BRL MR 1804

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MEMORANDUM REPORT NO. 1804

DATA REDUCTION FROM PHOTOGRAPHS OF BLASTS

by

Alvars Celmiņš

November 1966

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U. S. ARMY MATERIEL COMMAND
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

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Computing Laboratory

RDT & E Project No. 1P014501A14B

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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ABSTRACT

This report gives a detailed description of a code for reduction of data from blast photographs. The code furnishes data about the orientation of the camera, computes scale factors of the frames, expresses the refractive distortion of the Sun's image in proper angles and determines the parameters of a function $R_o(f)$, which gives the radius of the blast bubble as a function of the frame number f (i.e.,time). The distortion data are corrected by the code for a constant movement of the camera and can be used for the calculation of the air density within the blast bubble by other codes already in use.

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I. INTRODUCTION

Observations of light refraction by a blast bubble can be used to compute the air density within the bubble. The theory of such calculations is given in Reference 1.* The data for these calculations will usually be taken from photographs of the blast and consist of coordinates of a number of points measured on the photographs.

Manual processing of such data from photographs was found to be very time consuming. Therefore, an automatization of the whole process was suggested by Mr. Ethridge. The standards and general lines of such automatic data processing were discussed and fixed on 27 January 1966 by Mr. Ethridge, Mr. Poetschke, and the writer. The present report is a description of a possible computer program for the processing of photographs of the Sun behind a blast bubble.

The subject of the code described in this report is the transformation of coordinates, measured on the photographs, in position angles, which are necessary for the calculation of the refraction index within the bubble. Special emphasis is put onto a thorough handling of observation errors and the propagation of errors. The description of the program for the observation of the sun is sufficiently detailed for coding, though essentially machine independent.

II. CONCLUSIONS

The detailed elaboration of the program showed that a correct reduction of data from blast photographs require a quite complicated code. The estimated coding and testing time for such a routine may be about 1 programmer-year. These facts permit us to draw the following conclusions.

1. Reduction of existing blast photographs can be done faster manually than by a computer. (Because the machine program is not coded at present time.)

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* *Superscript numbers denote references which may be found on page 71.*

2. Manual reduction will furnish results inferior to those from machine reduction because the complicated mutual influence of data cannot be considered correctly during manual reduction. The significance of the machine program's superiority, however, cannot be estimated *a priori*.

3. Because of its complexity, the machine code is inflexible. Change of the technique of the experiments may require a reprogramming of large parts of the code.

III. RECOMMENDATIONS

1. The routines described in this report should be coded only if new blast observations are planned.

2. For the coding a programmer should be assigned who is familiar with the observation techniques. The same programmer should be charged with the caretaking and updating of the code.

3. A minute description of any new versions of the code should be prepared.

4. The standards of blast observations should be fixed well in advance of the experiments, so that corresponding changes of the code can be made and checked before the experiments.

5. It should be requested, that every photograph contains images of fiducials fixed inside the camera as well as images of camera independent fiducials.

IV. PROBLEM OUTLINE

The refraction index within a blast bubble can be computed if the following position angles and distances are given for a number of objects (see Figure 1 and Ref. 1):

δ - position angle of an object observed through the bubble.

δ' - position angle of the same object observed without the bubble.

δ_o - angle corresponding to the apparent radius of the bubble.

D - distance from observer to the shot point.

A - distance from the observer to the object observed.

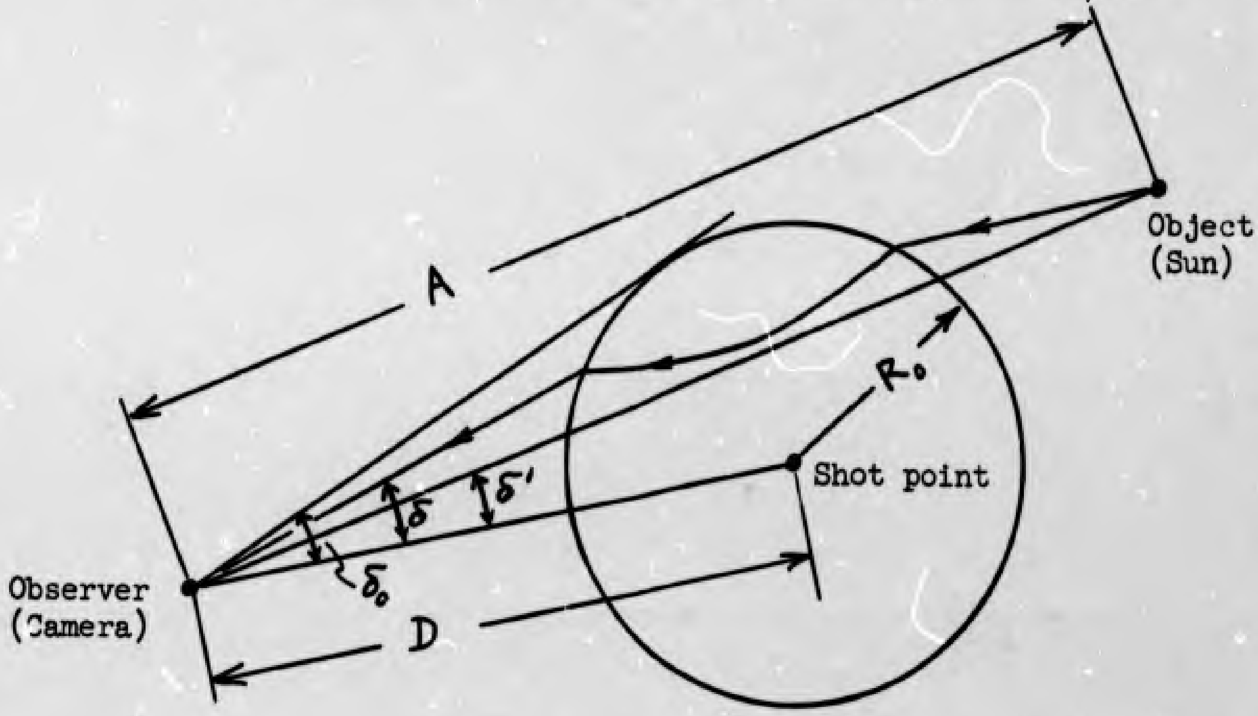


Figure 1

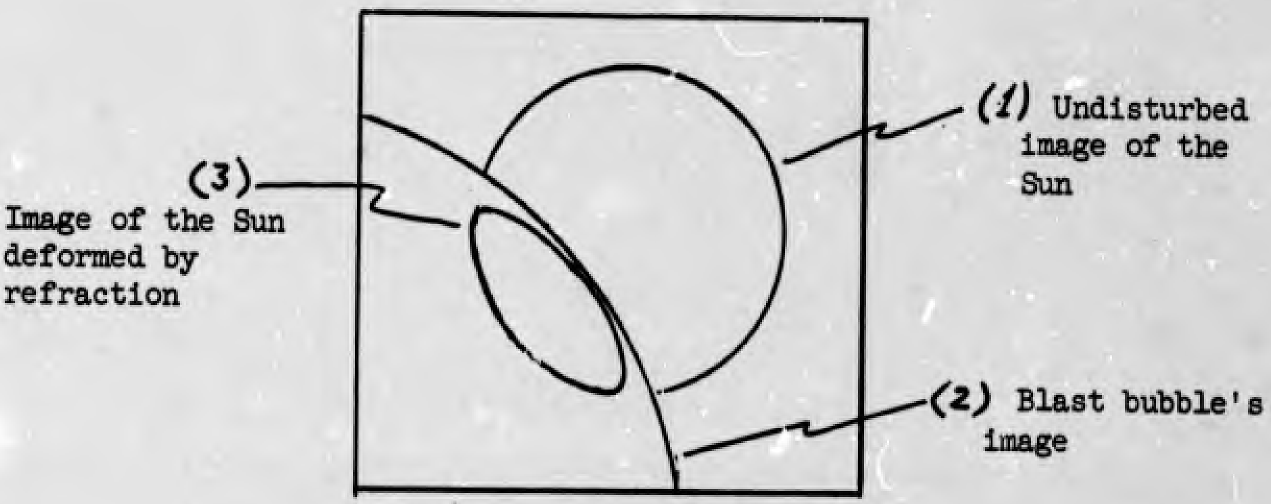


Figure 2

A photograph of the blast may look like Figure 2, where an observation of the Sun is shown as an example. (The code described in this report deals with such observations of the Sun only.) Obviously all data necessary can be extracted from such a photograph if we know the distance D between the camera and the shot point. The objects observed are, in this case, a number of points on the boundaries of the Sun's images. The processing of the photographs is as follows:

1. The coordinates of a number of points corresponding to the curves (1), (2) and (3) of Figure 2 are read. The coordinate system used is arbitrary, but it must be fixed to the camera. This can be achieved by using fiducials fixed inside the camera. In absence of such fiducials the sprocket holes of the film can be used instead.

2. The points of the curve (1) (i.e., the images of camera independent fiducials) are used to establish a conversion factor for converting lengths measured on the photograph into angles of sight. At the same time these points are used to fix the orientation of the camera.

3. The points of the curve (2) (i.e., of the blast bubble's boundary) are used to compute the coordinates of the shot point. In case the camera independent fiducial is the Sun, at this step also a possible rotation of the camera around the axis camera-Sun will be detected. This calculation step furnishes the value of δ_o (= angle corresponding to the apparent radius of the blast bubble; see Figure 1) and, in some cases, the value of D. Also the radius R_o of the blast bubble is calculated by this step.

4. The points of the curve (3) (i.e., of the refracted image) are used to compute the position angles δ and δ' (Figure 1). δ is obtained by expressing the distance from a point P of curve (3) to the shot point's image in radians. To obtain δ', first the undisturbed position P' of P is computed. δ' is then obtained by expressing the distance from P' to the shot point's image in radians.

V. BASIC ASSUMPTION AND VARIATIONS OF THE PROGRAM

The basic assumption for the calculation processes described in this program is, that all objects observed (including the camera independent fiducials) are located within a spherical angle of about 0.01 radians, if viewed from the camera. (The diameter of the Sun is $9.332 . 10^{-3}$ radians.) The consequences of this assumption is, that instead of dealing with angles of sight (δ_0, δ, δ', Figure 1) and spherical geometry, we can use a plane coordinate system fixed within the camera (i.e.,attached to the photograph) and transform the results into angles using a constant conversion factor.

This simplification of the computations may not be possible if photographic cameras with wide-angle lenses are used in order to cover large phases of the explosion by the same camera. In such cases it is recommended:

1. to have such fiducials fixed to the camera, which permit the computation of the location of the "center of the frame", i.e.,the image of the optical axis of the lens;

2. to have the focal length of the lens system measured independently of the blast experiments.

Instead of using plane corrdinates fixed in the camera, spherical coordinates with the origin in the lens system of the photographic camera should be used in these cases. The longitude φ may be counted from any camera-independent fiducial. The latitude θ of a point observed on the film can then be computed by

$$\theta = \text{arc tg} \frac{B}{F} , \tag{1}$$

where

B is the distance from the point observed to the "center of the frame", measured on the film,

and

F is the focal length of the camera's lens system.

Every point of the frame is then described by the polar coordinates θ and φ, instead of being described by plane coordinates fixed to the frame.

The formulae given in the Descriptions of the Algorithms must then be changed making use of spherical geometry.

VI. DATA FLOW AND DATA STANDARDS

A survey of data processing by the code is given by the Data Flow Chart. The general scheme of that chart may be the same even if the basic assumption about small angles of sight (Section V) cannot be made. The changes corresponding to wide-angle observations may affect some formulae of the Algorithms only.

The standards of the input data can be fixed independently of any assumption about the experiment. Therefore, in this Section a complete description of the input data and their documentation is given.

1. Photographs

The original data consist of a number of photographic films. The order of the film number is 10, the number of frames on every film is about 30. On each frame, 4 till 6 different kinds of points (curves) are obtainable, as well as some camera-fixed fiducials. The Figure 1 shows an example of such a photograph. The distinction between points of type (3) and type (4) is necessary for the present case where the Sun is observed. In case other objects are observed instead of the Sun, even more detailed distinctions might become necessary.

The coordinates of these 7 different types of points will be read using the equipment of Ballistic Measurements Laboratory. Each reading creates a punched card containing the coordinates of the point as well as an identification of the point. The identification will contain the film number, the frame number, the type of the point and the number of the point. The frames will be numbered starting with 10 in order to avoid numerical difficulties in the Algorithm 2.

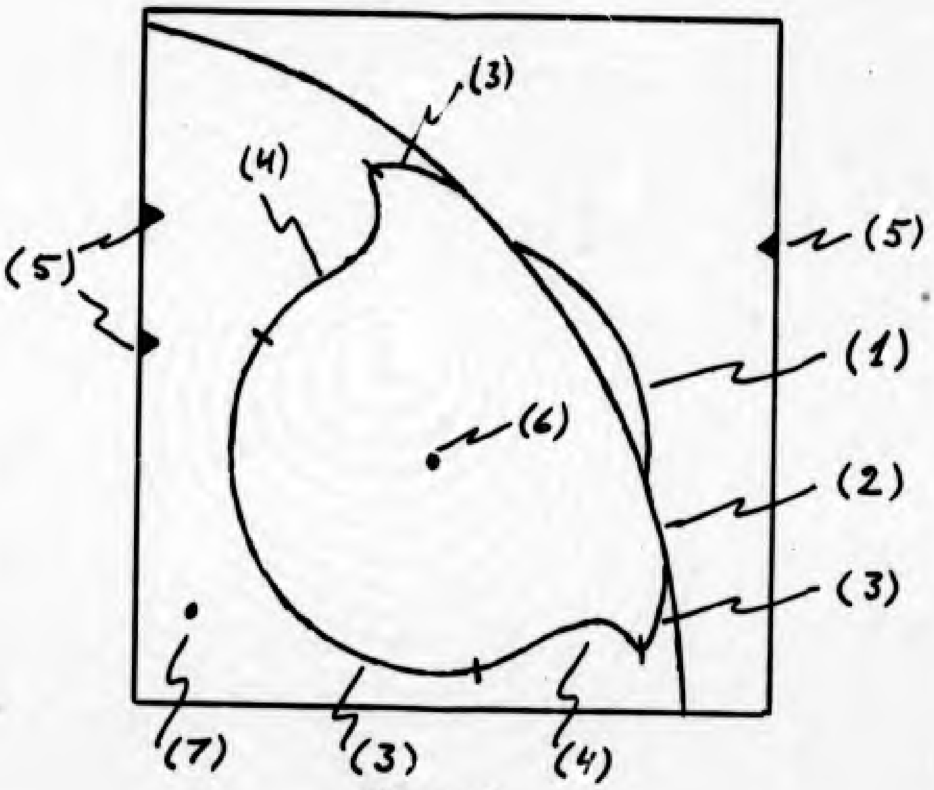


Figure 1

(1) Undisturbed camera-independent fiducials (Sun).

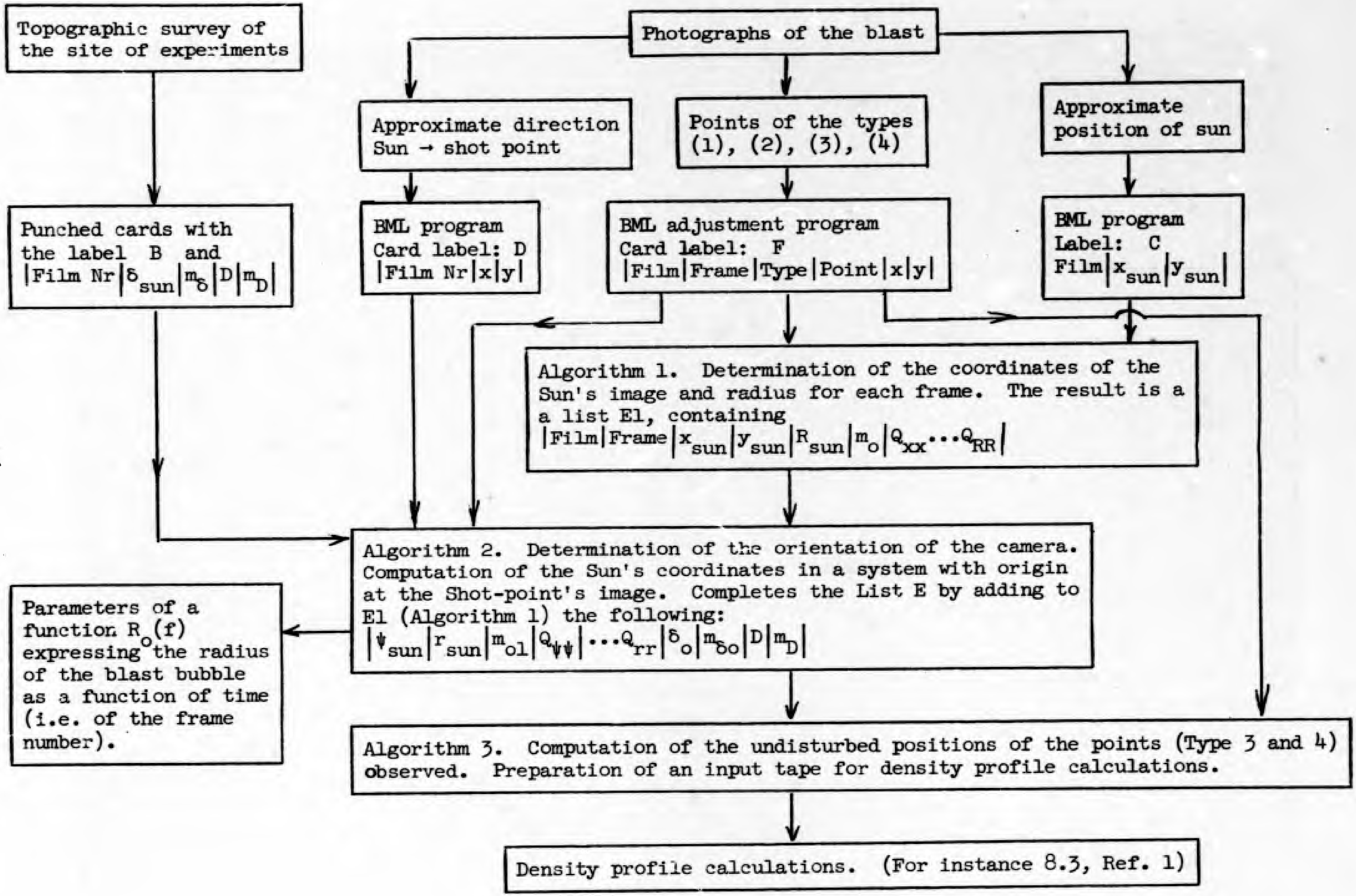
(2) Shock front (blast bubble's boundary).

(3), (4) Image of the object, disturbed by refraction. This curve is subdivided in 2 parts, depending on the original location of the points. If a point belongs to that part of the Sun's disk, which is on the shot-point's side, it is labeled as "Type (3) point", otherwise it is of "Type (4)". (Note, that Figure 2 of Section IV demonstrates a case, where no points of "Type (4)" can be observed.)

(5) Images of camera-fixed fiducials.

(6) Approximate position of the Sun's center. (See Card C in Section VI.2.)

(7) Approximate position of a point on the connection Sun-Shot point. (See Card D in Section VI.2).



Data Flow Chart.

2. BML Utility Programs

Utility programs existing in the Ballistic Measurements Laboratories will be used for the first adjustment of the readings. These programs adjust the data in such a way, that the camera-fixed fiducials (points of Type (5)) have the same coordinates on all frames of the same film. By this adjustment the effects of possible distortion of the films after their exposure are eliminated. The result is again a punched card for each point, containing the identification of the point and its adjusted coordinates.

In view of the later processing of these cards, a label will be punched in every card (see Data Flow Chart). The following labels will be used:

F - for all points of the types (1) through (4). These points are camera independent fiducials (type (1)), blast bubble's boundary (type (2)) and disturbed images (types (3) and (4)). The particular type is given by the identification of the point. There will be for each frame about 200 cards with the label F.

