Scientific Report No. 2

Contract N00140 66 C 0172

December 1966

• applied • mathematics • •

• pml • applied • mathematics • pml •

The Physical Interpretation of Weston's Toroidal Wave Functions

by

W. Williams and C.H. Sherman

for

U.S. Navy Underwater Sound Laboratory

PARKE MATHEMATICAL LABORATORIES, Inc.
One River Road • Carlisle, Massachusetts

46

PARKE MATHEMATICAL LABORATORIES, INCORPORATED
ONE RIVER ROAD • CARLISLE, MASSACHUSETTS

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

DEFENSE DOCUMENTATION CENTER (DDC)
CAMERON STATION
ALEXANDRIA, VIRGINIA 22314

Department of Defense contractors must be established for DDC services or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

All other persons and organizations should apply to the:

CLEARINGHOUSE FOR FEDERAL SCIENTIFIC
AND TECHNICAL INFORMATION (CFSTI)
SILLS BUILDING
5285 PORT ROYAL ROAD
SPRINGFIELD, VIRGINIA 22151

A limited number of copies are also available by writing to:

PARKE MATHEMATICAL LABORATORIES, INC.
ONE RIVER ROAD
CARLISLE, MASSACHUSETTS 01741

Scientific Report No. 2

Contract N00140 66 C 0172

December 1966

• • • applied • mathematics • •

The Physical Interpretation of Weston's Toroidal Wave Functions

by

W. Williams and C.H. Sherman

for

U.S. Navy Underwater Sound Laboratory

PARKE MATHEMATICAL LABORATORIES, Inc.
One River Road • Carlisle, Massachusetts

• p m l • applied • mathematics • p m l •

SCIENTIFIC REPORTS

their purpose and place

•••A•P•P•L•I•E•D•M•A•T•H•E•M•A•T•I•C•I•A•N•S•••

THE EVER INCREASING NEED TO BE INFORMED has caused an ever increasing volume of publications. The very quantity of these publications, however, makes it difficult to satisfy the need, because it becomes more and more arduous to locate the literature pertinent to any one subject; and because any one of us becomes more and more weary of having so much to read. Moreover, only a small minority of these publications may help satisfy the need, because an overwhelming majority is void of content.

Consequently, it is imperative to present to the public only manuscripts which do carry new information without undue verbiage; hoping to be judged not by the number of published pages but by their quality. In particular, papers in the field of Applied Mathematics ought to be published only if they contain one or more of the following items: new basic results, new methods, new applications, new numerical results, new presentation of difficult and important topics, up-to-date bibliographies; and if the number of their pages is not dictated by the desire of imposing upon the superficial reader.

To discharge our contractual obligations, we publish *Technical or Scientific Reports*, such as the one you now have in your hands. It has been our constant policy to see to it, that they satisfy the above strict criterion.

PARKE MATHEMATICAL LABORATORIES, INC.
bedford road, carlisle, massachusetts

PARKE MATHEMATICAL LABORATORIES, INCORPORATED
ONE RIVER ROAD • CARLISLE, MASSACHUSETTS

Abstract

Weston's toroidal wave functions are shown to be interpretable as the field resulting from familiar spherical sources distributed on a ring. For the case of acoustic sources the physical interpretation is extended to show how particular wave functions are associated with certain motions of a ring.

Table of Contents

BLANK PAGE

1.0 Introduction

In previous publications[1,2] the toroidal wave functions derived by Weston,[3,4] following his development of a method for solving the Helmholtz equation in nonseparable, rotational coordinate systems,[5] have been applied to the problem of the acoustic radiation from a thin torus as a first approximation to the more general problem of the sound field of the free-flooding ring transducer. The purpose of this report is to examine the structures of Weston's toroidal wave functions and to interpret these structures in terms which will facilitate application of the wave functions to physical problems.

2.0 Toroidal Wave Function Integral Representations - Review

The toroidal coordinates (s, η, ϕ) are related to the Cartesian coordinates (x, y, z) by

$$x = d\,\frac{(s^2-1)^{1/2}}{s-\cos\eta}\,\cos\phi\,, \tag{1a}$$

$$y = d\,\frac{(s^2-1)^{1/2}}{s-\cos\eta}\,\sin\phi\,, \tag{1b}$$

$$z = d\,\frac{\sin\eta}{s-\cos\eta}\,, \tag{1c}$$

in which ϕ is the usual azimuthal angle, the surfaces s = constant are tori centered on the origin and the surfaces η = constant are spheres centered on the Z -axis; these are pictured in Figure 1 for the half-plane $\phi = 90^{\circ}$. In this system $s = \infty$ is a ring of radius d lying in the x-y plane, $s = 1$ consists of the Z -axis and a surface at infinity and $\infty > s > 1$ are intermediate tori; η ranges from 0 to 2π and measures location around the s = constant tori.

Weston has shown[3] that solutions of the Helmholtz equation which are single-valued and continuous outside a torus and which satisfy the Sommerfeld radiation condition have the form

$$e^{im\phi} V^{M}_{M+2\ell}(s,\eta) \tag{2a}$$

and

$$e^{im\phi} W^{M}_{M+2\ell+1}(s,\eta), \tag{2b}$$

in which $m = 0, \pm 1, \pm 2, \ldots,$ $M = |m|$ and $\ell = 0, 1, 2, \ldots$ and where $V^{M}_{M+2\ell}(s,\eta)$ is an even function of η and $W^{M}_{M+2\ell+1}(s,\eta)$ is an odd function of η. These wave functions were defined in terms of the auxiliary functions $S^{M}_{N}(s,\eta)$ and $T^{M}_{N}(s,\eta)$ by the relations

$$V^{M}_{N}(s,\eta) = S^{M}_{N}(s,\eta) + i(-)^{N+1} S^{M}_{-N-1}(s,\eta) \tag{3a}$$

and

$$W^{M}_{N}(s,\eta) = T^{M}_{N}(s,\eta) + i(-)^{N+1} T^{M}_{-N-1}(s,\eta) \tag{3b}$$

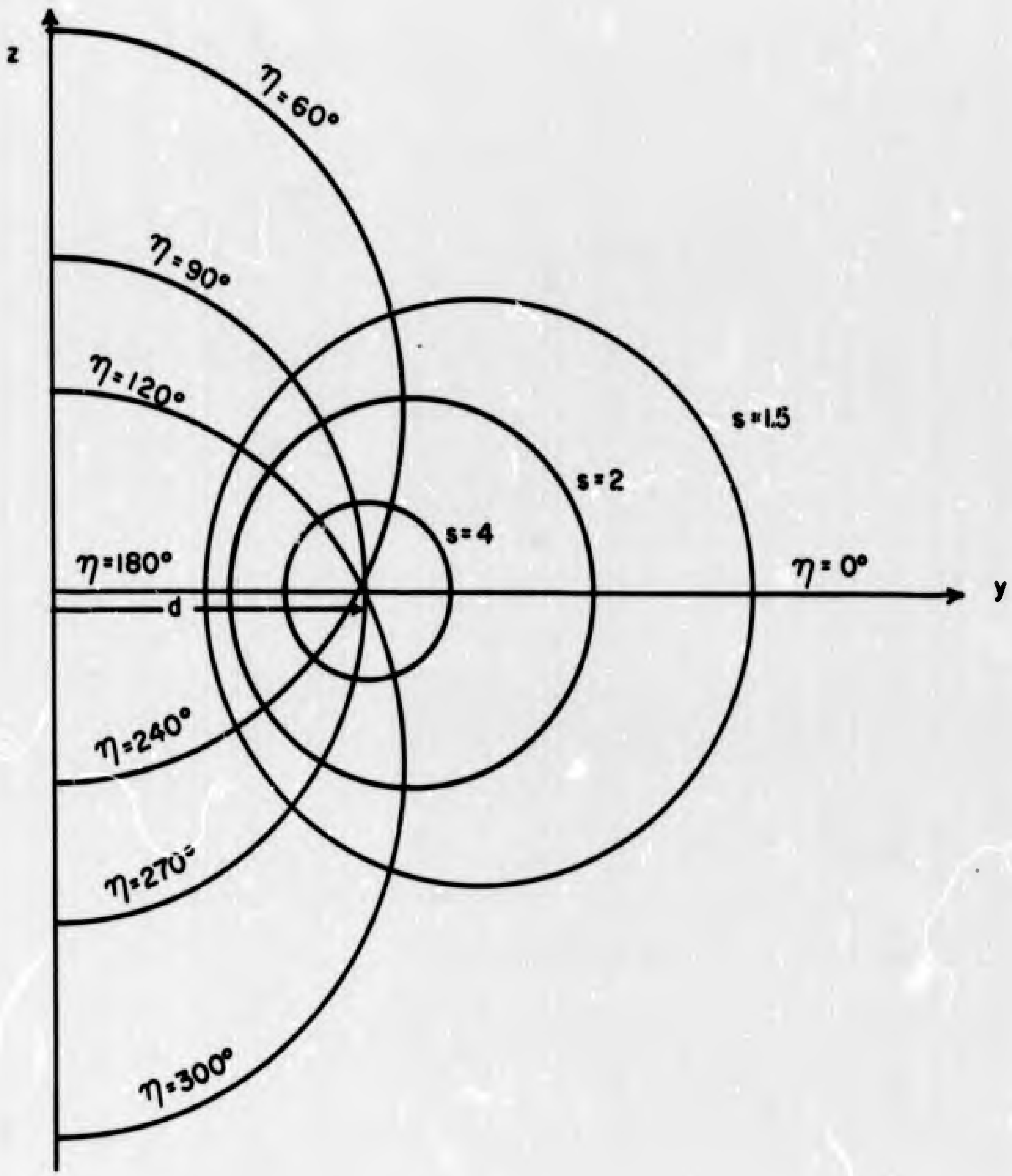


Figure 1

Toroidal Coordinates, showing the intersection of S = constant and η = constant surfaces with the half-plane $\phi = 90^{\circ}$.

where the auxiliary functions are given explicitly by the series representations

$$S_N^M(s,\eta) = C_N^M \sum_{p=0}^{\infty} \sum_{r=0} a_p^r \left(\frac{M+N+1}{2}\right)_r \left(\frac{M-N}{2}\right)_r P^{-M-r}_{p+\frac{N+M}{2}}(s) \tag{4a}$$

and

$$T_N^M(s,\eta) = \frac{C_N^M \sin\eta}{2^{1/2}(s-\cos\eta)^{1/2}} \sum_{p=0}^{\infty} \sum_{r=0} a_p^r \left(\frac{M+N+1}{2}\right)_r \left(\frac{M-N+1}{2}\right)_r P^{-M-r}_{p+\frac{N+M-1}{2}}(s) \tag{4b}$$

in which $(m)_n$ is the Pochhammer notation for $\Gamma(m+n)/\Gamma(m)$, $P_\ell^m(s)$ is the associated Legendre function and

$$C_N^M = \frac{\Gamma(N+M+1)\pi^{1/2}(-)^M(kd)^N}{\Gamma(N-M+1)\;2^{(M+N+2)/2}}, \tag{5a}$$

and

$$a_p^r = \frac{(kd)^{2p}(-)^p(s^2-1)^{r/2}}{p!\,2^p\,\Gamma(N+3/2+p)\,r!\,(s-\cos\eta)^{p+r+(N+M)/2}}, \tag{5b}$$

where k is the wave number $2\pi/\lambda$.

