MEMORANDUM
RM-3862-1-ARPA
OCTOBER 1964

A NEW APPROACH TO TRANSPORT THEORY

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(SECOND PRINTING)

This research is supported and monitored by the Advanced Research Projects Agency under Contract No. SD-79. Any views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of ARPA.

The RAND Corporation
Santa Monica, California

PREFACE

Many studies have been made concerning the use of invariant-imbedding principles to obtain integro-differential equations describing transport processes. In this Memorandum the author formulates particle-transport processes in terms of an abstract mathematical system. This study evolved from research in neutron-transport theory sponsored by the Advanced Research Projects Agency.

This version was rewritten in order to achieve greater clarity, and it differs from the original version by a change in notation and emphasis.

SUMMARY

A complete mathematical definition of an abstract linear transport process is given in terms of a new axiomatic system. After several preliminary theorems are proven, the basic algebraic equations relating the "transport operators" are derived, and the semigroup properties of these equations are indicated. These algebraic equations are then used to derive the standard differential equations describing a transport process.

The generalized invariant-imbedding equations are used to describe an energy-dependent, neutron-transport process.

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1. INTRODUCTION

The methods of invariant imbedding [1] have been utilized by Bellman, Kalaba, and Wing [1,2,3] to analyze one-dimensional neutron-transport processes arising from the chain reactions of nuclear-fission processes. The invariant imbedding technique allows one to replace the classical equations of transfer, which are constrained by two-point boundary conditions, with a system of differential equations with initial-value constraints. The relation between the invariant-imbedding approach to neutron transport theory and the circuit-theory formalism of transmission lines [4] was discovered by Redheffer [4,5,6], who investigated the algebraic structure of transfer processes. (Extensive references on neutron transport and transmission lines can be found in [3] and [5].) In this Memorandum, we propose a mathematical formulation of one-dimensional transport processes which yields the algebraic description of the process in terms of Redheffer's "scattering matrix" [5] as well as the differential equations arising from Bellman's invariant-imbedding approach to neutron transport. A result of our mathematical description is a demonstration of the nonsingularity of the "transmission operators" (See c. 2) which is independent of any assumption of differentialability of the process. We also apply our results to obtain integro-differential equations which completely describe one-dimensional energy-dependent neutron transport processes.

We shall let the one-dimensional neutron transport process serve as the motivating example for our mathematical system. The one-dimensional transport process consists of a finite real line segment, referred to as the <u>rod</u>, which represents the interacting medium through which the neutrons travel (see Fig. 1). The neutrons, considered as abstract point-particles, travel through the rod, moving either to the right or to the left, and interact with the medium, occasionally producing additional neutrons by nuclear fissions. For each point y of the rod, we let u(y) and v(y) represent the expected flux of neutrons passing through y to the right and to the left, respectively (see Fig. 1).

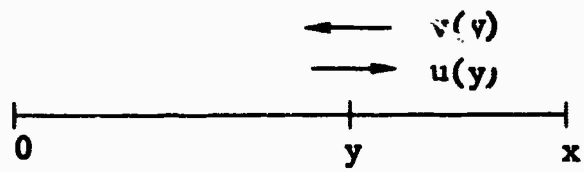

Fig. 1 – Rod Extending from 0 to x

To keep our discussion general, we shall assume that the fluxes u(y) and v(y) are elements of a Banach space. The transport process can then be described in terms of linear differential equations of the form

$$\text{(1.1)} \qquad \frac{d}{dy} u(y) = b_1(y)u(y) + a(y)v(y) ,$$

$$\frac{d}{dy} v(y) = -c(y)u(y) - b_2(y)v(y) ,$$

where $u(y)$ and $v(y)$ are the fluxes to the right and left at the point y, and $a(y), b_1(y), b_2(y)$, and $c(y)$ are linear operators, defined for each point y in the rod $[0,x]$.

The formulation of transport processes in terms of equations (1.1) has a major drawback. To study practical problems in which incident neutron beams initiate a transport process in the rod $[0,x]$, we must solve equation (1.1) subject to the boundary conditions

$$u(0) = u_{in}, \quad v(x) = v_{in} ,$$

where u_{in} and v_{in} are the incident fluxes to the right at 0 and to the left at x, respectively. (In many applications we pick u_{in} or v_{in} to be 0.) Thus we must solve equations of the form (1.1)—which may be 2n simultaneous ordinary differential equations [3] or integro-differential equations [2]—with <u>two-point boundary-value conditions</u>. This formulation is often unsatisfactory when numerical answers are desired, because of the difficulty of numerically solving two-point boundary-value problems of high dimensionality.

