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WASHINGTON 7, D.C.

THE MOMENT ACTING ON A RANKINE OVOID MOVING UNDER A FREE SURFACE

by

Hartley L. Pond

DDC

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LIST OF SYMBOLS

a	One half the distance between the source and sink of the Rankine Ovoid
b	Maximum radius of body
C_M	Moment coefficient $= \dfrac{M}{\frac{1}{2}\rho c^2 \pi b^2 \ell}$
c	Constant uniform stream velocity
d	Maximum diameter of body
F	Froude number based on length $= \dfrac{c}{\sqrt{g\ell}}$
$\overline{F}$	Force vector
f	Distance of source or sink below the undisturbed fluid surface
g	Acceleration of gravity
K_0, K_1	Modified Bessel functions of the second kind (Reference 13, p. 78)
k_0	g/c^2
L_{2n-1}	A function defined on page 15
ℓ	Over-all length of body
M, M_1, M_2	Moments
m	The strength of a source (a source of strength m emits a volume $4\pi m$ per unit time)
P	Placed before an integral sign means that the Cauchy principal value of the integral is to be taken
$\bar{q}$	Resultant fluid velocity vector at the location of a source due to all other sources
q_x, q_y, q_z	Magnitude of components of $\bar{q}$ in x, y, z directions, respectively
r	Radial distance
u, v, w	Magnitude of components of local velocity in x, y, z directions, respectively
x, y, z	Rectangular coordinates
α	$\dfrac{\text{depth of centerline of body}}{\text{diameter of body}} = \dfrac{f}{d}$
β	$\dfrac{\text{length of body}}{\text{diameter of body}} = \dfrac{\ell}{d}$
ϵ	$\dfrac{\text{distance between source and sink}}{\text{length of body}} = \dfrac{2a}{\ell}$

η	A parameter
λ_i	Christoffel number used in numerical quadrature formula
μ	Strength of doublet
ρ	Mass density of fluid
ϕ	Velocity potential $\left(u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y},\ w = -\frac{\partial \phi}{\partial z}\right)$
ω	$\frac{2\alpha}{\beta P^2}$

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ABSTRACT

The moment acting on a Rankine Ovoid moving under the free surface of a fluid of infinite depth is calculated. In order to account for the effect of the waves formed by the motion of the body a correction is given in the form of a second approximation.

INTRODUCTION

In the calculation of the resistance of bodies moving below a free surface, it is usually assumed that the singularity system (sources, sinks, doublets, etc.) used to represent the body can be taken, as a first approximation, to be the same as the singularity system representing the body in an unbounded fluid. In the present report, it is shown that in calculating the moment acting on a body moving below a free surface, it is necessary to modify the singularity system to account for the waves formed by the motion of the body. To the order of approximation considered in the present report the modification of the singularity system is such that it gives rise only to a couple. That is, there is no change in the horizontal and vertical forces acting on the body.

THE FIRST APPROXIMATION

With the usual assumptions that the wave slope is small, and that the velocity (due to the wave motion) of the fluid particles is sufficiently small so that the square of this velocity can be neglected in Bernoulli's equation (Reference 1*, page 1) the velocity potential of fluid motion (for an incompressible, non-viscous fluid) due to a source (Reference 1, page 404) located below the free surface of a uniform stream of infinite depth is (Reference 2, page 3)

*References are listed on page 17.

$$\phi(x,y,z) = cx + \frac{m}{r_1} - \frac{m}{r_2} - \frac{4k_0 m}{\pi} P\int_0^{\frac{\pi}{2}} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{-k(f-z)}\cos(kx\cos\theta)\cos(ky\sin\theta)}{k-k_0\sec^2\theta}\, dk$$

$$- 4k_0 m\int_0^{\frac{\pi}{2}} \sec^2\theta\, e^{-k_0(f-z)\sec^2\theta} \sin(k_0 x\sec\theta)\cos(k_0 y\sin\theta\sec^2\theta)\, d\theta \qquad [1]$$

where x, y, z are rectangular coordinates, z positive upwards (the undisturbed free surface is the xy-plane of Figure 1),

- f is the depth of the source below the undisturbed free surface,
- r_1^2 is equal to $x^2 + y^2 + (z + f)^2$
- r_2^2 is equal to $x^2 + y^2 + (z - f)^2$
- $\phi(x,y,z)$ is the velocity potential $\left(u = -\frac{\partial\phi}{\partial x},\ v = -\frac{\partial\phi}{\partial y},\ w = -\frac{\partial\phi}{\partial z}\right)$
- u, v, w are velocities in positive x, y, z directions,
- $-c$ is the uniform stream velocity,
- k_0 is equal to g/c^2
- g is the acceleration of gravity,
- m is the strength of the source (a source of strength m emits a volume $4\pi m$ per unit time), and
- P placed before an integral sign means that the Cauchy principal value is to be taken (Reference 3, page 128).