C - for the approximate position of the Sun. There will be one card with label C for each frame. (Point type (6)).

D - for the point of type (7), located approximately on the line connecting the image of the Sun with the image of the shot point. There will be one D-card for each frame.

Since the total number of F - cards may be up to 6,000 for each film, it is advisable to sort these cards according to the film number and handle the data of each film separately.

3. Geographic Coordinates

As mentioned in Section IV, Problem Outline, some knowledge about the locations of camera, shot point and objects observed is necessary in addition to the blast photograph. Particularly for each point observed, the corresponding values of D, A and δ' are needed. (See Figure 1, Section IV.) If the objects observed are at finite distances from the camera, these data can be obtained most easily by a topographic survey of the site of experiment. The values of D, A and δ', computed from the results of the survey will in general be correlated. The corresponding cofactors will be furnished by the evaluation of the results of the survey.

In the case considered here, the "objects" observed are not-identified points on the boundary of the Sun's image. The distance A is therefore constant for all objects. Since its value is by several orders bigger than the value of D, we can even assume, that A is infinite. (D is the order of 1 to 10 km, whereas $A = 1.495 . 10^8$ km.) The distance D can be obtained from a topographic survey of the site of experiment. The values of δ' cannot be obtained by a survey before the experiment because the points observed are not identified. However, the value of δ', corresponding to an observation δ can be computed if the angle δ_{sun} (Figure 2) is known.

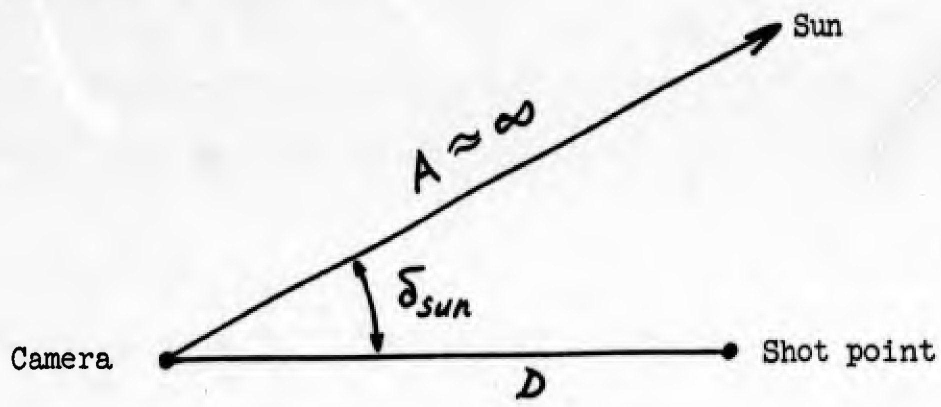


Figure 2

This will be done in Algorithm 3. The value of δ_{sun} can be computed with the aid of Solar Tables if the geographical coordinates of the camera and shot point and the time of the explosion are known. Under favorable circumstances δ_{sun} can also be computed from the blast photographs, if only D is given. (About particulars see the description of Algorithm 2).

The computation of all these quantities is not the subject of this report. We assume here, that δ_{sun} and D are obtained somehow and the results available in form of a punched card for each film, with the following contents:

Film Nr	δ_{sun}	m_δ	D	m_D	"B"

m_δ and m_D are the standard errors of δ_{sun} and D, respectively. A correlation between these values may exist, but will have no effect because of the assumption $A = \infty$. "B" is the label of the card.

In the general case, where A is finite, corresponding data for each object observed must be furnished. In this case the correlations between the quantities D, A and δ' must be considered. We need then the following information for every object observed:

Film Nr	Frame Nr	Point Nr	δ'	D	A	m_o	$Q_{11} \cdots Q_{33}$	"B"

The Q_{ik} are the cofactors of the 3 quantities δ', D and A. m_o is the standard error of unit weight, corresponding to the cofactors. "B" is the label of the list.

It is obvious, that in this general case 2 punched cards for each object are needed for all the information necessary.

If the objects observed are arranged by some pattern, it will be sufficient to furnish some parameters, which characterize the pattern. (In the present case these parameters are the values of δ_{sun} and D.) The Algorithms must be changed in such case accordingly.

4. Computer Algorithms

The processing of the data is subdivided into 3 Algorithms. (See Data Flow Chart.) The subdivision was chosen such, that each Algorithm is essentially independent of the others. Connections between the Algorithms are maintained by the data only. The programming standards and formulae necessary for the coding of the Algorithms are given in the corresponding descriptions (Sections VII, VIII and IX.)

Besides, of the 3 Algorithms indicated in the Data Flow Chart, also a general least squares subroutine is necessary. This subroutine, which must be able to handle correlated data, is described in the Appendix.

The machine program will need the following tape units:

Tape unit 1 - data tape for the least squares subroutine and output tape of Algorithm 3.

Tape unit 2 - output tape of Algorithm 2.

Tape unit 4 - tape for the storage of the cards F (see Section VI.2) in case of storage overflow.

Tape unit 7 - temporary storage tape for the least squares subroutine.

Tape unit 8 - "printer tape" to store output for a printer with 132 characters per line.

All tape units, except tape unit 8 will be used in binary mode. For particulars see the descriptions of the Algorithms.

VII. DESCRIPTION OF ALGORITHM 1

1. Purpose of the Algorithm

The purpose of the Algorithm 1 is twofold:

a) To compute a scale factor for the conversion of distances on the film in angles of sight;

b) To estimate the position and the orientation of the photographic camera.

2. Method Applied

2.1 Assumption.

a) We assume that the distance from the camera to the Sun is constant during the observations (i.e., for one film), and equal to $1.495 \cdot 10^{8}$ Km. Since the real value varies by about 1.5% during a year, the proper distance for the day of observations should be used instead of this average when reducing real data.

b) We assume that the only fiducial available is the Sun's image on some frames of the film under consideration.

c) We assume that the orientation of the camera is a linear function at time.

2.2 Method. On each frame the Sun's image is approximated by a circle using a least squares process. Because of the assumption a) the angle supported by the Sun's radius is constant and equal to

$$\text{arc sin} \frac{6.955}{1.495} 10^{-3} = 4.666 \cdot 10^{-3} [\text{rad}]. \quad * \qquad (1)$$

The results of the least squares approximation furnish therefore at the same time the scale factor required as well as the coordinates of the Sun's center on the frame. From these coordinates the orientation of the camera can be computed.

The orientation of the camera might not be constant. In order to compute the orientation of the camera for such frames on which the Sun cannot be seen, the results of the least squares evaluations of single frames are used to establish the Sun's coordinates as functions of the number of the frame. These functions are linear because of the assumption c).

Before these functions are used to compute the Sun's coordinates it is checked whether the coefficient of the linear term is significantly different from zero. If such is not the case, a constant orientation of the camera is assumed and a joint least squares approximation worked out for all frames containing the Sun's image.

* *As mentioned in 2.1 a) this angle varies during the year and its actual value of the day of observations should be used.*

3. Formulae for the Approximation of the Sun's Image

We assume that the coordinates (X_{j1}, X_{j2}) of $r > 3$ points on the boundary of the Sun's image are given. Call the coordinates of the center of the Sun's image (y_1, y_2) and the radius of the image y_3. Then for correct observations the following relations hold (see Figure 1)

$$
\begin{aligned}
F(\bar{x}_j;\bar{y}) &= F(x_{j1}, x_{j2}; y_1, y_2, y_3) \\
&= \sqrt{(x_{j1}-y_1)^2 + (x_{j2}-y_2)^2} - y_3 = 0 \quad (j=1,2,\ldots,r)
\end{aligned} \tag{2}
$$

The real observations, however, do not furnish the values x_{j1} and x_{j2}, but some approximations X_{j1} and X_{j2} due to observation errors. We assume that

$$
\begin{aligned}
x_{j1} &= X_{j1} + \xi_{j1} \\
& \qquad\qquad (j=1,2,\ldots,r) \\
x_{j2} &= X_{j2} + \xi_{j2}
\end{aligned} \tag{3}
$$

where ξ_{ji} are the (negative) errors of the observations. The observation of X_{ji} may have the weight p_{ji}. Then a least squares problem can be formulated as follows:

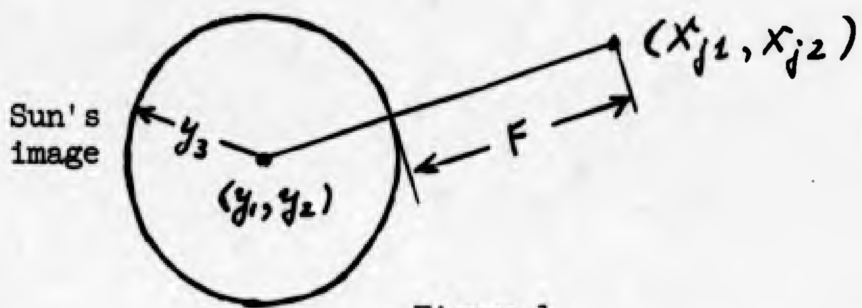


Figure 1

Find such y_1, y_2, y_3 and ξ_{j1}, ξ_{j2} $(j=1,2,\ldots,r)$, that

$$
\sum_{j=1}^{r} (p_{j1}\xi_{j1}^2 + p_{j2}\xi_{j2}^2) = \text{Min} \tag{4}
$$

and the r equations (2) are satisfied.

The solution of this problem is handled in detail in the Appendix where also a subroutine for the solution is described. In order to apply that subroutine to the present problem we need the partial derivatives of F and some approximations Y_1, Y_2, Y_3 of the unknown parameters y_1, y_2, y_3. The partial derivatives are, in the notation of the Appendix, as follows:

$$A_{j1} = \frac{X_{j1} - Y_1}{\sqrt{(X_{j1}-Y_1)^2 + (X_{j2}-Y_2)^2}} \quad (j=1,2,\ldots,r) \qquad (5)$$

$$A_{j2} = \frac{X_{j2} - Y_2}{\sqrt{(X_{j1}-Y_1)^2 + (X_{j2}-Y_2)^2}} \quad (j=1,2,\ldots,r) \qquad (6)$$

$$B_{j1} = - A_{j1} \qquad (j=1,2,\ldots,r) \qquad (7)$$

$$B_{j2} = - A_{j2} \qquad (j=1,2,\ldots,r) \qquad (8)$$

$$B_{j3} = - 1 \qquad (j=1,2,\ldots,r) \qquad (9)$$

The function F_j of the Appendix is

$$F_j = \sqrt{(X_{j1}-Y_1)^2 + (X_{j2}-Y_2)^2} - Y_3 \quad (j=1,2,\ldots,r) \qquad (10)$$

4. Formulae for the Trend Investigation

Assume, that we have obtained from the approximations described in the Section 3 the results of K frames. The result is a list of the following type with K lines:

Film number	Frame number f	y_1	y_2	y_3	m_o	Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	Q_{23}	Q_{31}	Q_{32}	Q_{33}

The Q_{ij} are elements of the cofactor matrix of the y_i and m_o is the corresponding standard error of weight one, as determined by the adjustments. The values of m_o are equal to one if the accuracies of the data X_{ij} (Section 3) were estimated correctly and the number of data was sufficiently large.

The list mentioned above includes only such frames, on which the undisturbed image of the Sun can be seen. In order to have the Sun's coordinates y_1 and y_2 available also for other frames we compute linear time functions as approximations for y_1 and y_2. (See assumption c Section 2.1.) The radius of the Sun's image y_3 is fixed for all frames by computing a weighted average from the list.

The time is represented in the present case by the number of the frame f. Hence the functions to be fixed are

$$\begin{aligned} y_1(f) &= a + A\cdot f \\ y_2(f) &= b + B\cdot f \\ y_3(f) &= c \end{aligned} \tag{11}$$

These functions depend on 5 parameters. Since the values of y_1, y_2 and y_3, given in the list, are correlated, also the corresponding functions (11) will be correlated. Their cofactors will be computed here in two steps, computing first the cofactors of the 5 parameters a, A, b, B and c and then the cofactors of the 3 functions.

First the following quantities are computed by summing over all K frames:

$$\begin{aligned} &S_{10} = \sum \frac{1}{m_o^2\, Q_{11}}\,; \quad S_{11} = \sum \frac{f}{m_o^2\, Q_{11}}\,; \quad S_{12} = \sum \frac{f^2}{m_o^2\, Q_{11}} \\ &S_{20} = \sum \frac{1}{m_o^2\, Q_{22}}\,; \quad S_{21} = \sum \frac{f}{m_o^2\, Q_{22}}\,; \quad S_{22} = \sum \frac{f^2}{m_o^2\, Q_{22}} \\ &S_{30} = \sum \frac{1}{m_o^2\, Q_{33}} \end{aligned} \tag{12}$$

With these sums the determinants of the normal equations for the 5 parameters are

$$D_1 = S_{10}\, S_{12} - S_{11}^2$$

$$D_2 = S_{20}\, S_{22} - S_{21}^2 \tag{13}$$

$$D_3 = S_{30}$$

Having the values (12) and (13) we compute for every frame the following expressions

$$S_a = \frac{1}{m_o^2\, Q_{11}\, D_1} (S_{12} - S_{11} \cdot f)$$

$$S_A = \frac{1}{m_o^2\, Q_{11}\, D_1} (S_{10} \cdot f - S_{11})$$

$$S_b = \frac{1}{m_o^2\, Q_{22} \cdot D_2} (S_{22} - S_{21} \cdot f) \tag{14}$$

$$S_B = \frac{1}{m_o^2\, Q_{22} \cdot D_2} (S_{20} \cdot f - S_{21})$$

$$S_c = \frac{1}{m_o^2\, Q_{33} \cdot D_3}$$

The 5 parameters of the 3 functions can then be obtained by computing the following sums over all K frames

$$a = \sum S_a \cdot y_1 \; ; \; A = \sum S_A \cdot y_1$$

$$b = \sum S_b \cdot y_2 \; ; \; B = \sum S_B \cdot y_2 \tag{15}$$

$$c = \sum S_c \cdot y_3$$

At the same time the corresponding cofactors can be computed by summing the expressions given in (16). Because of the symmetry of the cofactor matrix there are only 15 different cofactors:

$$
\begin{aligned}
&Q_{aa} = \sum m_o^2\, Q_{11}\, S_a^2 \;;\; Q_{aA} = \sum m_o^2\, Q_{11}\, S_a\, S_A \;;\; Q_{ab} = \sum m_o^2\, Q_{12}\, S_a\, S_b \\
&Q_{aB} = \sum m_o^2\, Q_{12}\, S_a\, S_B \;;\; Q_{ac} = \sum m_o^2\, Q_{13}\, S_a\, S_c \\
&Q_{AA} = \sum m_o^2\, Q_{11}\, S_A^2 \;;\; Q_{Ab} = \sum m_o^2\, Q_{12}\, S_A\, S_b \;;\; Q_{AB} = \sum m_o^2\, Q_{12}\, S_A\, S_B \\
&Q_{Ac} = \sum m_o^2\, Q_{13}\, S_A\, S_c \\
&Q_{bb} = \sum m_o^2\, Q_{22}\, S_b^2 \;;\; Q_{bB} = \sum m_o^2\, Q_{22}\, S_b\, S_B \;;\; Q_{bc} = \sum m_o^2\, Q_{23}\, S_b\, S_c \\
&Q_{BB} = \sum m_o^2\, Q_{22}\, S_B^2 \;;\; Q_{Bc} = \sum m_o^2\, Q_{23}\, S_B S_c \\
&Q_{cc} = 1
\end{aligned}
\tag{16}
$$

(Q_{cc} is equal to 1 and need not be computed.)

Before (15) and (16) are used to calculate the Sun's coordinates for all frames, it is checked whether the parameters A and B are significantly different from zero. In order to do this check, first the standard errors of weight one for the 3 functions (11) must be computed. These quantities are computed by the formulae

$$m_1 = \sqrt{\frac{1}{k-2} \sum^{k} (y_1 - a - Af)^2 \frac{1}{m_o^2\, Q_{11}}} \tag{17}$$

$$m_2 = \sqrt{\frac{1}{k-2} \sum^{k} (y_2 - b - Bf)^2 \frac{1}{m_o^2\, Q_{22}}} \tag{18}$$

$$m_3 = \sqrt{\frac{1}{k-1} \sum^{k} (y_3 - c)^2 \frac{1}{m_o^2\, Q_{33}}} \tag{19}$$

The sums in (17), (18) and (19) are extended over all k frames with the Sun's image.

The formula (11) for $y_1(f)$ can be used if

$$|A| > 3\, m_1 \sqrt{Q_{AA}} \tag{20}$$

and the corresponding formula for $y_2(f)$ can be used if

$$|B| > 3\, m_2 \sqrt{Q_{BB}}\ . \tag{21}$$

If (20) is not satisfied, a new approximation of $y_1(f)$ is computed, namely

$$y_1(f) = a. \tag{22}$$

The formulae for computing a and the cofactors can still be used, except that the following values must be assumed throughout the formulae.

$$A = 0;\ S_{11} = 0;\ S_{12} = 1;\ S_A = 0. \tag{23}$$

The value of Q_{aa} is in this case 1 and, therefo e, need not be computed. Instead of (17) we have then

$$m_1 = \sqrt{\frac{1}{k-1} \sum^{k} (y_1 - a)\ \frac{1}{m_o^2\ Q_{11}}} \tag{24}$$

If (21) is not satisfied, we assume, that $y_2(f)$ is a constant, replacing (11) by

$$y_2(f) = b. \tag{25}$$

The corresponding changes of the formulae for b and the cofactors are achieved by assuming

$$B = 0;\ S_{21} = 0;\ S_{22} = 1;\ S_B = 0. \tag{26}$$

In this case the value of Q_{bb} is 1 and instead of (18) the following formula is used for the computation of the standard error of unit weight

$$m_2 = \sqrt{\frac{1}{k-1} \sum^{k} (y_2 - b) \frac{1}{m_o^2 Q_{22}}} \; . \tag{27}$$

The cofactors of the 3 functions $y_1(f)$, $y_2(f)$ and $y_3(f)$ are finally given by the formulae

$$m_o = 1 \tag{28}$$

$$Q_{y1y1} = m_1^2 \, (Q_{aa} + 2Q_{aA}f + Q_{AA}f^2)$$

$$Q_{y1y2} = m_1 m_2 \, (Q_{ab} + (Q_{aB} + Q_{Ab})f + Q_{AB}f^2)$$

$$Q_{y1y3} = m_1 m_3 \, (Q_{ac} + Q_{Ac}f)$$

$$Q_{y2y2} = m_2^2 \, (Q_{bb} + 2Q_{bB}f + Q_{BB}f^2)$$

$$Q_{y2y3} = m_2 m_3 \, (Q_{bc} + Q_{Bc}f)$$

$$Q_{y3y3} = m_3^2 \, Q_{cc}$$

With these formulae and the corresponding expressions for $y_1(f)$, $y_2(f)$ and $y_3(f)$, these 3 functions and their cofactors can be computed for any f, i.e.,for every frame.

In case both conditions, (20) and (21) are not satisfied, we assume, that the Sun's coordinates are constant throughout the film. In this case it is simpler to repeat the least squares process of Section 3 using data from all frames of the film simultaneously instead of using the formulae of this Section.

5. Flow Chart of Algorithm 1

In this Section the Flow Chart of Algorithm 1 is presented. The mathematical formulae necessary for the calculations were derived in Sections 3, and 4.

The machine program for the Algorithm needs 2 tape units, namely

tape unit 1 as data tape for the least squares program and

tape unit 4 for the storage of the sorted cards F .

Besides these tape units, also the tape unit 7 is used by the least squares routine for temporary storage. The tape unit 4 might not be necessary if the number of cards F is so small that all data from cards F can be stored in the memory of the computer.

