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GENERALIZED LAW OF THE WALL AND EDDY VISCOSITY MODEL FOR WALL BOUNDARY LAYERS

GDALIA KLEINSTEIN
NEW YORK UNIVERSITY
BRONX, NEW YORK

Contract No. AF 33(615)-2215
Project No. 7064

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UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This interim report was prepared by Dr. Gdalia Kleinstein, Research Associate Professor of Aeronautics and Astronautics at New York University, and presents research carried out under a National Science Foundation Institutional Grant, "Statistical Methods and Their Application to Fluid Dynamics Problems", and Contract No. AF33(615)-2215, "Boundary Layer Characteristics for Hypersonic Flow in the Presence of Mass Addition", Project No. 7064.

ABSTRACT

Based on the logarithmic velocity distribution at distances not so far from the wall and a turbulent shear variation proportional to the cubic power of the wall distance in the immediate proximity of the wall, an analytical model for an eddy viscosity throughout the wall region is derived leading to a closed form expression for the velocity distribution. An extension of the above analysis to include shear variation results in a generalized law of the wall from which the effect of Reynolds number on the velocity distribution is derived. While many applications of the generalized solution are envisaged, as an example, the velocity distribution for the case of injection or suction is considered. The theoretical result is shown to be in agreement with available experimental data.

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NOMENCLATURE

$\bar{u}$	mean flow velocity (streamwise component)
u_τ	friction velocity (local) $u_\tau = \left[\tau(y)/\rho\right]^{\frac{1}{2}}$
u^*	wall friction velocity $u^* = (u_\tau)_w = \left[\tau(0)/\rho\right]^{\frac{1}{2}}$
u^+	dimensionless mean flow velocity $u^+ = \bar{u}/u^*$
u_τ^+	dimensionless friction velocity $u_\tau^+ = u_\tau/u^*$
ϵ	eddy viscosity
ϵ^+	dimensionless eddy viscosity ϵ/ν
y	normal coordinate
y^+	dimensionless normal coordinate $y^+ = yu^*/\nu$
v_w^+	dimensionless injection velocity $v_w^+ = v_w/u^*$

INTRODUCTION

The validity of the logarithmic nature of the law of the wall, which may be derived on the basis of substantially different underlying assumptions such as dimensional analysis as well as the Malkus theory[1], has been satisfactorily verified by a large volume of experimental data. Since the law of the wall as such does not hold through the transition region (buffer zone) and obviously not in the laminar sublayer, it has been desired to obtain a formulation which will connect continuously and smoothly the laminar sublayer velocity distribution to the law of the wall. Most successful has been Reichardt[2] who from considerations of the behavior of the velocity fluctuations in the immediate vicinity of the wall deduced that the eddy viscosity must grow at least as the cubic power of y. Accordingly, by choosing a function which satisfies the y^3 behavior near the wall; and, asymptotically approaches the required linear dependence on y, he was able to construct an eddy viscosity model which in turn led to a velocity distribution in agreement with experimental data from $y = 0$ to the law of the wall region. Other formulas for the eddy viscosity have been suggested by Deissler[3] and Van Driest[4] which, although successfully predict the velocity distribution, raise the objection that they do not satisfy the y^3 requirement of Reichardt's theoretical result.

In this paper it is shown that by taking the velocity as the independent variable, an eddy viscosity model can be derived from the law of the wall and Reichardt's y^3 decay requirement without introducing a new arbitrary function. The resulting eddy viscosity model yields an integrable closed form expression for the velocity distribution as compared with the models of Reichardt, Deissler and Van Driest which require a numerical integration.

By employing a simple transformation the above formulation is further extended to include the effect of shear variation. Consequently, a generalized form for the law of the wall is obtained; and in particular it is shown that for regions where the eddy viscosity is much larger than the kinematic viscosity the law of the wall has a universal nature independent of the shear distribution. The application of this result to pipe flow provides an explanation for the effect of Reynolds number on the velocity distribution which has been observed by Hinze[5]. As an additional example, the generalized formula has been applied to the problem of wall injection and suction where good agreement with experimental data has been demonstrated in regions close to the wall including the buffer layer.

ANALYSIS

1. Introduction

The total shear associated with the transport of momentum in the direction normal to the wall is given by

$$\tau = \mu\frac{\partial u}{\partial y} - \overline{\rho v' u'} \tag{1}$$

If a concept of an eddy viscosity is adopted the turbulent contribution may be expressed by the relation

$$-\overline{\rho v' u'} = \rho\epsilon\frac{\partial \bar{u}}{\partial y} \tag{2}$$

where Eq. (2) is taken as a definition of the eddy viscosity. In terms of the nondimensional quantities $y^+ = (\rho u^* y)/\mu$, $u^+ = \bar{u}/u^*$, $\epsilon^+ = \rho\epsilon/\mu$; where $u^* = (\tau_w/\rho)^{\frac{1}{2}}$, Eqs. (1) and (2) may be combined in the form

$$\frac{\tau}{\tau_w} = (1+\epsilon^+)\frac{\partial u^+}{\partial y^+} \tag{3}$$

or

$$(u_\tau^+)^2 = (1+\epsilon^+)\frac{\partial u^+}{\partial y^+} \tag{3'}$$

where $u_\tau^+ = u_\tau/u^* = [\tau/\tau_w]^{\frac{1}{2}}$ is the nondimensionalized friction velocity.

Transforming the variables from (x,y) to (x,u) yields Eq. (3′) in the desired form i.e.

$$\epsilon^+ + 1 = (u_\tau^+)^2 \frac{\partial y^+}{\partial u^+} \tag{4}$$

By dimensional analysis considerations, Prandtl in his momentum transport theory[6] proposed an eddy viscosity model which in the present notation takes the form

$$\epsilon^+ = k_1^2 (y^+)^2 \frac{du^+}{dy^+} \tag{5}$$

where partial derivatives have been replaced by an ordinary derivative to indicate a dependence on a single variable. Eliminating du^+/dy^+ from Eq. (5) by Eq. (3′) where ϵ^+ is taken much larger than unity yields an equivalent model in the form,

$$\epsilon^+ = k_1 u_\tau^+ y^+ \tag{6}$$

The model given by Eq. (6) has been previously used by Ferrari[7] in several problems and specifically in problems related to turbulent boundary layer separation. Eqs. (4) and (6) in conjunction with the requirement of $(y^+)^3$ decay at the wall constitute the basis for the derivation of the complete velocity distribution and eddy viscosity formulation given below.

2. Constant Shear Solution

If the shear is assumed constant, $u_\tau^+ = 1$, and Eqs. (4) and (6) become

$$\epsilon^+ = \frac{dy^+}{du^+} \tag{4'}$$

and

$$\epsilon^+ = k_1 y^+$$

In Eq. (4′) ϵ^+ is taken much larger than unity and a dependence on a single variable is assumed.

Eliminating ϵ^+ between Eqs. (4′) and (6′) and integrating yields, the logarithmic law

$$u^+ = \frac{1}{k_1} \ln y^+ + k_2' = \frac{1}{k_1} \ln k_2 y^+ \tag{7}$$

where $k_1 = 0.4$ is the von Karman universal constant, and k_2 is taken equal to 7.7 which is obtained from $k_2' = 5.1$.