The technique of invariant imbedding has been successfully used by Bellman, Kalaba, and Wing [1,2] to reduce the two-point boundary-value equations of neutron-transport processes to initial-value problems. We now briefly describe the invariant-imbedding approach. Consider a one-dimensional transport process on the rod [0, L] which is described by the linear operators $a(y)$, $b_1(y)$, $b_2(y)$, $c(y)$ (for $y \in [0,L]$) of equation (1.1). For each x in the interval $[0,L]$, we consider the transport process taking place on the shorter rod $[0, x]$ which is described by the same linear operators. We define the family of operators $\rho(x)$, $r(x)$, $\tau(x)$, and $t(x)$ as follows:

$\rho(x)\ \psi$ = the "reflected" flux leaving the rod $[0, x]$ to the right due to a flux ψ incident upon the rod to the left,

$\tau(x)\ \psi$ = the "transmitted" flux leaving the rod $[0,x]$ to the left due to a flux ψ incident upon the rod to the left,

$r(x)\ \varphi$ = the "reflected" flux leaving the rod $[0,x]$ to the left due to a flux φ incident upon the rod to the right,

$t(x)\ \varphi$ = the "transmitted" flux leaving the rod $[0,x]$ to the right due to a flux φ incident upon the rod to the right.

Thus in the particle-transport process on $[0,x]$, if the incident fluxes are given by

$$u(0) = \varphi, \quad v(x) = \psi,$$

then the "output" fluxes (see Fig. 2) are given by

$$u(x) = t(x)\,\varphi + \rho(x)\,\psi,$$
$$v(0) = r(x)\,\varphi + \tau(x)\,\psi.$$

Fig. 2

By considering the rod $[0, x + \Delta]$, and using the "particle counting" technique of Bellman _et al._ (see [1, 2, 3]), we arrive at the operator differential equations

$$(1.2)\qquad \frac{d}{dx}\,\rho(x) = a(x) + b_1(x)\rho(x) + \rho(x)b_2(x) + \rho(x)c(x)\rho(x),$$

$$\frac{d}{dx}\,\tau(x) = \tau(x)\ [b_2(x) + c(x)\rho(x)],$$

$$\frac{d}{dx}\,r(x) = \tau(x)c(x)t(x),$$

$$\frac{d}{dx}\,t(x) = [b_1(x) + \rho(x)c(x)]t(x),$$

with the _initial conditions_

$$(1.3)\qquad \rho(0) = r(0) = 0\ ,$$
$$\tau(0) = t(0) = I\ ,$$

where I is the identity operator.

The operators ρ, r, τ, and t, called the _transport operators_, play an important role in the theory.

By using the transport operators, one replaces the two-point boundary-value problems of classical transport theory by the differential equation (1.2) with the initial conditions (1.3).

2. THE MATHEMATICAL FORMULATION

We now give an axiomatic definition of transport processes from which we shall mathematically derive our desired results. To account for the "infinite-dimensional case," we shall consider the transport operators as linear operators on a Banach space. We shall closely follow the notation used by Redheffer [5,6].

Let $\mathcal{X}$ be a Banach space, over a real or a complex field. We shall refer to elements of $\mathcal{X}$ as fluxes in order to recall the neutron transport process discussed in the introduction. (When dealing with transmission lines, elements of $\mathcal{X}$ would represent electrical signals.) We shall also consider the Banach space $\mathcal{X}^2 = \mathcal{X} \times \mathcal{X}$, and we shall write elements of $\mathcal{X}^2$ in the form (u,v) or $\begin{bmatrix} u \\ v \end{bmatrix}$, where $u,v \in \mathcal{X}$. The norm in $\mathcal{X}^2$ is given by

$$||(u,v)|| = ||u|| + ||v||,$$

where $||u||$ is the given norm in $\mathcal{X}$.

<u>Definition 1</u>. Let the real interval $[x_0,x_1]$ be given. Define the space

$$D = \{(y_1,y_2) \in R^2 | x_0 \leq y_1 \leq y_2 \leq x_1\},$$

considered as a subspace of the metric space R^2.

Let S be a mapping of D into the space of bounded linear operators on $\mathcal{X}^2$. Thus for every (y_1,y_2) in D, we associate with it a bounded linear operator

$$S(y_1,y_2): \mathcal{X}^2 \to \mathcal{X}^2 .$$

We also write

$$(2.1) \qquad S(y_1,y_2)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} t(y_1,y_2)u + \rho(y_1,y_2)v \\ r(y_1,y_2)u + \tau(y_1,y_2)v \end{bmatrix}$$

$$= \begin{bmatrix} t(y_1,y_2) & \rho(y_1,y_2) \\ r(y_1,y_2) & \tau(y_1y_2) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

for all $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{X}^2$.

Since $S(y_1,y_2)$ is always assumed to be a bounded linear operator on $\mathcal{X}^2$, it follows that $t(y_1,y_2)$, $\tau(y_1,y_2)$, $r(y_1,y_2)$, and $\rho(y_1,y_2)$ are bounded linear operators on $\mathcal{X}$. We can abbreviate (2.1) by writing

$$S = \begin{bmatrix} t & \rho \\ r & \tau \end{bmatrix} .$$

We pause to describe the physical meaning that we attach to the above definition. Considering the neutron-transport model mentioned in the introduction, we regard each element (y_1,y_2) of D as the rod segment extending from y_1 to y_2 "imbedded" in the whole rod $[x_0,x_1]$. The linear operator $S(y_1,y_2)$ represents the transformation relating the "output fluxes" from the rod $[y_1,y_2]$ to the "incident fluxes." Specifically, we want to have

$$(2.2) \qquad \begin{bmatrix} u(y_2) \\ v(y_1) \end{bmatrix} = S(y_1,y_2) \begin{bmatrix} u(y_1) \\ v(y_2) \end{bmatrix} ,$$

where $u(y_i)$ and $v(y_i)$ are the fluxes to the right and to the left, respectively, at the point y_i, $i = 1,2$ (see Fig. 3). The reader should verify that $t(y_1,y_2)$, $\tau(y_1,y_2)$, $r(y_1,y_2)$, and $\rho(y_1,y_2)$ represent the transport operators (as defined in Sec. 1) for the rod $[y_1,y_2]$. The operator $S(y_1,y_2)$ is called the <u>scattering operator</u> in analogy with Redheffer's scattering matrix [5,6].