Comments to the Flow Chart.

a1. The sorted data may be stored on tape unit 4, if the number of cards F becomes too large. This might be the case if several films are processed in one run. In such case the cards are sorted according to the film number and stored in the same way on the tape. After the end of data of each film the sentinel (ENDbFILMbb) is written on the tape. The end of the data is indicated by the sentinel (TOTALbENDb). The tape recording is done in binary mode using the Binary Tape Output routine BT.WT.

a2. This printout starts on a new page and has the headline and formats as given on the next page.

```
DATE 12.12.1966,    TIME 12 HRS. 12.12 MIN.
Blank line
LEAST SQUARES APPROXIMATIONS OF THE SUN'S IMAGE FOR SINGLE FRAMES.
Blank line
  X  AND  Y  ARE THE COORDINATES OF THE SUN'S IMAGE IN MILLIMETERS
  R  IS THE RADIUS OF THE SUN'S IMAGE IN MILLIMETERS
  M  IS THE STANDARD ERROR OF WEIGHT ONE
  Q  ARE THE ELEMENTS OF THE COFACTOR MATRIX
Blank line
.................................................................
FILM NR. FRAME NR.      X            Y            R           M        QXX

  +123      +123     +1.1234+12  +1.1234+12  +1.1234+12  +1.12+12  +1.12+12
```

```
.......................................
  QXY       QXR       QYY       QYR       QRR

+1.12+12  +1.12+12  +1.12+12  +1.12+12  +1.12+12
```

b. The results of the approximations of Section 4 are printed with the following text:

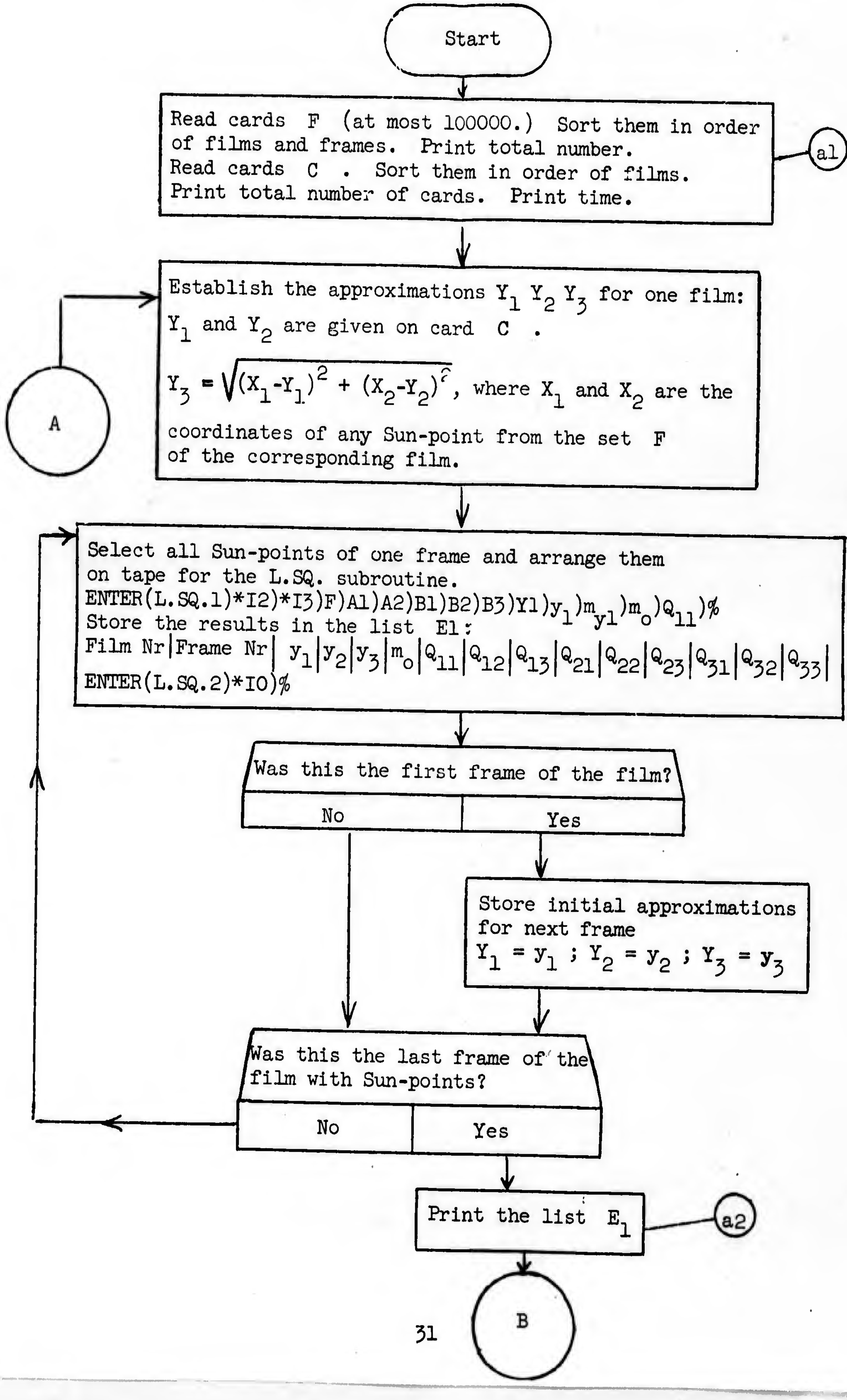
```
3 Blank lines
THE QUANTITIES  X, Y AND R  OF THE ABOVE LIST CAN BE APPROXIMATED BY
THE FOLLOWING LINEAR FUNCTIONS OF THE FRAME NUMBER  F.
Blank line
X   = (+1.1234+12) + (+1.1234+12)*F
Y   = (+1.1234+12) + (+1.1234+12)*F
R   =   1.1234+12
Blank line
THE CORRESPONDING STANDARD ERROR OF UNIT WEIGHT IS 1.1234+12
Blank line
THE COFACTORS OF THESE FUNCTIONS ARE
Blank line
QXX = (+1.123+12) + (+1.123+12)*F + (+1.123+12)*F**2
QXY = (+1.123+12) + (+1.123+12)*F + (+1.123+12)*F**2
QXR = (+1.123+12) + (+1.123+12)*F
QYY = (+1.123+12) + (+1.123+12)*F + (+1.123+12)*F**2
QYR = (+1.123+12) + (+1.123+12)*F
QRR =  +1.123+12
```

c. Depending on the corresponding branch, the following is printed after 3 blank lines:either

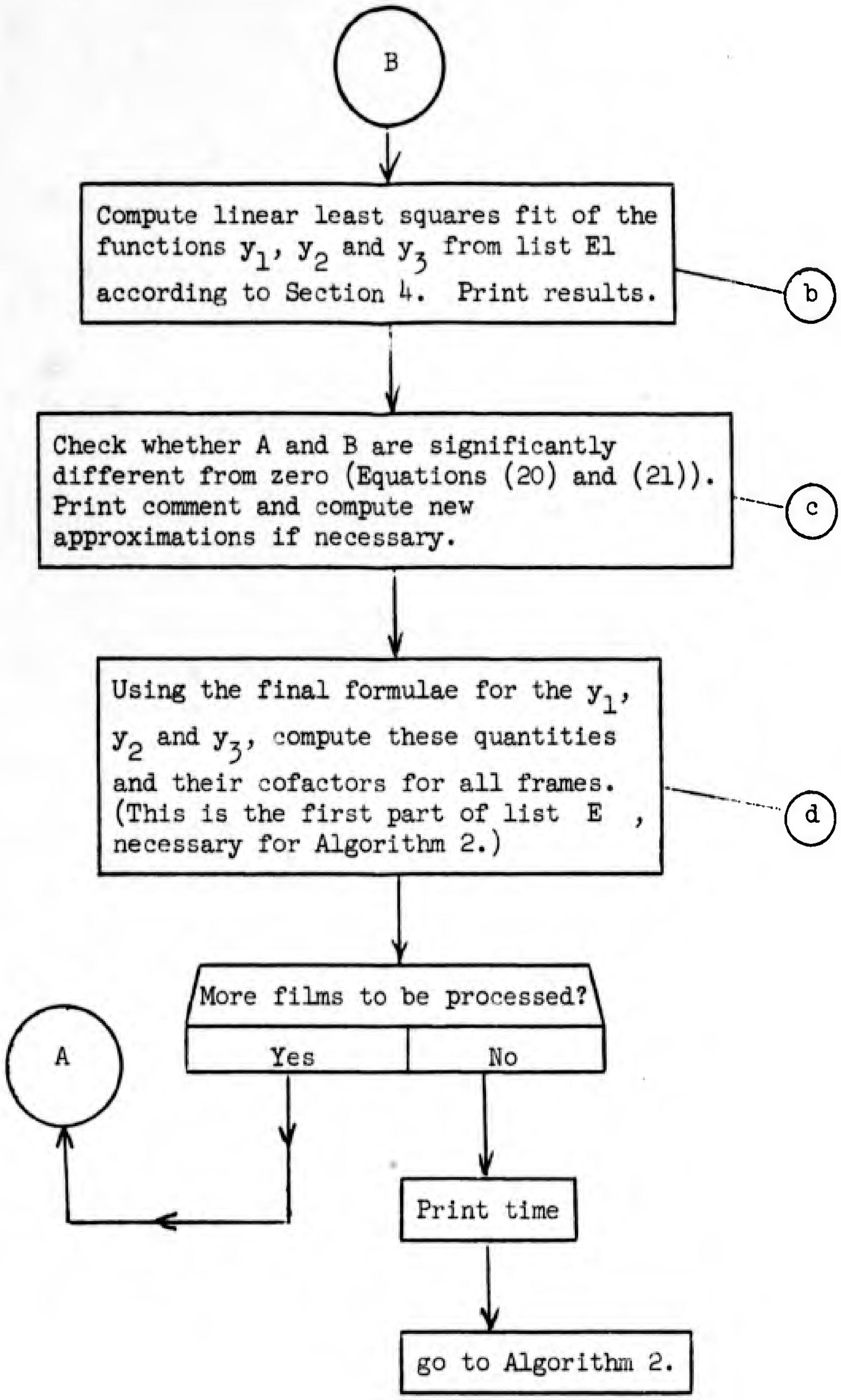
THESE FUNCTIONS WILL BE USED FOR FURTHER CALCULATIONS.

or

FLOW CHART OF ALGORITHM 1, PART 1



FLOW CHART OF ALGORITHM 1, PART 2



THE LINEAR TERM FOR X OR Y IS NOT SIGNIFICANTLY DIFFERENT FROM ZERO. THEREFORE NEW LEAST SQUARES APPROXIMATIONS WILL BE CALCULATED.

In this case the new approximations are printed with the same text as the first approximations. (See comment b.)

d. In case printing of these results is required the output starts on a new page with the following text and formats:

```
FINAL APPROXIMATION OF THE SUN'S IMAGE FOR THE FILM NR. +123.
Blank line
  X  AND  Y  ARE THE COORDINATES OF THE SUN'S IMAGE IN MILLIMETERS
  R  IS THE RADIUS OF THE SUN'S IMAGE IN MILLIMETERS
  M  IS THE STANDARD ERROR OF WEIGHT ONE
  Q  ARE THE ELEMENTS OF THE COFACTOR MATRIX
Blank line
.......................................................................
FRAME NR.      X            Y            R            M        QXX      QXY

  +123     +1.1234+12  +1.1234+12  +1.1234+12  +1.12+12  +1.12+12  +1.12+12

......................................
   QXR        QYY        QYR        QRR

+1.12+12  +1.12+12  +1.12+12  +1.12+12
```

VIII. DESCRIPTION OF ALGORITHM 2

1. Purpose of the Algorithm

The purpose of the Algorithm 2 is as follows:

a) To determine the coordinates of the shot point's image for each frame.

b) To determine the radius of the blast bubble's image for each frame.

For the evaluation of the observations all measurements must be expressed in a coordinate system with origin at the shot point's image. The task a) will provide the information necessary for a corresponding coordinate transformation.

The task b) will furnish data for the calculation of the expansion rate of the blast bubble as well as the value of δ_o necessary for the calculation of refraction coefficients.

2. Method Applied

2.1 Assumptions.

a) We assume, that approximate values of the angle δ_{sun} and the distance D (see Figure 1) are furnished by the cards B for each film.

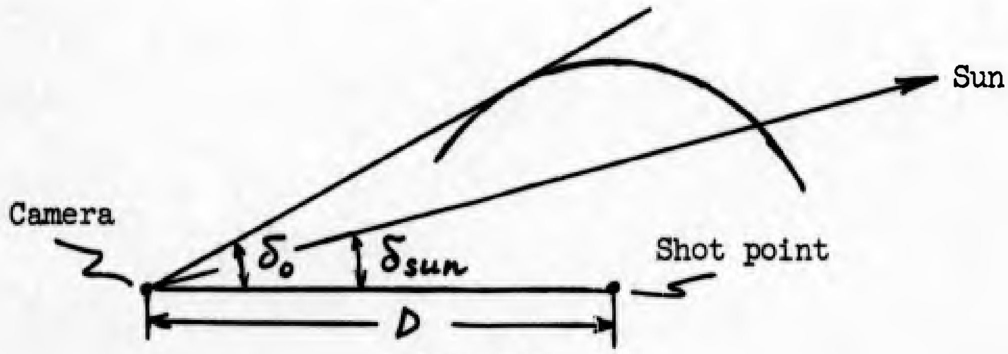


Figure 1

b) We assume, that on some frames the image of the blast bubble's boundary can be detected. Points of this image will be referred to as "shock-points".

c) We assume, that approximate coordinates of a point P on the image of the line Sun-shot point is provided for each film by the cards D .

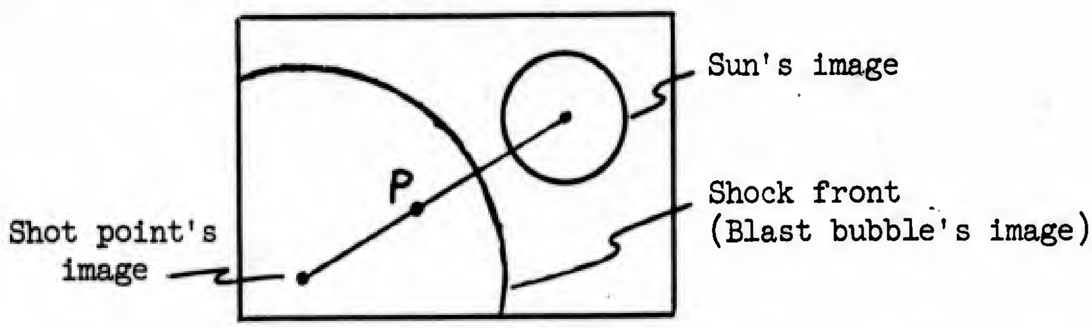


Figure 2

d) We assume, that Algorithm 1 has been completed and the corresponding results are in the storage together with the (sorted) data from cards F type,

e) We assume, that the orientation of the camera is a linear function of the time.

2.2 Method. On each frame we approximate the image of the blast bubble by a circle using a least squares approximation. This process will furnish simultaneously the coordinates of the shot point's image as well as the apparent radius of the blast bubble. As in Algorithm 1 a subsequent trend analysis will be carried out in order to detect constant movements of the camera. Some components of such movements were already determined by Algorithm 1 and their effects eliminated. The Algorithm 2 will determine a possible rotation of the camera around the axis camera-sun and a translation in direction of that axis and correct for the effects of such movements. These movements and the position of the camera are fully described by the 3 parameters which will be fixed by the Algorithm 2, namely the angle ψ_{sun}, the angle δ_{sun} and the apparent radius r_o. The physical meanings of these quantities are shown in the Figures 1 and 3. The radius r_{sun} and the

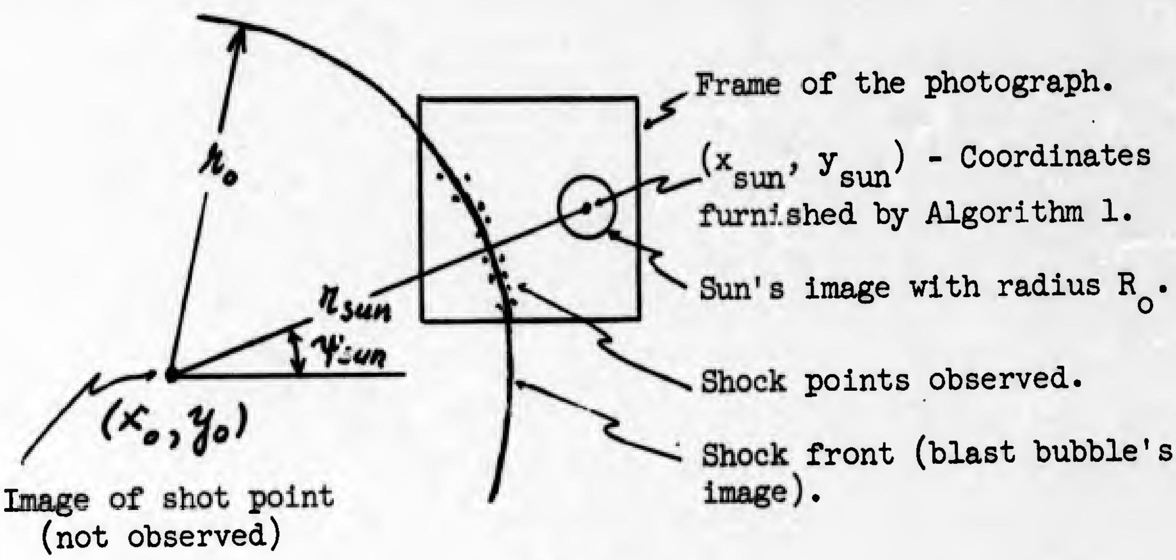


Figure 3

radius r_o are related to the angles δ_{sun} and δ_o by the formulae

$$\delta_{sun} = r_{sun} \cdot \frac{4.666 \cdot 10^{-3}}{R_{sun}} \ [rad]^{*} \qquad (1)$$

* *See footnote on page 21.*

$$\delta_o = r_o \cdot \frac{4.666.10^{-3}}{R_{sun}} \text{ [rad]},^{*} \qquad (2)$$

where R_{sun} is the radius of the Sun's image, determined by Algorithm 1.

All 3 parameters, δ_{sun}, δ_o and ψ_{sun} can be determined simultaneously by a least squares process for each frame. However, in many cases the shock points available might be unfavorably distributed (as in Figure 3) and the resulting r_{sun} (i.e., δ_{sun}) might not have the accuracy desired. In such cases the determination of r_{sun} is not included in the least squares process. Instead r_{sun} is considered constant for the calculation of the remaining 2 parameters. (An approximate value of r_{sun} is furnished by the cards B .) The Algorithm 2 compares the accuracy of r_{sun} obtained by the least squares evaluation of shock points with the accuracy of direct measurements (from cards B) and the more accurate value is then taken for the final evaluations. The remaining 2 parameters r_o and ψ_{sun} are then computed accordingly for each frame with shock points. From these 2 parameters ψ_{sun} might be a linear function of time (Section 2.1,e)), whereas r_o is a more complicated function. In order to make the computation process simpler, these functions of time are not determined simultaneously. Instead the linear function ψ_{sun} is determined independently of any assumptions about the behavior of r_o.

After the linear time functions (or constant values) r_{sun} and ψ_{sun} are computed for all frames new least squares approximations are carried out to determine the value of r_o for each frame with shock points. Since the value of r_o is needed for all other frames too, a corresponding time function representing r_o is obtained by another least squares process, similar to that described in Ref. 1, Section 8.2.

3. <u>Formulae for the Approximation of the Coordinates of the Shot Point</u>

3.1 <u>Introduction</u>. For the evaluation of refraction observations all points observed on a photograph must be expressed in terms of a coordinate system with origin at the image of the shot point. The

* *See footnote on page 21.*

readings of points of the photograph are done in some other convenient coordinate system, in which also the results of Algorithm 1 are expressed. In order to have the data necessary for a transformation from one coordinate system to the other we compute the coordinates of the shot point's image in the coordinate system of Algorithm 1. As indicated in Figure 3, the computation will be based on the observations of "shock points", i.e., points located on the blast bubble's boundary.