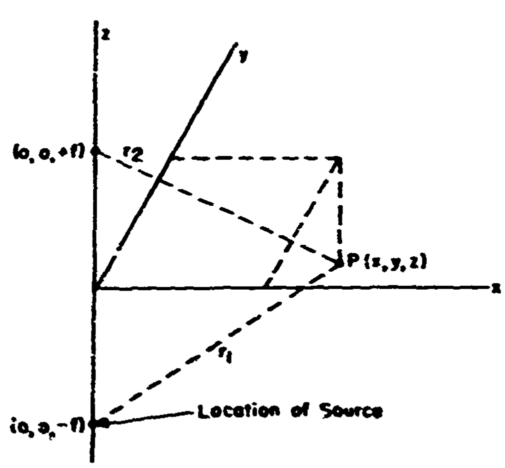

Figure 1

In the notation used the source is considered to be held stationary in a uniform stream. This is, of course, equivalent to having the source move with a uniform velocity through a stationary fluid. It is of interest to note that the first two terms of Equation [1] give the velocity potential of a source in an unbounded uniform stream.

If a source and equal sink are placed a given distance apart on a line parallel to a uniform stream in an infinite fluid, the resulting fluid motion is that for the flow of a uniform stream about an oval shaped body called the Rankine Ovoid (Reference 1, page 411). If the source and sink are near the free surface of a uniform stream the motion is that for the flow of a uniform stream (with a free surface) about a "distorted" Rankine Ovoid. The distortion is small when the source and sink are sufficiently far from the free surface of an infinitely deep stream. Using Equation [1] the velocity potential of the fluid motion about the "distorted" Rankine Ovoid with source at $(a,o,-f)$ and sink at $(-a,o,-f)$ is

$$
\begin{aligned}
\phi(x,y,z) = {} & cx + \frac{m}{r_1} - \frac{m}{r_3} - \frac{m}{r_2} + \frac{m}{r_4} \\
& - \frac{4k_0 m}{\pi} P\int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta \int_0^{\infty} \frac{e^{-k(f-z)} \cos[k(x-a)\cos\theta]\cos(ky\sin\theta)}{k - k_0\sec^2\theta} \, dk \\
& - 4k_0 m \int_0^{\frac{\pi}{2}} \sec^2\theta \, e^{-k_0(f-z)\sec^2\theta} \sin[k_0(x-a)\sec\theta]\cos(k_0 y\sin\theta\sec^2\theta) \, d\theta \qquad [2] \\
& + \frac{4k_0 m}{\pi} P\int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta \int_0^{\infty} \frac{e^{-k(f-z)} \cos[k(x+a)\cos\theta]\cos(ky\sin\theta)}{k - k_0\sec^2\theta} \, dk \\
& + 4k_0 m \int_0^{\frac{\pi}{2}} \sec^2\theta \, e^{-k_0(f-z)\sec^2\theta} \sin[k_0(x+a)\sec\theta]\cos(k_0 y\sin\theta\sec^2\theta) \, d\theta
\end{aligned}
$$

where, in addition to terms defined for Equation [1]

$$r_1^2 = (x-a)^2 + y^2 + (z+f)^2$$

$$r_2^2 = (x-a)^2 + y^2 + (z-f)^2$$

$$r_3^2 = (x+a)^2 + y^2 + (z+f)^2$$

$$r_4^2 = (x+a)^2 + y^2 + (z-f)^2$$

The first three terms of Equation [2] give the velocity potential of a source and equal sink in an unbounded uniform stream.

Lagally's theorem[4,5] may be applied to obtain the moment acting on the above "distorted" Rankine Ovoid. This theorem states that the forces acting on a body whose surface is a closed stream surface of the fluid motion, are given by the vectors

$$\overline{F} = -4\pi\rho m\bar{q}$$

where m is the strength of a source internal to the stream surface,

$\bar{q}$ is the resultant fluid velocity vector at the location of the source due to all other sources (its components in the x, y, z directions are q_x, q_y, q_z), and

ρ is the mass density of the fluid.

Thus, the force has the direction of $-\bar{q}$ and its line of action passes through the point at which the source is located. Each term of Equations [1] and [2] may be considered as the velocity potential due to a certain source or distribution of sources. Applying Lagally's theorem, the vertical force acting on the "distorted" Rankine Ovoid through the point (a,o,-f) is

$$Z(a,o,-f) = -4\pi\rho m q_z(a,o,-f) \qquad [3]$$

and the vertical force acting on the body through the point (-a,o,-f) is

$$Z(-a,o,-f) = -4\pi\rho(-m)q_z(-a,o,-f) \qquad [4]$$

Since there are no other internal sources for this body, Equations [3] and [4] give the only vertical forces acting on the body.