Then, by substituting the following integral representation for the associated Legendre function

$$P_\nu^\mu(s) = \frac{2^\mu(s^2-1)^{-\mu/2}}{\pi^{1/2}\,\Gamma(\frac{1}{2}-\mu)} \int_0^\pi \left[s+(s^2-1)^{1/2}\cos t\right]^{\nu+\mu} (\sin t)^{-2\mu}\, dt \tag{6}$$

$$\mathrm{Re}\,\mu < 1/2$$

into (4a) and (4b), Weston showed in the same paper after some straight-forward but tedious manipulation, that the auxiliary functions $S_N^M(s,\eta)$ and $T_N^M(s,\eta)$ could be given explicitly in terms of the integral representations

$$S_N^M(s,\eta) = \frac{(-2)^M \Gamma(N+M+1)}{\pi^{1/2}\Gamma(N-M+1)\Gamma(\frac{1}{2}+M)} \int_0^\pi {}_2F_1\left(\frac{M-N}{2}, \frac{M+N+1}{2}; \frac{1}{2}+M; z\right) j_N(x)\, z^{\frac{M}{2}} (\sin t)^M \, dt \tag{7a}$$

and

$$T_N^M(s,\eta) = \frac{\Gamma(N+M+1)\sin\eta}{\pi^{1/2}(-2)^M\Gamma(N-M+1)\Gamma(\frac{1}{2}+M)(s^2-1)^{1/2}} \int_0^\pi {}_2F_1\left(\frac{N-M+1}{2}, \frac{M+N+1}{2}; \frac{1}{2}+M; z\right) j_N(x)\, z^{\frac{M+N}{2}} (\sin t)^{M-1} \, dt \tag{7b}$$

in which ${}_2F_1(a,b;c;z)$ is the usual hypergeometric function, $j_N(x)$ is the spherical Bessel function of order N and where

$$z = \frac{(s^2-1)(\sin t)^2}{2(s-\cos\eta)[s+(s^2-1)^{1/2}\cos t]} \tag{8a}$$

and

$$X = (kd)2^{1/2}\left[\frac{s+(s^2-1)^{1/2}\cos t}{s-\cos\eta}\right]^{1/2}. \tag{8b}$$

Hence, since the spherical Hankel function of the first kind $h_N^{(1)}(X)$ is given by

$$h_N^{(1)}(X) \equiv j_N(X) + i(-)^{N+1} j_{-(N+1)}(X), \tag{9}$$

then the substitution of (7a) into (3a) with $N=2\ell+M$ and the substitution of (7b) into (3b) with $N=2\ell+M+1$ led directly to the following integral representations of the toroidal wave functions $V_{M+2\ell}^{M}(s,\eta)$ and $W_{M+2\ell}^{M}(s,\eta)$:

$$V_{M+2\ell}^{M}(s,\eta) = \frac{(-)^M(2M+2\ell)!}{\pi^{1/2}2^M(2\ell)!\,\Gamma(\frac{1}{2}+M)}\int_0^\pi {}_2F_1(-\ell, M+\ell+1/2; 1/2+M; z)\, h_{M+2\ell}^{(1)}(X)\, z^{M/2}(\sin t)^M dt \tag{10a}$$

and

$$W_{M+2\ell+1}^{M}(s,\eta) = \frac{(-)^M(2M+2\ell+1)!\,\sin\eta}{\pi^{1/2}2^M(2\ell+1)!\,\Gamma(\frac{1}{2}+M)(s^2-1)^{1/2}}\int_0^\pi {}_2F_1(-\ell, M+\ell+3/2; 1/2+M; z)\, h_{M+2\ell+1}^{(1)}(X)\, z^{\frac{M+1}{2}}(\sin t)^{M-1} dt. \tag{10b}$$

In a subsequent paper,[4] Weston employed the following integral representation for the associated Legendre function

$$P_\nu^{-m}(s) = \frac{\Gamma(\nu-m+1)}{2\pi\,\Gamma(\nu+1)} \int_{-\pi}^{\pi} e^{imt}\,[s+(s^2-1)^{1/2}\cos t]^\nu\,dt\,, \quad m>0 \tag{11}$$

in place of (6) and - with only brief hints as to the course of what must have been a circuital and tedious analysis* - stated that the auxiliary function $S^M_{-M-2\ell-1}(s,\eta)$ could be written as

$$S^M_{-M-2\ell-1}(s,\eta) = \frac{(2M+2\ell)!}{2\pi(2\ell)!(kd)^M}(-)^{M+\ell}\sum_{p=0}^{\ell}\frac{(-\ell)_p(M+\ell+1/2)_p}{p!}\left(\frac{2}{kd}\right)^p(-)^p\mathcal{S}^M_{-(p+1)}(s,\eta) \tag{12**}$$

in which $\mathcal{S}^M_{-(\ell+1)}(s,\eta)$ has the integral representation

$$\mathcal{S}^M_{-(\ell+1)}(s,\eta) = \int_{-\pi}^{+\pi} e^{iMt}\,z^\ell\,f_{-(\ell+1)}(X)\,dt \tag{13}$$

and where

$$X = (kd)\,2^{1/2}\left[\frac{s+(s^2-1)^{1/2}\cos t}{s-\cos\eta}\right]^{1/2}, \tag{14a}$$

as in (8b) and

$$z = \frac{[e^{it}(s^2-1)^{1/2}+s-\cos\eta]}{2^{1/2}(s-\cos\eta)^{1/2}[s+(s^2-1)^{1/2}\cos t]^{1/2}}\,. \tag{14b}$$

---

* See Appendix A for an a posteriori justification of Weston's development of (15) and (16) as an example of part of this analysis.

** Note that the roles of r and p in (4a) have been interchanged in (12).

Similarly, Weston stated that $S^M_{M+2\ell}(s,\eta)$ could be written as

$$S^M_{M+2\ell}(s,\eta) = \frac{(2M+2\ell)!}{2\pi(2\ell)!(kd)^M}(-)^\ell \sum_{p=0}^{\ell} \frac{(-\ell)_p(M+\ell+1/2)_p}{p!}\left(\frac{2}{kd}\right)^p \mathscr{S}^M_p(s,\eta) \quad (15)$$

in which $\mathscr{S}^M_\ell(s,\eta)$ has the integral representation

$$\mathscr{S}^M_\ell(s,\eta) = \int_{-\pi}^{\pi} e^{iMt} z^\ell j_\ell(X)\,dt. \quad (16)$$

Consequently, the original toroidal wave functions $V^M_{M+2\ell}(s,\eta)$ were decomposed into the new set $\mathscr{V}^M_\ell(s,\eta)$ by means of the relationship

$$V^M_{M+2\ell}(s,\eta) = \frac{(2M+2\ell)!}{2\pi(2\ell)!(kd)^M}(-)^\ell \sum_{p=0}^{\ell} \frac{(-\ell)_p(M+\ell+1/2)_p}{p!}\left(\frac{2}{kd}\right)^p \mathscr{V}^M_p(s,\eta) \quad (17)$$

where $\mathscr{V}^M_\ell(s,\eta)$ is defined by

$$\mathscr{V}^M_\ell(s,\eta) \equiv \mathscr{S}^M_\ell(s,\eta) + i(-)^{\ell+1}\mathscr{S}^M_{-(\ell+1)} \quad (18)$$

which leads, upon substituting (13) and (16) into (18), to the following integral representation

$$\mathscr{V}^M_\ell(s,\eta) = \int_{-\pi}^{\pi} e^{iMt} z^\ell h^{(1)}_\ell(X)\,dt. \quad (19)$$

In like manner, the toroidal wave functions $W^M_{M+2\ell+1}(s,\eta)$ were decomposed into the new set $\mathscr{W}^M_\ell(s,\eta)$ by means of the relationship

$$W^{M}_{M+2\ell+1}(s,\kappa)=\frac{(2M+2\ell+1)!}{(2\ell+1)!\,2\pi\,(kd)^{M-1}}\,(-)^{\ell}\sum_{p=0}^{\ell}\frac{(-\ell)_p\,(M+\ell+3/2)_p}{p!}\left(\frac{2}{kd}\right)^{p} w^{M}_{p}(s,\kappa) \tag{20}$$

where $w^{M}_{\ell}(s,\kappa)$ has the integral representation

$$w^{M}_{\ell}(s,\kappa)=\frac{\sin\kappa}{s-\cos\kappa}\int_{-\pi}^{\pi} e^{iMt}\,\mathfrak{z}^{\ell}\,X^{-1}\,h^{(1)}_{\ell+1}(X)\,dt. \tag{21}$$

The remaining sections of this report will examine the structures of (10a) and (10b), the integral representations of Weston's first wave functions, and of (19) and (21), the integral representations of his second wave functions, following some essential geometric interpretations.

3.0 Geometric Interpretations of the Arguments X, Z and $\mathfrak{z}$.

As in Figure 1, place the $s=\infty$ ring of radius d in the x-y plane and next consider two points: a field point which is arbitrarily placed in the x-Z plane at a distance R from the coordinate origin and a source point on the ring located azimuthally by the angle τ as shown in Figure 2. If the source point - field point separation is denoted by the vector $\vec{a}$, then the x, y and Z components of $\vec{a}$ are given immediately, using the toroidal coordinates of (1), by

