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Relationships Among Some Notions
of Bivariate Dependence

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RELATIONSHIPS AMONG SOME NOTIONS

OF BIVARIATE DEPENDENCE

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ABSTRACT

A random variable T is left tail decreasing in a random variable
S if $P[T \leq t \mid S \leq s]$ is non-increasing in s for all t, and right
tail increasing in S if $P(T > t \mid S > s)$ is non-decreasing in s for
all t. We show that either of these conditions implies that S,T are
associated, i.e. $\text{Cov}(f(S,T), g(S,T)) \geq 0$ for all pairs of functions
f,g which are non-decreasing in each argument. No two of these con-
ditions for bivariate dependence are equivalent. Applications of these
and other conditions for dependence in probability, statistics, and
reliability theory are considered in Lehmann (1966) *Ann. Math. Statist.*
and Esary, Proschan, and Walkup (1966) Boeing documents D1-82-0567,
D1-82-0578.

1. Introduction

Lehmann ([2], 1966) defines two random variables S, T to be
positively quadrant dependent if $P[S \leq s, T \leq t] \geq P[S \leq s] \cdot P[T \leq t]$
for all s, t; and T to be *positively regression dependent* on S
if $P[T \leq t \mid S = s]$ is non-increasing in s for all t (with
reference for the latter definition to Tukey, 1958, [3]). Esary, Proschan,
and Walkup ([1], 1966) define S, T to be *associated* if
$Cov[f(S,T), g(S,T)] \geq 0$ for all pairs of functions f, g which are
non-decreasing in each argument, and such that Ef(S,T), Eg(S,T),
Ef(S,T)g(S,T) exist. Lehmann also mentions the type of dependence
characterized by

(1.1) $P[T \leq t \mid S \leq s]$ is non-increasing in s for

all t.

If condition (1.1) holds, we say that T is *left tail decreasing* in
S.   A condition similar to (1.1) is

(1.2) $P[T > t \mid S > s]$ is non-decreasing in s for

all t.

If condition (1.2) holds, we say that T is *right tail increasing* in
S.

Among T positively regression dependent on S (we write
PRD{T|S}), T left tail decreasing in S (LTD{T|S}), and S, T positively

quadrant dependent $(PQD\{S,T\})$ the implications

$$PRD\{T|S\} \Rightarrow LTD\{T|S\} \Rightarrow PQD\{S,T\}$$

hold. The implications are strict, i.e., no two of the conditions are equivalent [2]. Among $PRD\{T|S\}$, S, T associated $(A\{S,T\})$, and $PQD\{S,T\}$ the strict implications

$$PRD\{T|S\} \Rightarrow A\{S,T\} \Rightarrow PQD\{S,T\}$$

hold [1].

In this note we study the unresolved relationships in this set of conditions for bivariate dependence, particularly the relationship of $LTD\{T|S\}$ and $RTI\{T|S\}$ to $A\{S,T\}$, and extend the structure of strict implications to

$$PRD\{T|S\} \Rightarrow \begin{bmatrix} LTD\{T|S\} \\ \text{and} \\ RTI\{T|S\} \end{bmatrix} \begin{array}{l} \Rightarrow LTD\{T|S\} \Rightarrow \\ \\ \Rightarrow RTI\{T|S\} \Rightarrow \end{array} A\{S,T\} \Rightarrow PQD\{S,T\}.$$

2. LTD, RTI, and PRD.

Condition $(1.1, LTD\{T|S\})$ can be restated as $P[T > t \mid S \le s]$ is non-decreasing in s for all t. Then by elementary manipulation condition (1.1) is equivalent to

(2.1) $P[T > t \mid S \le s_1] \le P[T > t \mid s_1 < S \le s_2]$ for all

t and $s_1 < s_2$.

Condition $(1.2, RTI\{T|S\})$ is equivalent to

(2.2) $P[T > t \mid s_1 < S \le s_2] \le P[T > t \mid s_2 < S]$ for all

t and $s_1 < s_2$.

These expressions give a convenient way of viewing the joint condition [LTD{T|S} and RTI{T|S}].

Using conditions (2.1) and (2.2) it is immediate that PRD{T|S} $\Rightarrow$ [LTD{T|S} and RTI{T|S}], since for any interval I

$$P[T > t \mid S \in I] = \int_{s \in I} P[T > t \mid S = s]dP[S \le s]/P[S \in I].$$

(cf. [2]).

It is known (e.g. see [1]) that all of the conditions for bivariate dependence considered in this note are equivalent for 2×2 distributions (we say that S, T have an $n \times m$ distribution if S has n values, T has m values). To show that the implication PRD{T|S} $\Rightarrow$ LTD{T|S} is strict Lehmann uses a 3×3 example. To show that the implication PRD{T|S} $\Rightarrow$ [LTD{T|S} and RTI{T|S}] is strict we use a 4×2 example, since PRD{T|S} $\Leftrightarrow$ [LTD{T|S} and RTI{T|S}] for any $3 \times m$ distribution by conditions (2.1) and (2.2). We let S take values $s_1 < s_2 < s_3 < s_4$, each with probability 1/4. We let T take values $t_1 < t_2$, with $P[T = t_2 \mid S = s_i] = p_i$. If $p_1 = .4$, $p_2 = .6$, $p_3 = .5$, $p_4 = .7$, we have [LTD{T|S} and RTI{T|S}] but not PRD{T|S}.