Depending on the availability of other data, the computation may be arranged differently:

a) It can be arranged so, that the angle δ_{sun} (i.e., r_{sun}, see Figures 1 and 3) is furnished by the shock point evaluations simultaneously with the 2 other parameters, ψ_{sun} and r_o. This case is treated in the Section 3.2.

b) If δ_{sun} is known rather accurate from direct measurements, the least squares approximations can be arranged so, that only the 2 parameters ψ_{sun} and r_o are computed, whereas r_{sun} is assumed fixed and known. This case is treated in the Section 3.3.

c) Finally, if both parameters ψ_{sun} and r_{sun} are known for all frames of a film, only one parameter r_o remains to be computed. This case is treated in the Section 3.4.

3.2 Three-Parameter Case. In this case we approximate the blast bubble's boundary on the photograph by a circle, which is described by the 3 parameters r_o, r_{sun} and ψ_{sun} (Figure 3). If (x_{j1}, x_{j2}) are the coordinates of a shock point, then obviously the following equation holds

$$F(x_{j1}, x_{j2}; r_o, \psi_{sun}, r_{sun}) =$$

$$= \sqrt{(x_{j1} - x_{sun} + r_{sun} \cos \psi_{sun})^2 + (x_{j2} - y_{sun} + r_{sun} \sin \psi_{sun})^2} - r_o = 0 \tag{3}$$

The values of x_{sun} and y_{sun} are known from Algorithm 1. If we have more than 3 shock points on the frame, the corresponding equations (3) constitute an overdetermined equation system for the 3 unknown parameters. Problems of this type are handled in the Appendix and the subroutine L.SQ. described in the Appendix can be used for the solution of the present problem. In order to have the same notation as in the Appendix we rename the parameters replacing

$$(r_o, \psi_{sun}, r_{sun})$$

by (4)

$$(y_1, y_2, y_3).$$

The observed coordinates of the j-th shock point we denote by (X_{j1}, X_{j2}), and their unknown corrections by (ξ_{j1}, ξ_{j2}). We have then the equations

$$\begin{aligned} x_{j1} &= X_{j1} + \xi_{j1} \\ x_{j2} &= X_{j2} + \xi_{j2} \qquad (j=1,2,\ldots,r). \end{aligned} \tag{5}$$

Initial approximations of the parameters y_i we denote by Y_i. The constraint function and its partial derivatives required by L.SQ. are then defined by the following formulae (6) through (11). In the Figure 4 geometrical interpretations of some of these quantities are given.

$$F_j = \sqrt{(X_{j1}-x_{sun}+Y_3 \cos Y_2)^2 + (X_{j2}-y_{sun}+Y_3 \sin Y_2)^2} - Y_1 \tag{6}$$

$$A_{j1} = \frac{X_{j1}-x_{sun}+Y_3 \cos Y_2}{\sqrt{(X_{j1}-x_{sun}+Y_3 \cos Y_2)^2 + (X_{j2}-y_{sun}+Y_3 \sin Y_2)^2}} = \cos \alpha_j \tag{7}$$

$$A_{j2} = \sin \alpha_j \tag{8}$$

$$B_{j1} = - 1 \tag{9}$$

$$B_{j2} = Y_3 \sin (\alpha_j - Y_2) \tag{10}$$

$$B_{j3} = \cos (\alpha_j - Y_2) \tag{11}$$

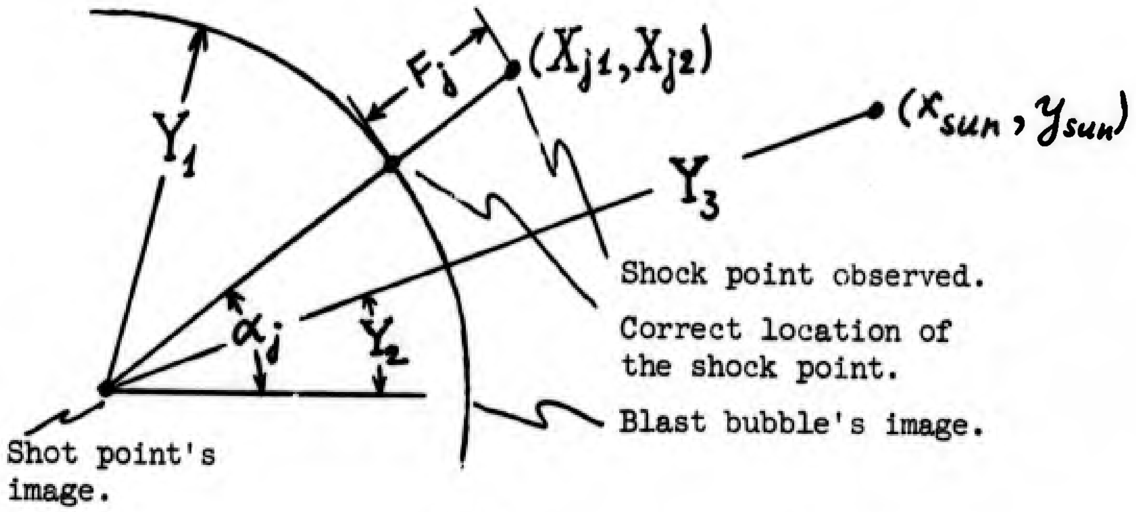


Figure 4

The initial approximation Y_3 is obtained from the card B . With (1) we have

$$Y_3 = \delta_{sun} \cdot \frac{R_{sun}}{4.666.10^{-3}} \text{ ,} \quad * \tag{12}$$

where R_{sun} is furnished by the Algorithm 1 and δ_{sun} by the card B . The factor $4.666.10^{-3}$ is correct, if δ_{sun} is expressed in radians. The dimension of Y_3 is the same as that of R_{sun}. The standard error of Y_3 is

$$m_{y3} = Y_3 \sqrt{\left(\frac{m_{\delta sun}}{\delta_{sun}}\right)^2 + \left(\frac{m_{Rsun}}{R_{sun}}\right)^2} \text{ .} \tag{13}$$

---

* *See footnote on page 21.*

The value of the standard error $m_{\delta sun}$ is furnished by the card B and the value of m_{Rsun} by the Algorithm 1 for all frames.

The initial approximation Y_2 is furnished by data from the card D . This card contains the coordinates (x_p, y_p) of the point P of Figure 2 (Section 2.1, c)). With these coordinates we obtain

$$\Phi_2 = \text{arc sin} \frac{y_{sun} - y_p}{\sqrt{(y_{sun}-y_p)^2 + (x_{sun}-x_p)^2}} \tag{14}$$

$$Y_2 = (1-\text{sgn}(x_{sun} - x_p))\ \text{sgn}(x_{sun} - x_p)(\Phi_2 + \frac{1}{2}\pi) + \Phi_2$$

This approximation may be used for the first frame of the film only. For subsequent frames a better approximation for Y_2 is the result y_2 of the previous frame's calculations.

The approximation Y_1 may be obtained by solving the equation (6) for Y_1. Thereby we substitute for Y_2 and Y_3 the approximations discussed above and for X_{j1} and X_{j2} any shock point coordinates of the particular frame, and set $F_j = 0$.

3.3 <u>Two-Parameter Case</u>. In case r_{sun} obtained by the least squares approximation of Section 3.2 is less accurate than the value furnished by card B with (12), the latter value is used for further calculations. In this case the least squares approximations of the blast bubble's image by a circle is repeated in a new fashion, considering only the 2 parameters r_o and ψ_{sun} (i.e., y_1 and y_2) as unknown. The formulae of the Section 3.2 remain unchanged. The least squares subroutine L.SQ. must now be called with m = 2 (m is the number of unknown parameters). Consequently the function B_{j3} (see (11)) is not needed for that subroutine.

3.4 <u>One-Parameter Case</u>. After fixing the 2 parameters ψ_{sun} and r_{sun} which determine the center of the blast bubble's image, the computation of the radius of the image is repeated for all frames with shock points. (See flow chart!) We have in this case only one free parameter, y_1 (or r_o), in the constraint function (3). Hence the least squares subroutine L.SQ. (Appendix) must be entered with m = 1. Except for this change, the evaluation process and the formulae of Section 3.2 remain the same ((10) and (11) are not needed.)

4. Trend Investigations

Assume, that the approximations described in Section 3.2 have been completed for K frames. The result is a list of the following type consisting of K lines:

Frame Nr. f	y_1	y_2	y_3	m_o	Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	Q_{23}	Q_{31}	Q_{32}	Q_{33}

According to Section 3.2 (4), the 3 parameters y_1, y_2 and y_3 are the quantities r_o, ψ_{sun}, and r_{sun}, respectively. Their values as well as the standard error of unit weight m_o and the cofactors $Q_{i\ell}$ are furnished by the least squares subroutine L.SQ. With 2.1.e. we assume, that y_2 and y_3 may be linear functions of time. The determination of these functions is the subject of this Section.

Time is represented in our case by the frame number f. Hence we have to determine the 4 parameters b, B, c and C of the 2 functions

$$\begin{aligned} y_2(f) &= b + B\cdot f \\ y_3(f) &= c + C\cdot f \end{aligned} \tag{15}$$

Since the values of y_2 and y_3 given in the list are correlated, also the functions (15) will be correlated. Their cofactors will be computed here by computing first the cofactors of the 4 parameters and then those of the 2 functions. The whole process is with little modifications the same as used for similar investigations in the Algorithm 1. It is a least squares approximation of the 2 linear functions (15), whereby the accuracies and cofactors of the data are properly used. First the following quantities are computed by summing over all K frames:

$$S_{20} = \sum \frac{1}{m_o^2 Q_{22}} ; \qquad S_{21} = \sum \frac{f}{m_o^2 Q_{22}} ; \qquad S_{22} = \sum \frac{f^2}{m_o^2 Q_{22}}$$

$$S_{30} = \sum \frac{1}{m_o^2 Q_{33}} ; \qquad S_{31} = \sum \frac{f}{m_o^2 Q_{33}} ; \qquad S_{32} = \sum \frac{f^2}{m_o^2 Q_{33}} \tag{16}$$

With these sums the determinants of the normal equations for the parameters b, B, c and C are

$$D_2 = S_{20} S_{22} - S_{21}^2$$

$$D_3 = S_{30} S_{32} - S_{31}^2 \tag{17}$$

With the values of (16) and (17) we compute for every frame in the list the following quantities

$$S_b = \frac{1}{m_o^2 Q_{22} D_2} (S_{22} - S_{21} f)$$

$$S_B = \frac{1}{m_o^2 Q_{22} D_2} (S_{20} \cdot f - S_{21}) \tag{18}$$

$$S_c = \frac{1}{m_o^2 Q_{33} D_3} (S_{32} - S_{31} \cdot f)$$

$$S_C = \frac{1}{m_o^2 Q_{33} D_3} (S_{30} \cdot f - S_{31})$$

The 4 parameters of the 2 functions (15) are then obtained by computing the following sums over the K frames

$$b = \sum S_b \cdot y_2 ; \qquad B = \sum S_B \cdot y_2$$

$$c = \sum S_c \cdot y_3 \qquad C = \sum S_C \cdot y_3 \tag{19}$$

Simultaneously with these sums, also corresponding sums representing the cofactors can be computed. Because of the symmetry of the cofactor matrix there are only 10 different cofactors in the present case, which are given by the sums (over the same K frames as (19))

$$Q_{bb} = \sum m_o^2 \, Q_{22} \, S_b^2 \; ; \qquad Q_{bB} = \sum m_o^2 \, Q_{22} \, S_b \, S_B$$

$$Q_{bc} = \sum m_o^2 \, Q_{23} \, S_b \, S_c \; ; \; Q_{bC} = \sum m_o^2 \, Q_{23} \, S_b \, S_C$$

$$Q_{BB} = \sum m_o^2 \, Q_{22} \, S_B^2 \; ; \qquad Q_{Bc} = \sum m_o^2 \, Q_{23} \, S_B \, S_c \tag{20}$$

$$Q_{BC} = \sum m_o^2 \, Q_{23} \, S_B \, S_C \; ; \; Q_{cc} = \sum m_o^2 \, Q_{33} \, S_c^2$$

$$Q_{cC} = \sum m_o^2 \, Q_{33} \, S_c \, S_C \; ; \; Q_{CC} = \sum m_o^2 \, Q_{33} \, S_C^2$$

In order to compute the accuracies and cofactors of the functions $y_2(f)$ and $y_3(f)$ we compute first the corresponding errors of unit weight:

$$m_2 = \frac{1}{K-2} \sum^{K} (y_2 - b - Bf)^2 \frac{1}{m_o^2 \, Q_{22}} \tag{21}$$

$$m_3 = \frac{1}{K-2} \sum^{K} (y_3 - c - C \cdot f)^2 \frac{1}{m_o^2 \, Q_{33}} \tag{22}$$

The cofactors of the functions $y_2(f)$ and $y_3(f)$ are then given by

$$m_o = 1 \tag{23}$$

$$Q_{y2y2} = m_2^2 \, (Q_{bb} + 2Q_{bB} f + Q_{BB} f^2) \tag{24}$$

$$Q_{y2y3} = m_2 m_3 \, (Q_{bc} + (Q_{bC} + Q_{Bc}) f + Q_{BC} f^2) \tag{25}$$

$$Q_{y3y3} = m_3^2\,(Q_{cc} + 2Q_{cC}f + Q_{CC}f^2) \qquad (26)$$

The standard errors of the functions are

$$m_{y2}(f) = m_o\,\sqrt{Q_{y2y2}} \qquad (27)$$

$$m_{y3}(f) = m_o\,\sqrt{Q_{y3y3}} \qquad (28)$$

According to Section 2.2 (see also the Flow Chart, box i) we will check whether the accuracy of the function $y_3(f)$ is better than that of direct observations. The standard error of the latter is given by (13). We check the usefulness of the function $y_3(f)$ by computing its standard error with (28) for all frames of the corresponding film (including frames without shock points) and compare these errors with the value given by (13). If any of the values (28) is larger than (13) we reject the 3-parameter approximations of Section 3.2 and proceed to the 2-parameter approximations of Section 3.3, as indicated by the Flow Chart.

In case the accuracy of $y_3(f)$ is sufficient (i.e.,if the errors computed by (28) are all smaller than the error computed by (13)), a check is made about the significances of the parameters B and C in (15). It is assumed, that B significantly differs from zero if

$$|B| > 3\cdot m_2\,\sqrt{Q_{BB}}\ . \qquad (29)$$

Also, C is assumed non-zero if

$$|C| > 3\cdot m_3\,\sqrt{Q_{CC}}\ . \qquad (30)$$

If (29) or (30), or both are not satisfied, the corresponding parameter is assumed to be zero. In these cases the constant terms in (15) are recalculated correspondingly. For this purpose the formulae (16) through (20) are used again, after substitution of the following values:

If $B = 0$, then

$$S_{21} = 0; \; S_{22} = 1; \; S_B = 0 \tag{31}$$

and

$$m_2 = \sqrt{\frac{1}{K-1} \sum^{K} (y_2 - b)^2 \frac{1}{m_o^2 \, Q_{22}}} \; . \tag{32}$$

If $C = 0$, then

$$S_{31} = 0; \; S_{32} = 1; \; S_C = 0 \tag{33}$$

and

$$m_3 = \sqrt{\frac{1}{K-1} \sum^{K} (y_3 - c)^2 \frac{1}{m_o^2 \, Q_{33}}} \tag{34}$$

The formulae (23) through (28) for the computation of the standard errors and cofactors can be used without alteration in these cases.

If the 2-parameter approximation of Section 3.3 is carried out for the K frames with shock points, the result will be the following list with K lines:

Frame Nr. f	y_1	y_2	m_o	Q_{11}	Q_{12}	Q_{21}	Q_{22}

Here y_1 stands for the radius r_o of the bubble's image and y_2 stands for the angle ψ_{sun}. (See Figures 3 and 4.) The standard errors of unit weight m_o and the cofactors Q_{js} are furnished by the least squares program L.SQ.

With Section 2.1.e. we assume that y_2 may be a linear function of the frame number f. Hence we have to determine the 2 parameters, b and B, of the function

$$y_2(f) = b + Bf \tag{35}$$

The formulae for the computation of these parameters and their standard errors are special cases of the formulae for the two functions y_2 and y_3 given above. The computation of b and B is done by computing first the 3 values S_{20}, S_{21} and S_{22} with (16) and the corresponding value of D_2 with (17). With these values we compute by summing over the K frames with shock points

$$b = \frac{1}{D_2} \sum^{K} (S_{22} - S_{21}\, f)\, y_2 \, \frac{1}{m_o^2\, Q_{22}} \tag{36}$$

$$B = \frac{1}{D_2} \sum^{K} (S_{20} \cdot f - S_{21})\, y_2 \, \frac{1}{m_o^2\, Q_{22}} \tag{37}$$

The cofactors of the parameters b and B are

$$Q_{bb} = \frac{1}{D_2^2} \sum^{K} \frac{(S_{22} - S_{21}\, f)^2}{m_o^2\, Q_{22}} \tag{38}$$

$$Q_{bB} = \frac{1}{D_2^2} \sum^{K} \frac{(S_{22} - S_{21}\, f)(S_{20}\, f - S_{21})}{m_o^2\, Q_{22}} \tag{39}$$

$$Q_{BB} = \frac{1}{D_2^2} \sum^{K} \frac{(S_{20}\, f - S_{21})^2}{m_o^2\, Q_{22}} \tag{40}$$

Using the values of b and B from (36) and (37) we compute m_2 with (21). The significance of B is checked by (29).

If (29) is satisfied, the formula (15) is used to compute the value of y_2 (i.e., ψ_{sun}) for all frames (including those with no shock points). The cofactor Q_{y2y2} and m_o are given by (24) and (23). The other cofactors are

$$Q_{y2y3} = 0 \,, \tag{41}$$

$$Q_{y3y3} = m_{y3}^2 , \tag{42}$$

where m_{y3} is given by (13).

In case the inequality (29) is not satisfied, we assume, that $B = 0$. The value of b is then computed by

$$b = \frac{\sum^{K} (y_2/m_o^2 \, Q_{22})}{\sum^{K} (1/m_o^2 \, Q_{22})} \tag{43}$$

The cofactors are

$$Q_{bb} = 1$$

$$Q_{bB} = Q_{BB} = 0$$

The value of m_3 is computed with (32). The cofactors of the two functions $y_2(f)$ and $y_3(f)$ are in this case given by (41), (42),

$$m_o = 1 ,$$

and (44)

$$Q_{y2y2} = m_2^2 .$$

5. <u>Distance to Shot Point</u>

After the value of r_{sun} has been fixed, either assuming a constant value or a linear time function, the distance D between the camera and the shot point can be computed accordingly. (See Sections 3.1 and 3.2 and Flow Chart, Box j and p.) This distance and its standard error are computed differently, depending on the process by which r_{sun} (i.e., δ_{sun}, see Figures 1 and 3) is computed. The following cases are possible:

a) The value of δ_{sun} furnished by the card B is more accurate than the value obtained by the process of Section 3.2.

In this case D together with its standard error are also taken from the card B. (See Flow Chart, Box j.)

b) The value of δ_{sun} is assumed to be constant, but different from that given by the card B.

In this case (Flow Chart, Box p) proceed as in Case a.

c) It is found, that δ_{sun} (i.e., r_{sun}) is a linear function of time.

From Figure 1 we deduce the equation

$$D \cdot \sin \delta_{sun} = \text{const.} \tag{45}$$

Hence, if δ_{sun} is not constant, also D must be a function of time. Since the values of D are needed for the computation of the blast bubble's (real) radius, the variation of D must be taken into account. The function δ_{sun} (f) was essentially determined in Section 4. Equation (15) with (1) yields

$$\delta_{sun} = \frac{4.666 . 10^{-3}}{R_{sun}} (c + C \cdot f) \quad * \tag{46}$$

(The frame number f is used here as a substitute for the time.) This equation furnishes together with (45)

$$D(f) = \frac{D_1 \cdot \sin \left(\frac{4.666 \cdot 10^{-3} \, (c + C \, f_1)}{R_{sun}} \right)}{\sin \left(\frac{4.666 \cdot 10^{-3} \, (c + Cf)}{R_{sun}} \right)} \, , \quad * \tag{47}$$

where D_1 is the value of D furnished by the card B and f_1 is the number of the first frame of the film. With (47) the value of D can be computed for all frames.