$$a_x=\frac{d(s^2-1)^{1/2}}{s-\cos\kappa}-d\cos\tau\,, \tag{22a}$$

$$a_y=-d\sin\tau\,, \tag{22b}$$

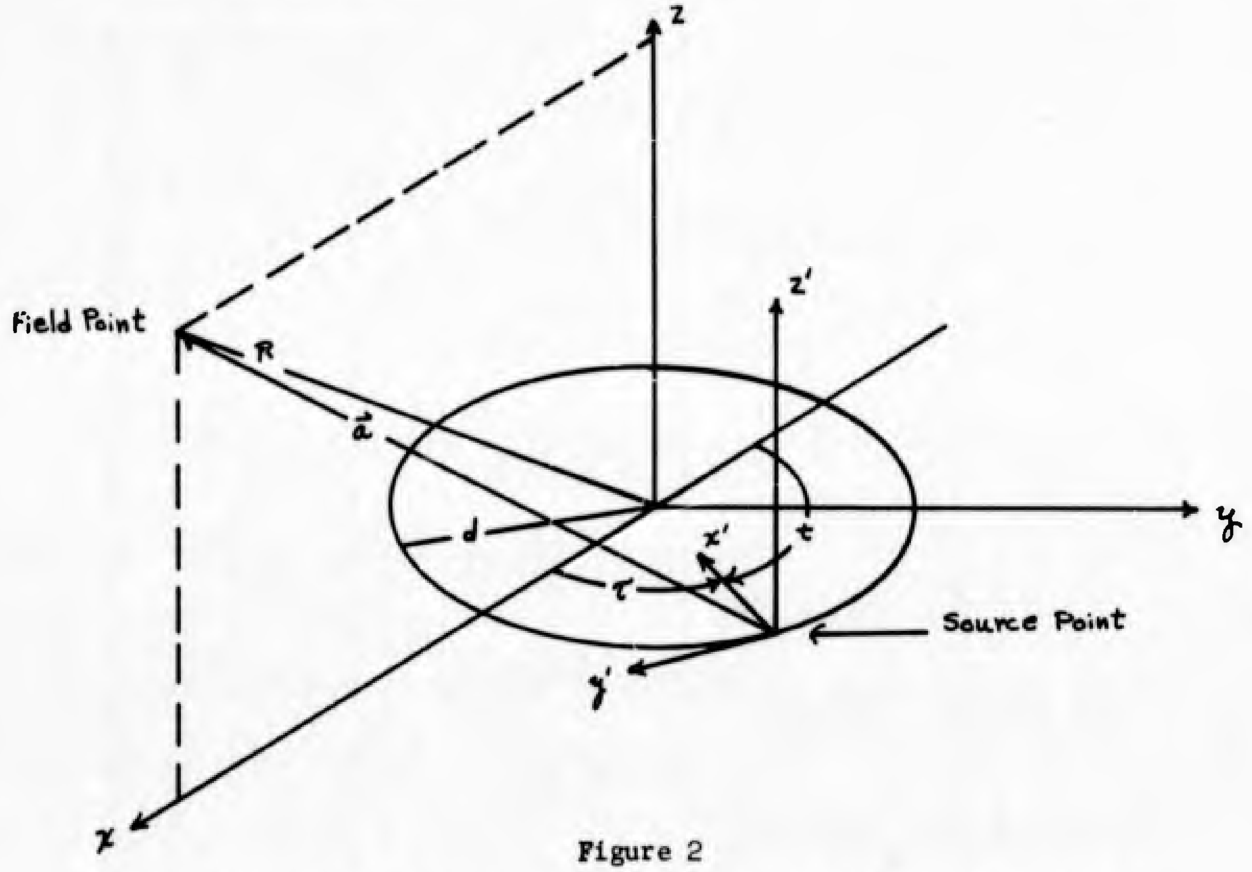


Figure 2

Coordinates and Notation for Interpretation of Wave Functions

and

$$a_z = \frac{d \sin \chi}{s - \cos \chi} \tag{22c}$$

from which $|a|$ is readily calculated to be

$$|a| = 2^{1/2} d \left[\frac{s - (s^2 - 1)^{1/2} \cos \tau}{s - \cos \chi} \right]^{1/2} . \tag{23}$$

By making the substitution $\tau = \pi - t$ - thereby locating the source point by the angle t with respect to the negative x-axis - (23) becomes

$$|a| = 2^{1/2} d \left[\frac{s + (s^2 - 1)^{1/2} \cos t}{s - \cos \chi} \right]^{1/2} , \tag{24}$$

which when substituted into either (8b) or (14a), leads directly to the identification

$$X = k|a|. \tag{25}$$

Thus X , the argument of the spherical Hankel functions in both sets of toroidal wave functions, is the product of the wave number k and the field point - source point separation. Next construct a local, right-handed coordinate system (x', y', z') at the source point such that y' is tangent to the ring, z' is perpendicular to the plane of the ring, and x' is directed radially inward as shown in Figure 2. The direction cosines of $\vec{a}$ in this local system, which has been rotated by $\pi + \tau = 2\pi - t$ with respect to the (x, y, z) system, are given by the relation

$$\begin{pmatrix} \alpha_{x'} \\ \alpha_{y'} \\ \alpha_{z'} \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x/|a| \\ a_y/|a| \\ a_z/|a| \end{pmatrix}, \tag{26}$$

or, in component form,

$$\alpha_{x'} = \frac{a_x}{|a|} \cos t - \frac{a_y}{|a|} \sin t , \tag{27a}$$

$$\alpha_{y'} = \frac{a_x}{|a|} \sin t + \frac{a_y}{|a|} \cos t , \tag{27b}$$

and

$$\alpha_z = \frac{a_z}{|a|} , \tag{27c}$$

which become, upon substituting (22) and (24) and combining terms,

$$\alpha_{x'} = \frac{(s^2-1)^{1/2}\cos t + (s - \cos\eta)}{2^{1/2}(s-\cos\eta)^{1/2}\left[s + (s^2-1)^{1/2}\cos t\right]^{1/2}} \tag{28a}$$

$$\alpha_{y'} = \frac{(s^2-1)^{1/2} \sin t}{2^{1/2}(s-\cos\eta)^{1/2}[s+(s^2-1)^{1/2}\cos t]^{1/2}} \tag{28b}$$

and

$$\alpha_{z'} = \frac{\sin\eta}{2^{1/2}(s-\cos\eta)^{1/2}[s+(s^2-1)^{1/2}\cos t]^{1/2}} \tag{28c}$$

Comparison of (8b) and (28b) leads immediately to the identification

$$Z^{1/2} = \alpha_{y'} , \tag{29}$$

or, that $Z^{1/2}$ whose square is the argument of the hypergeometric function in the first toroidal wave function set, is the cosine of the angle between the source point-field point line and the tangent to the ring at the source point.

Finally, it is noted that $\mathfrak{z}$ of (14) may be written as

$$\mathfrak{z} = \frac{(s^2-1)^{1/2}\cos t + (s-\cos\eta) + i(s^2-1)^{1/2}\sin t}{2^{1/2}(s-\cos\eta)^{1/2}[s+(s^2-1)^{1/2}\cos t]^{1/2}} \tag{30}$$

or, employing (28a) and (28b), as

$$\mathfrak{z} = \alpha_{x'} + i\,\alpha_{y'} . \tag{31}$$

Hence, if a local, spherical coordinate system θ', ϕ' is employed with

$$\alpha_{x'} = \sin\theta' \cos\phi' , \tag{32a}$$

$$\alpha_{y'} = \sin\theta' \sin\phi' = Z^{1/2} , \tag{32b}$$

and

$$\alpha_{z'} = \cos\theta' , \tag{32c}$$

then (31) becomes

$$z = \sin\theta'\cos\phi' + i\sin\theta'\sin\phi' \tag{33}$$

or

$$z = \sin\theta' e^{i\phi'} \tag{34}$$

With these geometric interpretations in hand, the structure of Weston's integral representations will now be examined.

4.0 The Structure of Weston's First Set of Toroidal Wave Functions

4.1 Interpretation of $V^{M}_{M+2\ell}(s,\kappa)$ for $M=0$ - In the course of calculating the acoustic radiation from a torus vibrating in an axially symmetric $(M=0)$ mode, $V^{0}_{2\ell}(s,\kappa)$ was observed to transform such that it could be interpreted "as an integral over a continuous distribution of spherical wave sources of order 2ℓ on a ring" $(S=\infty)$.*

This transformation is readily effected by setting $M=0$ in (10a) so that

$$V^{0}_{2\ell}(s,\kappa) = \frac{1}{\pi}\int_0^{\pi} {}_2F_1(-\ell,\ell+1/2;\,1/2;\,z)\, h^{(1)}_{2\ell}(x)\, dt. \tag{35}$$

But since**

$$P^{\mu}_{\mu+2s}(z) = \frac{(-)^s 2^{\mu}(1-z^2)^{\mu/2}\,\Gamma(s+\mu+1/2)}{\pi^{1/2}\,\Gamma(s+1)}\, {}_2F_1(-s,\, s+\mu+1/2;\, 1/2;\, z^2), \tag{36a}$$

which reduces, upon setting $s=\ell$ and $\mu=0$, to

---

* Ref. 2, p.8.

** See Ref. 6, p.216.

$$P^{\circ}_{2\ell}(z) = \frac{(-)^{\ell}(2\ell)!}{2^{2\ell}(\ell!)^2}\; {}_2F_1(-\ell, \ell+1/2;\ 1/2;\ z^2), \tag{36b}$$

then (35) becomes

$$V^{\circ}_{2\ell}(s,\eta) = \frac{(-)^{\ell}\, 2^{2\ell}(\ell!)^2}{\pi(2\ell)!} \int_0^{\pi} P^{\circ}_{2\ell}(z)\, h^{(1)}_{2\ell}(X)\, dt \tag{37a}$$

in which, as has been demonstrated in the previous section, the argument $z = Z^{1/2}$ is the cosine of the angle between the source point-field point line and the tangent to the $S = \infty$ ring at the source point,* and the argument X is the product of the wave number k and the source point-field point distance. Hence, the arguments of the Legendre function and spherical Hankel function in (37a) are physically reasonable for spherical waves emanating from the source point. Since the integrand is even in t,

$$V^{\circ}_{2\ell}(s,\eta) = \frac{(-)^{\ell}\, 2^{2\ell}(\ell!)^2}{2\pi(2\ell)!} \int_0^{2\pi} P^{\circ}_{2\ell}(z)\, h^{(1)}_{2\ell}(X)\, dt \tag{37b}$$

which led directly to the interpretation cited in Ref. 2

4.2 Interpretation of $V^{M}_{M+2\ell}(s,\eta)$ for $M=1$ - Similarly, by setting $M=1$ in (10a) so that

---

* It should be noted in passing that $Z^{1/2} = \sin\theta' \sin\phi'$ leads naturally to the value of $Z \sim \sin^2\theta \sin^2 t$, the far field approximation for Z given in (7.3) of Ref. 3. In this approximation $\vec{r}$ is parallel to $\vec{R}$ from which follows $\theta' \to \theta$ and $\phi' = t$.

$$V'_{2\ell+1}(s,\eta) = (-)\frac{(2\ell+2)!}{\pi(2\ell)!}\int_0^{\pi} \sin t \; {}_2F_1(-\ell, \ell+3/2; 3/2; z)\, z^{1/2}\, h^{(1)}_{2\ell+1}(x)\, dt\,, \tag{38}$$

and noting that*

$$P^{\mu}_{\mu+2s+1}(z) = \frac{(-)^{s} 2^{\mu+1}(1-z^2)^{\mu/2}\,\Gamma(s+\mu+3/2)}{\pi^{1/2}\,\Gamma(s+1)}\, z\; {}_2F_1(-s, s+\mu+3/2; 3/2; z^2) \tag{39a}$$

which reduces, upon setting $s=\ell$ and $\mu=0$ to

$$P^{0}_{2\ell+1}(z) = \frac{(-)^{\ell}(2\ell+1)!}{2^{2\ell}(\ell!)^2}\, z\; {}_2F_1(-\ell, \ell+3/2; 3/2; z^2) \tag{39b}$$

then (38) becomes, since again the integrand is even in t ,

$$V'_{2\ell+1}(s,\eta) = \frac{(-)^{\ell+1}\, 2^{2\ell+1}(\ell+1)!\,\ell!}{2\pi\,(2\ell)!}\int_0^{2\pi} \sin t \; P^{0}_{2\ell+1}(z)\, h^{(1)}_{2\ell+1}(x)\, dt \tag{40}$$

which leads to the interpretation of $V'_{2\ell+1}(s,\eta)$ as an integral over the $s=\infty$ ring of a continuous distribution of spherical wave sources of order $2\ell+1$ with strengths "modulated" by the factor $\sin t$.