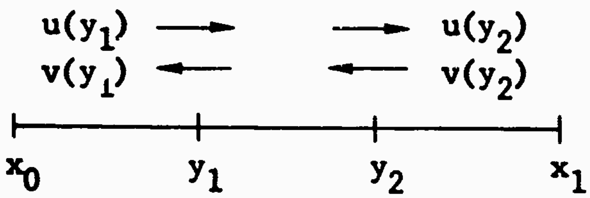

Fig. 3

Definition 2. For each $(y_1,y_2)\in D$, define the relation $G_{y_2}^{y_1} \subset \mathcal{X}^2 \times \mathcal{X}^2$ by

$$G_{y_2}^{y_1} = \Big\{(w_1,w_2) \,|\, (u_2,v_1) = S(y_1,y_2)(u_1,v_2),$$

where

$$(u_i,v_i) = w_i \in \mathcal{X}^2,\ i = 1,2\Big\} .$$

Each element of $G_{y_2}^{y_1}$ is thus an ordered pair, the first element representing the fluxes to the right and left, respectively, at the point y_1, and the second element representing the fluxes at the point y_2. We say that the relation $G_{y_2}^{y_1}$ relates the fluxes at the point y_2 to the fluxes at the point y_1, in the transport process under consideration. Note that we are not assuming that given the fluxes $u(y_1)$ and $v(y_1)$ at the point y_1, there would exist fluxes $u(y_1)$ and $v(y_2)$ satisfying equation (2.2). In the course of our mathematical analysis, we shall instead show that there is a one-one relationship between the fluxes at two different points, and equations (2.3) and (2.4) give an explicit formula for the fluxes at one point in terms of the fluxes at another point. Until we demonstrate this one-one correspondence, however, we shall have to be content with the somewhat clumsy representation in terms of the family of relations $G_{y_2}^{y_1}$.

We now review briefly some basic mathematical conventions to be used in the following discussion. The space of bounded linear operators is given the topology induced by the usual norm,

$$||T|| = \sup_{||w||=1} ||Tw||.$$

(This is called the uniform operator topology.) A linear operator on a linear space is said to be invertible when it is one-one and onto. If T is an invertible bounded linear operator on a Banach space, then the elementary theory of Banach spaces [7] tells us that the inverse T^{-1} is also bounded. The composition of two relations G_1 and G_2 is defined by

$$G_2G_1 = \{(\alpha,\beta) \mid \exists\, \gamma \text{ such that } (\alpha,\gamma)\in G_1 \text{ and } (\gamma,\beta)\in G_2\}$$

If the relations G_1 and G_2 are functions, then their composition, G_2G_1, is the usual composition of functions.

We are now ready to give a complete definition of a one-dimensional transport process.

Definition 3. Let S be described as in Definition 1. We say that S <u>describes a transport process</u> when the following three axioms are satisfied:

(i) $S(y,y) = I$ for all $y \in [x_0,x_1]$ (I is the identity operator);

(ii) S is continuous in the uniform operator topology;

(iii) $G_{y_3}^{y_2} G_{y_2}^{y_1} = G_{y_3}^{y_1}$ for $x_0 \le y_1 \le y_2 \le y_3 \le x_1$, where $G_{y_j}^{y_i}$ is given by Definition 2.

In the remaining discussion we shall always assume that S is given and describes a transport process.

Axiom (i) is equivalent to the statement that $r(y,y) = \rho(y,y) = 0$ and $t(y,y) = \tau(y,y) = I$. Axiom (iii) is really a shorthand notation for the following two statements:

(a) If $\begin{bmatrix} u_2 \\ v_1 \end{bmatrix} = S(y_1, y_2) \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} u_3 \\ v_2 \end{bmatrix} = S(y_2, y_3) \begin{bmatrix} u_2 \\ v_3 \end{bmatrix}$

then $\begin{bmatrix} u_3 \\ v_1 \end{bmatrix} = S(y_1, y_3) \begin{bmatrix} u_1 \\ v_3 \end{bmatrix}$.

(b) If $\begin{bmatrix} u_3 \\ v_1 \end{bmatrix} = S(y_1, y_3) \begin{bmatrix} u_1 \\ v_3 \end{bmatrix}$,

then $\exists \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in \mathcal{R}^2$ such that

$$\begin{bmatrix} u_2 \\ v_1 \end{bmatrix} = S(y_1, y_2) \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} \text{ and } \begin{bmatrix} u_3 \\ v_2 \end{bmatrix} = S(y_2, y_3) \begin{bmatrix} u_2 \\ v_3 \end{bmatrix} .$$

Figure 4 serves as an illustration of this cardinal axiom. Note that we do not assume that, for a given transport process in a rod $[y_1, y_3]$, there exist unique internal fluxes consistent with the process at an intermediate point y_2. We shall instead prove this uniqueness in the course of our mathematical analysis.

