv = -0,0433784$	$b_2v^2 = -6,23016$	$l_2vu = -2,70187$	$a_2vu = -2,9199$
$v^2 = 0,2698951$	$b_3u^2 = -0,00178$	$l_3v^3 = 0,04152$	$a_3v^2 = 0,0449$
$u^3 = 0,000582$	$b_4v^2u = -0,00034$	$l_4vu^2 = -0,00451$	$a_4vu^2 = -0,0043$
$u^2v = -0,003623$	$b_5u^3 = 0,00000$	$l_5v^3u = 0,00021$	$a_5v^3u = 0,0002$
$uv^2 = 0,022536$	$b_6v^4 = 0,00012$	$l_6vu^3 = 0,00001$	$a_6vu^3 = 0,0000$
$v^3 = -0,140214$	$b_7v^2u^2 = -0,00002$		
$u^3v = -0,0003$	$B_2 - B_1 = 264'',3568$	$L_2 - L_1 = -2271'',8151$	$A_2 - A_1 = -1530'',9676$
$u^2v^2 = 0,0019$	$B_1 = 42°19'53'',2714$	$L_1 = 45°04'55'',3915$	$A_1 \pm 180° = 99°07'50'',447$
$uv^3 = -0,0117$	$b = 4'24'',3568$	$l = -37'51'',8151$	$a = -25'30'',968$
$v^4 = 0,0728$	$B_2 = 42°24'17'',6282$	$L_2 = 44°27'03'',5764$	$A_2 = 98°42'19'',479$

§ 26. SCHREIBER-IZOTOV FORMULAS FOR CALCULATION OF GEODETIC COORDINATES OF 1ST ORDER TRIANGULATION POINTS

The geometric formulas (5.9) give ties between geodetic coordinates (B, L) and polar coordinates (s, A). In certain cases it is expedient for calculation of differences of latitudes, longitudes and azimuths to use right-angle spheroidal coordinates (p, q). The problem in this case is resolved in the following manner.

First obtain spheroidal coordinates p and q by the polar, then the differences of latitudes, longitudes and azimuths are expressed in functions of these coordinates. In other words, first of all resolve the right-angle spheroidal triangle P_1CP_2 (Fig. 69). Since in this method in the beginning it is necessary to determine the latitude of point C, this method is frequently called the <u>method of auxiliary point</u>. Principles of such an approach to resolution of a problem were proposed for the first time by Schreiber by whose name for the most part the formulas and method of a given resolution are called.

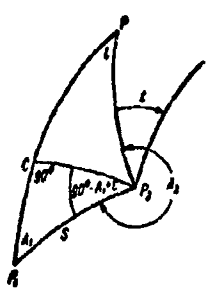


Fig. 69.

Both derivation, and the form of correction terms of the formulas, used in geodetic calculations in USSR, differ from corresponding Schreiber formulas. This circumstance should be underlined since essentially the formulas for solution of direct geodetic problem differ one from another in correction terms, the main terms almost always coincide. Schreiber formulas were obtained different ways and investigated by F. N. Krasovskiy, who established their fitness for geodetic work in USSR. Variant of these formulas was first obtained by A. A. Izotov and published in geodetic tables.

Formulas for spheroidal coordinates by polar were obtained in § 22 (4.34), and by designations in the preceding paragraph have the following form:

$$p = u + v^2 u \frac{3K_{p_1} + 3K_c + 2K_{p_2}}{24} + l_5,$$
$$q = v - vu^2 \frac{2K_{p_1} + K_c + K_{p_2}}{24} + l_5.$$

Here K_{p1} = K_c and K_{p2} are Gauss curvature of vertexes P_1, C and P_2. With accuracy up to small values of fifth order we can accept in these formulas that $K_{p1} = K_c = K_{p2} = \frac{1}{R_1^2}$, where R_1 – mean radius of curvature at initial point P_1, then:

$$\left.\begin{aligned} p &= u\left(1 + \frac{v^2}{3R_1^2}\right) + l_5 \\ q &= v\left(1 - \frac{u^2}{6R_1^2}\right) + l_5 \end{aligned}\right\}. \tag{5.10}$$

We calculate the difference of latitude of the given and auxiliary points, by designating the latitude of the later by B_0. For that we use the first formula from group (5.9).

In our case on the line of abscissas:

$$A = 0, \quad v = 0, \quad u = p,$$

therefore:

$$B_0 - B_1 = b = b_1 p + b_2 p^2 + b_3 p^3 + b_4 p^4 + l_5 \eta_1^2$$

or:

$$b = b_1 p\left(1 + \frac{b_2}{b_1} p + \frac{b_3}{b_1} p^2 + \frac{b_4}{b_1} p^3\right) + l_5 \eta_1^2.$$

The end term of this expression has the greatest value where p = s. Where $\frac{s}{R}$ = = 1/50, B = 60° we obtain:

$$(b_4 p^4)_{max} = \frac{p^4 \eta_1^2 t}{2N^4} V^2 \rho'' = \frac{1{,}7 \cdot 2 \cdot 10^5}{2 \cdot 625 \cdot 10^4 \cdot 6 \cdot 10^2} = \left(\frac{1{,}7}{37500}\right)''.$$

Consequently, even where s = 130-140 km this term can be dropped.

Introducing designations:

$$(1)_1 u = b_1 u = \beta; \quad \frac{\mu 10^8 b_2}{b_1} = -(4)_1; \quad \frac{\mu \cdot 10^8}{3R^2} = (5)_1 \text{ и } \frac{\mu 10^8 b_4}{b_1} = (6)_1$$

and converting to logarithmic form, we obtain:

$$\left.\begin{aligned} \lg b &= \lg \beta - (4)_1 u + (5)_1 u^2 + (6)_1 u^3 + l_5 \\ B_0 &= B_1 + b \end{aligned}\right\}. \tag{5.11}$$

Sign "1" for values in formula (5.11) in this case designates that it is taken from the tables by argument of latitude of first point. Main term in formulas (5.11) will be β, others are called correction terms and are expressed in eighth decimal place of a logarithm.

Having obtained the latitude of the origin of the ordinate q (i.e., point 0), we can transform formulas (5.9) for geodetic line OP_2 and obtain the difference of latitudes, longitudes and convergence of meridians of the unknown and auxiliary points.

For ordinates $A = 90^{\circ}$, $v = q$, $p = 0$, it therefore follows from formulas (5.9) that:

$$\left.\begin{aligned} &1.\ -d = B_2 - B_0 = b_2^0 q^2 + b_4^0 q^4 + l_6 \eta^2 \\ &2.\ l = L_2 - L_1 = l_1^0 q + l_3^0 q^3 + l_5^0 q^5 + l_6 \eta^2 \\ &3.\ t = A_2' - A_1 = a_1^0 q + a_3^0 q^3 + a_5^0 q^5 + l_6 \eta^2 \end{aligned}\right\}. \qquad (5.12)$$

Sign "0" indicates that these coefficients take according to the latitude of the base ordinates B_0.

Formula (5.10) and (5.12) can also be applied for calculation of the differences of latitudes, longitudes and azimuths with tables, containing coefficients, depending on the latitude. For logarithmic calculations of formula (5.12) it is necessary to transform.

After substitution of the value of coefficients b_2^0 and b_4^0 we obtain:

$$d = B_0 - B_2 = -b_2^0 q^2\left[1 + \frac{b_4^0}{b_2^0} q^2\right] = \frac{V_0^4 q^2 t_0 \rho''}{2c^2}\left[1 - \frac{q^2 V_0^2}{12c^2} \times \right.$$
$$\left. \times (1 + 3t_0^2 + \eta_0^2 - 9\eta_0^2 t_0^2)\right].$$

Let us designate:

$$\rho'' \frac{V_0}{c} q t = (2)_0 q \operatorname{tg} B_0 = \tau; \quad \rho'' \frac{V_0}{c} q \sec B_0 = \lambda,$$

then:

$$\lambda^2 - \tau^2 = \frac{V^2}{c^2} q^2 \rho''^2 = c_0^2,$$

therefore:

$$d = B_0 - B_2 = \frac{V_0^2}{2\rho''} \tau c_0 \left\{1 - \frac{c_0^2}{12\rho''^2} - \frac{1}{4}\frac{\tau^2}{\rho''^2} + \frac{\rho''^2 V_0^2 \tau^2}{12c^2 \rho''^2}(9t_0^2 - 1)\right\}.$$

Taking additional designations:

$$(3)_0 = \frac{V_0^2}{2\rho''}; \quad (8)_0 = \frac{(9t_0^2 - 1)\mu 10^8}{12\rho''^2}\eta_1^2, \quad \delta = (3)_0 \tau c_0$$

and converting to logarithmic form taking into account that:

$$c_0^2 = \lambda^2 - \tau^2, \quad \frac{\rho^2 V_0^2}{c^2}\sec^2 B_0 = \lambda^2,$$

we obtain:

$$\lg d = \lg \delta - \nu\tau^2 - \frac{1}{2}\nu\lambda^2 + (8)_0 \lambda^2 + l_6 \eta_1^2, \qquad (5.13)$$

where:

$$\nu = \frac{10^6 \mu}{6 \rho''^2}.$$

Thus, unknown latitude will be:

$$B_2 = B_0 - d = B_1 + b - d.$$

For logarithmic calculation of difference of longitudes we transform second expression from (5.12).

We have:

$$l = l_1^0 q \left(1 + \frac{l_3^0}{l_1^0} q^2 + \frac{l_7^0}{l_1^0} q^4\right) + l_6 \tau^2.$$

After substitution of values l_2^0, l_3^0 and l_7^0;

$$l = \frac{V_0 \sec B_0 \rho'' q}{c} \left\{1 - \frac{V_0^2 t_0^2}{3c^2} q^2 + \frac{V_0^4 t_0^2 q^4}{15c^4} (1 + 3t_0^2)\right\},$$

but:

$$q \frac{V_0 \sec B_0 \rho''}{c} = (2)_0 \sec B_0 q = \lambda; \quad \frac{V_0^2}{c^2} t_0^2 q_0^2 \rho''^2 = \tau^2.$$

Taking these designations and converting to logarithmic form, we obtain:

$$\lg l = \lg \lambda - \frac{10^6 \mu \tau^2}{3 \rho''^2} + \frac{\mu 10^6 V_0^2}{15 c^4} t_0^2 q^4 (1 + 3t_0^2) - \frac{10^6 \mu \tau^4}{18 \rho''^4}.$$

After reduction we designate

$$\frac{10^6 \mu}{90 \rho''^4} \cos^2 B_0 \sin^2 B_0 (6 + 13t_0^2) = (9)_0.$$

Then for final result we have:

$$\lg l = \lg \lambda - 2\nu\tau^2 + (9)_0 \lambda^4 + l_6 \tau^2. \qquad (5.14)$$

Third expression from equations (5.12) after substitution of values of coefficients a_1^0, a_3^0 and a_7^0 will be:

$$t = \frac{V_0 t_0}{c} q \rho'' \left\{1 - \frac{V_0^2}{6c^2} q (1 + 2t^2 + \eta^2) + \frac{V_0^4 q^4}{120 c^4} (1 + 20t^2 + 24t^4)\right\}.$$

Last term $\frac{V_0^4 q^4}{120 c^4}$ $(1 + 20\ t^2 + 24\ t^4)$ where $B = 45^\circ$; $\frac{s}{N} = \frac{1}{50}$ less $\left(\frac{1}{4000}\right)''$. However azimuths are calculated to a thousandth of a fraction of a second, consequently, even at 120-130 km this term can be dropped.

Converting to logarithmic form and considering designation $(7)_0 = -\frac{10^6 \mu}{6 c^4} \rho''^4 \cos^4 B$, we obtain:

$$\lg t = \lg \tau - \nu\tau^2 - \nu\lambda^2 + (7)_0 \lambda^4 + t_6. \qquad (5.15)$$

In accordance with Fig. 63

$$l_2 = 360° - (90° - t) - (90° - A_1 + \varepsilon)$$

or:

$$A_2 = A_1 \pm 180° + t - \varepsilon, \qquad (5.10)$$

where ε is a spherical excess of a right-angle spherical triangle P_1CP_2, calculated by the formula:

$$\varepsilon = \frac{b'' c_0''}{2\rho''}.$$

Thus, calculation of differences of latitudes, longitudes and azimuths by a method of auxiliary point is done by the formulas:

$$\lg b = \lg \beta - (4)_1 u + (5)_1 v^2 + (6)_1 u^2, \qquad (5.11)$$

$$\lg d = \lg \delta - \nu\tau^2 - \frac{1}{3}\nu\lambda^2 + (8)_0 \lambda^2, \qquad (5.13)$$

$$\lg l = \lg \lambda - 2\nu\tau^2 + (9)_0 \lambda^4, \qquad (5.14)$$

$$\lg t = \lg \tau - \nu\lambda^2 - \nu\tau^2 + (7)_0 \lambda^2, \qquad (5.15)$$

$$\varepsilon = \frac{b c_0}{2\rho''},$$

$$B_2 = B_1 + b - d = B_0 - d,$$

$$L_2 = L_1 + l,$$

$$A_2 = A_1 \pm 180° + t - \varepsilon.$$

In these formulas the following designations are made:

$$u = s\cos A_1, \qquad v = s\sin A_1,$$

$$\beta = (1)_1 u,$$

$$\delta = (3)_0 \tau c_0,$$

$$\lambda = c_0 \sec B_0,$$

$$\tau = c_0 \operatorname{tg} B_0.$$

In USSR these formulas are taken for the calculation of geodetic coordinates of 1st order triangulation points. For their application "Tables for Calculation of Geodetic Coordinates" were composed at TsNIIGAIK under direction of Professor A. A. Izotov, which are usually called "Geodetic Tables". In these tables, intended for logarithmic calculations, are given for every minute of latitude lg (1), lg (2), lg R with eight, lg (3) with six, lg (4) with five decimal places. Logarithmic corrections $(6)u^2$, $(7)\lambda^2$, $(8)\lambda^2$ and $(9)\lambda^4$ are given by the argument lg u and lg λ. There is a special table for obtaining corrections $\nu\tau^2$ and $\nu\lambda^2$.

Tables are composed very thoroughly, and are provided with explanatory texts and examples, facilitating application of the formulas.

Obtained formulas, due to the presence in them in a manner of argument of a tangent of latitude, become less exact in northern latitudes (70°-80°). For these latitudes they are applicable to distances of not more than 60-70 km. But this limitation does not have a great significance, since by adopted scheme of construction of 1st order triangulation in USSR, the sides, as a rule, should not exceed 25-30 km.

For distances up to 25-30 km at mean latitudes the derived formulas can be simplified by means of exception of correction terms $(6)_1 u^2$, $(7)_0 \lambda^2$, $(8)_0 \lambda^2$ and $(9)_0 \lambda^4$, after which they will take the following form:

$$\left.\begin{aligned} \lg b &= \lg \beta - (4)_1 u + (5)_1 v^2 \\ \lg d &= \lg \delta - \nu\tau^2 - \frac{1}{2}\nu\lambda^2 \\ \lg l &= \lg \lambda - 2\nu\tau^2 \\ \lg t &= \lg \tau - \nu\tau^2 - \nu\lambda^2 \end{aligned}\right\}. \qquad (6.17)$$

Formulas obtained in this paragraph for calculation of geodetic coordinates do not previde the control of calculations. Therefore calculations are made in two branches. In the second branch it is preferable to use other formulas, giving independent results. For that formulas are used with mean latitude and a mean azimuth, whose derivations will be given in the next paragraph. Let us note that the control of calculations by the Schreiber-Izotov formulas can be carried out by means of a fundamental equation of a geodetic in the form:

$$r_1 \sin A_1 = - r_2 \sin A_2.$$

Values r_1 and r_2 are extracted from D. A. Larin Tables. By a shown formula a simultaneous check for correctness in obtaining unknown latitude and azimuth is made, where during eight place calculations of azimuths are obtained with an accuracy of up to 0".001.

Examples of calculations by the formulas are given in "Practicum on Higher Geodesy" p. 278-282 and in "Geodetic Tables" p. 20-25.

§ 27. FORMULAS FOR MEAN LATITUDE AND MEAN AZIMUTH OR GAUSS FORMULAS

In factorization of differences of latitudes, longitudes and azimuths by sequences of s (§ 25) the derivatives were calculated by coordinates of initial point. As it is shown in formula (1.3), the Taylor line is doubly reduced, if instead of initial azimuths and latitudes the mean were taken. In this case all terms with even degrees drop from the series. With the same number of terms in series with application of mean arguments the accuracy becomes one order higher as compared to the accuracy of series, obtained by the initial arguments. The principle of mean arguments for calculation of geodetic coordinates were first applied by Gauss. He evolved a formula with mean arguments.

Let us designate:

$$\frac{B_2+B_1}{2}=B_m,\quad \frac{A_1\pm 180^\circ+A_2}{2}=A_m,$$

$$\left(\frac{d^iB}{ds^i}\right)_m=B^i_m,\quad \left(\frac{d^iL}{ds^i}\right)_m=L^i_m,\quad \left(\frac{d^iA}{ds^i}\right)_m=A^i_m.$$

$$(i=1,\ 2,\ 3\ldots)$$

Geodesic coordinates of initial point P_1 and polar coordinates (s, A) point P_2 are given. Let us divide s in half and designate the middle of the arc s by C (Fig. 70). The latitude of this point and azimuth of geodesic line in it, we will designate B_c and A_c.

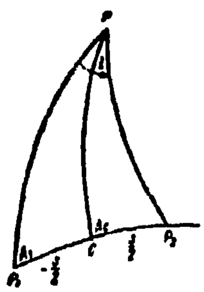


Fig. 70.

We will take point C for the origin of the polar coordinates. Then apply factorization of differences of latitudes, longitudes and azimuths by series of s, to the two sections of arc s, considering right section as positive, left as negative. In accordance with (5.2) we have:

$$B_2-B_C=\frac{s}{2}B'_C+\frac{s^2}{8}B''_C+\frac{s^3}{48}B'''_C+\frac{s^4}{384}B^{IV}_C+\frac{s^5}{3840}B^{V}_C+\ldots$$

$$B_1-B_C=-\frac{s}{2}B'_C+\frac{s^2}{8}B''_C-\frac{s^3}{48}B'''_C+\frac{s^4}{384}B^{IV}_C-\frac{s^5}{3840}B^{V}_C+\ldots.$$

Sum and difference of these expressions will be:

$$B_2+B_1-2B_C=\frac{s^2}{4}B''_C+\frac{s^4}{192}B^{IV}_C+\ \ldots$$

or:

$$B_m-B_C=\frac{s^2}{8}B''_C+\frac{s^4}{384}B^{IV}_C+\ldots+l_6. \qquad (5.18)$$

$$B_2-B_1=sB'_C+\frac{s^3}{24}B'''_C+\frac{s^5}{1920}B^{V}_C+\ldots+l_7. \qquad (5.19)$$

Analogously we find:

$$\left.\begin{aligned} L_2-L_1&=sL'_C+\frac{s^3}{24}L'''_C+\frac{s^5}{1920}L^{V}_C+\ldots+l_7\\ A'_2-A_1&=sA'_C+\frac{s^3}{24}A'''_C+\frac{s^5}{1920}A^{V}_C+\ldots+l_7\end{aligned}\right\}, \qquad (5.20)$$

$$A_m-A_C=\frac{s^2}{8}A''_C+\frac{s^4}{384}A^{IV}_C+\ldots+l_6. \qquad (5.21)$$

Series (5.18) and (5.21) show that the differences of mean arguments and azimuths of median point of arc are small values of the second order.

In series (5.19) and (5.20) we will express unknown B_c, A_c by B_m and A_m.

We have:

$$\left.\begin{aligned} B'_C&=\frac{\cos A_C}{M_C}=\varphi(B_C,\ A_C)=\varphi[B_m+(B_C-B_m),\ A_m+(A'_C-A_m)]\\ L'_C&=\frac{\sin A_C\sec B_C}{N_C}=\varphi_1(B_C,\ A_C)=\varphi_1[B_m+(B_C-B_m),\\ &\qquad A_m+(A_C-A_m)]\end{aligned}\right| \qquad (5.22)$$

$$A'_C = \frac{\sin A_C \operatorname{tg} B_C}{N_C} = \varphi_3(B_C, A_C) = \varphi_3[B_m + (B_C - B_m), \; A_m + (A_C - A_m)] \qquad (5.22)\ \text{cont}$$

Hence:

$$\left.\begin{aligned} B'_C &= B'_m + \left(\frac{\partial B'}{\partial B}\right)_m (B_C - B_m) + \left(\frac{\partial B'}{\partial A}\right)_m (A_C - A_m) + l_5 \\ L'_C &= L'_m + \left(\frac{\partial L'}{\partial B}\right)_m (B_C - B_m) + \left(\frac{\partial L'}{\partial A}\right)_m (A_C - A_m) + l_5 \\ A'_C &= A'_m + \left(\frac{\partial A'}{\partial B}\right)_m (B_C - B_m) + \left(\frac{\partial A'}{\partial A}\right)_m (A_C - A_m) + l_5 \end{aligned}\right\}.$$

Substituting values $(B_C - B_m)$ and $(A_C - A_m)$ from (5.18) and (5.21) in (5.22), we have:

$$B'_C = B'_m - \frac{s^2}{8}\left[\left(\frac{\partial B'}{\partial B}\right)_m B''_m + \left(\frac{\partial B'}{\partial A}\right)_m A''_m\right] + l_5,$$

$$A'_C = A'_m - \frac{s^2}{8}\left[\left(\frac{\partial A'}{\partial B}\right)_m B''_m + \left(\frac{\partial A'}{\partial A}\right)_m A''_m\right] + l_5.$$

With these values B'_C and A'_C formulas (5.19) and (5.20) will take the form:

$$\left.\begin{aligned} B_2 - B_1 = b &= sB'_m - \frac{s^3}{8}\left[\left(\frac{\partial B'}{\partial B}\right)_m B''_m + \left(\frac{\partial B'}{\partial A}\right)_m A''_m - \frac{B'''_m}{3}\right] + l_5 \\ L_2 - L_1 = l &= sL'_m - \frac{s^3}{8}\left[\left(\frac{\partial L'}{\partial B}\right)_m B''_m + \left(\frac{\partial L'}{\partial A}\right)_m A''_m - \frac{L'''_m}{3}\right] + l_5 \\ A_2 - A_1 = a &= sA'_m - \frac{s^3}{8}\left[\left(\frac{\partial A'}{\partial B}\right)_m B''_m + \left(\frac{\partial A'}{\partial A}\right)_m A''_m + \frac{A'''_m}{3}\right] + l_5 \end{aligned}\right\}. \qquad (5.23)$$

Entering here derivatives B'_m, B''_m, B'''_m, L'_m, L''_m, L'''_m, A'_m, A''_m, and A'''_m obtained from (5.2'), (5.4) and (5.5) by means of substitution of index "1" for index "m"; then partial derivatives have the form:

$$\left.\begin{aligned} &1.\ \left(\frac{\partial B'}{\partial B}\right)_m = -\frac{3\eta_m^2 t_m \cos A_m}{N_m}; \quad \left(\frac{\partial B'}{\partial A}\right)_m = -\frac{V_m^2}{N_m}\sin A_m \\ &2.\ \left(\frac{\partial L'}{\partial B}\right)_m = \frac{\sin A_m \sec B_m}{N_m V_m^2}\, t_m; \quad \left(\frac{\partial L'}{\partial A}\right)_m = \frac{\cos A_m \sec B_m}{N_m} \\ &3.\ \left(\frac{\partial A'}{\partial B}\right)_m = \frac{\sin A_m}{N_m V_m^2}(1 + \eta_m^2 + t_m^2); \quad \left(\frac{\partial A'}{\partial A}\right)_m = \frac{\cos A_m t_m}{N_m} \end{aligned}\right\}. \qquad (5.24)$$

Let us substitute the values of derivatives from (5.2'), (5.4), (5.5) and partial derivatives from (5.24), after reductions omitted here, we obtain:

$$\left.\begin{aligned} b'' &= \frac{s\cos A_m}{M_m}\rho''\Big\{1 + \frac{s^2}{24N_m^2}[\sin^2 A_m(2 + 3t_m^2 + 2\eta_m^2) + \\ &\quad + 3\eta_m^2\cos^2 A_m(t_m^2 - 1 - \eta_m^2 - 4\eta_m^2 t_m^2)]\Big\} + l_5 \\ l'' &= \frac{s\sin A_m \sec B_m}{N_m}\rho''\Big\{1 + \frac{s^2}{24N_m^2}[\sin^2 A_m t_m^2 - \cos^2 A_m(1 + \\ &\quad + \eta_m^2 - 9\eta_m^2 t_m^2)]\Big\} + l_5 \\ a'' &= \frac{s\sin A_m \operatorname{tg} B_m}{N_m}\rho''\Big\{1 + \frac{s^2}{24N_m^2}[\sin^2 A_m(2 + 2\eta_m^2 + t_m^2) + \\ &\quad + \cos^2 A_m(2 + 7\eta_m^2 + 9\eta_m^2 t_m^2 + 5\eta_m^4)]\Big\} + l_5 \end{aligned}\right\}. \qquad (5.25)$$

We designate:

$$\frac{s\cos A_m \rho''}{M_m} = (1)_m s\cos A_m = \beta_m,$$

$$\frac{s\sin A_m \sec B_m \rho''}{N_m} = (2)_m s\sin A_m \sec B_m = \lambda_m,$$

$$\frac{s\sin A_m \operatorname{tg} B_m \rho''}{N_m} = (2)_m s\sin A_m \operatorname{tg} B_m = \tau_m;$$

or:

$$s^2\cos^2 A_m = \frac{M_m^2}{\rho''^2}\beta_m^2 = \frac{N_m^2}{\rho''^2}\cdot\frac{\beta_m^2}{V_m^4},$$

$$s^2\sin^2 A_m = \frac{N_m^2\cos^2 B_m \lambda_m^2}{\rho''^2}.$$

by these designations expression (5.25) will take form:

$$b = \beta_m\left\{1 + \frac{\lambda_m^2 \cos^2 B_m}{24\rho''^2}(2 + 3t_m^2 + 2\eta_m^2) + \frac{\beta_m^2\eta_m^2}{8\rho''^2}\times\right.$$
$$\left.\times\left(\frac{t_m^2 - 1 - \eta_m^2 - 4\eta_m^2 t_m^2}{V_m^4}\right)\right\} + l_5.$$

$$l = \lambda_m\left\{1 + \frac{\lambda_m^2\sin^2 B_m}{24\rho''^2} - \frac{\beta_m^2}{24\rho''^2}\left(\frac{1+\eta_m^2 - 9\eta_m^2 t_m^2}{V_m^4}\right)\right\} + l_5.$$

$$a = \tau_m\left\{1 + \frac{\lambda_m^2\cos^2 B_m}{24\rho''^2}(2 + t_m^2 + 2\eta_m^2) + \frac{\beta_m^2}{24\rho''^2}\times\right.$$
$$\left.\times\left(\frac{2 + 7\eta_m^2 + 9\eta_m^2 t_m^2 + 5\eta_m^4}{V_m^4}\right)\right\} + l_5.$$

From the last two formulas we obtain by means of a division of the third by second:

$$\frac{a}{l} = \sin B_m\left\{1 + \frac{\lambda_m^2\cos^2 B_m}{24\rho''^2}(2 + t_m^2 + 2\eta_m^2) + \frac{\beta_m^2}{24\rho''^2}\times\right.$$
$$\times\left(\frac{2 + 7\eta_m^2 + 9\eta_m^2 t_m^2 + 5\eta_m^4}{V_m^4}\right) - \frac{\lambda_m^2\sin^2 B_m}{24\rho''^2} + \frac{\beta_m^2}{24\rho''^2}\times$$
$$\left.\times\left(\frac{1 + \eta_m^2 - 9\eta_m^2 t_m^2}{V_m^4}\right)\right\}$$

or

$$a = l\sin B_m\left\{1 + \frac{\lambda_m^2\cos^2 B_m V_m^2}{12\rho''^2} + \frac{\beta_m^2}{24\rho''^2}\left(\frac{3 + 8\eta_m^2 + 5\eta_m^4}{V_m^4}\right)\right\} + l_5.$$

Converting formulas for b, l and a, to logarithms we obtain:

$$\left.\begin{array}{l}\lg b = \lg\beta_m + \nu_2\lambda_m^2\cos^2 B_m + \nu_3\beta_m^2 + l_5\\ \lg l = \lg\lambda_m + \frac{1}{4}\nu\tau_m^2 - \nu_1\beta_m^2 + l_5\\ \lg a = \lg\tau_m + \nu_4\lambda_m^2\cos^2 B_m + \nu_5\beta_m^2 + l_5\\ \lg a = \lg l\sin B_m + \nu_4 l^2\cos^2 B_m + \nu_6 b^2\end{array}\right\}. \qquad (5.26)$$

where:

$$\nu = \frac{H'' s}{6\rho''^2},$$

$$\nu_1 = \frac{\nu}{4} \cdot \frac{(1 + \eta_m^2 - 9\eta_m^2 t_m^2)}{V_m^4},$$

$$\nu_2 = \frac{\nu}{4} (2 + 3t_m^2 + 2\eta_m^2),$$

$$\nu_3 = \frac{3}{4} \nu \frac{(t_m^2 - 1 - 4\eta_m^2 (t_m^2 - \eta_m^2)\eta_m^2}{V_m^4},$$

$$\nu_4 = \frac{\nu}{2} V_m^2,$$

$$\nu_5 = \frac{\nu}{4} \left(\frac{8 + \eta_m^2 + 5\eta_m^4}{V_m^4} \right).$$

Logarithms of values ν_1, ν_2, ν_3, ν_4, and ν_5 by argument of mean latitude are given in geodetic tables.

Formulas (6.26), namely those, which were obtained by Gauss twice in the second article of "Research in Higher Geodesy"[1] by conformal presentation of ellipsoid on a sphere by means of factorization in series by powers of s with mean arguments. Therefore they are called Gauss formulas.

In Gauss formulas small values are retained to third order inclusively, as in Schreiber formulas, but their advantage in comparison to Schreiber formulas with respect to accuracy consists in that the dropped terms in them, small values of fifth order are 10-15 times less than in Schreiber formulas. Effective correction terms in Gauss formulas have 4-5 times less correction terms than Schreiber formulas. Therefore Gauss formulas can be used for calculations of coordinates for greater distances, than Schreiber formulas. In identical requirements for accuracy of unknown values Gauss formulas are applicable for distances on the order of 200-230 km within latitudes of 60°-70°. If however in differences of latitudes and longitudes 0".001, and in azimuths 0".01, are retained, then these formulas can be used for distances between points up to 300-350 km. We mention in passing that with such distances, as a rule, necessity does not arise for calculation of differences of latitudes and longitudes to 0".0001. For distances on the order of 300-400 km it is sufficient to calculate differences of latitudes and longitudes to 0".001, and azimuths to 0".01. Then relative error $\frac{\Delta s}{s}$ in transmission of coordinates both for short, and long distances, will be of the same order.

Transmission of coordinates to still greater distances, i.e., to 400-500 km, in practice of contemporary geodetic work is encountered comparatively rarely. For that

[1] G. F. Gauss. Selected geodetic work: Vol. II. Higher Geodesy, M., Geodezizdat, 1958, p. 86.

case it is possible to use complete formulas with mean arguments, given in geodetic tables. They also are obtained by a method of factorization of differences of coordinates and azimuths in power series, but with retention of small values to fifth order inclusively. Therefore, avoiding repetition of preceding derivation, we show the formulas in their final form:

$$\left.\begin{aligned} \lg b &= \lg \beta_m + \nu_2 \lambda_m^2 \cos^2 B_m + \nu_3 \beta_m^2 + \varkappa_1 \beta^2 \lambda^2 + \varkappa_2 \lambda^4 \\ \lg l &= \lg \lambda_m + \frac{1}{4} \nu \tau_m^2 - \nu_1 \beta_m^2 + \varkappa_1 \beta^2 l^2 + \varkappa_3 \lambda^4 + \frac{\nu \beta^4}{15} \\ \lg a &= \lg \tau_m + \nu_4 \lambda_m^2 \cos^2 B_m + \nu_5 \beta_m^2 - \varkappa_4 \beta^2 \lambda^2 + \varkappa_5 \lambda^4 + \varkappa \beta^4 \end{aligned}\right\} . \qquad (5.26')$$

As compared to formulas (5.26) terms appearing here where $\varkappa_i$ ($i = 1, 2, \ldots, n$), whose logarithms are given in tables have the following expressions:

$$\begin{aligned} \varkappa &= \frac{10^8 \mu}{192 \rho''^2}; \\ \varkappa_1 &= \frac{2\varkappa}{15}(4 + 15t^2)\cos^2 B, \\ \varkappa_2 &= \frac{\varkappa}{15}(12t^2 + t^4)\cos^4 B, \\ \varkappa_3 &= \frac{\varkappa}{15}(14 + 40t^2 + 15t^4)\cos^4 B, \\ \varkappa_4 &= \frac{\varkappa}{4}\sin^2 B, \\ \varkappa_5 &= \frac{2\varkappa}{15}(7 - 6t^2)\cos^4 B. \end{aligned}$$

In calculations by the formulas (5.26') in general cases the method of successive approximations should be applied.

If there is no approximate value of mean azimuth and mean latitude, then in application of formulas (5.26') it is better in first approximation of the problem to resolve it by the Schreiber formulas. In this case the number of approximations will be cut in half. If in latitudes and longitudes only 0".001, and in azimuths 0".01, were retained then these formulas can be applied for distances of 500-600 km. Such distances in contemporary geodetic work are met in radar measurements. Thus, formula (5.26') meets the requirements, which arise during radar geodetic measurements.

For distances of 25-30 km formula (5.26) can be simplified by dropping small values of fourth order, i.e., terms with η^2. Then:

$$\begin{aligned} \nu_1 &= \frac{1}{4}\nu, \\ \nu_2 &= \frac{1}{2}\nu + \frac{3}{4}\nu t^2, \\ \nu_3 &= 0, \\ \nu_4 &= \frac{1}{2}\nu, \\ \nu_5 &= \frac{3}{4}\nu. \end{aligned}$$

Substituting new values ν_1, ν_2, ν_3, ν_4 and ν_5 in formulas (5.26), we obtain:

$$
\left.\begin{aligned}
\lg b &= \lg \beta_m + \frac{\nu}{2}\lambda_m^2 + \frac{\nu\tau_m^2}{4} + l_4 \\
\lg l &= \lg \lambda_m + \frac{1}{4}\nu\tau_m^2 - \frac{1}{4}\nu\beta_m^2 + l_4 \\
\lg a &= \lg \tau_m + \frac{1}{2}\nu\lambda_m^2 - \frac{1}{2}\nu\tau_m^2 + \frac{3}{4}\nu\beta_m^2 + l_4
\end{aligned}\right\}. \tag{5.27}
$$

Formulas (5.27) have been used for a long time in Russia in treatment of 1st order triangulation and only since 1904-1906 were replaced by formulas (5.17). This replacement happened after publication of geodetic tables of military geodesist Scharnhorst. Derivation and foundation of formulas (5.17) and (5.27) were first given by F. N. Krasovskiy.

All formulas with mean arguments have that general deficiency where during solution of direct geodetic problems it is necessary to apply a method of approximations, when a number of approximations is unknown beforehand. It is true, a number of approximations can be decreased, if cartographic materials for determination of approximate values of mean arguments are used. But such additional work is hardly desirable to a computer. Therefore most frequently these formulas are used for control, when mean arguments are known with sufficient degree of accuracy. But the methodical merit of these formulas remains in force.

Example of calculation by the formulas (5.26') is given in "Geodetic Tables" (p. 24), and for formulas (5.27) in "Practicum on Higher Geodesy" (p. 284).

§ 28. RESOLUTION OF DIRECT AND INVERSE GEODETIC PROBLEMS BY A METHOD OF CHORDS OF ELLIPSOID (STUDY OF M. S. MOLODENSKIY METHOD)

The idea of application of chords of ellipsoid for solution of geodetic problems is not new. As far back as 1799 Delambre developed a method of resolution of chord triangles for terrestrial spheroid. In 1869 famous geodesist Bremiker in work "Studien über Hohere Geodäsie" proposed elimination of use of geodetics by means of formation instead of spheroidal triangles of rectilinears from the chords and so to resolve geodetic problems, utilizing these triangles. This idea subsequently developed by him in the indicated work. Geodesists in the past both here and abroad have returned to the use of this idea in reference to particular problems. Recently this problem was raised again in the USSR and was originally developed by M. S. Molodenskiy.

The advantage of application of chords as compared to geodetics consists in that, independent of the distance between the points, finite formulas are obtained in closed form as a combination of elementary functions. As can be seen from preceding account, the application of geodetics during resolution of geodetic problems leads to formulas in the form of infinite series, in fact of fast convergency. However the advantage of the use of chords for resolution of principal geodetic problem disappears, as soon as necessity arises to have a length of arc between points on the surface of an ellipsoid. Chord and elliptic arc are connected among themselves by series (3.16). Transition from chord to arc again returns us to infinite series. Furthermore, in presentation of an ellipsoid on a sphere or plane application of geodetic lines gives general solution of the problem, while in application of chords this generalization is absent.

The theory of a method of resolution of geodetic problems, founded on application of chords between vertexes of triangles on an ellipsoid, is presented in M. S. Molodenskiy work "New Method of Resolution of Geodetic Problems". In this work formulas are given for solution of direct and inverse geodetic problems, differential formulas are studied, and formulas for reduction of measured directions to the surface reference-ellipsoid and transition from one geodetic system to another during conversion and reorientation of reference-ellipsoid are given. To this section of the course only the first problem pertains, which we now take up.

Let us assume that on a spheroid two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given, by their space right-angle coordinates, $\bar{s}$ — chord, connecting point P_1 and P_2.

We have:

$$\left.\begin{aligned} x &= N\cos B\cos L \\ y &= N\cos B\sin L \\ z &= \frac{b^2}{a^2} N\sin B \end{aligned}\right\}. \tag{2.16}$$

M. S. Molodenskiy considers the more general case, when points do not lie on the surface.

Further, $x_2 - x_1 = \bar{s}\cos\alpha$; $y_2 - y_1 = \bar{s}\cos\beta$; $z_2 - z_1 = \bar{s}\cos\gamma$, where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are direction cosines of chord $\bar{s}$. Let us designate them, according to M. S. Molodenskiy.

$$\begin{aligned} \cos\alpha &= l_{12}, \\ \cos\beta &= m_{12}, \\ \cos\gamma &= n_{12}. \end{aligned}$$

From differential geometry it is known that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Substituting values of x, y, z from (2.16) and combining plane yz with a plane of meridian of first point P_1, i.e., assuming $y_1 = 0$, we obtain:

$$
\left.\begin{aligned}
1.\ & \bar{s}l_{12} = x_2 - x_1 = N_2\cos B_2\cos l - N_1\cos B_1\\
2.\ & \bar{s}m_{12} = y_2 - y_1 = N_2\cos B_2\sin l\\
3.\ & \bar{s}n_{12} = z_2 - z_1 = \frac{b^2}{a^2}(N_2\sin B_2 - N_1\sin B_1)
\end{aligned}\right\} \qquad (5.28)
$$

where l is a difference of geodetic longitudes from meridian P_1. Raising (5.28) to a square and adding, we obtain:

$$
\bar{s}^2 = N_1^2 + N_2^2 - 2N_1N_2(\sin B_1\sin B_2 + \cos B_1\cos B_2\cos l) - \frac{a^4-b^4}{a^4}(N_2\sin B_2 - N_1\sin B_1)^2.
$$

Designating:

$$
\cos\psi = \sin B_1\sin B_2 + \cos B_1\cos B_2\cos l,
$$

for $\bar{s}^2$ by means of identity transformation we have:

$$
\bar{s}^2 = N_1^2 + N_2^2 - 2N_1N_2 + 2N_1N_2 - 2N_1N_2\cos\psi - \frac{a^4-b^4}{a^4}\times(N_2\sin B_2 - N_1\sin B_1)^2.
$$

Hence:

$$
\bar{s}^2 = 4N_1N_2\sin^2\frac{\psi}{2} - \frac{a^4-b^4}{a^4}(N_2\sin B_2 - N_1\sin B_1)^2 + (N_2 - N_1)^2. \qquad (5.29)
$$

here:

$$
\sin^2\frac{\psi}{2} = \sin^2\frac{1}{2}(B_2 - B_1) + \cos B_1\cos B_2\sin^2\frac{1}{2}l. \qquad (5.30)
$$

In (5.29) the first term of right side is main, and second and third are small values of the order of compression and the square of compression of terrestrial spheroid correspondingly.

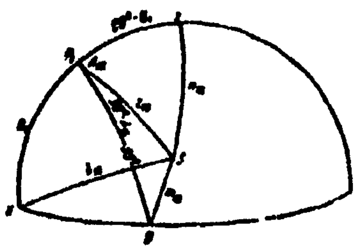


Fig. 71.

For Krasovskiy ellipsoid $\frac{a^4 - b^4}{a^4} = 0.013342041$.

Let us imagine a sphere of unit radius (Fig. 71), on which P_1 is a geodetic zenith of first point. Laying out from this point to the right, an arc, equal to $90 - B_1$, we obtain a pole, i.e., a point, corresponding to axis Oz. Let the direction of a chord from point P_1 to point P_2 intersect the sphere at point s, then spherical distance P_1s will be zenith distance of second point, angle sP_1z and an azimuth of direction of chord $\bar{s}_c$ from point P_1 to P_2.

Let us assume, that to directions of axes of coordinates O_x and O_y points x and y correspond on a sphere, then arcs sx, sy and sz on a sphere will be equal to cosines directing chords $\bar{s}$ from point P_1 to P_2. Line xyQ is a horizon of point P_1. Thus, we constructed geodetic horizontal system of coordinates, in which the position of points is determined by zenithal distances and azimuth of direction, i.e., z and A.

Let us determine cosines directing the chord.

From triangles P_1xs, P_1ys and P_1zs we have:

$$\left.\begin{array}{l} 1.\ l_{12} = \cos B_1 \cos z_{12} - \sin B_1 \sin z_{12} \cos A_{12} \\ 2.\ m_{12} = \sin z_{12} \sin A_{12} \\ 3.\ n_{12} = \sin B_1 \cos z_{12} + \cos B_1 \sin z_{12} \cos A_{12} \end{array}\right| \quad \left|\begin{array}{l} \sin B_1 \\ \\ \cos B_1 \end{array}\right| \begin{array}{l} \cos B_1 \\ \\ \sin B_1 \end{array} \tag{5.31}$$

If B_1, z_{12} and A_{12} (5.31) are given then they fully determine l_{12}, m_{12} and n_{12}. In order to resolve the inverse problem, i.e., to determine z_{12}, A_{12} by B_1, l_{12}, m_{12} and n_{12}, multiply the first expression (5.31) by $\sin B_1$, the third – by $\cos B_1$ and add. Then, conversely, multiply the first by $\cos B_1$, and third by $\sin B_1$. From the third subtract the first, dividing the difference by m_{12} term by term, then we obtain:

$$\cos z_{12} = \cos B_1\, l_{12} + \sin B_1\, n_{12}, \tag{5.32}$$

$$\operatorname{ctg} A_{12} = \frac{\cos B_1 n_{12} - \sin B_1 l_{12}}{m_{12}}. \tag{5.33}$$

Substituting in (5.33) n_{12} and l_{12} with the aid of (5.28) by geodetic coordinates, we obtain:

$$\operatorname{ctg} A_{12} = \frac{\cos B_1 (N_2 \sin B_2 - N_1 \sin B_1) \frac{b^2}{a^2} - \sin B_1 (N_2 \cos B_2 \cos l - N_1 \cos B_1)}{N_2 \cos B_2 \sin l},$$

but:

$$\frac{b^2}{a^2} = (1 - e^2),$$

therefore:

$$\operatorname{ctg} A_{12} = \frac{\cos B_1 \operatorname{tg} B_2}{\sin l} - \sin B_1 \operatorname{ctg} l + e^2 \left(\frac{N_1 \sin B_1 - N_2 \sin B_2}{N_2 \cos B_2 \sin l} \right) \cos B_1.$$

$$\operatorname{ctg} A_{12} = \frac{\sin (B_2 - B_1)}{\cos B_2 \sin l} + \sin B_1 \operatorname{tg} \frac{l}{2} - e^2 \frac{N_2 \sin B_2 - N_1 \sin B_1}{N_2 \cos B_2 \sin l} \cos B_1.$$

We designate:

$$\operatorname{ctg} a_{12} = \frac{\sin (B_2 - B_1)}{\cos B_2 \sin l} + \sin B_1 \operatorname{tg} \frac{l}{2},$$

then:

$$\operatorname{ctg} A_{12} = \operatorname{ctg} a_{12} + e^2 \frac{N_1 \sin B_1 - N_2 \sin B_2}{N_2 \cos B_2 \sin l} \cos B_1. \tag{5.34}$$

Changing indices 1 to 2, we obtain:

$$\operatorname{ctg} A_{21} = \operatorname{ctg} a_{21} + e^2 \frac{N_2 \sin B_2 - N_1 \sin B_1}{N_1 \cos B_1 \sin l} \cos B_2, \tag{5.35}$$

where:

$$\operatorname{ctg} a_{21} = \frac{\sin (B_1 - B_2)}{\cos B_1 \sin l} + \sin B_2 \operatorname{tg} \frac{l}{2}. \tag{5.35'}$$

Formulas (5.34) and (5.35) determine direct and back azimuths of a chord in relation to a plane of meridian and zenith of the first point P_1; z_{12} is calculated by $\bar{s}$ according to the formula, whose derivation is shown below.

Grid coordinates of points P_1 and P_2 must satisfy the equation for the ellipsoid: for P_1

$$\frac{x_1^2+y_1^2}{a^2}+\frac{z_1^2}{b^2}=1,$$

for P_2

$$\frac{(x_1+\bar{s}l_{12})^2}{a^2}+\frac{(y_1+\bar{s}m_{12})^2}{a^2}+\frac{(z_1+\bar{s}n_{12})^2}{b^2}=1.$$

Subtracting first from the second, we obtain

$$\bar{s}(1+e'^2n_{12}^2)+2\left(x_1l_{12}+y_1m_{12}+\frac{a^2}{b^2}z_1n_{12}\right)=0.$$

Substituting x_1, y_1 and z_1 by own values by geodetic coordinates we have:

$$\begin{gathered} x_1=N_1\cos B_1,\\ y_1=0,\\ z_1=(1-e^2)N_1\sin B_1, \end{gathered}$$

we find:

$$\bar{s}(1+e'^2n_{12}^2)=-2N_1(\cos B_1l_{12}+\sin B_1n_{12}).$$

Considering (5.32), we obtain:

$$\bar{s}(1+e'^2n_{12}^2)=-2N_1\cos z_{12}.$$

Hence:

$$\cos z_{12}=-\frac{\bar{s}}{2N_1}(1+e'^2n_{12}^2). \qquad (5.36)$$

Let us designate:

$$-\frac{1}{N_1}(1+e'^2n_{12}^2)=\frac{1}{R_2}, \qquad (5.37)$$

then:

$$\cos z_{12}=-\frac{\bar{s}}{2R_2}. \qquad (5.38)$$

Let us determine the geometric meaning of $\frac{1}{R_2}$. From (5.36) it follows that when $\bar{s}=0$, $z=90^{\circ}$; in this case from (5.31) we find that $n_{12}^0=\cos B_1\cos A_1$, then:

$$\frac{1}{R_2^0}=\frac{1}{N_1}(1+e'^2\cos^2B_1\cos^2A_1)=\frac{1}{N_1}(1+\eta_1^2\cos^2A_1),$$

i.e., we obtain formula (2.29). Consequently, by R_2 it is necessary to understand radius of curvature of normal section at current point, in this case at P_2. Therefore by the formula (5.38) it is possible to satisfy the calculation only by a method of successive approximations. In the first approximation, it is possible to naturally take:

$$\frac{1}{R_2} \approx \frac{1}{N_1} \text{ and } \sin z_{12} \approx 1.$$

Since $n_{12} = -n_{21}$, then, considering (5.37), we obtain

$$N_1 \cos z_{12} = N_2 \cos z_{21}. \tag{5.39}$$

Further, from second formulas (5.28) and (5.31) it follows that:

$$N_1 \sin z_{12} \sin A_1 \cos B_1 = -N_2 \sin z_{21} \sin A_2 \cos B_2. \tag{5.40}$$

The direct and inverse geodetic problems can be resolved by Molodenskiy method by the formulas (5.29), (5.34) and (5.35). Inverse problem is resolved directly by these formulas for significant $\bar{s}$, and the direct problem, according to method of approximations. However for the solution of the direct problem these formulas should be applied in a somewhat different form.

Dividing the first of equations (5.28) by the second, we obtain:

$$\operatorname{ctg} l = \frac{l_{12}}{m_{12}} + \frac{N_1 \cos B_1}{\bar{s}\, m_{12}}. \tag{5.41}$$

Multiplying the second of equations (5.28) by $\left(-\frac{\sin B_1}{\sin l}\right)$, and third by $\frac{a^2}{b^2} \cos B_2$ and adding, we obtain:

$$N_2 \sin(B_2 - B_1) = N_1 \sin B_1 \cos B_1 + n_{12}\bar{s}\frac{a^2}{b^2} \cos B_1 - m_{12}\frac{\bar{s} \sin B_1}{\sin l}. \tag{5.42}$$

The back azimuth of a cord is determined by formula (5.35).

By the formulas (5.41) and (5.42) l and ΔB are determined, but for application of these formulas in practice it is necessary to first calculate z_{12} approximations by formula (5.38). The number of approximations is determined depending upon the length of a chord. For distances $\bar{s} < 100$ km it is possible to be limited to only one approximation. For $\bar{s} > 100$ km by two-three or more, in order to retain fractions thousandth of a second in azimuths and eight decimal places for $\bar{s}$.