The presence of this "modulation factor" $\sin t$ is an essential feature of $V'_{2\ell+1}(s,\eta)$ which describes only the s,η dependence of the total toroidal wave function $e^{\pm i\phi}\, V'_{2\ell+1}(s,\eta)$. For if there is to be a ϕ-dependent field produced by the spherical sources distributed over the ring, then this ϕ-dependence must derive from a similar ϕ-dependence (or, equivalently, a t-dependence) of the source strengths themselves. Hence, in the axially symmetric case of (37b) where

---

* Ref. 6, p.216

$|m| = M = 0$, no modulation factor is required since $V^0_{2\ell}(s,\eta)$ is in fact the total wave function; in the case at hand, however, where $M=1$, the modulation factor is necessary.

4.3 Structure of $W^M_{M+2\ell+1}(s,\eta)$ for $M=0,1$ - Now, by setting $M=0$ in (10b) so that

$$W^0_{2\ell+1}(s,\eta) = \frac{\sin\eta}{(s^2-1)^{1/2}}\,\frac{1}{\pi}\int_0^{\pi} z^{1/2}\,{}_2F_1(-\ell,\ell+3/2,\,1/2;\,z)\,h^{(1)}_{2\ell+1}(X)\,(\sin t)^{-1}\,dt\,, \tag{41}$$

and setting $s=\ell$ and $\mu=1$ in (36a) which reduces to

$$P^1_{2\ell+1}(z) = \frac{(-)^{\ell}(2\ell+1)!}{2^{2\ell}(\ell!)^2}(1-z^2)^{1/2}\,{}_2F_1(-\ell,\ell+3/2;\,\tfrac{1}{2};\,z^2), \tag{42}$$

then (41) becomes, since the integrand is even in t,

$$W^0_{2\ell+1}(s,\eta) = \frac{(-)^{\ell}2^{2\ell}(\ell!)^2}{2\pi(2\ell+1)!}\int_0^{2\pi}\frac{P^1_{2\ell+1}(z)}{(1-z^2)^{1/2}}\,h^{(1)}_{2\ell+1}(X)\,\frac{z\sin\eta}{(s^2-1)^{1/2}\sin t}\,dt\,. \tag{43}$$

Similarly, by setting $M=1$ in (10b) so that

$$W^1_{2\ell+2}(s,\eta) = (-)\frac{(2\ell+3)!\sin\eta}{(2\ell+1)!\,(s^2-1)^{1/2}}\cdot\frac{1}{\pi}\int_0^{\pi} z\,{}_2F_1(-\ell,\ell+5/2,\,3/2;\,z)\,h^{(1)}_{2\ell+2}(X)\,dt \tag{44}$$

and setting $s=\ell$ and $\mu=1$ in (39a) which reduces to

$$P^1_{2\ell+2}(z) = \frac{(-)^{\ell}(2\ell+3)(2\ell+1)!}{2^{2\ell}(\ell!)^2}(1-z^2)^{1/2}z\,{}_2F_1(-\ell,\ell+5/2;\,3/2;\,z^2), \tag{45}$$

then

$$W^1_{2\ell+2}(s,\eta) = \frac{(-)^{\ell+1}2^{2\ell+1}(\ell+1)!\,\ell!}{2\pi(2\ell+1)!}\int_0^{2\pi}\sin t\,\frac{P^1_{2\ell+2}(z)}{(1-z^2)^{1/2}}\,h^{(1)}_{2\ell+2}(X)\,\frac{z\sin\eta}{(s^2-1)^{1/2}\sin t}\,dt\,. \tag{46}$$

However, the relations for z and $\cos\theta'$ in (28b) and (28c) show that

$$\frac{z \sin\eta}{(s^2-1)^{1/2} \sin t} = \cos\theta', \tag{47}$$

with which (43) and (46) become

$$W^0_{2\ell+1}(s,\eta) = \frac{(-)^{\ell}\, 2^{2\ell} (\ell!)^2}{2\pi\, (2\ell+1)!} \int_0^{2\pi} \frac{P'_{2\ell+1}(z)}{(1-z^2)^{1/2}}\, h^{(1)}_{2\ell+1}(X) \cos\theta' \, dt, \tag{48a}$$

and

$$W^1_{2\ell+2}(s,\eta) = \frac{(-)^{\ell+1}\, 2^{2\ell+1} (\ell+1)!\,\ell!}{2\pi\, (2\ell+1)!} \int_0^{2\pi} \sin t\, \frac{P'_{2\ell+2}(z)}{(1-z^2)^{1/2}}\, h^{(1)}_{2\ell+2}(X) \cos\theta' \, dt. \tag{48b}$$

As will be demonstrated presently, these wave functions, despite the structural similarities between $W^0_{2\ell+1}(s,\eta)$ and $W^1_{2\ell+2}(s,\eta)$ on the one hand and $V^0_{2\ell}(s,\eta)$ and $V^1_{2\ell+1}(s,\eta)$ on the other, cannot be simply interpreted as an integral over a continuous distribution of spherical wave sources except in the special case $\ell = 0$. This breakdown in interpretation is caused primarily by the presence of two distinct-and reasonable-coordinate systems in the integral representations.

The first of these systems is based on the identification of the argument of the Legendre Functions in (48) - as well as in (37b) and (40) - with the cosine of the angle between the tangent to the ring at the source point and the source point-field point separation. Hence a reasonable orientation of the local coordinate system would place the local polar axis along this tangent, the local x-axis perpendicular to the plane of the ring and the local y-axis radially inwards, so as to complete the right handed specification, thus identifying z with the cosine of the polar angle in this local system.* The second local coordinate system is based on the identification of the angle θ' in the $\cos\theta'$ factor of (48) as the polar angle in a system - the one described after (25) - with the local z-axis perpendicular to the plane of the ring (and thus parallel to the "master" z-axis), the local x-axis directed radially inward and the local y-axis tangent to the ring. That this coordinate system is also reasonable can be seen from the following interpretable $\ell = 0$ specializations of (48a) and (48b).

By setting $\ell = 0$ in (48a) and noting that $P_1'(z) = (1-z^2)^{1/2}$, then (48a) becomes

$$W_1^0(s,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta' \, h_1^{(1)}(x)\, dt \tag{49a}$$

or, since $P_1^0(\cos\theta') = \cos\theta'$

$$W_1^0(s,\eta) = \frac{1}{2\pi}\int_0^{2\pi} P_1^0(\cos\theta')\, h_1^{(1)}(x)\, dt. \tag{49b}$$

Similarly, by setting $\ell = 0$ in (48b) and noting that $P_2'(z) = 3z(1-z^2)^{1/2}$, then (48b) becomes, upon substituting $z = \sin\theta' \sin\phi'$ from (32b),

---

* Note that this coordinate system (local polar axis tangent to the ring etc.) behaves strangely in the $d \to 0$ limiting process. For all of the source point local polar axes, which are tangentially oriented around the ring, must be re-oriented perpendicular to the ring plane in order to coalesce into the "master" z-axis in the limit.

$$W_2^1(s,\eta) = \frac{2}{2\pi}\int_0^{2\pi} \sin t \ \ \sin\phi' \,(3\sin\theta'\cos\theta')\, h_2^{(1)}(X)\, dt \tag{50a}$$

or,

$$W_2^1(s,\eta) = \frac{2}{2\pi}\int_0^{2\pi} \sin t \left[\sin\phi' \, P_2^1(\cos\theta')\right] h_2^{(1)}(X)\, dt. \tag{50b}$$

Since $\sin\phi' P_2^1(\cos\theta')$ is the part of the (unnormalized) complex spherical harmonic which is odd in t, and since the even part will integrate to zero, (50b) can be written

$$W_2^1(s,\eta) = \frac{2}{2\pi i}\int_0^{2\pi} \sin t \ Y_2^1(\theta',\phi')\, h_2^{(1)}(X)\, dt. \tag{50c}$$

Consequently, it is seen that each of the extreme specializations $W_1^0(s,\eta)$ and $W_2^1(s,\eta)$ can be interpreted as an integral over the $s = -0$ ring of a continuous distribution of modulated spherical wave sources (whose arguments are referred to the (θ', ϕ') spherical coordinate system).

It should be noted at this point, for completeness, that the V-functions of (37b) and (40) can also be recast, in terms of the (θ', ϕ') coordinate system for this extreme specialization. For, by setting $\ell = 0$ in these two relations and observing that $P_0(z) = 1 = P_0(\cos\theta')$ in (37b) and $P_1^0(z) = z = \sin\phi' \sin\theta' = \sin\phi' P_1'(\cos\theta') = Y_1'(\theta',\phi')$, then

$$V_0^0(s,\eta) = \frac{1}{2\pi}\int_0^{2\pi} P_0^0(\cos\theta')\, h_0^{(1)}(X)\, dt \tag{51a}$$

and

$$V_1'(s,\eta) = \frac{2}{2\pi i}\int_0^{2\pi} \sin t \; Y_1'(\theta',\phi')\, h_1^{(1)}(x)\, dt\,, \tag{51b}$$

respectively; these are, of course, interpreted in exactly the same manner as $W_1^{\circ}(s,\eta)$ and $W_2'(s,\eta)$.

In view of this modest success in interpreting the $\ell = 0$, $M = 0, 1$ W-functions (and the $\ell = 0$, $M = 0, 1$ V-functions) after having transformed in effect, the arguments of the Legendre functions from a coordinate system in which the polar angle is referenced to the ring tangent to the (θ', ϕ') coordinate system, an attempt was made to effect this transformation directly in (48) by employing the general trans formation formula for spherical harmonics.* This attempt did not lead to useful results, for it became apparent in the course of this analysis that, except for the special $\ell = 0$ cases already discussed, the presence of the factor $(1-z^2)^{1/2}$ prevented the complete interpretation of the integrands.** Consequently, no further effort was expended on the W-functions.

4.4 Structure of $V^M_{M+2\ell}(s,\eta)$ for M Arbitrary - If $M \neq 0, 1$ in (10a), the conditions necessary for the reduction of the hypergeometric function to a Legendre function are not present. This follows from the following: whenever Gauss' hypergeometric series admits of a quadratic transformation, the hypergeometric equation

* Ref. 7, p.17.