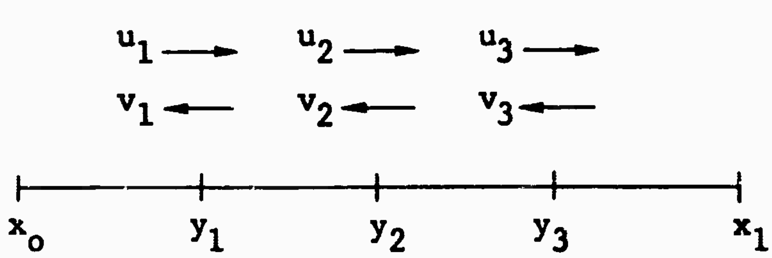

Fig. 4

<u>Lemma 1:</u> $\exists\Delta > 0$ <u>such that</u> $h \in [0,\Delta]$ <u>and</u> $(y,y+h) \in D$ <u>implies that</u> $t(y,y+h)$ <u>and</u> $\tau(y, y+h)$ <u>are invertible</u>.

<u>Proof</u>. Let $t(y_1,y_2) = I - Z(y_1,y_2)$. Since

$$\Big|\,||Z(y_1', y_2')|| - ||Z(y_1,y_2)||\,\Big| \leq ||\ t(y_1',y_2') - t(y_1,y_2)||$$

$$\leq ||S(y_1',y_2') - S(y_1,y_2)||,$$

the continuity of S (in the uniform operator topology) implies the continuity of $||Z||$. Since D is compact, we conclude that $||Z||$ is uniformly continuous on D. Therefore $\exists\,\delta_1 > 0$ such that $h \in [0,\delta_1]$ and $(y,y+h) \in D$ implies that $||Z(y,y+h)|| < 1$, and therefore that $t(y, y+h) = I - Z(y, y+h)$ is invertible (see [7], page 66). Similarly, $\exists\,\delta_2$ such that $h\in[0,\delta_2]$ and $(y, y+h) \in D$ implies that $\tau(y, y+h)$ is invertible. We then pick $\Delta = \min(\delta_1,\delta_2)$.

<u>Lemma 2</u>. <u>Let</u> $x_o \leq y_1 \leq y_2 \leq x_1$ <u>such that</u> $\tau(y_1,y_2)$ <u>and</u> $t(y_1,y_2)$ <u>are invertible</u>. <u>Then</u> $G_{y_2}^{y_1}$ <u>is a bijection</u> (<u>one-one correspondence) with range and domain</u> $\mathcal{X}^2$. <u>Writing</u> $((u_1,v_1), (u_2,v_2)) \in G_{y_2}^{y_1}$, <u>we have the formulas</u>

$$(2.3)\qquad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} t - \rho\tau^{-1} r & \rho\tau^{-1} \\ -\tau^{-1} r & \tau^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

and

$$(2.4)\qquad \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} t^{-1} & -t^{-1}\rho \\ rt^{-1} & \tau - rt^{-1}\rho \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

where $r = r(y_1,y_2)$, $t = t(y_1,y_2)$, <u>etc</u>.

The proof follows immediately from Definition 1.

Theorem 1. $G_{y_2}^{y_1}$ is a bijection with range and domain $\mathcal{X}^2$ for all $(y_1, y_2) \in D$.

Proof. Let (y_1, y_2) be given. Partition the interval $[y_1, y_2]$ with a set $\{\eta_i\}$, where $y_1 = \eta_o < \eta_1 < \dots \eta_n = y_2$, so that $\eta_i - \eta_{i-1} < \Delta$ for $1 \leq i \leq n$, with Δ picked as in Lemma 1. Thus $\tau(\eta_{i-1}, \eta_i)$ and $t(\eta_{i-1}, \eta_i)$ are invertible for $1 \leq i \leq n$. By Lemma 2, $G_{\eta_i}^{\eta_{i-1}}$ is a bijection for $1 \leq i \leq n$. Since $G_{\eta_i}^{\eta_o} = G_{\eta_i}^{\eta_{i-1}} G_{\eta_{i-1}}^{\eta_o}$, one establishes that $G_{\eta_i}^{\eta_o}$ is a bijection for all i, and in particular $G_{y_2}^{y_1} = G_{\eta_n}^{\eta_o}$ is a bijection.

Theorem 1 states that if we are given the fluxes in both directions at a point y in a rod in which a transport process is taking place, then we can uniquely determine the fluxes at any other point in the rod. We now state the fact that in any transport process with given incident fluxes, there exist unique fluxes at each interior point of the rod.

Corollary. Let $\begin{bmatrix} u_2 \\ v_1 \end{bmatrix} = S(y_1, y_2) \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}$; then for each $y \in [y_1, y_2]$ there exist unique vectors $u(y) \in \mathcal{X}$ and $v(y) \in \mathcal{X}$ such that

$$\begin{bmatrix} u(y) \\ v_1 \end{bmatrix} = S(y_1, y) \begin{bmatrix} u_1 \\ v(y) \end{bmatrix}$$

and

$$\begin{bmatrix} u_2 \\ v(y) \end{bmatrix} = S(y, y_2) \begin{bmatrix} u(y) \\ v_2 \end{bmatrix} .$$

<u>Proof</u>. Existence follows from axiom (iii), and uniqueness follows from Theorem 1.