Method of chords in that form, as proposed by M. S. Molodenskiy, fully resolves the problem on hand; besides the formulas are obtained in the form of closed combinations of elementary functions. For short distances these formulas are less convenient for practical calculations than the formulas of Schreiber, Izotov and Gauss, since here it is necessary to deal with a method of approximations and calculation of trigonometric functions of acute angles. As it is known, the interpolation by tables in these cases is very labor-consuming. Geodesists are familiar with this circumstance, therefore they always prefer to convert from trigonometric functions for acute angles to angles with the aid of series. Schreiber, Gauss formulas and others are built this. If however, from utilizing the Molodenskiy method we convert from trigonometric functions of acute angles to angles, then one of the advantages of chord method is lost. Comparison shows that the volume of calculations in application of presented Molodenskiy method is somewhat greater.

In the work of Candidate of Technical Sciences V. F. Yeremeyev "Formulas and Tables for Calculation of Geodetic Coordinates according to Molodenskiy Method"[1] are given practical formulas, and models of necessary tables for the resolution of geodetic problems and examples of calculations.

§ 29. THE INVERSE GEODETIC PROBLEM

The inverse geodetic problem is the determination of length of geodesic and its azimuths at at its terminal points by geodetic coordinates of these points.

This problem as compared to direct is in practice resolved less frequently. It can be resolved by any inverse formulas of direct problem, but the most rational resolution is obtained by the formulas with mean arguments.

Let us rewrite formula (5.26') in such a form:

$$\lg b = \lg (1)_m s \cos A_m + v_2 l^2 \cos^2 B_m + v_3 b^2 + x_1 b^2 l^2 + x_2 l^4,$$
$$\lg l = \lg (2)_m s \sin A_m \sec B_m + \frac{1}{4} v l^2 \sin^2 B_m - v_1 b^2 + x_1 b^2 l^2 + x_2 l^4 - \frac{x l^4}{15},$$
$$\lg a = \lg l \sin B_m + v_4 l^2 \cos^2 B_m + v_5 b^2 - x_4 b^2 l^2 + x_5 l^4 + x b^4.$$

In these formulas correction terms β_m, λ_m and τ_m are substituted correspondingly by b, l and $l \sin B_m$.

Let us take designations according to geodetic tables. Let:

$$\lg s \cos A_m = \lg Q,$$
$$\lg s \sin A_m = \lg P,$$
$$\Delta \lg (s \cos A_m) = -v_2 l^2 \cos^2 B_m - v_3 b^2 - x_1 b^2 l^2 - x_2 l^4,$$
$$\Delta \lg (s \sin A_m) = -\frac{1}{4} v l^2 \sin^2 B_m + v_1 b^2 - x_1 b^2 l^2 - x_2 l^4 + \frac{x l^4}{15},$$
$$\Delta \lg a = v_4 l^2 \cos^2 B_m + v_5 b^2 - x_4 b^2 l^2 + x_5 l^4 + x b^4.$$

With these designations:

$$\left.\begin{aligned} \lg Q &= \lg \frac{b}{(1)_m} + \Delta \lg (s \cos A_m) \\ \lg P &= \lg \frac{l \cos B_m}{(2)_m} + \Delta \lg (s \sin A_m). \end{aligned}\right\}; \qquad (5.43)$$

$$\left.\begin{aligned} &1.\ \lg a'' = \lg l \sin B_m + \Delta \lg a \\ &2.\ \lg \operatorname{tg} A_m = \lg P - \lg Q \end{aligned}\right\} \qquad (5.43')$$

[1] Works of TsNIIGAIK. Issue 121. M., Geodezizdat, 1957, p. 77-112.

$$3.\ \lg s = \lg P - \lg \sin A_m = \lg Q - \lg \cos A_m \\ A_1 = A_m - \frac{1}{2} a'' \\ A_2 = A_m + \frac{1}{2} a'' \pm 180^\circ \qquad (5.44)$$

These formulas are applicable to distances of 600-700 km. If, however the distances and azimuths are required to be known less precisely, for instance, a distance with accuracy up to decimeters, but azimuths to tenths of fractions of a second, then these formulas can be applied for distances on the order of 800-1000 km.

If in correction terms, the terms with factors κ_i ($i = 1, 2, 3, \ldots$), are dropped they will become useful for $s \leq 200$-250 km.

However in practice more frequently it is necessary to resolve problems for distances on the order of length of a side of 1st order triangulation. In such a case correction terms are greatly simplified and take the form of:

$$\Delta \lg (s \cos A_m) = -\frac{1}{2} \nu l^2 - \frac{1}{4} \nu l^2 \sin^2 B_m,$$
$$\Delta \lg (s \sin A_m) = \frac{1}{4} \nu b^2 - \frac{1}{4} \nu l^2 \sin^2 B_m,$$
$$\Delta \lg a = \frac{3}{4} \nu b^2 + \frac{1}{2} \nu l^2 \cos^2 B_m.$$

Consequently,

$$\lg a'' = \lg l \sin B_m + \Delta \lg a'' \\ \lg \operatorname{tg} A_m = \lg P - \lg Q \\ \lg s = \lg P - \lg \sin A_m = \lg Q - \lg \cos A_m \\ A_1 = A_m - \frac{1}{2} a'' \\ A_2 = A_m + \frac{1}{2} a'' \pm 180^\circ \qquad (5.45)$$

Errors in lg tg A_m are composed of errors lg b and lg l. Let us consider the problem of accuracy in obtaining azimuths by resolution of inverse problem.

We have

$$\lg \operatorname{tg} A_m = \mu \ln \operatorname{tg} A_m,$$

or

$$\Delta \lg (\operatorname{tg} A_m) = \mu \frac{\Delta A_m}{\operatorname{tg} A_m \cos^2 A_m} = \frac{\mu \Delta A_m}{\cos A_m \sin A_m} = \frac{2 \Delta A_m \mu}{\sin 2A_m},$$

whence

$$\Delta A_m = \Delta \lg (\operatorname{tg} A_m) \cdot \frac{\sin 2A_m}{2\mu} \rho'',$$

i.e.,

$$(\Delta A_m)_{max} = \Delta \lg (\operatorname{tg} A_m) \frac{1}{2\mu} \rho''.$$

Inasmuch as lg tg A_m is obtained as a difference of logarithms P and Q, then it is always possible to allow that the error can be equal to two-three digits of the last sign; and in case of eight-place calculations two-three digits of eighth decimal place. Consequently, it is possible to take:

$$\Delta \lg \operatorname{tg} A_m \approx 3 \cdot 10^{-8},$$

then

$$\Delta A_m'' \approx \frac{3 \cdot 10^{-8} \cdot 2 \cdot 10^{5}}{2 \cdot 0.43} \approx \left(\frac{3}{430}\right)''.$$

Thus, we arrive at a conclusion that azimuths from inverse geodetic problems at accepted accuracy of calculations can be obtained with accuracy of up to 0."01.

Examples of the resolution of inverse geodetic problem by the formulas (5.45) and (5.43)-(5.44) are given in "Practicum" (p. 286 and 288) and in "Geodesic Tables" (p. 22 and 25).

In this chapter are presented basic methods of calculation of geodetic coordinates, having practical and methodical value. For application of these methods in practice in USSR formulas and fundamental geodetic tables are developed. With the presence of these tables geodetic coordinates are calculated very simply and precisely.

The more practical for calculation of geodetic coordinates of 1st order triangulation points are Schreiber-Izotov formulas. By simplicity and accuracy these formulas completely answer theoretical and practical requirements for precise calculations of geodetic coordinates of 1st order triangulation points.

Gauss formulas although they ensure great accuracy and have simpler construction, are nevertheless in their application in practice are somewhat complicated from the method of approximations. Therefore they should be used for control at second hand, as was already indicated. For control of calculations of latitude and azimuth it is possible to use the fundamental equation of geodesic

$$r_1 \sin A_1 = -r_2 \sin A_2 \quad \text{or} \quad (2)_2 \cos B_1 \sin A_1 = -(2)_1 \cos B_2 \sin A_2.$$

In transmission of coordinates to distances on the order of 300-400 km formulas with mean arguments in combination with Schreiber-Izotov formulas should be used, , i.e., for obtaining coordinates for first approximation Schreiber-Izotov formulas should be applied. From this resolution of differences of latitudes, longitudes and azimuths will be obtained with an accuracy of up to 0."01. After that for obtaining unknown values with required accuracy it will be sufficient to make two approximations. However, such problems are met comparatively rarely in practice, and in every individual case a method of resolution, conforming with the requirements of accuracy should be established.

The inverse geodetic problem both for short, and long distances (600-700 km) are best resolved by the formulas with mean arguments.

Method of resolution of geodetic problems, proposed by Molodenskiy, yields to Schrieber methods and to mean arguments. Therefore it should hardly be mentioned for its application in mass geodetic calculations. Approach of M. S. Molodenskiy has methodical value, inasmuch as he expands our knowledge in an area of resolution of geodetic problems.

On calculation of geodetic coordinates there exists an extensive special literature. Scientific investigations in this direction are also being conducted at present. In particular, attempt is being made to apply to resolution of geodetic problems the methods of vector analysis. First investigations in this direction reveal advantages of methodical character.

For practical purposes tables of Bulgarian Academician V. K. Khristov should be published with proper changes and supplements, for nonlogarithmic calculations of geodetic coordinates.

Contemporary scheme for state 1st order triangulation of USSR anticipates construction of triangles with sides on an average of 20-25 km. In other measure such distances on Earth's surface in differences of latitudes, longitudes and azimuths corresponds to 700"-800"; for calculations of such lines with an accuracy of up to 0."0001 it is sufficient to apply tables with seven decimal places. Therefore it is expedient along with eight-place geodetic tables to have seven-place geodetic tables. They can also be used for educational purposes.

Such tables were composed by the author on a chair of higher geodesy.

CHAPTER VI

RESOLUTION OF GEODETIC PROBLEMS FOR LONG DISTANCES

§ 30. GENERAL CONSIDERATIONS

In resolution of geodetic problems short, medium and long distances are distinguished. Usually by short distances are implied lengths of sides of 1st order triangulation mean are lengths of diagonals of one or several sections of triangulation and long distances are on the order of radius of the Earth. Referring these distances to mean radius of the Earth, we obtain numerical characteristics of their order. Ratio of small distances to radius has an order e^2, of mean distances e. The relation of short distances e^2 is the value of first order, for the mean, and the value of the second order.

In derivation of formulas for transmission of coordinates to short distances power series were used in the preceding chapter, i.e., factorization in series by powers of s of the differences of latitudes, longitudes and azimuths. Such series quickly converge and give convenient formulas for practical calculations. When distances, close in length to the radius of the Earth R, or great R, the application of series is practically inexpedient, since they converge so slowly that it is difficult to establish, which terms must be retained, and which should be dropped. Where $\frac{s}{R} > 1$ subsequent terms of series by absolute value can be greater than the first. In other words, series by ascending powers of $\frac{s}{R}$ cannot be used for great distances in resolution of geodetic problems.

In resolution of geodetic problems for great distances series are also used however they, as a rule, are designed by ascending powers of e^2. We already encountered the use of such series in calculation of lengths of arc of the meridian. These series possess properties of geodetic series. They are sign changing and rapidly-converging.

In this and preceding paragraphs we consider geodetic problems, in which s can be as great as needed.

But for geodetic targets necessity of resolution of a problem for very great distances is very rarely encountered. Transmission of coordinates for great distances can arise, for instance, during connection of separated geodetic nets of continents. However in radar navigation and rocket technology the necessity for resolution of such problems appears frequently. Therefore the resolution of geodetic problems for great distances has actual practical and scientific value.

First general question, which appears in connection with transmission of coordinates for great distances, pertains to the uniqueness of solution. The direct geodetic problem is always resolved simply, if the difference of the longitudes of terminals of geodetic lines are less than 180°. This position is based on equation $r \sin A = c$, from which it follows that through every point on a spheroid under given azimuth A can pass only one geodesic. At the given length of line and azimuth coordinates of a second point are determined if coordinates of first point are known.

The inverse geodetic problem is also resolved single valued, if the shortest distance between two given points is determined. Uncertainty of resolution arises in cases, where the difference in longitudes is equal or is close to 180°.

If a bundle of geodesics was presented passing through point P_1, then for this bundle it is possible to expose an astroidal evolute, whose center coincides with point P_D, diametrically opposite P_1, where coordinates P_D will be:

$$B_D = -B_1,$$
$$L_D = L_1 \pm 180^\circ.$$

Evolute axes and, consequently, their vertexes fall onto a parallel and meridian of point P_D (Fig. 72). In Fig. 72 dotted lines depict evolutes of bundles of geodesics, emanating from points with latitudes 0°, 30° and 60°. Geodesics are depicted by straight lines, 0-180° line depicts rotation axis of a spheroid and line 90°-270° depicts the equator.

Outside the evolute of point P_1 a bundle of geodesics will form a field, in which through every point of a spheroid passes only one geodesic. From Fig. 72 it follows that the direct problem is resolved by single values for points, located outside the evolute of point P_1. However single value solutions of inverse problem

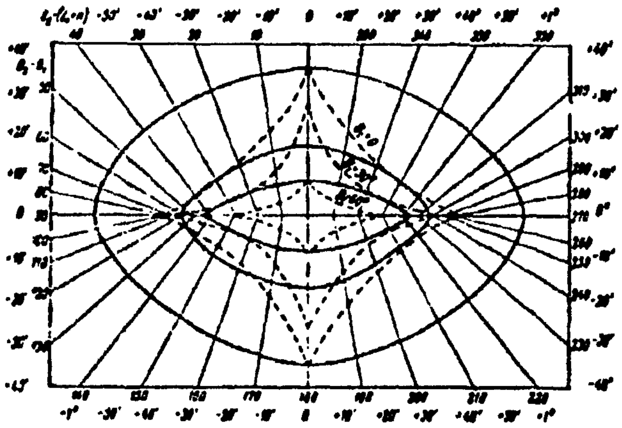


Fig. 72.

are possible on both the boundary and on vertexes of the evolute of point P_1, if only the shortest distance between points is determined. It is necessary to point out that these cases are borderline and each of them should be investigated individually. But if points P_1 fall on the axes of the evolute, located in west-east direction, then there are two lines of identical length. This case is very rare and for its resolution a special investigation is required. It is possible to determine single values of the solution of inverse geodetic problem by Fig. 72.

§ 31. LENGTH OF ARC OF GEODESIC AND THE DIFFERENCE OF LONGITUDES OF ITS TERMINAL POINTS

Let us consider on the surface of a spheroid and auxiliary sphere the unit radius corresponding to elementary right-angle triangles (Fig. 73).

On a spheroid (Fig. 73a)

$$\left.\begin{aligned} M dB &= ds\cos A \\ r dl &= ds\sin A \end{aligned}\right\}. \qquad (6.1)$$

On sphere (Fig. 73b).

$$\left.\begin{aligned} du &= d\sigma\cos A \\ \cos u\, d\omega &= d\sigma\sin A \end{aligned}\right\}. \qquad (6.2)$$

From (6.1) and (6.2)

$$\frac{ds}{d\sigma} = M\frac{dB}{du},$$
$$\frac{dl}{d\omega} = M\frac{dB}{du}\frac{\cos u}{r},$$

but

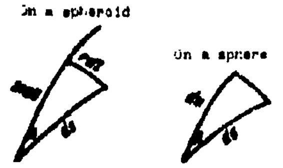
$$\frac{dB}{du} = \frac{a}{M}\sqrt{1-e^2\cos^2 u},$$

and

$$r = a\cos u.$$

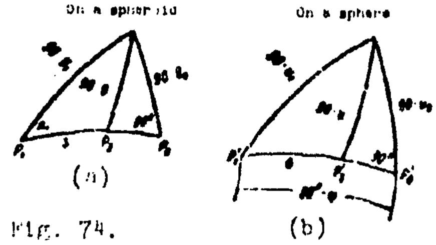
Therefore

$$\left.\begin{aligned} ds &= a\sqrt{1-e^2\cos^2 u}\, d\sigma \\ dl &= \sqrt{1-e^2\cos^2 u}\, d\omega \end{aligned}\right\}. \qquad (6.3)$$



(a) (b)

Fig. 73.



(a) (b)

Fig. 74.

Equations (6.3) are the basic first order differential equations for obtaining lengths of arc of a geodesic and differences of longitudes of its terminal points. However integrals of these differential equations are not undertaken in elementary functions. They must be found by means of factorization of subradical expression in power series.

For integration (6.3) we will accomplish the substitution of a variable, i.e., express u in corresponding formulas by σ.

Let us assume that geodesic s between given points P_1 and P_2 on ellipsoid and corresponding to arc of a great circle σ on an auxiliary sphere (Fig. 74) is given.

Let us extend arc s and σ to their intersection with the meridian at right angle. These points of intersection we will designate accordingly by P_0 and P_0'. Latitudes of points P_0 and P_0' we shall designate by B_0 and u_0, a connection between them will be determined by the well known formula:

$$\operatorname{tg} u_0 = \sqrt{1-e^2}\operatorname{tg} B_0$$

Let us designate arc $P_1'P_0'$ on all auxiliary sphere by $90^\circ - \varphi$.

From spherical right-angle triangles $P_2'P'P_0'$ and $P_1'P'P_0'$

$$\left.\begin{aligned}\cos u_0 &= \cos u \sin A_1\\ \cos A_1 &= \operatorname{ctg}(90^\circ - u_1)\operatorname{ctg}\varphi\end{aligned}\right\} \qquad (6.4)$$

or:

$$\operatorname{tg}\varphi = \frac{\operatorname{tg} u_1}{\cos A_1}. \qquad (6.5)$$

Further

$$\sin u_0 = \frac{\sin u_1}{\sin\varphi}. \qquad (6.6)$$

From triangle $P_2'P'P_0'$

$$\sin u = \sin u_0 \sin(\varphi + \sigma).$$

or:

$$\cos^2 u = 1 - \sin^2 u_0 \cdot \sin^2(\varphi + \sigma). \qquad (6.7)$$

Substituting (6.7) into the first of (6.3), we obtain:

$$ds = a\sqrt{1-e^2+e^2\sin^2 u_0 \sin^2(\varphi+\sigma)}\, d\sigma =$$

$$= a\sqrt{1-e^2}\cdot\sqrt{1+\frac{e^2}{1-e^2}\sin^2 u_0 \sin^2(\varphi+\sigma)}d\sigma$$

$$a\sqrt{1-e^2} = b,$$

Put:

$$\frac{e^2}{1-e^2} = e'^2,$$

designating

$$e'\sin u_0 = k,$$

we obtain

$$ds = b\sqrt{1+k^2\sin^2(\varphi+\sigma)}\, d\sigma \tag{6.7}$$

or

$$s = b\int_0^{\sigma'}\sqrt{1+k^2\sin^2\sigma'}\, d\sigma', \tag{6.8}$$

where

$$\sigma' = \varphi + \sigma, \quad d\sigma' = d\sigma.$$

Expression (6.8) is an elliptic integral of second type. Thus, the length of arc of a geodesic is an elliptic integral of second type. Since where $B = B_0$ a variable of σ' will equal zero, then the lengths of arcs s are considered from one meridian, and namely eastward.

Consequently,

$$s = bF(k, \sigma'), \tag{6.9}$$

where $F(k, \sigma')$ are tabulated elliptic integrals of second type.

If extreme latitude B_0 is given for the geodesic, then (6.9) gives length of arc between points with latitudes B_1 and B_0. But if, besides B_0 and B_1, there is also given latitude B_2, then length of arc between points P_1 and P_2 will be obtained as a difference of elliptic integrals, i.e.,

$$s = b(F(k, \sigma_2') - F(k, \sigma_1')). \tag{6.9'}$$

Formula (6.9') can only be used when detailed tables of elliptic integrals are available. However the application of tables of elliptic integrals, or integrals of Legendre, is hampered by the fact that in these tables interpolation must be conducted by two arguments, by k and by σ', frequently with four differences. For practical calculations this method of calculation of s requires considerable work, although geometrically it is simpler. In higher geodesy the preference is given to series, obtained by factorization into binomial series integral expression in the right part of the equation (6.8). Series thus obtained converge very rapidly and depend on only one argument.

§ 30. BESSEL METHOD FOR RESOLUTION OF DIRECT GEODETIC PROBLEM

For length of arc s in preceding paragraph we obtained:

$$s = b\int_0^{\sigma} \sqrt{1 + k^2 \sin^2 \sigma'}\, d\sigma'. \tag{6.8}$$

We have:

$$\sqrt{1 + k^2 \sin^2 \sigma'} = 1 + \frac{k^2}{2} \sin^2 \sigma' - \frac{k^4}{8} \sin^4 \sigma' + \ldots,$$

substituting:

$$\sin^2 \sigma' = \frac{1}{2} - \frac{1}{2} \cos 2\sigma',$$
$$\sin^4 \sigma' = \frac{3}{8} - \frac{1}{2} \cos 2\sigma' + \frac{1}{8} \cos 4\sigma',$$

. . . .

we obtain:

$$\sqrt{1 + k^2 \sin^2 \sigma'} = \left(1 + \frac{1}{4} k^2 - \frac{3}{64} k^4\right) - \left(\frac{1}{4} k^2 - \frac{1}{16} k^4\right) \cos 2\sigma' - \frac{k^4}{64} \cos 4\sigma' + \ldots$$

Designate:

$$\left.\begin{aligned} A &= 1 + \frac{1}{4} k^2 - \frac{3}{64} k^4 + \ldots \\ B &= \frac{1}{4} k^2 - \frac{1}{16} k^4 + \ldots \\ C &= \frac{k^4}{128} - \ldots \end{aligned}\right\} \tag{6.10}$$

Consequently,

$$s = b\int_0^{\sigma} (A - B\cos 2\sigma' - 2C\cos 4\sigma' + \ldots) \cdot d\sigma'.$$

Making term by term integration by the formulas:

$$\int_0^{\sigma} \cos 2\sigma'\, d\sigma' = \frac{1}{2} (\sin 2\sigma' - \sin 2\sigma'_0) = \sin \sigma \cos (2\varphi + \sigma), \tag{a}$$

$$\int_0^{\sigma} \cos 4\sigma'\, d\sigma' = \frac{1}{4} (\sin 4\sigma' - \sin 4\sigma'_0) = \frac{1}{2} \sin 2\sigma \cos (4\varphi + 2\sigma), \tag{б}$$

we obtain

$$s = b(A\sigma - B\sin \sigma \cos (2\varphi + \sigma) - C\sin 2\sigma \sin (4\varphi + 2\sigma)) \tag{6.11}$$

or:

$$\sigma'' = \frac{s}{bA}\rho'' + \frac{B}{A}\rho'' \sin\sigma\cos(2\varphi+\sigma) + \frac{C}{A}\rho'' \sin 2\sigma \cdot \cos(4\varphi+2\sigma). \qquad (6.12)$$

We designate:

$$\left.\begin{aligned} \alpha &= \frac{\rho''}{A} \\ \beta &= \frac{B}{A}\rho'' \\ \gamma &= \frac{C}{A}\rho'' \end{aligned}\right\}. \qquad (6.13)$$

Consequently,

$$\sigma'' = \alpha\frac{s}{b} + \beta\sin\sigma\cos(2\varphi+\sigma) + \gamma\sin 2\sigma\cos(4\varphi+2\sigma). \qquad (6.14)$$

Formula (6.11) is used in resolution of inverse geodetic problem and formula (6.14) in resolution of a direct geodetic problem.

Both by the formula (6.11), and by the formula (6.14) it is expedient to calculate according to the method of consecutive approximations.

From second equation (6.3)

$$\sqrt{1-e^2\cos^2 u} = 1 - \frac{e^2}{2}\cos^2 u - \frac{e^4}{8}\cos^4 u - \frac{e^6\cos^6 u}{16} - \ldots \qquad (6.15)$$

From (6.2) and (6.4)

$$d\omega = \frac{\cos u_0}{\cos^2 u}d\sigma. \qquad (6.16)$$

Substituting (6.15) and (6.16) in (6.3), we obtain:

$$dl = d\omega - e^2\cos u_0\left(\frac{1}{2} + \frac{e^2}{8}\cos^2 u + \frac{e^4}{16}\cos^4 u + \ldots\right)d\sigma. \qquad (6.17)$$

But from (6.7)

$$\left.\begin{aligned} \cos^2 u &= 1 - \sin^2 u_0 \sin^2\sigma' \\ \cos^4 u &= 1 - 2\sin^2 u_0\sin^2\sigma' + \sin^4 u_0 \sin^4\sigma' \end{aligned}\right\}. \qquad (6.18)$$

Further

$$\left.\begin{aligned} \sin^2\sigma' &= \frac{1}{2} - \frac{1}{2}\cos 2\sigma' \\ \sin^4\sigma' &= \frac{3}{8} - \frac{1}{2}\cos 2\sigma' + \frac{1}{8}\cos 4\sigma' \end{aligned}\right\}. \qquad (6.19)$$

Substituting (6.18) and (6.19) in (6.17), we obtain

$$dl = d\omega - e^2\cos u\,(A' + B'\cos 2\sigma' + C'\cos 4\sigma' + \ldots)\,d\sigma', \qquad (6.20)$$

where:

$$\left.\begin{aligned} A' &= \frac{1}{2} + \frac{e^2}{8} + \frac{e^4}{16} - \frac{e^2\sin^2 u_0}{16} - \frac{e^4}{16}\sin^2 u_0 + \ldots \\ B' &= \frac{e^2}{16}\sin^2 u_0 + \frac{e^4}{16}\sin^2 u_0 - \frac{e^4}{32}\sin^4 u_0 + \ldots \\ C' &= \frac{e^4\sin^4 u_0}{128} + \ldots \end{aligned}\right\}. \qquad (6.21)$$

Integrating (6.20) and taking into account ($\bar{a}$) and ($\bar{b}$), we obtain

$$l = \omega - e^2 \cos u_0 \left\{ A'\sigma + B' \sin\sigma \cos(2\varphi + \sigma) + \frac{C'}{2} \sin 2\sigma \cos(4\varphi + 2\sigma) \right\}.$$

Or, designating $\alpha' = A'e^2$, $\beta' = B'e^2\rho''$, $\gamma' = \frac{C'e^2}{2}\rho''$, we have finally:

$$l = \omega - \cos u_0 \{\alpha'\sigma + \beta' \sin\sigma \cos(2\varphi + \sigma) + \gamma' \sin 2\sigma \cos(4\varphi + 2\sigma)\}. \quad (6.22)$$

The final term of this formula can be a maximum of $\frac{e^4}{256} \sin^4 u_0$, and its numerical value is always less than 0"0002. Therefore it can be dropped in further reckoning.

In solution of inverse problem l is known, therefore ω is determined by a method of approximations by the formula:

$$\omega = l + \cos u_0 \{\alpha'\sigma + \beta' \sin\sigma \cos(2\varphi + \sigma)\}. \quad (6.23)$$

By the formula (6.23) it is also expedient to calculate by a method of consecutive approximations.

Formulas (6.14) and (6.22) are basic in resolution of straight geodetic problem by the Bessel method. In derivation of these formulas we did not impose any limitations on s with respect to its length, therefore they are applicable to any distances between points on a spheroid.

Bessel composed tables for values of: lg α, lg β, lg γ, lg α' and lg β', where the tables of values lg α, lg β and lg γ are composed by argument that lg k = lg e sin u_0, and lg α' and gl β' are by argument:

$$\lg k' = \lg e \frac{\sqrt{0,75} \sin u_0}{\sqrt{1 - 0,75e^2}}$$

In our designations α', β' and γ' are functions of k. Therefore, selecting lg α', lg β' and lg γ' from tables by argument of lg k, it is necessary to add constant value, equal to:

$$\lg \frac{e^2}{\sqrt[3]{1 - 0,75e^2}}.$$

Besides the shown Bessel tables,[1] in 1953, V. P. Morozov Doctor of Technical Sciences composed tables on Dimensions of Krasovskiy Ellipsoid by an argument: $\cos^2 u_0$ or $\sin^2 A_0$ for α, β, γ, α', and β'.[2]

---

[1]F. V. Bessel. Higher geodesy and method of least squares. Edited by G. V. Bagratuni. M., Geodezizdat, 1961, p. 272.

[2]V. P. Morozov. Formulas and tables for resolution of straight and inverse geodetic problems on the surface of earth's ellipsoid. (Publication of Military-Engineering academy imeni V. V. Kuybyshev, 1958.)

Professor N. A. Urmayev made his own tables,[1] composed for k and k' according to arguments by Bessel they are intended for nonlogarithmic calculations and require parabolic interpolation. These tables are very compact, and take up only a quarter of a page, but are somewhat inconvenient for interpolation. As Bessel tables, Urmayev tables are applicable to any reference-ellepsoids.

Expressions for coefficients A', B', C' are somewhat simplified if e is expressed by e' by formulas:

$$e'^2 = \frac{e^2}{1-e^2} = e^2 + e^4 + e^6 + \ldots$$

$$\frac{e'}{e+e'} = \frac{\frac{e}{\sqrt{1-e^2}}}{e + \frac{e}{\sqrt{1-e^2}}} = \frac{1}{1+\sqrt{1-e^2}} = \frac{1}{2}\left(1 + \frac{e^2}{4} + \frac{e^4}{8} + \ldots\right).$$

then:

$$\left.\begin{aligned} A' &= \frac{e'}{e+e'} - \frac{e'^2}{16}\sin^2 u_0 + \frac{3}{128}e'^4\sin^4 u_0 \\ B' &= \frac{e'^2}{16}\sin^2 u_0 - \frac{e'^4}{32}\sin^4 u_0 \\ C' &= \frac{e'^4}{256}\sin^4 u_0 \end{aligned}\right\}. \qquad (6.24)$$

Consequently,

$$\left.\begin{aligned} \alpha' &= \frac{e^2 e'}{e+e'} - \frac{e^2 e'^2}{16}\sin^2 u_0 + \frac{3}{128}e^2 e'^4\sin^4 u_0 \\ \beta' &= \frac{e^2 e'^2}{16}\sin^2 u_0 - \frac{e^2 e'^4}{32}\sin^4 u_0 \\ \gamma' &= \frac{e^2 e'^4}{256}\sin^4 u_0 \end{aligned}\right\} \qquad (6.25)$$

In resolution of direct geodetic problem according to Bessel method it is expedient to hold to the following order.

1. Calculation of reduced latitude of first point by a given geodesic:

$$\operatorname{tg} u_1 = \sqrt{1-e^2}\,\operatorname{tg} B_1.$$

2. Determination of auxiliary values u_0 and φ by the formulas in (6.4) and (6.5).

3. Calculation of arguments k, k' and selection from Bessel of Urmayev tables of lg α, lg β, lg γ, lg α', lg β'.

4. Calculation of spherical distance σ by the formulas (6.14), where for reduction of quantity of approximations the first approximation should be calculated

[1] N. A. Urmayev. Spheroidal geodesy. Editorial-Publishing Department VTS, M., 1955.

by the formula:

$$\sigma_1 = \frac{\rho''}{b} s V_1,$$

where $V_1 = \sqrt{1 + e'^2 \cos^2 B_1}$ is taken from geodetic tables for B_1.

6. Resolution of spherical triangle $P_1'P'P_2'$ (Fig. 74b) by Napier's analogies and finding of A_2, u_2 and ω.

7. Transition from u_2 and ω to B_2 and L by formulas:

$$\operatorname{tg} B_2 = \frac{\operatorname{tg} u_2}{\sqrt{1 - e^2}},$$

$$l = \omega - \cos u_0 [\alpha'\sigma + \beta' \sin\sigma \cos(2\varphi + \sigma) + \ldots].$$

In resolution of the direct problem by Bessel method a necessity arises for determination of quarters for auxiliary values φ and u_0. For that data in Table 5 can be used.

Table 5

A_1	u_0	φ
0	90°	u_1
90°	u_1	90°
180°	90°	$180° - u_1$
270°	$180° - u_1$	90°

§ 33. FORMULAS OF PROFESSOR A. M. VIROVETS

From preceding paragraph it follows that integration of equations (6.3) for spherical arc leads to Bessel formulas. However this is not the only way to integration of these equations. Integration can also be accomplished by reduced latitude. For that it is necessary that $d\sigma$ and $d\omega$ be expressed by du according to corresponding formulas.

We have

$$du = d\sigma \cos A, \tag{6.2}$$

$$\cos u \sin A = c = \cos u_0. \tag{6.26}$$

From (6.26)

$$\cos A = \frac{\sqrt{\cos^2 u - c^2}}{\cos u}. \tag{6.27}$$

From (6.2) and (6.27)

$$d\sigma = \frac{\cos u \, du}{\sqrt{\cos^2 u - c^2}}. \tag{6.28}$$

Further

$$d\omega \cos u = d\sigma \sin A. \tag{6.29}$$

Substituting value of sin A from (6.26) and $d\sigma$ from (6.28), we obtain:

$$d\omega = \frac{c \, du}{\cos u \sqrt{\cos^2 u - c^2}}. \tag{6.30}$$

Replacing $d\sigma$ and $d\omega$ by (6.28) and (6.30) in (6.3), we obtain

$$
\left.
\begin{aligned}
ds &= \frac{a\sqrt{1-e^2\cos^2 u}\,\cos u\,du}{\sqrt{\cos^2 u - c^2}} \\
dl &= \frac{c\sqrt{1-e^2\cos^2 u}\,du}{\cos u\sqrt{\cos^2 u - c^2}}
\end{aligned}
\right\}. \qquad (6.31)
$$

We obtained differential equations, in which right side-function are only u. Integration of these equations is carried out by means of factorization in series:

$$
\sqrt{1-e^2\cos^2 u} = 1 - \frac{e^2}{2}\cos^2 u - \frac{1}{8}e^4\cos^4 u - \frac{1}{16}e^6\cos^6 u \;\ldots \qquad (6.32)
$$

Replacing in (6.31) $\sqrt{1-e^2\cos^2 u}$ by series (6.32) and being limited by terms with e^4, we arrive at the following integrals for s and l:

$$
s = a\left\{\int_{u_1}^{u_2}\frac{\cos u\,du}{\sqrt{\cos^2 u - c^2}} - \frac{1}{2}e^2\int_{u_1}^{u_2}\frac{\cos^3 u\,du}{\sqrt{\cos^2 u - c^2}} - \frac{1}{8}e^4\int_{u_1}^{u_2}\frac{\cos^5 u\,du}{\sqrt{\cos^2 u - c^2}}\right\}. \qquad (6.33)
$$

$$
l = c\left\{\int_{u_1}^{u_2}\frac{du}{\cos u\sqrt{\cos^2 u - c^2}} - \frac{1}{2}e^2\int_{u_1}^{u_2}\frac{\cos u\,du}{\sqrt{\cos^2 u - c^2}} - \frac{1}{8}e^4\int_{u_1}^{u_2}\frac{\cos^3 u\,du}{\sqrt{\cos^2 u - c^2}}\right\}. \qquad (6.34)
$$

Substituting $\sqrt{\cos^2 u - c^2} = z$ these integrals are reduced to tabular form.

We have:

$$
\left.
\begin{aligned}
&\int\frac{\cos u\,du}{\sqrt{\cos^2 u - c^2}} = -\arcsin\frac{t}{k} + C_1 \\
&\int\frac{\cos^3 u\,du}{\sqrt{\cos^2 u - c^2}} = \frac{1}{2}\left\{\sin u\,t - k^2\arcsin\frac{t}{k}\right\} + C_2 \\
&\int\frac{\cos^5 u\,du}{\sqrt{\cos^2 u - c^2}} = \sin u\left\{\frac{1}{4}t^3 + \frac{1}{8}(5c^2+3)t\right\} - \left[c^4 + \frac{3}{8}\times\right. \\
&\left.\times(1-c^2)^2\arcsin\frac{t}{k}\right] + C_3 \\
&\int\frac{du}{\cos u\sqrt{\cos^2 u - c^2}} = -\frac{1}{c}\,\mathrm{arc\,tg}\left(\frac{c\cdot\sin u}{t}\right) + C_4
\end{aligned}
\right\}. \qquad (6.35)
$$

In (6.35) designations are taken:

$$
t = \sqrt{\cos^2 u - c^2}, \quad k = \sqrt{1-c^2}.
$$

We designate:

$$
\left.
\begin{aligned}
\frac{c\sin u}{t} &= \mathrm{tg}\,\beta \\
\frac{t}{k} &= \sin\alpha
\end{aligned}
\right\}, \qquad (6.36)
$$

then

$$
\left.
\begin{aligned}
\mathrm{arc\,tg}\left(\frac{c\sin u}{t}\right) &= \beta \\
\arcsin\frac{t}{k} &= \alpha
\end{aligned}
\right\}. \qquad (6.37)
$$

We have also

$$\left.\begin{aligned} \sin u &= k\cos\alpha \\ t\sin u &= \frac{1}{2}k^2\sin 2\alpha \\ t^2\sin u &= \frac{1}{2}k^3\sin 2\alpha\sin^2\alpha \end{aligned}\right\}. \qquad (6.38)$$

Substituting the value of integrals in (6.35) taking into account (6.36), (6.37) and (6.38), we obtain:

$$\begin{aligned} s = -a\Big\{&\Big[1-\frac{1}{4}e^2(1+e^2)-\frac{1}{64}e^4(3+2e^2+3e^4)\Big](\alpha_2-\alpha_1)+ \\ &+\Big[\frac{1}{8}e^2(1-e^2)+\frac{1}{128}e^4(3+2e^2-5e^4)\Big](\sin 2\alpha_2-\sin 2\alpha_1)+ \\ &+\Big[\frac{1}{64}e^4(1-e^2)\Big](\sin 2\alpha_2\sin^2\alpha_2-\sin 2\alpha_1\sin^2\alpha_1)\Big\}. \end{aligned} \qquad (6.39)$$

$$\begin{aligned} l = (\beta_2-\beta_1)+e\Big\{&\Big[\frac{1}{2}e^2+\frac{1}{16}e^4(1+e^2)\Big](\alpha_2-\alpha_1)- \\ &-\Big[\frac{1}{32}e^4(1-e^2)^2\Big](\sin 2\alpha_2-\sin 2\alpha_1)\Big\}. \end{aligned} \qquad (6.40)$$

In expressions for s and l terms with e^6 are dropped, which gives an error for s less than 0.1 m and in l less than 0."001.

Formula (6.39) is applied in solution of inverse problem, for solution of the direct problem it is necessary to obtain from this formula $(\alpha_2 - \alpha_1)$

$$\begin{aligned} (\alpha_2-\alpha_1)'' = -\frac{\rho''}{a}s&\Big[1+\frac{1}{4}e^2(1+e^2)+\frac{1}{64}e^4(7+10e^2+7e^4)\Big]- \\ -\rho''&\Big[\frac{1}{8}e^2(1-e^2)+\frac{1}{128}e^4(7+2e^2-9e^4)\Big](\sin 2\alpha_2-\sin 2\alpha_1)- \\ -\rho''&\Big[\frac{1}{64}e^4(1-e^2)^2\Big](\sin 2\alpha_2\sin^2\alpha_2-\sin 2\alpha_1\sin^2\alpha_1). \end{aligned} \qquad (6.41)$$

From (6.36) and (6.38)

$$\operatorname{tg}\alpha = \frac{t}{\sin u} = \operatorname{ctg} u\cos A,$$

but

$$\operatorname{ctg} u = \frac{\operatorname{ctg} B}{\sqrt{1-e^2}},$$

where B is geodetic latitude, therefore:

$$\operatorname{tg}\alpha = \frac{\operatorname{ctg} B\cos A}{\sqrt{1-e^2}}. \qquad (6.42)$$

Formula (6.42) is applicable to any point.

For P_1

$$\operatorname{tg}\alpha_1 = \frac{\operatorname{ctg} B_1\cos A_1}{\sqrt{1-e^2}}. \qquad (6.42')$$

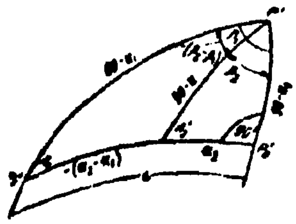


Fig. 75.

For P_2

$$\operatorname{tg}\alpha_2 = \frac{\operatorname{ctg} B_2 \cos A_2}{\sqrt{1-e^2}}. \qquad (6.42'')$$

Angles β_1 and β_2 in accordance with (6.37) are determined from equalities:

$$\left.\begin{aligned}\operatorname{tg}\beta_1 &= e\operatorname{ctg}\alpha_1\\ \operatorname{tg}\beta_2 &= e\operatorname{ctg}\alpha_2\end{aligned}\right\}. \qquad (6.43)$$

Geometric values of α_1, α_2, β_1 and β_2 are shown in Fig. 75.

In resolution of direct problem it is necessary to determine B_2 and A_2.

From (6.36) and (6.38), omitting details of computations,

$$\operatorname{tg} u = \frac{1}{e}\sqrt{1-e^2}\sin\beta$$

or

$$\operatorname{tg} B = \frac{1}{e}\cdot\frac{\sqrt{1-e^2}}{\sqrt{1-e^2}}\sin\beta. \qquad (6.44)$$

Formula (6.44) is applicable for any point on a spheroid.

For points P_1 and P_2 we have:

$$\operatorname{tg} B_1 = \frac{1}{e}\frac{\sqrt{1-e^2}}{\sqrt{1-e^2}}\sin\beta_1,$$
$$\operatorname{tg} B_2 = \frac{1}{e}\frac{\sqrt{1-e^2}}{\sqrt{1-e^2}}\sin\beta_2$$

hence:

$$\operatorname{tg} B_2 = \operatorname{tg} B_1\frac{\sin\beta_2}{\sin\beta_1}. \qquad (6.45)$$

From (6.27)

$$\frac{1}{\cos^2 A} = \frac{\cos^2 u}{\cos^2 u - e^2} = 1 + \operatorname{tg}^2 A$$

or:

$$\operatorname{tg} A = \frac{e}{\sqrt{\cos^2 u - e^2}}. \qquad (6.46)$$

From (6.46) and (6.36) it follows that:

$$\operatorname{tg} A = \frac{e}{\sqrt{1-e^2}}\frac{1}{\sin u},$$

hence in points P_1 and P_2

$$\left.\begin{aligned}\operatorname{tg} A_1 &= \frac{e}{\sqrt{1-e^2}}\frac{1}{\sin u_1}\\ \operatorname{tg} A_2' &= \frac{e}{\sqrt{1-e^2}}\frac{1}{\sin u_2}\end{aligned}\right\}. \qquad (6.46')$$

Consequently,

$$\operatorname{tg} A_2' = \operatorname{tg} A_1 \frac{\sin \alpha_1}{\sin \alpha_2}. \tag{6.47}$$

while:

$$A_2 = A_2' \pm 180°.$$

By the formulas of Professor A. M. Virovets straight geodetic problem is resolved in the following sequence:

$$1. \quad c = \cos u_0 = \frac{\cos B_1}{\sqrt{1 - e^2 \sin^2 B_1}} \sin A_1$$

$$2. \quad \operatorname{tg} \alpha_1 = \frac{\operatorname{ctg} B_1}{\sqrt{1 - e^2}} \cos A_1.$$

$$3. \quad (\alpha_2 - \alpha_1)'' = p s + q \sin(\alpha_2 - \alpha_1) \cos(\alpha_2 + \alpha_1) + r (m_2 - m_1).$$

$(\alpha_2 - \alpha_1)$ is obtained by means of consecutive approximations.

$$4. \quad \alpha_2 = \alpha_1 + (\alpha_2 - \alpha_1),$$

$$5. \quad \operatorname{tg} \beta_1 = \cos u_0 \operatorname{ctg} \alpha_1,$$

$$6. \quad \operatorname{tg} \beta_2 = \cos u_0 \operatorname{ctg} \alpha_2,$$

$$7. \; l = (\beta_2 - \beta_1) + q' \cos u_0 (\alpha_2 - \alpha_1) + r' c \sin(\alpha_2 - \alpha_1) \cos(\alpha_2 + \alpha_1),$$

$$8. \quad \operatorname{tg} B_2 = \operatorname{tg} B_1 \frac{\sin \beta_2}{\sin \beta_1},$$

$$9. \quad \operatorname{tg} A_2' = \operatorname{tg} A_1 \frac{\sin \alpha_1}{\sin \alpha_2}.$$

In these formulas following designations are taken:

$$\left.\begin{aligned}
p &= -\frac{\rho''}{a}\left\{1 + \frac{1}{4} e^2 (1 + c^2) + \frac{1}{64} e^4 (7 + 10c^2 + 7c^4) + \;\ldots\right\} \\
q &= -\rho''\left\{\frac{1}{4} e^2 (1 - c^2) + \frac{1}{64} e^4 (7 + 2c^2 - 9c^4) + \ldots\right\} \\
r &= -\rho''\left\{\frac{1}{64} e^4 (1 - c^2)^2 + \;\ldots\right\} \\
q' &= \frac{1}{2} e^2 + \frac{1}{16} e^4 (1 + c^2) + \;\ldots \\
r' &= -\rho''\left\{\frac{1}{16} e^4 (1 - c^2) + \;\ldots\right\} \\
m_1 &= \sin 2\alpha_1 \sin^2 \alpha; \quad m_2 = \sin 2\alpha_2 \sin^2 \alpha_2
\end{aligned}\right\}. \tag{6.48}$$

Given formulas in somewhat different way were first obtained by Professor A. M. Virovets in 1935.[1] If they are compared with Bessel formulas, then it is possible to expose the following coincidences:

$$1. \quad \sigma = -(\alpha_2 - \alpha_1),$$

$$2. \quad \omega = -(\beta_2 - \beta_1),$$

$$3. \quad \alpha_1 = 90° - \varphi,$$

$$4. \quad \sin m = \cos u_0, \text{ т. е. } m = 90° - u_0,$$

$$5. \quad p = -\frac{a}{b},$$

$$6. \quad \varphi = \alpha'.$$

The more essential is the fact that series by Bessel are constructed by powers

[1] A. M. Virovets. Resolution of direct geodetic problem for significant distances between geodesic points. Journal "Geodesist," No. 4, 1935, p. 16-21.

of $k^2 = e'^2 \sin^2 u_0$, whereas by A. M. Virovets by powers of e^2. But this circumstance has more fundamental than practical value, since the difference in series is beyond the limits of accuracy for unknown values, which is considered in calculations.

For calculation by the formulas of A. M. Virovets detailed tables and instructions were composed by the author in 1955. Tables contain natural meaning of values $\frac{1}{W}$, p, q, q', r' and m. Instructions and tables are published in works of TsNIIGAiK, issue No. 93. In the same text resolution of inverse problem which will be discussed in § 37 was developed.

Investigations in recent years, among them those of an author show that in application of formulas of A. M. Virovets and tables for them it is expedient to make a number of changes for the purpose of excluding negative arcs as, for instance, $-(\alpha_2 - \alpha_1)$, $-(\beta_2 - \beta_1)$, i.e., a formula should be transformed for them. This requirement leads to another composition of formulas. It is possible that it is better to have tables by argument cos u_0. These problems require special investigation with which the author is occupied at present.

Simple comparison of formulas of Bessel and A. M. Virovets leads to a thought that they are invariants, since in basis of their derivation fundamental equation of geodesic is assumed. It is necessary to consider that other variants of these formulas are possible, which will be obtained by integration of basic differential equations (6.3) by means of replacement of a variable. In the following paragraph results of investigations are presented of certain foreign geodesists concerning this question.

§ 34. LEVALLOIS-DUPUY METHOD

Above we had:

$$\left.\begin{aligned} ds &= a\sqrt{1-e^2\cos^2 u}\,d\sigma \\ d\lambda &= \sqrt{1-e^2\cos^2 u}\,d\omega \end{aligned}\right|. \tag{6.3}$$

Replacing e^2 by e'^2 in the formula:

$$e^2 = \frac{e'^2}{1+e'^2},$$

we obtain:

$$ds = b\sqrt{1+e'^2\sin^2 u}\,d\sigma.$$

From triangle EPP_1 (Fig. 76)

$$\sin u = \sin\sigma\cos A_0. \tag{6.49}$$

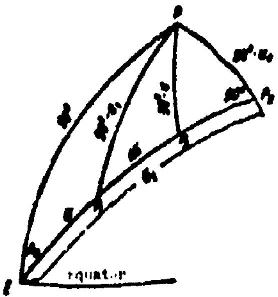


Fig. 76.

therefore:

$$ds = b\sqrt{1+e'^2\cos^2 A_0\sin^2\sigma}\,d\sigma. \qquad (6.49')$$

Here σ is a spherical distance from equator to current point on auxiliary sphere along the arc of great circle, corresponding to geodesic on a spheroid:

$$\sqrt{1+e'^2\cos^2 A_0\sin^2\sigma} = 1+\frac{e'^2}{2}\cos^2 A_0\sin^2\sigma-\frac{e'^4}{8}\cos^4 A_0\sin^4\sigma+ + \frac{1}{16}e'^6\cos^6 A_0\sin^6\sigma - \ \dots$$

or:

$$s = b\left\{\int_{\sigma_1}^{\sigma_2}\left(1+\frac{e'^2}{2}\cos^2 A_0\sin^2\sigma-\frac{e'^4}{8}\cos^4 A_0\sin^4\sigma+\frac{1}{16}e'^6\cos^6 A_0\sin^6\sigma\right)d\sigma\right\}$$

Let us designate:

$$e'^2\cos^2 A_0 = k^2,$$

$$\frac{1}{2}\int_0^\sigma \sin^2\sigma\,d\sigma = J_2,$$

$$\frac{1}{8}\int_0^\sigma \sin^4\sigma\,d\sigma = J_4,$$

$$\frac{1}{16}\int_0^\sigma \sin^6\sigma\,d\sigma = J_6.$$

Consequently,

$$\frac{s}{b} = \sigma + k^2\Delta J_2 - k^4\Delta J_4 + k^6\Delta J_6 + \ \dots \qquad (6.50)$$

where

$$\Delta J_i = \frac{1}{n}\int_{\sigma_1}^{\sigma_2}\sin^i\sigma\,d\sigma \quad (n = 2,\ 8,\ 16,\ \dots;\ i = 2,\ 4,\ 6,\ \dots),$$

J_2, J_4, J_6 are Wallace integrals,

ΔJ_2, ΔJ_4 and ΔJ_6 are differences of these integrals.