** Since $(1-z^2)^{1/2}$ is not present in $V^{\circ}_{2\ell}(s,\eta)$ and $V'_{2\ell+1}(s,\eta)$ their integrands can be transformed entirely into the (θ',ϕ') system; this was not done because a satisfying interpretation based on structural simplicity was already in hand.

reduces to Legendre's Equation;* and, considering the hypergeometric function $F(a, b; c; z)$, that the necessary and sufficient conditions for the existence of a quadratic transformation are that any two of the numbers $\pm(1-c)$, $\pm(a-b)$ and $\pm(a+b-c)$ must be equal or any one must equal 1/2;** and that examination of the hypergeometric function ${}_2F_1(-\ell, \ell+M+1/2; 1/2+M; z)$ of $V^M_{M+2\ell}(s, \eta)$ (and also, incidentally, of the hypergeometric function ${}_2F_1(-\ell, \ell+M+3/2; 1/2+M; z)$ of $W^M_{M+2\ell+1}(s, \eta)$) shows that only $M=0$ and $M=1$ satisfy these conditions. However, although it was recognized that a manipulation of the integrand of $V^M_{M+2\ell}(s, \eta)$ would not lead to a "simple" Legendre function, it was hoped that the result might be interpretable (in either coordinate system). That this hope was unrealistic is demonstrated by the following analysis, which is included for completeness.

Using the relation***

$$ {}_2F_1(a, b; c+n; z) = \frac{(c)_n}{(c-a)_n (c-b)_n} (1-z)^{n+c-a-b} \frac{d^n}{dz^n}\left[(1-z)^{a+b-c}\, {}_2F_1(a, b; c; z)\right] \tag{52a} $$

which becomes, upon substituting $a=-\ell$, $b=\ell+M+1/2$, $c=1/2$, and $n=M$,

$$ {}_2F_1(-\ell, \ell+M+1/2; 1/2+M; z) = \frac{\Gamma(\frac{1}{2}+\ell)\Gamma(-\ell-M)\Gamma(\frac{1}{2}+M)}{\Gamma(\frac{1}{2}+\ell+M)\Gamma(-\ell)\Gamma(\frac{1}{2})} \frac{d^M}{dz^M}\left[(1-z)^M\, {}_2F_1(-\ell, \ell+M+1/2; 1/2; z\right] \tag{52b} $$

---

* Ref. 8, p.121
** Ibid, p.64
*** Ibid, p.102

and (36a) which, upon substituting $\mu = M$ and $s = \ell$ becomes,

$$ {}_2F_1\left(-\ell, \ell+M+{}^1\!/_2; {}^1\!/_2; \zeta^2\right) = (-)^\ell \frac{\pi^{1/2}\,\Gamma(\ell+1)}{2^M (1-\zeta^2)^{M/2}\,\Gamma(\ell+M+{}^1\!/_2)} P^M_{M+2\ell}(\zeta) $$

then (10a) can be written, after some tedious, but straightforward, manipulation of the Γ-functions, as

$$ V^M_{M+2\ell}(s,\eta) = \frac{(-)^\ell\, 2^{2\ell+2M}\, \ell!\,(\ell+M)!}{\pi\,(2\ell+2M)!} \int_0^\pi \sin^M t\; \zeta^{M/2} \frac{d^M}{d\zeta^M}\left[(1-\zeta^2)^{M/2} P^M_{M+2\ell}(\zeta)\right] h^{(1)}_{M+2\ell}(X)\, dt \tag{53} $$

the integrand of which can be transformed in two steps.

First, since*

$$ \frac{d}{dz}\left[(1-z^2)^{\frac{m}{2}} P^m_n(z)\right] = -(n-m+1)(n+m)(1-z^2)^{\frac{m-1}{2}} P^{m-1}_n(z) \tag{54a} $$

then it follows, after $r-1$ differentiations that

$$ \frac{d^r}{dz^r}\left[(1-z^2)^{m/2} P^m_n(z)\right] = (-)^r \frac{(n-m+r)!}{(n-m)!} \frac{(n+m)!}{(n+m-r)!} (1-z^2)^{\frac{m-r}{2}} P^{m-r}_n(z), \tag{54b} $$

and second, since

$$ \frac{d^r}{d\zeta^r} f(z) \equiv \frac{d^r}{d(z^2)^r} f(z) \equiv \left(\frac{2}{z}\frac{d}{dz}\right)^r f(z) = \frac{1}{2^r z^r} \sum_{q=0}^{r-1} \frac{(-)^q}{2^q} \frac{(r+q-1)!}{(r-q-1)!\,q!} \frac{1}{z^q} \frac{d^{r-q}}{dz^{r-q}} f(z), \tag{55} $$

---

* Ref. 9, p.1326

then, for $M > 0$,

$$\frac{d^M}{dz^M}\left[(1-z^2)^{M/2} P^M_{M+2\ell}(z)\right] = \frac{(-)^M (2\ell+2M)!}{2^M (2\ell)!} z^{-M/2} \sum_{q=0}^{M-1} \frac{(M-1+q)!(M+2\ell-q)!\,2^{-q}}{(M-1-q)!(M+2\ell+q)!\,q!} \left[\frac{(1-z^2)^{1/2}}{z}\right]^q P^q_{M+2\ell}(z). \tag{56}$$

Hence, (53) becomes, since the integrand is even in t,

$$V^M_{M+2\ell}(s,\kappa) = \frac{(-)^{\ell+M}\, 2^{M+2\ell}\, \ell!(\ell+M)!}{(2\ell)!\, 2\pi} \int_0^{2\pi} \sin^M t \left\{\sum_{q=0}^{M-1} \frac{(M-1+q)!(M+2\ell-q)!\,2^{-q}}{(M-1-q)!(M+2\ell+q)!\,q!} \left[\frac{(1-z^2)^{1/2}}{z}\right]^q P^q_{M+2\ell}(z)\right\} h^{(1)}_{M+2\ell}(\chi)\, dt. \tag{57}$$

Here again, the presence of the factor $(1-z^2)^{q/2}$ apparently prevents any further useful transformation of the integrand and, hence, an interpretation of the general V-function in terms of spherical wave functions.

5.0 The Structure of Weston's Second Set of Toroidal Wave Functions.

5.1 Interpretation of $\nu^M_\ell(s,\kappa)$ - As (14) indicates, the $V^M_{M+2\ell}(s,\kappa)$ can be written as a linear combination of the new wave functions $\nu^M_\ell(s,\kappa)$ in the form

$$V^M_{M+2\ell}(s,\eta) = \frac{(-)^\ell (2M+2\ell)!}{2\pi (2\ell)!\,(kd)^M} \sum_{p=0}^{\ell} a^\ell_p\, v^M_p(s,\eta) \tag{58}$$

where

$$a^\ell_p = \frac{(-\ell)_p\,(M+\ell+1/2)_p}{p!} \left(\frac{2}{kd}\right)^p \tag{59}$$

and where, following (19)

$$v^M_\ell(s,\eta) = \int_{-\pi}^{\pi} e^{iMt} z^\ell h^{(1)}_\ell(x)\,dt. \tag{60}$$

Thus, for example, $V^M_{M+2\ell}(s,\eta)$ becomes, for $\ell = 0$,

$$V^M_M(s,\eta) = \frac{1}{2\pi}\frac{(2M)!}{(kd)^M}\int_{-\pi}^{\pi} e^{iMt} h^{(1)}_0(x)\,dt \tag{61a}$$

and, for $\ell = 1$

$$V^M_{M+2}(s,\eta) = -\frac{1}{2\pi}\frac{(2M+2)!}{2!\,(kd)^M}\left\{\int_{-\pi}^{\pi} e^{iMt} h^{(1)}_0(x)\,dt - \frac{2M+3}{kd}\int_{-\pi}^{\pi} e^{iMt} z\, h^{(1)}_1(x)\,dt\right\} \tag{61b}$$

If the relation for z, given by (34) is employed then (60) becomes

$$v^M_\ell(s,\eta) = \int_{-\pi}^{\pi} e^{iMt} e^{i\ell\phi'} (\sin\theta')^\ell\, h^{(1)}_\ell(x)\,dt. \tag{62a}$$

But since (apart from multiplicative factors), $e^{il\phi'}(\sin\theta')^{l} \sim e^{il\phi'}P_l^l(\cos\theta') \sim Y_l^l(\theta',\phi')$, then (62a) is seen to be of the form

$$\mathscr{v}_{2l}^{M}(s,\eta) \sim \int_{-\pi}^{\pi} e^{iMt}\, Y_l^l(\theta',\phi')\, h_l^{(1)}(X)\, dt\,; \qquad (62b)$$

which is readily interpreted as an integral over the $s=\infty$ ring of a continuous distribution of spherical wave sources (whose arguments are referred to the (θ',ϕ') coordinate system) of order l and degree l modulated by e^{iMt}. Since the $\mathscr{v}_l^M(s,\eta)$ can all be interpreted in this way, it is clear now that, although $V_{M+2l}^{M}(s,\eta)$ cannot generally be interpreted as a single integral over a distribution of spherical wave sources, it can be interpreted as a sum of such integrals.

5.2 Interpretation of $\mathscr{w}_l^M(s,\eta)$ - Similarly, if one inserts into (21) the relation for $\mathfrak{z}$, given by (34) and observes, using (24) and (25) and (28c), that

$$\frac{\sin\eta}{(s-\cos\eta)X} = \frac{\cos\theta'}{kd}\,, \qquad (63)$$

then

$$\mathscr{w}_l^M(s,\eta) = \int_{-\pi}^{\pi} e^{iMt}\, e^{il\phi'}\, \frac{\cos\theta'}{kd}\,(\sin\theta')^{l}\, h_{l+1}^{(1)}(X)\, dt\,. \qquad (64a)$$

But since (apart from multiplcative factors). $e^{il\phi'}\cos\theta'(\sin\theta')^{l} \sim e^{il\phi'}P_{l+1}^{l}(\cos\theta') \sim Y_{l+1}^{l}(\theta',\phi')$ then (64a) is seen to be of the form

$$\mathscr{w}_l^M(s,\eta) \sim \int_{-\pi}^{\pi} e^{iMt}\, Y_{l+1}^{l}(\theta',\phi')\, h_{l+1}^{(1)}(X)\, dt \qquad (64b)$$

which is interpreted as an integral over the $s=\infty$ ring of a continuous distribution of spherical wave sources (whose arguments are referred to the (θ', ϕ') coordinate system) of order $\ell+1$ and degree ℓ modulated by e^{iMt}. Thus the $\mathcal{N}_\ell^M(s,\eta)$ and the $\mathcal{W}_\ell^M(s,\eta)$ can all be interpreted in terms of spherical waves.