<u>Theorem 2</u>. $\tau(y_1,y_2)$ <u>and</u> $t(y_1,y_2)$ <u>are invertible for all</u> $(y_1,y_2) \in \mathcal{S}$.

<u>Proof</u>. Let (y_1,y_2) be given, and pick an arbitrary $\varphi \in \mathcal{X}$. Since $(0, \varphi) \in \mathcal{X}^2$, by Theorem 1 there exists a unique (u, v) such that $((0, \varphi), (u, v)) \in G_{y_2}^{y_1}$. But this means that $\varphi = \tau(y_1,y_2)\, v$, which establishes that $\tau(y_1,y_2)$ is invertible.

<u>Corollary</u>. <u>The equations (2.3) and (2.4) are valid for all</u> $(y_1, y_2) \in D$.

3. THE ALGEBRAIC FORMALISM

The family of operators

$$\hat{S}(y_1,y_2) = \begin{bmatrix} t^{-1} & -t^{-1}\rho \\ rt^{-1} & \tau - rt^{-1}\rho \end{bmatrix}, \tag{3.1}$$

where $r = r(y_1,y_2)$, etc., was defined by Redheffer [5] in order to linearize the relationship between the scattering operators. The significance of $\hat{S}$ is obtained from equation (2.4), which becomes

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \hat{S}(y_1,y_2) \begin{bmatrix} u_2 \\ v_2 \end{bmatrix},$$

where u_i and $v_i (i = 1,2)$ satisfy the equation

$$\begin{bmatrix} u_2 \\ v_1 \end{bmatrix} = S(y_1,y_2) \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}.$$

Furthermore, it is clear from (2-3) that $\hat{S}(y_1,y_2)$ is invertible (for any $(y_1,y_2) \in D$), and its inverse is given by

$$\hat{S}^{-1} = \begin{bmatrix} t - \rho\tau^{-1} & \rho\tau^{-1} \\ -\tau^{-1}r & \tau^{-1} \end{bmatrix} . \tag{3.2}$$

Additionally, we can consider the "hat" operator abstractly by writing

$$\widehat{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D-CA^{-1}B \end{bmatrix},$$

which is defined whenever A is invertible. It is then easy to verify that

$$\hat{\hat{S}} = S.$$

Defining $\bar{S} = \hat{S}^{-1}$, we similarly have

$$\bar{\bar{S}} = S.$$

We can now express axiom (iii) in terms of the composition of linear operators. From equation (2.4), we obtain the functional equations

$$\hat{S}(y_1,y_3) = \hat{S}(y_1,y_2)\, \hat{S}(y_2,y_3) \; , \tag{3.3}$$

$$\bar{S}(y_1,y_3) = \bar{S}(y_2,y_3)\, \bar{S}(y_1,y_2) \; ,$$

$$\text{for } x_0 \le y_1 \le y_2 \le y_3 \le x_1 \; .$$

<u>Theorem 3</u>. <u>Let</u> $x_0 \leq y_1 \leq y_2 \leq y_3 \leq x_1$, <u>and write</u>

$$S_\alpha = S(y_1,y_2),\ S_\beta = S(y_2,y_3),\ \text{and}\ S_{\alpha\beta} = S(y_1,y_3)\ .$$

<u>Then</u> $I - r_\beta \rho_\alpha$ <u>and</u> $I - \rho_\alpha r_\beta$ <u>are invertible, and</u>

$$\begin{aligned}
(3.4) \qquad \rho_{\alpha\beta} &= \rho_\beta + t_\beta \rho_\alpha (I - r_\beta \rho_\alpha)^{-1} \tau_\beta\ ,\\
\tau_{\alpha\beta} &= \tau_\alpha (I - r_\beta \rho_\alpha)^{-1} \tau_\beta\ ,\\
r_{\alpha\beta} &= r_\alpha + \tau_\alpha r_\beta (I - \rho_\alpha r_\beta)^{-1} t_\alpha,\\
t_{\alpha\beta} &= t_\beta (I - \rho_\alpha r_\beta)^{-1} t_\alpha\ .
\end{aligned}$$

<u>Proof</u>. Equations (3.3) yield the identities

$$t_{\alpha\beta}^{-1} = t_\alpha^{-1}(I-\rho_\alpha r_\beta)\, t_\beta^{-1}$$

and

$$\tau_{\alpha\beta}^{-1} = \tau_\beta^{-1}(I-r_\beta \rho_\alpha)\,\tau_\alpha^{-1}\ ,$$

which demonstrate the existence of $(I - r_\beta \rho_\alpha)^{-1}$ and $(I - \rho_\alpha r_\beta)^{-1}$ and yield the equations of $\tau_{\alpha\beta}$ and $t_{\alpha\beta}$. The equations for $\rho_{\alpha\beta}$ are obtained from the fact that

$$\begin{bmatrix} t_{\alpha\beta} & \rho_{\alpha\beta} \\ r_{\alpha\beta} & \tau_{\alpha\beta} \end{bmatrix} = S_{\alpha\beta} = \hat{\hat{S}}_{\alpha\beta} = \widehat{(\hat{S}_\alpha \hat{S}_\beta)}\ .$$