Carrying out the partial differentiations indicated in Equations [3] and [4], the following expressions are obtained

$$Z(a,o,-f) = -4\pi\rho m\left[\frac{m}{4f^2} - \frac{fm}{4(a^2+f^2)^{3/2}} + \frac{4k_o m}{\pi} P\int_0^{\frac{\pi}{2}}\sec^2\theta d\theta\int_0^{\infty}\frac{ke^{-2fk}}{k-k_o\sec^2\theta}dk\right.$$
$$- \frac{4k_o m}{\pi} P\int_0^{\frac{\pi}{2}}\sec^2\theta d\theta\int_0^{\infty}\frac{ke^{-2fk}\cos[2ak\cos\theta]}{k-k_o\sec^2\theta}dk \qquad [3']$$
$$\left. - 4k_o^2 m\int_0^{\frac{\pi}{2}}\sec^4\theta e^{-2fk_o\sec^2\theta}\sin[2ak_o\sec\theta]d\theta\right]$$

$$Z(-a,o,-f) = +4\pi\rho m\left[-\frac{m}{4f^2} + \frac{fm}{4(a^2+f^2)^{3/2}} - \frac{4k_o m}{\pi} P\int_0^{\frac{\pi}{2}}\sec^2\theta d\theta\int_0^{\infty}\frac{ke^{-2kf}}{k-k_o\sec^2\theta}dk\right.$$
$$+ \frac{4k_o m}{\pi} P\int_0^{\frac{\pi}{2}}\sec^2\theta d\theta\int_0^{\infty}\frac{ke^{-2kf}\cos[2ak\cos\theta]}{k-k_o\sec^2\theta}dk \qquad [4']$$
$$\left. - 4k_o^2 m\int_0^{\frac{\pi}{2}}\sec^4\theta e^{-2fk_o\sec^2\theta}\sin[2ak_o\sec\theta]d\theta\right]$$

Hence, the moment acting on the body about the center of buoyancy, $M_1 = a[Z(a,o,-f) - Z(-a,o,-f)]$, is

$$M_1 = 32\pi\rho a m^2 k_0^2 \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-2fk_0 \sec^2\theta} \sin[2ak_0 \sec\theta] d\theta \qquad [5]$$

(In the above notation a positive moment acts to raise the nose of the body.) Making the substitution $t = \tan\theta$, this equation may be written as

$$M_1 = 32\pi\rho a m^2 k_0^2 \int_0^{\infty} (1+t^2) e^{-2fk_0(1+t^2)} \sin[2ak_0\sqrt{1+t^2}] dt$$

For Rankine Ovoids with sufficiently large length-diameter ratios

$$m = \frac{b^2 c}{4}$$

where b is the maximum radius of the Rankine Ovoid. With this substitution the expression for M_1 becomes

$$M_1 = \pi\rho b^4 c^2 k_0 (2ak_0) e^{-2fk_0} \int_0^{\infty} e^{-2fk_0 t^2} (1+t^2) \sin[2ak_0\sqrt{1+t^2}] dt \qquad [5']$$

THE SECOND APPROXIMATION

Equation [5] gives the moment acting on a "distorted" Rankine Ovoid. To obtain a closer approximation to the moment acting on an undistorted Rankine Ovoid placed below the surface of a uniform stream the simple distribution of a source and equal sink on the axis of the closed stream surface representing the Rankine Ovoid must be modified. This might be done by an extension of the method of images as applied by Havelock in the case of a circular cylinder. This method would require finding the image system of sources within the closed stream surface due to the source system above the free surface. However, instead of attempting to obtain this image system exactly an approximate image system will be sought.

It has been shown by von Kármán[6] that for a body of revolution with its axis parallel to a uniform stream the effect of superimposing a flow perpendicular to the axis may be obtained approximately by a suitable distribution of doublets (Reference 6, page 12) along the axis of the body between the limits of the source-sink distribution which defines the body in the uniform stream. The doublets are oriented so that their axes are opposite in direction to the transverse flow and their strength per unit distance along the axis of the body is

$$\mu = -\frac{1}{2}r^2 w \qquad [6]$$

where r is the radius of the body at the position of the doublet, and w is the superimposed transverse velocity.

Therefore in the case of the Rankine Ovoid moving below a free surface, the effect of the velocity induced by the free surface may be accounted for by a suitable distribution of doublets along the axis of the stream surface between the source and sink. It will be apparent that for the calculation of moments (see below) only the vertical component of the induced velocity need be considered. This vertical velocity is obtained from Equation [2] by evaluating $\frac{\partial \phi}{\partial z}$ at points along the line between the source and sink. Since the vertical velocity does not change very rapidly with depth, this calculation of the vertical velocity is probably satisfactory. For Rankine Ovoids with fairly large length-diameter ratios the diameter of the body is nearly constant in the region between the source and sink. Thus, as a further simplification, the radius r in Equation [6] will be considered constant and equal to the maximum radius of the body. The desired doublet distribution is now given by

$$\mu = -\frac{1}{2}b^2 w \qquad [7]$$

where b is the maximum radius of the undistorted Rankine Ovoid, and w is the vertical component of the induced velocity calculated from Equation [2].

An application of Lagally's theorem shows that a body whose surface is a closed stream surface of the fluid motion experiences a moment if the axis of any internal doublet is normal to the direction of a superimposed uniform stream. (The doublet, however, causes no resultant force on the body.) This moment is given by (Reference 5, page 13)

$$M = -4\pi\rho\mu c \qquad [8]$$

where ρ is the mass density of the fluid,

μ is the doublet strength, and

c is the uniform stream velocity.