Using u^+ as the independent variable Eq. (7) is written as

$$y^+ = \frac{1}{k_2} \exp(k_1 u^+) \tag{8}$$

and correspondingly the expression for ϵ^+ becomes

$$\epsilon^+ = \frac{k_1}{k_2} \exp(k_1 u^+) \tag{9}$$

In order to extend the above solution to the wall (i.e. $y^+ = 0$) the behavior of $\overline{\rho v'u'}$ and $\bar{u}$ must be specified in this region. For the immediate vicinity of the wall, Reichardt[2] has shown that the turbulent shear varies proportionally to at least the cubic power of the distance from the wall, and since for the same region the mean velocity $\bar{u}$ varies linearly with the same dimension it follows from the definition $-\overline{\rho v'u'} = \rho\epsilon\partial\bar{u}/\partial y$ that

$$\rho\epsilon \sim y^3 \sim \bar{u}^3 \tag{10}$$

Now including a correction term in Eq. (9), ϵ^+ may be written in the form

$$\epsilon^+ = \frac{k_1}{k_2}[\exp(k_1 u^+) + f(k_1 u^+)] \tag{11}$$

where f must satisfy the asymptotic condition

$$\lim_{y^+ \to \infty} f(k_1 u^+)\exp(-k_1 u^+) = 0 \tag{12'}$$

and, in accordance with Eq. (10), the boundary conditions

$$f(0) = f'(0) = f''(0) = -1 \tag{12''}$$

While Eq. (12) obviously does not define f uniquely, an exponential representation for f in the form $c_1 \exp(c_2 k_1 u^+)$ where c_1 and c_2 are arbitrary constants is clearly excluded by Eqs. (12′) and (12″). Taking then the highest degree polynomial which satisfies both conditions and at the same time ˙ es not introduce new arbitrary constants yields,

$$f(k_1 u^+) = -\left[1 + (k_1 u^+) + \frac{1}{2}(k_1 u^+)^2\right] \tag{13}$$

Accordingly, the modified expression for ϵ^+ becomes,

$$\epsilon^+ = \frac{k_1}{k_2}\left\{[\exp(k_1 u^+)] - [1 + (k_1 u^+) + \frac{1}{2}(k_1 u^+)^2]\right\} \tag{14}$$

Inserting Eq. (14) into Eq. (4) with $u_\tau^+ = 1$, and integrating subject to the boundary condition $y^+ = 0$ yields the velocity distribution

$$y^+ = u^+ + \frac{1}{k_2}\left\{[\exp(k_1 u^+)] - [1 + (k_1 u^+) + \frac{1}{2}(k_1 u^+)^2 + \frac{1}{6}(k_1 u^+)^3]\right\} \tag{15}$$

A plot of Eq. (15) is given in Fig. (1) where the classical law of the wall and the laminar velocity distribution for the sublayer are included for comparison. Fig. (2) is a reproduction of Fig. (3) of Lindgren's recent report on experiments in turbulent pipe flow of distilled water[8]. The solid line is a plot of Eq. (15) using the $\log_{10} y^+$ representation. The agreement with these measurements is surprisingly good within the range $0 < u^+ < 18$. The deviation beyond this region, although small in the beginning, becomes more and more significant as the wake region is penetrated.

A comparison between ϵ^+ as given by Eq. (14) and the measurement of

Abbrecht and Churchill[9] are shown in Fig. (3). While the agreement is by no means as satisfactory as in the case of the velocity distribution the deviation assumes a maximum value of less than 10% around $y^+ = 75$ and then starts to drop again. In the velocity distribution, where y^+ is obtained as an integral quantity of ϵ^+ with respect to u^+, this deviation becomes completely insignificant.

3. Variable Shear Solution

The velocity distribution obtained in the previous section was based on the constant shear assumption. The result obtained is "universal" in the sense that independent of the Reynolds number, y^+ is a function of u^+ only. In order to obtain a more general form for the law of the wall the simplification introduced into the analysis by taking $u_\tau^+ = 1$ in Eq. (6) will be dropped. Thus, following the same procedure as in the previous case, Eqs. (4) and (5) are written first for the region where $\epsilon^+ >> 1$, namely

$$\epsilon^+ = u_\tau^{+2} \frac{dy^+}{du^+} \tag{16}$$

$$\epsilon^+ = k_1 u_\tau^+ y^+ \tag{17}$$

Introducing a new variable

$$U = \int_0^{u^+} \frac{du'}{u_\tau^+} \tag{18}$$

into Eqs. (16) and (17) and eliminating ϵ^+ results in the equation

$$k_1 y^+ = \frac{dy^+}{dU} \tag{19}$$

which upon integration gives,

$$U = \frac{1}{k_1} \ln y + K_2' = \frac{1}{k_1} \ln K_2 y^+$$

A comparison of this result with the known solution for the $u_\tau^+ = 1$ case yields $K_2 = k_2$; and the logarithmic law becomes

$$U = \frac{1}{k_1} \ln k_2 y^+ \tag{20}$$

Thus, for regions where $\epsilon^+ >> 1$ the logarithmic law of the wall holds independent of the particular shear distribution provided U replaces u^+

as the variable. Taking U as the independent variable,

$$y^+ = \frac{1}{k_2}\exp(k_1 U) \tag{21}$$

and

$$\epsilon^+ = u_\tau^+ \frac{k_1}{k_2}\exp(k_1 U) \tag{22}$$

Now, since independent of the nature of the shear distribution there always exists a narrow region close enough to the wall where U may be taken equal to u^+, it follows from Reichardt condition that

$$\epsilon^+ \sim (y^+)^3 \sim (u^+)^3 \sim (U)^3$$

and consequently by the identical arguments of the previous section,

$$\epsilon^+ = u_\tau^+ \frac{k_1}{k_2}\left\{[\exp(k_1 U)] - [1 + (k_1 U) + \frac{1}{2}(k_1 U)^2]\right\} \tag{23}$$

Integrating the equation

$$\epsilon^+ + 1 = u_\tau^+ \frac{dy^+}{dU}$$

subject to the boundary condition $y^+ = 0$ at $U = 0$ yields the generalized law of the wall in the form,

$$y^+ = \int_0^U \frac{dU'}{u_\tau^+} + \frac{1}{k_2}\left\{[\exp(k_1 U)] - [1 + (k_1 U) + \frac{1}{2}(k_1 U)^2 + \frac{1}{6}(k_1 U)^6]\right\} \tag{24}$$

It shoule be noted here that there is a fundamental difference between Eq. (24) and Eq. (15). While Eq. (15) represents y^+ as a function of u^+ only Eq. (24) may depend on the Reynolds number as for example in a pipe where the shear is given by

$$\frac{\tau}{\tau_w} = 1 - \frac{2y}{d} \tag{25}$$

Now, if Eq. (24) is to predict the correct velocity distribution for a pipe it remains to explain the apparently good agreement of Eq. (15) with experimental data where the shear was assumed constant. Since in pipe flow seventy to eighty percent of the centerline velocity is attained within a region close to the wall, corresponding to 2y/d of the order of fifteen to twenty percent, and, since beyond this region the flow field is definitely within the region of the law of the wake it is required (1) to show that Eq. (24) and Eq. (15) do coincide within the range $0 < 2y/d < 0.20$

to a close approximation independent of the shear variation and (2) to determine the Reynolds number dependence of the law of the wall in a pipe.