Tables of Wallace integrals were composed and are used in France. They are simpler than Legendre tables of elliptic integrals, since they are composed on one argument. Tables enable to obtain correction terms of formula (6.50) with precision of $0''.0001$. In USSR these tables have not been used up till now.

Integrals $\int_{\sigma_1}^{\sigma_2}\sin^n\sigma\,d\sigma$ $(n = 1,\ 2,\ 3\ \dots)$ can be expressed by formulas:

$$\left.\begin{aligned}
&\int_{\sigma_1}^{\sigma_2} \sin^2\sigma\, d\sigma = \frac{1}{2}(\sigma_2-\sigma_1) - \frac{1}{2}\sin(\sigma_2-\sigma_1)\cos(\sigma_2+\sigma_1)\\
&\int_{\sigma_1}^{\sigma_2} \sin^4\sigma\, d\sigma = \frac{3}{8}(\sigma_2-\sigma_1) - \frac{1}{2}\sin(\sigma_2-\sigma_1)\cos(\sigma_2+\sigma_1) +\\
&+\frac{1}{16}\sin 2(\sigma_2-\sigma_1)\cos 2(\sigma_2+\sigma_1)
\end{aligned}\right\}. \qquad (6.51)$$

Therefore

$$\frac{s}{b} = A\sigma - B\sin\sigma\cos(2\sigma_1+\sigma) + C\sin 2\sigma\cos(4\sigma_1+2\sigma). \qquad (6.52)$$

Here:

$$\left.\begin{aligned}
&\sigma = \sigma_2-\sigma_1\\
&A = 1 + \frac{k^2}{4} - \frac{3}{64}k^4 + \dots\\
&B = \frac{k^2}{4} - \frac{k^4}{16} + \dots\\
&C = \frac{k^4}{128} + \dots
\end{aligned}\right\}. \qquad (6.53)$$

Formula (6.52) can be used in resolution of the inverse problem.

As before we will designate:

$$\alpha = \frac{\rho''}{bA},\quad \beta = \rho''\frac{B}{A},\quad \gamma = \rho''\frac{C}{A}.$$

then from (6.52)

$$\sigma = \frac{1}{\alpha}\left(s - \beta\sin\sigma\cos(2\sigma_1+\sigma) + \gamma\sin 2\sigma\cos(4\sigma_1+\sigma)\right). \qquad (6.54)$$

In other words, we arrived at Bessel formula.

From (6.29), replacing:

$$\sin A = \frac{\sin A_0}{\cos u},$$

$$d\omega = \frac{\sin A_0}{\cos^2 u}\, d\sigma$$

and designating:

$$U = \sqrt{1 - e^2\cos^2 u},$$

we obtain

$$\omega - l = \int_{\sigma_1}^{\sigma_2}\left(1 - \frac{1}{U}\right)\frac{\sin A_0}{\cos^2 u}\, d\sigma.$$

But

$$1 - \frac{1}{U} = \frac{1}{2}e^2\cos^2 u + \frac{1}{8}e^4\cos^4 u + \frac{1}{16}e^6\cos^6 u,$$

and

$$\cos^2 u = 1 - \cos^2 A_0\sin^2\sigma,$$

therefore

$$l = m - \sin A_0 \left\{ \int_{\sigma_1}^{\sigma_2} A' d\sigma - B' \cos^2 A_0 \int_{\sigma_1}^{\sigma_2} \sin^2\sigma d\sigma + C' \cos^4 A_0 \int_{\sigma_1}^{\sigma_2} \sin^4\sigma d\sigma \right\}.$$

Or, accomplishing term by term integration, we obtain:

$$l = m - \sin A_0 \left\{ A'\sigma - B' \cos^2 A_0 \Delta J_2 + C' \cos^4 A_0 \Delta J_4 \right\}. \qquad (6,55)$$

Here

$$A' = \frac{e'^2}{2} + \frac{e'^4}{8} + \frac{e'^6}{16},$$
$$B' = \frac{e'^4}{8} + \frac{e'^6}{8},$$
$$C' = \frac{e'^6}{16}.$$

Considering former designation $k^2 = e'^2 \cos^2 A_0$, we have:

$$l = m - \sin A_0 (A'\sigma - \bar{B}k^2 \Delta J_2 + \bar{C}k^4 \Delta J_4), \qquad (6,55')$$

where

$$\bar{B} = \frac{B'}{e'^2}, \quad \bar{C} = \frac{C'}{e'^4}.$$

Formulas (6,55) and (6,55') are obtained in such a form by French geodesist Levallois and Dupuy [?].[1] Here, as compared to Bessel and A. M. Virovets formulas, a new item is the introduction of Wallace integrals. With availability of Wallis tables of integrals, this method can be used on a par with Bessel and Virovets methods.

§ 35. HELMERT METHOD

From preceding account it is clear that series for s and l by Bessel and Levallois-Dupuy [?] work on ascending powers of $k = e' \sin u_0 = e' \cos A_0$. Therefore from the point of view of convergence these series are equivalent.

Helmert, for acceleration of convergence of series and convenience of solution of inverse problem introduced parameter k_1 instead of k, which he determined the following manner.

Let us assume that

$$\operatorname{tg} \varepsilon = k \qquad (6,56)$$

then:

$$\operatorname{tg}^2 \frac{\varepsilon}{2} = k_1. \qquad (6,57)$$

[1]Geodetic Bulletin, No. 48, 1959, p. 30-38.

or:

$$k_1 = \frac{1}{4}k^2 + \frac{1}{8}k^4 + \frac{5}{64}k^6 + \frac{7}{128}k^8. \qquad (6.57')$$

With Helmert parameter the differential of arc of geodesic from (6.5) will take form of:

$$ds = a\frac{\sqrt{1-e^2}}{1-k_1}\sqrt{1+k_1^2+2k_1\cos 2\sigma'}\,d\sigma'. \qquad (6.58)$$

Integral of lengths will be:

$$s = a\frac{\sqrt{1-e^2}}{1-k_1}\int_0^{\sigma'}\sqrt{1+k_1^2+2k_1\cos 2\sigma'}\,d\sigma'. \qquad (6.59)$$

With respect to geometric value of σ' it should be noted that the arc of the great circle on a sphere and a geodesic on an ellipsoid have two characteristic points, point of intersection with equator, in which the azimuth is equal to A_0, and point of intersection with meridian, where azimuth is equal to 90^0: the later one has maximum latitude along the entire extent of arc.

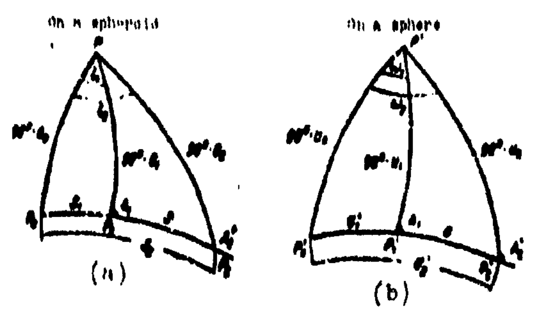


Fig. 77.

In our designations the maximum latitudes u_0 are on a sphere, and B_0 is on a spheroid.

Arc σ', as in Bessel method, is counted from point P_0 (Fig. 77a and b).

Term by term integration factorized in binomial series of subintegral expression (6.59) gives:

$$s = b\frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\sigma' + \left(\frac{1}{2}k_1 - \frac{3}{16}k_1^3\right)\sin 2\sigma' - \frac{1}{16}k_1^2\sin 4\sigma' + \right.$$
$$\left. + \frac{1}{48}k_1^3\sin 6\sigma' + \ldots\right\} \qquad (6.60)$$

Equation (6.60) gives distance of point P_1 from point P_0 along the geodesic. If we apply (6.60) to two points of geodesic, we will find the shortest distance between them, the geodesic arc. Leaving designation s for this arc, we obtain by (6.60):

$$s = b\frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\sigma + \left(k_1 - \frac{3}{8}k_1^3\right)\cos 2\sigma_m\sin\sigma - \frac{1}{8}k_1^2\cos 4\sigma_m\sin 2\sigma + \right.$$
$$\left. + \frac{1}{24}k_1^3\cos 6\sigma_m\sin 3\sigma\right\}. \qquad (6.60')$$

where:

$$\left.\begin{aligned} \sigma_m &= \frac{1}{2}(\sigma_1' + \sigma_2') \\ \sigma &= (\sigma_2' - \sigma_1') \end{aligned}\right\} . \qquad (6,61)$$

In resolution of the direct problem σ is usually unknown. Helmert means of artificial transformations with the help (6,60) determines σ. Schematically this derivation consists in the following.

Will designate:

$$s_1' = \frac{s_1(1-k_1)}{b(1+\frac{1}{4}k_1^2)},$$

$$s_2' = \frac{s_2(1-k_1)}{b(1+\frac{1}{4}k_1^2)}.$$

Consequently,

$$s' = s_2' - s_1' = \frac{s(1-k_1)}{b(1+\frac{1}{4}k_1^2)}.$$

For current point of geodesic Helmert finds:

$$\sigma_i' = s_i' - \left(\frac{1}{2}k_1 - \frac{9}{32}k_1^2\right)\sin 2s_i' + \frac{5}{16}k_1^2 \sin 4s_i' - \; . \; . \; . \; ,$$

where σ_1' and s_1' are counted from points P_0' and P_0. Taking differences of σ_1 and σ_{1-1}', we obtain spherical distance between points P_1' and P_2'.

$$\sigma_2' - \sigma_1' = \sigma' = s' - \left(k_1 - \frac{9}{16}k_1^2\right)\cos 2s_m' \sin s' + \frac{5}{8}k_1^2 \cos 4s_m' \sin 2s'. \qquad (6,62)$$

Here

$$\left.\begin{aligned} s' &= s_2' - s_1' \\ s_m' &= \frac{1}{2}(s_2' + s_1') \end{aligned}\right\} . \qquad (6,62')$$

Formula (6,62) is used for resolution of direct geodetic problem; where, as can be seen from (6,62), for the determination of σ' the method of approximations is not required.

For differences of longitudes by means of substitution of variables u and ω from (6,5) we obtain:

$$dl = d\lambda - \cos u_0 \left(1 - \sqrt{1-e^2}\right) \frac{\sqrt{1 + k^2 \cos^2 \sigma'}\, d\sigma'}{1 - \sin^2 u_0 \cos^2 \sigma'}. \qquad (6,63)$$

Omitting details of computations of integration, from (6,63) for l with Helmert parameter we have:

$$l = \lambda - \frac{1}{2}e^2 \cos u_0 \left\{\left(1 + n - \frac{1}{2}k_1 - \frac{1}{4}k_1^2\right)\sigma' - \right.$$
$$\left. - \frac{1}{2}k_1 \cos 2\sigma_m' \sin \sigma' + \frac{1}{8}k_1^2 \cos 4\sigma_m' \sin 2\sigma'\right\}. \qquad (6,64)$$

In (6,64) following is taken:

$$\sigma'_m = \frac{1}{2}(\sigma'_2 + \sigma'_1); \quad l = L_2 - L_1;$$
$$\sigma' = \sigma'_2 - \sigma'_1; \quad \lambda = \lambda_2 - \lambda_1.$$

Sequence of resolution of the direct problem according to Helmert method is approximately the same as the Bessel method.

1. Calculation of a given latitude by the formula:

$$\operatorname{tg} u_1 = \sqrt{1 - e^2} \operatorname{tg} B_1.$$

2. Calculation of auxiliary values u_0, σ'_1 and λ_1 from solution of spherical triangle $P'_0 P' P'_1$ by the formulas:

$$\left.\begin{aligned} \operatorname{tg} \sigma'_1 &= -\frac{\cos A_1}{\operatorname{tg} u_1} \\ \cos u_0 &= \cos u_1 \sin A_1 \\ \operatorname{tg} \lambda_1 &= \frac{\operatorname{tg} \sigma'_1}{\cos u_0} \end{aligned}\right\}. \qquad (6.65)$$

For control, the following formulas should be used:

$$\left.\begin{aligned} \cos \lambda_1 &= \frac{\operatorname{tg} u_1}{\operatorname{tg} u_0} \\ \sin u_0 &= \frac{\sin u_1}{\cos \sigma'_1} \end{aligned}\right\}. \qquad (6.65')$$

3. Calculation of σ' by the formula (6.62). Where this calculation is conducted in the following sequence.

First s' is obtained in degrees by the formula:

$$(s')^\circ = \frac{s}{bP_1},$$

then:

$$2(s'_1)^\circ = 2(\sigma'_1)^\circ + \bar{k}_1 \sin 2\sigma'_1 + P_2 \sin 4\sigma'_1.$$

Here:

$$\bar{k}_1 = k_1 \rho^\circ,$$

$$P_1 = \frac{1}{\rho^\circ} \frac{1 + \frac{1}{4}\frac{\bar{k}_1^2}{\rho^2}}{1 - \frac{\bar{k}_1}{\rho}}, \quad P_2 = -\frac{1}{6\rho} \bar{k}_1^2.$$

Having s' and s'_1, we obtain s'_2 and s'_m.

After that apply formula (6.62) is used in the form:

$$\sigma' = s' + Q_1 \cos 2s'_m \sin s' + Q_2 \cos 4s'_m \sin 2s' + Q_3 \cos 6s'_m \sin 3s';$$

coefficients Q_1, Q_2, Q_3 are equal:

$$Q_1 = -\left(\bar{K}_1 - \frac{9}{16} \cdot \frac{\bar{K}_1^2}{\rho^2}\right),$$

$$Q_2 = \frac{5}{8} \frac{\bar{K}_1^2}{\rho},$$

$$Q_3 = -\frac{29}{48} \frac{\bar{K}_1^2}{\rho^2}.$$

Coefficients $\bar{K}_1$, P_2, Q_1, Q_2 and Q_3 are taken from the tables.[1] Tables for $\bar{K}_1$ are composed for each ellipsoid and P_2, Q_1, Q_2 and Q_3 are taken as functions of $\bar{K}_1$. Consequently, with availability of table for $\bar{K}_1$ tables for P_2, Q_1, Q_2 and Q_3 are applicable for any reference-ellipsoid.

4. Calculation of latitude B_2 and back azimuth A_2.

After finding σ' calculate σ_2' by the formula:

$$\sigma_2' = \sigma_1' + \sigma'.$$

It is for the determined with control:

$$\left.\begin{aligned} \operatorname{tg} \lambda_2 &= \frac{\operatorname{tg} \sigma_2'}{\cos u_0} \\ \cos u_2 &= \frac{\sin \sigma_2'}{\sin \lambda_2} \\ \cos A_2 &= -\operatorname{tg} \sigma_2' \operatorname{tg} u_0 \\ \operatorname{tg} B_2 &= \frac{\operatorname{tg} u_2}{\sqrt{1-e^2}} \end{aligned}\right\}. \qquad (6.66)$$

5. Calculation of differences of longitudes by the formula (6.64). For practical application this formula is recommended to be transformed thus:

$$l = \lambda - \cos u_0 \left(R_1 \sigma' - R_2 \cos 2\sigma_m' \sin \sigma' + R_3 \cos 4\sigma_m' \sin 2\sigma' + \ldots\right), \qquad (6.64')$$

where:

$$\left.\begin{aligned} R_1 &= \frac{e^2}{2}\left(1 + n - \frac{1}{2} \frac{\bar{K}_1}{\rho} - \frac{1}{4} \frac{\bar{K}_1^2}{\rho^2}\right) \\ R_2 &= \frac{e^2}{2} \frac{\bar{K}_1}{2} \\ R_3 &= \frac{e^2}{16} \frac{\bar{K}_1^2}{\rho} \end{aligned}\right\}. \qquad (6.67)$$

Coefficients R_1, R_2 and R_3 are taken from tables and also by argument $\bar{K}_1$. By the formula (6.64') l is obtained in degree measure.

In resolution of geodetic problem difficulties arise in determination of the sign and quarters for auxiliary values λ and σ'. Table 6 gives orientation for determination of quarters.

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[1]Shown tables are composed [illegible] dimensions of Hayford ellipsoid by German geodesist Baudmuller (?) and are found in his work on "Formulas and tables for calculation of direct and inverse geodetic problems for long distances for international ellipsoid." Munich, 1955.

Table [illegible]

A quarter	a	σ' and λ
I	>0	$270° < \sigma', \lambda < 360°$
	<0	$180° < \sigma', \lambda < 270°$
II	>0	$0 < \sigma', \lambda < 90°$
	<0	$90° < \sigma', \lambda < 180°$

Methods presented in preceding paragraphs do not exhaust all possibilities for resolution of problem by equations (6.5). They differ essentially by a method of integration of equations (6.3) and modification of series for σ and l by means of introduction of parameters k or k_1. Differences in methods of Bessel, A. M. Virovets, Levallois-Dupuy (?) and Helmert with respect to accuracy and speed of resolution can appear imperceptible, if all possibilities for simplification of calculations, taking into account peculiarities and structure of formulas are fully used.

On a basis of considered in this chapter methods there is Clairaut's (?) equation and ensuing from it position, that for every geodesic on a spheroid there corresponds a definite arc of great circle on a sphere of arbitrary radius and corresponding points of this arcs of latitude are equal to reduced latitudes, and azimuths to azimuths of geodesics. In this interpretation, first determined by Bessel, the matter is not about the presentation of a spheroid on a sphere, as it is incorrectly treated by certain authors, and all the more so not about spherical resolution of a problem, but about a very important interpretation of geometric properties of geodesic on a spheroid. The shown property of geodesic leads to the fact that, if one were to connect two pairs of mutually corresponding points on a spheroid and a sphere with northern poles, a mutually corresponding geodetic and spherical polar triangles will be obtained, which will be right-angle, when azimuth in one of a pair of points is equal to 90°.

From the point of view of rapidity of convergence of series and uniformity of resolution direct and inverse geodetic problems by Helmert method should be given preference to one before the other, if sufficiently detailed tables for coefficients $\overline{K}_1$, P_2, Q_1, Q_2, Q_3, R_1, R_2 and R_3 are available.

§ 36. INVERSE GEODETIC PROBLEM

Transmission of geodetic coordinates for distances of thousand kilometers up till now is used in world practice of geodetic work only in particular problems.

Therefore methods of resolution of this problem presented in preceding paragraphs have so far only theoretical value, which allows deeper study of geometry of the surface of terrestrial spheroid.

However resolution of inverse geodetic problem - determination of distance and azimuths according to coordinates of two points, has both theoretical, and practical value. Development of rocket technology, radar navigation, international broadcasting and air navigation requires determination of distances between very distant points of earth's surface and directions between these points. It is true, now these requirements can be satisfied by methods of approximation, but in time, when a single world geodetic net will be created, exact resolutions of this problem will be needed.

For the last 10-15 years in USSR and abroad significant scientific investigations were conducted and published in series of works in an area of resolution of this problem of spheroidal geodesy. In these works, part of which will be considered further, new methods of resolution of the problem were offered, they were investigated and evaluated on the basis of contemporary requirements for ideas and methods, by the greatest geodesists of the past century.

In connection with development of computer technology a necessity appeared for creation of methods, useful for application of electronic-computers. In this case the more important is not quantity arithmetical actions, but convenience of programming. In other words, in spheroidal geo y necessity arose for creation of methods and formulas for resolution of geodetic problems with the help of new means of computer technology. This eliminates necessity for special geodetic tables, all resolution is reduced to composition of program for the computer.

But, of course, from this it does not follow that it is yet necessary to completely depart from former approaches. Well developed former methods will be used for a long time and in certain particular cases can be the most practical.

§ 37. INVERSE PROBLEM BY THE BESSEL AND A. M. VIROVETS FORMULAS

Bessel did not leave any instructions for resolution of inverse geodetic problem. The method of resolution of geodetic problem was developed by Bessel under the following circumstances. In 1831-1834 Bessel and General I. Ya. Bayer (?) carried out Prussian measurements between Trunts and Memel', whose characteristic peculiarity was in the fact that it was conducted indirectly in regard to the meridian. Bessel posed a problem; whereby the length of geodesic, astronomical azimuth and latitude of initial point he was to calculate geodesic latitude and azimuth at terminal of

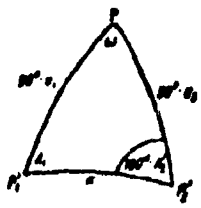

Fig. 78.

the arc (Memel') and to compare the calculated values with those obtained for that point by astronomical observations, of latitude and azimuth. From differences of these values he obtained correction for semiaxis of ellipsoid and compression.

Given below is a method which is a combination of different proposals for application of Bessel formulas to resolution of inverse geodetic problem.

The simplest way of resolution of a given problem is composed of the following actions:

a) from given geodetic latitude B_1 and B_2 convert to corresponding reduced latitudes;

b) taking in the first approximation $\omega = l$, resolve auxiliary spherical triangle $P_1'P'P_2'$ (Fig. 78) and determine approximate values σ_0 and A_{10}

With these σ_0 and A_{10} again calculate ω_1 by the formula

$$\omega_1 = l + \alpha' \sigma_0 \cos u_0.$$

Obtaining ω_1, repeat resolution of triangle $P_1'P'P_2'$ and find more exact value of σ and A_1. Having latest values, calculate anew the ω by a complete formula

$$\omega = l + \cos u_0 (\alpha'\sigma + \beta' \sin\sigma \cos(2\varphi + \sigma)). \qquad (6.23)$$

Such approximations depending upon required accuracy of resolution are made several times, but not more than three. After obtaining precise values σ, A_1, A_2, u_0 and φ by the formula (6.11), calculate s. By this method unknown values can be obtained practically with any degree of accuracy.

Given scheme of resolution of inverse problem is applicable for formulas of Professor A. M. Virovets.

From triangles $P_1'P'P_0'$ and $P_2'P'P_0'$ (Fig. 75) we have:

$$\cos\beta_1 = \operatorname{tg} u_1 \operatorname{ctg} u_0, \qquad (a)$$

$$\cos\beta_2 = \operatorname{tg} u_2 \operatorname{ctg} u_0. \qquad (b)$$

Excluding from these formulas cotangent u_0 and constituting derivative proportion, we obtain:

$$-\operatorname{ctg}\frac{\beta_2+\beta_1}{2} = \operatorname{tg}\frac{\beta_2-\beta_1}{2} \cdot \frac{\sin(u_2+u_1)}{\sin(u_2-u_1)}. \qquad (6.68)$$

Formula (6.68) can be used with the method of approximations. In the first approximation $\beta_2 - \beta_1 = -l$, let us find first approximation for β_1 and β_2 and then by (a) and (b) determine u_0. With these values β_1, β_2 and u_0 calculate second approximation

$$\beta_2-\beta_1=l-q'(\alpha_2-\alpha_1)\cos u_0,$$

where

$$\begin{gathered}\operatorname{tg}\alpha_1=\cos u_0\operatorname{ctg}\beta_1,\\ \operatorname{tg}\alpha_2=\cos u_0\operatorname{ctg}\beta_2,\\ \beta_2=\beta_1+(\beta_2-\beta_1).\end{gathered}$$

Approximations depending upon requirements for accuracy of resolution are repeated 2-3 times.

After obtaining final values α_1, α_2, p, q and r unknown s, A_1 and A_2 are calculated by the formulas:

$$s=\frac{1}{p}\{(\alpha_2-\alpha_1)-q\sin(\alpha_2-\alpha_1)\cos(\alpha_2+\alpha_1)-r(m_2-m_1)\}, \tag{6.69}$$

$$\left.\begin{gathered}\operatorname{tg}A_1=\operatorname{ctg}u_0\operatorname{cosec}\alpha_1\\ \operatorname{tg}A_2'=\operatorname{ctg}u_0\operatorname{cosec}\alpha_2\\ A_2=A_2'\pm 180^\circ\end{gathered}\right\}. \tag{6.70}$$

According to Helmert method inverse problem is resolved in the same sequence, as by the Bessel and A. M. Virovets formulas.

1. Marking on a small-scale map by coordinates the given points with accuracy of up to one degree or more exactly remove from this map the value u_0 and σ_0' the spherical distance between points.

2. Obtain first approximation by these values

$$\lambda_{(1)}=l+(1+n)\frac{e^2}{2}\cos u_0\sigma_0'.$$

Further calculate value λ and $\lambda_{(2)}$.

$$-\operatorname{ctg}\frac{\lambda_2-\lambda_1}{2}=\operatorname{tg}\frac{\Delta\lambda_{(1)}}{2}\frac{\sin(u_1+u_2)}{\sin(u_2-u_1)}, \tag{6.71}$$

$$\left.\begin{gathered}\operatorname{tg}u_0=\frac{\operatorname{tg}u_1}{\cos\lambda_1}=\frac{\operatorname{tg}u_2}{\cos\lambda_2}\\ \cos\sigma_1'=\frac{\sin u_1}{\sin u_0}\\ \cos\sigma_2'=\frac{\sin u_2}{\sin u_0}\\ \sigma'=\sigma_2'-\sigma_1'\end{gathered}\right\}. \tag{6.72}$$

3. Having value of point 2, calculate λ by complete formula

$$\lambda_{(2)}=l+\cos u_0(R_1\sigma'-R_2\cos 2\sigma_m'\sin\sigma'+R_3\cos 4\sigma_m'\sin 2\sigma'). \tag{6.73}$$

Calculation by the formula (6.73) is repeated if great accuracy of resolution is required. However in overwhelming majority of cases this approximation is sufficient

4. Having λ, calculate unknown A_1, A_2 and s by the formulas

$$\operatorname{ctg} A_1 = -\operatorname{tg} u_0 \sin \sigma'_1,$$
$$\operatorname{ctg} A_2 = \operatorname{tg} u_0 \sin \sigma'_2,$$
$$\sigma = \sigma' + Q_0 \cos 2\sigma'_m \sin \sigma' + P_2 \cos 4\sigma'_m \sin 2\sigma',$$

Where:

$$Q_0 = \left(\bar{k}_1 - \frac{3}{8}\frac{\bar{k}_1^2}{\rho}\right),$$
$$s = \rho'' b P_1.$$

Method of approximations is the most universal in resolution of inverse problems for long distances: it is easy to limit quantity of approximations, considering that where $s \approx 10{,}000$ km first approximation $(\lambda - \omega)$ gives error in a less than [illegible] km, second up to 100 m and third to 0.3 m. In other words, third approximation in practice is fully sufficient for precise resolutions. In approximation calculations it is sufficient to do just the second approximation.

§ 58. METHODS OF REDUCTION OF QUANTITY OF APPROXIMATIONS

As can be seen from the above, in resolution of inverse problem the main factor is finding λ a difference of longitudes on auxiliary sphere. For reduction of approximations in determination of λ both graphic, and analytic methods can be used.

From graphic method, first of all use of maps of different scales is recommended on which the given points are marked by coordinates. On these maps can be found approximate values of s, A_1, A_2, and, consequently, u_0 it is then possible to proceed immediately with calculation of second approximation.

It is possible also to use the following graphic method.

On tracing paper, at determined scale draw a bundle of geodesics and lines $\sigma' =$ const; on the same scale on usual drafting paper draw a graticule. On the graticule place two given points. For determination of u_0 and σ'_1 put graticule on a tracing paper in such a manner that image of equators coincides. After that determine approximately the position of the geodesic, passing through the two given points, and thus find u_0, $(\sigma'_1)_0$, $(\sigma'_2)_0$ and σ'_0. By these data approximations by the formula (6.73) are carried out.

Analytic methods of acceleration of approximations are more universal and possess great possibilities. Let us consider some of them.

We will convert second equation (6.3), replacing in it the reduced latitude of geodesic.

We have:

$$d\omega = dlV,\qquad (6,74)$$

where

$$V = \sqrt{1+e'^2\cos^2 B}.$$

Applying to integral (6,74) Lagrange theorem about mean value of function, we obtain:

$$\omega = lV_m.\qquad (6,74')$$

Value V_m can be obtained differently: it is possible to approximately take it as equal:

$$\text{a)}\ V'_m = \sqrt{1+e'^2\cos^2 B_m},\quad B_m = \frac{1}{2}(B_1+B_2),$$
$$\text{b)}\ V''_m = \frac{1}{2}(V_1+V_2),\quad V_{1,2} = \sqrt{1+e'^2\cos^2 B_{1,2}},$$
$$\text{c)}\ V'''_m = \frac{1}{6}(V_1+4V'_m+V_2).$$

Then we will obtain three equivalent approximate formulas for calculation of difference of longitudes on auxiliary sphere:

$$\left.\begin{aligned}\omega' &= lV'_m\\ \omega'' &= lV''_m\\ \omega''' &= lV'''_m\end{aligned}\right\}.\qquad (6,75)$$

Formulas (6,75) are applicable for distances on the order of 1000-2000 km. For these distances they give good first approximation, which for corresponding requirements for accuracy of calculations is fully sufficient. For great distances these formulas are applicable for calculation of first approximation.

Fig. 79.

In formula (6,74) drop the terms with e'^4

$$\omega^{IV} = l + k_0\frac{e'^2}{2}\cos u_0 \sin\sigma',\qquad (6,75')$$

where:

$$k_0 = \frac{\sigma'}{\sin\sigma'},\qquad (6,76)$$

σ' – approximate value σ.

From triangle $P'_1P'P'_2$ (Fig. 79)

$$\sin A_1 = \frac{\sin\omega}{\sin\sigma'}\cos u_2 = \frac{\sin l\cos u_2}{\sin\sigma'}.$$

Further

$$\cos u_0 = \cos u_1 \sin A_1$$

or

$$\cos u_0 = \frac{\sin l \cos u_1 \cos u_2}{\sin \sigma'},\qquad (6.77)$$

Substituting (6.77) for (6.75'), we obtain

$$\omega^{IV} \approx l + k_0 \frac{e^2}{2} \cos u_1 \cos u_2 \sin l. \qquad (6.78)$$

Coefficient k_0 has values, given in Table 7.

Table 7

σ'	k_0	σ'	k_0	σ'	k_0
0°	1,00	60°	1,21	110°	2,04
10	1,00	70	1,30	120	2,42
20	1,02	80	1,42	130	2,96
30	1,05	90	1,57	140	3,80
40	1,09	100	1,77	150	4,30
50	1,14				

As can be seen from Table 7, for distances up to [illegible] km during approximate calculations coefficient $k_0 \approx 1$. For distances:

from 6000 to 8000 km $k_0 \approx 1,5$,
" 8000 " 10000 " $k_0 \approx 2,0$,
" 10000 " 12000 " $k_0 \approx 2,5$,
" 12000 " 13000 " $k_0 \approx 3,0$.

Spherical distance σ for determination of k_0 must be known very approximately, with accuracy up to 1°, for which it is possible to use a map.

For distances up to 6000 km formula (6.78) can be simplified by means of replacement of given latitude by geodesics and to take $e^2 \approx 2\alpha$ (here α — compression of ellipsoid)

$$\omega^{IV} = l + \alpha \sin l \cos B_1 \cos B_2, \qquad (6.78')$$

where:

$$\alpha\rho'' \cong \frac{e^2\rho''}{2} \cong 667'',$$

or

$$\omega^{IV} = l + 667'' k_0 \sin l \cos B_1 \cos B_2. \qquad (6.78'')$$

In Table 8 data are given, which presents the accuracy of formulas (6.75) and (6.78).

Table 8

s, km	A_m^0	B^0 / l^0	ω'	ω''	ω'''	ω^{IV}	$\omega_{accurate}$
2500	57°	8°/42°	−41°11'23"	11'24"	10'50"	11'09"	−42°11'13"
4000	55	30/10	10 00 40	00 43	00 30	00 31	10 00 33
8000	53	31/155	−155 42 12	42 50	44 27	33 02	−155 32 48
10000	11	00/105	105 34 08	24 05	30 55	26 26	105 27 20

From this table it follows that formula (6.78') gives better first approximation. Therefore it should be used for resolution of inverse problem for long distances, especially during approximation calculations.

Having obtained ω, it is possible to calculate A_1, A_2' and σ' by different groups of formulas.

First group of formulas:

$$\left.\begin{aligned}
\operatorname{ctg} A_1 &= \operatorname{tg} u_2 \cos u_1 \operatorname{cosec} \omega - \sin u_1 \operatorname{ctg} \omega \\
\operatorname{ctg} A_2' &= \sin u_2 \operatorname{ctg} \omega - \operatorname{tg} u_1 \cos u_2 \operatorname{cosec} \omega \\
\operatorname{ctg} \sigma_1' &= \frac{\cos A_1}{\operatorname{tg} u_1} \\
\operatorname{ctg} \sigma_2' &= \frac{\cos A_2'}{\operatorname{tg} u_2} \\
\sigma' &= \sigma_2' - \sigma_1'
\end{aligned}\right\}. \tag{6.79}$$

Second group of formulas:

$$\left.\begin{aligned}
x_1 &= \cos u_1 \sin u_2 - \sin u_1 \cos u_2 \cos \omega \\
x_2 &= \cos u_1 \sin u_2 \cos \omega - \sin u_1 \cos u_2 \\
y_1 &= \cos u_2 \sin \omega \\
y_2 &= \cos u_1 \sin \omega \\
\operatorname{tg} A_1 &= \frac{y_1}{x_1} \\
\operatorname{tg} A_2' &= -\frac{y_2}{x_2} \\
\sin \sigma &= \frac{x_1}{\cos A_1} = \frac{y_1}{\sin A_1}
\end{aligned}\right\}. \tag{6.80}$$

Third group of formulas:

$$\left.\begin{aligned}
u_m &= \frac{1}{2}(u_1 + u_2) \\
\Delta u &= u_2 - u_1 \\
A_m &= \frac{1}{2}(A_1 + A_2 \pm 180^\circ) \\
\Delta A &= (A_2 - A_1 \pm 180^\circ) \\
\operatorname{tg} A_m &= \operatorname{tg} \frac{\omega}{2} \operatorname{cosec} \frac{\Delta u}{2} \cos u_m \\
\operatorname{tg} \frac{\Delta A}{2} &= \operatorname{tg} \frac{\Delta u}{2} \operatorname{tg} u_m \operatorname{tg} A_m \\
\operatorname{tg} \frac{\sigma}{2} &= \frac{1}{\operatorname{tg} u_m} \sin \frac{\Delta A}{2} \operatorname{cosec} \frac{\Delta A}{2} \quad \text{or} \\
\operatorname{tg} \frac{\sigma}{2} &= \frac{\operatorname{tg} \frac{\Delta u}{2} \cos \frac{\Delta A}{2}}{\cos A_m} \\
A_1 &= A_m - \frac{\Delta A}{2} \\
A_2 &= A_m + \frac{\Delta A}{2} \pm 180^\circ
\end{aligned}\right\}. \tag{6.81}$$

In formulas (6.79) unknown values are determined mainly through tangents or cotangents that ensures least error during interpolation of tables of trigonometric functions. Formulas (6.80) should not be used when σ, is close to 90°. Formulas (6.81) are convenient for logarithmic calculations.

Unknown distance between given points P_1 and P_2 after obtaining σ', σ'_1 and σ'_2 can be calculated by the Helmert formula:

$$s = \sigma' + Q_4 \cos(2\sigma'_1 + \sigma') \sin \sigma' + P_2 \cos(4\sigma'_1 + 2\sigma') \sin 2\sigma' +$$
$$+ Q_5 \cos(6\sigma'_1 + 3\sigma') \sin 3\sigma'.$$

Values Q_4, P_2 and Q_5 are taken from the tables by argument for $\overline{k}_1$

$$P_2 = -\frac{1}{8\rho}\overline{k}_1^2,$$
$$Q_4 = \left(\overline{k}_1 - \frac{3}{8\rho^2}\overline{k}_1^3\right).$$
$$Q_5 = \frac{\overline{k}_1^3}{24\rho^2}.$$

During recent years scientific investigations appeared, directed towards substitution of approximation method by finding λ by a straight line resolution of the problem.

Schematically this way of resolution of the problem can be shown in the following manner.

Let us assume that

$$\lambda = l + x,$$

where:

$$x = f_1(u_0, \sigma'_1, \sigma'_2)$$

and:

$$\sigma'_1 = \sigma'_1(u_1, u_0),$$
$$\sigma'_2 = \sigma'_2(u_2, u_0).$$

Consequently,

$$x = f_1(u_1, u_2, u_0),$$

but

$$u_0 = u_0(u_1, A_1).$$

Therefore:

$$x = f_2(u_1, u_2, A_1).$$

but:

$$A_1 = A_1(u_1, u_2, \lambda),$$

or:

$$A_1 = A_1(u_1, u_2, l + x).$$

Thus, finally:

$$x = f_3(u_1, u_2, l + x).$$

American geodesist E. Sodano[1] by means of very complicated analytic transformations managed to obtain a formula for λ, which does not require method of approximations. Sodano formulas are still very complicated for calculations, they resolve question only in principle, but for practical calculations are not useful.

In spite of abundance of printed matter on resolution of geodetic problems for long distances, this problem cannot be considered finally solved. Inasmuch as at present time mainly approximate solutions of problem are required, then depending upon requirements for accuracy different methods can be offered, basically leaning on strict methods of resolution, the main ones are presented in this chapter. Mathematical and geodetic bases of resolution of these problems are founded on Bessel methods, compared to them remaining proposals are only modifications, essentially simplifying calculation and improving convergence of used series.

Numerical examples of resolution of geodetic problems, direct and inverse, are given according to Bessel method on p. 292-296 of "Practicum on higher geodesy" and by the formulas of Professor A. M. Virovets on p. 20-29 Issue 93 Works of TsNIIGAiK.

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[1]G. V. Bagratuni. Review of methods of resolution of inverse geodetic problem for long distances from material of General Assembly of International Geodetic and Geophysical Union. Izvestiya MBO, 1960, No. 4.

CHAPTER VII

IMAGE OF A TERRESTRIAL SPHEROID ON A SPHERE

§ 39. GENERAL BASES OF IMAGE OF ONE SURFACE ON ANOTHER

To depict one surface on another means finding a law, in accordance with which each point of one surface should correspond to a fixed point on another surface. In other words, in projecting surfaces an established point must conform to both surfaces.

Let us assume that coordinates of points of first surface are expressed by parameters u and v, and second — by u' and v', then

$$\left.\begin{aligned} u' &= f_1(u, v) \\ v' &= f_2(u, v) \end{aligned}\right\}. \tag{7.1}$$

Since the image should satisfy definite geometric conditions, then function f_1 and f_2 cannot be arbitrary, their form is determined by assignment of conditions to the image.

From equations (7.1)

$$\left.\begin{aligned} du' &= \frac{\partial u'}{\partial u} du + \frac{\partial u'}{\partial v} dv \\ dv' &= \frac{\partial v'}{\partial u} du + \frac{\partial v'}{\partial v} dv \end{aligned}\right\}. \tag{7.2}$$

Let us find geometric values of partial derivatives (7.2). We will not disturb generalization of reasonings, if we assume, the correspondence between two pairs of variables (u, v) and (u', v') is established on the same surface. For this surface the square of lineal element in Gaussian form and curvilinear coordinates (u, v) and (u', v') have the form:

$$\left.\begin{aligned} ds^2 &= E\,du^2 + 2F\,du\,dv + G\,dv^2 \\ ds^2 &= E'\,du'^2 + 2F'\,du'\,dv' + G'\,dv'^2 \end{aligned}\right\} \tag{7.3}$$

From Fig. 80 it follows that through every point P pass four parametric lines, for which u, v, u', v' are corresponding constants. Elements of these lines, corresponding to differentials du, dv, du' and dv', are equal

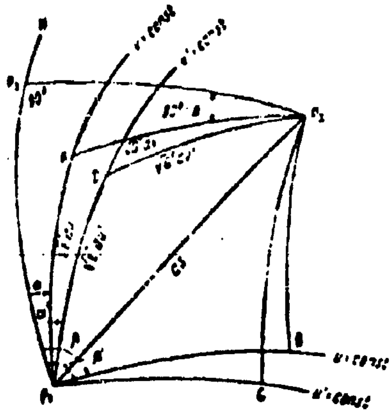

Fig. 80.

$$\sqrt{E}\,du,\ \sqrt{G}\,dv,\ \sqrt{E'}\,du',\ \sqrt{G'}\,dv'.$$

Let us draw from point P_1 an arbitrary direction P_1N and designate angles, formed by these directions and coordinate lines, by α, β, α', β', and coordinate angles by ω and ω'. We drop from point P_2 a perpendicular P_2P_2' on line P_1N and find its length.

Projection of broken $P_2AP_1P_2'$ and $P_2DP_1P_2'$ on line $P_2'P_2$ are equal among themselves and are equal to the length of the perpendicular, i.e.,

$$\sqrt{E}\,du\sin\alpha + \sqrt{G}\,dv\sin\beta = \sqrt{E'}\,du'\sin\alpha' + \sqrt{G'}\,dv'\sin\beta'. \tag{7.4}$$

Obviously,

$$\omega = \beta - \alpha,$$
$$\omega' = \beta' - \alpha'.$$

Additionally we designate $\beta - \alpha' = \gamma$, then:

$$\left.\begin{aligned} \alpha &= \beta - \omega = \alpha' + \gamma - \omega \\ \beta &= \alpha' + \gamma \\ \beta' &= \alpha' + \omega' \end{aligned}\right\}. \tag{7.5}$$

We will copy (7.4), expressing in it α, β and β' by α', γ, ω and ω' by (7.5), then

$$\begin{aligned} &\sqrt{E}\,du\sin(\alpha' + \gamma - \omega) + \\ &+ \sqrt{G}\,dv\sin(\alpha' + \gamma) = \\ &= \sqrt{E'}\,du'\sin\alpha' + \\ &+ \sqrt{G'}\,dv'\sin(\alpha' + \omega'). \end{aligned} \tag{7.4'}$$

Replacing α' throughout in (7.4') by the formula

$$\alpha' = \beta' - \omega',$$

we obtain:

$$\begin{aligned} \sqrt{E}\,du\sin(\beta' + \gamma - \omega - \omega') + \sqrt{G}\,dv\sin(\beta' + \gamma - \omega') = \sqrt{E'}\,du'\sin(\beta' - \omega') + \\ + \sqrt{G'}\,dv'\sin\beta'. \end{aligned} \tag{7.4''}$$

Direction P_1N is selected absolutely arbitrarily and equalities (7.4') and (7.4") obviously, do not depend on it. Therefore we can take for (7.4') $\alpha' = 0$, and for (7.4") $\beta' = 0$. Consequently,

$$\left.\begin{aligned} \sqrt{E'}\,du'\sin\omega' &= \sqrt{E}\,du\sin(\omega+\omega'-\gamma)+\sqrt{G}\,dv\sin(\omega'-\gamma) \\ \sqrt{G'}\,dv'\sin\omega' &= \sqrt{E}\,du\sin(\gamma-\omega)+\sqrt{G}\,dv\sin\gamma \end{aligned}\right\}\begin{aligned} &\text{when } \beta'=0, \\ &\text{when } \alpha'=0. \end{aligned}$$

Hence:

$$\left.\begin{aligned} du' &= \sqrt{\frac{E}{E'}}\frac{\sin(\omega+\omega'-\gamma)}{\sin\omega'}du+\sqrt{\frac{G}{E'}}\frac{\sin(\omega'-\gamma)}{\sin\omega'}dv \\ dv' &= \sqrt{\frac{E}{G'}}\frac{\sin(\gamma-\omega)}{\sin\omega'}du+\sqrt{\frac{G}{G'}}\frac{\sin\gamma}{\sin\omega'}dv \end{aligned}\right\}. \qquad (7.6)$$

From comparison (7.2) and (7.6) it follows, that:

$$\left.\begin{aligned} \frac{\partial u'}{\partial u} &= \sqrt{\frac{E}{E'}}\frac{\sin(\omega+\omega'-\gamma)}{\sin\omega'}; \quad \frac{\partial u'}{\partial v} = \sqrt{\frac{G}{E'}}\frac{\sin(\omega'-\gamma)}{\sin\omega'} \\ \frac{\partial v'}{\partial u} &= \sqrt{\frac{E}{G'}}\frac{\sin(\gamma-\omega)}{\sin\omega'}; \quad \frac{\partial v'}{\partial v} = \sqrt{\frac{G}{G'}}\frac{\sin\gamma}{\sin\omega'} \end{aligned}\right\}. \qquad (7.7)$$

Equations (7.7) are justified for any system of curvilinear coordinates. However the more important is the case of orthogonal systems, when $\omega = \omega' = 90^\circ$.

Besides:

$$\begin{aligned} \frac{\partial u'}{\partial u} &= \sqrt{\frac{E}{E'}}\sin\gamma; \quad \frac{\partial u'}{\partial v} = \sqrt{\frac{G}{E'}}\cos\gamma; \\ \frac{\partial v'}{\partial u} &= -\sqrt{\frac{E}{G'}}\cos\gamma; \quad \frac{\partial v'}{\partial v} = \sqrt{\frac{G}{G'}}\sin\gamma. \end{aligned}$$

Excluding from these equations $\sin\gamma$ and $\cos\gamma$, we obtain:

$$\left.\begin{aligned} \sqrt{E'G}\frac{\partial u'}{\partial u} &= \sqrt{EG'}\frac{\partial v'}{\partial v} \\ \sqrt{E'E}\frac{\partial u'}{\partial v} &= -\sqrt{G'G}\frac{\partial v'}{\partial u} \end{aligned}\right\}. \qquad (7.8)$$

Equations (7.8) are fundamental relationships of the theory of image of one surface on another; they express point conformity between two surfaces, if systems (u, v) and (u', v') are referred to different surfaces and give transformation of curvilinear coordinates, if systems (u, v) and (u', v') are selected on the same surface. These equations we will subsequently use very frequently, since they give general solution of the problem of one image of one surface on another and transformation of curvilinear coordinates on a given surface.

Equations (7.8) by their construction resemble known conditions of Cauchy-Riemann (?). In fact, where $E = G = E' = G' = 1$ from (7.8) it follows that:

$$\left.\begin{aligned} \frac{\partial u'}{\partial u} &= \frac{\partial v'}{\partial v} \\ \frac{\partial u'}{\partial v} &= -\frac{\partial v'}{\partial u} \end{aligned}\right\}. \qquad (7.8')$$

Equations (7.8) or (7.8') possess the property of symmetry, i.e., they are equally suitable for resolution both of direct, and inverse problem of image transfer of surfaces and transformation of systems of coordinates.

Equations (7.1) conclude all possible images of one surface on another. In spheroidal geodesy such images are used, they preserve similarity of geometric figures in their infinitesimal parts. Such images are called conformal. Similarity of figures, as it is known from geometry, have a place, if lines, forming arbitrarily small figure on one surface, are proportional to corresponding lines on second surface, and the angles, included between the lines of the first surface, are equal to angles between corresponding lines on second surface.

Inasmuch as the simpler surfaces are plane and sphere, then in spheroidal geodesy conformal projections of a spheroid are used mainly on a plane and a sphere. Conformal projection of a spheroid on a sphere is used, as noted in Chapter V, in resolution of direct and inverse geodetic problems. Very frequently the conformal projection of a spheroid on a sphere is used as a step during complicated mathematical calculations on a surface of a prolate spheroid, furthermore, with the help of the image of ellipsoid on a sphere there is established a degree of geometric proximity of terrestrial ellipsoid to a globe.

Gauss was the first to develop the theory and practice of conformal representation of an ellipsoid on a sphere for geodetic purposes in his work "on research in higher geodesy."[1] Gauss approach even now did not lose its value in spheroidal geodesy. Although we now have many other methods of resolution of this problem and thorough supplements to Gauss theory at our disposal, his work in this area still remains a classical heritage in spheroidal geodesy.

§ 40. CONFORMAL REPRESENTATION OF ELLIPSOID ON SPHERE BY GAUSS

In presenting the question about representation of an ellipsoid on a sphere Gauss applied a theory of analytic functions, specially developed by him. Here a somewhat different mathematical approach is applied, in which elements of the theory of analytic functions are absent.

Square of lineal element of a spheroid will be expressed as:

$$ds^2 = M^2 dB^2 + N^2 \cos^2 B\, dL^2 = N^2 \cos^2 B \left(\frac{M^2 dB^2}{N^2 \cos^2 B} + dL^2 \right),$$

[1]K. F. Gauss. Selected geodetic works. T. N. M., Geodezizdat, 1958, p. 63-91.

and analogously for sphere

$$ds^2 = R^2 dU^2 + R^2 \cos^2 U d\lambda^2 = R^2 \cos^2 U \left(\frac{dU^2}{\cos^2 U} + d\lambda^2\right),$$

where R — radius of a sphere (while undetermined),

U — latitude on a sphere,

λ — longitude on a sphere.

Let us introduce the designations:

$$\left.\begin{aligned} \frac{M dB}{N \cos B} &= d\psi, \\ \frac{dU}{\cos U} &= d\psi', \end{aligned}\right\} \qquad (7.9)$$

then

$$\left.\begin{aligned} ds^2 &= N^2 \cos^2 B (d\psi^2 + dL^2) \\ ds^2 &= R^2 \cos^2 U (d\psi'^2 + d\lambda^2) \end{aligned}\right\}. \qquad (7.10)$$

If you compare (7.10) with general recording of the square of lineal element by Gauss, it would be easy to establish that in our case F = 0, E = G. Curvilinear coordinates, for them, F = 0, E = G, are called isometric ("uniform") coordinates. Thus, (ψ, L) are isometric coordinates on a spheroid and (ψ', λ), on a sphere. Isometric coordinates on given surface form a network of squares, if coordinate line u = const and v = const are broken down in uniform sections $\Delta u = \Delta v$.

Values ψ, ψ' are called isometric latitudes on a spheroid and on a sphere correspondingly.