Hence, from Equations [7] and [8] the moment per unit length for the Rankine Ovoid (due to the uniform stream c) has the algebraic sign of the vertical velocity and is given by

$$M = 2\pi\rho c b^2 w \qquad [9]$$

and the total moment* on the body is

$$M_z = 2\pi\rho c b^2 \int_{-\infty}^{\infty} w \, dx \qquad [10]$$

where, using Equation [2], the velocity w is

$$\begin{aligned} w = w(x,o,-f) &= -\left.\frac{\partial\phi(x,y,z)}{\partial z}\right]_{\substack{x=x\\y=o\\z=-f}} \\ &= 2fm[(x-a)^2 + (2f)^2]^{-3/2} - 2fm[(x+a)^2 + (2f)^2]^{-3/2} \\ &+ \frac{4k_0 m}{\pi} P\int_0^{\frac{\pi}{2}} \sec^2\theta d\theta \int_0^{\infty} \frac{ke^{-2fk}\cos[k(x-a)\cos\theta]}{k-k_0\sec^2\theta} dk \\ &- \frac{4k_0 m}{\pi} P\int_0^{\frac{\pi}{2}} \sec^2\theta d\theta \int_0^{\infty} \frac{ke^{-2fk}\cos[k(x+a)\cos\theta]}{k-k_0\sec^2\theta} dk \\ &+ 4k_0^2 m \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-2fk_0\sec^2\theta} \sin[k_0(x-a)\sec\theta] d\theta \\ &- 4k_0^2 m \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-2fk_0\sec^2\theta} \sin[k_0(x+a)\sec\theta] d\theta \end{aligned} \qquad [11]$$

The first four terms of Equation [11] give that part of w which is skew symmetric about $x = 0$, and the last two the symmetric part. Since the integrals of the skew symmetric terms vanish, Equation [10] now becomes

$$\begin{aligned} M_z = 16\pi\rho c b^2 k_0^2 m \Bigg[&\int_0^{\infty}\int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-2fk_0\sec^2\theta} \sin[k_0(x-a)\sec\theta] d\theta \, dx \\ &- \int_0^{\infty}\int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-2fk_0\sec^2\theta} \sin[k_0(x+a)\sec\theta] d\theta \, dx \Bigg] \end{aligned} \qquad [12]$$

The double integrals in Equation [12] may be expressed in a more convenient form. Consider, for example, the second of these integrals and let

---

*The longitudinal velocity due to the wave system is small compared with the uniform stream velocity c and is not considered in calculating M_z. In addition the mutual actions of the sources and doublets within the closed stream surface representing the body give no resultant force or moment (Reference 5, pages 3 to 5).

$$I_2(x) = \int_0^{\frac{\pi}{2}} \sec^4\theta \, e^{-2fk_0\sec^2\theta} \sin[k_0(x+a)\sec\theta]\,d\theta$$

The substitution $t = \tan\theta$ transforms this integral into

$$I_2(x) = \int_0^{\infty} (1+t^2) e^{-2fk_0(1+t^2)} \sin[k_0(x+a)\sqrt{1+t^2}]\,dt$$

The second double integral of Equation [12] can then be written as

$$J_2 = \int_0^{a}\int_0^{\infty} (1+t^2) e^{-2fk_0(1+t^2)} \sin[k_0(x+a)\sqrt{1+t^2}]\,dt\,dx$$

Interchanging the order of integration (Reference 7, page 277) the expression for J_2 becomes

$$J_2 = \frac{e^{-2fk_0}}{k_0}\int_0^{\infty} e^{-2fk_0t^2} \sqrt{1+t^2} \cos[k_0a\sqrt{1+t^2}]\,dt$$

$$- \frac{e^{-2fk_0}}{k_0}\int_0^{\infty} e^{-2fk_0t^2} \sqrt{1+t^2} \cos[2k_0a\sqrt{1+t^2}]\,dt$$

Designating the first integral of Equation [12] by J_1, and proceeding as above, it is found that

$$J_1 = \frac{e^{-2fk_0}}{k_0}\int_0^{\infty} e^{-2fk_0t^2} \sqrt{1+t^2} \cos[k_0a\sqrt{1+t^2}]\,dt$$

$$- \frac{e^{-2fk_0}}{k_0}\int_0^{\infty} e^{-2fk_0t^2} \sqrt{1+t^2}\,dt$$

Hence Equation [12] may be written as (putting $m = \frac{b^2c}{4}$ as in Equation [5'])

$$M_2 = 4\pi\rho b^4c^2k_0e^{-2fk_0}\int_0^{\infty} e^{-2fk_0t^2} \sqrt{1+t^2}(\cos[2k_0a\sqrt{1+t^2}] - 1)\,dt \qquad [13]$$