The standard error of D(f) can be computed from the standard errors of the quantities D_1, R_{sun}, c and C. Of these quantities c and C are correlated and their cofactors given by (20). The standard

* *See footnote on page 21.*

error $m_D(f)$ of $D(f)$ is with these values given by

$$m_D^2 = \left(\frac{\partial D}{\partial D_1}\right)^2 m_{D1}^2 + \left(\frac{\partial D}{\partial R_{sun}}\right)^2 m_{Rsun}^2$$

$$+ \left\{\left(\frac{\partial D}{\partial c}\right)^2 Q_{cc} + \left(\frac{\partial D}{\partial c} + \frac{\partial D}{\partial C}\right) Q_{cC} + \left(\frac{\partial D}{\partial C}\right)^2 Q_{CC}\right\} m_3^2 \qquad (48)$$

In this formula

m_{D1} = standard error of D_1 furnished by the card B

m_{Rsun} = standard error of the radius of the Sun's image R_{sun}, furnished by Algorithm 1.

m_3 = standard error given by (22)

The partial derivatives in (48) may be computed using the program of Ref. 2 and, therefore, need not be given explicitly.

The correlation between R_{sun} and the constants c and C is not considered here nor in further error formulae, in order to avoid too complicated error formulae. Such a correlation exists, because r_{sun} was computed using the values x_{sun} and y_{sun} from Algorithm 1 (see Section 3.2), and the latter values are correlated with R_{sun}.

6. Radius of Blast Bubble

The last quantity which is computed by the Algorithm 2 is the radius r_o of the blast bubble's image. (See Flow Chart, Part 3.) This radius is known with the calculations of Section 3 for all frames with shock points. Since r_o is needed for other frames too, we will determine a function $r_o(f)$ which permits to calculate the value of r_o for any frame number f. This function will be established essentially by the process described in Reference 1, Section 8.2. Particularly, the real radius R_o of the blast bubble will be approximated by the following function of the frame number f:

$$R_o(f) = y_1 + y_2 f + y_3 \ln f + y_4 \frac{1}{f} \,. \qquad (49)$$

Data available for the determination of the 4 parameters y_1 through y_4 in (49) are a list of the K frames with shock points and some parameters. Of these parameters we will use the following

Frame Nr f	R_{sun}	m_{Rsun}	r_o	m_{ro}	D	m_D

The values of R_{sun} with their standard errors m_{Rsun} are furnished by Algorithm 1. The values of r_o and m_{ro} are computed by the process described in Section 3.4 (see Flow Chart, Box r). The values of D and m_D are computed by the process described in Section 5 (see Flow Chart, Box j and p). We complete the list by adding a column with the values of δ_o, computed with (2) and a second column with the standard errors of δ_o, computed with

$$m_{\delta o} = |\delta_o| \sqrt{\frac{m_{Rsun}^2}{R_{sun}^2} + \frac{m_{ro}^2}{r_o^2}} \quad . \tag{50}$$

With these values we can compute for each of the K frames the corresponding value of R_o by

$$R_o = D \cdot \sin \delta_o \tag{51}$$

(see Figure 1). The approximation of R_o by the function (49) is established using the least squares program L.SQ. of Appendix 1. As independent observations we consider for this purpose the triplets (f, δ_o, D). In order to have the same symbols here as in Appendix 1, we rename these quantities, replacing

$$(f, \delta_o, D)$$

by (52)

$$(x_1, x_2, x_3) \; .$$

The standard errors e_2 and e_3 of x_2 and x_3, necessary for the least squares program are those of δ_o and D, respectively. The coordinate

x_1, that is f, should be considered as exact. We assign therefore to the numbers f an "error" of 10^{-6}. Since the values of f are 3 digit numbers, this assumption makes the relative errors of x_1 by some orders smaller than those of x_2 and x_3. As a consequence, the corrections of f, computed by the least squares program will be negligibly small.

The constraint function for the least squares program is

$$F_j = X_{j3} \sin X_{j2} - Y_1 - Y_2 X_{j1} - Y_3 \ell n X_{j1} - Y_4 \frac{1}{X_{j1}} \tag{53}$$

and the partial derivatives are

$$\begin{aligned}
A_{j1} &= - Y_2 - Y_3 \frac{1}{X_{j1}} + Y_4 \frac{1}{X_{j1}^2} \\
A_{j2} &= X_{j3} \cos X_{j2} \\
A_{j3} &= \sin X_{j2} \\
B_{j1} &= - 1 \\
B_{j2} &= - X_{j1} \\
B_{j3} &= - \ell n \, X_{j1} \\
B_{j4} &= - \frac{1}{X_{j1}} \, .
\end{aligned} \tag{54}$$

Similarly as in the processes of Section 4, also here only such parameters y_ℓ will be used finally, which are significantly different from zero. Therefore, first the parameters y_1, y_2 and y_3 only are computed (assuming $y_4 = 0$ in (49)). If the resulting y_3 does not satisfy the condition

$$|y_3| > 3 \cdot m_{y3} \tag{55}$$

(m_{y3} is the standard error of y_3, furnished by the least squares program), we assume, that $y_3 = 0$. In this case a new, 2-parameter approximation is computed, using the parameters y_1 and y_2 only.

If (55) is satisfied, it is checked whether $y_4 \neq 0$. This is done by computing a 4-parameter approximation and checking the significance of y_4. If

$$|y_4| > 3 \cdot m_{y4} , \tag{56}$$

the 4-parameter approximation is considered as final. Otherwise the 3-parameter approximation, which was computed first, is considered as the final approximation.

After the parameters y_1 through y_4 are fixed, the values of δ_o are computed with

$$\delta_o(f) = \text{arc sin} \frac{R_o(f)}{D(f)} \tag{57}$$

for all frames. In order to obtain a smooth function $\delta_o(f)$ we use in (57) for D(f) the original values from Section 5. (For the K frames with shock points we are furnished with corrected D-values by the least squares routine. These values will not be used for further calculations.) Accordingly the standard error of $\delta_o(f)$ is given by

$$m_{\delta o}(f) = \sqrt{\left(\frac{\partial\delta_o}{\partial R_o}\right)^2 m_{Ro}^2(f) + \left(\frac{\partial\delta_o}{\partial D}\right)^2 m_D^2} \tag{58}$$

The partial derivatives in (58) can be computed by the program of Reference 2 and need not be given explicitely here. m_D is furnished for all frames by the calculations of Section 5. m_{Ro} is given by the formula

$$\begin{aligned} m_{Ro}^2 = m_o^2 \big(Q_{11} &+ 2Q_{12} \cdot f + 2Q_{13}\, \ell nf + 2Q_{14} \frac{1}{f} \\ &+ Q_{22}\, f^2 + 2Q_{23}\, f \ell nf + 2Q_{24} \\ &+ Q_{33}\, (\ell nf)^2 + 2Q_{34} \frac{\ell nf}{f} \\ &+ Q_{44} \frac{1}{f^2}\big), \end{aligned} \tag{59}$$

where m_o and the Q_{ik} are furnished by the least squares program for fixing the parameters y_1 through y_4 in (49).

7. Flow Chart of Algorithm 2

In this Section the flow chart of Algorithm 2 is presented. The mathematical formulae necessary for the computations were derived in Sections 2 through 6. The following comments to the flow chart will indicate the particular Sections.

The machine program for the Algorithm needs 2 tape units, namely

tape unit 1 as data tape for the least squares program and

tape unit 2 to store the results computed by the Algorithm 2.

Besides these tape units, also the tape unit 7 is used for temporary storage by the least squares routine.

Comments to the Flow Chart.

a. The tape unit 1 is rewound in order to use it for storage of input data for the least squares subroutine L.SQ. However, (if other data are on the tape) rewinding is not necessary if provisions are taken to place the tape at the correct position when L.SQ. is entered. (See Comment b.)

The contents of the cards B and D are explained in the general Data Flow Chart. There will be one card B and one card D for each film. No special sequence of these cards is assumed. Instead data will be read, until for each film of the List F (in storage), one B card and one D card are read. If there are not sufficient cards or more than one card for some films, print the message.

Blank line.
FOR FILMS WITH THE FOLLOWING IDENTIFICATIONS EITHER TOO MANY OR NO
B- OR D-CARDS ARE AVAILABLE
(Film number +123)
The List F and the List E1 (result of Algorithm 1) is then reduced by dropping all data corresponding to these films. In case no film has correct data from these cards, print

Blank line
EXIT TO NEXT PROGRAM FROM ALGORITHM 2 BECAUSE NONE OF THE FILMS HAS CORRECT DATA FROM B- AND D-CARDS.
Blank line
DATE 12.12.1966 TIME 12 HRS 12.12 MIN.

(b.) The backspacing of tape unit 1 covers all data collected for the last film. Backspacing by K files means, therefore, backspacing by $S_1 + S_2 + \ldots + S_K + K$ blocks (S_i = number of shock points on the i-th frame). Rewind if tape 1 was rewound by a.

(c.) The approximations of r_{sun} and ψ_{sun} are given by (12) and (13) of Section 3.1.

(d.) The values to be stored are the shock point coordinates, their errors and identifications. About the format of the tape, see the Appendix, Section 11. K is the number of frames with shock points for the film in work. S is the number of shock points on the particular frame. (Hence S is the number of data sets for the L.SQ. program.)

(e.) Print:

Blank line
FILM WITH THE IDENTIFICATION (+123) CANNOT BE PROCESSED BECAUSE THERE ARE LESS THAN 5 FRAMES WITH SHOCK POINTS ON THE FILM.

Reduce the List E by destroying all data from the corresponding film. Reduce correspondingly the lists established by reading the cards B and D.

(f.) The initial approximations of the parameters are for the first file (frame) computed by (12) and (14), Section 3.1. For subsequent files of the film in work take the results of the previous file as initial values. Iterate 5 times for each file by backspacing tape unit 1 after returning from L.SQ.1 accordingly and replacing the initial parameter values by the results. After L.SQ.1 has been entered 5 times, enter L.SQ.2 and do not backspace tape unit 1. Then proceed in the same manner using now data from next file.

The result will be the following Working List , W , consisting of $K > 5$ lines.

Frame Nr f	x_{sun}	y_{sun}	R_{sun}	r_o	ψ_{sun}	r_{sun}	m_o	$Q_{11} \cdots Q_{33}$

$$\underbrace{x_{sun} \mid y_{sun} \mid R_{sun}}_{\substack{\text{Established by} \\ \text{Algorithm 1.} \\ \text{Taken from List} \\ \text{E1}}} \mid \underbrace{r_o \mid \psi_{sun} \mid r_{sun} \mid m_o \mid Q_{11} \cdots Q_{33}}_{\substack{\text{first established by} \\ \text{the loop f.}}}$$

For the storing of the List W it can be assumed, that $K < 40$. The values established by this loop will be checked and possibly changed by later parts of Algorithm 2.

g. Computation of the linear functions for ψ_{sun} and r_{sun} is discussed in Section 4. For the computation, the Working List W will be used. The result of the computation is

b	B	c	C	Q_{bb}	...	Q_{CC}	m_2	m_3

- m_3 → error of r_{sun}
- m_2 → error of ψ_{sun}
- $Q_{bb} \cdots Q_{CC}$ → 16 cofactors
- c, C → parameters of r_{sun}
- b, B → parameters of ψ_{sun}.

These results will be checked and might be changed by later parts of Algorithm 2.

h. Print:

Blank line
EXIT FROM ALGORITHM 2 TO NEXT PROGRAM BECAUSE NO FILM HAS MORE THAN 5 FRAMES WITH SHOCK POINTS. SUCH FILMS CANNOT BE PROCESSED.
Blank line
DATE 12.12.1966 TIME 12 HRS 12.12 MIN.

In case of exit to Algorithm 3, write on the tape unit 2 (which by now contains the final list E) the sentinel END LIST E by

```
          ENTER (BT.WR)2)ELE)%
          GO TO (Algorithm 3)
    ELE   ALFNENDbLISTbE .
```

i. The standard error of the linear function $r_{sun}(f) = c + Cf$ is

$$m_{rsun}(f) = m_3 \sqrt{Q_{cc} + 2Q_{cC}\, f + Q_{CC} f^2} \quad ,$$

where m_3, Q_{cc}, Q_{cC} and Q_{CC} are furnished by Box g. This standard error is compared for each frame (i.e., for each f-value present in the List E of the film in work) with the standard error of r_{sun}, computed with (13).

j. In this case the value of r_{sun} is computed with (12) and that of m_{rsun} with (13). For D and m_D the constant values from card B are stored. The results of Box g are modified by setting $m_3 = m_{rsun}$, $Q_{cc} = 1$ and of the other cofactors all such cofactors equal to zero, which have an index c or C.

From the List E, the first part E1 is furnished by Algorithm 1. This part contains the following data:

Film Nr	Frame Nr	x_{sun}	y_{sun}	R_{sun}	m_o	$Q_{xx} \dots Q_{RR}$

By Algorithm 2 this list will be completed by adding the following quantities to the list

ψ_{sun}	r_{sun}	m_o	$Q_{\psi\psi}$	$Q_{r\psi}$	Q_{rr}	δ_o	m_δ	D	m_D

At the present step only the following constants are added to the List E.

r_{sun} - computed with (12)

$m_o = 1$

$Q_{rr} = m^2_{rsun}$ - computed with (13)

$Q_{r\psi} = 0$

D and m_D - from card B

k. This is the approximation of Section 3.3. The parameters are computed by the iteration process described in comment f. The subroutines for the functions F_j, A_{jk}, $B_{j\ell}$ must be modified in such manner, that they can be called with 2 x-arguments and 2 y-arguments. (The third y-argument is the constant given by (12).) The result is a modified Working List W. The changes against the first values (see Comment f) are as follows:

$$r_{sun} = \text{const. (computed with (12))}$$

$$m_o = 1$$

$$Q_{13} = Q_{23} = Q_{31} = Q_{32} = 0$$

$$Q_{33} = m^2_{rsun} \text{ (computed with (13))}$$

The least squares program L.SQ. furnishes now with the values of the 2 parameters r_o and ψ_{sun}, also a value of the standard error of weight one M_o and the cofactors q_{11}; q_{12}; q_{21}; q_{22}. In the Working List W the following cofactors are stored with r_o and ψ_{sun}:

$$Q_{11} = M_o^2 \, q_{11}$$

$$Q_{12} = Q_{21} = M_o^2 \, q_{12}$$

$$Q_{22} = M_o^2 \, q_{22}$$

l. This computation is described in the Section 4. Data for the computation of the parameters b and B is the Working List W. (y_2 of Section 4 corresponds to ψ_{sun} in the List W.) The result are the following 6 quantities:

b	B	Q_{bb}	Q_{bB}	Q_{BB}	m_2

This is a further modification of the results of Box g. (These results were first modified by Box j.)

m. The trend of ψ_{sun} is established by checking whether B is significantly different from zero, according to Section 4, (29).

(n.) This branch is entered if (29) is not satisfied. The new value of b is computed with (43), taking the values of y_2, m_o and Q_{22} from the Working List W. ($y_2 = \psi_{sun}$). The results of Box g are now modified further by setting

$$b \text{ - computed with (43)}$$

$$B = 0$$

$$Q_{bb} = 1$$

$$Q_{bB} = Q_{BB} = 0$$

$$m_2 \text{ - computed with (32).}$$

(o.) The trend investigation follows the lines of Section 4, where the corresponding recalculations of some of the parameters are described. The result is either a modification of the results of Box g or a confirmation of those results.

(p.) The results obtained so far (Box g and o) permit the calculation of r_{sun} by

$$r_{sun} = c + Cf$$

for all values of f. The values of m_o and the cofactors are with Section 4

$$m_o = 1 \quad (23)$$

$$Q_{\psi\psi} = Q_{y2y2} \text{ with (24)}$$

$$Q_{r\psi} = Q_{y2y3} \text{ with (25)}$$

$$Q_{rr} = Q_{y3y3} \text{ with (26).}$$

The cofactors Q_{bb} through Q_{cc} (see (24), (25), (26)) are computed in Box g and possibly modified in o. The computation of the distance D is described in the Section 5. About the List E see Comment j.

q. The function ψ_{sun} is computed by $\psi_{sun} = b + Bf$. In case this box is entered from the branch j, also the quantity $Q_{\psi\psi}$ is computed at this point with (24).

r. The computation of r_o is described in Section 3.4. The least squares subroutine L.SQ. is now entered with m = 1 (i.e., one parameter). The other parameters $y_2 = \psi_{sun}$ and $y_3 = r_{sun}$ are fixed for each frame by previous calculations. The computation of r_o is done applying the iteration described in Comment f. The results, namely the values of r_o and their standard errors are stored in a list W_1, consisting of K lines (K is the number of frames with shock points for the film processed; $4 < K < 40$):

Frame Nr f	r_o	m_{ro}	D	m_D

The values of D and m_D can be obtained from the List E, where these values are stored by Box j or p.

s. The computation and storage of δ_o and the other data are described in Section 6. The least squares process is in this case not iterated. Immediately after entering L.SQ.1, the routine L.SQ.2 is entered and the data tape on tape unit 1 backspaced for the calculations u and v. The results of L.SQ.1 (parameters and cofactors) are saved for possible use in x.

t. The significance of y_3 is checked by (55).

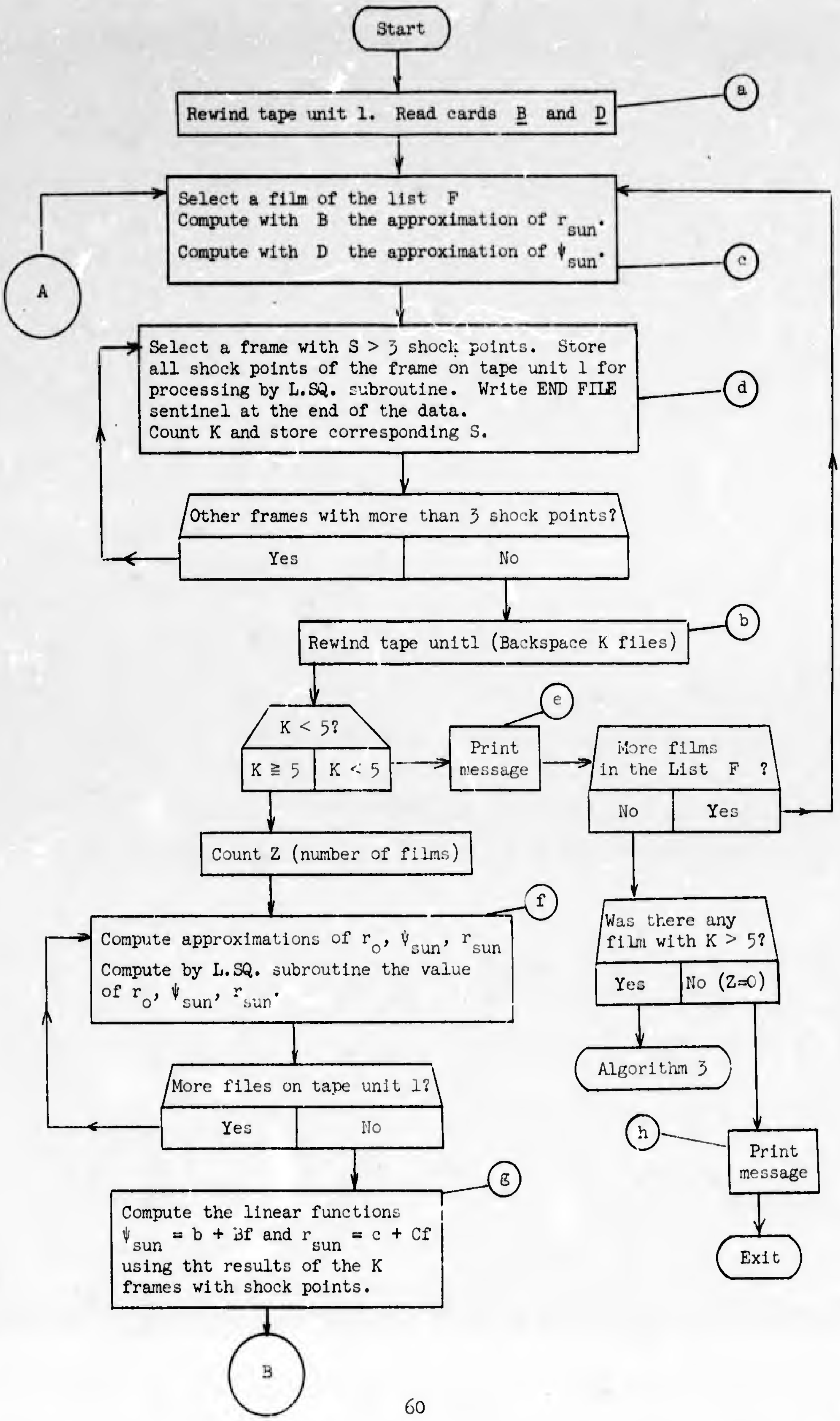
u. See Section 6. In this case $y_3 = y_4 = 0$, and also the cofactors with index 3 or 4 are assumed zero for later calculations.

v. See Section 6.