From (7.9)

$$\psi = \int_0^B \frac{M dB}{N \cos B} = \int_0^B \frac{(1-e^2)\, dB}{(1-e^2 \sin^2 B)\cos B} = \int_0^B \frac{(1-e^2\sin^2 B - e^2\cos^2 B)\, dB}{(1-e^2\sin^2 B)\cos B} =$$

$$= \int_0^B \frac{dB}{\cos B} - \int_0^B \frac{e^2 \cos B\, dB}{1-e^2\sin^2 B}.$$

For reduction of second integral to the right part to the tabular form let us introduce a new variable under condition:

$$e \sin B = \sin \varphi.$$

Then

$$e \cos B\, dB = \cos\varphi\, d\varphi \text{ and } 1 - e^2 \sin^2 B = \cos^2\varphi.$$

Consequently,

$$\psi = \int_0^B \frac{dB}{\cos B} - e \int_0^{\varphi} \frac{d\varphi}{\cos\varphi}.$$

or:

$$\psi = \ln \operatorname{tg}\left(45^\circ - \frac{B}{2}\right) - \ln \operatorname{tg}^{e}\left(45^\circ + \frac{\varphi}{2}\right).$$

but:

$$\operatorname{tg}^{e}\left(45^\circ + \frac{\varphi}{2}\right) = \left(\frac{1 + e\sin B}{1 - e\sin B}\right)^{\frac{e}{2}}.$$

therefore

$$\psi = \ln \operatorname{tg}\left(45 + \frac{B}{2}\right)\left(\frac{1 - e\sin B}{1 + e\sin B}\right)^{\frac{e}{2}} \tag{7.11}$$

or

$$\bar{e}^{\psi} = \operatorname{tg}\left(45^\circ + \frac{B}{2}\right)\left(\frac{1 - e\sin B}{1 + e\sin B}\right)^{\frac{e}{2}}, \tag{7.11'}$$

$\bar{e}$ – in the left part of equation (7.11') is the base of natural functions.

For a sphere where e = 0 from (7.11) it follows

$$\psi' = \ln \operatorname{tg}\left(45^\circ + \frac{U}{2}\right). \tag{7.11''}$$

Values ψ and ψ' are shown in special tables: for instance, in Cartographic tables of TsNIIGAiK for 1°; in Cartographic tables of the Hydrographic administration VMS ψ are given for 1'.[1]

Let us apply to lineal elements (7.10) equations (7.8). Assuming that:

$$\left.\begin{array}{l} E' = G' = R^2\cos^2 U, \quad u' = \psi' \text{ and } v' = \lambda \\ E = G = N^2\cos^2 B, \quad u = \psi \text{ and } v = L \end{array}\right\}. \tag{7.12}$$

We have

$$\left.\begin{array}{l} \dfrac{\partial \psi'}{\partial \psi} = \dfrac{\partial \lambda}{\partial L} \\ \dfrac{\partial \psi'}{\partial L} = -\dfrac{\partial \lambda}{\partial \psi} \end{array}\right\}. \tag{7.12'}$$

where:

$$\left.\begin{array}{l} \psi' = f_1(\psi, L) \\ \lambda = f_2(\psi, L) \end{array}\right\} \tag{7.13}$$

For integration of equations (7.12') it is necessary to set definite geometric

[1]In work of Danish geodetic institute Geodetic Tables on international ellipsoid difference (B - ψ) are given six decimal places. Copenhagen, 1956.

conditions, which simultaneously will determine the form of functions f_1 and f_2. Our aim is to obtain conformal image of ellipsoid on a sphere. Let us set a condition, that parallels on an ellipsoid corresponded to parallels on a sphere. Then, for strength of conformity, meridians of an ellipsoid must correspond to the meridians of a sphere. This means that

$$\left.\begin{array}{l}\psi' = f_1(\varphi) \\ \lambda = f_2(L)\end{array}\right\}. \tag{7.13'}$$

The simpler solution is obtained, when arbitrary functions f_1 and f_2 possess equal to themselves arguments, i.e.,

$$\left.\begin{array}{l}\psi' = f_1(\varphi) = \psi \\ \lambda = f_2(L) = L\end{array}\right\}. \tag{7.13''}$$

Such approach is expedient, if it is required to transfer all surface of an ellipsoid to a sphere which is done in resolution of various cartographic problems. In resolution of geodetic problems only comparatively small parts of the surface of an ellipsoid studied. In these cases it is more profitable to introduce indefinite constant coefficients, with whose help it is possible to use the image of small parts of the surface in the most profitable manner.

Let:

$$\lambda = f_2(L) = \alpha L, \tag{7.14}$$

where α is a constant.

Consequently,

$$\left.\begin{array}{l}\dfrac{\partial\lambda}{\partial L} = \alpha \\ \dfrac{\partial\lambda}{\partial\psi} = 0\end{array}\right\}. \tag{7.14'}$$

Under these conditions from (7.12') it follows that:

$$\frac{\partial\psi'}{\partial\psi} = \alpha$$

or

$$\psi' = \alpha\varphi + C.$$

Without disturbing the generalization of resolution, we assume that:

$$C = \alpha k,$$

k is also a constant.

We have:

$$\psi' = \alpha(\psi + k). \tag{7.15}$$

Formulas (7.14) and (7.15) give the law of transfer or transformation of isometric coordinates of a spheroid to isometric coordinates of a sphere in conformal projection of first to a second.

Formulas (7.14) and (7.15) contain two arbitrary constants α and k. Furthermore, radius of a sphere R remains unknown. The honor of artificial selection of constants belongs to Gauss, who proposed the selection of constants in such a manner that the scale of image m deviated from a unit by a small value of third order, not counting the factor e^2.

In conformal projection scale $m = \frac{d\sigma}{ds}$ does not depend on direction and is a function of the latitude:

$$m = m(B) = m[B_0 + (B - B_0)]. \tag{7.16}$$

Applying Maclaurin series to (7.16), we obtain:

$$m = m_0 + (B - B_0)m_0' + \frac{(B - B_0)^2}{2!}m_0'' + \frac{(B - B_0)^3}{3!}m_0''' + \dots,$$

where B_0 — the latitude of central parallel of depicted part of a surface, $m^l = \frac{d^l m}{dB^l}$. Derivatives m_0', m_0'', ... are calculated by latitude B_0.

For determination of constants fo R, α and k let us set the following conditions: let scale m on central parallel be equal to one, but on other parallels it deviates from one by values of third order, considering difference $(B - B_0)$ a value of first order. These conditions analytically are expressed by equations:

$$\left.\begin{aligned} &1.\ m_0 = 1 \\ &2.\ m_0' = 0 \\ &3.\ m_0'' = 0 \end{aligned}\right\} \tag{7.17}$$

and

$$m = 1 + \frac{(B - B_0)^3}{3!}m_0''' + \dots + l_4. \tag{7.18}$$

Element of parallel of a spheroid is equal to N cos B d L, and the meridian — MdB; these elements on a sphere will be R cos Udλ and RdU.

By condition of conformity:

$$m = \frac{RdU}{MdB} = \frac{R\cos U d\lambda}{N\cos B\, dL} = \frac{\alpha R\cos U}{N\cos B}, \tag{7.19}$$

hence:

$$\frac{dU}{dB} = \frac{\alpha M\cos U}{N\cos B}, \tag{7.19'}$$

$$\frac{dm}{dB} = m' = \alpha R\left(-\frac{\sin U dU}{N\cos B dB} + \frac{M\sin B\cos U}{N^2\cos^2 B}\right)$$

or:

$$m' = \alpha R\left(-\alpha\frac{M\cos U}{N^2\cos^2 B}\sin U + \frac{M\cos U\sin B}{N^2\cos^2 B}\right).$$

By condition $m_0' = 0$

consequently,

$$-\alpha\frac{M_0\cos U_0\sin U_0}{N_0^2\cos^2 B_0} + \frac{M_0\cos U_0\sin B_0}{N_0^2\cos^2 B_0} = 0$$

or

$$\alpha\sin U_0 = \sin B_0. \tag{7.20}$$

In order that $m_0'' = 0$, would be sufficiently needed:

$$\frac{d}{dB}(-\alpha\sin U + \sin B) = 0,$$

or

$$-\alpha\cos U_0\frac{dU}{dB} + \cos B_0 = -\alpha^2\frac{M_0}{N_0}\frac{\cos^2 U_0}{\cos B_0} + \cos B_0.$$

Consequently,

$$\alpha^2\frac{M_0\cos^2 U_0}{N_0\cos B_0} = \cos B_0 \tag{7.20'}$$

or:

$$\alpha^2\cos^2 U_0 = \frac{N_0\cos^2 B_0}{M_0} = \frac{(1-e^2\sin^2 B_0)\cos^2 B_0}{1-e^2}.$$

But by the formula (7.20):

$$\alpha^2 - \alpha^2\sin^2 U_0 = \alpha^2 - \sin^2 B_0,$$

therefore:

$$\alpha^2 = \sin^2 B_0 + \frac{(1-e^2\sin^2 B_0)\cos^2 B_0}{1-e^2} = 1 + \frac{e^2\cos^4 B_0}{1-e^2} = 1 + e'^2\cos^4 B_0.$$

Finally:

$$\alpha = \sqrt{1 + e'^2\cos^4 B_0}. \tag{7.21}$$

By equation (7.21) where a given B_0 determines α; from (7.20) find U_0. Having B_0 and U_0 by the formulas (7.11) and (7.11'), we determine ψ and ψ'. With ψ, ψ' and α from (7.15) we find second constant of projection k by the formula:

$$k = \frac{\psi'}{\alpha} - \psi. \tag{7.22}$$

From condition $m_0 = 1$ it follows that:

$$\frac{\alpha R\cos U_0}{N_0\cos B_0} = 1,$$

hence, taking into account (7.20'),

$$R_0 = \frac{N_0 \cos B_0}{\alpha \cos U_0} = \frac{N_0 \cos B_0}{\sqrt{\frac{N_0}{M_0}} \cos B_0} = \sqrt{N_0 M_0}. \tag{7.23}$$

Thus, radius of sphere R_0 is equal to mean radius of curvature of a spheroid at latitude of central parallel B_0. Frequently the latitude of the central parallel is called normal. Inasmuch as central parallel can be selected, considering the benefits of the resolution of the problem on hand, it is better to call it standard parallel.

Omitting details of calculations, let us record the approximate value of third derivative of the scale by latitude:

$$m_0''' = -\frac{2e^2(1-e^2)\sin 2B_0}{(1-e^2\sin^2 B_0)^2} + \ldots$$

Consequently,

$$m = 1 - \frac{e^2(1-e^2)\sin 2B_0}{3(1-e^2\sin^2 B_0)^2}(B-B_0)^3 + \ldots + l_4. \tag{7.24}$$

For numerical calculations this formula can be used in the form:

$$m = 1 - \frac{e^2}{3}\sin 2B_0 (B-B_0)^3 + l_4. \tag{7.24'}$$

Where $B_0 = 45^{\circ}$, differences $(B - B_0) = 1/40$, which will correspond to differences of latitude approximately by $1^{\circ}.5$

$$\frac{e^2}{3}\sin 2B_0 (B-B_0)^3 \approx \frac{1}{3.150.40.40.40} = \frac{1}{288\ 10^5}.$$

Hence follows a very important derivation that if a maximum distortion of lineal elements is allowed by the value of $\frac{1}{3\cdot 10^7}$ at the edge of a belt three degrees wide along a latitude, then the scale within the limits of this belt can be considered constant and equal to 1. In this case it will not be necessary to introduce corrections in the measured elements, i.e., in lengths and direction. Thus, we see that sufficiently significant parts of a surface of a terrestrial spheroid can be replaced by spherical with the help of a properly selected radius. As can be seen from (7.23) this radius should be the mean radius of curvature of an ellipsoid on the standard parallel.

As a result of cited investigations the following plan for the solution of geodetic problems is obtained from given geodetic coordinates of the first point of triangulation we convert to spherical coordinates by the formulas (7.14) and (7.15); if the triangulation is located within the limits of a three-degree latitudinal belt, then it is taken as lying on the surface of a sphere and having the same angles and sides, as on an ellipsoid; by these data we calculate the latitude and longitude of points of triangulation on a sphere; then from them we convert to geodetic coordinates of a spheroid. Transition from geodetic coordinates to spherical and conversely should be accomplished with help of special tables. Gauss composed such tables.

Such method of calculation of geodetic coordinates was applied in the past in Russia in accomplishment of land exploitation work in Transcaucasus. At present this method does not have practical value, but presents a methodical interest for spheroidal geodesy it gives clear example of geometric approach to resolution of geodetic problem. Benefits of such approach become perceptible, when the dimensions of depicted territories are such that within its limits the scale of image can be taken as equal to one and, thus, eliminates the reduction problem.

§ 41. APPLICATION OF CONFORMAL REPRESENTATION OF ELLIPSOID ON A SPHERE TO RESOLUTION OF DIRECT GEODETIC PROBLEM

In preceding paragraph it is shown that in the part of a surface of a terrestrial spheroid, limited by parallels, whose difference of latitudes does not exceed 3° (with accuracy up to $1 \cdot 10^{-8}$), it can be taken as spherical. Radius of a sphere is equal to the mean radius of curvature of a spheroid on the standard parallel. Using this important derivation, it is more expedient to resolve spherodal problems by means of representation of a spheroid on a sphere. using the sphere as an intermediate instance in mathematical derivations.

Area of application of this method is quite extensive, but for the illustration of basic idea it is sufficient to consider one classical example, solved by Gauss. We have in mind a derivation of formulas with mean arguments for resolution of the direct geodesic problem for distances, not exceeding 25-30 km. In this case the area of representation by latitude will be less than 1°, and the scale of the image will be equal to 1 everywhere.

Therefore:

$$\left.\begin{array}{l} 1.\ M dB = R dU \\ 2.\ N \cos B = \alpha R \cos U \end{array}\right\}. \qquad (7.25)$$

Changing from differentials dB and dU to finite differences and substituting current parallel by the standard, we obtain:

$$\left.\begin{array}{r} M_0 b = R u \\ N_0 \cos B_0 = \alpha R_0 \cos U_0 \end{array}\right\}. \qquad (7.25')$$

We have:

$$\sin B_0 = \alpha \sin U_0, \tag{7.20}$$

B_0 – latitude of median point of arc s on a spheroid,

U_0 – mean latitude on a sphere, equal to $\frac{U_1 + U_2}{2} = U_0 = U_m$.

From (5.18) it follows that $(B_0 - B_m)$ is small value of the second order; with accuracy up to values of third order, it can be taken as:

$$\left.\begin{array}{l} 1.\ u = \frac{\Delta l_m}{R_m} b + \ldots + l_3 \\ 2.\ N_m \cos B_m = \alpha R_m \cos U_m + \ldots + l_3 \\ 3.\ \sin B_m = \alpha \sin U_m + l_3 \end{array}\right| . \tag{7.26}$$

On a sphere polar spheroidal triangle P_1PP_2 (Fig. 81a) will correspond polar spherical triangle $P_1'P'P_2'$ (Fig. 81b). Applying to spherical triangle $P_1'P'P_2'$ Gauss-Delambre formula, we obtain:

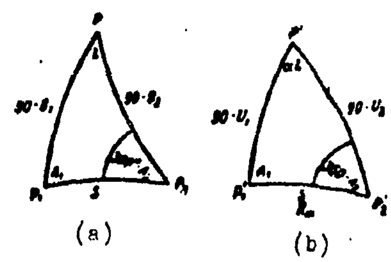

Fig. 81.

$$\left.\begin{array}{l} \sin \frac{s}{2R_m} \sin A_m = \sin \frac{\alpha l}{2} \cos U_m \\ \sin \frac{s}{2R_m} \cos A_m = \cos \frac{\alpha l}{2} \sin \frac{u}{2} \\ \cos \frac{s}{2R_m} \sin \frac{t}{2} = \sin \frac{\alpha l}{2} \sin U_m \\ \cos \frac{s}{2R_m} \cos \frac{t}{2} = \cos \frac{\alpha l}{2} \cos \frac{u}{2} \end{array}\right| . \tag{7.27}$$

Here $t = A_2 - A_1 \pm 180°$, $A_m = \frac{A_1 + A_2 \pm 180°}{2}$.

Arranging sines and cosines of acute angles in series and retaining in them small values of third order inclusively, from (7.27) we obtain

$$\left.\begin{array}{l} \frac{s}{R_m}\left(1 - \frac{s^2}{24R_m^2}\right)\sin A_m = \alpha l \cos U_m \left(1 - \frac{\alpha^2 l^2}{24}\right) \\ \frac{s}{R_m}\left(1 - \frac{s^2}{24R_m^2}\right)\cos A_m = u \left(1 - \frac{u^2}{24}\right)\left(1 - \frac{\alpha^2 l^2}{8}\right) \\ t\left(1 - \frac{t^2}{24}\right)\left(1 - \frac{s^2}{8R_m^2}\right) = \alpha l \sin U_m \left(1 - \frac{\alpha^2 l^2}{24}\right) \\ \left(1 - \frac{s^2}{8R_m^2}\right)\left(1 - \frac{t^2}{8}\right) = \left(1 - \frac{\alpha^2 l^2}{8}\right)\left(1 - \frac{u^2}{8}\right) \end{array}\right| . \tag{7.27'}$$

In all correction terms, in parentheses, with error in values of fourth order it is possible to accept that $\alpha^2 = 1$.

From the last equation it follows with the same accuracy, that:

$$\frac{s^2}{R_m^2} = l^2 + u^2 - t^2. \tag{7.28}$$

Substituting by the formulas (7.26) spherical elements by spheroidal in (7.27') and expressing differences of latitudes, longitudes, and azimuths in seconds, we obtain:

$$b'' = \frac{s}{M_m}\rho''\cos A_m\left(1 + \frac{l''^2}{12\rho''^2} + \frac{t''^2}{24\rho''^2}\right),$$

$$l'' = \frac{s}{N_m}\rho''\sin A_m \sec B_m\left(1 - \frac{b''^2}{24\rho''^2} + \frac{t''^2}{24\rho''^2}\right),$$

$$t'' = l''\sin B_m\left(1 + \frac{b''^2}{8\rho''^2} + \frac{l''^2}{12\rho''^2} - \frac{t''^2}{12\rho''^2}\right).$$

Introducing known designations:

$$\frac{\rho''}{M_m} = (1)''_m, \quad \frac{\rho''}{N_m} = (2)''_m,$$

$$(1)_m s\cos A_m = \beta''_m,$$

$$(2)_m s\sin A_m \sec B_m = \lambda''_m,$$

$$(2)_m s\sin A_m \operatorname{tg} B_m = \tau''_m,$$

after transition to logarithms we obtain:

$$\left.\begin{aligned} \lg b'' &= \lg\beta''_m + \frac{1}{4}\nu\tau''^2_m + \frac{1}{2}\nu\lambda''^2_m \\ \lg l'' &= \lg\lambda''_m + \frac{1}{4}\nu\tau''^2_m - \frac{1}{4}\nu\beta''^2_m \\ \lg t'' &= \lg\tau''_m + \frac{3}{4}\nu\beta''^2_m + \frac{1}{2}\nu\lambda''^2_m - \frac{1}{2}\nu\tau''^2_m \end{aligned}\right\}. \qquad (7.29)$$

In formulas (7.29)

$$\nu = \frac{10^6\mu}{6\rho''^2},$$

in correction terms it is taken that $b'' = \beta''_m$, $l'' = \lambda''_m$ and $t'' = t''_m$.

Formulas (7.29) were already obtained in Chapter V under number (5.27). Here the object was to show, how the problem in question can be resolved with application of conformal representation of an ellipsoid on a sphere according to Gauss.

§ 42. CERTAIN OTHER METHODS OF REPRESENTATION OF AN ELLIPSOID ON A SPHERE

From preceding paragraphs of this chapter it follows that for representing an ellipsoid on a sphere we are free to select from three parameters or constants, one of which is the radius of a sphere. Frequently in representing an ellipsoid on a sphere for geodetic purposes, it is expedient to take a sphere with unit radius, but with remaining two parameters, to act in conformity with problem at hand. It is absolutely clear that the variants of representation are many and the problem will consist in selection of the most suitable for a given purpose.

Let us consider the more important and simple in geodetic sense of representation of a spheroid on a sphere of unit radius.

1. Spherical Representation

Let us assume that on a sphere of unit radius the geodetic coordinates, latitude and longitude of points coincide with geodetic coordinates of a spheroid, i.e.,

$$\left.\begin{matrix} u = B \\ \lambda = L \end{matrix}\right\}. \tag{7.30}$$

Then scales of representation:

$$m = \begin{cases} \frac{1}{N} & \text{— on parallel,} \\ \frac{1}{M} & \text{— on meridian,} \\ \frac{\sin^2 A}{N} + \frac{\cos^2 A}{M} & \text{— in the direction with azimuth A,} \end{cases}$$

where N — radius of curvature of first vertical,

M — radius of curvature of meridian.

Value inverse to scale of representation

$$\frac{1}{m} = \begin{cases} N & \text{— on parallel,} \\ M & \text{— on meridian,} \\ N\sin^2 A' + M\cos^2 A' & \text{— by direction with azimuth } A' \text{ on a sphere.} \end{cases}$$

Let us designate:

ds — element of geodesic arc on a spheroid,

$d\sigma$ — element of great circle on a sphere.

Consequently,

$$m = \frac{d\sigma}{ds},$$

or:

$$ds = \frac{d\sigma}{m} = (N\sin^2 A' + M\cos^2 A')\,d\sigma. \tag{7.31}$$

But, as it is known, on a sphere

$$\sin A' = \frac{\sin A'_s}{\cos B},$$

A'_s — azimuth of the great circle arc at point of its intersection with equator.

Further,

$$N = \frac{a}{W}, \quad M = \frac{a(1-e^2)}{W^3}.$$

Substituting these values in (7.31) and converting to integral, we obtain:

$$s = \left(\frac{b^2}{a}\cos^2 A'_s + a\sin^2 A'_s\right)\int_{\sigma_A}^{\sigma_B}\frac{d\sigma}{W^3}.$$

$$W^{-3} = 1 + \frac{3}{2}e^2\sin^2 B + \frac{15}{8}e^4\sin^4 B + \ldots$$

Introducing this expression W^{-3} for integral and satisfying integration with the help of Wallace integrals, we obtain:

$$s = a(1-\eta_1^2)(\Delta\sigma + 3\eta_1^2\Delta W_2 + 15\eta_1^4\Delta W_4 + 35\eta_1^6\Delta W_6 + \ldots). \qquad (7.32)$$

Here:

$$\eta_1^2 = e'^2\cos^2 A,$$

$$\Delta W_i = \frac{1}{n}\int_{\sigma_A}^{\sigma_B}\sin^i B\,d\sigma \quad (n = 2,\ 8,\ 16;\ i = 2,\ 4,\ 6).$$

$$\Delta\sigma = \int_{\sigma_A}^{\sigma_B} d\sigma.$$

Let us designate elementary arc

$d\sigma_n$ – parallels on a sphere,

$d\sigma_m$ – meridian on a sphere,

ds_n – parallels on a spheroid,

ds_m – meridian on a spheroid.

We have:

$$d\sigma_n = m_n ds_n;\quad d\sigma_m = m_m ds_m;\quad m_n = \frac{1}{N};\quad m_m = \frac{1}{M}.$$

On sphere	On ellipsoid
$d\sigma_n = d\sigma\sin A'$,	$ds_n = ds\sin A$,
$d\sigma_m = d\sigma\cos A'$,	$ds_m = ds\cos A$.

Therefore:

$$\operatorname{tg} A' = \frac{d\sigma_n}{d\sigma_m} = \frac{\frac{ds}{N}\sin A}{\frac{ds}{M}\cos A} = \frac{M}{N}\operatorname{tg} A,$$

or:

$$\operatorname{tg} A' = V^{-2}\operatorname{tg} A. \qquad (7.33)$$

By this very simple formula we calculate azimuths of arcs of the great circles on a sphere of normals. Value V^{-2} by argument of given latitude can be taken from geodetic tables, where lg V are given with a large number of decimal places.

From (7.33) with accuracy up to small values of the second order we have:

$$\left.\begin{aligned}(A-A')'' &\approx \rho''\frac{e'^2}{2}\cos^2 B\sin 2A\\ (A-A')''_{\max} &\approx \rho''\frac{e'^2}{2}\approx 666'',7\approx 11',1\end{aligned}\right\}. \qquad (7.33')$$

2. Equal-Spacing Representation

Element of parallel on a sphere = a cos u dλ.

Element of parallel on a spheroid = N cos B dl.

Consequently, $n = \frac{a \cos u \, d\lambda}{N \cos B \, dl} = 1.$

Consequently, condition of equal spacing along parallels is contained in equations:

$$\left.\begin{array}{l} 1.\ \operatorname{tg} u = \sqrt{1-e^2}\operatorname{tg} B \\ 2.\ \lambda = l \end{array}\right\}, \tag{7.34}$$

where u — given latitude.

The first of (7.34) after differentiation gives:

$$\frac{du}{dB} = \frac{\sqrt{1-e^2}}{W^2} = \frac{1}{VW}. \tag{7.35}$$

On an ellipsoid:

$$\operatorname{tg} A = \frac{N \cos B \, dl}{M dB}, \tag{a}$$

on a sphere:

$$\operatorname{tg} \alpha = \frac{\cos u \, d\lambda}{du}, \tag{b}$$

where:

$$\cos u = \frac{\cos B}{W}. \tag{c}$$

From formulas (a), (b), and (c) it follows, that

$$\operatorname{tg} \alpha = V^{-1} \operatorname{tg} A. \tag{7.36}$$

With accuracy up to small values of the second order

$$(A-\alpha)'' = \rho'' \frac{e'^2}{4} \cos^2 B \sin 2A, \tag{7.36'}$$

$$(A-\alpha)''_{max} = \rho'' \frac{e'^2}{4} = \frac{2 \cdot 10^5}{6 \cdot 10^2} \approx 333''.3 \approx 5'.6.$$

We find expression for length of arc of a geodesic through arc of a great circle.

Element of geodetic longitude:

on a spheroid   on a sphere

$$dl = \frac{ds}{N} \sin A \sec B, \qquad d\lambda = \frac{d\sigma}{a} \sin \alpha \sec u.$$

Consequently,

$$\frac{ds}{d\sigma} = \frac{N \cos B \sin \alpha}{a \cos u \sin A}. \tag{7.37}$$

But N cos B = a cos u, therefore:

$$\frac{ds}{d\sigma}=\frac{\sin\alpha}{\sin A}. \qquad (7.37')$$

From (7.36)

$$1+\operatorname{tg}^2 A=1+V^2\operatorname{tg}^2\alpha,$$

or

$$\frac{1}{\cos^2 A}=\frac{1}{\cos^2\alpha}(1+e'^2\cos^2 B\sin^2\alpha).$$

Substituting

$$e'^2=\frac{e^2}{1-e^2};\quad \cos^2 B=\frac{(1-e^2)\cos^2 u}{1-e^2\cos^2 u},$$

we obtain

$$\frac{\cos^2\alpha}{\cos^2 A}=1+\frac{e^2\sin^2\alpha\cos^2 u}{1-e^2\cos^2 u}.$$

But on a sphere:

$\cos u\sin\alpha=\sin\alpha_{э}$,

where α – azimuth of great circle arc at its point of intersection with equator.

Therefore:

$$\frac{\cos^2\alpha}{\cos^2 A}=\frac{1+e^2(\sin^2\alpha_0-\cos^2 u)}{1-e^2\cos^2 u}. \qquad (7.38)$$

From (7.36)

$$V^2\frac{\sin^2\alpha}{\sin^2 A}=\frac{\cos^2\alpha}{\cos^2 A}=\frac{1+e^2(\sin^2\alpha_0-\cos^2 u)}{1-e^2\cos^2 u}. \qquad (7.38')$$

It is known, that

$$V^2=\frac{1}{1-e^2\cos^2 u}.$$

From (7.38') after extraction of a square root

$$\frac{\sin\alpha}{\sin A}=\sqrt{1+e^2(\sin^2\alpha_0-\cos^2 u)}. \qquad (7.39)$$

For a sphere

$$\sin u=\sin\sigma\cos\alpha_0$$

or:

$$\cos^2 u=1-\sin^2\sigma\cos^2\alpha_0. \qquad (7.40)$$

Substituting (7.39) and (7.40) in (7.37), we obtain:

$$ds=a\sqrt{1-e^2\cos^2\alpha_0\cos^2\sigma}\cdot d\sigma. \qquad (7.41)$$

Let us designate

$$e^2\cos^2\alpha_0=k_0^2. \qquad (7.42)$$

Arranging $\sqrt{1-k_0^2\cos^2\sigma}$ in series and integrating term by term, we obtain

$$\frac{s}{b} = F_0\sigma - F_1 \sin\sigma \cos(2\sigma_1+\sigma) - F_2 \sin 2\sigma \cos(4\sigma_1+2\sigma). \tag{7.43}$$

Here

$$\left.\begin{aligned} F_0 &= 1 - \frac{1}{4}k_0^2 - \frac{3}{64}k_0^4 - \frac{5}{256}k_0^6 \\ F_1 &= \frac{1}{4}k_0^2 + \frac{1}{16}k_0^4 + \frac{15}{512}k_0^6 \\ F_2 &= \frac{1}{128}k_0^4 + \frac{3}{512}k_0^6 \end{aligned}\right\}. \tag{7.44}$$

For coefficients F_0, F_1, F_2 special tables can be composed by argument k_0, they will have the same form, as tables for Bessel method.

3. Understanding of Aposphere

In representation of an ellipsoid on a sphere geodetic problems are resolved simply, if small parts of a surface are depicted. In this case the scale of image is close to a unit and the question about introduction of reduction does not arise. In representation of significant parts of an ellipsoid complicated reduction problem is inevitabe. In connection with this a new problem appeared about representation of an ellipsoid on such a surface, where reduction problem was also simply resolved, as in representation of small parts of an ellipsoid on a sphere.

In 1947 English geodesist M. Hotine[1] proposed to use an image of an ellipsoid on aposphere for geodetic purposes. Aposphere is the surface of a prolate, whose axis coincides with the axis of rotation of an ellipsoid, but meridians are determined from an equation:

$$r^* = R^* \operatorname{sch} \alpha(\psi + c). \tag{7.45}$$

where r^* — radius of parallel of aposphere; ψ — isometric latitude; sch - hyperbolic secant; R^*, α and c — constant images.

For determination of constants conditions are made.

1. On central parallel of depicted territory radii of parallels of a spheroid and aposphere are equal, i.e.,

$$r^* = r_0. \tag{7.46}$$

2. Geodetic latitudes are determined from equation

$$\frac{dr^*}{MdB} = -\sin B, \tag{7.47}$$

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[1] T. Hotine. The orthomorphic projection of the Spheroid, Empire Survey Review, 1946-1947, No. 62-66.

are also are equal. Consequently, a spheroid and an aposphere along this parallel have general meridional tangency and radius of curvature of the first vertical N.

3. Curvatures of meridian sections along this parallel, equal to $\frac{1}{M}$ are identical, therefore tangency occurs both along meridian, and parallel.

These three conditions fully determine R^*, α and c. Omitting details of calculations, which on the whole coincide with analogous calculations in representation of an ellipsoid on a sphere by Gauss, give following final results:

$$\left.\begin{array}{l} 1.\ \alpha^2 = 1 + e'^2\cos^4 B_0 \\ 2.\ \frac{R^{*2}}{\alpha^2} = R_0^2 \\ 3.\ \alpha\, th\, \alpha(\psi_0 + c) = \sin B_0 \end{array}\right\}. \qquad (7.48)$$

Sign "0" indicates that corresponding values are referred to latitude of central parallel.

From characteristic function (7.45) of aposphere ensue the following properties.

1. Gauss curvature of aposphere, equai to $\frac{\alpha^2}{R^{*2}}$ $\frac{1}{R^2}$, is constant for all points on the surface, therefore it may be developed into a sphere of radius R without distortions just as cone and cylinder on a plane. Equation of this sphere can be represented in the form:

$$r^* = R\, \mathrm{sch}\, \psi_c, \qquad (7.49)$$

ψ_c — isometric latitude on a sphere.

Parametric lines of aposphere, meridians and parallels, convert to a sphere without distortion.

2. Isometric coordinates of a sphere, as in the case of representation of ellipsoid on a sphere by Gauss, are determined from equations:

$$\left.\begin{array}{l} \lambda_c = \alpha\lambda, \\ \psi_c = \alpha(\psi + c) \end{array}\right\}. \qquad (7.50)$$

Since scale m is constant everywhere, then:

$$mr^* = \alpha mr. \qquad (7.51)$$

From formulas (7.50) and (7.51) it follows that any expression, determining projection of meridians and parallels of a sphere of radius R on a plane, as function of values λ_c and ψ_c, in accuracy is applicable for projection of an aposphere on a plane, if ψ_c is substituted for ψ and λ_c for λ. This position is equally applicable with respect to both lengths and angles. Character of projection from such substitution is not changed.

3. Main radii of curvature M and N at any point of aposphere just as on a

spheroid (with the exception of poles), are not equal. Therefore selection of constant R^*, α, c can be carried out so that where insignificant areas the scale will remain factually constant and close to a unit, i.e., at points with equal ψ values of r^* and r will be almost equal.

After projection of ellipsoid on an aposphere it is possible to resolve geodetic problems on the aposphere. However this problem was not developed in detail up till now. But use of aposphere as intermediate instance during projection of an ellipsoid on a plane renders geometric clarity of resolution of the problem. In this case proceed thus.

1. Depict surface of an ellipsoid on aposphere which is reduced to determination of constant parameters R^*, α and c.

2. Aposphere is developed on a sphere, i.e., the law of transition of isometric coordinates ψ and λ of aposphere to ψ_c and λ_c of a sphere, is established.

3. Project sphere on a plane of a chosen projection. With suitable selection of parameters scale of the image at any point will be little different from the scale of the image of a sphere on a plane. Therefore reduction in angles and lengths will be small.

CHAPTER VIII

GEODETIC PROJECTIONS

§ 43. BASIC POSITIONS AND DETERMINATION

Engineering geodetic works, intended for geodetic guarantee of construction of tunnels, irrigating systems, thermal and hydro-electric stations, airports, railroads, highways, superhighways, bridges, industrial and agricultural projects and so forth, are as a rule, executed in a comparatively small areas. State topographic surveys, especially large-scale, being developed gradually, also embrace in every stage only small parts of the terrain.

System of coordinates and mathematical treatment of materials of limited geodetic nets, made for indicated purposes, have to be of the simpler type. For engineering-geodetic work it is inexpedient to use a system of geodetic coordinates, in spite of the fact that they are general for all the surface of the terrestrial spheroid, since they are obtained by means of relatively complicated calculations and moreover are in arc form, but linear values of arc units change with change of latitude of the place. The simpler form is the grid system of coordinates on a plane, which however, is not directly connected with the surface of terrestrial spheroid. Investigation of curvature of the surface of a spheroid shows that only very small sections of it can be taken as a plane. Thus, for instance, if one were to determine lineal elements of geodetic nets with accuracy of up to 0.4 mm, then only a section of earth's surface of 5 km radius can be taken as a plane. Therefore application of plane grid coordinates in geodetic work is only possible by means of projection of parts of the surface of a reference-ellipsoid on a plane. Selection of projection for converting geodetic construction from ellipsoid to a plane presents theoretically and practically an important problem for spheroidal geodesy.

Projections of reference-ellipsoid on planes, taken for conversion and treatment of geodetic measurements, are called geodetic. In distinction from cartographic projections, where the main problem consists of representation of earth's surface on a paper (plane), geodetic projections give methods of exact conversion of elements of surface of an ellipsoid (lines, angles) to a plane. Many cartographic and geodetic projections can be offered. In selection of geodetic projections initial conditions are: amount of distortions and simplicity of their calculation. It is quite clear that the less the distortion in a given projection, the greater the territory where it can be applied. However minimum of distortions and simplicity of their calculation in general are incompatible in geodetic projections. Characteristic peculiarity of geodetic projections is in the fact that for translation and treatment of every geodetic net the whole process of application of projection is wholly repeated.

Distortions are inevitable in any-projection, therefore the main requirement in selection of geodetic projection should be considered the ease and convenience of calculation of distortions. However this requirement still does not determine the character and form of projection.

Geodetic construction, as a rule, is developed by means of measurement of angles of geometric figures, and linear measurements are made, for instance, in triangulation only for assignment of scale of the net. Thus, in selection of projection a condition should be set, that angles of geodetic nets during their translation from an ellipsoid to a plane of projection preserve their values. Such projections, where equality of angles is observed, are called equiangular or conformal in mathematical cartography.

For geodesy conformal projections possess a very important property, they preserve similarity in infinitesimal parts. However there is an infinite number of conformal projections of an ellipsoid to a plane. Problem in general consists in selection from them of one, that best satisfies the geographic disposition of a given area and is convenient for practical calculations.

Territory of the Soviet Union extends approximately 45° in latitude, and nearly 150° in longitude. Mathematical cartography recommends in general that in representation of a territory, stretched along longitudes, the use of conical projections. Therefore, it would seem that for geodetic work in USSR one should take some conical conformal projection. However investigations show that transition from ellipsoidal elements to a plane is very complicated in conical projections. Besides the, central parallel of an area, whose image, as a rule, is taken for the axis of ordinates, in conical projections will be a circumference. Due to this it is necessary to divide the depicted area by meridians into smaller sections, within the limits of which the representation of a central parallel can be taken for a straight line. This produces great inconveniences, particularly in significant removal from the central meridi of the country. As can be seen from preceding chapter constant conical projections change with the change of central parallel.

The above fundamental considerations formed the basis for selection of geodetic projection for the USSR. Selection, in 1928-1930 fell on Gauss-Kruger conformal projection, which up to that time had comparatively small application in geodetic work in USSR and abroad.

Gauss-Kruger projection, initially called "Method of Projection, Hanover State Survey," was developed and introduced in thirtieth years of the past century by Gauss during survey of the Hanover-Duchy. However during his life Gauss did not publish this work. Ideas and individual investigations of Gauss in the form of miscellaneous notes were revealed in his literary heritage by Kruger and published in IX volume of the works of Gauss. All this material is translated into Russian language and published in the second volume "of Selected geodetic compositions" of Gauss.[1] Kruger service is in the fact that he developed and systematically expounded the theory and practice of this projection in his work "Konforme Abbidung des Erdellipsoids in der Ebene," (Conformal representation of terrestrial ellipsoid on a plane. Leipzig, 1912).

Gauss-Kruger projection (transverse-cylindrical for a sphere) is used in separation of the surface of a reference-ellipsoid into coordinate zones, bounded by meridians and spreading from North to South Poles.

Gauss-Kruger projection is determined by the following conditions:

1. Gauss-Kruger projection is conformal, i.e., the scale of the image is constant at a given point and consequently, depends only on coordinates of a point.

2. Axial meridian of each zone is depicted on a plane by a straight line, taken as an axis of abscissas.

Origin of coordinates in each zone is selected at a point of intersection of

[1]K. F. Gauss. Selected geodetic compositions. Vol. II, "Higher geodesy." Edited and with introduction by G. V. Bagratuni. M., Geodezizdat, 1958, p. 149-171.

the image of the axial meridian with the image of the equator. Axis of ordinates coincides with the image of equator.

3. The scale of the image on axial meridian is equal to 1, i.e., axial meridian is depicted on a plane in full-size. Thus, for points of axial meridian the abscissas are equal to arcs of meridian, counted from equator.

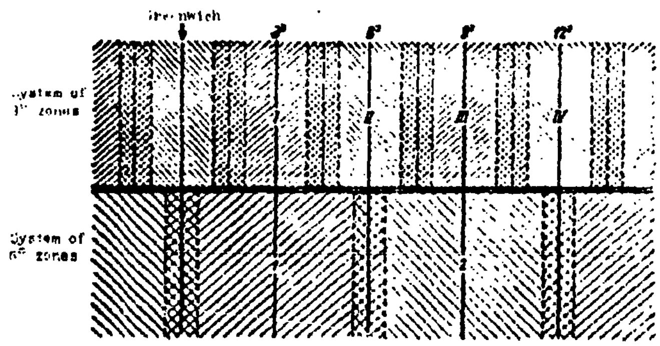


Fig. 82.

Width of grid coordinate zones is established, proceeding from values of linear distortions and taking into account the convenience of practical application of formulas. In USSR two systems of coordinate zones are used: six-degree and three-degree zones (Fig. 82). Axial meridians of six-degree zones coincide with central meridians on map sheets, scale 1:1,000,000 and ordinal number of a zone is determined by the formula

$$n = N - 30,$$

where N — number of column of map sheet 1:1,000,000.

Longitudes of axial meridians of six-degree zones are determined by the formula

$$L_0 = 6n - 3.$$

Within the limits of USSR, abscissas of Gauss-Kruger coordinates, counted from the image of equator to north, are positive, ordinates are also positive eastward, they are negative westward from axial meridian. In order not to deal with negative values, at points of axial meridian ordinates of 500,000 m with obligatory indication ahead of a number of coordinate zone are conditionally added.

System of three-degree zones is used for large-scale surveys and treatment of materials of numerical surveys. Axial meridians of three-degree zones are selected so that they either coincide with central meridians of individual meridians of map sheets of 1:1,000,000 scale. Coincidence of central and axial meridians occurs through every three-degree zone, therefore half of them coincides with central, and half with individual meridians of the 1:1,000,000-map sheets. Longitudes of axial meridians of three-degree zones are determined by the formula

$$L_0 = 3k,$$

k - number corresponding to three-degree zone.

In three-degree zones rule of signs for abscissas and ordinates is the same, as in six-degree zones, but conditional increase of ordinates is not applied.

System of coordinate zones in geodetic work of USSR is firmly fixed, numbers and axial meridians are predetermined. In Table 9 all data, pertaining to coordinate zones of USSR on Gauss-Kruger projection is given.

Table 9

Six-Degree zones			Three-Degree zones			
Zone Numbers	Longitude of axial meridians	Number of columns of 1:1,000,000 maps	Zone Numbers	Longitude of axial meridians	Zone Numbers	Longitude of axial meridians
5	27°	35	8	24°	34	102°
6	33	36	9	27	35	105
7	39	37	10	30	36	108
8	45	38	11	33	37	111
9	51	39	12	36	38	114
10	57	40	13	39	39	117
11	63	41	14	42	40	120
12	69	42	15	45	41	123
13	75	43	16	48	42	126
...	...	...	...	...	...	...
22	129	52	25	75	51	153
23	135	53	26	78	52	156
24	141	54	27	81	53	159
25	147	55	28	84	54	162
26	153	56	29	87	55	165
27	159	57	30	90	56	168
28	165	58	31	93	57	171
29	171	59	32	96	58	174
30	177	60	33	99	59	177

From preceding account it follows that of Gauss-Kruger projection allows to establish uniformity in calculation of plane conformal coordinates for all of the USSR, but these coordinates are calculated in a definite zone, where each zone has its own system of coordinates. Therefore in their practical application it is necessary to recompute coordinates from one zone to one adjacent to it. In connection with this in USSR overlap of zones by 37'.5 in longitude is established: each six-degree western zone overlaps eastern by 30', and eastern the

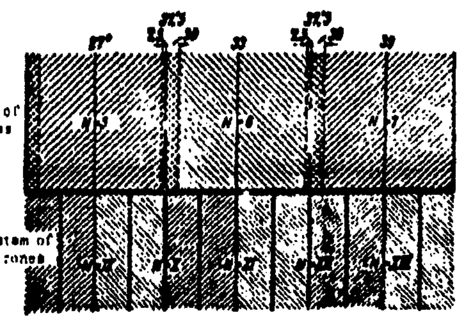


Fig. 85.

western – by 7.5' (Fig. 83).[1] Consequently, coordinates of points of geodetic nets located in an overlap band are given, in a system of two adjacent zones.

In geodetic work of special assignment, for instance in a survey of cities, construction of tunnels, construction of industrial and agricultural projects and so forth, for the purpose of decrease or exception of influence of distortions of projection, deviation is allowed from conventional scheme of application of projection and coordinates of Gauss-Kruger. In these cases the origin of coordinates and axial meridian are selected in the center of the object; however coordinates of points of basic geodetic net must also be calculated in corresponding six or three-degree zones. In selection of sectional beginning of grid coordinates and sectional axial meridian they should be so calculated so that distortion of projection would not be taken into consideration.

Basic designations and values, used in transition from an ellipsoid to a plane, in Gauss-Kruger projection are shown below in an example of translation of a triangulation triangle.

In Fig. 84 QP is axial meridian of a zone; P_1P is a meridian of a point P_1; P_1T is a geodetic parallel. $P_1P_2P_3$ is a triangulation triangle, whose sides s, s_1 and s_2 are geodesics; A is azimuth of a geodesic s; t is a geodetic convergence of meridians on an ellipsoid; l is difference of geodetic longitudes. Geodetic coordinates of point P_1 (B and L) are considered given.

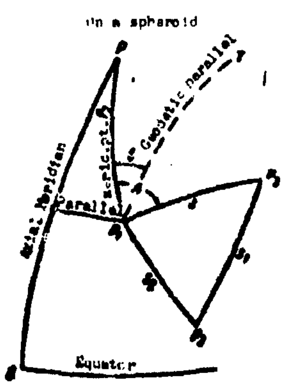


Fig. 84.

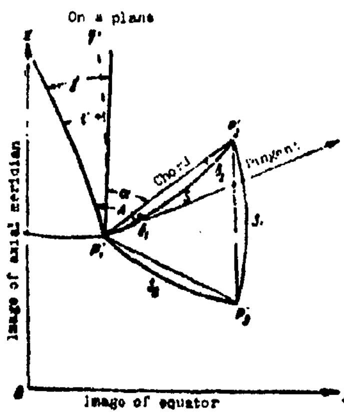


Fig. 85.

In Fig. 85 a geodetic triangle $P_1P_2P_3$ is depicted (Fig. 84) on a plane of Gauss-Kruger projection. OX is image of axial meridian; $P_1'X$ is meridian of point P_1'; $P_1'P_2'P_3'$ is an image of spheroidal triangle $P_1P_2P_3$. Straight lines, connecting points P_1', P_2' and P_3' are chords of images of geodesics s, s_1 and s_2; P_1T' is a line,

[1]At present zone overlap is established at 1° along longitude: western and eastern zones are mutually overlap by 30'.

parallel to axial meridian. Due to conformity the angles between the corresponding lines on a plane are preserved, therefore angles at vertexes P_1, P_2 and P_3 of the spheroidal triangle are equal to angles of a plane triangle $P_1'P_2'P_3'$, formed by curves by images of the sides of a triangle on a plane. Angle between the chord and the line, parallel to axial meridian, is called directional angle (grid azimuth) on a plane and is designated α; it is counted off by the same rule, as azimuth: the angle between tangent to image of meridian of a given point and line, parallel to axial meridian, is called Gauss convergence of meridians or convergence of meridians on a plane and is designated γ: the angle between chord and image of geodesic is called correction for curvature of image of geodesic or reduction of direction and is designated δ; these corrections are small, but are computed with great accuracy (to $0\overset{''}{.}001$).

Difference γ - t is small value of fourth order and is equal to: $+2/3 l^3 \eta^2 \sin B \cos^2 B + \dots$

Order of translation of support geodetic net from an ellipsoid to a plane in Gauss-Kruger projection consists of the following stages:

1. From geodetic coordinates of initial point of a net convert to Gauss-Kruger grid coordinates; simultaneously calculate Gauss convergence of meridians γ.

2. From length of geodesic and its azimuth at initial point convert to length and directional angle of the chord.

3. From angles between geodesics convert to angles between chords of their image on a plane.

Satisfying these actions, obtain geodetic net of rectilinear triangles on a plane, then equate it by a method of least squares and calculate grid coordinates of all vertexes.

§ 44. MATHEMATICAL BASES OF GAUSS-KRUGER PROJECTION

To depict conformally the surface of a terrestrial spheroid on a plane — means to establish regular conformity between points of a surface and a plane in such a manner that the corresponding angles of small geometric figures of a spheroid and a plane are equal, and the sides are proportional. In theory of geodetic projections the main object is the establishment of an indicated point of conformity, i.e., in determination of coordinates on a plane by geodetic requirements and conversely.

General equations of point conformity can be expressed by the following functional dependencies:

$$\left.\begin{aligned} x &= f_1(B, L) \\ y &= f_2(B, L) \end{aligned}\right\}. \tag{8.1}$$

where B, L are geodetic coordinates, the latitude and longitude of a depicted point, and x, y are its grid plane coordinates on selected projection.

In Gauss-Kruger projection, where depicted part of a surface of a spheroid is broken down into zones, it is expedient to replace geodesic longitudes in (8.1) by differences of longitudes of given and axial meridian, designating them by $l = L - L_0$; mathematical reckonings are simplified, if geodetic latitude B in (8.1) is expressed by isometric latitude, designating it q. Dependence between q and B is obtained in preceding chapter by the formulas (7.11) and (7.11').

Let us assume that indicated transformations are already carried out, then the equations (8.1) will take the form:

$$\left.\begin{aligned} x &= x(q, l) \\ y &= y(q, l) \end{aligned}\right\}. \tag{8.2}$$

System of coordinates (q, l) on ellipsoid possesses a property where dq = dl the surface is broken up into a net of infinitesimal squares. Areas of these squares, naturally are not equal among themselves, since they depend on position of squares on the surface, whose curvature changes from point to point. Such coordinate net is called <u>isometric</u>, and the system (q, l) is <u>isometric system of coordinates</u> on a spheroid. Only grid coordinates on a plane, being also isometric, create a network of equal squares.

Isometric coordinates possess symmetry, i.e., in permutation of coordinates isometric network does not change. By means of conversion of equations (8.2) with respect to q and l it is possible to arrive at:

$$\left.\begin{aligned} q &= q(x, y) \\ l &= l(x, y) \end{aligned}\right\}. \tag{8.3}$$

Equations (8.2) and (8.3) express in general form the point conformity between surface of a spheroid and a plane and determine grid coordinates (x, y) by required (q, l). Form of functions (8.2) and (8.3) is determined by required conditions which should satisfy the image of a spheroid on a plane.

From equations (8.2) and (8.3) by means of differentiation we obtain

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial q}dq + \frac{\partial x}{\partial l}dl \\ dy &= \frac{\partial y}{\partial q}dq + \frac{\partial y}{\partial l}dl \end{aligned}\right\}. \tag{8.4}$$

$$\left.\begin{aligned} dq &= \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy \\ dl &= \frac{\partial l}{\partial x}dx + \frac{\partial l}{\partial y}dy \end{aligned}\right\}. \qquad (8.5)$$

Partial derivatives in (8.4) and (8.5) have to satisfy fundamental equations of transformation of coordinates (7.8), which were obtained in preceding chapter. They have the form:

$$\left.\begin{aligned} \sqrt{E'G}\frac{\partial u'}{\partial u} &= \sqrt{EG'}\frac{\partial v'}{\partial v} \\ \sqrt{E'E}\frac{\partial u'}{\partial v} &= -\sqrt{G'G}\frac{\partial v'}{\partial u} \end{aligned}\right\}. \qquad (8.6)$$

Here E, E', G, G' are coefficients of first quadratic form of Gauss on required two surfaces (u, v) and (u', v') are curvilinear coordinates on these surfaces. Let us consider equations (8.6) between isometric coordinates of spheroid and a plane.