In terms of the nondimensionalized variables, the friction velocity for the pipe is given by

$$u_\tau^+ = (1 - \frac{2y}{d})^{\frac{1}{2}} \approx (1 - \frac{2y^*}{d^*})^{\frac{1}{2}} \tag{26}$$

which for small values of the ratio 2y/d may also be written as,

$$u_\tau^+ = 1 - y^*/d^* + 0[(y^*/d^*)^2] \tag{26'}$$

For the zero order solution of (24), designated by y_o^+, let $u_\tau^+ = (u_\tau^+)_o = 1$. Correspondingly $U = u^+$; the integral term reduces to U and therefore to u^+; and, thus the velocity distribution of Eq. (15) is recovered. For the first order solution let

$$u_\tau^+ = (u_\tau^+)_1 = 1 - \frac{y_o^+}{d^+}$$

Hence,

$$U = \int_0^{u^+} \frac{du'}{u_\tau^+} = \int_0^{u^+} (1 + \frac{y_o^+}{d^+})du' = u^+ +$$

$$+ (d^+)^{-1}\Big[\frac{1}{2}(u^+)^2 + \frac{1}{k_1 k_2}\Big\{[\exp(k_1 u^+)] - \Big[1 + (k_1 u^+) + \frac{1}{2}(k_1 u^+)^2 + \frac{1}{6}(k_1 u^+)^3 + + \frac{1}{24}(k_1 u^+)^4\Big]\Big\}\Big] \tag{27}$$

and taking into account that $y_o^+(u^+) \equiv y_o^+(U)$,

$$\underline{I} = \int_0^U \frac{dU'}{u_\tau} = \int_0^U (1 + \frac{y_o^+}{d^+})dU' = U +$$

$$+ (d^+)^{-1}\Big[\frac{1}{2}U^2 + \frac{1}{k_1 k_2}\Big\{[\exp(k_1 U)] - [1 + (k_1 U) + \frac{1}{2}(k_1 U)^2 + \frac{1}{6}(k_1 U)^3 + \frac{1}{24}(k_1 U)^4]\Big\}\Big] \tag{28}$$

With U as a known function of u^+ and $\underline{I}$ resolved in terms of U, Eqs. (27) and (28) in conjunction with Eq. (27) completely describe the dependence of the velocity distribution on the Reynolds number $d^+ = u^* d/\nu$.

As an immediate consequence of this solution it is apparent that for any fixed value of u^+, y_1^+ approaches the constant shear solution y_o^+ as the Reynolds number d^+ approaches infinity. This result may also be directly inferred from Eq. (20) since a fixed u^+ implies a fixed y_o^+.

In order to assess the maximum deviation between y^+ and y_o^+ it is necessary to consider the asymptotic region where independent of how large d^+ is, the ratio $2y_o^+/d^+$ may assume values up to 0.20. For the asymptotic region Eq. (27) and (28) reduce respectively to

$$U = u^+ + \frac{1}{d^+}\frac{1}{k_1 k_2}\exp(k_1 u^+) \tag{27'}$$

and

$$\underline{I} = U + \frac{1}{d^+}\frac{1}{k_1 k_2}\exp(k_1 U) \tag{28'}$$

Substituting Eq. (28′) in Eq. (24) yields

$$y^+ = U + \left(\frac{1}{d^+ k_1} + 1\right)\frac{1}{k_2}\exp(k_1 U) \tag{29}$$

where for the time being quantities of the order of $(d^+)^{-1}$ are retained with respect to U.

Using Eq. (27) to eliminate U from Eq. (29) yields

$$y^+ = u^+ + \frac{1}{d^+}\frac{1}{k_1 k_2}\exp(k_1 u^+) + \left(\frac{1}{d^+ k_1} + 1\right)\frac{1}{k_2}\exp(k_1 u^+)\exp\left[\frac{1}{d^+}\frac{1}{k_2}\exp(k_1 u^+)\right].$$

Now, since asymptotically $y_o^+ = \frac{1}{k_2}\exp(k_1 u^+)$ the above equations may be written as

$$y_1^+ = u^+ + \frac{1}{k_1}\frac{y_o^+}{d^+} + \left(\frac{1}{d^+ k_1} + 1\right)y_o^+\exp\left(\frac{y_o^+}{d^+}\right) =$$

$$= u^+ + \frac{1}{k_1}\frac{y_o^+}{d^+} + \left(\frac{1}{d^+ k_1} + 1\right)y_o^+\left(1 + \frac{y_o^+}{d^+} + 0\left[\left(\frac{y_o^+}{d^+}\right)^2\right]\right)$$

$$= (u^+ + y_o^+) + \frac{y_o^+}{d^+}\left[y_o^+ + \frac{2}{k_1}\right] + 0\left[\left(\frac{y_o^+}{d^+}\right)^2\right]$$

But since $y_o^+ >> u^+$ and $y_o^+ >> \frac{2}{k_1}$ the asymptotic value of y_1^+ becomes

$$y_1^+ = y_o^+\left[1 + \frac{y_o^+}{d^+}\right] + 0\left[\left(\frac{y_o^+}{d^+}\right)^2\right] \tag{30}$$

While as far as y_1^+ is concerned, this deviation may be as large as ten percent, as far as the velocity distribution is concerned where u^+ is taken proportional to $\ell n y^+$, the deviation $\Delta u^+/u^+$ given by

$$\frac{\Delta u^+}{u^+} = \frac{\ln y^+ - \ln y_o^+}{\ln y_o^+} \sim \frac{y_o^+/d^+}{\ln y_o^+}$$

is approximately within one percent.

A different interpretation of this result may be given as follows. Let y^+ be taken as a function of d^+ then at a fixed value of y_o^+

$$\frac{y_1^+}{y_2^+} = (1 + \frac{y_o^+}{d_1^+})(1 - \frac{y_o^+}{d_2^+}) = 1 + y_o^+(\frac{d_2^+ - d_1^+}{d_1^+ d_2^+})$$

Thus, for $d_2^+ > d_1^+ \Rightarrow y_1^+ > y_2^+$. Furthermore, since at increasing values of y_o^+ the difference between y_1^+ and y_2^+ increases, it follows that as the Reynolds number increases the slope of the straight line in the $u^+ \sim \ln y^+$ representation increases and correspondingly the intersection with the $y^+ \leq 1$ line occurs at a lower value of the u^+ coordinate. In terms of the notation

$$u^+ = \frac{1}{k_1}\ln y^+ + k_2'$$

it means that $1/k_1$ increases while k_2' decreases with an increasing Reynolds number. This dependence on the Reynolds number has been shown to exist in Nikuradse's data by Hinze[5].