Equations (3.4) are the algebraic equations discovered by Redheffer [5]. We can now express our functional equation in the form

$$S(y_1,y_3) = S(y_1,y_2) * S(y_2,y_3),$$

where the *-multiplication is defined by

$$\begin{bmatrix} t & \rho \\ r & \tau \end{bmatrix} * \begin{bmatrix} t' & \rho' \\ r' & \tau' \end{bmatrix}$$

$$= \begin{bmatrix} t'(I-\rho r')^{-1} t & \rho' + t'\rho(I-r'\rho)^{-1}\tau' \\ r+\tau r'(I-\rho r')^{-1} t & \tau(I-r'\rho)^{-1}\tau' \end{bmatrix}. \tag{3.5}$$

We remark that the *-multiplication is associative, and that

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} * S = S * \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = S,$$

so that $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ is a two-sided identity. The space of all bounded linear operators on $\mathcal{X}^2$ together with the *-operation forms a local semigroup with identity. The product

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} * \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

is defined whenever $I - B_1C_2$ is invertible. (We also need the invertibility of $I - C_2B_1$, but the existence of one inverse implies the other, for $I + C_2(I - B_1C_2)^{-1}B_1$ is easily verified to be the inverse of $I - C_2B_1$.)

4. THE DIFFERENTIAL EQUATIONS

Let A be a function from a real interval into the space of bounded linear operators on a Banach space. Then A is defined to be differentiable at ξ_o if

$$\lim_{h\to 0} \frac{1}{h}[A(\xi_o + h)x - A(\xi_o)x]$$

exists for all x in the Banach space. The derivative $A'(\xi_o)$ is defined by equating $A'(\xi_o)x$ to this limit. We say that A is differentiable on a set Ω if A is differentiable at every point of Ω. (We are defining differentiability in the strong operator topology.)

We now state without proof the following fact about bounded linear operators:

Theorem 4. *Let A and B be functions mapping a real interval into the space of bounded linear operators on a Banach space. If A and B are continuous (in the uniform operator topology) on the interval and differentiable (in the strong operator topology) on a subset Ω of the interval, we can then state the following:*

(i) *AB is continuous (in the uniform operator topology) and differentiable on Ω (in the strong operator topology), and the derivative of AB is given by $A'B + AB'$.*

(ii) If $A(\xi)$ is also invertible for all ξ in the interval, then A^{-1} is continuous (in the uniform operator topology) and differentiable on Ω (in the strong operator topology), and the derivative of A^{-1} is given by $-A^{-1}A'A^{-1}$.

Theorem 5. Suppose that the (one-sided) derivatives $\frac{\partial}{\partial h} S(y,y+h)|_{h=0}$ and $\frac{\partial}{\partial h} S(y-h,y)|_{h=0}$ exist for all y in a given interval and that

$$\frac{\partial}{\partial h} S(y,y+h)|_{h=0} = \frac{\partial}{\partial h} S(y-h,y)|_{h=0} = \begin{bmatrix} b_1(y) & a(y) \\ c(y) & b_2(y) \end{bmatrix} .$$

Then, for fixed y_0, the function $S(y) = S(y_0,y)$ is differentiable on the given interval and satisfies the differential equation

$$S' = \begin{bmatrix} (b_1 + \rho c)t & a+b_1\rho + \rho b_2 + \rho c\rho \\ \tau c t & \tau(b_2 + c\rho) \end{bmatrix}, \tag{4.1}$$

where we let $S = \begin{bmatrix} t & \rho \\ r & \tau \end{bmatrix}$ as usual.

The proof follows immediately from the preceding theorem and the functional equations

$$S(y_0,y+h) = S(y_0,y) * S(y, y+h)$$

and

$$\hat{S}(y_0, y-h) = \hat{S}(y_0,y)\,\bar{S}(y-h,y),$$

$$\bar{S}(y_0, y-h) = \hat{S}(y-h,y)\,\bar{S}(y_0,y) .$$

Equation (4.1) is the well-known invariant-imbedding equation (see [5]) and is identical to equation (1.2). The equations (1.1) for the "internal fluxes" also follow immediately from the equations

$$\begin{bmatrix} u(y+h) \\ v(y+h) \end{bmatrix} = \bar{S}(y,\ y+h) \begin{bmatrix} u(y) \\ v(y) \end{bmatrix}$$

and

$$\begin{bmatrix} u(y-h) \\ v(y-h) \end{bmatrix} = \hat{S}(y-h, y) \begin{bmatrix} u(y) \\ v(y) \end{bmatrix}$$

5. APPLICATION

We now indicate an application of the operator differential equation (1.2). (Other applications can be found in [1,6]). We consider a one-dimensional neutron-transport process in which the energy of the neutrons is allowed to vary continuously. The following functions shall represent the physical parameters that describe the neutron transport process:

$\left.\begin{matrix} \sigma_\ell(y,\ \mu) \\ \sigma_r(y,\ \mu) \end{matrix}\right\}$ = the reciprocal of the mean free path of a neutron of energy μ travelling to the $\begin{Bmatrix} \text{left} \\ \text{right} \end{Bmatrix}$ at the point y in the rod,