Square of lineal element of a spheroid has the following form in geodetic coordinates:

$$ds^2 = M^2 dB^2 + r^2 dl^2 = r^2(dq^2 + dl^2), \qquad (8.7)$$

$$dq = \frac{MdB}{r} \text{ when } q = \int \frac{MdB}{r}. \qquad (8.8)$$

Let us assume that system (u', v') coincides with system (x, y), i.e., E' = G' = = 1, and system (u, v) — with system (q, l), hence E = G = r^2, then from (8.6) for our case:

$$\left.\begin{aligned} \frac{\partial x}{\partial q} &= \frac{\partial y}{\partial l} \\ \frac{\partial x}{\partial l} &= -\frac{\partial y}{\partial q} \end{aligned}\right\}. \qquad (8.9)$$

Let us assume now that systems of coordinates (x, y) and (q, l) correspondingly coincide with (u, v) and (u', v'), then from (8.6) we obtain absolutely symmetric (8.9) equations in the form:

$$\left.\begin{aligned} \frac{\partial q}{\partial x} &= \frac{\partial l}{\partial y} \\ \frac{\partial q}{\partial y} &= -\frac{\partial l}{\partial x} \end{aligned}\right\}. \qquad (8.10)$$

Equations (8.9) and (8.10) are fundamental equations of conformal transformation of isometric coordinates. Their integration is made under initial conditions, which are set for representation of an ellipsoid on a plane or conversely. These equations are called conditions of Cauchy-Riemann in the theory of analytic functions; they are fundamental interrelationships as in a theory of analytic functions, just as in conformal representation of surfaces.

1. Formulas for Calculation of Gauss-Kruger Coordinates by Geodetic Coordinates

In Gauss-Kruger projection the axial meridian is depicted by a straight line into a natural value, i.e., for points of axial meridian abscissas are equal to arcs of meridian, but ordinates are zero. If we designate the arc of meridian by X, then for points of axial meridian where $l = 0$ we obtain:

$$\left.\begin{aligned} x &= X \\ y &= 0 \end{aligned}\right\}. \qquad (8.11)$$

In addition, positive l has to correspond to positive y and to negative l to negative y; to positive and negative l only positive x corresponds. These conditions fully determine Gauss-Kruger projection.

Following power series satisfy the set conditions for Gauss-Kruger projection

$$\left.\begin{aligned} x &= X + a_2 l^2 + a_4 l^4 + a_6 l^6 + \cdots \\ y &= b_1 l + b_3 l^3 + b_5 l^5 + b_7 l^7 + \cdots \end{aligned}\right\}, \qquad (8.12)$$

where a_2, a_4, a_6 ... b_1, b_3, b_5, b_7 ... are functions of geodetic latitude of a given point.

From (8.12) where $l = 0$ we have $y = 0$ and $x = X$; with negative value of l ordinate y is negative, and abscissa x is positive. These conditions are fully sufficient for integration of equations (8.9) with the help of series (8.12).

From (8.12) it follows:

$$\left.\begin{aligned} \frac{\partial x}{\partial q} &= \frac{dX}{dq} + l^2 \frac{da_2}{dq} + l^4 \frac{da_4}{dq} + l^6 \frac{da_6}{dq} + \cdots \\ \frac{\partial x}{\partial l} &= 2a_2 l + 4a_4 l^3 + 6a_6 l^5 + \cdots \\ \frac{\partial y}{\partial q} &= l \frac{db_1}{dq} + l^3 \frac{db_3}{dq} + l^5 \frac{db_5}{dq} + l^7 \frac{db_7}{dq} + \cdots \\ \frac{\partial y}{\partial l} &= b_1 + 3b_3 l^2 + 5b_5 l^4 + 7b_7 l^6 + \cdots \end{aligned}\right\}. \qquad (8.13)$$

For determination of coefficients a_2, a_4, a_6 ... b_1, b_3, b_5 ... substitute obtained partial derivatives (8.13) in (8.9).

We have:

$$\left.\begin{aligned} b_1 &= \frac{dX}{dq}; & 4a_4 &= -\frac{db_3}{dq} \\ 2a_2 &= -\frac{db_1}{dq}; & 5b_5 &= \frac{da_4}{dq} \\ & & 6a_6 &= -\frac{db_5}{dq} \\ 3b_3 &= \frac{da_2}{dq}; & 7b_7 &= \frac{da_6}{dq} \end{aligned}\right\}. \qquad (8.14)$$

X is arc of meridian, whose element is dX = MdB,

then:

$$dq = \frac{MdB}{N\cos B} = \frac{dX}{r},$$

therefore:

$$\frac{dX}{dq} = r, \tag{8.15}$$

r — radius of parallel.

With this value $\frac{dX}{dq}$ formulas (8.14) will take form:

$$\left.\begin{aligned} b_1 &= r, & & \\ 2a_2 &= -\frac{dr}{dq}, & 120\,b_5 &= \frac{d^4r}{dq^4} \\ 6b_3 &= -\frac{d^2r}{dq^2}, & 720\,a_6 &= -\frac{d^5r}{dq^5} \\ 24a_4 &= \frac{d^3r}{dq^3}, & 5040\,b_7 &= \frac{d^6r}{dq^6} \end{aligned}\right\}. \tag{8.16}$$

Derivatives $\frac{d^ir}{dq^i}$ (i = 1, 2, 3 ...) have the following values:

$$\left.\begin{aligned} &1.\ \frac{dr}{dq} = -N\cos B\sin B \\ &2.\ \frac{d^2r}{dq^2} = -N\cos^3 B(1 - t^2 + \eta^2) \\ &3.\ \frac{d^3r}{dq^3} = N\cos^3 B\sin B(5 - t^2 + 9\eta^2 + 4\eta^4) \\ &4.\ \frac{d^4r}{dq^4} = N\cos^5 B(5 - 18t^2 + t^4 + 14\eta^2 - 58\eta^2t^2 + 13\eta^4 - 64\eta^4t^2) \\ &5.\ \frac{d^5r}{dq^5} = -N\cos^5 B\sin B(61 - 58t^2 + t^4 + 270\eta^2 - 330\eta^2t^2) \\ &6.\ \frac{d^6r}{dq^6} = -N\cos^7 B(61 - 479t^2 + 179t^4 - t^6) \end{aligned}\right\}. \tag{8.17}$$

Here, as before, t = tg B; $\eta^2 = e'^2\cos^2 B$; e' are second meridian eccentricity.

In value $\frac{d^5r}{dq^5}$ terms with η^4 are dropped, and in $\frac{d^6r}{dq^6}$ terms with η^2 are dropped. Substituting values of derivatives $\frac{d^ir}{dq^i}$ in (8.16), we obtain the following system of formulas for coefficients of power series (8.12):

$$\left.\begin{aligned} a_2 &= \frac{N}{2}\cos B\sin B \\ a_4 &= \frac{N\cos^3 B\sin B}{24}(5 - t^2 + 9\eta^2 + 4\eta^4) \\ a_6 &= \frac{N\cos^5 B\sin B}{720}(61 - 58t^2 + t^4 + 270\eta^2 - 330\eta^2t^2) \\ b_1 &= N\cos B \\ b_3 &= \frac{N\cos^3 B}{6}(1 - t^2 + \eta^2) \\ b_5 &= \frac{N\cos^5 B}{120}(5 - 18t^2 + t^4 + 14\eta^2 - 58\eta^2t^2 + 13\eta^4 - 64\eta^4t^2) \\ b_7 &= \frac{N\cos^7 B}{5040}(61 - 479t^2 + 179t^4 - t^6) \end{aligned}\right\}. \tag{8.17'}$$

Consequently,

$$x = X + \frac{l''^2}{2\rho''^2} N \cos B \sin B + N\frac{l''^4}{24\rho''^4} \sin B \cos^3 B (5 - t^2 + 9\eta^2 + 4\eta^4) + $$
$$+ \frac{l''^6}{720\rho''^6} N \sin B \cos^5 B (61 - 58t^2 + t^4 + 270\eta^2 - 330\eta^2 t^2) \qquad (8.18)$$

$$y = \frac{l''}{\rho''} N \cos B + N \frac{l''^3}{6\rho''^3} \cos^3 B (1 - t^2 + \eta^2) + \frac{l''^5}{120\rho''^5} \ldots \cos^5 B (5 - 18t^2 +$$
$$+ t^4 + 14\eta^2 - 58\eta^2 t^2 + 13\eta^4 - 64\eta^4 t^2) +$$
$$+ \frac{l''^7}{5040\rho''^7} N \cos^7 B (61 - 479t^2 + 179t^4 - t^6). \qquad (8.19)$$

Formulas (8.18) and (8.19) possess high accuracy and can be applied for differences of longitudes $l \approx 3\text{-}4^{\circ}$, i.e., for a system of six-degree zones. Naturally, for three-degree zones these formulas can be simplified namely: in formula for x terms $l^4\eta^4$ and l^6, and for y, terms with $l^5\eta^2$ and l^7 can be dropped, then for such a case we have:

$$\left.\begin{aligned} x &= X + \frac{l''^2}{2\rho''^2} N \sin B \cos B + \frac{l''^4}{24\rho''^4} \sin B \cos^3 B (5 - t^2 + 9\eta^2) \\ y &= \frac{l''}{\rho''} N \cos B + \frac{l''^3}{6\rho''^3} N \cos^3 B (1 - t^2 + \eta^2) + \\ &+ \frac{l''^5}{120\rho''^5} N \cos^5 B (5 - 18t^2 + t^4) \end{aligned}\right\} \qquad (8.18')$$

9. Formulas for Calculation of Geodetic Coordinates by Gauss-Kruger Coordina'

In order to obtain formulas for calculation of geodetic coordinates by Gauss-Kruger coordinates, it is necessary to integrate differential equations (8.10) under following initial conditions: with $y = 0$ should be $l = 0$, $x_0 = X_0$ and $q = q_0$. Considering the symmetry of projections with respect to axial meridian and the fact that sign l always corresponds to sign y and with any sign of y value q is positive, we have:

$$\left.\begin{aligned} q &= q_0 + a_2' y^2 + a_4' y^4 + a_6' y^6 + \cdots \\ l &= b_1' y + b_3' y^3 + b_5' y^5 + b_7' y^7 + \cdots \end{aligned}\right\}. \qquad (8.19')$$

Here coefficients a_2', a_4', a_6', b_1', b_3', b_5', b_7' are functions of latitude of the base of ordinates y. Let us designate this latitude by B_0, it is obtained by required x, if x is considered an arc of axial meridian. B_0 is calculated by x according to tables for arcs of meridians.

From (8.19')

$$
\left.
\begin{aligned}
\frac{\partial q}{\partial x} &= \frac{dq_0}{dx} + y^2\frac{da_2'}{dx} + y^4\frac{da_4'}{dx} + \cdots \\
\frac{\partial q}{\partial y} &= 2a_2' y + 4a_4' y^3 + 6a_6' y^5 + \cdots \\
\frac{\partial l}{\partial x} &= y\frac{db_1'}{dx} + y^3\frac{db_3'}{dx} + y^5\frac{db_5'}{dx} + \cdots \\
\frac{\partial l}{\partial y} &= b_1' + 3b_3' y^2 + 5b_5' y^4 + \cdots
\end{aligned}
\right\} \qquad (8.20)
$$

In accordance with equations (8.10) and (8.20) we have the following equations for determination of coefficients: a_{2i}', b' $(i = 1, 2, \ldots)$

$$
\begin{aligned}
&\frac{dq_0}{dx} + y^2\frac{da_2'}{dx} + y^4\frac{da_4'}{dx} + y^6\frac{da_6'}{dx} + \cdots = b_1' + \\
&\quad + 3b_3' y^2 + 5b_5' y^4 + 7b_7' y^6 + \cdots \\
&y\frac{db_1'}{dx} + y^3\frac{db_3'}{dx} + y^5\frac{db_5'}{dx} + y^7\frac{db_7'}{dx} = \\
&\quad = -(2a_2' y + 4a_4' y^3 + 6a_6' y^5 + \cdots),
\end{aligned}
$$

whence

$$
\left.
\begin{array}{ll}
1.\ b_1' = \dfrac{dq_0}{dx}, & \\
2.\ 3b_3' = \dfrac{da_2'}{dx}, & 5.\ 2a_2' = -\dfrac{db_1'}{dx} \\
3.\ 5b_5' = \dfrac{da_4'}{dx}, & 6.\ 4a_4' = -\dfrac{db_3'}{dx} \\
4.\ 7b_7' = \dfrac{da_6'}{dx}, & 7.\ 6a_6' = -\dfrac{db_5'}{dx}
\end{array}
\right\} \qquad (8.20')
$$

As before,

$$
dx = dX = MdB, \quad dq = \frac{MdB}{N\cos B}, \quad dq_0 = \left(\frac{MdB}{N\cos B}\right)_0 = \frac{dX_0}{r_0};
$$

sign "0" here designates that the corresponding values are referred to latitude B_0 base of ordinate y.

Consequently,

$$
\frac{dq_0}{dx} = \frac{1}{N_0\cos B_0} = \frac{1}{r_0},
$$

$$
\left.
\begin{aligned}
&1.\quad b_1' = \frac{1}{r_0} \\
&2.\quad 2a_2' = \frac{1}{r_0^2}\left(\frac{dr}{dx}\right)_0 \\
&3.\quad 6b_3' = -\left\{\frac{2}{r_0^3}\left(\frac{dr}{dx}\right)_0^2 - \frac{1}{r_0^2}\left(\frac{d^2r}{dx^2}\right)_0\right\} \\
&4.\quad 24a_4' = -\left\{\frac{6}{r_0^4}\left(\frac{dr}{dx}\right)_0^3 - \frac{6}{r_0^3}\left(\frac{dr}{dx}\right)_0\left(\frac{d^2r}{dx^2}\right)_0 + \frac{1}{r_0^2}\left(\frac{d^3r}{dx^3}\right)_0\right\} \\
&5.\quad 120b_5' = \left\{\frac{24}{r_0^5}\left(\frac{dr}{dx}\right)_0^4 - \frac{36}{r_0^4}\left(\frac{dr}{dx}\right)_0^2\left(\frac{d^2r}{dx^2}\right)_0 + \right. \\
&\qquad\left. + \frac{8}{r_0^3}\left(\frac{dr}{dx}\right)_0\left(\frac{d^3r}{dx^3}\right)_0 + \frac{6}{r_0^3}\left(\frac{d^2r}{dx^2}\right)_0^2 - \frac{1}{r_0^2}\left(\frac{d^4r}{dx^4}\right)_0\right\}
\end{aligned}
\right\} \qquad (8.21)
$$

$$6.\quad 720a_6' = \Big[\frac{120}{r_0^6}\left(\frac{dr}{dx}\right)_0^6 - \frac{240}{r_0^5}\left(\frac{dr}{dx}\right)_0^3\left(\frac{d^2r}{dx^2}\right)_0^2 +$$
$$+\frac{90}{r_0^4}\left(\frac{dr}{dx}\right)_0\left(\frac{d^2r}{dx^3}\right)_0^2 + \frac{60}{r_0^4}\left(\frac{dr}{dx}\right)_0^2\left(\frac{d^3r}{dx^3}\right)_0 -$$
$$-\frac{20}{r_0^3}\left(\frac{d^3r}{dx^3}\right)_0\left(\frac{d^3r}{dx^3}\right)_0 - \frac{10}{r_0^3}\left(\frac{dr}{dx}\right)_0\left(\frac{d^4r}{dx^4}\right)_0 - \frac{1}{r_0^2}\left(\frac{d^5r}{dx^5}\right)_0\Big] \qquad (8.21)\ \text{cont.}$$

but dr = -M sin BdB, dx = MdB, therefore:

$$\left.\begin{aligned}
\left(\frac{dr}{dx}\right)_0 &= -\sin B_0\\
\left(\frac{d^2r}{dx^2}\right)_0 &= -\cos B_0\left(\frac{dB}{dx}\right)_0\\
\left(\frac{d^3r}{dx^3}\right)_0 &= \sin B_0\left(\frac{dB}{dx}\right)_0^2 - \cos B_0\left(\frac{d^2B}{dx^2}\right)_0\\
\left(\frac{d^4r}{dx^4}\right)_0 &= \cos B_0\left(\frac{dB}{dx}\right)_0^3 + 3\sin B_0\left(\frac{dB}{dx}\right)_0\left(\frac{d^2B}{dx^2}\right)_0 - \cos B_0\left(\frac{d^3B}{dx^3}\right)_0\\
\left(\frac{d^5r}{dx^5}\right)_0 &= -\sin B_0\left(\frac{dB}{dx}\right)_0^4 + 6\cos B_0\left(\frac{dB}{dx}\right)_0^2\left(\frac{d^2B}{dx^2}\right)_0 +\\
&+ 4\sin B_0\left(\frac{dB}{dx}\right)_0\left(\frac{d^3B}{dx^3}\right)_0 + 3\sin B_0\left(\frac{d^2B}{dx^2}\right)_0 - \cos B_0\left(\frac{d^4B}{dx^4}\right)_0
\end{aligned}\right\} \qquad (8.22)$$

In derivatives $\frac{d^ir}{dx^i}$ (i = 1, 2, 3, 4, 5) and $\frac{d^nB}{dx^n}$ (n = 1, 2, 3, 4) have the following values:

$$\left.\begin{aligned}
\left(\frac{dB}{dx}\right)_0 &= \frac{V_0^3}{c}\\
\left(\frac{d^2B}{dx^2}\right)_0 &= -\frac{3V_0^4}{c^2}\eta_0^2 t_0\\
\left(\frac{d^3B}{dx^3}\right)_0 &= -\frac{3V_0^5}{c^3}\eta_0^2(1+\eta_0^2-t_0^2-5\eta_0^2t_0^2)\\
\left(\frac{d^4B}{dx^4}\right)_0 &= \frac{6V_0^6\eta_0^2t_0}{c^4}(1+t_0^2+5\eta_0^2+5\eta_0^2t_0^2)
\end{aligned}\right\} \qquad (8.23)$$

Last derivative is taken in "spherical presentation," i.e., in its calculation it is taken η_0^2 = const.

Calculating derivatives $\frac{d^ir}{dx^i}$ by (8.23) and substituting them in (8.21), for coefficients of power series (8.19) we obtain:

$$\left.\begin{aligned}
b_1' &= \frac{1}{r_0} = \frac{1}{N_0\cos B_0} = \frac{\sec B_0}{N_0}\\
b_3' &= -\frac{\sec B_0}{6N_0^3}(1+2t_0^2+\eta_0^2)\\
b_5' &= \frac{\sec B_0}{120N_0^5}(5+28t_0^2+24t_0^4+6\eta_0^2+8\eta_0^2t_0^2)\\
a_2' &= -\frac{t_0\sec B_0}{2N_0^2}\\
a_4' &= \frac{t_0\sec B_0}{24N_0^4}(5+6t_0^2+\eta_0^2-4\eta_0^4)\\
a_6' &= -\frac{t_0\sec B_0}{720N_0^6}(61+180t_0^2+120t_0^4+46\eta_0^2+48\eta_0^2t_0^2)
\end{aligned}\right\} \qquad (8.24)$$

Substituting these values of coefficients in (8.19), we obtain

$$q = q_0 - y^2 \frac{t_0 \sec B_0}{2N_0^2} + y^4 \frac{t_0 \sec B_0}{24N_0^4}(5 + 6t_0^2 + \eta_0^2 - 4\eta_0^4) -$$
$$- y^6 \frac{t_0 \sec B_0}{720N_0^6}(61 + 180t_0^2 + 120t_0^4 + 46\eta_0^2 + 48\eta_0^2 t_0^2). \qquad (8.25)$$

$$l = y \frac{\sec B_0}{N_0}\rho'' - y^3 \frac{\sec B_0}{6N_0^3}\rho''(1 + 2t_0^2 + \eta_0^2) +$$
$$+ y^5 \frac{\sec B_0}{120N_0^5}\rho''(5 + 28t_0^2 + 24t_0^4 + 6\eta_0^2 + 8t_0^2\eta_0^2). \qquad (8.26)$$

Formula (8.26) is final, and formula (8.25) must be converted in such a way that from q and q_0 we shift correspondingly to B and B_0. We have:

$$B = B(q), \quad B_0 = B(q_0)$$

We designate:

$$\Delta q = q_0 - q,$$

then:

$$B = B_0 - \Delta q\left(\frac{dB}{dq}\right)_0 + \frac{\Delta q^2}{2!}\left(\frac{d^2B}{dq^2}\right)_0 - \frac{\Delta q^3}{3!}\left(\frac{d^3B}{dq^3}\right)_0 + \cdots, \qquad (8.27)$$

where:

$$\left.\begin{aligned}
\left(\frac{dB}{dq}\right)_0 &= V_0^2 \cos B_0 \\
\left(\frac{d^2B}{dq^2}\right)_0 &= -\cos B_0 \sin B_0 (1 + 4\eta_0^2 + 3\eta_0^4), \\
\left(\frac{d^3B}{dq^3}\right)_0 &= -\cos^3 B_0 (1 - t_0^2 + 5\eta_0^2 - 13\eta_0^2 t_0^2 + 7\eta_0^4 - 27t_0^2\eta_0^4) \\
\left(\frac{d^4B}{dq^4}\right)_0 &= \cos^3 B_0 \sin B_0 (5 - t_0^2 + 56\eta_0^2 - 40t_0^2\eta_0^2)
\end{aligned}\right\}. \qquad (8.28)$$

Substituting value Δq^i and $\frac{d^iB}{dq^i}$ (i = 1, 2, 3) from (8.25) and (8.28) to (8.27), for unknown latitudes we have:

$$B = B_0 - y^2 \frac{t_0 V_0^2}{2N_0^2}\rho'' + y^4 \frac{t_0}{24N_0^4}\rho''(5 + 3t_0^2 + 6\eta_0^2 - 6\eta_0^2 t_0^2) -$$
$$- y^6 \frac{t_0}{720N_0^6}\rho''(61 + 90t_0^2 + 45t_0^4). \qquad (8.29)$$

We designate:

$$\bar{a}_2 = -\frac{t_0 V_0^2}{2N_0^2}\rho'',$$
$$\bar{a}_4 = \frac{t_0}{24N_0^4}\rho''(5 + 3t_0^2 + 6\eta_0^2 - 6\eta_0^2 t_0^2),$$
$$\bar{a}_6 = -\frac{t_0}{720N_0^6}\rho''(61 + 90t_0^2 + 45t_0^4),$$

then:

$$B = B_0 + \bar{a}_2 y^2 + \bar{a}_4 y^4 + \bar{a}_6 y^6. \qquad (8.29')$$

Formulas (8.18), (8.19), (8.26) and (8.29') resolve direct and inverse problems of Gauss-Kruger projection, if it is understood that by direct problem the determination of x, y by B and l, and under inverse the determination B and l by x and y. Obtained formulas appear very bulky, but their right parts are functions of either B, or B_0. Therefore with the presence of corresponding tables for coefficients $a_2, a_4, a_6, \bar{a}_2, \bar{a}_4, \bar{a}_6, b_1, b_3, b_5, b_1', b_3', b_5'$ the problem of calculation of x, y or B and l is resolved rather simply. Such tables exist in USSR both for logarithmic calculation (F. N. Krasovskiy and A. A. Izotov Tables), and for nonlogarithmic calculation (D. A. Larin Tables). Practice of calculations with these tables will be discussed in § 47.

3. Convergence of Meridians on a Plane

Convergence of meridians on a plane or Gauss approach in Gauss-Kruger projection is called an angle between tangent to image of a given meridian and a line, parallel to image of axial meridian.

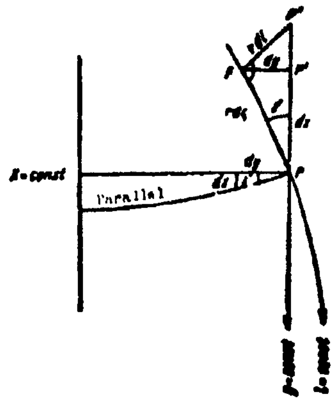


Fig. 86.

From elementary triangle (Fig. 86) with sides dx and dy we have

$$\operatorname{tg}\gamma = \frac{dx}{dy} = \frac{\frac{\partial x}{\partial l}}{\frac{\partial y}{\partial l}}. \tag{8.30}$$

From (8.12)

$$\left.\begin{aligned} \frac{\partial x}{\partial l} &= 2a_2 l + 4a_4 l^3 + 6a_6 l^5 + \cdots \\ \frac{\partial y}{\partial l} &= b_1 + 3b_3 l^2 + 5b_5 l^4 + \cdots \end{aligned}\right\}. \tag{8.31}$$

Consequently:

$$\operatorname{tg}\gamma = \frac{2a_2 l + 4a_4 l^3 + 6a_6 l^5 + \dots}{b_1 + 3b_3 l^2 + 5b_5 l^4} = \frac{2a_2 l}{b_1}\left(1 + \frac{2a_4}{a_2} l^2 + \frac{3a_6 l^4}{a_2}\right)\left(1 + 3\frac{b_3}{b_1} l^2 + \frac{5b_5}{b_1} l^4\right)^{-1}.$$

Breaking down last factor by binomial and retaining small values to fifth power inclusive, we obtain:

$$\operatorname{tg}\gamma = \frac{2a_2 l}{b_1}\left\{1 + l^2\left(\frac{2a_4}{a_2} - \frac{3b_3}{b_1}\right) + l^4\left(\frac{3a_6}{a_2} - \frac{5b_5}{b_1} + \frac{9b_3^2}{b_1^2} - \frac{6a_4}{a_2}\cdot\frac{b_3}{b_1}\right)\right\}.$$

Substituting values a_2, a_4, a_6, b_1, b_3 and b_5 from (8.17'), we obtain:

$$\operatorname{tg}\gamma = l\sin B + \frac{1}{3}\sin B\cos^2 B(1 + t^2 + 3\eta^2 + 2\eta^4)l^3 + $$
$$+ \frac{1}{15}\sin B\cos^4 B(2 + 4t^2 + 2t^4 + 15\eta^2)l^5 + \cdots \qquad (8.32)$$

Convergence of meridians $\gamma \leq 3^{\circ}$, therefore for calculation of this value it is expedient to replace tangent of a small angle by an angle in the formula:

$$\gamma = \operatorname{tg}\gamma - \frac{1}{3}\operatorname{tg}^3\gamma + \frac{1}{5}\operatorname{tg}^5\gamma - \cdots \qquad (8.33)$$

Finally we obtain in seconds

$$\gamma'' = l''\sin B + \frac{1}{3}\frac{\sin B\cos^2 B}{\rho''^2}(1 + 3\eta^2 + 2\eta^4)l''^3 +$$
$$+ \frac{1}{15}\frac{\sin B\cos^4 B}{\rho''^4}(2 - t^2 + 15\eta^2 - 15t^2\eta^2)l''^5. \qquad (8.33')$$

We designate:

$$\left.\begin{aligned} c_1 &= \sin B \\ c_3 &= \frac{1}{3}\frac{\sin B\cos^2 B}{\rho''^2}(1 + 3\eta^2 + 2\eta^4) \\ c_5 &= \frac{1}{15}\frac{\sin B\cos^4 B}{\rho''^4}(2 - t^2 + 15\eta^2 - 15\eta^2 t^2) \end{aligned}\right\}, \qquad (8.34)$$

then:

$$\gamma = c_1 l + c_3 l^3 + c_5 l^5. \qquad (8.35)$$

Formula (8.35) by its construction coincides with formulas (8.12). All of them are convenient for nonlogarithmic calculation with tables of coefficients a, b and c.

In resolution of inverse problem of projection γ can be expressed as a function of grid coordinates (x, y). From small right-angle triangle PFP" (Fig. 86), whose sides are elementary arcs of meridian and parallels, we obtain:

$$\operatorname{tg}\gamma = \frac{r\,dl}{r\,dq} = \frac{\frac{\partial l}{\partial x}}{\frac{\partial q}{\partial x}}. \qquad (8.36)$$

From (8.19'):

$$\frac{\partial l}{\partial x} = y\frac{db_1'}{dx} + y^3\frac{db_3'}{dx} + y^5\frac{db_5'}{dx} + \cdots,$$
$$\frac{\partial q}{\partial x} = \frac{da_0}{dx} + y^2\frac{da_2'}{dx} + y^4\frac{da_4'}{dx} + \cdots$$

Consequently,

$$\operatorname{tg}\gamma = \frac{y\frac{db_1'}{dx} + y^3\frac{db_3'}{dx} + y^5\frac{db_5'}{dx} + \cdots}{\frac{da_0}{dx} + y^2\frac{da_2'}{dx} + y^4\frac{da_4'}{dx} + \cdots};$$

or, considering (8.24) we obtain

$$\operatorname{tg}\gamma = -\frac{2a_2' y + 4a_4' y^3 + 6a_6' y^5 + \dots}{b_1' + 3b_3' y^2 + 5b_5' y^4 + \dots} =$$

$$= -\frac{2a_2' y}{b_1'}\left(1 + \frac{2a_4'}{a_2'} y^2 + \frac{3a_6'}{a_2'} y^4\right)\left(1 + \frac{3b_3'}{b_1'} y^2 + \frac{5b_5'}{b_1'} y^4\right)^{-1}.$$

Breaking down last factor with negative power by binomial theorem and retaining only the terms with y^4, we obtain

$$\operatorname{tg}\gamma = -\frac{2a_2' y}{b_1'}\left\{1 + y^2\left(\frac{2a_4'}{a_2'} - \frac{3b_3'}{b_1'}\right) + \right.$$

$$\left. + y^4\left(\frac{3a_6'}{a_2'} - \frac{5b_5'}{b_1'} + 9\frac{b_3'^2}{b_1'^2} - \frac{6a_4' b_3'}{a_2' b_1'}\right)\right\}.$$

Substituting values a_2', a_4', a_6', b_1', b_3', b_5' from (8.24), we obtain

$$\operatorname{tg}\gamma = \frac{t_0}{N_0} y - \frac{t_0}{3N_0^3}(1 - \eta_0^2 - 2\eta_0^4) y^3 + \frac{2t_0}{15N_0^5}(1 + t_0^2 + 3\eta_0^2 t_0^2) y^5. \qquad (8.37)$$

Changing from tangent to angle by the formula (8.33), we obtain:

$$\gamma = \frac{t_0}{N_0} y - \frac{t_0}{3N_0^3}(1 + t_0^2 - \eta_0^2 - 2\eta_0^4) y^3 +$$

$$+ \frac{t_0}{15N_0^5}(2 + 5t_0^2 + 3t_0^4 + 2\eta_0^2 + \eta_0^2 t_0^2) y^5. \qquad (8.38)$$

Sign "0" means that corresponding values pertain to latitude of the base of ordinate y. We designate:

$$c_1' = \frac{t_0}{N_0},$$

$$c_3' = -\frac{t_0}{3N_0^3}(1 + t_0^2 - \eta_0^2 - 2\eta_0^4),$$

$$c_5' = \frac{t_0}{15N_0^5}(2 + 5t_0^2 + 3t_0^4 + 2\eta_0^2 + \eta_0^2 t_0^2),$$

then:

$$\gamma = c_1' y + c_3' y^3 + c_5' y^5 + \dots \qquad (8.39)$$

Formula (8.39) is applicable in resolution of inverse problem of projection, but in its resolution we at first determine B and l by x and y, therefore γ can be calculated by (8.35) after calculation of B and l. This approach is recommended by D. A. Larin in "Tables for Gauss-Kruger coordinates." With such procedure necessity for tables for c_1', c_3' and c_5', is eliminated, this leads to decrease in volume of tables. However it must be borne in mind that in this case all errors in determination of B and l will in the corresponding manner reflect on determination of γ and there will be no control. Therefore it is expedient to preserve independence of determination of B, l and γ in resolution of inverse problem of projection. For this it is necessary to additionally place tables for c_1', c_3' and c_5' into "Tables of Gauss-Kruger and D. A. Larin coordinates."

Gauss convergence of meridians tables are necessary for transition from azimuth A of a given direction on an ellipsoid to grid azimuth on a plane.

$$\left.\begin{array}{l}\alpha = A - \gamma - \delta \\ A = \alpha + \gamma + \delta\end{array}\right\}. \qquad (8.40)$$

4. Scale of Image

Let us assume that ds is lineal element on an ellipsoid, and dS — on plane, then scale of image is:

$$m = \frac{dS}{ds}. \qquad (8.41)$$

Hence:

$$m^2 = \frac{dS^2}{ds^2} = \frac{dx^2 + dy^2}{r^2(dq^2 + dl^2)} = \frac{1}{r^2}\,\frac{\left(\frac{\partial x}{\partial l}\right)^2 + \left(\frac{\partial y}{\partial l}\right)^2}{1 + \left(\frac{\partial q}{\partial l}\right)^2}.$$

On ellipsoid:

$$\frac{\partial q}{\partial l} = 0,$$

therefore:

$$m^2 = \frac{1}{r^2}\left[\left(\frac{\partial x}{\partial l}\right)^2 + \left(\frac{\partial y}{\partial l}\right)^2\right]. \qquad (8.41')$$

From (8.12)

$$\frac{\partial x}{\partial l} = 2a_2 l + 4a_4 l^3 + \ldots$$

$$\frac{\partial y}{\partial l} = b_1 + 3b_3 l^2 + 5b_5 l^4 + \ldots$$

Retaining small values to l^4,

$$\left(\frac{\partial x}{\partial l}\right)^2 = 4a_2^2 l^2 + 16a_2 a_4 l^4 + \ldots,$$

$$\left(\frac{\partial y}{\partial l}\right)^2 = b_1^2 + 6b_1 b_3 l^2 + 10b_1 b_5 l^4 + 9b_3^2 l^4 + \ldots$$

Consequently,

$$m^2 = \frac{b_1^2}{r^2}\left\{1 + \left(4\frac{a_2^2}{b_1^2} + 6\frac{b_3}{b_1}\right)l^2 + \left(16\frac{a_2 a_4}{b_1^2} + 10\frac{b_5}{b_1} + 9\frac{b_3^2}{b_1^2}\right)l^4\right\}.$$

Substituting values b_1, b_3, b_5, a_2, a_4 (by 8.17') in expression for m^2, we have:

$$m^2 = 1 + V^2\cos^2 B\, l^2 + \frac{\cos^4 B}{3}(2 - t^2 + 5\eta_i^2 - 7\eta_i^2 t^2)\, l^4;$$

with the same accuracy after extraction of square root by means of factorization in binominal series by the formula

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$$

we obtain:

$$m = 1 + \frac{V^2\cos^2 B}{2} l^2 + \frac{\cos^4 B}{24}(5 - 4t^2 + 14\eta_i^2 - 28\eta_i^2 t^2)\, l^4 + \dots \tag{8.42}$$

Where $l = 1/20$, $B =$ [illegible] terms with η^2 in (8.42) are negligibly small. For instance:

$$\frac{\cos^4 B}{24}(14\eta_i^2 - 28\eta_i^2 t^2)\, l^4 < 1 \cdot 10^{-8}.$$

Dropping terms with η^2 in (8.42), we obtain:

$$m = 1 + \frac{V^2\cos^2 B}{2} l^2 + \frac{\cos^4 B}{24}(5 - 4t^2)\, l^4 + \dots \tag{8.43}$$

Designating:

$$d_2 = \frac{\cos^2 B}{2} V^2,$$
$$d_4 = \frac{\cos^4 B}{24}(5 - 4t^2),$$

we finally have:

$$m = 1 + d_2 l^2 + d_4 l^4 + \dots, \tag{8.44}$$

For calculation by formula (8.44) it is necessary to have tables for d_2 and d_4 by argument of latitude B.

In practice the more commonly used is the scale formula as a function of Gauss-Kruger grid coordinates.

For obtaining the shown formula let us express in (8.43) l^2 and l^4 by y^2 and y^4 by means of (8.26) then:

$$\left.\begin{aligned} l^2 &= \frac{y^2\sec^2 B_0}{N_0^2} - \frac{y^4\sec^2 B_0}{3N_0^4}(1 + 2t_0^2) + \dots \\ l^4 &= \frac{y^4\sec^4 B}{N_0^4} + \dots \end{aligned}\right\}. \tag{8.45}$$

Further, omitting details of calculations:

$$\left.\begin{aligned} \cos^2 B &= \cos^2 B_0\left(1 + \frac{y^2 t_0^2}{N_0^2} + \dots\right) \\ V^2 &= V_0^2 + \frac{y^2\eta^2 t^2}{N_0^2} + \dots \end{aligned}\right\}. \tag{8.46}$$

Substituting (8.45) and (8.46) in (8.43) and dropping terms with $\eta^2 y^4$, we obtain:

$$m = 1 + \frac{y^2 V_0^2}{2N_0^2} + \frac{y^4}{24N_0^4} + \ldots$$

But $\frac{V_0^2}{N_0^2} = \frac{1}{R_0^2}$, therefore with accepted accuracy

$$m = 1 + \frac{y^2}{2R_0^2} + \frac{y^4}{24R_0^4} + \ldots \qquad (8.47)$$

Sign "0" as before means that these values pertain to latitude of base of ordinate y. For symmetry we designate

$$d_2' = \frac{1}{2R_0^2}, \quad d_4' = \frac{1}{24R_0^4}. \qquad (8.48)$$

then:

$$m = 1 + d_2' y^2 + d_4' y^4 + \ldots \qquad (8.49)$$

In conclusion of paragraph we give summary of formulas for resolution of inverse problem of Gauss-Kruger projection.

<u>Direct problem</u>. Given: B, l and A; determine x, y, γ and m:

$$\left.\begin{array}{l} 1.\ x = X + a_2 l^2 + a_4 l^4 + a_6 l^6 \\ 2.\ y = b_1 l + b_3 l^3 + b_5 l^5 \\ 3.\ \gamma = c_1 l + c_3 l^3 + c_5 l^5 \\ 4.\ m = 1 + d_2 l^2 + d_4 l^4 \end{array}\right\}. \qquad (8.50)$$

<u>Inverse problem</u>: Given x, y, σ; determine B, l, γ and m:

$$\left.\begin{array}{l} 1.\ B = B_0 + \bar{a}_2 y^2 + \bar{a}_4 y^4 + \bar{a}_6 y^6 \\ 2.\ l = b_1' y + b_3' y^3 + b_5' y^5 \\ 3.\ \gamma = c_1' y + c_3' y^3 + c_5' y^5 \\ 4.\ m = 1 + d_2' y^2 + d_4' y^4 \end{array}\right\}. \qquad (8.51)$$

Formulas (8.50) and (8.51) are symmetric with respect to l and y and are very convenient for nonlogarithmic calculation. Tables for calculation of Gauss-Kruger coordinates must contain coefficients of these formulas, depending on latitude. Coordinates are calculated with accuracy of up to one millimeter, latitudes and longitudes up to 0".0001, and γ up to 0".001. Scale of image is calculated for one unit of eighth decimal place. Such accuracy of calculations are ensured both by reduced formulas (8.50) and (8.51), and by existing tables.

§ 45. REDUCTION PROBLEM OF GAUSS-KRUGER PROJECTION

Reduction problem is understood to be translation of distances and directions from ellipsoid to a plane. Reduction of distances consists of finding the difference of the length of geodesic and chord of image of geodesic, connecting two adjacent points of triangulation. Reduction of directions consists of determination of correction for curvature of conformal image of geodesic on a plane. After introduction of these reductions in measured values we obtain a triangulation network, reduced from ellipsoid to a plane.

In order to have clear concept about the reduction values, we will first find their approximate analytic expressions and numerical characteristics. Let us assume that as before ds is the element of arc of geodesic on a spheroid; dS is the image ds on a plane, then:

$$m = \frac{dS}{ds} \text{ when } dS = m\,ds,$$

whence:

$$S = \int_0^s m\,ds,$$

m — scale of image, which is the function of coordinates of a given point.

Value of scale changes from point to point, this change in small sections is comparatively small and quite regular. Therefore on the basis of Legendre theorem on mean values we can accept:

$$S = m_c s, \tag{8.52}$$

where m_c is the value of a scale at a certain point, intermediate between given ones. In our case, knowing the character of change m, we can take m for median point or for point with mean latitude B_m. The later is convenient for practical application.

By (8.47)

$$m = 1 + \frac{y^2}{2R^2} + \frac{y^4}{24R^4}.$$

in accordance with above we can take:

$$m_c = 1 + \frac{y_m^2}{2R_m^2} + \ldots, \tag{8.53}$$

$$y_m = \frac{y_1 + y_2}{2}.$$

$\frac{1}{R_m^2}$ is Gaussian curvature at a point of mean-latitude.

From (8.52) and (8.53):

$$S = s\left(1 + \frac{y_m^2}{2R_m^2}\right). \tag{8.52'}$$

Formula (8.52') is approximate and gives main term of reduction of distances.

Value $\frac{s y_m^2}{2R_m^2}$ is the value of linear distortion of a given line. Where y_m = [illegible] km, s = [illegible] km (value of ordinate in six-degree zone at edge of zone at mean latitudes and the length of a side of 1st order triangulation) and R_m = 6400 km,

$$\Delta S = \frac{s y_m^2}{2R_m^2} = \frac{30 \cdot 4 \cdot 10^8}{2 \cdot 64 \cdot 64 \cdot 10^6} \approx \frac{30}{32 \cdot 64} \approx 15 \text{ м},$$

or in relative form $\frac{\Delta S}{S} \approx \frac{1}{2000}$.

From this calculation it follows that linear distortion at the edge of the zone in Gauss-Kruger projection is significant enough, to make it necessary to introduce corrections not only in lengths of initial sides of triangulation of all classes, but also in lengths of polygonometry and even theodolite movements.

Let us assume that ABCD is small trapezoid, formed by geodesics on an ellipsoid (Fig. 87); $A_1B_1C_1D_1$ is its image on a plane (Fig. 88). We will unite point C_1 and D_1 by chord d, and angles between arc D_1C_1 and chord at points D_1 and C_1 will be designated by δ_1 and δ_2 correspondingly, then:

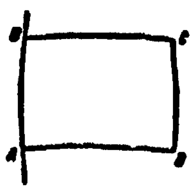


Fig. 87.

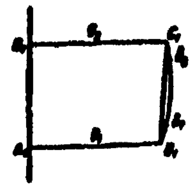


Fig. 88.

$$A_1B_1 = x_2 - x_1;\ B_1C_1 = y_2;\ A_1D_1 = y_1.$$

Sum of the angles of spheroidal trapezoid ABCD is equal $(360 + \varepsilon)$, where ε is spherical excess of this figure; the sum of angles in a plane figure $A_1B_1C_1D_1$ is equal to $(360 + \delta_1 + \delta_2)$. By condition of conformity

$$360 + \varepsilon = 360 + \delta_1 + \delta_2.$$

Consequently,

$$\varepsilon = \delta_1 + \delta_2,$$

but:

$$\varepsilon = \frac{P}{R^2}\rho''.$$

P — are of trapezoid $A_1B_1C_1D_1$, equal:

$$P = (x_2 - x_1)\frac{y_2 + y_1}{2},$$

therefore:

$$\delta_1 + \delta_2 = \varepsilon = \frac{x_2 - x_1}{2R^2}(y_1 + y_2)\rho''$$

or:

$$\delta_1 + \delta_2 = \rho'' \frac{x_2 - x_1}{R^2} y_m,$$

where:

$$y_m = \frac{y_1 + y_2}{2}.$$

Under δ_1 and δ_2 in preceding expressions it is necessary to understand their absolute value.

Considering δ_1 and δ_2 as corrections and taking approximately $\delta = \delta_2 = \delta$, we obtain:

$$\delta = \rho'' \frac{(x_2 - x_1)}{2R^2} y_m. \tag{8.64}$$

Where $y_m = 200$ км, $x_2 - x_1 = 30$ км and $R = 6400$ км

$$\delta = \frac{2 \cdot 10^5 \cdot 30 \cdot 200}{2 \cdot 64 \cdot 64 \cdot 10^4} \approx 15''.$$

Thus, the mean value of reduction of direction at the edge of six-degree zone in 1st order triangulation is less than 15".

After these preliminary calculations let us turn to derivation of formulas for calculation of reduction of directions and lengths.

1. Derivation of Formulas for Reduction of Distances

Let us assume that in Fig. 89 A_1B_1 is an image of geodetic arc on a plane; d — chord, subtending this arc; δ on an angle between chord and initial element of arc A_1B_1, then:

Fig. 89.

$$d = \int \cos\delta \, dS.$$

According to preceding calculation, δ_{Max} is less than 15", therefore:

$$\cos\delta = 1 - \frac{\delta^2}{2} = 1 - \frac{1}{4 \cdot 10^8} + \ldots$$

With error in value of $\frac{1}{4 \cdot 10^8}$ it is possible to take $\cos\delta = 1$, then:

$$d = S.$$

This is a very important derivation, showing that where distances are on the order of a side of 1st order triangulation difference d − S can be disregarded in any precise calculations.

From (8.41)

$$s = \int_0^S \frac{dS}{m}. \tag{8.65}$$

From (8.47) for current point with ordinate y:

$$m = 1 + \frac{y^2}{2R^2} + \frac{y^4}{24R^4}$$

or:

$$\frac{1}{m} = 1 - \frac{y^2}{2R^2}; \tag{8.56}$$

R and y pertain to current point whose latitude is B.

We have:

$$\frac{1}{R^2} = \frac{1}{R_1^2} + (B - B_1)\frac{d}{dB}\left(\frac{1}{R^2}\right)_1 + \ldots = \frac{1}{R_1^2}\left(1 - \frac{4(B - B_1)}{V_1^2}\eta_1^2 t_1\right),$$

where:

$$B - B_1 = \frac{x - x_1}{M},$$

therefore:

$$\frac{1}{R^2} = \frac{1}{R_1^2}\left[1 - \frac{4(x - x_1)}{R_1}\eta_1^2 t_1\right]. \tag{8.57}$$

Further:

$$\left.\begin{aligned} x &= x_1 + S\cos\alpha_1 \\ y &= y_1 + S\sin\alpha_1 \end{aligned}\right\} \tag{8.58}$$

Substituting (8.57) and (8.58) in (8.56), we obtain:

$$\frac{1}{m} = 1 - \frac{(y_1 + S\sin\alpha_1)^2}{2R_1^2}\left(1 - \frac{4S\cos\alpha_1}{R_1}\eta_1^2 t_1\right),$$

or:

$$\frac{1}{m} = k_0 + Sk_1 + S^2k_2 + S^3k_3, \tag{8.59}$$

where:

$$\left.\begin{aligned} k_0 &= 1 - \frac{y_1^2}{2R_1^2}, \\ k_1 &= -\frac{y_1\sin\alpha_1}{R_1^2} + \frac{2y_1^2\cos\alpha_1}{R_1^3}\eta_1^2 t_1 \\ k_2 &= -\frac{\sin^2\alpha_1}{2R_1^2} + \frac{4y_1\sin\alpha_1\cos\alpha_1}{R_1^3}\eta_1^2 t_1 \\ k_3 &= \qquad\qquad + \frac{2\sin^2\alpha_1\cos\alpha_1}{R_1^3}\eta_1^2 t_1 \end{aligned}\right\}. \tag{8.60}$$

Here $t_1 = \mathrm{tg}\, B_1$, $\eta_1^2 = e'^2\cos^2 B_1$ and sign "1" means that these values pertain to latitude B_1.

Substituting (8.59) in (8.55) and integrating term by term from 0 to S, we obtain:

$$s = S\left(k_0 + k_1\frac{S}{2} + k_2\frac{S^2}{3} + k_3\frac{S^3}{4}\right). \qquad (8.61)$$

Formula (8.61) can be obtained by somewhat different means from (8.60) in the following manner:

$$\left.\begin{array}{l}
\text{for initial point where } S = 0,\ \dfrac{1}{m_1} = k_0 \\
\text{for median point where } S = \dfrac{S}{2},\ \dfrac{1}{m_m} = k_0 + k_1\dfrac{S}{2} + k_2\dfrac{S^2}{4} + \\
\qquad + k_3\dfrac{S^3}{8} \\
\text{for finite point where } S = S,\ \dfrac{1}{m_2} = k_0 + k_1 S + k_2 S^2 + k_3 S^3,
\end{array}\right\} \qquad (8.62)$$

hence

$$s = \frac{S}{6}\left(\frac{1}{m_1} + \frac{4}{m_m} + \frac{1}{m_2}\right). \qquad (8.61')$$

Substituting values $\frac{1}{m_1}$, $\frac{1}{m_m}$ and $\frac{1}{m_2}$ from (8.62) in (8.61'), we again obtain (8.61). Formula (8.61') can be obtained by (8.47), by passing calculation of coefficients k_0, k_1, k_2, k_3, i.e., proceeding from (8.61'), considering that:

$$\left.\begin{array}{l}
m_1 = 1 + \dfrac{y_1^2}{2R_1^2} + \dfrac{y_1^4}{24R_1^4} \\
m_m = 1 + \dfrac{y_m^2}{2R_m^2} + \dfrac{y_m^4}{24R_m^4} \\
m_2 = 1 + \dfrac{y_2^2}{2R_2^2} + \dfrac{y_2^4}{24R_2^4}
\end{array}\right\}. \qquad (8.63)$$

Substituting value k_0, k_1, k_2, k_3 in (8.61) and replacing in them:

$$S\sin\alpha_1 = y_2 - y_1, \quad S\cos\alpha_1 = x_2 - x_1,$$

we obtain:

$$s = S\left\{1 - \frac{y_1^2 + y_1 y_2 + y_2^2}{6R_1^2} + \frac{(x_2 - x_1)(y_1^2 + 2y_1 y_2 + 3y_2^2)\,\eta_1^2 t_1}{6R_1^3}\right\}. \qquad (8.64)$$

If however R_1 is replaced by R_m by the formula:

$$\frac{1}{R_1^2} = \frac{1}{R_m^2}\left[1 + \frac{2(x_2 - x_1)}{R_1}\eta_1^2 t_1\right],$$

we obtain:

$$s = S\left\{1 - \frac{(y_1^2 + y_1 y_2 + y_2^2)}{6R_m^2} + \frac{(x_2 - x_1)(y_2^2 - y_1^2)}{6R_m^3}\eta_1^2 t_1\right\}. \qquad (8.65)$$

This formula possesses high accuracy and can be used for $S \leq 75$ km and $y \leq 300$ km. In practice such cases rarely occur; for usual sides of triangulation this formula should be simplified.