In order to see the effect of the Reynolds number on the laminar sublayer a series expansion of the solution in this neighborhood is considered. Accordingly

$$y^+ = u^+ + \frac{1}{k_2}\frac{1}{24}(k_1 u^+)^4 + (d^+)^{-1}[(u^+)^2 + \frac{1}{k_1 k_2}\frac{11}{120}(k_1 u^+)^5 + \ldots] \tag{31}$$

By comparing this result to the strictly laminar distribution

$$u^+ = y^+ - (d^+)^{-1} y^{+2}$$

which for large Reynolds numbers and $y/d < 1/2$ may be written in the form,

$$y^+ = u^+ + (d^+)^{-1} u^{+2} + O[(d^+)^{-2}$$

the extent of the laminar sublayer may be obtained from the condition

$$\frac{1}{k_2}\frac{1}{24}(k_1 u^+)^4 << (d^+)^{-1}(u^+)^2$$

Taking as the definition of the edge of the laminar sublayer the point where

$$\frac{1}{k_2}\,\frac{1}{24}(k_1 u^+_{S.L.})^4 = 0.1(d^+)^{-1}(u^+_{S.L.})^2 \tag{32}$$

yields, after introducing the numerical constants

$$u^+_{S.L.} \simeq 27(d^+)^{\frac{1}{2}}$$

and by substitution into Eq. (31)

$$y^+_{S.L.} \simeq 27\ \left(\frac{u^* d}{\nu}\right)^{-\frac{1}{2}}$$

or

$$\frac{y_{S.L.}}{d} \simeq 27\ \left(\frac{u^* d}{\nu}\right)^{-3/2} \tag{33}$$

Hence, it follows that the laminar sublayer has a thickness proportional to the inverse three half power of the Reynolds number based on the frictional velocity, and the diameter d.

4. The Law of the Wall for Injection or Suction

The shear distribution in the wall region with injection or suction is determined for a flat plate with zero pressure gradient from the equation,

$$v\frac{du}{wdy} = \frac{1}{\rho}\,\frac{d\tau}{dy} \tag{34}$$

Integrating (34) yields

$$\frac{\tau}{\tau_w} = \frac{\rho v_w u}{\tau_w} + 1$$

which in the present notation may be written as

$$(u^+_\tau)^2 = v^+_w u^+ + 1 \tag{35}$$

where $v^+_w = v_w/u^*$

From Eq. (18)

$$U = \int_0^{u^+} \frac{du'}{d\tau^+} = \int_0^{u^+} \frac{du'}{(v^+_w u^+ + 1)^{\frac{1}{2}}} = \frac{2}{v^+_w}\left[(1+v^+_w u^+)^{\frac{1}{2}} - 1\right] \tag{36}$$

and the integral term in Eq. (27) becomes,

$$I = \int_0^U \frac{du'}{u_\tau^+} = \int_0^{u^+} \frac{du'}{(u_\tau^+)^2} = \frac{1}{v_w^+}\ln[1+v_w^+u^+] = \frac{2}{v_w^+}\ln[1+\frac{v_w^+U}{2}] \tag{37}$$

Substituting Eq. (36) into Eq. (24) yields the law of the wall for injection or suction in the form,

$$y^+ = \frac{2}{v_w^+}\ln[1+\frac{v_w^+U}{2}] + \frac{1}{k_2}\left\{[\exp(k_1 U)] - [1+(k_1 U)+\frac{1}{2}(k_1 U)^2+\frac{1}{6}(k_1 U)^3]\right\} \tag{38}$$

where U is given by Eq. (36). Now, either from the asymptotic form of Eq. (38) or directly from Eq. (20), the region outside the laminar sublayer and the buffer layer is described by the equation

$$U = \frac{1}{k_1}\ln k_2 y^+$$

Written explicitly in terms of u^+ as determined by Eq. (36), Eq. (20) becomes

$$\frac{2}{v_w^+}[(1+v_w^+u^+)^{\frac{1}{2}}-1] = \frac{1}{k_1}\ln k_2 y^+ \tag{35}$$

which is identically the modified law of the wall as suggested by Stevenson[10] based on his experimental data.

In Fig. (4) the law of the wall for injection or suction is shown for four different cases (a) $v_w^+ = 0.45$ (b) $v_w^+ = 0.064$ (c) $v_w^+ = -0.051$ and (d) $v_w^+ = -0.066$. In Figs. (4a,b) the data of Stevenson is included while in Figs. (4c,d) the data from Black and Sarnecki[11], as measured by Black is presented. While in either the large injection or the large suction cases agreement is satisfactory, in the law v_w cases where experimental scatter seems to be more pronounced, the agreement is at most fair.

It is important to note that as the injection rate increases the buffer region swells on the lower side of the distribution while the wake component appears at lower values of the parameter U, thus leaving an extremely narrow region for the logarithmic region. It seems however that as long as such a region exists regardless of how narrow it is, the buffer layer is satisfactorily represented by the present theory.

SUMMARY

A generalized law of the wall and an eddy viscosity model for wall boundary layers has been derived for regions of the flow field which depend only on a single space variable. The derivation is based on Reichardt's result, viz., that the turbulent shear decays as the cubic power of the distance from the wall and on the validity of the logarithmic law of the wall for a constant shear flow field.

Considering the velocity as the independent variable and expressing ϵ as a function thereof yields, in the constant shear case, a closed form solution for the velocity distribution. The resulting distribution is shown to be in good agreement with experimental data. Included in the comparison are some recent experiments where special care has been taken in order to obtain accurate measurements in the region very close to the wall.

By introducing a transformation, an extension of the above formulation to variable shear has been obtained. The generalized form has been then applied to the investigation of two problems (1) the effect of Reynolds number $(u^* d/\nu)$ on the velocity distribution in a pipe and, (2) the derivation of the law of the wall for boundary layers with injection or suction.

In the first problem, it has been demonstrated that, although to a very small extent there is nonetheless a steepening effect on the slope of the logarithmic portion of the law of the wall with an increase in the Reynolds number. This result is in agreement with Hinze's analysis of Nikurudse's data. Furthermore, an expression for the thickness of the laminar sublayer has been obtained which shows that $(y_{S.L}/d)$ is inversely proportional to the 3/2 power of the Reynolds number $(u^* d/\nu)$.

In the second problem a law of the wall for injection or suction has been derived in terms of the parameter v_w^+. Good agreement with available experimental data has been indicated and a tendency for a

decrease in the extent of the logarithmic portion of the law of the wall with increase of injection has been observed.

From the experimental data considered, it seems that the model proposed above correctly predicts the velocity distribution in the wall region for a variable shear flow. Obviously, a more complete description requires the matching of these results with an eddy viscosity for the outer region.