$\left.\begin{matrix} f_\ell(y,\ \mu,\ \nu) \\ f_r(y,\ \mu,\ \nu) \end{matrix}\right\}$ = the expected energy distribution (in μ) of neutrons travelling to the $\begin{Bmatrix} \text{left} \\ \text{right} \end{Bmatrix}$ due to a fission at y of a neutron of energy ν travelling to the $\begin{Bmatrix} \text{left} \\ \text{right} \end{Bmatrix}$,

$\left.\begin{matrix} g_\ell(y,\ \mu,\ \nu) \\ g_r(y,\ \mu,\ \nu) \end{matrix}\right\}$ = the expected energy distribution (in μ) of neutrons travelling to the $\begin{Bmatrix} \text{right} \\ \text{left} \end{Bmatrix}$ due to a fission at y of a neutron of energy ν travelling to the $\begin{Bmatrix} \text{left} \\ \text{right} \end{Bmatrix}$.

If we represent neutron fluxes as energy-distribution functions $u(\nu)$ and $v(\nu)$, the above definitions yield the integro-differential equations

$$
\begin{aligned}
\frac{\partial}{\partial y} u(y, \mu) = & \int \sigma_r(y, \nu) f_r(y, \mu, \nu) u(y, \nu) d\nu \\
& - \sigma_r(y, \mu)\, u(y, \mu) \\
& + \int \sigma_\ell(y, \nu) g_\ell(y, \mu, \nu) v(y, \nu) d\nu, \\
-\frac{\partial}{\partial y} v(y, \mu) = & \int \sigma_r(y, \nu) g_r(y, \mu, \nu) u(y, \nu) d\nu \\
& + \int \sigma_\ell(y, \nu) f_\ell(y, \mu, \nu) v(y, \nu) d\nu \\
& - \sigma_\ell(y, \mu) v(y, \mu),
\end{aligned}
$$

where the integrations are over the allowable energy range.

Comparing equation (5.1) with (1.1), we see that the operators $a(y)$, $b_1(y)$, $b_2(y)$, $c(y)$ are given by

$$
\begin{aligned}
a(y)\, \Psi(\mu) &= \int \sigma_\ell(y, \nu) g_\ell(y,\mu,\nu)\, \Psi(\nu) d\nu, \\
c(y)\, \Psi(\mu) &= \int \sigma_r(y, \nu) g_r(y,\mu,\nu)\, \Psi(\nu) d\nu, \\
b_1(y)\, \Psi(\mu) &= \int \sigma_r(y, \nu) f_r(y,\mu,\nu)\, \Psi(\nu) d\nu \\
&\quad - \sigma_r(y,\mu)\, \Psi(\mu), \\
b_2(y)\, \Psi(\mu) &= \int \sigma_\ell(y,\nu) f_\ell(y,\mu,\nu)\, \Psi(\nu) d\nu \\
&\quad - \sigma_\ell(y,\mu)\, \Psi(\mu),
\end{aligned}
\tag{5.2}
$$

where Ψ is an energy distribution function representing a neutron flux.

We then represent our transport operator as follows:

$$\rho(x)\,\psi(\mu) = \int R(x,\mu,\nu)\,\psi(\nu)d\nu,$$

$$r(x)\,\psi(\mu) = \int r(x,\mu,\nu)\,\psi(\nu)d\nu,$$

(5.3)

$$\tau(x)\,\psi(\mu) = T_o(x,\mu)\,\psi(\mu) + \int T(x,\mu,\nu)\,\psi(\nu)d\nu,$$

$$t(x)\,\psi(\mu) = t_o(x,\mu)\,\psi(\mu) + \int t(x,\mu,\nu)\,\psi(\nu)d\nu.$$

We can now apply our operator differential equations (1.2) to obtain the differential equations for the "reflection" and "transmission" functions defined by (5.3); thus:

(5.4)

$$\begin{aligned}
\frac{\partial}{\partial x} R(x,\mu,\nu) &= \sigma_\ell(x,\nu)g_\ell(x,\mu,\nu) \\
&\quad - [\sigma_\ell(x,\nu) + \sigma_r(x,\mu)]R(x,\mu,\nu) \\
&\quad + \int \sigma_\ell(x,\nu)f_\ell(x,\alpha,\nu)R(x,\mu,\alpha)d\alpha \\
&\quad + \int \sigma_r(x,\alpha)f_r(x,\mu,\alpha)R(x,\alpha,\nu)d\alpha \\
&\quad + \iint R(x,\mu,\beta)\sigma_r(x,\alpha)g_r(x,\beta,\alpha)R(x,\alpha,\nu)d\alpha d\beta, \\
\frac{\partial}{\partial x} T_o(x,\mu) &= -\sigma_\ell(x,\mu)T_o(x,\mu), \\
\frac{\partial}{\partial x} t_o(x,\mu) &= -\sigma_r(x,\mu)t_o(x,\mu), \\
\frac{\partial}{\partial x} T(x,\mu,\nu) &= \int T(x,\mu,\alpha)\sigma_\ell(x,\nu)f_\ell(x,\alpha,\nu)d\alpha \\
&\quad - T(x,\mu,\nu)\sigma_\ell(x,\nu) \\
&\quad + \iint T(x,\mu,\alpha)\sigma_r(x,\beta)g_r(x,\alpha,\beta)R(x,\beta,\nu)d\alpha d\beta \\
&\quad + T_o(x,\mu)[\sigma_\ell(x,\nu)f_\ell(x,\mu,\nu) + \int \sigma_r(x,\alpha) \\
&\qquad g_r(x,\mu,\alpha)R(x,\alpha,\nu)d\alpha],
\end{aligned}$$