Where $x_2 - x_1 = 40$ km, $y_2 - y_1 = 30$ km, $y_1 = 245$, $y_2 = 275$, $\eta_2 = 0.0024$, $R_m = 6435$ km the end term of formula (8.65) is less than 0.1 mm. Therefore for usual sides of first-order triangulation, i.e., where $s = 20\text{-}25$ km formula (8.65) should be used in the form:

$$S = d = s\left[1 + \frac{1}{6R_m^2}(y_1^2 + y_1 y_2 + y_2^2)\right] \tag{8.66}$$

or, considering that:

$$y_1 = y_m - \frac{1}{2}\Delta y;\ y_2 = y_m + \frac{1}{2}\Delta y;\ y_1^2 + y_2^2 = 2y_m^2 + \frac{\Delta y^2}{2},$$

$$S = s\left(1 + \frac{y_m^2}{2R_m^2} + \frac{\Delta y^2}{24R_m^2}\right) \tag{8.67}$$

or:

$$\lg S = \lg s + \frac{\mu}{2R_m^2} y_m^2 + \frac{\mu}{24R_m^2}\Delta y^2. \tag{8.68}$$

Designate:

$$\lg m_1 = \mu\frac{y_1^2}{2R_m^2}; \qquad \lg m_m = \mu\frac{(y_1 + y_2)^2}{8R_m^2}; \qquad \lg m_2 = \mu\frac{y_2^2}{2R_m^2}.$$

then from (8.61')

$$\lg S = \lg s + \frac{1}{6}(\lg m_1 + 4\lg m_m + \lg m_2). \tag{8.69}$$

Formula (8.69) possesses both high accuracy, and convenience for calculations, but in practice formula (8.68) is applied more frequently. For 2nd order triangulation and lower it is recommended that the following formula be used:

$$\lg d = \lg s + \frac{\mu}{2}\left(\frac{y_m}{R_m}\right)^2. \tag{8.70}$$

For introduction of corrections to sides of polygonmeteric movements following formula should be used:

$$\Delta S = \frac{S y_m^2}{2R^2}. \tag{8.71}$$

where ΔS — correction, S — length of side of movement. Value of $\frac{y_m^2}{2R^2}$ is usually given in tables by argument y_m.

2. Reduction of Directions

In transition from ellipsoid on a plane geodesics, connecting points of support of nets on an ellipsoid, are depicted by curves, angles among them, by condition of conformity, are preserved. However on a plane geodetic nets are formed by chords of images of geodesics, for this it is necessary to introduce in each direction a correction for transition from an arc to a chord. These corrections, or reductions numerically are equal to angles between arc and chord, subtending it, and are called the corrections for curvature of the image of a geodesic on a plane.

At each point on a Gauss-Kruger plane infinitesimal element of geodesic, when b = 0, equality has a place,

$$\alpha_i = A_i - \gamma_i. \tag{8.72}$$

α_1 is grid azimuth on a plane, A_1 is azimuth on an ellipsoid and γ_1 is convergence of meridians on a plane (Fig. 90).

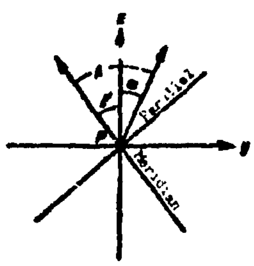


Fig. 90.

For two adjacent points from (8.72)

$$\left.\begin{array}{l}\alpha_1 = A_1 - \gamma_1 \\ \alpha_2 = A_2 - \gamma_2\end{array}\right\}. \tag{8.73}$$

Hence:

$$\alpha_2 - \alpha_1 = A_2 - A_1 - (\gamma_2 - \gamma_1).$$

or:

$$d\alpha = dA - d\gamma. \tag{8.74}$$

It is known that:

$$dA = dl \sin B. \tag{8.75}$$

By third formula (8.50)

$$d\gamma = c_1 dl + dc_1 l + 3c_3 l^2 dl + \ldots \tag{8.76}$$

Coefficients c_1, c_3 and c_5 are functions of latitude, where c_3 and c_5 change so slowly with change of latitude that in (8.76) they can be taken for constants and a term with l^4 can in general be dropped. Substituting (8.75) and (8.76) in (8.74) and remembering that c_1 = sin B, we obtain

$$d\alpha = -l\cos B\,dB - l^2 \sin B \cos^2 B\,(1 + 3\eta^2)\,dl. \tag{8.76'}$$

Geodetic coordinates B and l are function of grid, therefore:

$$\left.\begin{array}{l} B = B(x, y),\ dB = \dfrac{\partial B}{\partial x}dx + \dfrac{\partial B}{\partial y}dy \\ l = l(x, y),\quad dl = \dfrac{\partial l}{\partial x}dx + \dfrac{\partial l}{\partial y}dy \end{array}\right|. \tag{8.76''}$$

Determine dB and dl from (8.76") and substitute them in (8.76'). It is known that:

$$\frac{\partial B}{\partial x} = \frac{1}{M},\ \frac{\partial B}{\partial y} = -\frac{yt}{R^2},$$
$$\frac{\partial l}{\partial x} = 0,\ \frac{\partial l}{\partial y} = \frac{1}{N\cos B} = \frac{1}{r}.$$

Hence:

$$\left.\begin{aligned} dB &= \frac{dx}{M} - \frac{yt\,dy}{R^2} \\ dl &= \frac{dy}{r} \end{aligned}\right\}. \qquad (8.77)$$

Substituting (8.77) in (8.76'), we obtain:

$$d\alpha = -\frac{l\cos B_1}{M}dx + \frac{l\sin B_1}{R^2}y\,dy - \frac{l^2\sin B_1\cos B_1}{N}(1+3\eta_1^2)\,dl.$$

Approximation:

$$l = \frac{y}{N\cos B}, \quad l^2 = \frac{y^2}{N^2\cos^2 B}.$$

Therefore:

$$d\alpha = -\frac{y\,dx}{R^2} + \frac{y^2t\,dy}{R^2N} - \frac{y^2t_1}{N^3}(1+3\eta_1^2)\,dy$$

or

$$d\alpha = -\frac{y\,dx}{R^2} - \frac{2y^2\eta^2t\,dy}{N^3}.$$

Last term is a small value of third order, therefore in it $R^3 = N^3$, can be taken, then:

$$d\alpha = -\frac{y\,dx}{R^2} - \frac{2y^2\eta^2t\,dy}{R^3}. \qquad (8.78)$$

Equation (8.78) is justified for any point on a plane. In it term with y^3 is not considered, it changes so slowly that in integration (8.78) it can be taken for a constant and considered only in the final formula. Consequently, for determination of analytic expression of correction for curvature it is necessary to integrate differential equation (8.78).

For integration (8.78) let us consider two points on a plane with coordinates (x_1, y_1) and (x_2, y_2); where distance between them is s (Fig. 91).

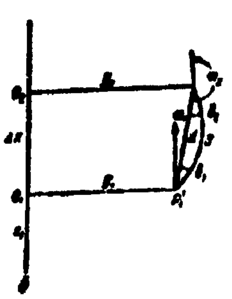


Fig. 91.

Degree of curvature at current point is equal to:

$$\frac{1}{\rho} = \frac{d\alpha}{ds}. \qquad (8.79)$$

Let us assume that the origin of coordinates will be point P_1'; axis of abscissas will be directed along the chord, and axis of ordinates, perpendicular to the chord. We will designate new coordinates by p and q. In this system of coordinates:

$$\frac{1}{\rho} = \frac{\frac{d^2q}{dp^2}}{\left[1+\left(\frac{dq}{dp}\right)^2\right]^{3/2}}.$$

In our selection of coordinates $\frac{dq}{dp} = \mathrm{tg}\,\delta$. The acuteness of an angle δ it can be taken that $\mathrm{tg}\,\delta = \delta$, then $\frac{dq}{dp} = \delta$.

Consequently,

$$\frac{1}{\rho} = \frac{d^2q}{dp^2}. \qquad (8.80)$$

Considering (8.79), (8.80) and (8.78), we obtain:

$$\frac{1}{\rho} = \frac{d^2q}{dp^2} = \frac{d\alpha}{ds} = \frac{d\alpha}{dp} = -\frac{y}{R^2}\frac{dx}{dp} - 2y^2\eta_1^2 t\,\frac{dy}{dx}. \qquad (8.81)$$

Let us express $\frac{1}{R^2}$ by $\frac{1}{R_1^2}$ by formula:

$$\frac{1}{R^2} = \frac{1}{R_1^2}\left(1 - \frac{4(x_2 - x_1)}{R_1}\eta_1^2 t_1\right). \qquad (8.82)$$

Further:

$$\left.\begin{aligned} x &= x_1 + p\cos\alpha_1; \quad \frac{dx}{dp} = \cos\alpha_1 \\ y &= y_1 + p\sin\alpha_1; \quad \frac{dy}{dp} = \sin\alpha_1 \end{aligned}\right\}. \qquad (8.83)$$

Substituting (8.82), (8.83) in (8.81), we obtain:

$$-\frac{d^2q}{dp^2} = k_0' + k_1'p + k_2'p^2. \qquad (8.84)$$

where:

$$\left.\begin{aligned} k_0' &= \frac{\cos\alpha_1}{R_1^2} + \frac{2y_1^2\eta_1^2 t_1 \sin\alpha_1}{R_1^2} \\ k_1' &= \frac{\sin\alpha_1\cos\alpha_1}{R_1^2} + \frac{4y_1\eta_1^2 t_1}{R_1^2}(\sin^2\alpha_1 - \cos^2\alpha_1) \\ k_2' &= \qquad\qquad \frac{2\eta_1^2 t_1}{R_1^2}\sin\alpha_1(\sin^2\alpha_1 - 2\cos^2\alpha_1) \end{aligned}\right\}. \qquad (8.84')$$

Integrals (8.84) are equal:

$$\left.\begin{aligned} -\frac{dq}{dp} &= k_0'p + \frac{k_1'}{2}p^2 + \frac{k_2'}{3}p^3 + c_1 \\ -q &= \frac{k_0'}{2}p^2 + \frac{k_1'}{6}p^3 + \frac{k_2'}{12}p^4 + c_1 p + c_0. \end{aligned}\right\}. \qquad (8.85)$$

At point P_1', where $p = 0$, $\frac{dq}{dp} = \mathrm{tg}\,\delta_1 = \delta_1$; at point P_2', where $p = s$, $\frac{dq}{dp} = \mathrm{tg}\,\delta_2 = \delta_2$. Further, where $p = 0$ and $q = 0$ from (8.85) it follows:

$$\left.\begin{aligned} -\delta_1 &= c_1 \\ c_0 &= 0 \end{aligned}\right\}. \qquad (8.86)$$

Let us resolve equation (8.85) for point P_2', i.e., $p = s = d$, $q = 0$, then:

$$\delta_2 = k_0' s + \frac{k_1'}{2} s^2 + \frac{k_2'}{3} s^3 - \delta_1,$$

$$0 = k_0' \frac{s^2}{2} + \frac{k_1'}{6} s^3 + \frac{k_2' s^4}{12} - \delta_1 s.$$

Consequently:

$$\delta_1 = \frac{k_0'}{2} s + \frac{k_1'}{6} s^2 + \frac{k_2'}{12} s^3,$$

$$\delta_2 = \frac{k_0'}{2} s + \frac{k_1'}{3} s^2 + \frac{k_2'}{4} s^3.$$

Substituting values of coefficients k_0', k_1', k_2' from (8.84'), we obtain:

$$\left.\begin{aligned} \delta_1 &= \frac{(x_2 - x_1)(2y_1 + y_2)}{6R_1^2} - \frac{\eta^2 t (x_2 - x_1)^2 (y_1 - y_2)}{3R_1^2} + \frac{\eta^2 t}{6R_1^3} \times \\ &\quad \times (y_2 - y_1)(3y_1^2 + 2y_1 y_2 + y_2^2) \\ \delta_2 &= \frac{x_2 - x_1}{6R_1^2}(y_1 + 2y_2) - \frac{\eta^2 t}{3R^2}(x_2 - x_1)^2 (y_1 + 3y_2) + \\ &\quad + \frac{\eta^2 t}{6R_1^3}(y_2 - y_1)(y_1 + 2y_1 y_2 + 3y_2^2) \end{aligned}\right\} \qquad (8.87)$$

In formulas (8.87) term $\frac{y^3(x_2 - x_1)}{6R^4}$ is not considered, which, although small, has order approximately the same as the last terms (8.87). Considering this term and changing from R_1 to R_m by formula:

$$\frac{1}{R_1^2} = \frac{1}{R_m^2}\left(1 + \frac{2\eta^2 t (x_2 - x_1)}{R_m}\right).$$

we obtain:

$$\left.\begin{aligned} \delta_1'' &= \frac{(x_2 - x_1)(2y_1 + y_2)}{6R_m^2}\rho'' - \frac{y_m^3 (x_2 - x_1)}{6R_m^4}\rho'' + \\ &+ \rho'' \frac{\eta_m^2 t_m (x_2 - x_1)^2 y_1}{3R_m^3} + \rho'' \eta_m^2 t_m \frac{(3y_1^2 + 2y_1 y_2 + y_2^2)(y_2 - y_1)}{6R_m^3}; \\ \delta_2'' &= \frac{(x_2 - x_1)(y_1 + 2y_2)}{6R_m^2}\rho'' - \frac{y_m^3 (x_2 - x_1)}{6R_m^4}\rho'' - \rho'' \frac{\eta_m^2 t_m}{3R_m^3} \times \\ &\times (x_2 - x_1)^2 y_2 + \rho'' \frac{\eta_m^2 t_m}{6R_m^3}(y_2 - y_1)(y_1^2 + 2y_1 y_2 + 3y_2^2) \end{aligned}\right\}. \qquad (8.88)$$

Formulas (8.88) possess high accuracy and can be recommended for precise calculations. Where $y_1 \leq 200$ km and $s \leq 40$-50 km errors in b_1 and b_2 are less than $0\overset{''}{.}0001$.

Contemporary scheme for the development of 1st order triangulation in USSR anticipates construction of triangles with sides 20-25 km. For this scheme of triangulation formulas (8.88) can be somewhat simplified. Let us express y_1 and y_2 by y_m by formulas:

$$y_1 = y_m - \frac{\Delta y}{2}, \quad y_2 = y_m + \frac{\Delta y}{2}, \quad y_1^2 + y_2^2 = 2y_m^2 + \frac{\Delta y^2}{2}.$$

Terms with $\Delta x^2 \eta_m^2$ and $\Delta y^2 \eta_m^2$ due to their smallness in general can be dropped;

[illegible] error in [illegible] of less than 0".0001. Having made indicated transforma[illegible], we [illegible] the following final form [illegible]:

$$\left.\begin{aligned}\delta_1'' &= \rho''\frac{(x_2-x_1)}{2R_m^2}\left(y_m-\frac{\Delta y}{6}\right)-\rho''\frac{(x_2-x_1)}{6R_m^4}y_m^3+\rho''\frac{\eta_m^2}{R_m^3}\times\\ &\quad\times t_m y_m^2(y_2-y_1)\\ \delta_2'' &= -\rho''\frac{(x_2-x_1)}{2R_m^2}\left(y_m+\frac{\Delta y}{6}\right)+\rho''\frac{(x_2-x_1)}{6R_m^4}y_m^3-\rho''\frac{\eta_m^2}{R_m^3}\times\\ &\quad\times t_m y_m^2(y_2-y_1).\end{aligned}\right\}\quad ([illegible])$$

$$\left.\begin{aligned}\alpha_1 &= A_1-\gamma_1-\delta_1\\ \alpha_2 &= A_2-\gamma_2+\delta_2\end{aligned}\right\}\quad ([illegible])$$

[illegible] are used in reduction of directions in 1st order triangulation, [illegible] order triangulation, these formulas should be simplified, namely, to drop [illegible] the last two terms and to use them in the form:

$$\left.\begin{aligned}\delta_1 &= \rho''\frac{(x_2-x_1)}{2R_m^2}\left(y_m-\frac{\Delta y}{6}\right)\\ \delta_2 &= -\rho''\frac{(x_2-x_1)}{2R_m^2}\left(y_m+\frac{\Delta y}{6}\right)\end{aligned}\right\}.\quad ([illegible])$$

[illegible] less exact calculations and in 3rd order triangulation formulas for [illegible] of reductions of directions should be used in the following form:

$$\delta_1 = \delta_2 = \delta = \rho''\frac{\Delta x\, y_m}{2R_m^2}.\quad ([illegible])$$

[illegible] calculations by the formulas ([illegible]) and ([illegible]) approximate x and y are required. Let us determine the necessary accuracy of approximate coordinates. [illegible] it is sufficient to investigate the main terms of formulas ([illegible]) and ([illegible]). [illegible]:

$$d(\lg S-\lg s)=\frac{\mu\, y_m}{R_m^2}\,dy.$$

[illegible]:

$$d(\lg S-\lg s)\quad 1\cdot10^{-8},$$
$$y_m = 200\ \text{км},$$
$$R = 6400\ \text{км},$$

[illegible]:

$$dy=\frac{1\cdot10^{-8}\,(6400)^2}{0\cdot 43200}\approx 4\cdot10^{-3}\approx 4\ \text{м}.$$

Further:

$$d\delta''=\rho''\frac{d\,x}{2R_m^2}\,y_m+\rho''\frac{\Delta x}{2R_m^2}\,dy;$$

where:

$$d\Delta x = dy = dz$$

$$dz = \frac{2d\delta'' R_m^2}{\rho''(y_m + \Delta x)}.$$

Let $d\delta'' = 0'',001$, $y_m = 200$ *км*, $\Delta x = 30$ *км*, then

$$dz = \frac{2 \cdot 10^{-3}\, 4 \cdot 10^{9}}{2 \cdot 10^{5} 23 \cdot 10} \approx \frac{4}{23} \cdot 10^{-2} \approx 1{,}7 \text{ м}.$$

Consequently, for calculation of reduction of distances and direction, in 1st order triangulation grid coordinates must be known with accuracy of up to [illegible] m.

Calculated by the formulas (8,89) reductions are subtracted from given directions. In translation of triangle from ellipsoid to a plane, reduction is introduced into each angle equal to difference of reductions for corresponding directions; the sum of reduction of angles of a triangle must be equal to its spherical excess, taken with reverse sign.

If reduction of angles is designated by r_1'', r_2'' and r_3'', then the indicated condition will be expressed by equation:

$$r_1'' + r_2'' + r_3'' = -\varepsilon''. \qquad (8,93)$$

If on a plane we have adjusted net of triangulation, then, introducing into adjusted plane angles reductions with reverse sign, we obtain adjusted angles on an ellipsoid. This circumstance makes it possible to pass from adjusted nets on a plane to corresponding nets on an ellipsoid and conversely.

§ 4[illegible]. DIRECT AND INVERSE GEODETIC PROBLEMS WITH GAUSS-KRUGER COORDINATES ON AN ELLIPSOID

In preceding paragraphs accepted scheme was presented for transition from an ellipsoid to a plane, when geodetic network is first reduced on a plane, and after its adjustments are made plane coordinates are calculated. However Gauss-Kruger coordinates can be calculated by given elements on an ellipsoid, by passing the reduction stage. In certain cases such a way of calculating is more expedient, for instance, when triangulation is adjusted on a surface of a reference-ellipsoid, this can be done in 1st order triangulation.

1. Direct Problem

Let us take given Gauss-Kruger coordinates x_1, y_1 of point P_1 (Fig. [illegible]), geodesic arc P_1P_2 [illegible], its directional angle on an ellipsoid. It is required to calculate by these data of Gauss-Kruger coordinates point $P_2(x_2, y_2)$ and back grid azimuth.

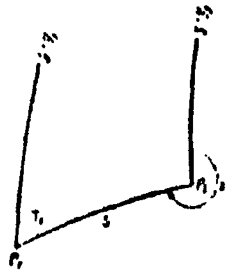


Fig. [illegible]

[illegible] designate:

$$\Delta x = x_2 - x_1, \qquad ([illegible])$$

$$\left.\begin{aligned}\Delta y &= y_2 - y_1,\\ \Delta T &= T_2^* - T_1.\end{aligned}\right\} \qquad ([illegible])$$

T_2^* [illegible] $T_2 \pm 180°$. [illegible] is called convergence of lines of ordinates.

We will use Taylor series for determination of increments.

We have:

$$\left.\begin{aligned}x_2 &= x_1 + \Delta x = x_1 + s\left(\frac{dx}{ds}\right)_1 + \frac{s^2}{2}\left(\frac{d^2x}{ds^2}\right)_1 + \frac{s^3}{6}\left(\frac{d^3x}{ds^3}\right)_1 + \ldots\\ y_2 &= y_1 + \Delta y = y_1 + s\left(\frac{dy}{ds}\right)_1 + \frac{s^2}{2}\left(\frac{d^2y}{ds^2}\right)_1 + \frac{s^3}{6}\left(\frac{d^3y}{ds^3}\right)_1 + \ldots\\ T_2^* &= T_1 + \Delta T = T_1 + s\left(\frac{dT}{ds}\right)_1 + \frac{s^2}{2}\left(\frac{d^2T}{ds^2}\right)_1 + \ldots\end{aligned}\right\} \qquad ([illegible])$$

[illegible] the following basic formulas for calculation of derivatives, included [illegible]:

$$m^2ds^2 = dx^2 + dy^2, \qquad ([illegible])$$

[illegible]:

$$\left.\begin{aligned}\frac{dx}{ds} &= m\cos T,\\ \frac{dy}{ds} &= m\sin T\\ \frac{dT}{ds} &= \frac{1}{m}\left(\frac{\partial m}{\partial x}\frac{dy}{ds} - \frac{\partial m}{\partial y}\frac{dx}{ds}\right)\end{aligned}\right\}. \qquad ([illegible])$$

[illegible]:

$$\left.\begin{aligned}\frac{d^2x}{ds^2} &= \frac{1}{m}\left[\frac{\partial m}{\partial x}\left(\frac{dx}{ds}\right)^2 + 2\frac{\partial m}{\partial y}\frac{dx}{ds}\frac{dy}{ds} - \frac{\partial m}{\partial x}\left(\frac{dy}{ds}\right)^2\right]\\ \frac{d^2y}{ds^2} &= \frac{1}{m}\left[-\frac{\partial m}{\partial y}\left(\frac{dx}{ds}\right)^2 + 2\frac{\partial m}{\partial x}\frac{dx}{ds}\frac{dy}{ds} + \frac{\partial m}{\partial y}\left(\frac{dy}{ds}\right)^2\right]\end{aligned}\right\}. \qquad ([illegible])$$

Calculations of derivatives, included in series ([illegible]) to fifth order inclusively, were made by V. K. Khristov.[1] Here these calculations are dropped. Introducing [illegible] $u = s\cos T_1$, $v = s\sin T_1$, we get the final formulas with retention of small values to third order inclusively

$$\left.\begin{aligned}x_2 &= x_1 + u + \frac{u}{2R_1^2}y_1^2 + \frac{u}{24R_1^4}y_1^4 + \frac{uv}{R_1^2}y_1 + \frac{uv^2}{2R_1^2},\\ y_2 &= y_1 + v + \frac{v}{2R_1^2}y_1^2 + \frac{v}{24R_1^4}y_1^4 - \frac{u^2}{2R_1^2}y_1 +\end{aligned}\right\} \qquad (8,100)$$

[1] V. K. Khristov, Transverse coordinates on a spheroid. M., Geodezizdat, 1957, p. 140-152.

$$+\frac{v^2}{2R_1^2}y_1-\frac{y^2v}{6R_1^2}+\frac{v^3}{6R_1^2}$$

$$\left.\begin{aligned}T_2'&=T_1-\frac{u}{R_1^2}y_1-\frac{u}{6R_1^4}y_1^3-\frac{2u}{R_1^2}y_1^2\eta_1^2t_1-\frac{uv}{2R_1^2}\\ T_2&=T_2'\pm180^\circ\end{aligned}\right\}\quad(8,100)$$

R_1 — mean radius of curvature at point i_1.

In order to facilitate calculation by the formulas (8,100), there should be tables for $\frac{1}{R_1^2}$ by argument x_1 for 100 km, values $\frac{1}{R_1}\eta_1^2$ by x_1 for 1000 km, and value $\frac{1}{R_1}$ ... be considered constant. With indicated tables, which are small by volume, calculation by the formulas (8,100) would be very simple.

2. Inverse Problem

In resolution of inverse problem Gauss-Kruger coordinates are given for two points i_1 and i_2 (Fig. 95). It is required to find distance between them by a geodesic and grid azimuths T_1 and T_2. Since coordinates of terminals are given, it is expedient to obtain formula with mean arguments, i.e., to introduce coordinates $x_m = \frac{1}{2}(x_1 + x_2)$ and $y_m = \frac{1}{2}(y_1 + y_2)$. Idea for derivation of formulas is analogous to that, which was used in derivation of Gauss formulas for resolution of direct and inverse geodetic problems in Chapter V. If the coordinates of a mean point and grid azimuth in it were designated x_0, y_0 and T_0 (Fig. 94), then it is necessary at first to look for differences $x_0 - x_m$, $y_0 - y_m$, $T_0 - T_m$. All these conclusions are given in above indicated book by V. K. Khristov. Inasmuch as they do not have fundamental value, dropping them, we will write final formulas, retaining small values to third order inclusively.

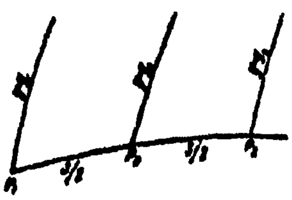


Fig. 95.

$$\left.\begin{aligned}s\cdot\sin T_m&=\Delta y-\frac{\Delta y}{2R_m^2}y_m^2+\frac{5}{24}\cdot\frac{\Delta y}{R_m^4}y_m^4-\frac{\Delta x^2\Delta y}{12R_m^2}-\frac{\Delta y^3}{24R_m^2}\\ s\cdot\cos T_m&=\Delta x-\frac{\Delta x}{2R_m^2}y_m^2+\frac{5}{24}\frac{\Delta x}{R_m^4}y_m^4+\frac{\Delta x\Delta y^2}{24R_m^2}\end{aligned}\right\}.\quad(8,101)$$

$$\Delta T=-\frac{\Delta x}{R_m^2}y_m+\frac{\Delta x}{3R_m^4}y_m^3-\frac{2\Delta x}{R_m^2}y_m^2\eta_m^2t_m.\quad(8,102)$$

$$\left.\begin{aligned}T_1&=T_m'-\frac{\Delta T}{2}\\ T_2&=T_m'\pm180^\circ+\frac{\Delta T}{2}\end{aligned}\right\}.\quad(8,103)$$

[illegible] about tables for resolution of a direct problem equally pertains to resolution of inverse problem, some tables will be required. Calculations by formulas (8.1[illegible]), and (8.101) and (8.10[illegible]) are useful for computers.

§ 47. [illegible]ACTICE OF APPLICATION OF GAUSS-KRUGER [illegible]

At present material of geodetic measurements of USSR, with the exception of 1st order triangulation, are processed in Gauss-Kruger projection with calculation of grid coordinates of vertexes of control geodetic networks. In certain cases adjustment of 1st order triangulation is also carried out on a plane.

[illegible] time of introduction of this projection in USSR great experience was [illegible], and the practice of application of Gauss-Kruger coordinates was brought to a great perfection. During that time many auxiliary means in the form of tables, [illegible] circuits, nomographs and others were prepared. From these means the tables are the most essential. We will cite the most important of them.

1. [illegible] Krasovskiy and A. A. Izotov. "Tables for logarithmic calculation of Gauss-Kruger coordinates within limits of latitudes [illegible]° to [illegible]°," Ellipsoid of [illegible] Krasovskiy, 1946.

2. "Tables for calculation of plane conformal Gauss coordinates within limits of latitudes [illegible]° to 80°," second publication, 1948, are composed under direction of D. A. Larin.

[illegible] "Table of Gauss-Kruger coordinates for latitudes 30° to 80° for 5' and for longitudes 0° to $6\frac{1}{2}$° for $7\frac{1}{2}$' and tables of dimensions of frames and areas of trapezoids of topographic surveys," 1947, are composed under direction of A. M. Virovets.

4. A. M. Virovets, "Tables for construction of frames of trapezoids of topographic surveys for scales of 1:5000 and 1:2000," Krasovskiy Ellipsoid, [illegible].

[illegible] A. M. Virovets and F. N. Rabinovich. "Tables for conversion of grid coordinates," 19[illegible].

[illegible] V. P. Morozov. "Tables for conversion of plane grid coordinates from one six-degree zone to another by the formulas with constant coefficients," [illegible] VIA, 19[illegible]3.

From tables of foreign authors for calculation of Gauss-Kruger coordinates great joint labor of Hungarian academy of sciences under direction of academician, A. Tarczi-Hornoch and Bulgarian academy of sciences under direction of academician, of V. K. Hristov: "Tables for F. N. Krasovskiy ellipsoid," Budapest, 19[illegible], [illegible].

This work contains many tables and examples for resolution of calculating problems of spheroitic geodesy with the help of computing machine, including all problems, connected with application of Gauss-Kruger coordinates for latitudes 40° to [illegible]°. In the work considerable space is occupied by tables for conversion of coordinates from one zone to another by various formulas.

Explanations of tables are composed in Russian, English and German languages.

1. F. N. Krasovskiy and A. A. Izotov Tables

Tables, intended for logarithmic calculations are composed of two groups of formulas. If difference of longitudes l is greater than $1°30'$, formulas of the first group are recommended:

$$\left.\begin{aligned} \sin u &= \sin l \cos B \\ \lg \operatorname{tg} \gamma &= \lg \operatorname{tg} l \sin B + (5)\,u''^2 \\ \lg m &= (3)\,u''^2 + (8)\,u''^4 \\ \lg y &= \lg\left(\frac{N}{\rho''}u''\right) + \frac{1}{2}\lg m \\ \lg(x-X) &= \lg\left(\frac{N}{\rho''}u'' \operatorname{tg}\frac{\gamma}{2}\right) + (4)\,u''^2 \\ x &= X + (x-X) \end{aligned}\right\}. \qquad (8.104)^{1}$$

Values (3), (4), (5) and (8) are functions of latitude, and their logarithms are given in tables in one degree.

Formulas (8.104) for all latitudes of the USSR give high accuracy and error of calculations of coordinates x, y do not exceed 2 mm, and convergence of meridians on a plane – 0".001. However their deficiency is in the fact that it is necessary to deal with logarithms of sines and tangents of acute angles. Recommendations given in tables remove this deficiency to a significant degree.

If the difference of longitudes l is less than $1°30'$, then following simplified formulas are recommended:

$$\left.\begin{aligned} p &= \frac{N}{\rho''}\, l'' \cos B \\ \lg y &= \lg p + (\mathrm{VI})\,p^2 \\ \lg \gamma &= \lg l \sin B + (\mathrm{V})\,p^2 \\ \lg(x-X) &= \lg\frac{p\gamma''}{2\rho''} - (\mathrm{IV})\,p^2 \\ \lg m &= (\mathrm{III})\,p^2, \\ x &= X + (x-X) \end{aligned}\right\}. \qquad (8.105)$$

Logarithms of values (III), (IV) and (V) are given for latitudes for a degree, lg (VI) – for 10'. Values X and lg $\frac{N}{\rho''}$ are given for every minute of latitude. First

[1]Derivation of formulas (8.104) is given by N. A. Urmayev in "Spheroitic geodesy," RIO VTS, 1955, p. 155-157.

in millimeters, accurate to nine decimal points.

For calculation of geodetic coordinates by grid coordinates two groups of formulas are recommended.

If point is removed from axial meridian from 1°30' to 2°30' along longitude, then following formulas are recommended:

$$\left.\begin{aligned}
\sigma_1'' &= \frac{\rho''}{N_1} y \\
\lg m &= (3)_1 \sigma_1''^2 - (8)_1 \sigma''^4 \\
\lg \sigma &= \lg \sigma_1'' - \frac{1}{3} \lg m \\
\lg l &= \lg \sigma \sec B_1 \\
\lg \operatorname{tg} \gamma &= \lg \sin \sigma \operatorname{tg} B_1 + (7)_1 \sigma''^2 \\
\lg (B_1 - B)'' &= \lg \left(\frac{\rho''}{M_1} y \operatorname{tg} \frac{\gamma}{2} \right) - (6)_1 \sigma''^2 \\
B &= B_1 - (B_1 - B).
\end{aligned}\right\} \qquad (8,106)$$

Lower sign "1" means that corresponding value pertains to latitude B_1 — latitude of base of ordinate y, is obtained if x is considered as an arc of meridian. Values $(5)_1$, $(6)_1$, $(7)_1$ and $(8)_1$ are functions of latitude B_1, their logarithms are given in tables of one degree. With these formulas unknown latitudes and longitudes are obtained with accuracy of up to 0".0001, convergence of meridians to 0".001. Deficiency of these formulas is the same as that of formulas (8,104).

If longitude of points of axial meridian being determined is less than 1°30', then for calculation of geodetic coordinates simplified formulas are recommended:

$$\left.\begin{aligned}
\lg l'' &= \lg \left(\frac{\rho''}{N_1} y \sec B_1 \right) - (VIII)_1 y^2 \\
\lg \gamma'' &= \lg l'' \sin B_1 - (VII)_1 y^2 \\
\lg (B_1 - B'') &= \lg \frac{y \gamma''}{2M_1} - (IV)_1 y^2 \\
\lg m &= (III)_1 y^2 \\
B &= B_1 - (B_1 - B)
\end{aligned}\right\} \qquad (8,107)$$

Values $(III)_1$, $(IV)_1$, $(VII)_1$ and $(VIII)_1$ are functions of latitude B_1, their logarithms are given in tables by argument B_1.

Calculation of reductions of distances and directions by tables is recommended to be carried out by formulas (8,88) and (8,89). For convenience of calculations the tables give $\lg \frac{\rho''}{2R_m^2}$, $\lg \frac{\rho''}{2N_m^2}$ and lg R for a degree of latitude.

F. N. Krasovskiy and A. A. Izotov Tables ensure high accuracy of translation of geodetic control networks from an ellipsoid to a plane and conversely. They are provided with numerical examples and the necessary explanations.

7. B. A. Larin Tables

Tables are intended for nonlogarithmic calculation of Gauss-Kruger coordinates and geodetic coordinates.

For calculation of Gauss-Kruger coordinates by geodetic coordinates the following formula is recommended:

$$\left.\begin{aligned} x - X &= a_2 l^2 + a_4 l^4 + a' k_6, \\ y &= b_1 l + b_3 l^3 + b' k_5, \\ \gamma &= c_1 l + c_3 l^3 + c' k_5. \end{aligned}\right\} \qquad (8.108)$$

As can be seen, formulas (8.108) coincides with formulas (8.59), the difference is only in last terms, which in (8.108) have the form:

$$a' = a_6 (4 \times 3600)^6; \qquad b' = b_5 (4 \times 3600)^5;$$
$$k_5 = \frac{l^5}{(4 \times 3600)^5}; \qquad k_6 = \frac{l^6}{(4 \times 36000)^6}.$$
$$c' = c_5 (4 \times 3600)^5.$$

In tables are given natural values of coefficients a_2, a_4, b_1, b_3 by argument of latitude for every minute and a', b' and c' for a degree; k_5 and k_6 are also taken from tables by argument l.

Formula (8.108) can be applied for differences of longitudes to 4°.

Besides indicated values, the tables contain arc of meridians X with accuracy of up to one millimeter for each minute of latitude. Coefficient $b_1 = \frac{N}{\rho''} \cos B$ allows to calculate the arc of parallel very simply:

$$s' = b_1 l'.$$

For calculation of geodetic coordinates by x and y the following formulas are recommended:

$$\left.\begin{aligned} B_0 - B &= A_2 y^2 + A_4 y^4 + A' k_6 \\ l &= y : (b_1 + B_3 y^2) 10^8 + B' k_5 \end{aligned}\right\}. \qquad (8.109)$$

Coefficients A_2, A_4 are the same, as a_2' and a_4' in formulas (8.61). Formula for l is converted in such a manner that it is possible to use value b_1 both for direct and inverse problems. Coefficients A_2, A_4, b_1, B_3 and B' are functions of latitude of a base of ordinate, which is designated by B_f in tables.

$$A' = A_6 \frac{(4 \times 3600)^6}{\rho^6} N^6 \cos^6 B_f,$$
$$B' = B_5 \frac{(4 \times 3600)^5}{\rho^5} N^5 \cos^5 B_f.$$

Natural values A_2, A_4 and B_3 are given for every minute of latitude, A' and B' — for a degree.

It is proposed that convergence of meridians on a plane be calculated by a third formula ([illegible]) after obtaining B and L. This should be considered deficiency of tables, since independence disappears for obtaining these three values. It is better to apply formula 3 from ([illegible]) for calculations, supplementing the table by coefficients c_1', c_2' and c_4' in subsequent editions.

Reduction of distances and directions should be calculated by the same formulas as in tables of F. N. Krasovskiy and A. A. Izotov. Tables are provided with explanatory text, true, very laconic, and examples of calculations.

D. A. Larin Tables resolve the problem set before them with high degree of accuracy. At present they obtained the greatest dissemination in geodetic work in USSR, for being the most economic.

6. Professor A. M. Virovets Tables

First set of tables contains values of Gauss-Kruger coordinates and convergence of meridians on a plane of vertices of angles of surveying trapezoids on a scale of 1:[illegible],000, and dimensions of frames and areas of trapezoids on scales from 1:10,000 to 1:200,000 inclusively.

Second set of tables are intended for obtaining grid coordinates (x, y) vertices of angles of surveying trapezoids on scales of 1:2000 and 1:5000; in these tables are shown directly and coordinates are not reduced, but they are composed in such a way that with their help unknown coordinates are found very simply.

Tables on latitudes embrace ten belts of one million map: R, Q, P, O, N, M, L, K, J, I. In them are also given dimensions of frames and areas of trapezoids on a scale of 1:2000 and 1:5000.

As the first set, the second set of tables contains values, using them it is possible to convert coordinates from one zone to another with accuracy sufficient for topographic work.

Thus, first and second tables of A. M. Virovets are intended to ensure application of Gauss-Kruger projection and coordinates in topographic work of USSR. Basic questions of application of plane coordinates are developed in them with definite sequence and necessary accuracy. Coordinates of vertexes of trapezoids are obtained with accuracy of up to 0.2 m with these tables.

Examples of treatment of triangulation on a plane in Gauss-Kruger coordinates are given in "Practicum on higher geodesy" (p. 83-87), and also in Krasovskiy-Izotov Tables (p. 27-39, examples I-V) and D. A. Larin (p. 7-10).

§ 68. CALCULATION AND DRAWING OF KILOMETER GRID; INSERTION OF GEOGRAPHIC NET ONTO GRID ON GAUSS-KRUGER PROJECTION

On topographic maps, for convenience of their use, as a rule, kilometer grids are drawn with given intervals depending upon purpose and scale of the maps. Kilometer grid and frame can be drawn simultaneously. If kilometer grid is drawn, then from it, by coordinates of vertices of angles of trapezoid it is possible to construct its frame. If however frame of a trapezoid is to be constructed, then from them it is easy to calculate and construct the kilometer grid. If a frame on sheet is given by lines of abscissas and ordinates, then a necessity can arise for determination on a plane of a topographic map of exits of meridians and parallels, i.e., insertion of graticule on a grid. For resolution of this problem additional calculations and construction are required.

Let us assume that Fig. 94 depicts a sheet of topographic map, grid coordinates of sheet corners are given, it is required to find grid coordinates of points F_1, F_2, E_1 and E_2 by geodetic coordinates B and l. For determination of coordinates of points F_1 and F_2 longitude l meridian $F_1 F_2$ and abscissa of lines AD and BC are used. Considering abscissa of line AD the arc of meridian from equator to a given point, we find from tables of A. M. Virovets latitude B_1 of point F_1; having latitude B_1 and l, we determine ordinate of point F_1 by the formula:

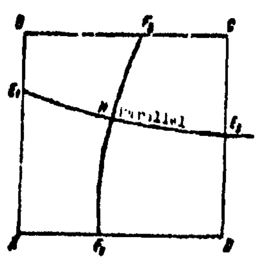


Fig. 94.

$$y = \frac{N_1}{\rho''} l \cos B_1 \left[1 + \frac{l''^2 \cos^2 B_1}{6\rho''^2}(1 + 2t_1^2)\right], \qquad (8.110)$$

or:

$$\lg y = \lg \frac{N_1 l \cos B_1}{\rho''} + (\mathrm{VIII})_1 l^2. \qquad (8.110')$$

Formula (8.110) is obtained from (8.26) by means of simple conversions. Calculation by the formula (8.110) is made by first or second A. M. Virovets tables depending upon the scale of the map. By the formula (8.110) ordinate F_2 is calculated, where in this case the latitude is calculated by abscissa of line BC.

For calculation of abscissa of point E_1 we have latitude of parallel E_1E_2 and ordinate of western frame of the trapezoid, consequently, we can find latitude of the base of ordinate B_0 by the formula:

$$B_0 = B + \frac{y^2}{2R^2}\rho'' \qquad (8.111)$$

having latitude B_0, we find arc of meridian from tables, i.e., abscissa of point P_1. Similar calculations are necessary for finding of abscissa P_2. It is recommended to determine one more point on parallel P_1P_2, for instance H, in order to consider curvature of this line. Point H can be given its latitude B and longitude, rounded to 7.5'. Then by A. M. Virovets tables we can find abscissa and ordinate of this point. For calculation of grid coordinates of exit of meridians and parallels, or in general, of points with arbitrary values of geodetic coordinates, in A. M. Virovets tables there are given special formulas and additional tables. For purely topographic work they should be the only ones to be used. In other words, the problem of insertion of graticule over grid is wholly resolved with the help of A. M. Virovets tables. They give the values of convergence of meridians for drawing on a map, lines, whose azimuths are given, and with the help of tables of values m - 1 can be found the lengths of lines on an ellipsoid by measured lengths on map and conversely.

§ 59. CONVERSION OF COORDINATES FROM ZONE TO ZONE

The presence of coordinate zones in application of Gauss-Krüger projection evokes necessity for resolution of additional problem conversion of grid coordinates from one zone to another. This problem most frequently occurs in junctions of zones in fulfillment of various geodetic and topographic work.

In USSR it is accepted that coordinates of points of state geodetic network, located in "overlap," are given in the systems of two adjacent zones. "Overlap" of zones, within whose limits points have coordinates in systems of two zones, stretches by 37.5' in longitude; a system of coordinates of the western zone overlaps eastern by 30' of longitude, and eastern overlaps western by 7.5' (Fig. [illegible]).[1]

However the indicated rule "of double" calculation of coordinates does not exclude necessity for special calculations for conversion of coordinates. The following cases of conversion of coordinates, are possible from one six-degree to another six-degree, from three-degree to three-degree, from six-degree to three-degree zones and conversely.

The simpler way of resolution of the problem consists of converting to geodetic coordinates from grid coordinates, and calculating grid coordinates from geodetic coordinates in a system of desired zone, the problem is resolved by these means with any desired degree of accuracy.

[1] See note on p. [illegible].

For a small number of points the indicated method can be fully useful.

However with a significant number of points the application of this method leads to unnecessary expenditure of calculating labor, since here double transition is actually accomplished. Naturally a necessity arises for a development of a method, whereby grid coordinates (x, y) in a given zone, coordinates (x', y') in system of another zone can be directly calculated.

Let us take Gauss-Kruger grid coordinates (x, y) and (x', y') in systems of adjacent zones for point of ellipsoid with isometric coordinates (q, L), consequently:

$$\left.\begin{aligned} x &= x(q, l_1) \\ y &= y(q, l_1) \end{aligned}\right\}. \qquad (8.112)$$

$$\left.\begin{aligned} x' &= x'(q, l_2) \\ y' &= y'(q, l_2) \end{aligned}\right\}. \qquad (8.113)$$

where $l_1 = L - L_0$, $l_2 = L - L_0'$, when L_0 and L_0' are longitudes of axial meridians of two adjacent zones. Difference $(L_0 - L_0')$ is always the given value; designating it by n, we have:

$$l_2 = l_1 + n. \qquad (8.114)$$

Excluding from (8.112) and (8.113) q and l, we obtain functional dependence between systems of grid conformal coordinates, i.e.,

$$\left.\begin{aligned} x' &= f_1(x, y) \\ y' &= f_2(x, y) \end{aligned}\right\}. \qquad (8.115)$$

Equations (8.115) in general form give formulas for conversion of coordinates from one system to another.

Problem, thus, consists of determining functions of f_1 and f_2 and by doing so finding the formulas for their calculation. For resolution of this problem we will introduce an auxiliary point $P_0(x_0, y_0)$ under the condition that:

$$\left.\begin{aligned} x &= x_0 + \Delta x; \qquad x_0' = f_1(x_0, y_0) \\ y &= y_0 + \Delta y; \qquad y_0' = f_2(x_0, y_0) \end{aligned}\right\}. \qquad (8.116)$$

Consequently, auxiliary point has coordinates in second zone (x_0', y_0'); Δx and Δy have the usual for two adjacent points of triangulation values, i.e., not more than 20-25 km.

Let us extend (8.115) in a series of ascending powers Δy.

We have:

$$\left.\begin{aligned} x' &= a + a'\Delta y + a''\Delta y^2 + a'''\Delta y^3 + \ldots \\ y' &= b + b'\Delta y + b''\Delta y^2 + b'''\Delta y^3 + \ldots \end{aligned}\right\}. \qquad (8.117)$$

Coefficients a, a', a'', ... b, b', b'', ... functions of x in the form of:

$$a = f_1(x, y_0);\quad b = f_2(x, y_0);\quad a^i = \frac{1}{i!}\frac{\partial^i f_1}{\partial y^i};\quad b^i = \frac{1}{i!}\frac{\partial^i f_2}{\partial y^i}$$

Let us expand them also in power series by ascending powers of x, then:

$$\left.\begin{aligned} a &= a_0 + a_1\Delta x + a_2\Delta x^2 + a_3\Delta x^3 + a_4\Delta x^4 + \dots \\ a' &= a'_0 + a'_1\Delta x + a'_2\Delta x^2 + a'_3\Delta x^3 + a'_4\Delta x^4 + \dots \\ a'' &= a''_0 + a''_1\Delta x + a''_2\Delta x^2 + a''_3\Delta x^3 + a''_4\Delta x^4 + \dots \\ a''' &= a'''_0 + a'''_1\Delta x + a'''_2\Delta x^2 + a'''_3\Delta x^3 + a'''_4\Delta x^4 + \dots \end{aligned}\right\}. \qquad (8.118)$$

$$\left.\begin{aligned} b &= b_0 + b_1\Delta x + b_2\Delta x^2 + b_3\Delta x^3 + b_4\Delta x^4 + \dots \\ b' &= b'_0 + b'_1\Delta x + b'_2\Delta x^2 + b'_3\Delta x^3 + b'_4\Delta x^4 + \dots \\ b'' &= b''_0 + b''_1\Delta x + b''_2\Delta x^2 + b''_3\Delta x^3 + b''_4\Delta x^4 + \dots \\ b''' &= b'''_0 + b'''_1\Delta x + b'''_2\Delta x^2 + b'''_3\Delta x^3 + \dots \end{aligned}\right\}. \qquad (8.119)$$

Substituting (8.118) and (8.119) in (8.117) and considering that $a_0 = x'_0$ and $b_0 = y'_0$, we obtain:

$$\begin{aligned} x' - x'_0 = \Delta x' &= a_1\Delta x + a_2\Delta x^2 + a_3\Delta x^3 + a_4\Delta x^4 + (a'_0 + a'_1\Delta x + \\ &+ a'_2\Delta x^2 + a'_3\Delta x^3)\Delta y + (a''_0 + a''_1\Delta x + a''_2\Delta x^2)\Delta y^2 + \\ &+ (a'''_0 + a'''_1\Delta x)\Delta y^3 + \dots, \end{aligned} \qquad (8.120)$$

$$\begin{aligned} y' - y'_0 = \Delta y' &= b_1\Delta x + b_2\Delta x^2 + b_3\Delta x^3 + b_4\Delta x^4 + (b'_0 + b'_1\Delta x + \\ &+ b'_2\Delta x^2 + b'_3\Delta x^3)\Delta y + (b''_0 + b''_1\Delta x + b''_2\Delta x^2)\Delta y^2 + \\ &+ (b'''_0 + b'''_1\Delta x)\Delta y^3 + \dots \end{aligned} \qquad (8.121)$$

Formulas both in (8.117), and in (8.120) and (8.121) are for conversion of coordinates from one zone to another. Difference between these formulas is in the fact that the first are called formulas with variable coefficients, where special tables are required for their application, the second are called formulas with constant coefficients; they are convenient for calculations with the aid of computers.