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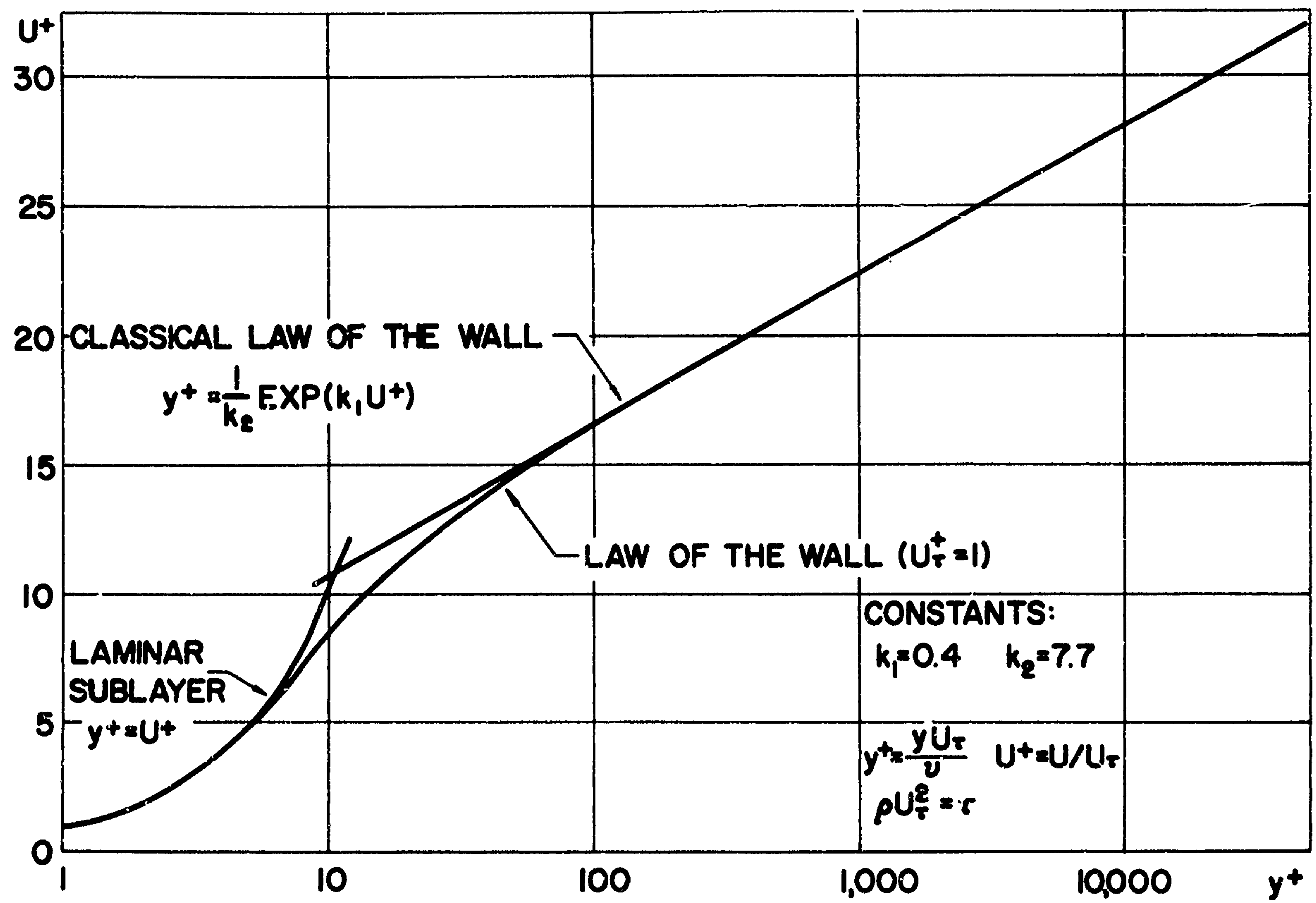

FIG. 1 THE LAW OF THE WALL FOR A CONSTANT SHEAR

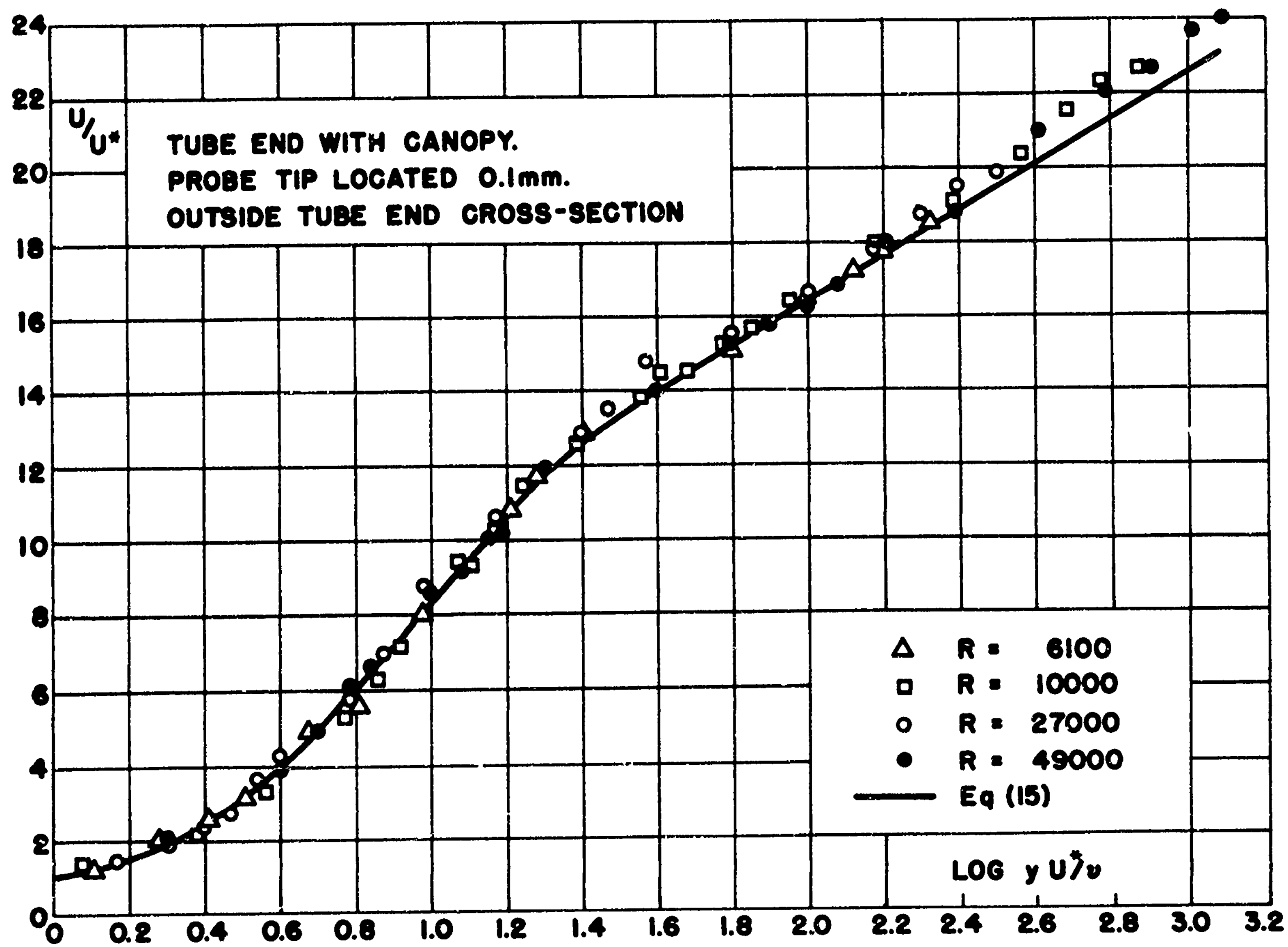

FIG. 2 A COMPARISON BETWEEN THE PRESENT ANALYSIS AND THE EXPERIMENTS OF LINDGREN[9]