A curious feature of the second set of wave functions appears when they are ordered as shown in Table 1 following the usual ordering of the complex spherical harmonic $Y_\ell^{m'}(\theta', \phi')$. Table 1 uses the notation

$$T_{\ell,M}^{m'}(s,\eta) = \int_{-\pi}^{\pi} e^{iMt}\, Y_\ell^{m'}(\theta', \phi')\, h_\ell^{(1)}(x)\, dt\,, \quad -\ell \leq m' \leq \ell\,, \tag{65}$$

such that

$$\mathcal{N}_\ell^M(s,\eta) \sim T_{\ell,M}^{\ell}(s,\eta), \tag{66a}$$

and

$$\mathcal{W}_\ell^M(s,\eta) \sim T_{\ell+1,M}^{\ell}(s,\eta); \tag{66b}$$

but the other $T_{\ell,M}^{m'}$ are not included in Weston's set of wave functions. In the light of the spherical wave interpretation the $T_{\ell,M}^{m'}$ functions for $m' \neq \ell$ or $\ell-1$ appear to be legitimate toroidal wave functions of the same general character as the $\mathcal{N}_\ell^M$ and $\mathcal{W}_\ell^M$. Weston has shown[4] that for large s, $\mathcal{N}_\ell^M$ and $\mathcal{W}_\ell^M$ become proportional to $\cos \ell\eta$ and $\sin(\ell+1)\eta$, respectively. Thus the $\mathcal{N}_\ell^M$ and $\mathcal{W}_\ell^M$ functions form a complete set on the surface of a sufficiently thin torus. The question of completeness under more general conditions is still under investigation.

Table 1

Ordering of $v_\ell^M(s,\eta)$ and $w_\ell^M(s,\eta)$ for $\ell = 0,1,2,3$.

$Y_\ell^{m'}$	$T_{\ell,M}^{m'}$	$v_\ell^M,\ w_\ell^M$
Y_0^0	$T_{0,M}^0$	v_0^M
Y_1^{-1}	$T_{1,M}^{-1}$	—
Y_1^0	$T_{1,M}^0$	w_0^M
Y_1^1	$T_{1,M}^1$	v_1^M
Y_2^{-2}	$T_{2,M}^{-2}$	—
Y_2^{-1}	$T_{2,M}^{-1}$	—
Y_2^0	$T_{2,M}^0$	—
Y_2^1	$T_{2,M}^1$	w_1^M
Y_2^2	$T_{2,M}^2$	v_2^M
Y_3^{-3}	$T_{3,M}^{-3}$	—
Y_3^{-2}	$T_{3,M}^{-2}$	—
Y_3^{-1}	$T_{3,M}^{-1}$	—
Y_3^0	$T_{3,M}^0$	—
Y_3^1	$T_{3,M}^1$	—
Y_3^2	$T_{3,M}^2$	w_2^M
Y_3^3	$T_{3,M}^3$	v_3^M

6.0 Identification of the Ring Motions Associated with the Toroidal Wave Functions

It has been found in (62b) and (64b) that each member of Weston's second set of toroidal wave functions can be interpreted as an integral over the $s = \infty$ ring of a continuous, modulated distribution of spherical wave sources. Thus each wave function represents the field produced by spherical sources of given order distributed around a ring. In acoustic applications each source can be considered a small sphere undergoing a specific motion, and in some cases such moving spheres on a ring can be associated with easily visualizable motions of a thin, solid, elastic torus.

For example, the lowest order wave function,

$$\mathcal{N}_0^M (s, \eta) = \int_{-\pi}^{\pi} e^{iMt} h_0(X)\, dt, \tag{67a}$$

represents the (s, η) dependence of the sound field radiated by pulsating spheres (monopoles) on a ring with their strengths modulated by e^{iMt}. The unmodulated case,

$$\mathcal{N}_0^0 (s, \eta) = \int_{-\pi}^{\pi} h_0(X)\, dt, \tag{67b}$$

which gives the sound field radiated by a ring of spheres each pulsating with the same amplitude, also gives the sound field radiated by a uniformly pulsating thin torus.

The toroidal wave functions related to first order spherical harmonics are the easiest to associate with motions of a torus because they include the rigid body motions. Thus the function

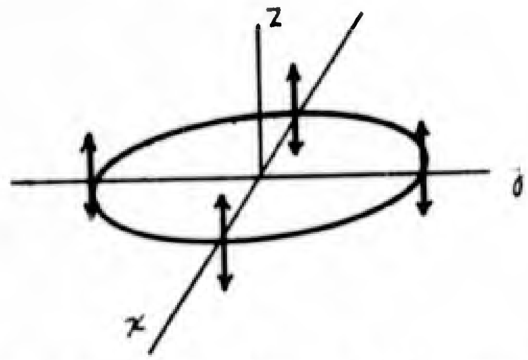
$$w_o^M(s,\eta) = \int_{-\pi}^{\pi} e^{iMt}\, P_1(\cos\theta')\, h_1(x)\, dt \tag{68a}$$

represents the (s,η) dependence of the sound field radiated by a ring of oscillating spheres (dipoles) with their vibration axes perpendicular to the plane of the ring. When the whole ring vibrates vertically as shown in Fig. 3a (rigid body translation along z -axis) the sound field it produces is given by (68a) with $M = 0$.

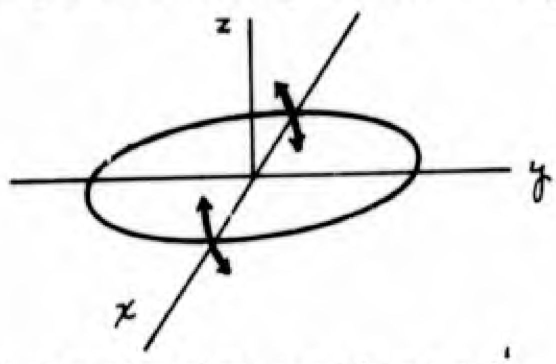
For $M = 1$ (68a) gives

$$w_o^1(s,\eta) = \int_{-\pi}^{\pi} e^{it}\, P_1(\cos\theta')\, h_1(x)\, dt. \tag{68b}$$

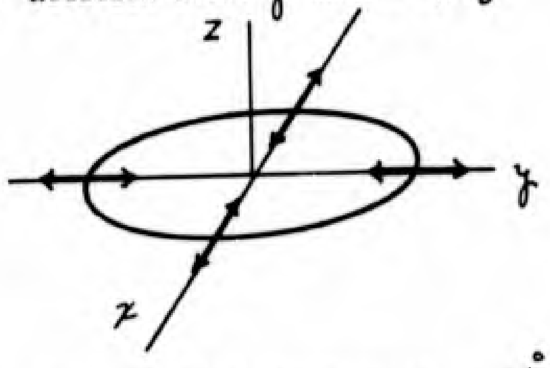
The part of (68b) involving $\sin t$ vanishes because the integrand is an odd function of t . The other part gives the (s,η) dependence of the sound field which results when the vertical motion of the ring is modulated by $\cos t$, in other words when the ring wobbles about the y -axis as shown in Fig. 3b (rotation about y -axis). Note that in this case the appropriate ϕ dependence of the field is $\cos\phi$, and the total wave function is $\cos\phi\, w_o^1(s,\eta)$. The similar motion in which the ring wobbles about the x -axis (rotation about x -axis) produces the wave function $\sin\phi\, w_o^1(s,\eta)$. Thus the function $w_o^1(s,\eta)$ is, in one case, the field in the x-z plane, while, in another case it is the field in the y-z plane. However, for the present discussion the functions must be regarded as the field in the x-z plane, since X , θ' and ϕ' were previously interpreted for a field point in the x-z plane and for t measured from the negative x-axis (see Fig. 2). This



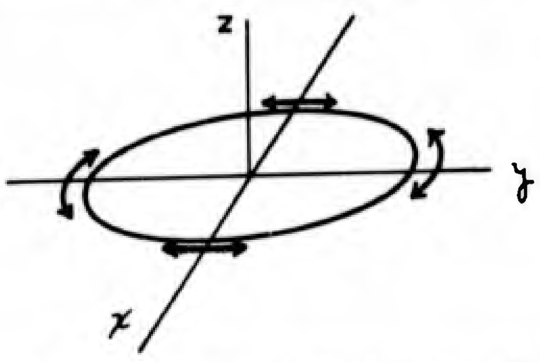
(a) Translation along z-axis — w_0^0



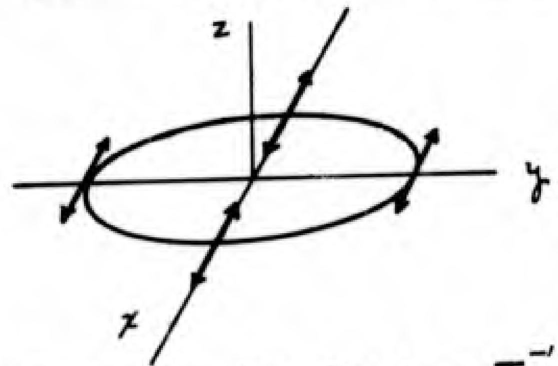
(b) Rotation about y-axis — w_0^1



(c) Uniform radial stretching — v_1^0



(d) Rotation about z-axis (non-radiating) — v_1^1



(e) Translation along x-axis — $T_{1,1}^{-1}$

Figure 3

Ring Motions Associated with the Toroidal Wave Functions which are Related to First Order Spherical Wave Functions is no loss of generality, but it is part of the structure of the wave functions which affects the details of their physical interpretation. For example, the vanishing of the part of (68b) involving $\sin t$ can be understood physically, since the $\sin t$ modulation corresponds to a ring wobbling about the x-axis which produces zero sound field in the $x-z$ plane. Note also that $w_0^M(s,\eta)$ vanishes in the $x-y$ plane for all M, because it is an odd function of η.

The next wave function in this group,

$$N_1^M(s,\eta) = \int_{-\pi}^{\pi} e^{iMt} P_1^1(\cos\theta')e^{i\phi'} h_1(x)\,dt$$

$$= \int_{-\pi}^{\pi} e^{iMt} \sin\theta' \cos\phi' \, h_1(x)\,dt + i\int_{-\pi}^{\pi} e^{iMt} \sin\theta' \sin\phi' \, h_1(x)\,dt, \tag{69a}$$

is the field radiated by other ring motions made up from oscillating spheres with their vibration axes in the plane of the ring. The first term in (69a) corresponds to vibration axes in the radial direction, since $\sin\theta' \cos\phi'$ is the cosine of the angle measured from the x' axis in Fig. 2. For $M=0$, this is a uniform stretching motion as illustrated in Fig. 3c. The second term in (69a) is associated with vibration axes tangent to the ring, and, for $M=0$, this corresponds to a rotational vibration of the ring in its own plane as depicted in Fig. 3d (rotation about z-axis). This latter motion has no velocity component normal to the ring, and therefore it radiates no sound. Consistent with no radiation, the second integral in (69a) vanishes for $M=0$, because of its odd integrand. Note that

$$\int_{-\pi}^{\pi} \sin\theta' \sin\phi' \, h_1(x)\, dt = 0,$$

is associated with a ring motion which produces zero sound field everywhere, while the vanishing integral from (68b),

$$\int_{-\pi}^{\pi} \sin t \; P_1(\cos\theta')\, h_1(x)\, dt = 0,$$

is associated with a ring motion which produces zero sound field in the x- z plane.