If in the example above $p_1 = .4$, $p_2 = .6$, $p_3 = .5$, $p_4 = .5$, we have LTD{T|S} but not RTI{T|S}. If $p_1 = .5$, $p_2 = .5$, $p_3 = .4$, $p_4 = .6$, we have RTI{T|S} but not LTD{T|S}.

3. LTD, RTI, and A.

By elementary manipulation condition (2.1, LTD{T|S}) is equivalent

to

(3.1) $P[t < T, S \leq s_1] \cdot P[T \leq t, s_1 < S \leq s_2] \leq P[T \leq t, S \leq s_1] \cdot P[t < T, s_1 < S \leq s_2]$

for all t and $s_1 < s_2$.

Condition (2.2, RTI$\{T|S\}$ is equivalent to

(3.2) $P[t < T, s_1 < S \leq s_2] \cdot P[T \leq t, s_2 < S] \leq P[T \leq t, s_1 < S \leq s_2] \cdot P[t < T, s_2 < S]$

for all t and $s_1 < s_2$.

In [1] it is shown that association (A$\{S,T\}$) is equivalent to

(3.3) $P[\gamma(S,T) = 1, \delta(S,T) = 0] \cdot P[\gamma(S,T) = 0, \delta(S,T) = 1]$

$\leq P[\gamma(S,T) = 0, \delta(S,T) = 0] \cdot P[\gamma(S,T) = 1, \delta(S,T) = 1]$

for all pairs γ, δ of binary functions which are non-decreasing

in each argument.

A function is *binary* if it takes only the values 0 and 1.

We consider the 3×3 distribution

$T = t_3$	p_{13}	0	1/4
$T = t_2$	0	1/4	0
$T = t_1$	1/4	0	p_{31}
	$S = s_1$	$S = s_2$	$S = s_3$

where $s_1 < s_2 < s_3$ and $t_1 < t_2 < t_3$. If $p_{13} = p_{31} = 1/8$, we have
A$\{S,T\}$ but neither LTD$\{T|S\}$ nor RTI$\{T|S\}$ (cf. [1]). If $p_{13} = 0$,
$p_{31} = 1/4$, we have LTD$\{T|S\}$ but not RTI$\{T|S\}$. If $p_{13} = 1/4$, $p_{31} = 0$,
we have RTI$\{T|S\}$ but not LTD$\{T|S\}$ (cf. [2]).

We now proceed toward a proof of the implication $RTI\{T|S\} \Rightarrow A\{S,T\}$.
Once this is accomplished, the implication $LTD\{T|S\} \Rightarrow A\{S,T\}$ follows,
since $LTD\{T|S\} \Leftrightarrow RTI\{-T|-S\} \Rightarrow A\{-S,-T\} \Leftrightarrow A\{S,T\}$.

Given random variables S, T we choose fixed $s_1 < s_2 < \cdots < s_n$
and $t_1 < t_2 < \cdots < t_m$. We define discrete random variables S^*, T^* by

$$
\begin{aligned}
S^* = 0 \ &\text{if} \quad S \le s_1 \qquad & T^* = 0 \ &\text{if} \quad T \le t_1 \\
1 \ &\text{if} \ s_1 < S \le s_2 \qquad & 1 \ &\text{if} \ t_1 < T \le t_2 \\
&\ \ \vdots & &\ \ \vdots \\
n \ &\text{if} \ s_n < S \qquad & m \ &\text{if} \ t_m < T.
\end{aligned}
$$

It is shown in [1] that $A\{S,T\}$ is equivalent to $A\{S^*,T^*\}$ for all
choices of n, m and $s_1,\ldots,s_n$, $t_1,\ldots,t_m$. It is clear that
$RTI\{T|S\} \Rightarrow RTI\{T^*|S^*\}$. Thus we only need to show that $RTI\{T^*|S^*\} \Rightarrow A\{S^*,T^*\}$.

Justified by the preceding remarks, *we assume from now on that* S
is discrete with the values $0, 1, \ldots, n$ *and that* T *is discrete with the*
values $0, 1, \ldots, m$. Also from now on we make the convention that γ, δ
are binary, non-decreasing functions of $s = 0, 1, \ldots, n$ and $t = 0, 1, \ldots, m$.

We say that (s_0,t_0) is a *boundary point* of $\{\gamma = 0\} = \{(s,t) \mid \gamma(s,t) = 0\}$
if $\gamma(s_0,t_0) = 0$ and $\gamma(s_0+1,t_0+1) = 1$.

Lemma 1.