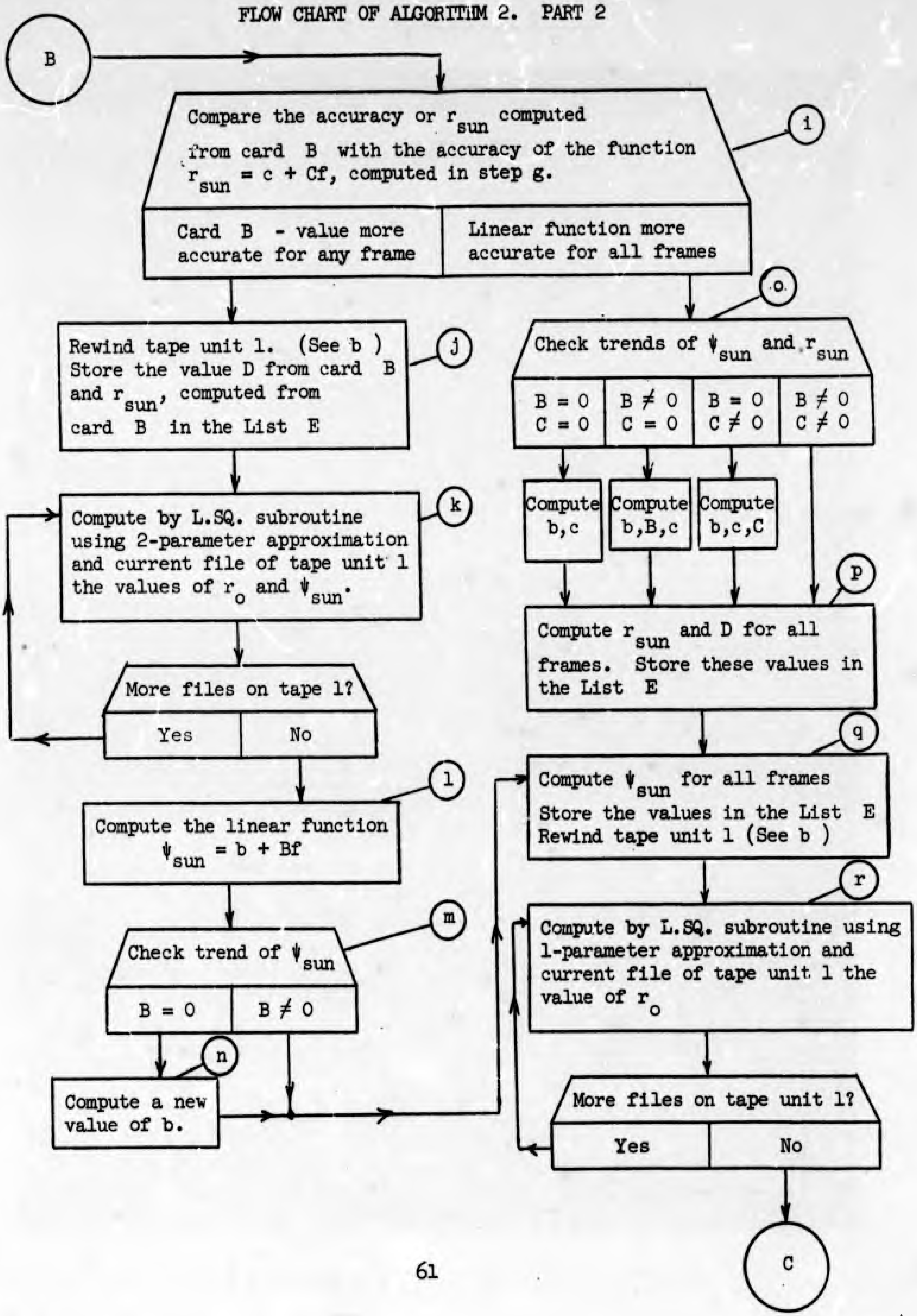
w. The significance of y_4 is tested by (56).

x. In this case y_4 and the corresponding cofactors are assumed to be zero. For further calculations use the results of Box s.

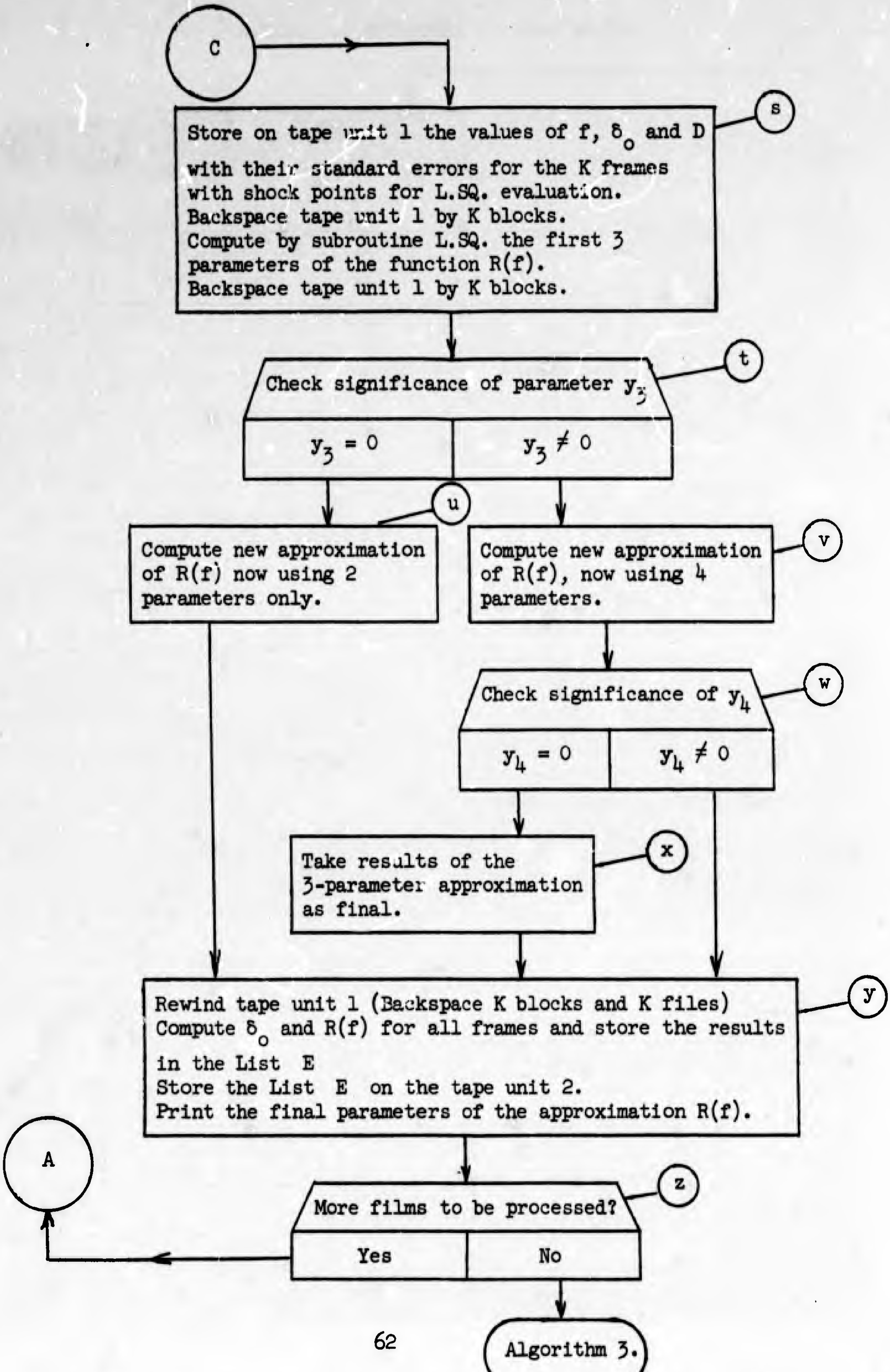
FLOW CHART OF ALGORITHM 2. PART 1



FLOW CHART OF ALGORITHM 2. PART 2



FLOW CHART OF ALGORITHM 2. PART 3



(y.) Compute R_o, δ_o and their standard errors with (49), (57), (58) and (59). These values complete the List E (see Comment j). Store the List E on tape unit 2 in binary form using the Binary Tape Output Routine (ENTER(BT.WR)2)...). Do not rewind or backspace tape unit 2 at this point, because data of other films may be stored on the same tape. The final parameters of the approximation R_o are printed with the following text:

```
New page.
THE RADIUS R OF THE BLAST BUBBLE CAN BE APPROXIMATED BY THE FOLLOWING
FUNCTION OF THE FRAME NUMBER F
Blank line.
R = Y1 + Y2*F + Y3*LOG(F) + Y4/F
Blank line
THE PARAMETERS Y HAVE THE FOLLOWING VALUES IN METERS
Blank line
(+1.12345+12)  (+1.12345+12) ...
Blank line
THE STANDARD ERROR OF WEIGHT ONE IS (+1.12345+12)
Blank line
THE COFACTOR MATRIX OF THE PARAMETERS IS
```

$$\begin{matrix} (Q_{11}) & \cdots & (Q_{14}) \\ \cdot & \cdots & \cdot \\ (Q_{41}) & \cdots & (Q_{44}) \end{matrix}$$

The parameter values given in this printout are obtained from the results of the least squares routine L.SQ. by correcting the dimension of these results. (All computations in Algorithm 2 are done in mm and radians.) The values of the parameters y_ℓ and the value of the standard error of unit weight m_o are divided by 1000 before printing. The values of the cofactors are divided by 10^6 before printing.

(z.) In case of exit to Algorithm 3 place the END LIST E sentinel on tape unit 2. (See Comment h.)

IX. DESCRIPTION OF ALGORITHM 3

1. Purpose of the Algorithm

The purpose of Algorithm 3 is to prepare from adjusted photographic data a tape, which can be used as input tape for the computation of the refraction index. (Such computation can be done, for instance, by a program of the type described in Ref. 1, Section 8.3.) Particularly Algorithm 3 furnishes for every point observed through the blast bubble the angles δ, δ' and δ_0 (see Figure 1.) In case of a finite distance to the object, also the values of D and A (Fig. 1) will be furnished. Hence in the general case Algorithm 3 processes the adjusted photographic data, which are the results of Algorithm 1 and 2, together with data about the geometry of the experiment. In

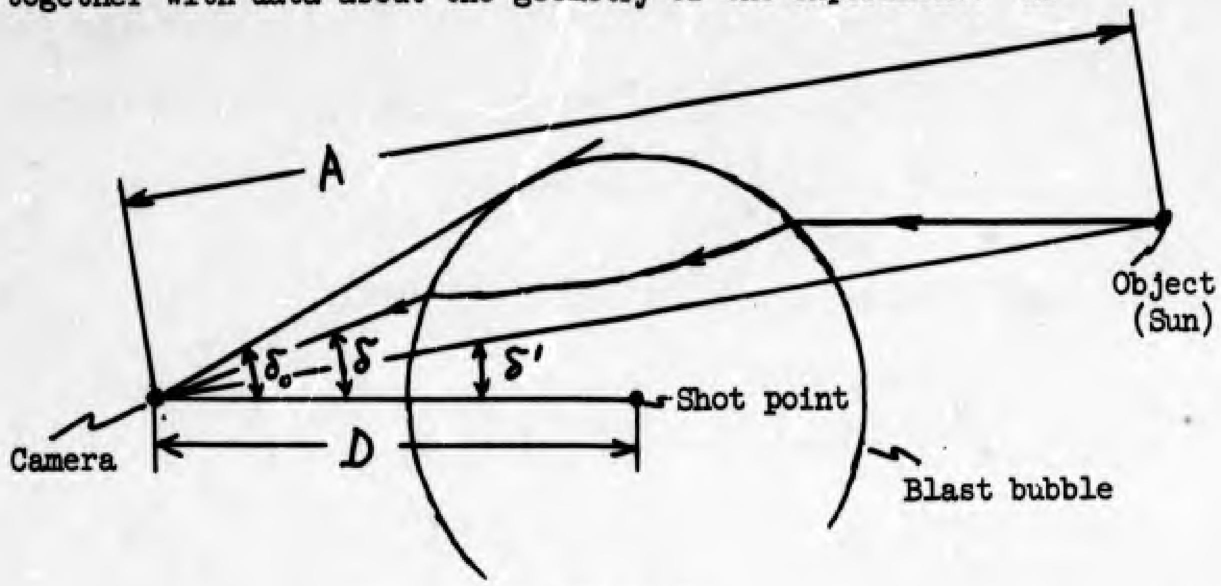


Figure 1

the case considered here, the object (Sun) is assumed at infinite distance from the photographic camera.

2. Method Applied

2.1 Assumptions.

a) We assume, that Algorithm 2 has been completed and the results are available either on tape unit 2 or in storage in form of the List E. This list contains the following data for each frame:

Film Nr.	Frame Nr f	x_{sun}	y_{sun}	R_{sun}	m_o	Q_{xx} ... Q_{RR}

ψ_{sun}	r_{sun}	m_{o1}	$Q_{\psi\psi}$	$Q_{\psi r}$	Q_{rr}	δ_o	$m_{\delta o}$	D	m_D

b) We assume, that coordinates of the "disturbed points" of the List F are available either in storage or on tape unit 4 or in punched cards. (These points are of the type "3" or "4", see Section VI.1). This list contains for each point the following data:

Film Nr	Frame Nr f	Point Nr and type	ξ	η	m_ξ	m_η	"F"

(The last column is the label of the list.)

c) In case of objects at finite distances we assume, that the List B is available in storage (or on cards). This list contains for each film the following data

Film Nr	δ_{sun}	m_δ	D	A	Q_{DD}	Q_{DA}	Q_{AA}	"B"

(The last column is the label of the list.)

2.2 Method. For each point (ξ,η) of the type "3" or "4" of the List F, the corresponding undisturbed position is determined by computing the intersection of a straight line through the shot point's image with the Sun's image. From there the corresponding position angles δ and δ' and their errors and cofactors are computed numerically. The method requires the evaluation of some functions and their partial derivatives and matrix multiplication.

3. Formulae

The position angle δ corresponding to the point (ξ,η) observed is a function of 7 arguments (see Figure 2):

$$\delta = F(\xi;\ \eta;\ x_{sun}, y_{sun}, R_{sun};\ \psi_{sun}, r_{sun}) = \tag{1}$$

$$= \frac{4.666.10^{-3}}{R_{sun}} \sqrt{(\xi - x_{sun} + r_{sun}\cos\psi_{sun})^2 + (\eta - y_{sun} + r_{sun}\sin\psi_{sin})^2}\,.^{*}$$

Some of these arguments may be correlated. The groups of possibly correlated arguments are separated in (1) by semicolons.

* *See footnote on page 21.*

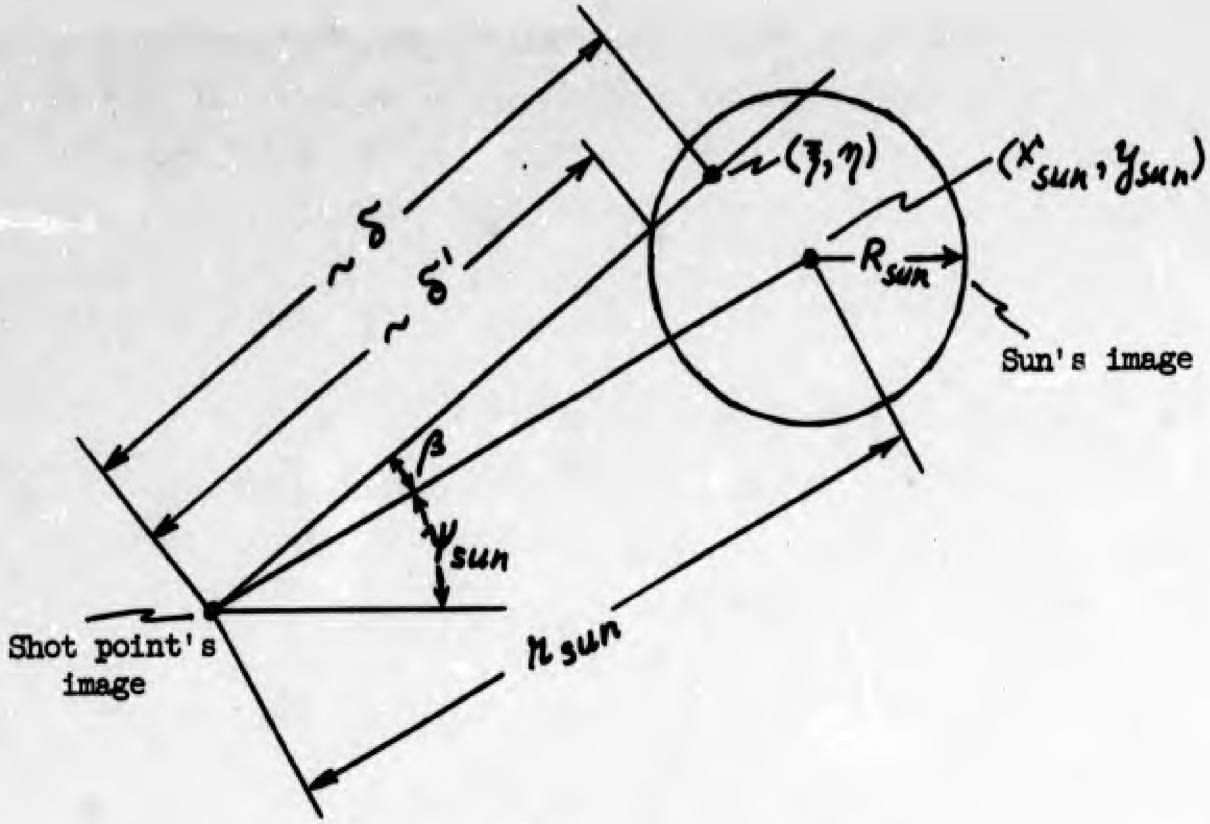


Figure 2

The corresponding undisturbed position angle δ' is a function of the same 7 arguments:

$$\delta' = G(\xi;\ \eta;\ x_{sun},\ y_{sun},\ R_{sun};\ \psi_{sun},\ r_{sun}) =$$

$$= \frac{4.666.10^{-3}}{R_{sun}} \left(r_{sun} \cos\beta \pm \sqrt{R_{sun}^2 - r_{sun}^2 \sin^2\beta}\ \right) * \tag{2}$$

with

$$\cos\beta = \frac{r_{sun} + (\xi - x_{sun}) \cos\psi_{sun} + (\eta - y_{sun}) \sin\psi_{sun}}{\sqrt{(\xi - x_{sun} + r_{sun}\cos\psi_{sun})^2 + (\eta - y_{sun} + r_{sun}\sin\psi_{sun})^2}} \tag{3}$$

and

---

* *See footnote on page 21.*

$$\sin\beta = \frac{-(\xi - x_{sun})\sin\psi_{sun} + (\eta - y_{sun})\cos\psi_{sun}}{\sqrt{(\xi - x_{sun} + r_{sun}\cos\psi_{sun})^2 + (\eta - y_{sun} + r_{sun}\sin\psi_{sun})^2}} \,. \tag{4}$$

The sign in (2) is positive for observations of type "4" and negative for observations of type "3".