$$\frac{\partial}{\partial x} t(x, \mu, \nu) = \int \sigma_r(x, \alpha) f_r(x, \mu, \alpha) t(x, \alpha, \nu) d\alpha$$
$$- \sigma_r(x, \mu) t(x, \mu, \nu)$$
$$+ \iint R(x, \mu, \alpha) \sigma_r(x, \beta) g_r(x, \alpha, \beta) t(x, \beta, \nu) d\alpha d\beta$$
$$+ t_o(x, \nu) [\sigma_r(x, \nu) f_r(x, \mu, \nu)$$
$$+ \int R(x, \mu, \alpha) \sigma_r(x, \nu) g_r(x, \alpha, \nu) d\alpha] ,$$

$$\frac{\partial}{\partial x} r(x, \mu, \nu) = \iint T(x, \mu, \alpha) \sigma_r(x, \beta) g_r(x, \alpha, \beta) t(x, \beta, \nu) d\alpha d\beta$$
$$+ T_o(x, \mu) \int \sigma_r(x, \alpha) g_r(x, \mu, \alpha) t(x, \alpha, \nu) d\alpha$$
$$+ t_o(x, \nu) \int T(x, \mu, \alpha) \sigma_r(x, \nu) g_r(x, \alpha, \nu) d\alpha$$
$$+ \sigma_r(x, \nu) (g_r(x, \mu, \nu) T_o(x, \mu) t_o(x, \nu),$$

The initial conditions are

$$(5.5) \quad \begin{aligned} &R(0, \mu, \nu) = r(0, \mu, \nu) = T(0, \mu, \nu) = t(0, \mu, \nu) = 0, \\ &T_o(0, \mu) = t_o(0, \mu) = 1. \end{aligned}$$

We see that $T_o(x, \mu)$ gives the fraction of the neutrons of energy μ incident to the left that leave the reactor without entering into any fission reactions, and $t_o(x, \mu)$ represents the similar fraction for neutrons incident to the right. While these rather formidable-looking equations are difficult to obtain directly from physical considerations, they follow from our operator equations (1.2) with surprisingly little mathematical manipulation.

6. REMARKS

We have given an axiomatized description of one-dimensional transport processes and have shown how our axiomatized system leads to the desired algebraic and differential equations of transport theory. One could also start with the differential equations (1.2) of the transport process and then verify the algebraic equations (3.4). This approach has been used by Redheffer [8].

In Sec. 5 we used the generalized invariant-imbedding equations (1.2) to describe an energy-dependent neutron-transport process. We remind the reader that when we consider neutron transport processes, our formalism deals only with rods of less than "critical length." When a rod reaches critical length, an infinite number of neutrons is produced, and the fluxes can no longer be considered as elements of the Banach space $\mathcal{X}$. To compute the critical length by the invariant-imbedding method, we find the point where the solution of the first equation of (1.2) becomes "infinite," as described in [1] and [3].

We conclude by remarking that our mathematical method of approach in this Memorandum is based on the idea of applying the invariant-imbedding concept to <u>families of operators</u>. It seems likely that this technique may have further applications in the domain of mathematical physics.

REFERENCES

1. Bellman, R., R. Kalaba, and G. M. Wing, "Invariant Imbedding and Mathematical Physics. I. Particle Processes," J. Mathematical Phys., Vol. 1, 1960, pp. 280-308.
2. Bellman, R., R. Kalaba, and G. M. Wing, "Invariant Imbedding and Neutron Transport Theory. II. Functional Equations," J. Math. Mech., Vol. 7, 1958, pp. 741-756.
3. Bellman R., and R. Kalaba, "Transport Theory and Invariant Imbedding," Proceedings of Symposia in Applied Mathematics, Vol. 11, 1961, pp. 206-218.
4. Redheffer, R., "Remarks on the Basis of Network Theory," J. Math. and Phys., Vol.28, 1950, pp. 237-258.
5. Redheffer, R., "On the Relation of Transmission-Line Theory to Scattering and Transfer," J. Math. and Phys., Vol. 41, 1962, pp. 1-41.
6. Redheffer, R., "Difference Equations and Functional Equations in Transmission-line Theory," in E.F.Beckenbach, Modern Mathematics for the Engineer, Second Series, Ch. 12, McGraw-Hill Book Co. Inc., 1961.
7. Loomis, L. H., An Introduction to Abstract Harmonic Analysis, D. Von Nostrand Co., Inc., Princeton, N. J., 1953.
8. Redheffer, R., "The Mycielski-Paszkowski Diffusion Problem," J. Math. Mech., Vol. 9, 1960, pp. 607-621.