For $M=1$, (69a) gives, omitting the terms which integrate to zero,

$$N_1'(s,\eta) = \int_{-\pi}^{\pi} \cos t \; \sin\theta' \cos\phi' \, h_1(x)\, dt - \int_{-\pi}^{\pi} \sin t \; \sin\theta' \sin\phi' \, h_1(x)\, dt. \qquad (69b)$$

This expression represents a distribution of spheres some of which are oscillating radially, modulated by $\cos t$, while the others are oscillating tangentially, modulated by $-\sin t$; it is associated with a non-rigid motion of the ring.

The remaining wave function in the second group of Table 1 is

$$T_{1,M}^{-}(s,\eta) = \int_{-\pi}^{\pi} e^{iMt} \, P_1'(\cos\theta')\, e^{-i\phi'} \, h_1(x)\, dt. \qquad (70a)$$

For $M=0$ (70a) is associated with essentially the same ring motions as (69a). The only difference is that the terms which correspond to rotational vibration in the plane of the ring have opposite signs in the two cases.

However, for $M=1$, (70a) gives, again omitting the terms which integrate to zero,

$$T_{1,1}^{-1}(s,\chi) = \int_{-\pi}^{\pi} \cos t \sin\theta' \cos\phi' \, h_1(X)\,dt + \int_{-\pi}^{\pi} \sin t \sin\theta' \sin\phi' \, h_1(X)\,dt. \qquad (70b)$$

This expression, which differs from (69b) only in that both terms here have the same sign, can be interpreted with the help of Fig. 2 as follows: a point on the ring at t undergoes radial motion (positive inward) with amplitude $\cos t$ and simultaneously undergoes tangential motion (positive clockwise) with amplitude $\sin t$. In the resultant motion every point of the ring moves parallel to the x-axis with the same amplitude. Thus the whole ring vibrates in its own plane parallel to the x-axis as shown in Fig. 3e (translation along x-axis).

This kind of rigid body motion, when parallel to the y-axis (translation along y-axis), is connected with the terms which vanished from (69b), because such motion produces zero sound field in the x-z plane. Similarly, the kind of motion given by (69b) can be associated with the terms which vanished from (70b). However, if only (69b) were available the wave functions would include only one of these two kinds of motion. The translational motions associated with (69b) and (70b) complete the list of rigid body motions (three translations and three rotations). No attempt will be made to describe the motions associated with the higher order wave functions.

The interpretation of the toroidal wave functions in terms of the ring motions which produce them is helpful in physical applications.

Earlier work on acoustic radiation from a thin torus[1,2] involved those wave functions of Weston's which are even in η for $M=0$. The motion of the thin torus which was specified as an approximate model for free-flooding ring transducers was a uniform pulsation coupled with the uniform radial stretching illustrated in Fig. 3c. Application of these boundary conditions showed that only the functions for $\ell=0$ and $\ell=1$ were required for a thin torus. This result is clear in the light of the physical interpretation given here since the pulsation produces v_0^0, while the radial stretching produces v_1^0.*

7.0 Conclusions

This study of Weston's toroidal wave functions has led to an interpretation of the wave functions which is helpful in the analysis of physical problems. In addition the interpretation suggests the possibility that wave functions for other coordinate systems in which the Helmholtz equation does not separate could be constructed in a simple way. It would be imagined that spherical wave sources were distributed over the spatial singularities natural to the given coordinate system. The fields produced by these sources at a given field point would then be integrated, and the result would be an integral representation for a wave function in the given coordinate system.

Acknowledgements

The writers are grateful to N.G. Parke, L. Calabi, J. Riley, W. Hartnett and J.L. Butler for stimulating discussions during the course of this study.

* Weston's first set of wave functions were used in references 1 and 2. However, V_0^0 is proportional to v_0^0 and V_1^0 is a linear combination of v_0^0 and v_1^0.

Appendix A

A Posteriori Justification of Weston's Second Integral Representation of $S^{M}_{M+2\ell}(s,\eta)$.

If the integral representation of the Legendre function (11) and (5a) and (5b) are inserted into (4a), N is specialized to $M+2\ell$ and the terms collected so as to correspond to the arrangement of (15), then $S^{M}_{M+2\ell}(s,\eta)$ becomes

$$S^{M}_{M+2\ell}(s,\eta) = \frac{(2\ell+2M)!}{(2\ell)!\,2\pi}(kd)^{-M}(-)^{\ell}\sum_{r=0}^{\ell}\frac{(-\ell)_r(\ell+M+\tfrac{1}{2})_r}{r!}\left(\frac{2}{kd}\right)^{r}\cdot \tag{A1}$$

$$\int_{-\pi}^{\pi}\left(\frac{kd}{2}\right)^{r}e^{iMt}\left[\frac{(s^2-1)^{1/2}e^{it}}{s-\cos\eta}\right]^{r}\frac{\pi^{1/2}}{2}\sum_{p=0}^{\infty}(-)^{\ell+M+p}\left[\frac{(kd)^2[s+(s^2-1)^{1/2}\cos t]}{2(s-\cos\eta)}\right]^{p+\ell+M}\frac{(p+\ell-r)!}{p!}\,dt$$

which, since

$$\left(\frac{X}{2}\right)^2 = \frac{(kd)^2}{2}\,\frac{[s+(s^2-1)^{1/2}\cos t]}{s-\cos\eta},$$

can be written as

$$S^{M}_{M+2\ell}(s,\eta) = \frac{(2\ell+2M)!}{(2\ell)!\,2\pi}(kd)^{-M}(-)^{\ell}\sum_{r=0}^{\ell}\frac{(-\ell)_r(M+\ell+\tfrac{1}{2})_r}{r!}\left(\frac{2}{kd}\right)^{2}\mathcal{S}^{M}_{r}(s,\eta) \tag{A2}$$

where

$$\mathcal{S}^{M}_{r}(s,\eta) = \int_{-\pi}^{\pi}e^{iMt}\left[\frac{\frac{kd}{2}(s^2-1)^{1/2}e^{it}}{s-\cos\eta}\right]^{r}\frac{\pi^{1/2}}{2}\sum_{p=0}^{\infty}\frac{(-)^{\ell+M+p}\left(\frac{X}{2}\right)^{2(\ell+M+p)}}{(\ell+M+p)!\,\Gamma(\ell+M+p+\tfrac{1}{2}+\ell)}\,\frac{(p+\ell-r)!}{p!}\,dt. \tag{A3}$$

The first step in verifying the Weston development consists in $\ell-r$ successive applications of the operator $\left(\frac{d}{xdx}\right)$ on the series

$$\sum \equiv \frac{\pi^{1/2}}{2} \sum_{p=0}^{r} \frac{(-)^{\ell+M+p}\left(\frac{x}{2}\right)^{2(p+\ell-r)}}{(\ell+M+p)!\,\Gamma(\ell+M+p+3/2+\ell)} \tag{A4}$$

to yield

$$\sum{}' \equiv \frac{\pi^{1/2}}{2} \sum_{p=0}^{\infty} \frac{(-)^{\ell+M+p}\left(\frac{x}{2}\right)^{2(\ell+M+p)}}{(\ell+M+p)!\,\Gamma(\ell+M+p+3/2+\ell)} \frac{(p+\ell-r)!}{p!} = \left(\frac{x}{2}\right)^{2(\ell+M)} 2^{(\ell-r)} \left(\frac{d}{xdx}\right)^{\ell-r} \sum . \tag{A5}$$

But, if $\beta = \ell+M+p$, then (A4) can be rewritten as

$$\sum = \left(\frac{x}{2}\right)^{-2(M+r)-\ell} \left[\frac{\pi^{1/2}}{2} \sum_{\beta=0}^{\infty} \frac{(-)^{\beta}\left(\frac{x}{2}\right)^{2\beta+\ell}}{\beta!\,\Gamma(\beta+3/2+\ell)} - \frac{\pi^{1/2}}{2} \sum_{\beta=0}^{\ell+M-1} \frac{(-)^{\beta}\left(\frac{x}{2}\right)^{2\beta+\ell}}{\beta!\,\Gamma(\beta+3/2+\ell)}\right] \tag{A6}$$

so that (A5) can be written as

$$\sum{}' = \left(\frac{x}{2}\right)^{2(\ell+M)} 2^{\ell-r} \left(\frac{d}{xdx}\right)^{\ell-r} \left\{\left(\frac{x}{2}\right)^{-2(M+r)} \left[2^{\ell} \frac{j_\ell(x)}{x^\ell} - \frac{\pi^{1/2}}{2} \sum_{\beta=0}^{\ell+M-1} \frac{(-)^{\beta}\left(\frac{x}{2}\right)^{2\beta}}{\beta!\,\Gamma(\beta+3/2+\ell)}\right]\right\} . \tag{A7}$$

For the case $r=\ell$ (A7) becomes immediately

$$\sum_{r=\ell}{}' = 2^{\ell} \frac{j_\ell(x)}{x^\ell} - \frac{\pi^{1/2}}{2} \sum_{\beta=0}^{\ell+M-1} \frac{(-)^{\beta}\left(\frac{x}{2}\right)^{2\beta}}{\beta!\,\Gamma(\beta+3/2+\ell)} \tag{A8a}$$

and for the cases $r \neq \ell$ (A7) becomes after some tedious analysis

$$\sum_{r\neq\ell}{}' = \frac{(-)^{\ell-r}(\ell-r)!}{(M+r-1)!} \sum_{\gamma=0}^{\ell-r} \frac{(-)^{\gamma}(\ell+M-1-\gamma)!}{(\ell-r-\gamma)!\,\gamma!} \left\{\frac{(-)^{\gamma}\, 2^{\ell-\gamma}}{x^{\ell-\gamma}}\, j_{\ell+\gamma}(x) - \frac{\pi^{1/2}}{2} \sum_{\beta=\gamma}^{\ell+M-1} \frac{(-)^{\beta}\left(\frac{x}{2}\right)^{2\beta}}{(\beta-\gamma)!\,\Gamma(\beta+3/2+\ell)}\right\} . \tag{A8b}$$

Consequently, (A1) can be written as

$$S^{M}_{M+2\ell}(s,\eta) = \frac{(2\ell+2M)!}{(2\ell)!\,2\pi}(kd)^{-M}(-)^{\ell}\sum_{r=0}^{\ell} a^{\ell}_{r}\int_{-\pi}^{\pi} e^{iMt}\left[\frac{kd/2\,(s^2-1)^{1/2}e^{it}}{s-\cos\chi}\right]^{r}\Sigma'\,dt \qquad \text{(A9)}$$

in which

$$a^{\ell}_{r} \equiv \frac{(-\ell)_r\,(\ell+M+1/2)_r}{r!}\left(\frac{2}{kd}\right)^{r} = \frac{(-)^r\,\ell!}{(\ell-r)!\,r!}\cdot\frac{\Gamma(\ell+M+r+1/2)}{\Gamma(\ell+M+1/2)}\left(\frac{2}{kd}\right)^{r}; \qquad \text{(A10)}$$

some special values of a^{ℓ}_{r} are given in Table A1.