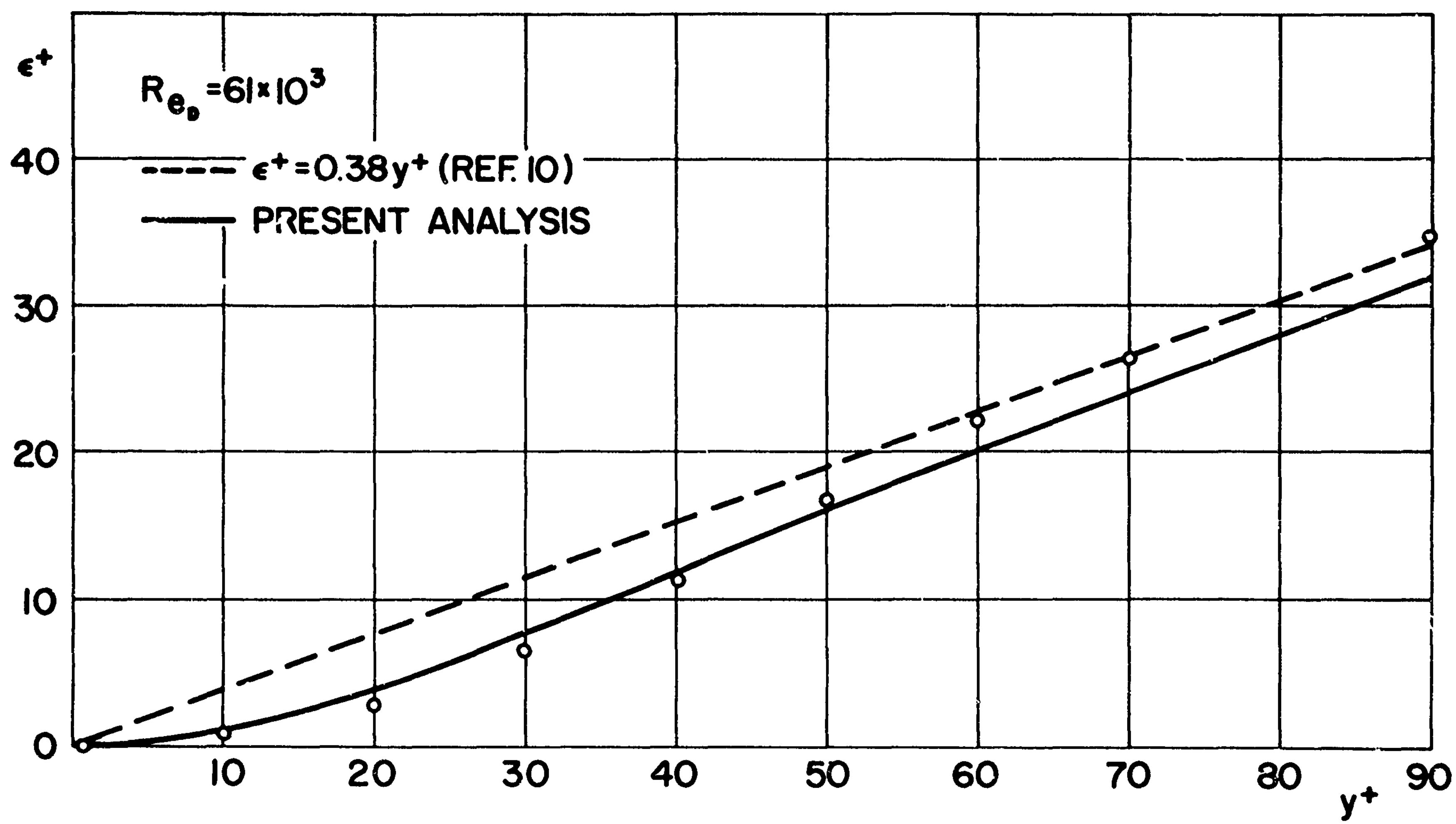

FIG. 3 VARIATION OF THE EDDY VISCOSITY IN THE WALL REGION CLOSE TO THE WALL (ABBRECHT AND CHURCHILL[10])