To (8.117), (8.120) and (8.121) conditions ensuing from (8.9) are applicable. We have:

$$\begin{aligned} \frac{\partial x'}{\partial x} &= \frac{\partial a}{\partial x} + \Delta y\frac{\partial a'}{\partial x} + \Delta y^2\frac{\partial a''}{\partial x} + \dots, \\ \frac{\partial x'}{\partial \Delta y} &= a' + 2a''\Delta y + 3a'''\Delta y^2, \\ \frac{\partial y'}{\partial x} &= \frac{\partial b}{\partial x} + \Delta y\frac{\partial b'}{\partial x} + \Delta y^2\frac{\partial b''}{\partial x}, \\ \frac{\partial y'}{\partial \Delta y} &= b' + 2b''\Delta y + 3b'''\Delta y^2. \end{aligned}$$

But by condition:

$$\begin{aligned} \frac{\partial x'}{\partial x} &= \frac{\partial y'}{\partial \Delta y}, \\ \frac{\partial x'}{\partial \Delta y} &= -\frac{\partial y'}{\partial x}. \end{aligned}$$

Consequently:

$$\left.\begin{array}{l} b' = \dfrac{\partial a}{\partial x}; \quad 2b'' = \dfrac{\partial a'}{\partial x}; \quad 3b''' = \dfrac{\partial a''}{\partial x} \\ a' = -\dfrac{\partial b}{\partial x}; \quad 2a'' = -\dfrac{\partial b'}{\partial x}; \quad 3a''' = -\dfrac{\partial b''}{\partial x} \end{array}\right| . \qquad (8.122)$$

From (8.118), (8.119) and (8.122) ensues:

$$\left.\begin{array}{lll} b'_0 = a, & a'_0 = -b_1, & \\ b'_1 = 2a_2, & a'_1 = -2b_2, & \\ b'_2 = 3a_3, & a'_2 = -3b_3, & \\ 2b''_0 = a'_1, & a'_3 = -4b_4, & 3a'''_1 = -2b''_2 \\ b''_1 = a'_2, & 2a''_0 = -b'_1, & \\ 2b''_2 = 3a'_3, & a''_1 = -b'_2, & \\ 3b'''_0 = a''_1, & 2a''_2 = -3b'_3, & \\ 3b'''_1 = 2a''_2, & 3a'''_0 = -b''_1. & \end{array}\right| . \qquad (8.123)$$

Expressions (8.122) and (8.123) show, how the coefficients are tied among themselves, in sines of (8.117), (8.120) and (8.121). They are fully determined, if a and b are given.

Formulas (8.120) and (8.121) give general expression for conversion of Gauss-Kruger coordinates from zone to zone. They are too complicated for practical use, but contain coordinates of auxiliary point $P_0(x_0, y_0)$ and with expedient selection, simple and convenient for practical application formulas can be obtained.

Let us assume that $x = x_0$ or $\Delta x = 0$, then under this condition from (8.120) and (8.121) we obtain:

$$\left.\begin{array}{l} x' = x'_0 + a'_0 \Delta y + a''_0 \Delta y^2 + a'''_0 \Delta y^3 + \dots \\ y' = y'_0 + b'_0 \Delta y + b''_0 \Delta y^2 + b'''_0 \Delta y^3 + \dots \end{array}\right| . \qquad (8.124)$$

Coefficients a_0^i, b_0^i ($i = {}'$, $''$, $'''$ ...), as a function of abscissa $x = x_1$ or latitude B_1, corresponding to x, if x is considered as an arc of meridian, they can be tabulated by argument x or B_1. Auxiliary coordinates x'_0 and y'_0 can be calculated by general formulas (8.12); they are given in tables along with a_0^i and b_0^i.

Omitting details of computations for obtaining a_0^i and b_0^i, which are given in the mentioned work of V. K. Khristov (p. 215), we give their final values as function B_1:

$$\left.\begin{array}{c} a'_0 = -2n \sin B_1 - \dfrac{2n^3 \sin B_1 \cos^2 B_1}{3} (1 - 2t_1^2 + 3\eta_1^2) \\ a''_0 = \dfrac{3n^2 \sin B_1 \cos B_1}{N_1} (1 + \eta_1^2) + \dfrac{n^4 \sin B_1 \cos^3 B_1}{2N_1} (1 - 13t_1^2) \\ a'''_0 = -\dfrac{n \sin B_1}{3N_1^2} (1 + 5\eta_1^2) \end{array}\right| . \qquad (8.125)$$

$$\left.\begin{aligned} b_0' &= 1 - 2n^2\sin^2 B_1 - \frac{2n^4\cos^4 B_1}{2}(2l_1^2 - l_1^4 - 6\eta_1^2 l_1^2) \\ b_0' &= -\frac{n\cos B_1(1+\eta_1^2)}{N_1} + \frac{n^3\cos^3 B_1}{6N_1}(1+3l_1^2) \\ b_0'' &= \frac{n^3\cos^3 B_1}{3N_1^2}(3-4l_1^2) \end{aligned}\right\} \qquad (8.1[illegible])$$

l — difference of longitudes of axial meridians of adjacent zones, expressed in radian measure.

Formulas (8.117) with variable coefficients were first used in "Tables for conversion of Gauss-Kruger grid coordinates," published in 1934 using Bessel ellipsoid.

Principle of construction of formulas (8.1[illegible]) with constant coefficients is applied in joint work of Professors A. M. Virovets and P. N. Rabinovich "Tables for conversion of grid coordinates," (Krasovskiy Ellipsoid, M, Geodezizdat, 19[illegible]). In these Tables formulas for calculation have the form of:

$$\left.\begin{aligned} x_0 &= X_0 + a\Delta y + b\Delta y^2 + c \\ y_0 &= \Delta y + a_1\Delta y + b_1\Delta y^2 + c_1 \end{aligned}\right\}, \qquad (8.1[illegible])$$

where:

$$c = D\Delta y^3 + E\Delta y^4; \quad c_1 = D_1\Delta y^3 + E_1\Delta y^4.$$

A. M. Virovets and P. N. Rabinovich Tables consist of three parts. First part contains values y_0, X_0, a, a_1, b and b_1 by argument x_1 is an abscissa of a given point; second and third parts give values of c and c_1 by arguments x_1 and Δy.

A. M. Virovets and P. N. Rabinovich Tables are universal and are useful for conversion of coordinates from six-degree zone to six-degree zone, and from six-degree to three-degree zones and conversely. Calculations are made with computer and for resolution of one problem approximately is required 15 minutes. Converted coordinates have error not greater than 2 cm, fully permissible for all topographic and certain geodetic work. Where great accuracy is required, first method of resolution of considered problem should be used. Tables are supplied with necessary explanations and characteristic examples for practical purposes. From six-degree zone to six-degree coordinates are converted by means of consecutive transition, first to three-degree zone, and from three-degree to six-degree zones.

Besides these tables, which have the greatest application, we shall indicate still others which are in use:

1. [illegible] "Tables for conversion of Gauss-Kruger coordinates from six-degree zone to adjacent six-degree zone."

2. Ye. Ye. [illegible] "Tables for conversion of Gauss-Kruger grid coordinates from one six-degree zone to another six-degree zone."

3. V. I. Morozov "Tables for conversion of plane grid coordinates from one six-degree zone to another by formulas with constant coefficients."

From the work of geodesists of socialist countries on [illegible] coordinates one should note major work of academician A. [illegible] and [illegible] — "Tables for Krasovskiy ellipsoid," Budapest, [illegible], where within the limits of latitudinal band of Bulgaria and Hungary numerous tables for conversion of coordinates both for constant and variable coefficients are given.

C H A P T E R IX

SHORT SURVEY OF GEODESIC PROJECTIONS

§ 50. GENERAL REMARKS

Gauss-Kruger projection and coordinates are used in geodetic work of USSR and in majority of socialist countries, in addition, the practical application of projection in these countries is carried out as a single program and a scheme. At present the geodetic work of socialist countries occupies conspicuous place in world geodetic activity. If one were to consider that in the future the weight of this work will be even greater, then it will be clear that in time the coordinate system of Gauss-Kruger can be converted into a world system of grid coordinates. At present no other system of grid coordinates has such wide application in geodetic work.

However in many European, American and African countries other geodetic projections are used, which have their own peculiarities. In order to objectively judge mathematical, and geodetic merits and deficiencies of these projections and mainly to compare them with Gauss-Kruger projection, it is necessary to become briefly acquainted with their mathematical and geodetic bases.

In selection of one or another projection the geographic configuration of a given country, the accuracy of geodetic and topographic work, the simplicity of mathematical basis of projection and convenience of its application are taken into account. No one projection can completely satisfy all given requirements, however majority of used geodetic projections to one degree or another satisfy the main conditions.

The most essential of them is the conformity of image. Conformal projections possess precious properties for geodetic work, they preserve similarity in small parts of depicted figures. Therefore as time goes by less application is found for nonconformal projections.

Geodetic projections can be determined by different methods, but in all cases they have to satisfy the following equations:

$$\left.\begin{array}{l} x = x(B, l) \\ y = y(B, l) \end{array}\right\}, \qquad (9.1)$$

where (x, y) are grid coordinates in the projection and (B, l) are geodetic coordinates. The form of function of (9.1) in the end result determines the merits and deficiencies of a given projection, therefore we will mainly consider these functions for each projection.

§ 51. SOL'DNER PROJECTIONS AND COORDINATES

Sol'dner coordinates in initial stage of development of higher geodesy and geodetic work played a definite role and were widely used in Germany, in France, and in prerevolutionary Russia prior to adoption of Gauss-Kruger coordinates in USSR. At present the Sol'dner projection and coordinates have only historical value for USSR geodetic work, however in the west these coordinates are still used. Furthermore, Sol'dner projection presents certain methodical interest.

During application of Sol'dner coordinates the Earth is assumed to be a sphere. The surface of the sphere is divided by meridians into coordinate zones of determined width, as in Gauss-Kruger projection. The central meridian of the zone is the axial meridian. Coordinate lines in Sol'dner system are great circles, perpendicular to axial meridian, and small circles, parallel to axial meridian. As abscissa of certain point P_1 serves as arc of meridian from equator to base of ordinate of this point, ordinate of distance along the arc of great circle is from axial meridian to a given point P_1 (Fig. 95).

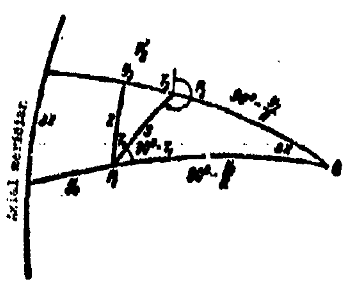


Fig. 95.

Positive abscissas — to north, positive ordinates — eastward and negative — westward from axial meridian. Thus, the system of count of Sol'dner coordinates is similar to Gauss-Kruger coordinates.

In Fig. 95 following designations are made: s is distance between given points P_1 and P_2; T_1 and T_2 are grid azimuths of arc s in its finite points P_1 and P_2; g is arc of small circle, parallel to axial meridian.

Let us assume that R is radius of a sphere.

From triangle P_1P_2Q we have:

$$\left.\begin{aligned} \sin\frac{y_2}{R} &= \cos\frac{s}{R}\sin\frac{y_1}{R} + \sin\frac{s}{R}\cos\frac{y_1}{R}\sin T_1 \\ \sin\frac{\Delta x}{R} &= \frac{\sin\frac{s}{R}}{\cos\frac{y_2}{R}}\cos T_1 \end{aligned}\right\}. \qquad (9.2)$$

Considering $\frac{y}{R}$, $\frac{\Delta y}{R}$ and $\frac{s}{R}$ small values of first order, trigonometric functions of these values are set in series and retain in them small values to third order inclusively, from (9.2) without detailed calculations we obtain

$$\left.\begin{aligned} y_2 &= y_1 + v - \frac{u^2}{2R^2}y_1 - \frac{vu^2}{6R^2} \\ x_2 &= x_1 + u + \frac{u}{2R^2}y_2^2 - \frac{v^2u}{6R^2} \end{aligned}\right\}. \qquad (9.3)$$

Here $u = s\cos T_1$, $v = s\sin T_1$.

After these preliminary remarks we will consider the Sol'dner projection and coordinates on a plane.

Let us assume that axial meridian is depicted on a plane by straight line by a line to full scale; great circles, perpendicular to axial meridian, will be depicted by straight lines, perpendicular to image of axial meridian on a plane and distant one from the other by the value of the difference of abscissas. Small circles, parallel to axial meridian, also will be depicted by straight lines, parallel to image of axial meridian and distant one from another at a distance, equal to difference of their ordinates (Fig. 96).

With such construction, obviously, coordinates on a sphere and plane will be equal, i.e., we use spherical coordinates on a plane.

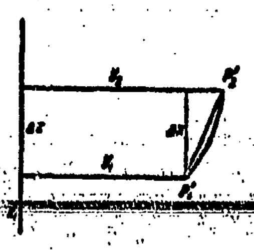


Fig. 96.

If our construction is viewed from the point of view of image of sphere on a plane, then it is easy to notice that the projection is produced on a cylindrical surface, coinciding with the sphere along the axial meridian. Great circles, perpendicular to axial meridian, are depicted by forming the cylinder; small circles, parallel to axial meridian, — intersect the cylinder. Sol'dner projection is a simple evenly spaced cylindrical projection.

Let us study linear and angular distortions of Sol'dner projection.

Let us assume that distance between points on a plane is s_0, then:

$$s_0^2 = \Delta x^2 + \Delta y^2. \qquad (9.4)$$

From (9.3) by means of squaring and addition we obtain:

$$\Delta x^2 + \Delta y^2 = u^2 + v^2 + \frac{u^2 y_2^2}{R^2} - \frac{vu^2 y_1}{R^3}.$$

Or, considering (9.4), we obtain:

$$s_0^2 - s^2 = \frac{u^2 y_2^2}{R^2} - \frac{vu^2 y_1}{R^3}.$$

Second term of right side is small as compared to first, therefore, without disturbing the generalization of reasoning, it can subsequently be dropped lower, then:

$$s_0 - s = \frac{u^2 y_2^2}{(s_0 + s) R^2}.$$

Difference ($s_0 - s$) is small value of at least the second order, therefore in denominator of right side we can take $s_0 \approx s$, then:

$$\nu = \frac{s_0 - s}{s} = \frac{y_2^2 \cos^2 T}{2R^2}. \qquad (9.5)$$

Formula (9.5) gives relative linear distortion of Sol'dner projection; it shows that the projection is nonconformal, since distortion depends on direction, i.e., from grid azimuth.

From (9.5) it follows also that the maximum distortion takes place in the direction of the axis of abscissas; it is equal to:

$$\nu_{max} = \frac{y^2}{2R^2}. \qquad (9.6)$$

Let us find reduction of direction. Designating directional angle on the plane through T^0, from (9.3), after simple conversions, we have with former accuracy:

$$\operatorname{tg} T_1^0 = \operatorname{tg} T_1 \left(1 - \frac{y_2^2}{2R^2} - \frac{u^2 y_1}{2R^2 v}\right).$$

Second term in the right part in parentheses is small in comparison to first. Dropping it, we obtain:

$$\operatorname{tg} T_1^0 - \operatorname{tg} T_1 = -\frac{y_2^2}{2R^2} \operatorname{tg} T_1$$

or:

$$\frac{\sin (T_1 - T_1^0)}{\cos T_1^0 \cos T_1} = \frac{y_2^2}{2R^2} \operatorname{tg} T_1.$$

Hence:

$$\delta_z'' = (T_1 - T_1^0)'' = \frac{y_2^2 \sin 2T_1^0}{4R^2} \rho''. \tag{9.7}$$

In order to compare values of linear distortions and reduction of directions in Sol'dner and Gauss-Kruger projections, we have:

$$\left.\begin{aligned} v_z &= \frac{y^2 \cos^2 T}{2R^2} \\ v_g &= \frac{y^2}{2R^2} \\ \delta_z &= \frac{y^2 \sin 2T_1^0}{4R^2} \rho'' \\ \delta_g &= \frac{y \Delta x}{2R^2} \rho'' \end{aligned}\right\} \tag{9.8}$$

Sign z designates that corresponding values pertain to Sol'dner projection, and g to Gauss-Kruger.

From formulas (9.8) it is simple to conclude that linear distortions in Sol'dner projection in general, are less than in the Gauss-Kruger projection, as:

$$v_z = v_g \cos^2 T.$$

However the great merit of Gauss-Kruger projection remains in the fact that distortion in it does not depend on direction.

This advantage is revealed especially clearly during work of materials of polygonometeric and theodolite runs. In Gauss-Kruger projection it is not necessary to reduce angles of runs, and correction by the formula is introduced into lengths of sides

$$\Delta s = \frac{s y_m^2}{2R^2},$$

where y_m can practically preserve the same value for all sides of runs and even series of runs. In Sol'dner projection, applying formula:

$$\Delta s = \frac{y_m^2 \cos^2 T}{2R^2},$$

it is necessary to calculate correction for every line of a run.

Reduction of directions in Sol'dner projection in main term does not depend on distance between points, but depends on departure from axial meridian, and in value they significantly exceed corresponding values of reductions of Gauss-Kruger projection. If one were to set a condition, so that in Sol'dner projection no correction is introduced into measured angles of polygonometeric and theodolite runs which has a place in Gauss-Kruger projection, then it is necessary to significantly limit the width of coordinate zones (in this case they should not exceed 50-60 km).

With the increase of width of zones to a shown limit a necessity arises for calculation of spheroidness of Earth and introduction of additional corrections. This complicates still more the application of Sol'dner projection and coordinates for countries with great land areas. Therefore the Sol'dner projection in reference to large areas yields in all ratios to the Gauss-Kruger projection.

§ 52. THE LAMBERT PROJECTION

The Lambert projection is a conical conformal projection, used in geodetic work in France, United States and other countries. The central line of projection is a standard parallel with a width B_0. Usually the standard parallel is chosen in such a manner that it passes through the center of depicted territory. If the scale of the image on this parallel is equal to a unit, then the projection is called "conical conformal projection with one standard parallel." If however the scale on two parallels is equal to a unit, then the projection is called "conical conformal projection with two standard parallels."

Let us assume that in Fig. 97 straight line OS depicts axial meridian, and curved OP_0 (circumference of radius $\rho_0 = N_0 \operatorname{ctg} B_0$) depicts central or standard parallel. All meridians are depicted by straight lines in this projection, distant one from another by an angle $\gamma = l \sin B_0$, where l is a difference of longitudes of a given meridian and axial meridian, B_0 is a latitude of the standard parallel. Parallels are depicted by curves of concentric arcs with center at S and radii $(\rho_0 - d)$, where d is a distance between given parallel and the standard parallel (Fig. 97).

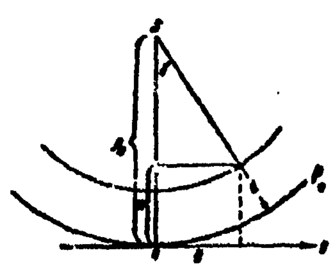


Fig. 97.

As a rule, origin of coordinates is selected at point O; the axis of abscissas are directed toward north along the axial meridian; the axis of ordinates is on a tangent to image of standard parallel at point O toward east. Scale along the standard parallel frequently is taken as equal not to a unit, but to $m_0 = 0.999$. Then:

$$\rho_0 = m_0 N_0 \operatorname{ctg} B_0. \tag{9.9}$$

Plane coordinates and convergence of meridians on a plane in Lambert projection are calculated by the following formulas:

$$\left.\begin{aligned} y &= (\rho_0 - d)\sin\gamma \\ x &= d + y\,\mathrm{tg}\,\frac{\gamma}{2} \\ \gamma &= (L - L_0)\sin B_0 \end{aligned}\right\}, \tag{9.10}$$

where L_0 is a longitude of axial meridian;

$$\alpha = A - \gamma + \delta. \tag{9.11}$$

Lambert projection is conical, therefore reduction problem here is resolved in a very complicated manner. In order to have a concept on the degree of complexity of indicated problem and to compare its formulas with corresponding formulas of Gauss-Kruger projection, listed below without derivations are the formulas for reduction of distances and directions (Fig. 98).

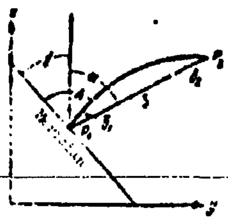


Fig. 98.

Reduction of distances

$$\lg S = \lg m_0 s + \frac{\mu}{2R_0^2}\left\{X_m^2 + \frac{(X_2 - X_1)^2}{12} + \frac{\mathrm{tg}\,B_0\,X_1X_2X_m}{3N_0} + \frac{s^2\,\mathrm{tg}\,B_0 X_m}{6N_0} - \right.$$
$$\left. - \frac{4e'^2\,\mathrm{tg}\,B_0\cos^2 B_0 X_1^3}{3N_0} + \frac{(5 + 3\,\mathrm{tg}^2 B_0)X_1^3(2X_2 - X_1)}{12N_0^2}\right\}^1. \tag{9.12}$$

Reduction of directions

$$\delta'' = \frac{x_2y_1 - x_1y_2 + \rho_0(x_2 - x_1)}{2\rho\sin 1''}\left\{A\ln\frac{\rho}{\rho_0} + B\ln^2\frac{\rho}{\rho_0} + \right.$$
$$\left. + C\ln^3\frac{\rho}{\rho_0} + D\ln^4\frac{\rho}{\rho_0}\right\}^1. \tag{9.13}$$

In these formulas:

s — distance on spheroid along the geodesic arc,

S — distance on a plane by chord,

B_0 — latitude of standard parallel,

μ — modulus of common logarithms,

e' — second meridian eccentricity,

X_1 and X_2 — distances of given points along meridian from standard parallel,

$$X_m = \frac{X_1 + X_2}{2},$$

$$A = -\mathrm{ctg}^2 B_0(1 + \eta_0^2),$$

$$B = -\mathrm{ctg}^2 B_0(1 + 3\eta_0^2 + 2\eta_0^4),$$

$$C = \frac{1}{3}\mathrm{ctg}^4 B_0(1 - 2\,\mathrm{tg}^2 B_0 + 4\eta_0^2 - 14e'^2\sin^2 B_0),$$

$$D = \frac{1}{3}\mathrm{ctg}^4 B_0(2 - \mathrm{tg}^2 B_0 + 15\eta_0^2 - 15e'^2\sin^2 B_0).$$

[1]G. Bomford. Geodesy. M., Geodezizdat, 1958, p. 185.

Formulas for calculation of reductions of distances and directions in Lambert projection are so complicated that their use for geodetic networks on a plane becomes absolutely inexpedient. Therefore during practical application of this projection for treatment of triangulation it should be conducted on the surface of an ellipsoid and then by the formula (9.10) grid coordinates should be calculated. Here we were again convinced that conformal conical projections are complicated in the part of resolution or reduction problem and therefore are unfit for use in geodetic work on such area, as the USSR.

Other deficiency of conical conformal projections consists in that with the change of standard parallel the constant projections are also changed. This projection is convenient only for topographic work, where for reduction it is sufficient to take only the main terms of formulas (9.12), (9.13).

§ 53. STEREOGRAPHIC GEODETIC PROJECTIONS

In mathematical cartography the class of conformal azimuthal and perspective projections of the sphere on a plane, whose point of view is on obtained surface, is called stereographic. These projections possess two very important properties for geodetic work, they are equiangular and are able to transmit all circles both infinitesimal, and finite in the form of circles.

However the surface of a sphere is the only one of curved surfaces, whose perspective is strictly conformal. Any perspective of significant parts of a surface of the spheroid distorts angles and does not transmit circles by circles.

However geodetic use of stereographic projections is characterized by a peculiarity that they are used for limited areas, and therefore preserve their valuable qualities for geodetic work. Due to these reservations of strict determination in the common form stereographic projection of ellipsoid on a plane does not exist. Under stereographic projection of ellipsoid a projection, is understood to possess above-indicated properties of stereographic projection of a sphere and turning to such $\alpha = 0$, where α — compression of ellipsoid.

There are many determinations of stereographic projections of ellipsoid on a plane. In geodesy there are known stereographic projections determined by Gauss, Russell, Heyvelink (so-called Dutch projection) and others. All of them correspond to horizontal stereographic projection of a sphere, i.e., projection with freely selected central point which is especially valuable for resolution of problems of higher geodesy.

In order to have clear geometric presentation of stereographic projections, let us first consider the stereographic projection of a sphere.

Let us assume that on the surface of a sphere P_1 and P_2 are two infinitely close points with latitudes φ and $\varphi + \Delta\varphi$, R a radius of sphere, C a point of view, PT a tangent at point P. Points P_1' and P_2' projections of points P_1 and P_2 on an image plane (Fig. 99).

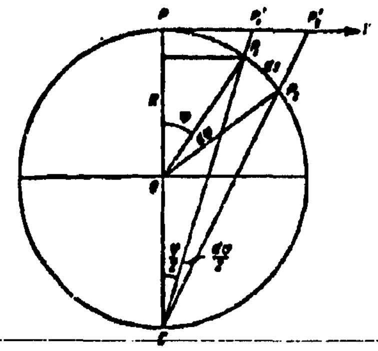


Fig. 99.

From triangles CPP_1' and CPP_2'

$$\left.\begin{aligned} PP_1' &= 2R\,\mathrm{tg}\,\frac{\varphi}{2} \\ PP_2' &= 2R\,\mathrm{tg}\,\frac{\varphi + d\varphi}{2} \end{aligned}\right\}. \qquad (9.14)$$

$$P_1'P_2' = PP_2' - PP_1' = 2R\left(\mathrm{tg}\,\frac{\varphi + d\varphi}{2} - \mathrm{tg}\,\frac{\varphi}{2}\right) = \frac{2R\sin\frac{d\varphi}{2}}{\cos\frac{\varphi}{2}\cos\frac{\varphi + d\varphi}{2}}.$$

Accepting by smallness $d\varphi$ $\sin d\varphi = 0$ and $\cos d\varphi = 1$, we obtain:

$$P_1'P_2' = \frac{R\,d\varphi}{\cos^2\frac{\varphi}{2}}. \qquad (9.15)$$

Scale of projected image will be:

$$m = \frac{P_1'P_2'}{P_1P_2} = \frac{\dfrac{R\,d\varphi}{\cos^2\frac{\varphi}{2}}}{R\,d\varphi} = \frac{1}{\cos^2\frac{\varphi}{2}},$$

where on a sphere $\varphi = \frac{s}{R}$, s is an arc of meridian, hence:

$$m = \frac{1}{\cos^2\frac{s}{2R}}.$$

Factoring $\cos^{-2}\frac{s}{2R}$ into binominal series and being limited by main term of these series, we obtain:

$$m = 1 + \frac{s^2}{4R^2} + \ldots \qquad (9.16)$$

Corresponding formula in Gauss-Kruger projection has the form:

$$m_g = 1 + \frac{s^2}{2R^2} + \ldots \qquad (9.16')$$

From comparison of formulas (9.16) and (9.16') it follows that distortions of stereographic projections are identical in all radial directions, and by dimensions they are half as large as distortions in Gauss-Kruger projection. In only one particular case, when x = y, the distortions in both projections are identical. However limiting distortions, determining dimension of areas, in stereographic projection are about half that of Gauss-Kruger projection. This position is just for all shown projections in determinations of Gauss, Russell, and Geyvelink.

If point F (Fig. 99) is taken as a beginning of grid coordinates, axis abscissa is directed along tangent FT, and axis of ordinates is perpendicular to FT, then when y = 0

$$x = 2R\,\mathrm{tg}\frac{s}{2R}, \tag{9.17}$$

where s — arc of meridian between parallel of given point and parallel of the beginning of coordinates.

1. Russell Projection

Of the stereographic geodetic projections the greater application was obtained by the projection determined by French geographer-geodesist Russell. In 1922 he generalized the formula (9.17) for surface of ellipsoid and offered stereographic projection, which is characterized by the following properties:

1) conformal image; 2) projection is symmetric relative to axial meridian and 3) abscissas of points of axial meridian by analogy with stereographic projection of a sphere are determined by formula:

$$x_R = 2R_0\,\mathrm{tg}\frac{X - X_0}{2R_0}, \tag{9.18}$$

where:

R_0 — mean radius of curvature at the origin of the coordinates;

X — arc of meridian from equator to a parallel with latitude B;

X_0 — arc of meridian from equator to parallel of the origin of coordinates.

In Gauss-Kruger projection it is quite the same where origin of coordinates is taken at axial meridian. Therefore let us assume that in this case the origin of coordinates of Gauss-Kruger projection coincides with the central point of Russell projection. In this case, obviously, $X - X_0$ will be abscissa of points of axial meridian in a system of Gauss-Kruger coordinates. We will designate it x_g, then:

$$x_R = 2R_0\,\mathrm{tg}\frac{x_g}{2R_0}, \tag{9.19}$$

$\frac{x_g}{2R_0}$ — small value; with the help of a series for tangents of this value we obtain:

$$x_R = x_g + \frac{x_g^3}{12R_0^2} + \frac{x_g^5}{120R_0^4} + \dots \tag{9.20}$$

To each point of ellipsoid with isometric coordinates (q, l) will correspond a point with grid coordinates (x_R, y_R) in Russell projection and the point with coordinates (x_g, y_g) in Gauss-Kruger projection. Otherwise, each point with coordinates (x_R, y_R) of Russell projection will correspond a point with coordinates (x_g, y_g) in Gauss-Kruger projection. This functional dependence can be expressed by equation in a form:

$$x_R + iy_R = f(x_g + iy_g). \tag{9.21}$$

Factoring right side of equation (9.21) by line of Taylor's method and dividing real and assumed parts we obtain:

$$x_R = f(x_g) - \frac{y_g^2}{2} f''(x_g) + \frac{y_g^4}{24} f^{IV}(x_g) + \dots$$

$$y_R = y_g f'(x_g) - \frac{1}{6} y_g^3 f'''(x_g) + \dots$$

When $y_R = 0$ and $y_g = 0$ then:

$$x_R = f(x_g) = 2R_0 \operatorname{tg} \frac{x_g}{2R_0} = x_g + \frac{x_g^3}{12R_0^2} + \frac{x_g^5}{120R_0^4} + \dots \tag{9.21'}$$

we have:

$$\left.\begin{aligned} f'(x_g) &= \frac{dx_R}{dx_g} = 1 + \frac{x_g^2}{4R_0^2} + \frac{x_g^4}{24R_0^4} + \dots \\ f''(x_g) &= \frac{d^2x_R}{dx_g^2} = \frac{x_g}{2R_0^2} + \frac{x_g^3}{6R_0^4} + \dots \\ f'''(x_g) &= \frac{d^3x_R}{dx_g^3} = \frac{1}{2R_0^2} + \frac{x_g^2}{3R_0^4} + \dots \end{aligned}\right\}. \tag{9.21''}$$

Substituting values of derivatives $f^i(x_g)$ in equation for x_R and y_R, we obtain:

$$\left.\begin{aligned} x_R &= x_g + \frac{x_g^3}{12R_0^2} - \frac{x_g y_g^2}{4R_0^2} + \dots \\ y_R &= y_g + \frac{y_g x_g^2}{4R_0^2} - \frac{y_g^3}{12R_0^2} \end{aligned}\right\}. \tag{9.22}$$

For obtaining Russell coordinates by a geodetic we will use a method, developed by academician V. K. Khristov.[1] For Russell coordinates we will record a known equation of conformal mapping:

$$x_R + iy_R = F[q_0 + (\Delta q + il)].$$

[1] Zeitschr. f. Vermessung. 1937, p. 84-89.

Here $\Delta q = q - q_0$, q_0 — values of isometric latitude in origin of coordinates in a system of Russell coordinates, where $x_R = 0$, $F(q_0) = 0$. We will expand the analytic function F by powers of $(\Delta q + il)$. We have:

$$x_R + iy_R = F(q_0) + (\Delta q + il)F'(q_0) + \frac{(\Delta q + il)^2}{2!}\cdot F''(q_0) + \\ + \frac{(\Delta q + il)^3}{3!}F'''(q_0) + \ldots \tag{9.23}$$

For calculation of derivatives we have a formula:

$$F'(q) = \frac{dx_R}{dq} = \frac{dx_R}{dx_g}\cdot\frac{dx_g}{dq},$$

$$F''(q) = \frac{d^2x_R}{dq^2} = \frac{d^2x_R}{dx_g^2}\left(\frac{dx_g}{dq}\right)^2 + \frac{dx_R}{dx_g}\cdot\frac{d^2x_g}{dq^2},$$

$$F'''(q) = \frac{d^3x_R}{dq^3} = \frac{d^3x_R}{dx_g^3}\left(\frac{dx_g}{dq}\right)^3 + 3\frac{d^2x_R}{dx_g^2}\cdot\frac{dx_g}{dq}\cdot\frac{d^2x_g}{dq^2} + \\ + \frac{dx_R}{dx_g}\cdot\frac{d^3x_g}{dq^3}.$$

These derivatives have to be calculated when $x_R = F(q_0) = 0$. Taking into account (9.20) and (8.17'), we obtain

$$F(q_0) = 0,$$

$$F'(q_0) = b_1^0 = N_0\cos B_0,$$

$$F''(q_0) = 2a_2^0 = -N_0\sin B_0\cos B_0,$$

$$F'''(q_0) = \frac{b_3^0}{2R^2} - b_3^0 = -\frac{1}{2}N_0\cos^3 B_0(1 - 2\mathrm{tg}^2 B_0 + \eta_0^2).$$

Substituting these values of derivative in (9.23), we obtain

$$x_R + iy_R = N_0\cos B_0(\Delta q + il) - \frac{1}{2}N_0\sin B_0\cos B_0(\Delta q + il)^2 - \\ - \frac{1}{12}N_0\cos^3 B_0(1 - 2\mathrm{tg}^2 B_0 + \eta_0^2)(\Delta q + il)^3 + \ldots$$

Or, after separation of real and assumed parts,

$$x = N_0\cos B_0\Delta q - \frac{1}{2}N_0\sin B_0\cos B_0\Delta q^2 + \frac{1}{2}N_0\sin B_0\cos B_0 l^2 - \\ - \frac{1}{12}N_0\cos^3 B_0(1 - 2\mathrm{tg}^2 B_0 + \eta_0^2)\Delta q^3 + \frac{1}{4}N_0\cos^3 B_0(1 - 2\mathrm{tg}^2 B_0 + \\ + \eta_0^2)\Delta q l^2 + \ldots, \tag{9.24}$$

$$y = N_0\cos B_0 l - N_0\sin B_0\cos B_0\Delta q l - \frac{1}{4}N_0\cos^3 B_0 \times \\ \times (1 - 2\mathrm{tg}^2 B_0 + \eta_0^2)\Delta q^2 l + \frac{1}{12}N_0\cos^3 B_0(1 - 2\mathrm{tg}^2 B_0 + \eta_0^2)l^3 + \ldots \tag{9.24}$$

Sign "0" here means that these values pertain to latitude of origin of coordinates. In these formulas Δq is usually expressed by ΔB by the formula (8.27); omitting details of calculations, we will record final formulas for x_R and y_R with substitution of Δq by corresponding dependence by ΔB.

$$x_R = N_0(1-\eta_0^2+\eta_0^4)\Delta B + \frac{1}{2}N_0 \operatorname{tg} B_0(3\eta_0^2-6\eta_0^4)\Delta B^2 + $$
$$+\frac{1}{2}N_0 \sin B_0 \cos B_0 l^2 + \frac{1}{12}N_0(1+4\eta_0^2-6\operatorname{tg}^2 B_0 \eta_0^2)\Delta B^3 +$$
$$+\frac{1}{4}N_0 \cos^2 B_0(1-2\operatorname{tg}^2 B_0+2\operatorname{tg}^2 B_0 \eta_0^2)\Delta B l^2 + \ldots \quad (9.26)$$

$$y_R = N_0 \cos B_0 l - N_0 \sin B_0 \cos B_0 (1-\eta_0^2+\eta_0^4)\Delta B l -$$
$$-\frac{1}{4}N_0 \cos B_0(1-\eta_0^2+6\operatorname{tg}^2 B_0 \eta_0^2)\Delta B^2 l + \frac{1}{12}N_0 \cos^3 B_0 \times$$
$$\times(1-2t_0+\eta_0^2)l^3 - \ldots \quad (9.27)$$

Calculation of Russell coordinates by the formulas (9.26) and (9.27) has that peculiarity which at a given origin of coordinates of values, appearing at differences of latitudes and longitudes, are constant, once and for all calculated. Therefore calculation is reduced to remultiplication of various degrees of differences of latitude and longitudes to constant coefficients. Due to this expression (9.26) and (9.27) are called formulas with constant coefficients in literature. Certain authors try to obtain similar formulas for Gauss-Kruger coordinates. However formulas with constant coefficients in Gauss-Kruger projection do not have practical benefits, as compared to the usual. In practice, if Russell coordinates are calculated with accuracy of up to 0.1 m, prepared tables, are used with whose help the required coordinates are obtained by interpolation. Calculation by the formulas (9.26) and (9.27) is used in rare cases, when it is required to have coordinates with accuracy of up to 1 cm.

Reductions of directions are calculated by approximate formula, whose derivation is shown below.

Let us assume that on an ellipsoid two points of triangulation are given P_1 and P_2 (Fig. 100). We will join them with origin of coordinates O by geodesic.

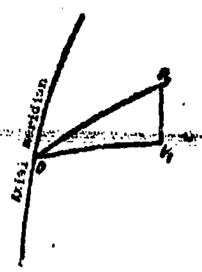


Fig. 100.

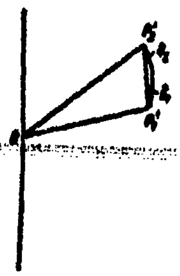


Fig. 101.

The sum of the angles of the triangle OP_1P_2 will be $180^\circ + \varepsilon$, ε is a spherical excess of this triangle. Let us assume that triangle $OP_1'P_2'$ (Fig. 101) corresponds to spherical triangle OP_1P_2 on a plane in Russell projection. The sum of the angles of the triangle $OP_1'P_2'$ will be $180^\circ + \delta_1 + \delta_2$, where δ_1 and δ_2 are corrections for curvature of the image of geodesic or reduction in directions.

Due to conformity of image:

$$180^\circ + \varepsilon = 180^\circ + \delta_1 + \delta_2$$

or

$$\varepsilon = \delta_1 + \delta_2$$

Let us approximate that $\delta_1 = \delta_2 = \delta$, then:

$$\delta = \frac{\varepsilon}{2}.$$

In enumerated formulas under δ_1, δ_2 and δ their absolute values are implied.

The spherical excess, as it is known, is equal to:

$$\varepsilon = \frac{P}{R^2}.$$

P – area of triangle $OP_1'P_2'$, which by coordinates of vertexes of a triangle is expressed by formula:

$$P = \frac{x_2 y_1 - x_1 y_2}{2},$$

therefore:

$$\delta'' = \rho'' \frac{x_2 y_1 - x_1 y_2}{4R^2}. \qquad (9.28)$$

Formula (9.28) gives main term for the reduction of direction.

We have:

$$y_1 = y_m - \frac{\Delta y}{2}; \quad y_2 = y_m + \frac{\Delta y}{2}.$$

then:

$$\delta'' = \rho'' \frac{\Delta x y_m}{4R^2} - \frac{x_m \Delta y}{4R^2} \rho''. \qquad (9.29)$$

Corresponding values in Gauss-Kruger projection are expressed by formula:

$$\delta_k'' = \rho'' \frac{\Delta x y_m}{2R^2}. \qquad (9.30)$$

From comparison of formulas (9.29) and (9.30) it follows that the reduction of directions in Russell projection is half as big as in Gauss-Kruger projection. This conclusion is quite correct with respect to main terms of formulas for the reduction of directions.

Russell projection is convenient for countries of round outline and comparatively small areas. It was used in geodetic work in Poland and Rumania, up to the Second World War, and in France since 1924.

Russell projection has certain advantages in comparison to Gauss-Kruger projection with respect to values of reduction in lengths and direction. But the calculations are somewhat more complicated than in Gauss-Kruger projection. The use of Russell projection in special geodetic work is profitable in cases, where necessity arises to introduce corrections in angles of polygonometeric and theodolite runs on Gauss-Kruger projection. In this case using Russell projection necessity of introduction of corrections in lengths of lines, and in directions is not needed.

Gauss-Kruger projection is universal and is useful for any countries and continents, but the Russell projection is only for small round shaped countries.

During planning of engineering construction and translation of a project to nature it is very important not to introduce into measured geodetic values of corrections in transition from ellipsoid to a plane. In this respect stereographic projection, especially for limited areas, has indubitable advantages over Gauss-Kruger projection.

In using stereographic projections in state work frequently the scale at central point is taken of equal value, to a smaller unit, i.e., $m_0 < 1$. Due to this, x and y coordinates, calculated by the formulas (9.26) and (9.27), should be multiplied by m_0, then the scale at any point will be:

$$m = \left(1 + \frac{s^2}{4R_v^2}\right) m_0.$$

Such scale decrease leads to redistribution of distortions, and maximum value of distortion falls to central area, but on area edges the scale becomes close to one. Prior to World War Poland adopted a scale with central point at $m_0 = 0.9995$. In this case the scale is equal to one at points within a radius of 285 km. In other words, the radius of an area of application of stereographic projection with $m_0 = 0.9995$ is nearly 300 km.

2. The Gauss Projection

Gauss made a more complex determination of stereographic projection of an ellipsoid on a plane. He proposed calculation of abscissa of points of axial meridian by a formula below:

$$\frac{x}{k} \approx \frac{g \operatorname{tg} \frac{\varphi}{2} - \operatorname{tg} \frac{\varphi_0}{2}}{1 + g \operatorname{tg} \frac{\varphi}{2} \operatorname{tg} \frac{\varphi_0}{2}}, \qquad (9.31)$$

where:

$$k = 2N_0, \quad \Phi = 90 - B, \quad \Phi_0 = 90 - B_0,$$

$$g = \frac{\left(\frac{1 + e\cos\Phi}{1 - e\cos\Phi}\right)^{\frac{1}{2}e}}{\left(\frac{1 + e\cos\Phi_0}{1 - e\cos\Phi_0}\right)^{\frac{1}{2}e}},$$

e — eccentricity of ellipsoid, sign "0" signifies, that a given value pertains to central point.

For any point of depicted area its stereographic coordinates are determined by a full equation:

$$\frac{-x + iy}{k} = \frac{ge^{il}\operatorname{tg}\frac{\Phi}{2} - \operatorname{tg}\frac{\Phi_0}{2}}{1 + ge^{il}\operatorname{tg}\frac{\Phi_0}{2}\operatorname{tg}\frac{\Phi}{2}}, \tag{9.32}$$

$$l = L - L_0, \quad i = \sqrt{-1}.$$

We will omit details of derivations and will give a formula, by which grid coordinates in Gauss stereographic projection are calculated:

$$\left.\begin{aligned} \frac{x}{k} &= \frac{1}{D_1}\left(\operatorname{ctg}\frac{\Phi_0}{2}\cos^2\frac{\Phi}{2} - g^2\operatorname{tg}\frac{\Phi_0}{2}\sin^2\frac{\Phi}{2}\right) - \operatorname{ctg}\Phi_0 \\ \frac{y}{k} &= g\,\frac{\sin\Phi\sin l}{D_1} \\ D_1 &= 2\cos^2\frac{\Phi_0}{2}\cos^2\frac{\Phi}{2} + g\sin\Phi_0\sin\Phi\cos l + 2g^2\sin^2\frac{\Phi_0}{2}\sin^2\frac{\Phi}{2} \end{aligned}\right\}. \quad (9.33)[1]$$

By these formulas abscissas are positive northward, and ordinates eastward.

The scale of image at main point is the same as in Russell projection only in expression of scale in Russell projection it is necessary to replace R_0 by N_0, therefore:

$$m = 1 + \frac{x^2 + y^2}{4N_0^2}. \tag{9.34}$$

Distortion of lengths in Gauss projection is the same, as in Russell projection. For reduction of directions by analogy with Russell projection we have:

$$\delta_g^{\prime\prime} = \frac{x_2y_1 - x_1y_2}{4N_0^2}\rho^{\prime\prime}. \tag{9.35}$$

Let us give in Table 10 the basic characteristics of Gauss-Kruger, Gauss and Russell projections.

This table shows that distortion of lengths in Gauss-Kruger projection is mostly larger than in stereographic projections; and the reduction of directions in

[1]L. Kruger. Zur stereographischen Projection, 1922, p. 8.

Table 10

Designation of characteristics	Gauss-Kruger projection	Stereographic projections	
		Gauss	Russell
Distortion of lengths	$\frac{s y_m^2}{2R_m^2}$	$\frac{s(y_m^2+x_m^2)}{4N_m^2}$	$\frac{s(y_m^2+x_m^2)}{4R_0^2}$
Reduction of lengths ($\lg d - \lg s$)	$\frac{\mu y_m^2}{2R_m^2}+\frac{\mu}{24}\left(\frac{\Delta y}{R_m}\right)^2$	$\frac{\mu(x_m^2+y_m^2)}{4N^2}+\mu\frac{d^2}{48N^2}$	$\frac{\mu(x_m^2+y_m^2)}{4R_0^2}+\mu\frac{d^2}{48R_0^2}$
Reduction of directions δ	$\frac{(x_2-x_1)y_m}{2R_m^2}\rho''$	$\frac{x_2y_1-x_1y_2}{4N^2}\rho''$	$\frac{x_2y_1-x_1y_2}{4R_0^2}\rho''$

stereographic projections is half as large. But stereographic projections yield to Gauss-Kruger projection with respect to simplicity of formulas. The application of stereographic projections is expedient for areas of round outline, while Gauss-Kruger projection is universal.

§ 54. CONCLUSION ON GEODETIC PROJECTIONS

Short survey of geodetic projections permits a comparatively easy answer to a very important question: how well is selection of Gauss-Kruger projection and coordinates for the USSR is founded.

The selection of geodetic projection stems from dimensions and configuration of the country; besides an effort is made to adopt a single system for all the country. For small countries with round configuration it is expedient to select some stereographic projection; when the area extends from south to north along a meridian it should be a cylindrical projection; when it is from west to east along a parallel it should be conical.

For such states as USSR, Chinese Peoples Republic, United States and others the question of selection of a single system of plane coordinates of any projection for entire country is generally dropped. Here a problem appears about expedient division of territory into coordinate zones, with the smallest possible number of them, with a single system of coordinates in zones for the convenience of practical calculations during transition from ellipsoid to a plane and conversely.

Gauss-Kruger coordinates are characterized by the following important properties for large areas:

1. Scale of image and convergence of meridians increases eastward and westward from the axial meridian comparatively slowly and are functions of ordinates of a point at a given latitude.

2. Coordinate zones are two angles of meridian directions, extending from southern to northern poles, and are symmetric in relation to axial meridian.

3. Systems of coordinates in all zones are similar; besides a number of coordinate zones for large areas and even for all the surface of the Earth is comparatively small.

4. Formulas for resolution of direct and inverse problems of projection, are simple power series of a similar form and are functions of not more than two arguments. With availability of special tables, identical for all zones, the calculations are made very simply and with necessary accuracy.

From mathematical side the advantages of Gauss-Kruger coordinates are easily revealed by comparison of basic characteristic functions of geodetic projections. These functions are usually given for points of axial meridian.

1. Gauss-Kruger projection $x_{G-K} = X$.

2. Sol'dner projection $x_{Z} = X$.

3. Russell stereographic projection $x_R = 2R_0 \operatorname{tg} \frac{X - X}{2R_0}$.

4. Gauss stereographic projection $x_G = \frac{g \operatorname{tg} \frac{1}{2}\phi - \operatorname{tg} \frac{1}{2}\phi_0}{1 + g \operatorname{tg} \frac{1}{2}\phi_0 \operatorname{tg} \frac{1}{2}\phi}$.

5. Lambert conical projection $x_L = N_0 \operatorname{ctg} B_0$

From these formulae it follows that the more simple characteristic functions are of Gauss-Kruger and Sol'dner projections, but the last one is not conformal. This property of Gauss-Kruger projection allows transfer of origin of coordinates along axial meridian and to take it at any point, this can lead to simplification of formulae without damage to their generalization.

From geodetic projections only the Gauss-Kruger projection can be applied for all the surface of the globe, if, of course, all the countries will adopt the same reference ellipsoid. We must assume that in the future a question can appear about a single system of rectangular coordinates for all the Earth. As Academician V. K. Khristov points out, it would be possible in such a case to avoid negative abscissas for southern hemisphere, if a length of square of a meridian is added to points of axial meridian, that is to take:

$$x_g = X + Q.$$

This means that length of arc of meridian should be measured from South Pole.

In propagation of Gauss-Kruger coordinates to large areas a definite system is required. USSR has the greatest experience in use of Gauss-Kruger coordinates both in geodetic, and cartographic work. This experience should be considered in all cases, however, it is already used in certain countries. Here is what German geodesist Kneiss, coauthor of the last (tenth) edition of a well known "Instruction on Geodesy" Jordan, in Volume IV of this work writes:

"In USSR for the purpose of cartography Gauss-Kruger projection is also used with three degree and six degree zones. The Soviet designations for six-degree zones, for coordinates and corresponding grid on maps are very expedient, for the same reason they were also adopted in other countries, among them in Germany, for special maps, made during the war."[1]

[1]Jordan-Eggert-Kneiss. Handbuch der Vermessungskunde. T. IV, 1958, p. 1150.

CHAPTER X

DIFFERENTIAL FORMULAS

§ 55. DETERMINATIONS

By differential formulas of spheroidal geodesy are meant such, with whose help corrections of calculated geodetic coordinates and azimuths for a change of initial geodetic data such as: initial geodesic coordinates, azimuths, distances, major semiaxis and compression of reference ellipsoid are taken into consideration. In accordance with this two forms of differential formulae are distinguished. Those, which give indicated corrections for change of initial geodetic coordinates, distances and azimuths, are called differential formulas of first type, and those, which give corrections for change of major semiaxis and compression of reference-ellipsoid, are called differential formulas of second type. It is clear, that these terms are conditional, but they have now become conventional.

Causes, for changes of initial geodetic data, are as a rule, unknown beforehand. For instance, initial geodetic coordinates can change after general adjustment of astronomic geodetic net of the country. Errors in initial data can be revealed during calculations of triangulation, even gross errors are possible in initial data, of coordinates and azimuths.

When, reference-ellipsoid adopted in a given country is replaced by another, more suitable for the area of the country, it becomes necessary to recompute the coordinates of points on a new reference-ellipsoid. In this case it is necessary to use differential formulas of both first, and second type.