The correlation between δ and δ' will be computed considering also the correlations between the arguments. To this end we compute first the cofactor matrix Q of the 7 arguments. With the data from the List E (see Section 2.1, a) we define for each frame the matrix

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{xx} & Q_{xy} & Q_{xR} & 0 & 0 \\ 0 & 0 & Q_{xy} & Q_{yy} & Q_{yR} & 0 & 0 \\ 0 & 0 & Q_{xR} & Q_{yR} & Q_{RR} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{\psi\psi} & Q_{\psi r} \\ 0 & 0 & 0 & 0 & 0 & Q_{\psi r} & Q_{rr} \end{bmatrix} \tag{5}$$

and a diagonal matrix M with the elements

$$(1,\ 1,\ m_o^2,\ m_o^2,\ m_o^2,\ m_{ol}^2,\ m_{ol}^2) \tag{6}$$

We obtain then a matrix Q_2 by the multiplication

$$Q_2 = M \cdot Q_1 \tag{7}$$

The matrix Q_2 is symmetric and has in the first 2 rows only in the diagonal non-zero elements, namely ones. The cofactor matrix Q of the arguments for a given point (ξ,η) is obtained from Q_2 by replacing these first two diagonal elements by the squares of the standard errors of ξ and η, respectively.

In order to obtain the cofactors of δ and δ' we compute first the partial derivatives of the functions (1) and (2) with respect to all 7 arguments. We denote these derivatives by attaching indices 1 through 7 to the function names F and G and define the vectors $\overline{F}$ and $\overline{G}$ with the partial derivatives as components:

$$\overline{F}' = (F_1, F_2, \ldots, F_7) \tag{8}$$

$$\overline{G}' = (G_1, G_2, \ldots, G_7) \tag{9}$$

The cofactors and the standard error of unit weight of the functions δ and δ' are then

$$Q_{\delta\delta} = \overline{F}' \, Q \, \overline{F} \tag{10}$$

$$Q_{\delta\delta'} = \overline{G}' \, Q \, \overline{F} \tag{11}$$

$$Q_{\delta'\delta'} = \overline{G}' \, Q \, \overline{G} \tag{12}$$

$$m_o = 1 \tag{13}$$

Routines for the computation of the partial derivatives of F and G can be obtained using the machine program of Ref. 2. Therefore, explicit formulae for these derivatives are not needed.

4. Flow Chart of Algorithm 3

The computer program of Algorithm 3 uses the following tape units:

tape unit 1 - to store the results of the Algorithm 3.

tape unit 2 - input tape with the List E, prepared by Algorithm 2.

tape unit 4 - input tape with the List F, prepared by Algorithm 1 (if the whole List F is not kept in the storage).

Comments to the Flow Chart.

a. In some cases the computation of an intersection may not be possible due to observation errors. (The radicand in (2) is then negative.) In these cases print

DELTA-PRIME OF THE OBSERVATION
FILM = 123, FRAME = 123, POINT = 1234
CANNOT BE COMPUTED
and proceed to next point.

(b.) Storage on the tape is done using the Binary Tape Output Routine and a format consistent with the requirements of the least squares subroutine COLS for correlated observations (Appendix). The format required is

$$\left|\delta_o\right|\delta\left|\delta'\right|7\text{ zeros}\left|1(=m_o)\right|m^2_{\delta o}\left|10\text{ zeros}\right|Q_{\delta\delta}\left|Q_{\delta\delta'}\right|$$

$$\left|8\text{ zeros}\right|Q_{\delta\delta'}\left|Q_{\delta'\delta'}\right|77\text{ zeros}\left|id_1\right|id_2\left|\right. .$$

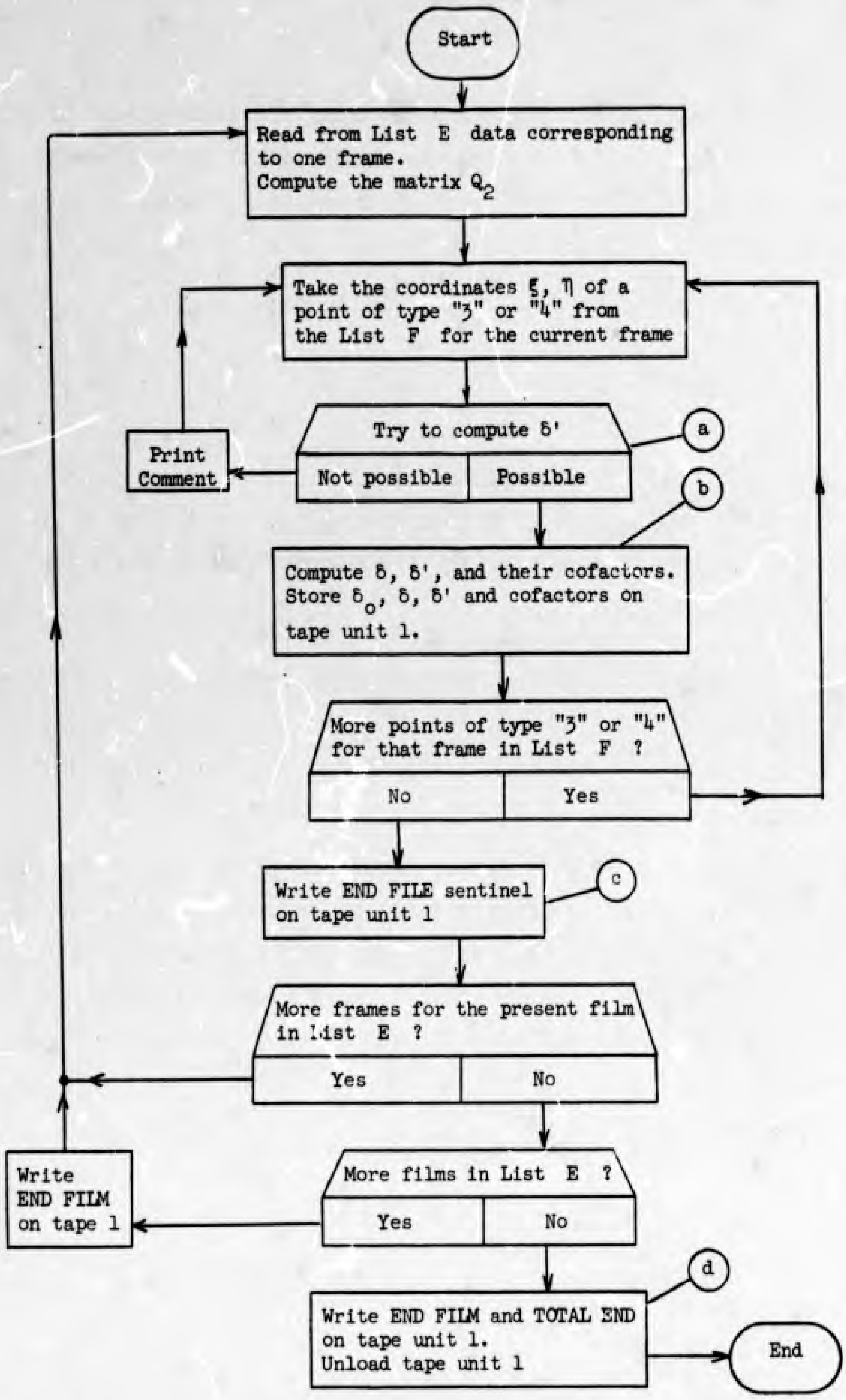
(In case of objects at finite distances, the contents of the tape must be supplemented accordingly.) The identification, consisting of 20 characters, gives the numbers of film, frame and point in the form

$$\left|\underset{\longleftarrow id_1 \longrightarrow}{\text{FI = 123bFR =}}\right|\underset{\longleftarrow id_2 \longrightarrow}{\text{123bP = 1234}}\left.\right|$$

(c.) This is the standard sentinel for the subroutine COLS (Appendix).

(d.) Instead of unloading the tapes and proceeding to the next program, at this point the computation of the refraction index can start. This computation may follow the lines of Ref. 1, Section 8.3, with the exception, that the more general least squares routine COLS, capable of handling correlated data, should be used, instead of L.SQ. (Appendix).

FLOW CHART OF ALGORITHM 3



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APPENDIX

GENERAL LEAST SQUARES PROBLEM

1. Problem

Suppose we are given a functional relationship

$$F(x_1, x_2, \dots x_n ; y_1, y_2, \dots y_m) = 0 \tag{1}$$

between n + m variables. Assume that the first n variables x_i can be determined by observations. The problem is to determine the values of the remaining m variables y_i. If F satisfies certain conditions, the m unknowns y_i can be computed if m sets $\{x_i\}$ are known. We assume that we know from observations r sets of $\{x_i\}$ and $r > m$. Then the r corresponding equations (1) constitute an overdetermined equation system for $\{y_i\}$ and we will apply the least squares method to the observations.

2. Linearization of the Problem

We denote

$$\{x_{ji}\} = \bar{x}_j = \bar{X}_j + \bar{\xi}_j = \{X_{ji}\} + \{\xi_{ji}\} \quad \text{with} \quad \begin{array}{l} j = 1,2,\dots,r \\ i = 1,2,\dots,n \end{array} \tag{2}$$

Here $\bar{X}_j$ are the r measured sets and $\bar{\xi}_j$ their (unknown) corrections. The x_j are the correct values. The weights of the measurements may be given by

$$\bar{p}_j = \{p_{ji}\} \quad \text{with} \quad \begin{array}{l} j = 1,2,\dots,r \\ i = 1,2,\dots,n \end{array} \tag{3}$$

The unknowns $\{y_\ell\}$ we denote by

$$\{y_\ell\} = \bar{y} = \bar{Y} + \bar{\eta} = \{Y_\ell\} + \{\eta_\ell\} \quad \text{with } \ell = 1,2,\dots,m \tag{4}$$

In (4) $\bar{Y}$ denotes an approximation to the unknown correct value $\bar{y}$ and $\bar{\eta}$ is the (unknown) correction of $\bar{Y}$. As stated in Section 1, we assume that

$$r > m \tag{5}$$

In order to linearize the problem we assume that F can be replaced with sufficient accuracy by the linear terms of its Taylor expansion in the vicinity of the r places defined by the sets $\{\bar{X}_j, \bar{Y}\}$. With the notations

$$F_j = F(\bar{X}_j, \bar{Y}) \qquad j = 1,2,\ldots,r \quad (6)$$

$$A_{ji} = \left\{\frac{\partial F}{\partial x_i}\right\}_{\substack{\bar{x} = \bar{X}_j \\ \bar{y} = \bar{Y}}} \qquad \begin{matrix} j = 1,2,\ldots,r \\ i = 1,2,\ldots,n \end{matrix} \quad (7)$$

$$B_{j\ell} = \left\{\frac{\partial F}{\partial y_\ell}\right\}_{\substack{\bar{x} = \bar{X}_j \\ \bar{y} = \bar{Y}}} \qquad \begin{matrix} j = 1,2,\ldots,r \\ \ell = 1,2,\ldots,m \end{matrix} \quad (8)$$

we can write then instead of F = 0 the linear equations

$$F_j + \sum_{i=1}^{n} A_{ji}\xi_{ji} + \sum_{\ell=1}^{m} B_{j\ell}\eta_\ell = 0 \qquad (j = 1,2,\ldots,r) \quad (9)$$

These are r equations for the $r \cdot n + m$ unknowns ξ_{ji} and η_ℓ.

3. Introduction of Correlates

In accordance with the least squares principle we will determine the corrections ξ_{ij} such that

$$[\bar{p}\,\bar{\xi}^2] = \sum_{j=1}^{r} \sum_{i=1}^{n} p_{ji}\xi_{ji}^2 = \min. \quad (10)$$

Instead of minimizing (10) we minimize the following function W, which we obtain from (10) by adding the r expressions (9), multiplied with some factors $-2k_j$.

$$W = \left[\bar{p}\,\bar{\xi}^2\right] - 2k_1\left(F_1 + \sum_{i=1}^{n} A_{1i}\xi_{1i} + \sum_{\ell=1}^{m} B_{1\ell}\eta_\ell\right) -$$

$$- 2k_2\left(F_2 + \sum_{i=1}^{n} A_{2i}\xi_{2i} + \sum_{\ell=1}^{m} B_{2\ell}\eta_\ell\right) - \ldots \tag{11}$$

$$\ldots - 2k_r\left(F_r + \sum_{i=1}^{n} A_{ri}\xi_{ri} + \sum_{\ell=1}^{m} B_{r\ell}\eta_{r\ell}\right) = \min$$

The quantities k_j $(j = 1,2,\ldots,r)$ are called correlates.

4. Normal Equations

Setting the n•r derivatives of W with respect to the ξ_{ji} equal to zero we obtain the following equations

$$p_{ji}\xi_{ji} - k_j A_{ji} = 0 \qquad \begin{matrix} j = 1,2,\ldots,r \\ i = 1,2,\ldots,n \end{matrix} \tag{12}$$

or

$$\xi_{ji} = k_j \frac{A_{ji}}{p_{ji}} \tag{13}$$

or

$$\begin{aligned} \bar{\xi}_j &= k_j \left\{\frac{A_{ji}}{p_{ji}}\right\} \\ &= k_j \left\{\frac{A_{j1}}{p_{j1}}, \frac{A_{j2}}{p_{j2}}, \ldots \frac{A_{jn}}{p_{jn}}\right\} \end{aligned} \tag{14}$$

Substituting these values into the r equations (9) we obtain the equations

$$F_j + k_j \sum_{i=1}^{n} \frac{A_{ji}^2}{p_{ji}} + \sum_{\ell=1}^{m} B_{j\ell}\eta_\ell = 0 \tag{15}$$

We define g_j by the equation

$$\frac{1}{g_j} = \sum_{i=1}^{n} \frac{A_{ji}^2}{p_{ji}} \qquad (j = 1,2,\ldots,r) \tag{16}$$

and have then

$$\frac{k_j}{g_j} + \sum_{\ell=1}^{m} B_{j\ell}\eta_\ell + F_j = 0 \qquad (j = 1,2,\ldots,r) \tag{17}$$

The derivatives of W with respect to the m unknown η_ℓ furnish the equations

$$\sum_{j=1}^{r} k_j B_{j\ell} = 0 \qquad (\ell = 1,2,\ldots,m) \tag{18}$$

If we substitute the expressions (17) for k_j into (18) we obtain the following normal equations for the unknowns η_ℓ:

$$\sum_{\ell=1}^{m} \left(\sum_{j=1}^{r} g_j B_{j\ell} B_{js}\right) \eta_\ell + \sum_{j=1}^{r} g_j F_j B_{js} = 0 \quad (s = 1,2,\ldots,m) \tag{19}$$

5. Observation Errors

The standard error of a measurement of weight one is given by the formula

$$m_o = \sqrt{\frac{\left[\bar{p}\,\bar{\xi}^2\right]}{r - m}} \tag{20}$$

where $\left[\bar{p}\,\bar{\xi}^2\right]$ is given by (10). The numerator of the radicand can be computed in many ways using the following formulae:

$$\sum_{j}^{r}\sum_{i}^{n} p_{ji}\xi_{ji}^2 = \sum_{j}^{r}\sum_{i}^{n} p_{ji}k_j^2 \frac{A_{ji}^2}{p_{ji}^2} \qquad \text{(with (13))}$$

$$= \sum_{j}^{r} k_j^2 \frac{1}{g_j} \qquad \text{(with (16))}$$

$$= \sum_{j}^{r} k_j(-F_j - \sum_{\ell}^{m} B_{j\ell}\eta_\ell) \qquad \text{(with (17))}$$

$$= - \sum_{j}^{r} k_jF_j - \sum_{\ell}^{m} (\sum_{j}^{r} k_jB_{j\ell})\, \eta_\ell$$

$$= - \sum_{j}^{r} k_jF_j \qquad \text{(with (18))}$$

$$= \sum_{j}^{r} g_jF_j^2 + \sum_{j}^{r}\sum_{\ell}^{m} g_jF_jB_{j\ell}\eta_\ell \qquad \text{(with (17))}$$

$$= \sum_{j}^{r} g_jF_j^2 + \sum_{\ell}^{m} (\sum_{j}^{r} g_jF_jB_{j\ell})\, \eta_\ell$$

Note that the factors of η_ℓ in the last expression are the negative right sides of the normal Equation (19).

The most important of these relations are

$$\sum_{j}^{r}\sum_{i}^{n} p_{ji}\xi_{ji}^2 = - \sum_{j}^{n} k_jF_j = \sum_{j}^{r} g_jF_j^2 + \sum_{\ell}^{m} (\sum_{j}^{r} g_jF_jB_{j\ell})\, \eta_\ell \qquad (21)$$

With the quantity m_o we can express the standard errors of the observations of X_{ji} by

$$m_{X_{ji}} = \frac{m_o}{\sqrt{p_{ji}}} \qquad \begin{pmatrix} j=1,2,\ldots,r \\ i=1,2,\ldots,n \end{pmatrix} \tag{22}$$

These errors should be of the same order as the observation errors estimated from the properties of the observation apparatus. (See Section 10 for controls of this type.)

For the present problem another set of observation errors is of equal importance. We consider the observations of the X_{ji} as independent of each other and have assumed that every observation X_{ji} enters in only one of the r equations $F = 0$ (equation (9)). With this assumption we can consider the values $F_j = F(\overline{X}_j,\overline{Y})$ as direct and independent observations with a standard error which follows from the standard errors of $\overline{X}_j$ by the law of error propagation, namely

$$m_{Fj} = \sqrt{\sum_{i=1}^{n} A_{ji}^2 \, m_{X_{ji}}^2} \tag{23}$$

With (22) and (16) we obtain from (23)

$$m_{Fj} = \frac{m_o}{\sqrt{g_j}} \qquad (j=1,2,\ldots,r) \tag{24}$$

6. Errors of the Unknown Parameters

In many applications we are interested mainly in the unknown parameters $\overline{y}$ or in some functions of the $\overline{y}$. In these cases we need the standard errors of $\overline{y}$ and of functions of $\overline{y}$.