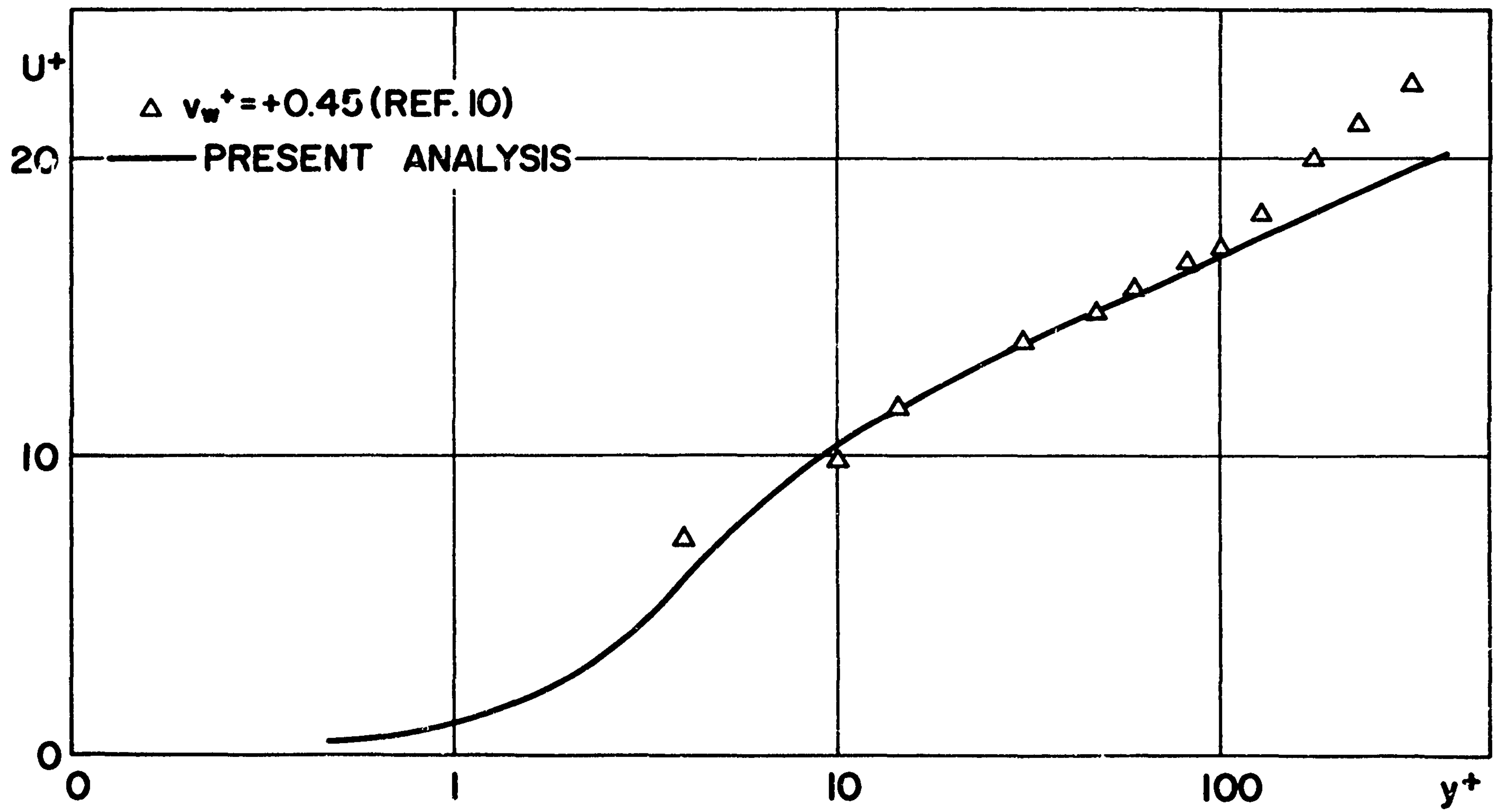

FIG. 4a LAW OF THE WALL FOR INJECTION OR SUCTION ($v_w^+ = +0.45$)

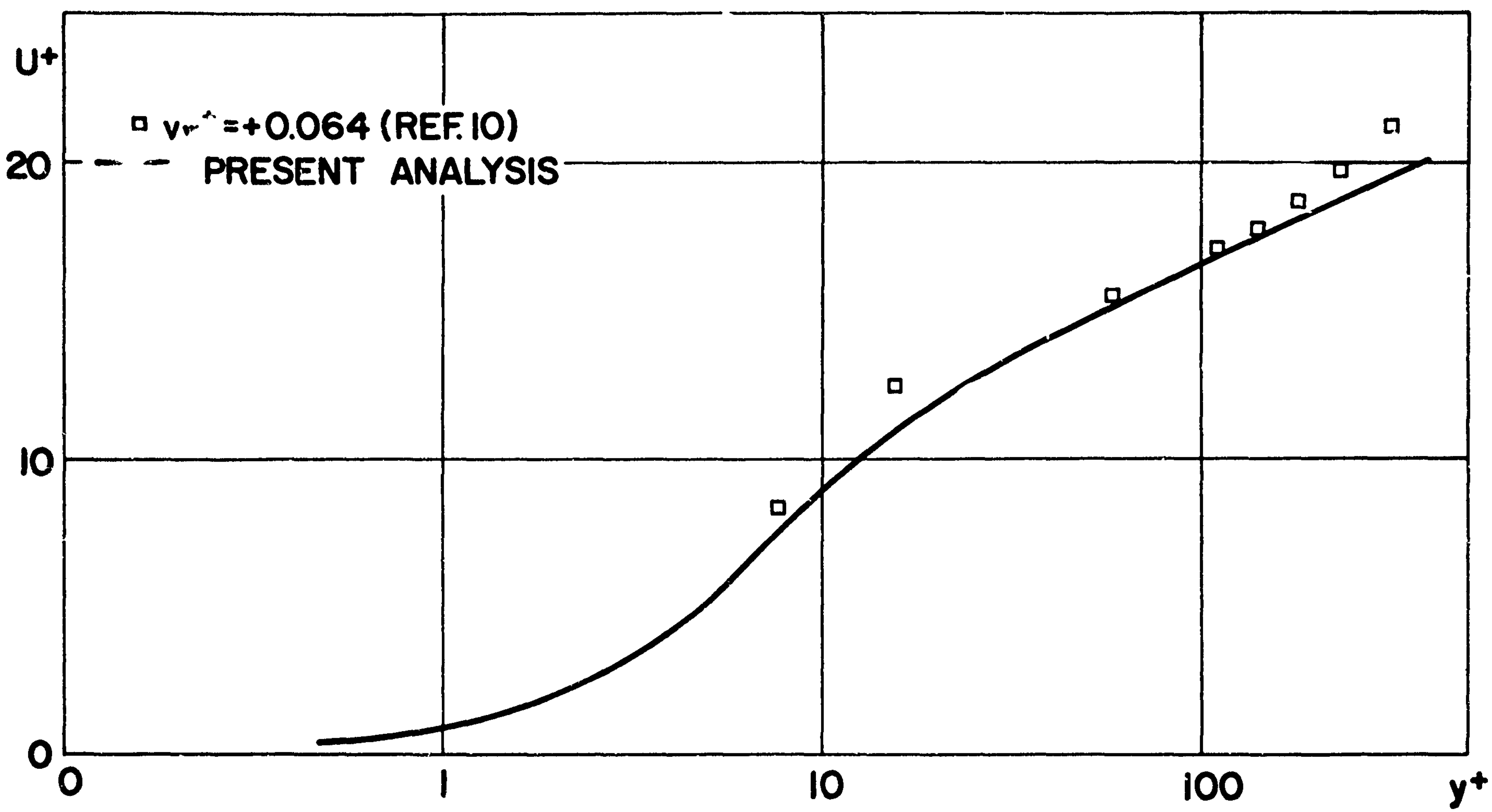

FIG. 4b LAW OF THE WALL FOR INJECTION OR SUCTION ($v_w^+ = +0.064$)