Prerevolutionary triangulation in Russia was computed on ellipsoids of Val'bek, Bessel, Clarke and even on "coordinating" ellipsoid. Therefore in using points of old triangulations, necessity arises for use of differential formulas for recomputation of coordinates of points of old triangulations on Krasovskiy reference-ellipsoid. Furthermore, differential formulas of the first type are used in adjusting astronomic geodetic net by a method of N. A. Urmayev, and formulas of the second type during composition of equations of triangulation.

Naturally, if geodetic coordinates and parameters of reference-ellipsoid are changed, then correspondingly grid coordinates have to be changed in USSR to Gauss-Kruger coordinates. Consequently, a necessity arises for obtaining differential formulas for grid coordinates for adopted projection of terrestrial ellipsoid on a plane. We will call such formulas differential formulas of the third type.

§ 56. DIFFERENTIAL FORMULAS OF THE FIRST TYPE

Let us assume that geodetic coordinates of the first point, by which distance and azimuth of coordinates of the second point were calculated obtained corresponding increases; then the change of coordinates of the second point and azimuth of geodesic of this point can be expressed by the following formulas:

$$\left.\begin{aligned} dB_2 &= \frac{\partial B_2}{\partial s}ds + \frac{\partial B_2}{\partial A_1}dA_1 + \frac{\partial B_2}{\partial B_1}dB_1 \\ dL_2 &= \frac{\partial L_2}{\partial s}ds + \frac{\partial L_2}{\partial A_1}dA_1 + \frac{\partial L_2}{\partial B_1}dB_1 + dL_1 \\ dA_2 &= \frac{\partial A_2}{\partial s}ds + \frac{\partial A_2}{\partial A_1}dA_1 + \frac{\partial A_2}{\partial B_1}dB_1 \end{aligned}\right\}. \tag{10.1}$$

Expressions (10.1) essentially are not total differentials, since here in a strictly mathematical sense there are no partial derivatives, and on the face of it partial changes ds, dA_1, dB_1 are not differentials, but certain given numbers, the values however, under the sign of partial derivatives are conversion factors. Therefore it would be more correct (10.1) to rewrite them in the following form:

$$\left.\begin{aligned} dB_2 &= dB_2^s + dB_2^A + dB_2^B \\ dL_2 &= dL_1 + dL_2^s + dL_2^A + dL_2^B \\ dA_2 &= dA_2^s + dA_2^A + dA_2^B \end{aligned}\right\}. \tag{10.2}$$

where signs from above are s, A_1 and B_1 mean that corresponding values consider only change of length, azimuth and latitude. We find these values from geometric relationships. We will define dB_2^s dL_2^s and dA_2^s. Let us assume that the geodesic between two points was changed by ds (Fig. 102), then from elementary right-angle triangle $P_2P_2'P_2''$ (Fig. 103) we have:

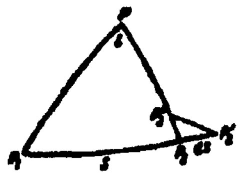


Fig. 102.

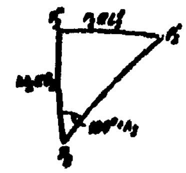


Fig. 103.

$$M_2 dB_2^s = ds\cos(180^\circ + A_2),$$
$$r_2 dL_2^s = ds\sin(180^\circ + A_2),$$
$$dA_2^s = dL_2^s \sin B_2,$$

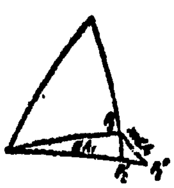
or

$$\left.\begin{aligned} dB_2^s &= -\frac{ds}{M_2}\cos A_2 \\ dL_2^s &= -\frac{ds}{r_2}\sin A_2 \\ dA_2^s &= -\frac{ds}{r_2}\sin A_2 \sin B_2 \end{aligned}\right\}. \qquad (10.3)$$

Formulas (10.3) give partial changes of latitude, longitude and azimuth at the change of s to ds.

Let us assume that now the initial azimuth was changed to dA_1 (Fig. 104), then in accordance with Fig. 105 we have:

$$-M_2 dB_2^{A_1} = -m\,dA_1 \sin A_2,$$
$$r_2 dL_2^{A_1} = -m\,dA_1 \cos A_2,$$



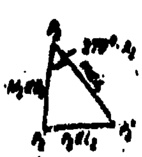
or

$$\left.\begin{aligned} dB_2^{A_1} &= \frac{m}{M_2}\sin A_2 dA_1 \\ dL_2^{A_1} &= -\frac{m}{r_2}\cos A_2 dA_1 \end{aligned}\right\}. \qquad (10.4)$$

Fig. 104.

Using (10.4), we find a change of second azimuth. Enter fundamental equation of a geodesic in the form of $r_1 \sin A_1 = -r_2 \sin A_2$, differentiating this and considering r_1 as a constant, we obtain:

$$r_1 \cos A_1 dA_1 = -dr_2 \sin A_2 - r_2 \cos A_2 dA_2,$$



hence:

$$dA_2 = -\frac{r_1 \cos A_1}{r_2 \cos A_2} dA_1 - \frac{dr_2}{r_2}\,\mathrm{tg}\,A_2.$$

Fig. 105.

But:

$$dr_2 = -M_2 \sin B_2 dB_2,$$

therefore taking into account the first from (10.4), we obtain:

$$dA_2^{A_1} = -\left[\frac{r_1 \cos A_1}{r_2 \cos A_2} - \frac{m}{r_2}\operatorname{tg} A_2 \sin A_2 \sin B_2\right] dA_1. \qquad (10.5)$$

In expressions (10.4) and (10.5) m is a reduced leng h of geodesic, for whose computation formula (3.45) should be used.

In order to find the influence of the change of initial latitude in geodetic coordinates of second point, we will use the following construction.

Let us assume that with constant s and A_1 the latitude B_1 is changed to dB_1 (Fig. 106), then P_1 will occupy on its meridian position P_1'. Let us take P_2 as origin of the polar geodetic coordinate, and radius, equal to s, and describe the geodetic circumference where $P_1P_1'' = m dA_2'$. We will transfer the geodesic $P_1''P_1'P_2$ parallel to itself in such a manner that point P_1'' coincided with P_1', then s, passing through P_2, will occupy a new position $P_1'P_2P_2''$, where by construction $P_1''P_1' = P_2P_2''$.

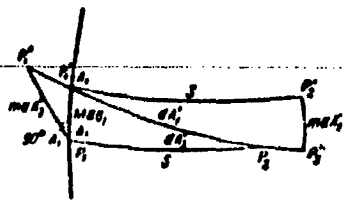


Fig. 106.

Taking now for the origin of polar coordinates point P_1', we will revolve the geodesic until it will not have an azimuth, equal to A_1 at point P_1'. During rotation of second end of the geodesic it will describe an arc of geodetic circumference and point P_2'' and occupy position P_2''. Obviously, P_2' is the unknown position P_2 at the change of B_1 to dB_1. This construction shows that the influence of dB_1 on final coordinates can be considered as a change of length of the geodesic to $ds' = P_2P_2''$ and the initial azimuth to dA_1'.

From the elementary right-angle triangle $P_1P_1''P_1'$

$$P_1P_1'' = m dA_2' = M_1 dB_1 \sin A_1, \quad P_1''P_1' = P_2P_2'' = M_1 dB_1 \cos A_1.$$

Applying equation (3.49):

$$\frac{d\theta}{d\sigma} = \frac{1}{m}\left(\frac{dm}{ds}\right)\sin\theta,$$

In our case, where $d\theta = dA_1'$, $d\sigma = M_1 dB_1$ and $\theta = A_1$.

$$dA_1' = \frac{1}{m}\left(\frac{dm}{ds}\right)_2 \sin A_1 M_1 dB_1,$$

$$dA_2' = \frac{P_1P_1''}{m} = \frac{M_1 \sin A_1 dB_1}{m},$$

$$P_2''P_2' = m dA_1' = M_1 \sin A_1 dB_1 \left(\frac{dm}{ds}\right)_2.$$

Passing from P_2 to P_2'' and from P_2'' to P_2', we obtain:

$$\left.\begin{aligned} dB_2^{n_1} &= -\frac{M_1}{M_2}\left\{\sin A_1 \sin A_2 \left(\frac{dm}{ds}\right)_2 + \cos A_1 \cos A_2\right\} dB_1 \\ dL_2^{n_1} &= \frac{M_1}{r_2}\left\{\sin A_1 \sin A_2 \left(\frac{dm}{ds}\right)_2 - \cos A_1 \sin A_2\right\} dB_1 \end{aligned}\right\}, \tag{10.6}$$

$$\begin{aligned} dA_2^{n_1} = \frac{M_1}{m}\Big\{&\sin A_1 - \frac{m}{r_2}\sin A_2 \cos A_1 \sin B_2 + \\ &+ \sin A_1\left[\frac{r_1 \cos A_1}{r_2 \cos A_2} - \frac{m}{r_2}\operatorname{tg} A_2 \sin A_2 \sin B_2\right]\Big\} dB_1. \end{aligned} \tag{10.7}$$

Sign "2" at derivative $\left(\frac{dm}{ds}\right)$ indicates that it is taken for point P_2.

Expressions (10.3), (10.4), (10.6) and (10.7) in totality strictly resolve the posed problem and are called differential formulas of first type.

These formulas are suitable for any s. For short distances, on the order of a side of 1st order triangulation, these formulas can be simplified, by taking $M_1 = M_2 = N_1 = N_2$ and $m = s - \frac{s^3}{6} \ldots$ But in practice it is better to use other formulas, which are obtained with the help of formulae with mean arguments.

Let us consider inverse problem of differential formulas: change of length of arc of geodesic and its azimuths, evoked by changes of latitude and longitude of terminal points.

Let us assume that arc s was changed to ds and A_1 to dA_1, then from (10.3), (10.4) and (10.5) we have:

$$\left.\begin{aligned} dB_2 &= \frac{m}{M_2}\sin A_2 dA_1 - \frac{\rho''}{M_2}\cos A_2 ds \\ dL_2 &= -\frac{m\cos A_2}{r_2} \cdot \frac{\rho'' \sin A_2 ds}{r_2} \end{aligned}\right| \begin{matrix} -M_2 \sin A_2 \\ r_2 \cos A_2 \end{matrix} \left| \begin{matrix} M_2 \cos A_2 \\ r_2 \sin A_2 \end{matrix} \right\| \tag{10.8}$$

$$\begin{aligned} dA_2 = &-\left[\frac{r_1 \cos A_1}{r_2 \cos A_2} + \frac{m}{r_2}\operatorname{tg} A_2 \sin A_2 \sin B_2\right] dA_1 - \\ &- \frac{\rho''}{r_2}\sin A_2 \sin B_2\, ds. \end{aligned} \tag{10.9}$$

From the first two equations (10.8) by means of multiplication by values, shown on the right, and addition we obtain:

$$\left.\begin{aligned} \rho'' ds &= -M_2 \cos A_2 dB_2 - r_2 \sin A_2 dL_2 \\ m dA_1 &= M_2 \sin A_2 dB_2 - r_2 \cos A_2 dL_2 \end{aligned}\right\} \tag{10.10}$$

Replacing in (10.9) ρ'' ds and mdA_1 through (10.10), we obtain:

$$\begin{aligned} m dA_2 = &-\left[\frac{r_1 \cos A_1}{r_2 \cos A_2} + \frac{m}{r_2}\operatorname{tg} A_2 \sin A_2 \sin B_2\right]\left[M_2 \sin A_2 dB_2 - r_2 \cos A_2 dL_2\right] + \\ &+ \frac{m \sin A_2 \sin B_2}{r_2}\left[M_2 \cos A_2 dB_2 - r_2 \sin A_2 dL_2\right]. \end{aligned}$$

Omitting details of calculations, this expression can be brought to the following form:

$$mdA_2 = M_2\left(\frac{dm}{ds}\right)_1 \sin A_2 dB_2 + r_1 \cos A_1 dL_2 \qquad (10.11)$$

Thus, as a result of change of geodetic coordinates of terminal point to dB_2 and dL_2 the length of geodesic and its azimuths change thus:

$$\left.\begin{aligned} \rho'' ds &= -M_2 \cos A_2 dB_2 - r_2 \sin A_2 dL_2 \\ mdA_1 &= M_2 \sin A_2 dB_2 - r_2 \cos A_2 dL_2 \\ mdA_2 &= M_2\left(\frac{dm}{ds}\right)_1 \sin A_2 dB_2 + r_1 \cos A_1 dL_2 \end{aligned}\right\}. \qquad (10.12)$$

At the change of coordinates of initial point to dB_1 and dL_1 formulae (10.12) preserve their strength with replacement of indices "1" to "2", i.e.,

$$\left.\begin{aligned} \rho'' ds &= -M_1 \cos A_1 dB_1 - r_1 \sin A_1 dL_1 \\ mdA_1 &= M_1\left(\frac{dm}{ds}\right)_2 \sin A_1 dB_1 + r_2 \cos A_2 dL_1 \\ mdA_2 &= M_1 \sin A_1 dB_1 - r_1 \cos A_1 dL_1 \end{aligned}\right\}. \qquad (10.13)$$

If however coordinates of initial and terminal points of geodesic are simultaneously changed then, taking into account $r_1 \sin A_1 = -r_2 \sin A_2$ from (10.12) and (10.13) it follows that:

$$\left.\begin{aligned} \rho'' ds &= -M_1 \cos A_1 dB_1 - M_2 \cos A_2 dB_2 - r_2 \sin A_2 (dL_2 - dL_1) \\ mdA_1 &= M_1\left(\frac{dm}{ds}\right)_2 \sin A_1 dB_1 + M_2 \sin A_2 dB_2 - r_2 \cos A_2 \times \\ &\times (dL_2 - dL_1) \\ mdA_2 &= M_1 \sin A_1 dB_1 + M_2\left(\frac{dm}{ds}\right)_1 \sin A_2 dB_2 + r_1 \cos A_1 \times \\ &\times (dL_2 - dL_1) \end{aligned}\right\}. \qquad (10.14)$$

Differential formulas (10.14) in simplified form are applied for adjusting astronomic geodetic nets by a method of N. A. Urmayev. All above obtained differential formulas are strict and are suitable for any s.

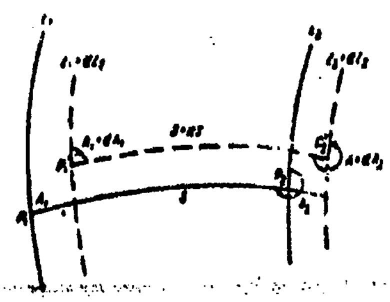


Fig. 107.

Here are differential formulas of first type in Helmert designations.

We have:

$$\left.\begin{aligned} dB_2 &= p_1 dB_1 + p_2 ds + p_3 dA_1 \\ dL_2 &= dL_1 + q_1 dB_1 + q_2 ds + q_3 dA_1 \\ dA_2 &= r_1 dB_1 + r_2 ds + r_3 dA_1 \end{aligned}\right\}. \qquad (10.15)$$

Here:

$$p_1 = -\frac{M_1}{M_2}\left[\sin A_1 \sin A_2 \left(\frac{dm}{ds}\right)_1 + \cos A_1 \cos A_2\right],$$

$$p_2 = -\frac{\cos A_2}{M_2}\rho'',$$

$$p_3 = \frac{m}{M_2}\sin A_2,$$

$$q_1 = \frac{M_1}{r_2}\left[\sin A_1 \sin A_2 \left(\frac{dm}{ds}\right)_1 - \cos A_1 \sin A_2\right].$$

$$q_3 = -\frac{\sin A_2}{r_2}\rho''$$

$$q_4 = -\frac{m}{r_2}\cos A_2,$$

$$r_1 = \frac{M}{m}\Big[\sin A_1 - \frac{m}{r_2}\sin A_2\cos A_1\sin B_2 + \sin A_1\Big(\frac{r_1\cos A_1}{r_2\cos A_2} -$$

$$- \frac{m}{r_2}\operatorname{tg} A_2\sin A_2\sin B_2\Big)\Big].$$

$$r_3 = -\frac{\sin A_2\sin B_2}{r_2}\rho'',$$

$$r_4 = -\left[\frac{r_1\cos A_1}{r_2\cos A_2} - \frac{m}{r_2}\operatorname{tg} A_2\sin A_2\sin B_2\right].$$

Fig. 107 gives geometric representation of values, included in differential formulas of the first type.

§ 57. DIFFERENTIAL FORMULAS OF SECOND TYPE

We must find changes of differences of latitudes, longitudes and azimuths, caused by changes of major semiaxis of adopted reference-ellipsoid to da and compression to $d\alpha$. In the common form we may assume that the shown differences, of the functions of major semiaxis and compression of ellipsoid are:

$$b = b(a, \alpha),$$
$$l = l(a, \alpha),$$
$$t = t(a, \alpha)$$

or:

$$\left.\begin{aligned} \delta b &= \frac{\partial b}{\partial a}da + \frac{\partial b}{\partial \alpha}d\alpha \\ \delta l &= \frac{\partial l}{\partial a}da + \frac{\partial l}{\partial \alpha}d\alpha \\ \delta t &= \frac{\partial t}{\partial a}da + \frac{\partial t}{\partial \alpha}d\alpha \end{aligned}\right\}. \qquad (10.16)$$

For b, l and t formulae (5.9) were obtained. Retaining in them small values up to second order inclusively, we have:

$$\left.\begin{aligned} b &= b_1 u + b_2 v^2 + b_3 u^2 + l_0 \\ l &= l_1 v + l_2 uv + l_0 \\ t &= a_1 v + a_2 uv + l_0 \end{aligned}\right\}. \qquad (10.17)$$

In these formulas:

$$\left.\begin{aligned} &u = s\cos A, \quad v = s\sin A \\ &b_1 = \frac{V^2}{N}, \quad b_2 = -\frac{V^2}{2N^2}\operatorname{tg} B, \quad b_3 = -\frac{3\eta^2\operatorname{tg} B V^2}{2N^2} \\ &l_1 = \frac{1}{N\cos B}, \quad l_2 = \frac{2\operatorname{tg} B}{N^2\cos B} \\ &a_1 = \frac{\operatorname{tg} B}{N}, \quad a_2 = \frac{1+\eta^2+\operatorname{tg}^2 B}{2N^2} \\ &V^2 = 1+\eta^2, \quad \eta^2 = e'^2\cos^2 B \end{aligned}\right\}. \qquad (10.18)$$

From (10.17) we have:

$$\frac{\partial b}{\partial a}=\frac{db_1}{da}u+\frac{db_2}{da}v^2+\frac{db_3}{da}u^2,\quad \frac{\partial b}{\partial \alpha}=\frac{db_1}{d\alpha}u+\frac{db_2}{d\alpha}v^2+\frac{db_3}{d\alpha}u^2.$$

$$\frac{\partial l}{\partial a}=\frac{dl_1}{da}v+\frac{dl_2}{da}uv,\quad \frac{\partial l}{\partial \alpha}=\frac{dl_1}{d\alpha}v+\frac{dl_2}{d\alpha}uv,$$

$$\frac{\partial t}{\partial a}=\frac{da_1}{da}v+\frac{da_2}{da}uv,\quad \frac{\partial t}{\partial \alpha}=\frac{da_1}{d\alpha}v+\frac{da_2}{d\alpha}uv.$$

Omitting calculation of derivatives, we will record final results, retaining in them as before small values of the second order inclusively:

$$\left.\begin{aligned}
\delta b'' &= -\left[b''-\frac{3}{2}\operatorname{tg}B\,\eta_1^2\frac{b''^2}{\rho''}-\frac{l''^2\cos^2 B\operatorname{tg}B}{2}\,\frac{l''^2}{\rho''}\right]\frac{da}{a}+\\
&+\left[b''\cos^2 B\left(2-t^2+\eta_1^2+\frac{7}{2}\eta_1^2\operatorname{tg}^2 B\right)-\frac{3b''^2\cos^2 B\operatorname{tg}B}{2\rho''}\times\right.\\
&\times\left(2-2\eta_1^2+2t^2\eta_1^2\right)+\frac{l''^2\cos^4 B\operatorname{tg}B}{2\rho''}\left(\operatorname{tg}^2 B+\frac{1}{2}\eta_1^2\operatorname{tg}^2 B+\right.\\
&\left.\left.+\frac{1}{2}\eta_1^2\operatorname{tg}^4 B\right)\right]d\alpha\\
\delta l'' &= -\left[l''+\frac{l''b''}{\rho''}\operatorname{tg}B\left(1-\eta_1^2\right)\right]\frac{da}{a}-\left[l''\cos^2 B\left(\operatorname{tg}^2 B-\right.\right.\\
&\left.-\frac{1}{2}\eta_1^2\operatorname{tg}^2 B+\frac{1}{2}\eta_1^2\operatorname{tg}^4 B\right)+\frac{l''b''}{\rho''}\cos^2 B\operatorname{tg}B\left(\operatorname{tg}^2 B-\right.\\
&\left.\left.-\frac{3}{2}\eta_1^2\operatorname{tg}^2 B+\frac{1}{2}\eta_1^2\operatorname{tg}^4 B\right)\right]d\alpha\\
\delta t'' &= -\left[l''\cos B\operatorname{tg}B+\frac{b''l''}{\rho''}\cos B\left(1+\operatorname{tg}^2 B-\eta_1^2\operatorname{tg}^2 B\right)\right]\frac{da}{a}-\\
&\quad-\left[l''\cos^3 B\operatorname{tg}B\left(\operatorname{tg}^2 B-\frac{1}{2}\eta_1^2\operatorname{tg}^2 B+\frac{1}{2}\eta_1^2\operatorname{tg}^4 B\right)-\right.\\
&\left.-\frac{b''l''}{\rho''}\cos^3 B\left(1-\operatorname{tg}^2 B-\operatorname{tg}^4 B+\frac{1}{2}\eta_1^2+2\eta_1^2 t^2\right)\right]d\alpha
\end{aligned}\right\} \qquad (10.19)$$

Differential formulas (10.19) are used in calculation of corrections in differences of latitudes, longitudes and azimuths for a change of major semiaxis by da and compression by dα on the adopted reference-ellipsoid. These formulae possess high accuracy for distances of about 200-300 km, i.e., for diagonals of 1st order triangulation figures. Therefore they should be recommended for degree measurements, when parameters of terrestrial ellipsoid from astronomic geodetic nets are determined. For 1st order triangulation, where limiting lengths of sides do not exceed 60-70 km, these formulas are excessively exact and are bulky.

Inasmuch as differential formulas at distances of 50-70 km are more frequently used, special formulas, computed for mass application are shown below.

§ 58. JOINT DIFFERENTIAL FORMULAS OF FIRST AND SECOND TYPE FOR 1ST ORDER TRIANGULATION

Let us assume that initial data for computation of coordinates of triangulation points were changed simultaneously:

$$
\begin{gathered}
B_1 \text{ to } dB_1;\quad s \text{ to } ds;\quad a \text{ to } da;\\
L_1 \text{ to } dL_1;\quad A_1 \text{ to } dA_1;\quad \alpha \text{ to } d\alpha;
\end{gathered}
$$

It is required to find changes B_2, L_2 and A_2.

It can be accepted that:

$$
\left.
\begin{aligned}
&B_2 = B(B_1, s, A_1, a, \alpha) \text{ and } B_2 + \delta B_2 = B(B_1 + dB_1, s + ds, A_1 +\\
&+ dA_1, a + da, \alpha + d\alpha)\\
&L_2 = L(B_1, s, A_1, a, \alpha) \text{ and } L_2 + \delta L_2 = L(B_1 + dB_1, s + ds, A_1 +\\
&+ dA_1, a + da, \alpha + d\alpha)\\
&A_2 = A(B_1, s, A_1, a, \alpha) \text{ and } A_2 + \delta A_2 = A(B_1 + dB_1, s + ds, A_1 +\\
&+ dA_1, a + da, \alpha + d\alpha)
\end{aligned}
\right\} . \qquad (10.20)
$$

From (10.20)

$$
\left.
\begin{aligned}
\delta B_2 &= dB_1 + \frac{\partial B}{\partial B_1} dB_1 + s\frac{\partial B}{\partial s}\frac{ds}{s} + \frac{\partial B}{\partial A_1} dA_1 + a\frac{\partial B}{\partial a}\frac{da}{a} + \frac{\partial B}{\partial \alpha} d\alpha\\
\delta L_2 &= dL_1 + \frac{\partial L}{\partial B_1} dB_1 + s\frac{\partial L}{\partial s}\frac{ds}{s} + \frac{\partial L}{\partial A_1} dA_1 + a\frac{\partial L}{\partial a}\frac{da}{a} + \frac{\partial L}{\partial \alpha} d\alpha\\
\delta A_2 &= dA_1 + \frac{\partial A}{\partial B_1} dB_1 + s\frac{\partial A}{\partial s}\frac{ds}{s} + \frac{\partial A}{\partial A_1} dA_1 + a\frac{\partial A}{\partial a}\frac{da}{a} + \frac{\partial A}{\partial \alpha} d\alpha
\end{aligned}
\right\} . \qquad (10.20')
$$

For computation of partial derivatives from B, L and A we will take these functions in the form of main terms of formulas with mean arguments:

$$
\left.
\begin{aligned}
b &= \frac{W^3_m}{a(1-e^2)} s\cos A_m + l_3\\
l &= \frac{W_m}{a} s\sin A_m \sec B_m + l_3\\
t &= \frac{W_m}{a} s\sin A_m \operatorname{tg} B_m + l_3
\end{aligned}
\right\} . \qquad (10.21)
$$

In computation of partial derivatives we will retain in them only small values of first order. Since partial derivatives have in (10.20') factors dB_1, $\frac{ds}{s}$, dA_1, $\frac{da}{a}$ and $d\alpha$ are small values of the second order, then final formulas for δB_2, δL_2 and δA_2 will be exact to small values of third order inclusively. For differentiation by latitude W_m will be considered constant since change in W_m by latitude for usual sides of 1st order triangulation shows only at sixth decimal point.

From (10.21) with shown reservations we have:

$$
\begin{array}{l|l|l}
\frac{\partial B}{\partial B_1} = 0, & \frac{\partial L}{\partial B_1} = \frac{l}{2}\,\mathrm{tg}\,B_m, & \frac{\partial A}{\partial B_1} = \frac{t}{\sin 2B_m}, \\
\frac{\partial B}{\partial s} = \frac{b}{s}, & \frac{\partial L}{\partial s} = \frac{l}{s}, & \frac{\partial A}{\partial s} = \frac{t}{s}, \\
\frac{\partial B}{\partial A_1} = -\frac{b\,\mathrm{tg}\,A_m}{2}, & \frac{\partial L}{\partial A_1} = \frac{l}{2}\,\mathrm{ctg}\,A_m', & \frac{\partial A}{\partial A_1} = \frac{t}{2}\,\mathrm{ctg}\,A_m, \\
\frac{\partial B}{\partial a} = -\frac{b}{a}, & \frac{\partial L}{\partial a} = -\frac{l}{a}, & \frac{\partial A}{\partial a} = -\frac{t}{a}, \\
\frac{\partial B}{\partial \varepsilon} = b(2 - 3\sin^2 B_m'). & \frac{\partial L}{\partial \varepsilon} = -l\sin^2 B_m' & \frac{\partial A}{\partial \varepsilon} = -t\sin^2 B_m'.
\end{array}
$$

Consequently,

$$
\left.
\begin{array}{l}
dB_2 = dB_1 + b\,\frac{ds}{s} - b\,\frac{\mathrm{tg}\,A_m\,dA_1}{2\rho''} - b\,\frac{da}{a} + b(2 - 3\sin^2 B_m)\,d\varepsilon, \\
dL_2 = dL_1 + l\,\frac{ds}{s} + \frac{l\,\mathrm{tg}\,A_m}{2\rho''}\,dB_1 + \frac{l\,\mathrm{ctg}\,A_m}{2\rho''}\,dA_1 - l\,\frac{da}{a} - \\
\quad - l\sin^2 B_m'\,d\varepsilon, \\
dA_2 = dA_1 + t\,\frac{ds}{s} + \frac{t}{\rho''\sin 2B_m}\,dB_1 + t\,\frac{\mathrm{ctg}\,A_m}{2\rho''}\,dA_1 - t\,\frac{da}{a} - \\
\quad - t\sin^2 B_m\,d\varepsilon.
\end{array}
\right\} \qquad (10.22)
$$

Obtained formulae are suitable for distances on an order of the length of a side of 1st order triangulation. They are convenient for calculation by computers with retention of five decimal places. Actual corrections have to be rounded to $0.''001$, since these formulas do not give great accuracy, inasmuch as coefficients for dB_1, dA_1, ... $d\alpha$ are erroneous for values e^2b, e^2l, e^2t. These formulas are fully suitable for any calculations for topographic and cartographic purposes.

During recomputation of coordinates from one ellipsoid to another, if this recomputation is made for a system of interconnected points, it is necessary to use differential formulas of first and second type simultaneously. Consequently, formulae (10.22) fully resolves this problem, if higher accuracy of recomputation is not required than that, which can be obtained from formulas (10.22).

§ 59. DIFFERENTIAL FORMULAS OF THIRD TYPE (FOR GAUSS-KRUGER COORDINATES)

Let us assume that simultaneously geodetic coordinates B and l, major semiaxis a and compression α of adopted reference-ellipsoid changed their values; it is required to find changes in Gauss-Kruger coordinates, i.e., changes in x and y.

For Gauss-Kruger coordinates take formulas (8.12)

$$
\begin{aligned}
x &= X + a_2 l^2 + a_4 l^4 + \dots \\
y &= b_1 l + b_3 l^3 + \dots
\end{aligned}
$$

We have:

$$\left.\begin{aligned} dx &= dX + \left(l^2 \frac{\partial a_2}{\partial B} + l^4 \frac{\partial a_4}{\partial B}\right) dB + l^2 \left(\frac{\partial a_2}{\partial a} da + \frac{\partial a_2}{\partial \alpha} d\alpha\right) + \\ &+ (2a_2 l + 4a_4 l^3)\, dl \\ dy &= \left(l \frac{\partial b_1}{\partial B} + l^3 \frac{\partial b_3}{\partial B}\right) dB + l \left(\frac{\partial b_1}{\partial a} da + \frac{\partial b_1}{\partial \alpha} d\alpha\right) + (b_1 + \\ &+ 3b_3 l^2)\, dl \end{aligned}\right\}. \qquad (10.23)$$

where:

$$dX = \frac{\partial X}{\partial B} dB + \frac{\partial X}{\partial a} da + \frac{\partial X}{\partial \alpha} d\alpha.$$

We designate:

$$\left.\begin{aligned} \delta X &= \frac{\partial X}{\partial B} dB \\ \delta x &= \frac{\partial X}{\partial a} da + \frac{\partial X}{\partial \alpha} d\alpha \end{aligned}\right\}. \qquad (10.24)$$

then:

$$dX = \delta X + \delta x. \qquad (10.25)$$

Partial derivatives, entered in (10.23), have the following values:

$$\begin{array}{l|l} \dfrac{\partial a_2}{\partial B} = \dfrac{N \cos 2B}{2}, & \dfrac{\partial b_1}{\partial B} = -M \sin B, \\ \dfrac{\partial a_4}{\partial B} = -\dfrac{N \cos^4 B}{24}(5 - 18t^2 + t^4); & \dfrac{\partial b_3}{\partial B} = -\dfrac{N \cos^3 B \sin B}{6}(5 - t^2 + 5\eta_1^2). \\ \dfrac{\partial a_2}{\partial a} = \dfrac{a_2}{a}, & \dfrac{\partial b_1}{\partial a} = \dfrac{b_1}{a}, \\ \dfrac{\partial a_2}{\partial \alpha} = \dfrac{a_2}{W^2} \sin^2 B, & \dfrac{\partial b_1}{\partial \alpha} = \dfrac{b_1 \sin^2 B}{W^2}. \end{array}$$

Substituting values of partial derivatives in (10.23), we obtain:

$$\left.\begin{aligned} dx &= \delta X + \delta x + \left[\frac{l^2 N \cos 2B}{2} - \frac{l^4 N \cos^4 B}{24}(5 - 18t^2 + t^4)\right] dB + \\ &\quad + a_2 l^2 \left(\frac{da}{a} + \sin^2 B\, d\alpha\right) + (2a_2 l + 4a_4 l^3)\, dl \\ dy &= -\left[lM \sin B + \frac{Nl^3}{6} \sin B \cos^2 B (5 - t^2 + 5\eta_1^2)\right] dB + \\ &\quad + b_1 l \left(\frac{da}{a} + \sin^2 B\, d\alpha\right) + (b_1 + 3b_3 l^2)\, dl \end{aligned}\right\} \qquad (10.26)$$

In practical application of formulas (10.26) it should be borne in mind that δX is change X, during change of latitude B to dB (taken from tables of arcs of meridians); δx is change X due to change a to da and α to dα (found in comparison of tabular arcs of meridians for the same latitude of both reference-ellipsoids).

During calculation of partial derivatives following simplifications are made: changes b_3 and a_4, evoked by changes of a and α, are so small that the following is adopted:

$$\frac{\partial a_4}{\partial a} = \frac{\partial b_3}{\partial a} = \frac{\partial a_4}{\partial \alpha} = \frac{\partial b_3}{\partial \alpha} = 0;$$

derivative $\frac{\partial a_4}{\partial B}$ is taken in "spherical presentation," i.e., where $\eta_1^2 = \text{const}$.

In application of these formulas, tables should be composed for values, depending on latitudes. Such tables are sufficient for a degree of latitude with 4–5 decimal places. Values l, l^2, l^3, l^4, dB and dL must be expressed in radians.

Formulas (10.26) have accuracy to small values of third order inclusively and can be used during precise geodetic computations.

Formulas (10.26) consist of two parts: first part expresses change of rectangular coordinates for changes in geodetic coordinates, and second part for change of major semiaxis and compression of adopted reference-ellipsoid. When necessary these formulas can be easily broken down into two independent parts. One will take into consideration the influence of a change of geodetic coordinates, and the other, change in dimensions and compression of reference-ellipsoid.

§ 60. DIFFERENTIAL FORMULAS FOR CALCULATION OF JOINT INFLUENCE OF VARIATION OF PARAMETERS AND ORIENTATION OF REFERENCE-ELLIPSOID

In certain technical questions necessity arises for resolution of geodetic problems between points on earth's surface, when their coordinates refer to different and differently oriented bodies of the Earth's reference-ellipsoids. In this case geodetic coordinates of points contain additional errors, evoked by difference of major semiaxis, compression and orientation of reference-ellipsoids.

The problem of determination of shown influences is analogous to that, which appears during substitution of adopted reference-ellipsoid with simultaneous change of geodetic coordinates of initial point of triangulation of a given country. In each case it is necessary to determine the influence of variation of parameters and orientation of ellipsoid to coordinates of points of geodetic construction. Below mentioned derivation is done according to the method of Professor A. A. Izotov.[1]

Let us assume that B and L are geodetic coordinates of a point of state triangulation on an ellipsoid with parameters a and α; h is height of geoid above reference-ellipsoid at this point; (x, y, z) are space rectangular coordinates of a point with origin of coordinates in center of ellipsoid (a, α). We designate variation of parameters of reference-ellipsoid and geodetic coordinates correspondingly: δa, $\delta\alpha$, δB, δL, δh. The connection between geodetic and

[1] A. A. Izotov. "Shape and dimensions of the Earth by contemporary data." M., Geodezizdat, 1950, p. 66-67.

rectangular space coordinates is shown by formulas (2.16), which, taking into account the value of h it is expedient to record in the form of:

$$\left.\begin{aligned} x &= N\cos B\cos L + h\cos B\cos L \\ y &= N\cos B\sin L + h\cos B\sin L \\ z &= N(1-e^2)\sin B + h\sin B \end{aligned}\right\}. \qquad (10.27)$$

N — radius of curvature of first vertical,

h — height of a given point above reference-ellipsoid.

If the parameters of ellipsoid were to change to values δa and $\delta\alpha$, and the geodetic coordinates to δB, δL and δh, then corresponding changes of rectangular coordinates, as functions B, L, h, a and α, it can be calculated by the formulas that:

$$\left.\begin{aligned} \delta x &= \frac{\partial x}{\partial B}\delta B + \frac{\partial x}{\partial L}\delta L + \frac{\partial x}{\partial h}\delta h + \frac{\partial x}{\partial a}\delta a + \frac{\partial x}{\partial \alpha}\delta\alpha \\ \delta y &= \frac{\partial y}{\partial B}\delta B + \frac{\partial y}{\partial L}\delta L + \frac{\partial y}{\partial h}\delta h + \frac{\partial y}{\partial a}\delta a + \frac{\partial y}{\partial \alpha}\delta\alpha \\ \delta z &= \frac{\partial z}{\partial B}\delta B + \frac{\partial z}{\partial L}\delta L + \frac{\partial z}{\partial h}\delta h + \frac{\partial z}{\partial a}\delta a + \frac{\partial z}{\partial \alpha}\delta\alpha \end{aligned}\right\}. \qquad (10.28)$$

From (10.27) after differentiation:

$$\begin{aligned} &\frac{\partial x}{\partial B} = -(M+h)\sin B\cos L, \quad \frac{\partial x}{\partial L} = -(N+h)\cos B\sin L, \\ &\frac{\partial y}{\partial B} = -(M+h)\sin B\sin L, \quad \frac{\partial y}{\partial L} = (N+h)\cos B\cos L, \\ &\frac{\partial z}{\partial B} = (M+h)\cos B, \qquad \frac{\partial z}{\partial L} = 0, \\ &\frac{\partial x}{\partial h} = \cos B\cos L, \qquad \frac{\partial x}{\partial a} = \frac{N}{a}\cos B\cos L, \quad \frac{\partial x}{\partial \alpha} = M\cos B\cos L\sin^2 B, \\ &\frac{\partial y}{\partial h} = \cos B\sin L, \qquad \frac{\partial y}{\partial a} = \frac{N}{a}\cos B\sin L, \quad \frac{\partial y}{\partial \alpha} = M\cos B\sin L\sin^2 B, \\ &\frac{\partial z}{\partial h} = \cos B, \qquad \frac{\partial z}{\partial a} = \frac{N}{a}(1-e^2)\sin B, \quad \frac{\partial z}{\partial \alpha} = M\sin^2 B - 2N\sin B. \end{aligned}$$

Substituting values of partial derivatives in (10.28), we find:

$$\left.\begin{aligned} \delta x &= -(M+h)\sin B\cos L\,\delta B - (N+h)\cos B\sin L\,\delta L + \\ &+ \cos B\cos L\,\delta h + N\cos B\cos L\,\delta\bar{a} + M\cos B\cos L\sin^2 B\,\delta\alpha \\ \delta y &= -(M+h)\sin B\sin L\,\delta B + (N+h)\cos B\cos L\,\delta L + \\ &+ \cos B\sin L\,\delta h + N\cos B\sin L\,\delta\bar{a} + M\cos B\sin L\sin^2 B\,\delta\alpha \\ \delta z &= (M+h)\cos B\,\delta B + \sin B\,\delta h + N(1-e^2)\sin B\,\delta\bar{a} - \\ &- M(1+\cos^2 B - e^2\sin^2 B)\sin B\,\delta\alpha. \end{aligned}\right\}. \qquad (10.29)$$

Here:

$$\delta\bar{a} = \frac{\delta a}{a}.$$

Resolving these equations relatively δB, δL and δh and considering that the influence h on geodetic coordinates is negligibly small, we obtain:

$$
\left.
\begin{aligned}
&M\,\delta B = -\sin B\cos L\,\delta x - \sin B\sin L\,\delta y + \cos B\,\delta z + \\
&+ Ne^2\sin B\cos B\,\delta\bar{\alpha} + M\sin 2B\,\delta\alpha \\
&r\,\delta L = -\sin L\,\delta x + \cos L\,\delta y \\
&\delta h = \cos B\cos L\,\delta x + \cos B\sin L\,\delta y + \sin B\,\delta z - \\
&- N(1 - e^2\sin^2 B)\,\delta\bar{\alpha} + M\sin^2 B\,\delta\alpha
\end{aligned}
\right\}. \qquad (10.30)
$$

Here r is a radius of parallel at a given point. In (10.30) terms with e^2 in $\delta\alpha$ were dropped.

It is evident that for calculations by (10.30) it is necessary to know δx, δy and δz. These values are determined from the following considerations: the axis of rotation of ellipsoids (a, α) and (a + δa, α + δα) after their orientation in a body of the Earth will be parallel to the axis of the world and, consequently, between themselves, therefore corresponding to them axis of coordinates of systems (x, y, z) and (x + δx, y + δy, z + δz) will also be parallel, i.e., formulas (10.29) are justified for any point. Applicable to initial point of triangulation if h = 0, these formulas will be in the form of:

$$
\left.
\begin{aligned}
&\delta x = \delta x_0 = -M_0\sin B_0\cos L_0\,\delta B_0 - N_0\cos B_0\sin L_0\,\delta L_0 + \\
&+ \cos B_0\cos L_0\,\delta h_0 + N_0\cos B_0\cos L_0\,\delta\bar{\alpha} + M_0\cos B_0\cos L_0\sin^2 B_0\,\delta\alpha \\
&\delta y = \delta y_0 = -M_0\sin B_0\sin L_0\,\delta B_0 + N_0\cos B_0\cos L_0\,\delta L_0 + \\
&+ \cos B_0\sin L_0\,\delta h_0 + N_0\cos B_0\sin L_0\,\delta\bar{\alpha} + \\
&+ M_0\cos B_0\sin L_0\sin^2 B_0\,\delta\alpha \\
&\delta z = \delta z_0 = M_0\cos B_0\,\delta B_0 + \sin B_0\,\delta h_0 + N_0(1 - e^2) \times \\
&\times \delta\bar{\alpha} + M_0(1 + \cos^2 B_0 - e^2\sin^2 B_0)\,\delta\alpha
\end{aligned}
\right\}. \qquad (10.31)
$$

Values with sign "0" pertain to the initial point, but with δB_0, δL_0 and δh_0 it is necessary to understand the difference in orientation of two ellipsoids. In formulas (10.30) with δx, δy and δz it must be implied that it is δx_0, δy_0 and δz_0.

Formulas (10.30) and (10.31) jointly resolved an important geodetic problem: with them it is possible to compute the correction to geodetic coordinates variation of parameters of adopted reference-ellipsoid and its orientation to the surface of the Earth. Values δB_0, δL_0 and δh_0 can be considered as an error of orientation of the second ellipsoid in relation to the first, in latitude, longitude and height. Values δB_0, δL_0 and δh_0 are only known in a case, where ties between different geodetic systems of coordinates exist. In the absence of these ties, formulas (10.30) and (10.31) can be used for approximate calculations and precomputations of accuracy in resolution of inverse geodetic problems.

Approximate calculations must be made in determination of expected accuracy of distances and azimuths, obtained from resolution of inverse geodetic problem, if points, between which the problem is being resolved belong to different geodetic systems of coordinates. Contemporary astronomic geodetic nets of various countries and continents in most cases do not have geodetic ties among themselves. Therefore in determination of limiting values bB_0, bL_0, bh_0 it is necessary to follow derivations, obtained by F. N. Krasovskiy on the basis of investigation of general deviations of ellipsoid from geoid.

"General deviations of geoid from ellipsoid are accompanied by general deviations of plumb lines. The greater value of such general deviations of plumb lines probably does not exceed 8"."[1]

Thus, in the absence of geodetic ties the problem of determination of bB_0, bL_0, bh_0 remains on the whole, unsolved. However by values of general deviations of the geoid from ellipsoid it is possible to precompute the expected accuracy of unknown values, obtained from resolution of inverse geodetic problem. This problem was studied in detail by the author, and obtained results are published in an article: "On accuracy of distances and azimuths, obtained from solution of inverse geodetic problem."[2]

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[1] F. N. Krasovskiy. Instruction on higher geodesy. Ch. N. M., Geodezizdat, 1942.

[2] "Geodesy and Aerial Photography" No. 3, 1959, p. 79-83.

CONCLUSION

Bases of spheroidal geodesy were embodied in the first half of 19th century and were developed during 19th and 20th centuries in works of the greatest geodesists – Gauss, Bessel, Struve, Helmert, Jordan, Krasovskiy and others. Mathematical apparatus of spheroidal geodesy was developed with the development of theory of surfaces, differential geometry, variable calculus and in general, with progress in the area of differential and integral calculus. Carrying out close contact between geodesy and mathematics, scientists reached brilliant successes in solution of spheroidal geodesy problems. It is justly considered that spheroidal geodesy is one of the most scientifically worked out divisions of higher geodesy. Theoretical and practical resolution of many of its problems by classical methods of mathematics is carried out to perfection.

However with development of physics, technology and mathematics new calculus appeared, possessing great potentialialities for scientific generalization and geometric clarity. There is in prospect a vector and tensor calculus. A new apparatus is presently widely used in many areas of science and technology, reducing to simplicity and clarity of presentation, complex problems and simultaneously creating a possibility for profound scientific generalizations and deductions. New calculus frees us from artificial constructions, unavoidable in application of systems of coordinates; geometrical solids and physical phenomena in vectors and tensors are studied in their natural state.

In spheroidal geodesy the new calculus is also forging a path for itself, but so far it is not widely used. There are a series of investigations and individual attempts of expounding spheroidal geodesy with the aid of the new mathematical apparatus. These first steps clearly show that future mathematical apparatus of spheroidal geodesy is vector and tensor calculus. Nevertheless, at present, the investigation in this area still did not lead to final results, which could be utilized for educational purposes. Time is required and further deep investigations before the new mathematical apparatus will fully show its advantages in resolution of problems of spheroidal geodesy over the methods of classical mathematics.

This is why in this book apparatus of vector and tensor calculus is not used for expounding bases of spheroidal geodesy. One of the scientific problems in area of spheroidal geodesy consists in application of this apparatus. Both in USSR, and abroad scientific work in this direction is conducted more or less intensely.

From the above it does not follow that classical apparatus of spheroidal geodesy is not in a position to resolve arising new problems. However it does not possess that depth of penetration, which is peculiar to the new apparatus.

Characteristic peculiarity of methods of spheroidal geodesy consists in that they are calculated mainly for treatment of material of 1st order triangulation. Besides the length of side of triangulation, along with square of eccentricity of a spheroid, are considered very small values of first order in comparison to mean radius of Earth. However contemporary radar technical means allow creation of geodetic nets with sides 500-600 km, and in prospect up to 800-1000 km. Thus, a picture looms of world geodetic nets with long sides and realization of geodetic ties between nets of individual countries and continents.

In connection with such prospective development of geodetic work, the first problem advanced is of great distances not as a particular problem, but as a basis, on which the theory of spheroidal geodesy is based. In Chapter VI basic methods are presented for the resolution of geodetic problems for great distances, but they do not exhaust the problem on the whole. Thorough investigations concerning this problem are being conducted. Resolution of geodetic problems for long distances on the surface of the ellipsoid is one of the fundamental scientific problems of spheroidal geodesy. "Surmounting long distances" in connection with the development of rocket technology and artificial cosmic bodies occurs with extraordinary speed in our time. The research in spheroidal geodesy is confronted with complex problems, whose resolution will require new powerful mathematical and geodetic means.

All geodetic measurements up till now have been done on the surface of the Earth, therefore for mathematical treatment of their results various surface systems of curvilinear coordinates are taken. Rocket technology, artificial cosmic bodies and radar technical means create absolutely new conditions for geodetic measurements, since they can also be produced in space. Obviously for treatment of material for such measurements systems of surface coordinates are inexpedient. Here it will be profitable to apply space systems of coordinates with origin in the center of a spheroid or at a given point on the surface. Development and application of such systems of coordinates is the future problem of spheroidal geodesy. Further, there arises a reduction problem, as a result of measurements, carried out at the surface of the Earth, to be projected on the surface of the reference-ellipsoid.

It was not difficult to see from Chapter III that a connection of points on the surface of the ellipsoid by geodetics and formation of figures from these lines lead to the fact that the difference of latitudes, longitudes and azimuths is not expressed in a closed form by elementary functions, but presents elliptic integrals, whose practical application evokes great difficulties. Due to this it is necessary to replace them by infinite series, the general term of which, as a rule, remains unknown, investigation of convergence however is a difficult problem. One of the problems of spheroidal geodesy is that, in order to investigate the question such as in what problems it is expedient to apply geodesics, and in what problems to teach about normal sections and chords of ellipsoid. Although this problem is not new, it is not completely solved under contemporary conditions.

Calculating work in spheroidal geodesy occupies a significant place. Contemporary computer technology is being developed at a rapid rate. Due to this formula and methods which actual calculations are made, must be basically changed. At present a number of problems in spheroidal geodesy are resolved on high speed computers. For machines the meaning is not complexity of formulas or quantity of arithmetical operations, which are characteristic for logarithmic calculation, but convenience of programming. Special investigations are required, in order to establish type of machines and accuracy of calculations; simultaneously, construction of formulas convenient for programming is required.

All geodetic projections are developed applicable to treatment of geodetic nets with short sides, layed out in comparatively small areas. The problem of long distances poses a problem in absolutely new fashion regarding selection of a surface of projection. Projecting on a plane of any projection at long distances will lead to prohibitive distortions and large deformations of geodetic construction. In a given case it might be expedient to utilize the properties of aposphere, where each point of the Gauss curvature coincides with similar curvature of the spheroid. Simultaneously with this it is necessary to investigate possible increase of width of zones in a system of Gauss-Kruger coordinates by means of introduction of supplemental condition for selection of characteristic functions. From mathematical side these problems are complex and require deep and manifold investigations for their solution.

Among the scientific problems of spheroidal geodesy not the least important place is occupied by questions of proper system of designations and special terminology. In spheroidal geodesy mathematical symbolism is mainly used, but up to present time this symbolism, stranded in the initial stage of its development, is too bulky. However the problem of creation of special symbolism for spheroidal geodesy must be resolved parallel with the development of the most scientific discipline in the course of resolution of theoretical and practical problems of higher geodesy.

Noted above are only the major scientific problems of contemporary spheroidal geodesy. With the development of geodetic work and new geodetic technology, also the requirements of adjacent disciplines, appear more and more new problems in the area of spheroidal geodesy. The worst error is that affirmation, in conformance with which it is believed that the problems of spheroidal geodesy were solved by the greatest mathematicians of the past so thoroughly that for the share of our generation only the current problems of daily practical activity remain.

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