The second approximation to the total moment acting on the undistorted Rankine Ovoid about its center of buoyancy is now given as the sum of the moments given by Equations [5'] and [13]. This sum may be written as

$$
\begin{aligned}
M &= M_1 + M_2 \\
&= \pi\rho b^4 c^2 k_0 (2ak_0) e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} (1+t^2) \sin[2ak_0\sqrt{1+t^2}]\,dt \\
&\quad + 4\pi\rho b^4 c^2 k_0 e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} \sqrt{1+t^2} (\cos[2ak_0\sqrt{1+t^2}]-1)\,dt
\end{aligned}
\qquad [14]
$$

or

$$
\begin{aligned}
M = \pi\rho b^4 c^2 k_0 e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} \sqrt{1+t^2} \Big[& 2ak_0\sqrt{1+t^2}\, \sin[2ak_0\sqrt{1+t^2}] \\
& + 4\cos[2ak_0\sqrt{1+t^2}] - 4 \Big]\,dt
\end{aligned}
\qquad [14']
$$

A moment coefficient is given by

$$
C_M = \frac{M}{\frac{1}{2}\rho c^2 \pi b^2 \ell} \qquad [15]
$$

where ρ is the mass density of the fluid,
c is the uniform stream velocity,
πb^2 is the maximum cross-sectional area, and
ℓ is the length of the body.

Then, for Rankine Ovoids with sufficiently large length-diameter ratios* the moment coefficient can be written as

$$
\begin{aligned}
C_M &= C_{M_1} + C_{M_2} \\
&= \frac{e^{-\frac{2a}{\beta F^2}}}{2\beta^2 F^2} \int_0^\infty e^{-\frac{2a}{\beta F^2} t^2} \sqrt{1+t^2} \left\{ \frac{\epsilon}{F^2} \sqrt{1+t^2} \sin\left[\frac{\epsilon}{F^2}\sqrt{1+t^2}\right] \right\} dt \\
&\quad + \frac{e^{-\frac{2a}{\beta F^2}}}{2\beta^2 F^2} \int_0^\infty e^{-\frac{2a}{\beta F^2} t^2} \sqrt{1+t^2} \left\{ 4\cos\left[\frac{\epsilon}{F^2}\sqrt{1+t^2}\right] - 4 \right\} dt \qquad [16] \\
&= \frac{e^{-\frac{2a}{\beta F^2}}}{2\beta^2 F^2} \int_0^\infty e^{-\frac{2a}{\beta F^2} t^2} \sqrt{1+t^2} \left\{ \frac{\epsilon}{F^2}\sqrt{1+t^2}\, \sin\left[\frac{\epsilon}{F^2}\sqrt{1+t^2}\right] + 4\cos\left[\frac{\epsilon}{F^2}\sqrt{1+t^2}\right] - 4 \right\} dt
\end{aligned}
$$

---

*For the present the term "sufficiently large" means 10 or greater. Rankine Ovoids with other length-diameter ratios will be investigated in a later report.

where $\beta = \dfrac{\text{length of body}}{\text{diameter of body}} = \dfrac{\ell}{d}$,

$$\alpha = \frac{\text{depth of centerline of body}}{\text{diameter of body}} = \frac{f}{d},$$

$$\epsilon = \frac{\text{distance between source and sink}}{\text{length}} = \frac{2a}{\ell}, \text{ and}$$

$$F = \frac{c}{\sqrt{g\ell}}$$

From Equation [16] it can be seen that C_M approaches zero as F becomes either very small or very large. The two cases of low and high speeds may also be approximated by considering the free surface as a rigid wall for very low speeds and by neglecting gravity forces for very high speeds. In each case the moment coefficient is zero.

Writing Equation [14'] in the form

$$M = d^4\left[\frac{\pi\rho g e^{-\frac{2fg}{c^2}}}{16}\right]\int_0^\infty e^{-\frac{2fg}{c^2}t^2}\sqrt{1+t^2}\left\{\frac{\epsilon\ell g}{c^2}\sqrt{1+t^2}\,\sin\left[\frac{\epsilon\ell g}{c^2}\sqrt{1+t^2}\right]\right.$$

$$\left. + 4\cos\left[\frac{\epsilon\ell g}{c^2}\sqrt{1+t^2}\right] - 4\right\} dt$$

it is evident that for a given length, depth, and speed the moment M is approximately proportional to the fourth power of the diameter for Rankine Ovoids with length-diameter ratios of 10 or greater. The approximation depends on ϵ. For a length-diameter ratio of 10.5 ϵ is 0.95 and as the length-diameter ratio is increased ϵ approaches 1 as a limit.

The integrals appearing above may be evaluated either by numerical quadrature or by expanding the integrand in an infinite series and integrating term by term. The series expansion is rapidly convergent for $F > 1$ (i.e., for large speeds of advance) while for $F < 1$ it is more convenient to use a Gauss type quadrature formula. The two methods are described in the Appendix.

Figures 3 and 4 indicate the values of C_{M_1} and $C_M = C_{M_1} + C_{M_2}$ given by Equation [16] for a Rankine Ovoid with a length-diameter ratio of 10.5 at submergences of 1.5 and 2.8 diameters. The dotted line in Figure 4 shows the values of C_M obtained from the approximate formula for C_M given by Equation [21] of the Appendix.