Let (s_2,t_2) be a boundary point of both $\{\gamma = 0\}$ and $\{\delta = 0\}$.
Then $RTI\{T|S\}$ implies

(3.4) $P[\gamma(S,T) \neq \delta(S,T), s_1 < S \leq s_2] \cdot P[\gamma(S,T) \neq \delta(S,T), s_2 < S]$

$\leq P[\gamma(S,T) = 0, \delta(S,T) = 0, s_1 < S \leq s_2] \cdot P[\gamma(S,T) = 1, \delta(S,T) = 1, s_2 < S]$

for all $s_1 < s_2$.

Proof.

Since $\gamma(s_2,t_2) = \delta(s_2,t_2) = 0$, $\gamma(s,t) = \delta(s,t) = 0$ for all
$s \leq s_2$, $t \leq t_2$. Since $\gamma(s_2+1,t_2+1) = \delta(s_2+1,t_2+1) = 1$, $\gamma(s,t) = \delta(s,t) = 1$
for all $s_2 < s$, $t_2 < t$. Thus $\{\gamma(s,t) \neq \delta(s,t), s_1 < s \leq s_2\}$
$\subset \{s_1 < s \leq s_2, t_2 < t\}$ and $\{\gamma(s,t) \neq \delta(s,t), s_2 < s\} \subset \{s_2 < s, t \leq t_2\}$.
Also $\{\gamma(s,t) = 0, \delta(s,t) = 0, s_1 < s \leq s_2\} \supset \{s_1 < s \leq s_2, t \leq t_2\}$ and
$\{\gamma(s,t) = 1, \delta(s,t) = 1, s_2 < s\} \supset \{s_2 < s, t_2 < t\}$. Inequality (3.4)
follows from condition (3.2, RTI$\{T|S\}$).$||$

For fixed s either (a) $\gamma(s,t) \geq \delta(s,t)$ for all t, or
(b) $\gamma(s,t) \leq \delta(s,t)$ for all t. It is clear that we can find an
alternating partition of $[0,n]$, i.e. a partition of $[0,n]$ into intervals
I_1, I_2, ..., I_k such that either (a) holds for all $s \in I_j$, or (b)
holds for all $s \in I_j$, $j = 1$, ..., k, and such that if (a) holds on I_j
(or (b) holds on I_j), then (b) holds on I_{j+1} ((a) holds on
I_{j+1}), $j = 1$, ..., $k - 1$.

Lemma 2.

Let I_1, I_2, ..., I_k be an alternating partition of $[0,n]$. Let
$s_j = \max\{s | s \in I_j\}$, $t_j = \max\{t | \gamma(s_j,t) = \delta(s_j,t) = 0\}$. Then the points
(s_j,t_j), $j = 1$, ..., $k - 1$, are boundary points of both $\{\gamma = 0\}$ and
$\{\delta = 0\}$.

Proof.

Suppose, to fix a case, that (a) holds on I_j. Then $\gamma(s_j,t_j) = 0$, $\delta(s_j,t_j) = 0$ and $\gamma(s_j,t_j+1) = 1$ from the definition of s_j and t_j. Since $\gamma(s_j+1,t_j+1) \geq \gamma(s_j,t_j+1)$, $\gamma(s_j+1,t_j+1) = 1$. Then since (b) holds on I_{j+1}, $\delta(s_j+1,t_j+1) \geq \gamma(s_j+1,t_j+1)$, so that finally $\gamma(s_j+1,t_j+1) = 1$, $\delta(s_j+1,t_j+1) = 1.||$

Theorem.

$RTI\{T|S\}$ *implies* $A\{S,T\}$.

Proof.

With reference to condition (3.3, $A\{S,T\}$) let $p_{ij} = P[\gamma(S,T) = i, \delta(S,T) = j]$, $i,j = 0,1$. Let $I_1, I_2, \ldots, I_k$ be an alternating partition of $[0,n]$. Let $a_j = P[\gamma(S,T) \neq \delta(S,T), s \in I_j]$, $b_j = P[\gamma(S,T) = 0, \delta(S,T) = 0, s \in I_j]$, and $c_j = P[\gamma(S,T) = 1, \delta(S,T) = 1, s \in I_j]$, $j = 1, \ldots, k$. In view of Lemma 2 we can apply Lemma 1 (with the interval $(s_1,s_2]$ of Lemma 1 taken to be I_j) to obtain

$$a_j(a_{j+1}+\ldots+a_k) \leq b_j(c_{j+1}+\ldots+c_k) \quad j = 1, \ldots, k - 1.$$

Now $p_{10} = \sum_{j=1}^{k} e_j a_j$, $p_{01} = \sum_{j=1}^{k} (1-e_j)a_j$, where $e_j = 1$ if $\gamma \geq \delta$ on I_j, $e_j = 0$ if $\gamma \leq \delta$ on I_j. Also $p_{00} = \sum_{j=1}^{k} b_j$ and $p_{11} = \sum_{j=1}^{k} c_j$. Then

$$p_{10}p_{01} \leq \sum\sum_{i<j} a_i a_j \leq \sum\sum_{i<j} b_i c_j \leq p_{00}p_{11}.$$

Thus condition (3.3, $A\{T|S\}$) is verified.$||$

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