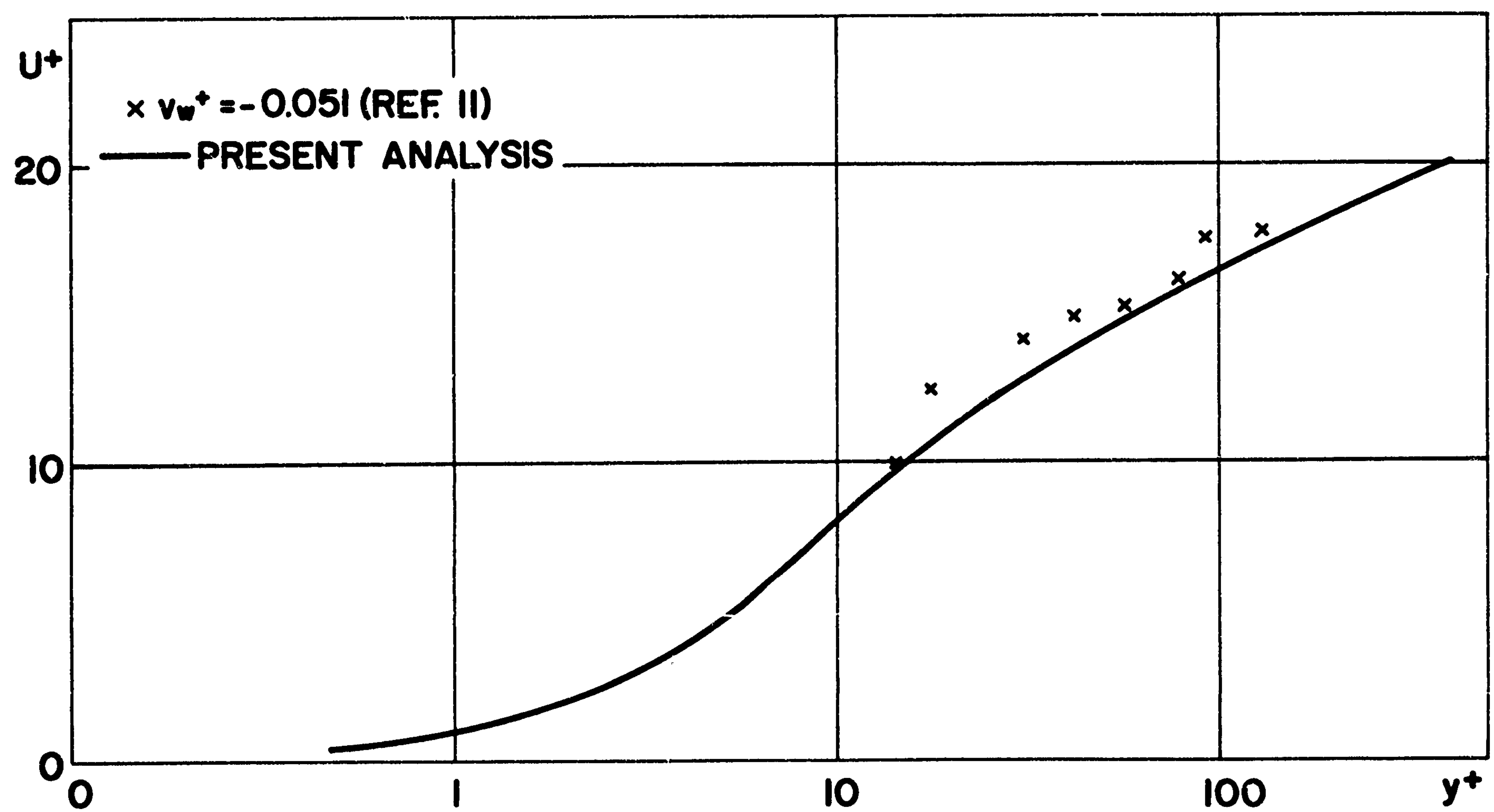

FIG. 4c LAW OF THE WALL FOR INJECTION OR SUCTION ($v_w^+ = -0.051$)

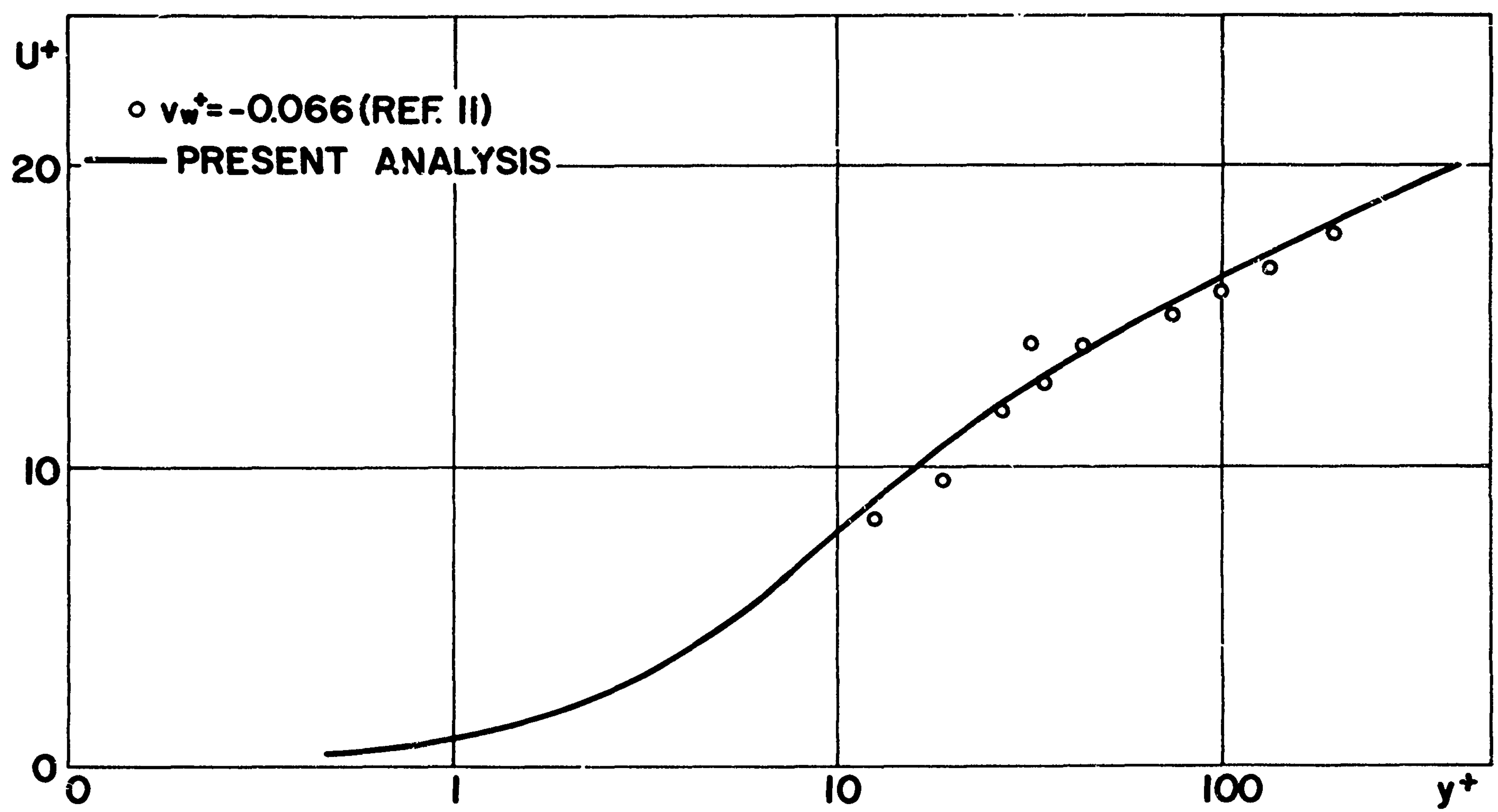

FIG. 4d LAW OF THE WALL FOR INJECTION OR SUCTION ($v_w^+ = -0.066$)

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13. ABSTRACT

Based on the logarithmic velocity distribution at distances not so far from the wall and a turbulent shear variation proportional to the cubic power of the wall distance in the immediate proximity of the wall, an analytical model for an eddy viscosity throughout the wall region is derived leading to a closed form expression for the velocity distribution. An extension of the above analysis to include shear variation results in a generalized law of the wall from which the effect of Reynolds number on the velocity distribution is derived. While many applications of the generalized solution are envisaged, as an example, the velocity distribution for the case of injection or suction is considered. The theoretical result is shown to be in agreement with available experimental data.

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
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wall boundary layer						
law of the wall						
pipe flow						
injection or suction						

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