Wetted Surface Coef., $\frac{\text{Surface area of ovoid}}{\pi d l} = 0.972$

Prismatic Coefficient, $\frac{\text{Volume of ovoid}}{\pi b^2 l} = 0.926$

$\frac{x}{l}$	$\frac{y}{l}$
0.000	0.00000
0.005	0.01670
0.010	0.02374
0.020	0.03164
0.050	0.04174
0.100	0.04597
0.150	0.04696
0.200	0.04730
0.250	0.04745
0.300	0.04752
0.350	0.04758
0.400	0.04760
0.450	0.04761
0.500	0.04762

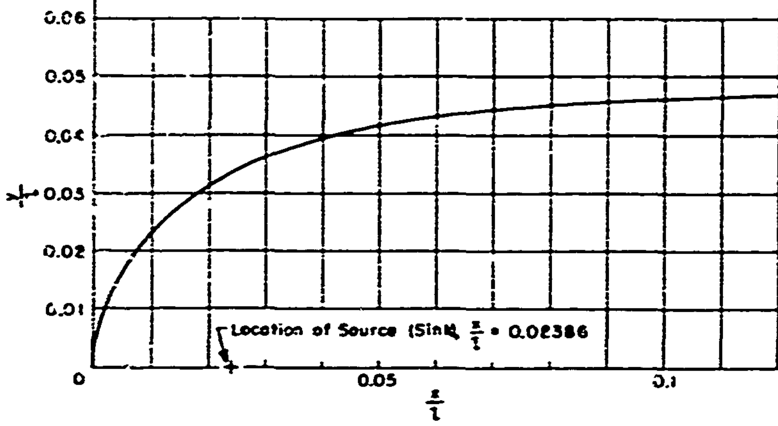

Figure 2 - Offsets of Rankine Ovoid with Length-Diameter Ratio of 10.5

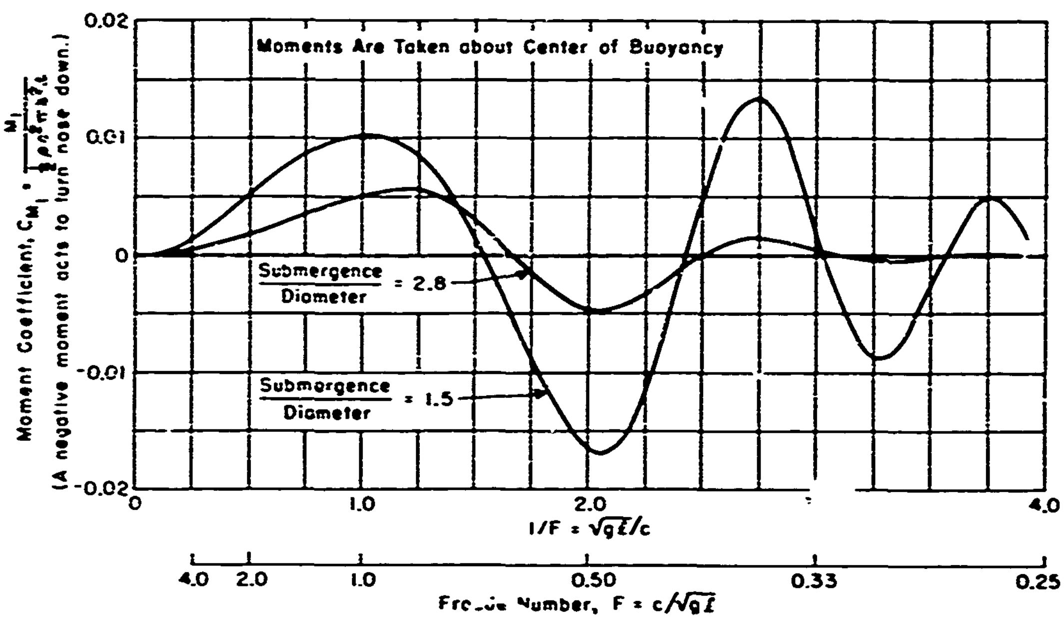

Figure 3 - Moment Coefficient C_{M_1} for Rankine Ovoid, $\frac{\text{Length}}{\text{Diameter}} = 10.5$