Table A1. Special Values of a^{ℓ}_{r}

$\ell \backslash r$	0	1	2
0	1	—	—
1	1	$-\frac{(2M+3)}{kd}$	—
2	1	$-\frac{2(2M+5)}{kd}$	$\frac{(2M+5)(2M+4)}{(kd)^2}$

The second, and final step, in verifying the Weston development consists in evaluating (A9) for special values of ℓ with M arbitrary.

Case I: $\ell = 0$

Setting $\ell = 0$ in (A9) and in (A8a) leads immediately to

$$S^{M}_{M}(s,\eta) = \frac{1}{2\pi}\frac{(2M)!}{(kd)^M}\int_{-\pi}^{\pi} e^{iMt}\left\{ j_0(\chi) - \frac{\pi}{2}^{1/2}\sum_{\beta=0}^{M-1}\frac{(-)^{\beta}\left(\frac{\chi}{2}\right)^{2\beta}}{\beta!\,\Gamma(\beta+3/2)}\right\}dt, \qquad \text{(A11a)}$$

or, trivially,

$$S_m^M(s,\eta) = \frac{1}{2\pi}\frac{(2M)!}{(kd)^M}\left\{\int_{-\pi}^{\pi} e^{iMt} j_0(X)\,dt - \frac{\pi^{1/2}}{2}\sum_{\beta=0}^{M-1}\frac{(-\beta)}{\beta!\,\Gamma(\beta+3/2)}\int_{-\pi}^{\pi} e^{iMt}\left(\frac{X}{2}\right)^{2\beta} dt\right. . \tag{A11b}$$

Since $\left(\frac{X}{2}\right)^{2\beta}$ can be expanded into the Fourier cosine-series

$$\left(\frac{kd}{2}\right)^{2\beta}\left[\frac{s+(s^2-1)^{1/2}\cos t}{s-\cos\eta}\right]^{\beta} = \sum_{\alpha=0}^{\beta} A_\alpha \cos\alpha t \tag{A12}$$

then the second integral in (A11b) is seen to be of the form

$$\int_{-\pi}^{\pi} e^{i\delta t}\cos\alpha t = \int_{-\pi}^{\pi}\cos\delta t\cos\alpha t + i\int_{-\pi}^{\pi}\sin\delta t\cos\alpha t = 0 \qquad \delta\neq\alpha \tag{A13}$$

where the second integral vanishes because sin δt is odd and where the first integral vanishes unless $\alpha=\delta$. Hence, since $\beta\neq M$ in (A11b), then

$$S_m^M(s,\eta) = \frac{1}{2\pi}\frac{(2M)!}{(kd)^M}\int_{-\pi}^{\pi} e^{iMt} j_0(X)\,dt \tag{A14}$$

which checks with (15) and (16) for $\ell=0$.

Case II: $\ell=1$

Setting $\ell=1$ in (A9), (A8a) and (A8b) leads to

$$S_{M+2}^M(s,\eta) = -\frac{1}{2\pi}\frac{(2M+2)!}{2!\,(kd)^M}\left\{\int_{-\pi}^{\pi} e^{iMt}\frac{(-)}{(M-1)!}\sum_{\gamma=0}^{1}\frac{(-)^\gamma(M-\gamma)!}{(1-\gamma)!\,\gamma!}\left[\frac{(-)^\gamma 2^{1-\gamma}}{X^{1-\gamma}} j_{1+\gamma}(X) - \right.\right. \tag{A15a}$$

$$\left. - \frac{\pi^{1/2}}{2}\sum_{\beta=\gamma}^{M}\frac{(-)^\beta\left(\frac{X}{2}\right)^{2\beta}}{(\beta-\gamma)!\,\Gamma(\beta+5/2)}\right] dt - \frac{(2M+3)}{kd}\int_{-\pi}^{\pi} e^{iMt}\left(\frac{kd}{2}\right)\left[\frac{(s^2-1)^{1/2}e^{it}}{s-\cos\eta}\right].$$

$$\left.\left[\frac{2}{X} j_1(X) - \frac{\pi^{1/2}}{2}\sum_{\beta=0}^{M}\frac{(-)^\beta\left(\frac{X}{2}\right)^{2\beta}}{\beta!\,\Gamma(\beta+5/2)}\right] dt\right\},$$

or

$$S^{M}_{M+2}(s,\eta) = -\frac{1}{2\pi}\frac{(2M+2)!}{2!(kd)^M}\left\{\int_{-\pi}^{\pi} e^{iMt}\frac{(-)}{(M-1)!}\left[M!\frac{2}{X}\,j_1(X) + (M-1)!\,j_2(X)\right]dt\right. - \tag{A15b}$$

$$-\int_{-\pi}^{\pi} e^{iMt}\frac{(-)}{(M-1)!}\sum_{\gamma=0}^{1}\frac{(-)^{\gamma}(M-\gamma)!}{(1-\gamma)!\,\gamma!}\cdot\frac{\pi^{1/2}}{2}\sum_{\beta=\gamma}^{M}\frac{(-)^{\beta}\left(\frac{X}{2}\right)^{2\beta}}{(\beta-\gamma)!\,\Gamma(\beta+5/2)}\,dt -$$

$$\left.-(2M+3)\int_{-\pi}^{\pi} e^{iMt}\left[\frac{(s^2-1)^{1/2}e^{it}}{s-\cos\eta}\right]\frac{j_1(X)}{X}\,dt + (2M+3)\int_{-\pi}^{\pi} e^{iMt}\left[\frac{(s^2-1)^{1/2}e^{it}}{s-\cos\eta}\right]\frac{\pi^{1/2}}{8}\sum_{\beta=0}^{M}\frac{(-)^{\beta}\left(\frac{X}{2}\right)^{2\beta}}{\beta!\,\Gamma(\beta+5/2)}\,dt\right\}.$$

But, the second integral, which can be written as

$$\int_{-\pi}^{\pi} e^{iMt}\frac{(-)}{(M-1)!}\left\{M!\left[\frac{\pi^{1/2}}{2}\sum_{\beta=0}^{M-1}\frac{(-)^{\beta}\left(\frac{X}{2}\right)^{2\beta}}{\beta!\,\Gamma(\beta+5/2)} + \frac{\pi^{1/2}}{2}\frac{(-)^{M}\left(\frac{X}{2}\right)^{2M}}{M!\,\Gamma(M+5/2)}\right]\right.$$

$$\left.-(M-1)!\left[\frac{\pi^{1/2}}{2}\sum_{\beta=1}^{M-1}\frac{(-)^{\beta}\left(\frac{X}{2}\right)^{2\beta}}{(\beta-1)!\,\Gamma(\beta+5/2)} + \left(\frac{\pi}{2}\right)^{1/2}\frac{(-)^{M}\left(\frac{X}{2}\right)^{2M}}{(M-1)!\,\Gamma(M+5/2)}\right]\right\}dt,$$

vanishes - since the second and fourth terms cancel and since, by (A13), the first and third terms vanish. Similarly the fourth integral in (A15b) vanishes by (A13) so that

$$S^{M}_{M+2}(s,\eta) = -\frac{1}{2\pi}\frac{(2M+2)!}{2!(kd)^M}\int_{-\pi}^{\pi} e^{iMt}\left\{-\frac{2M}{X}j_1(X) - j_2(X) - (2M+3)\left[\frac{(s^2-1)^{1/2}e^{it}}{s-\cosh\eta}\right]\frac{j_1(X)}{X}\right\}dt. \tag{A16}$$

But, from the recursion relation for the spherical Bessel functions,

$$\left(\frac{2n+1}{X}\right)j_n(X) = j_{n-1}(X) + j_{n+1}(X), \tag{A17}$$

and from (8b) the defining equation for X, (A16) can be written as

$$S_{M+2}^{M}(s,\chi) = -\frac{1}{2\pi}\frac{(2M+2)!}{2!(kd)^{M}}\int_{-\pi}^{\pi} e^{iMt}\left\{ j_0(X) - \frac{(2M+3)}{kd}\left[\frac{(s^2-1)^{1/2}e^{it} + s - \cos\chi}{2^{1/2}(s-\cos\chi)^{1/2}\left[s+(s^2-1)^{1/2}\cos t\right]^{1/2}}\right] j_1(X)\right\} dt \tag{A18a}$$

or, employing (14b) the defining equation for ζ,

$$S_{M+2}^{M}(s,\chi) = -\frac{1}{2\pi}\frac{(2M+2)!}{2!(kd)^{M}}\left\{\int_{-\pi}^{\pi} e^{iMt} j_0(X)\,dt - \frac{2M+3}{kd}\int_{-\pi}^{\pi} e^{iMt}\zeta\, j_1(X)\,dt\right\} \tag{A18b}$$

which checks with (15) and (16) for $\ell = 1$.

A similar treatment has been carried out for $\ell = 2$ * and again (15) and (16) are checked. Consequently, although at the time of this report no general reduction had been found for arbitrary ℓ, it can be concluded that Weston's general expression for $S_{M+2\ell}^{M}(s,\chi)$, (15), is correct.

---

* In this treatment for $\ell = 2$, as in the treatment for $\ell = 1$, the "unwanted" terms either vanish by (A13), when $\alpha \neq \delta$ or exactly cancel each other when $\alpha = \delta$. This same feature will undoubtedly hold for all values of ℓ.

References

1) C.H. Sherman and N.G. Parke, J. Acoust. Soc. Am. 38, 715-722 (1965).

2) C.H. Sherman and N.G. Parke, "Toroidal Wave Functions and Their Application to the Free-Flooding Ring Transducer", Parke Mathematical Laboratories, Scientific Report No. 2, Contract N140(70024)75686B (December, 1964).

3) V.H. Weston, Quart. Appl. Math. 16, 237-254 (1958).

4) V.H. Weston, J. Math. and Phys. 39, 64-71 (1960).

5) V.H. Weston, Quart. Appl. Math. 15, 420-425 (1957)

6) C. Snow, "Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory", U.S. Government Printing Office, 1952.

7) M.E. Rose, "Multipole Fields", John Wiley and Sons, Inc., New York, 1955.

8) A. Erdelyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, "Higher Transcendental Functions" Volume I, McGraw-Hill Book Company, Inc., New York, 1953.

9) P.M. Morse and H. Feshbach, "Methods of Theoretical Physics", McGraw-Hill Book Company, Inc., New York, 1953.