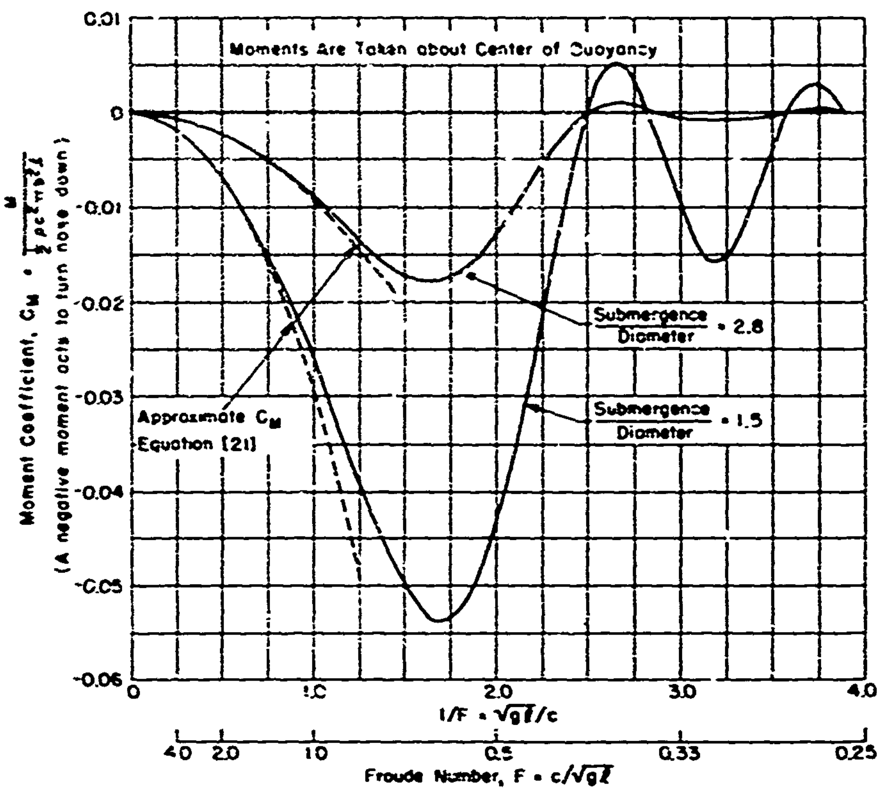

Figure 4 - Moment Coefficient $C_M = C_{M_1} + C_{M_2}$ for Rankine Ovoid, $\frac{\text{Length}}{\text{Diameter}} = 10.5$

CONCLUSIONS

The difference between the first and second approximations for the moment acting on the Rankine Ovoid moving below a free surface indicates that it is essential to modify the singularity system representing a body in an infinite fluid in order to account for the effect of the waves formed by the motion of the body if the correct moment acting on the body is to be obtained. To the order of approximation considered the modification of the singularity system is such that it gives rise only to a couple.

ACKNOWLEDGMENT

The numerical calculations required for this report were carried out by Mrs. Helen W. Henderson.

APPENDIX

EVALUATION OF INTEGRALS APPEARING IN TEXT

As mentioned in the report the integrals appearing in the expressions for moments can be evaluated either by numerical quadrature or by expanding the integrand in an infinite series and integrating term by term. An efficient method of numerical quadrature for integrals of the type required is given by the Gauss-Christoffel formula

$$\int_{-\infty}^{\infty} e^{-x^2} G(x)dx \approx \sum_{i=0}^{n} \lambda_i G(x_i) \qquad [17]$$

where the x_i's are roots of the Hermite polynomials of the n^{th} order and the λ_i's are called Christoffel numbers. The theory of this method of numerical quadrature is given in References 8 and 9, and tables of the x_i's and λ_i's for various values of n are given in References 10 and 11.

Since the x_i's are symmetrically spaced on either side of $x = 0$ and the λ_i's are symmetric about $x = 0$, the formula can be used when $G(x)$ is symmetric for the evaluation of integrals of the form

$$\int_{0}^{\infty} e^{-x^2} G(x)dx$$

In this case only the positive x_i's of Equation [17] are taken.

As an application of this formula consider Equation [16] for the moment C_M. Making the transformation

$$p = \sqrt{\frac{2\alpha}{\beta P^2}}\, t$$

the resulting expression for C_M is

$$C_M = s\int_{0}^{\infty} e^{-p^2} G(p)dp \qquad [18]$$

where $s = \dfrac{e^{-\omega}}{2\beta^2 \epsilon \sqrt{\omega}}$,

$\omega = \dfrac{2\alpha}{\beta P^2}$,

$G(p) = \theta^2 \sin\theta + 4\theta(\cos\theta - 1)$, and

$$\theta = \frac{\beta\epsilon\sqrt{\omega}}{2\alpha}\sqrt{\omega+p^2}$$

As a particular example consider the Rankine Ovoid described on page 10 and let the Froude number be F = 0.7 and the depth of submergence be 1.5 diameters. The constant s is equal to 0.0034808, and applying the quadrature formula (with n = 4) the result is*

$$C_M = 0.0034808\left[\lambda_0 G(p_0) + \lambda_1 G(p_1) + \cdots + \lambda_4 G(p_4)\right]$$

$$= 0.0034808\left[0.36012\ G(0) + 0.43265\ G(0.72355)\right.$$

$$\left. + 0.088475\ G(1.4086) + 0.0049436\ G(2.2666) + 0.000039607\ G(3.1910)\right]$$

$$= 0.0034808\left[-2.5525 - 7.3905 - 3.5838 - 0.037864 - 0.00047394\right]$$

$$= -0.047217$$

Repeating this process for n = 7 the result is $C_M = -0.047207$

No general expression for the error in this method of quadrature is available. However, for the case considered here, a check can be obtained since the integral can be expressed as an alternating series from which the correct answer may be determined to any desired accuracy. This series expansion is discussed in the next few paragraphs.

The corresponding series expansion for C_M is obtained by expanding the integrand in an infinite series and then (since the series is uniformly convergent) inverting the order of integration and summation. Thus, substituting the series

$$\sqrt{1+t^2}\left\{\frac{\epsilon}{p^2}\sqrt{1+t^2}\sin\left[\frac{\epsilon}{p^2}\sqrt{1+t^2}\right] + 4\cos\left[\frac{\epsilon}{p^2}\sqrt{1+t^2}\right] - 4\right\}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2n-4}{(2n)!}\left(\frac{\epsilon}{p^2}\right)^{2n}(1+t^2)^{n+\frac{1}{2}}$$

*Values of λ_i and p_i are taken from Reference 10.

in Equation [16], the result is

$$C_M = \frac{e^{-\frac{2\alpha}{\beta F^2}}}{2\beta^2 P^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-4)}{(2n)!} \left(\frac{\epsilon}{P^2}\right)^{2n} \int_0^{\infty} e^{-\frac{2\alpha}{\beta F^2} t^2} (1+t^2)^{\frac{2n+1}{2}} dt \qquad [19]$$

Havelock has observed (Reference 12, page 284) that integrals of the form

$$L_{2n+1} = \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{\frac{2n+1}{2}} dt$$

where n is an integer, can be expressed in terms of modified Bessel functions of the second kind of orders zero and one. One procedure for doing this depends on the formula*

$$L_{2n+1} = \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{\frac{2n+1}{2}} dt = \frac{n+\eta}{\eta} \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{\frac{2n-1}{2}} dt$$

$$- \frac{2n-1}{2\eta} \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{\frac{2n-3}{2}} dt$$

For $2n+1 = 3$ this formula gives

$$L_3 = \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{3/2} dt = \frac{1+\eta}{\eta} \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{1/2} dt - \frac{1}{2\eta} \int_0^{\infty} e^{-\eta t^2} (1+t^2)^{-1/2} dt$$

$$= \frac{1+\eta}{\eta} L_1 - \frac{1}{2\eta} L_{-1}$$

and continuing the procedure,

$$L_5 = \frac{2+\eta}{\eta} L_3 - \frac{3}{2\eta} L_1$$

$$\vdots$$

$$L_{2n+1} = \frac{n+\eta}{\eta} L_{2n-1} - \frac{2n-1}{2\eta} L_{2n-3}$$

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*This reduction formula was shown to the author by Dr. J.W. Wrench, Jr. It may be derived by taking the derivative with respect to t of

$$t(1+t^2)^{\frac{2n+1}{2}} e^{-\eta t^2}$$

rearranging terms, and then integrating between the limits 0 and ∞.

Hence, from L_1 and L_{-1} all the other L's may be determined. The transformation $t = \sinh\mu$ applied to the integrals for L_1 and L_{-1} transforms them into known expressions for modified Bessel functions of the second kind K_0 and K_1 (Reference 13, page 181). The result is

$$L_{-1} = \frac{1}{2} e^{\eta/2} K_0\left(\frac{\eta}{2}\right)$$

$$L_1 = \frac{1}{4} e^{\eta/2}\left[K_0\left(\frac{\eta}{2}\right) + K_1\left(\frac{\eta}{2}\right)\right]$$

Since the functions $K_0(x)$ and $K_1(x)$ have been tabulated,[13,14] the L-functions can be determined. The expression for C_M now becomes

$$C_M = \frac{e^{-\frac{2\alpha}{\beta F^2}}}{2\beta^2 F^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-4)}{(2n)!}\left(\frac{\epsilon}{F^2}\right)^{2n} L_{2n+1}\left(\frac{2\alpha}{\beta F^2}\right) \qquad [20]$$

For the same example considered on pages 13 and 14 the sum of the first 11 terms of this series is -0.0472070 and the 12^{th} term is -0.0000007. Hence to 4 significant figures the value of C_M is -0.04721. This shows that the quadrature formula with $n = 7$ gives the correct answer to 4 significant figures and for $n = 4$ gives an answer that is only in error by 1 in the fourth significant figure. Since 3 significant figures give sufficient accuracy for any application, the quadrature formula with $n = 4$ was used in calculating the results given in Figures 3 and 4.

Noting that the term for $n = 2$ is zero in the series [20] it appears that, for sufficiently large F, the first term of the series may be a good estimate of C_M. Figure 4 compares the correct C_M with the values obtained from the first term of Equation [20] for a Rankine Ovoid with a length-diameter ratio of 10.5 submerged 1.5 diameters and 2.8 diameters. The figure shows that for Froude numbers F greater than 1 the first term of Equation [20] is a good approximation to C_M.

Thus for values of the parameters not too different from those in the example, the approximate value of the moment coefficient for Rankine Ovoids is

$$C_M = -\frac{\beta\epsilon^2}{64\alpha^3}\,\omega^2 e^{-\frac{\omega}{2}}\left[\omega K_0\left(\frac{\omega}{2}\right) + (1+\omega)K_1\left(\frac{\omega}{2}\right)\right] \qquad [21]$$

where
$$\omega = \frac{2\alpha}{\beta F^2}$$

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