The standard errors of the parameters $\overline{y}$ are the same as those of η (see (4)). The latter can be computed by the following consideration. We assign to the observation set j (consisting of the n observations X_{ji}) the error

$$v_j = -\sum_{i=1}^{n} A_{ji}\xi_{ji} = -\frac{k_j}{g_j} \tag{25}$$

and the weight g_j (see equation (16)). The equations (17) have then the form

$$v_j = \sum_{\ell=1}^{m} B_{j\ell}\eta_\ell + F_j \qquad (j=1,2,\ldots,r) \tag{26}$$

We consider these equations as error equations for the observations F_j and determine η_ℓ such that $[gv^2]$ assumes a minimum. This requirement is equivalent to the requirement $[\bar{p}\bar{\xi}^2] = \min$ (equation (10)) because with (13) and (16) we have the relation

$$g_j v_j^2 = \frac{k_i^2}{g_j} = \sum_{i=1}^{n} p_{ji}\xi_{ji}^2 \qquad (j=1,2,\ldots,r). \tag{27}$$

The normal equations corresponding to this problem are identical to the equations (19). We will write them here in matrix form and define therefore first the following matrices

$$\bar{B} = \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \cdot & \cdots & \cdot \\ B_{r1} & \cdots & B_{rm} \end{pmatrix} \tag{28}$$

$$\bar{G} = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & g_r \end{pmatrix} \tag{29}$$

$$\bar{F} = \begin{pmatrix} F_1 \\ F_2 \\ \cdot\cdot \\ F_r \end{pmatrix}, \quad \bar{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdot\cdot \\ \eta_m \end{pmatrix} \tag{30}$$

The normal equations (19) take with these notations the form

$$\bar{B}'\,\bar{G}\,\bar{B}\,\bar{\eta} = -\,\bar{B}'\,\bar{G}\,\bar{F}\,. \tag{31}$$

Let the inverse matrix to the matrix of the normal equations (19) or (31) be

$$\bar{Q} = (\bar{B}' \; \bar{G} \; \bar{B})^{-1} \tag{32}$$

$\bar{Q}$ is called the matrix of cofactors of the η_ℓ

The solution $\bar{\eta}$ of the normal equations is then

$$\bar{\eta} = - \bar{Q} \; \bar{B}' \; \bar{G} \; \bar{F} = \bar{H} \; \bar{F}, \tag{33}$$

where $\bar{H}$ represents the product $- \bar{Q} \; \bar{B}' \; \bar{G}$. The components F_j of $\bar{F}$ are independent "observations" with the standard errors m_{Fj} from equation (24). The standard errors of the components η_ℓ of $\bar{\eta}$ therefore can be obtained from the errors m_{Fj} by applying the law of error propagation on the equation (33). If h_{rt} are the elements of the matrix $\bar{H}$, then the standard error $m_{y\ell}$ of η_ℓ is given by

$$m_{\eta\ell}^2 = m_o^2 \left(\frac{h_{\ell 1}^2}{g_1} + \frac{h_{\ell 2}^2}{g_2} + \ldots + \frac{h_{\ell r}^2}{g_r}\right) \tag{34}$$

The expression in parenthesis in (34) is the ℓ-th diagonal term of the matrix

$$\begin{aligned} \bar{H} \; \bar{G}^{-1} \; \bar{H}' &= (\bar{Q} \; \bar{B}' \; \bar{G}) \; \bar{G}^{-1} \; (\bar{G} \; \bar{B} \; \bar{Q}\,) = \\ &= \bar{Q} \; (\bar{B}' \; \bar{G} \; \bar{B}) \; \bar{Q} = \bar{Q} \end{aligned} \tag{35}$$

Hence the standard error $m_{y\ell}$ of η_ℓ (or y_ℓ) can be expressed by

$$m_{\eta\ell} = m_o \cdot \sqrt{Q_{\ell\ell}}\,, \tag{36}$$

where $Q_{\ell\ell}$ is the ℓ-th diagonal element in the matrix $\bar{Q}$ inverse to the matrix of the normal equations (19) and m_o is the error of a unit weight observation, given by (20).

7. Functions of the Parameters

Assume that we use the parameters y_ℓ to compute some other function, say

$$T = T(y_1, y_2, \ldots, y_m) \tag{37}$$

The standard error of T depends on the accuracies of the arguments y_ℓ. It can be computed by applying the law of error propagation on (37). However, since the y_ℓ are not independent of each other we must first express T in terms of the independent "observations" F_j. Thereby we can confine ourselves to the expression of the differential dT of T in terms of the differentials dF_j of F_j. First we introduce the vector $\bar{T}_y$ of the partial derivatives of T by

$$\bar{T}_y = \begin{pmatrix} T_{y1} \\ T_{y2} \\ \cdots \\ T_{ym} \end{pmatrix} = \begin{pmatrix} \partial T/\partial y_1 \\ \partial T/\partial y_2 \\ \cdots\cdots \\ \partial T/\partial y_m \end{pmatrix}. \tag{38}$$

With (38) the differential dT of T is

$$dT = \bar{T}'_y \, (d\bar{\eta}), \tag{39}$$

where the components of the vector $(d\bar{\eta})$ are the differentials $d\eta_\ell$ of η_ℓ. (According to (4) these differentials are equal to those of y_ℓ.) The η_ℓ are with (33) linear functions of the F_j. The differentials $d\eta_\ell$ are therefore the same linear functions (33) of the differentials dF_j. Substituting (33) into (39) we obtain

$$dT = - \bar{T}'_y \, \bar{Q} \, \bar{B}' \, \bar{G} \, (d\bar{F}) = - \bar{T}'_y \, \bar{H} \, (d\bar{F}). \tag{40}$$

or

$$dT = R_1 dF_1 + R_2 dF_2 + \dots + R_r dF_r, \tag{41}$$

where R_j is a component of the vector

$$\bar{R} = - \bar{H}' \, \bar{T}_y. \tag{42}$$

The standard error m_T of T follows from (41) and (24) by applying the law of error propagation:

$$m_T^2 = m_o^2 \left(\frac{R_1^2}{g_1} + \frac{R_2^2}{g_2} + \dots + \frac{R_r^2}{g_r}\right) =$$

$$= m_o^2 \bar{R}' \bar{G}^{-1} \bar{R} =$$

$$= m_o^2 (\bar{T}'_y \bar{H}) \bar{G}^{-1} (\bar{H}' \bar{T}_y) =$$

$$= m_o^2 \bar{T}'_y (\bar{H} \bar{G}^{-1} \bar{H}') \bar{T}_y =$$

$$= m_o^2 \bar{T}'_y \bar{Q} \bar{T}_y .$$

(For the last equation see (35).) Hence

$$m_T = m_o \cdot \sqrt{\bar{T}'_y \bar{Q} \bar{T}_y} \tag{43}$$

where $\bar{Q}$ is the inverse matrix to the matrix of the normal equations (19) and m_o is given by (20). Expressing the matrix product in (43) in terms of the elements of the matrices we obtain

$$m_T = m_o \sqrt{\sum_{\ell=1}^{m} \sum_{s=1}^{m} Q_{\ell s} T_{y\ell} T_{ys}} . \tag{44}$$

8. Errors of the Adjusted Observations

The standard errors of the adjusted observations $x_{ji} = X_{ji} + \xi_{ji}$ are seldom needed. The corresponding formulae will be noted here for sake of completeness.

First we note that with (13) the standard error of x_{ji} depends on the standard error of the correlate k_j by the formula

$$m_{xji} = \left|\frac{A_{ji}}{p_{ji}}\right| m_{kj} . \quad \begin{pmatrix} j=1,2,\dots,r \\ i=1,2,\dots,n \end{pmatrix} \tag{45}$$

Hence we may first compute the standard errors of the correlates k_j. We denote the vector of the correlates by

$$\bar{k} = \begin{pmatrix} k_1 \\ k_2 \\ \cdots \\ k_r \end{pmatrix} \tag{46}$$

and express the equations (17) in the form

$$\bar{k} = -\bar{G}\,\bar{F} - \bar{G}\,\bar{B}\,\bar{\eta}. \tag{47}$$

using the symbols (28), (29) and (30). Substituting (33) into (47) and defining $\bar{K}$ by $\bar{K} = \bar{G} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G}$ we obtain

$$\begin{aligned} \bar{k} &= -\bar{G}\,\bar{F} + \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G}\,\bar{F} = \\ &= -(\bar{G} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G})\,\bar{F} = -\bar{K}\,\bar{F}. \end{aligned} \tag{48}$$

Application of the law of error propagation on (48) yields the following formula for the standard error m_{kj} of the correlate k_j

$$m_{kj}^2 = m_o^2 \left(\frac{K_{j1}^2}{g_1} + \frac{K_{j2}^2}{g_2} + \ldots + \frac{K_{jr}^2}{g_r}\right), \tag{49}$$

where K_{jt} are the elements of the matrix $\bar{K}$, defined by (48). The expression in parenthesis in (49) is the j-th diagonal element of the matrix

$$\begin{aligned} \bar{K}\,\bar{G}^{-1}\,\bar{K}' &= (\bar{G} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G})\,\bar{G}^{-1}\,(\bar{G} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G}) = \\ &= (\bar{I} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}')\,(\bar{G} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G}) = \\ &= \bar{G} - 2\,\bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G} + \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G} = \\ &= \bar{G} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}'\,\bar{G} = \\ &= (\bar{I} - \bar{G}\,\bar{B}\,\bar{Q}\,\bar{B}')\,\bar{G} \end{aligned} \tag{50}$$

Hence the standard error m_{kj} of the correlate k_j is equal to m_o times the square root of the j-th diagonal element of the matrix (50). Expressing m_{kj} in by the elements of the matrices $\bar{G}$, $\bar{B}$ and $\bar{Q}$ we obtain

$$m_{kj} = m_o \sqrt{g_j} \cdot \sqrt{1 - g_j \sum_{\ell=1}^{m} \sum_{t=1}^{m} Q_{\ell t} B_{j\ell} B_{jt}} \tag{51}$$

With (45) we obtain from (51)

$$m_{xji} = m_o \left|\frac{A_{ji}}{p_{ji}}\right| \sqrt{g_j} \sqrt{1 - g_j \sum_{\ell=1}^{m} \sum_{t=1}^{m} Q_{\ell t} B_{j\ell} B_{jt}} \tag{52}$$

The observations X_{ji} had the standard errors m_{Xji}, defined by (22). In order to compare the error m_{xji} of the adjusted observations x_{ji} with the m_{Xji} we note the following form of (52):

$$m_{xji} = \frac{m_o}{\sqrt{p_{ji}}} \sqrt{\frac{A_{ji}^2/p_{ji}}{\sum_{s=1}^{n} \frac{A_{js}^2}{p_{js}}} \left(1 - g_j \sum_{\ell=1}^{m} \sum_{t=1}^{m} Q_{\ell t} B_{j\ell} B_{jt}\right)} \tag{53}$$

The first factor in (53) is the standard error of the original observation X_{ji}. The second factor which is obviously less than 1 represents the improvement of the accuracy.

9. Functions of the Parameters and of the Adjusted Observations

In some cases we might be interested in a function $U(x_1, \ldots x_n; y_1, \ldots y_m)$ at the observation points. After the computations of Section 4 we will naturally use for the arguments of U the adjusted observations $\bar{x}_j = \bar{X}_j + \bar{\xi}_j$, rather tha the original $\bar{X}_j$, and the final parameter values $\bar{y} = \bar{Y} + \bar{\eta}$. Since the $\bar{x}_j$ as well as the $\bar{y}$ are known only approximately, we may ask for the accuracy of the corresponding values of U.

An example of such a function U is the constraint function F. After calculating the $\bar{x}_j$ and $\bar{y}$ we may test the calculations by checking whether $F(\bar{x}_j, \bar{y}) = 0$ for all $j=1,2,\ldots,r$. For the purpose of the check we need to know how accurately these r equations must be satisfied.

In order to find an expression for the standard error of U we will proceed in the same manner as in Section 7. First we will express the differential of U in terms of differentials of the independent "observations" F_j and then apply the law of error propagation on the differentials.

For convenience we consider the values of U at all observation points simultaneously and use matrix algebra. We denote

$$U_j = U(\bar{x}_j, \bar{y}) \qquad (j=1,2,\ldots,r) \tag{54}$$

$$U_{xji} = \left\{\frac{\partial U}{\partial x_i}\right\}_{\bar{x}=\bar{x}_j} \qquad \begin{pmatrix} j=1,2,\ldots,r \\ i=1,2,\ldots,n \end{pmatrix} \tag{55}$$

$$U_{yj\ell} = \left\{\frac{\partial U}{\partial y_\ell}\right\}_{\bar{x}=\bar{x}_j} \qquad \begin{pmatrix} j=1,2,\ldots,r \\ \ell=1,2,\ldots,m \end{pmatrix} \tag{56}$$

The differential of U at $\bar{x}_j$ is

$$dU_j = \sum_{i=1}^{n} U_{xji}\, dx_{ji} + \sum_{\ell=1}^{m} U_{yj\ell}\, dy_\ell =$$

$$= \delta U_{jx} + \delta U_{jy}, \tag{57}$$

with obvious meaning of δU_{jx} and δU_{jy}. With (13) and (57) we have

$$\delta U_{jx} = \sum_{i=1}^{n} U_{xji} \frac{A_{ji}}{p_{ji}} \cdot k_j. \tag{58}$$

We introduce the symbol w_j by

$$\frac{1}{w_j} = \sum_{i=1}^{n} U_{xji} \frac{A_{ji}}{p_{ji}} \tag{59}$$

and the matrices

$$\overline{W} = \begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & w_r \end{pmatrix}, \tag{60}$$

$$\delta\overline{U}_x = \begin{pmatrix} \delta U_{1x} \\ \dots \\ \delta U_{rx} \end{pmatrix}, \quad \delta\overline{U}_y = \begin{pmatrix} \delta U_{1y} \\ \dots \\ \delta U_{ry} \end{pmatrix}, \tag{61}$$

and

$$d\overline{U} = \delta\overline{U}_x + \delta\overline{U}_y. \tag{62}$$

With these symbols we can express $\delta\overline{U}_x$ from (58), (46) and (48) by

$$\begin{aligned} \delta\overline{U}_x &= \overline{W}^{-1}\,\overline{k} = \\ &= (-\overline{W}^{-1}\,\overline{G} + \overline{W}^{-1}\,\overline{G}\,\overline{B}\,\overline{Q}\,\overline{B}'\,\overline{G})\,d\overline{F}. \end{aligned} \tag{63}$$

The value of $\delta\overline{U}_y$ can be found by applying the general formula (40) of Section 7. With the matrix

$$\overline{U}_y = \begin{pmatrix} U_{y11} & \dots & U_{y1m} \\ \dots & \dots & \dots \\ U_{yr1} & & U_{yrm} \end{pmatrix} \tag{64}$$

and (40) we obtain

$$\delta\overline{U}_y = -\,\overline{U}_y\,\overline{Q}\,\overline{B}'\,\overline{G}\,d\,\overline{F}. \tag{65}$$

The sum of (63) and (65) furnishes the differential $d\overline{U}$ in terms of $d\overline{F}$:

$$d\overline{U} = (\,-\overline{W}^{-1}\,\overline{G} + \overline{W}^{-1}\,\overline{G}\,\overline{B}\,\overline{Q}\,\overline{B}'\,\overline{G} - \overline{U}_y\,\overline{Q}\,\overline{B}'\,\overline{G})\,d\overline{F}. \tag{66}$$

We denote by v_{js} the elements of the matrix in parenthesis in (66), and have with (24) the following formula for the standard error m_{Uj} of U_j

$$m_{Uj} = m_o \sqrt{\frac{v_{j1}^2}{g_1} + \frac{v_{j2}^2}{g_2} + \ldots + \frac{v_{jr}^2}{g_r}} \; . \qquad (67)$$

Explicitely we obtain for the element v_{js} the expression

$$v_{js} = - \frac{g_j}{w_j} \delta_{js} + \frac{g_j}{w_j} \sum_{t=1}^{m} \sum_{\ell=1}^{m} Q_{\ell t} \, B_{j\ell} \, B_{st} \, g_s -$$

$$- \sum_{t=1}^{m} \sum_{\ell=1}^{m} Q_{\ell t} \, U_{yj\ell} \, B_{st} \, g_s$$

or

$$v_{js} = - \frac{g_j}{w_j} \delta_{js} + g_s \sum_{t=1}^{m} \sum_{\ell=1}^{m} Q_{\ell t} \, B_{st} \left(\frac{g_j}{w_j} B_{j\ell} - U_{yj\ell}\right), \qquad (68)$$

where δ_{js} is the Kronecker symbol

$$\delta_{js} = \begin{cases} 1 & \text{if} \quad j = s \\ 0 & \text{if} \quad j \neq s. \end{cases}$$

A special case of (67) and (68) is the above mentioned one where the function U is the constraint function F. If moreover F is linear in $\bar{x}$ and $\bar{y}$ (or, as assumed in Section 2, linear within sufficient accuracy), then we have the equations

$$\bar{W} = \bar{G} \qquad (69)$$

and

$$\bar{U}_y = \bar{B}. \qquad (70)$$

The equation (68) turns out to be simply

$$v_{js} = - \delta_{js} \qquad (71)$$

and, consequently,

$$m_{Uj} = \frac{m_o}{\sqrt{g_j}} = m_{Fj} \; . \qquad (72)$$

This means that in all practical cases the accuracy of the constraint function F computed at the adjusted observation places $\overline{X}_j$ is the same as the accuracy of the function F computed at the not adjusted observation places $\overline{x}_j$. Note that these accuracy considerations do not involve the systematic errors of $F(\overline{X}_j, \overline{Y})$ caused by the choice of the approximation $\overline{Y}$ for the parameters.

10. Controls of the Computation

The results of the computations must be controlled with respect to the following features:

a. Accuracy of the results computed.

b. Validity of the linearization (cf. Section 2).

c. Validity of the assumed relationship $F(\overline{x}, \overline{y}) = 0$.

d. Accuracy of input data.

a. Accuracy of the results computed.

This control should show whether the calculations are carried out with a sufficient number of digits. A control for the solution of the normal equations (19) is provided by the matrix $\overline{Q}$ (see Section 6). The solution $\overline{\eta}$ of (19) or (31) may be compared with the values obtained by the matrix multiplication (33). Both results should agree.

At the end of the calculations the last equation in (21) furnishes a thorough control of the computation of $\overline{\eta}$.

b. Linearization.

The linearization of the function $F(\overline{x}, \overline{y})$ with respect to the variables $\overline{y}$ is justified if the following equations hold within the accuracy required:

$$F(\overline{X}_j, \overline{Y} + \overline{\eta}) = F(\overline{X}_j, \overline{Y}) + \sum_{\ell=1}^{m} B_{j\ell}\, \eta_\ell \qquad (j=1,2,\ldots,r)$$

or

$$F(\overline{X}_j, \overline{y}) = F_j + \sum_{\ell=1}^{m} B_{j\ell}\, \eta_\ell \qquad (j=1,2,\ldots,r) \qquad (73)$$

This quantity was denoted in Section 6 by v_j and the relation

$$\sum_{j=1}^{r} g_j \, v_j^2 = \sum_{j=1}^{r} \sum_{i=1}^{n} p_{ji} \, \xi_{ji}^2 = [\bar{p} \, \bar{\xi}^2] \tag{74}$$

proved. Hence one possibility to check the validity of (73) is to compute the sum

$$\sum_{j=1}^{r} g_j \, F^2 \, (\bar{X}_j, \, \bar{y}) \tag{75}$$

and check whether it is equal to the other expressions for $[\bar{p} \, \bar{\xi}^2]$ in the equation (21). The values of those expressions are computed when doing the final accuracy check according to the previous Section a.

The linearization of the function $F(\bar{x}, \bar{y})$ with respect to the variables $\bar{x}$ can be checked most rigorously by computing the adjusted observations $\bar{x}_j = \bar{X}_j + \bar{\xi}_j$ and checking whether

$$F(\bar{x}_j, \, \bar{y}) = 0 \qquad (j=1,2,\ldots,r) \tag{76}$$

within the accuracy of the particular observation set. (See Section 9 for the accuracy of $F(\bar{x}_j, \bar{y})$.)

c. Validity of $F(\bar{x}, \bar{y}) = 0$.

A possible check of the validity of the assumed relationship $F(\bar{x}, \bar{y}) = 0$ between the variables $\bar{x}$ and $\bar{y}$ is given by consideration of standard errors. If a correct relationship is assumed (within the accuracy of the measurements), then the standard errors m_{Xij} from (22) must be of the same order as the standard observation errors, which may be estimated from the properties of the observation apparatus. If this condition is not satisfied, an erroneous relationship F was assumed or erroneous assumptions about the accuracies of the observations must have been made.

In general the components of $\bar{x}$ will have different dimensions and consequently the weights will be defined differently for each component. As a convenient standard we assume for the following considerations that the measurements X_{ji} are furnished together with their standard errors e_{ji} (i.e. mean square errors). We can then assign the weight $p_{ji} = 1/e_{ji}^2$ to the observation X_{ji}. With these weights the quantity m_o (see (20) or (21)) is dimensionless and should be of the order one. This requirement constitutes a first check of the validity of $F(\bar{x}, \bar{y}) = 0$.

The standard error of the function F due to the observation errors is equal to m_{Fj} (see (24)). If we assume that the standard errors e_{ji} are correct, then m_{Fj} is the standard deviation of $F(\bar{x}_j, \bar{y})$ from zero (see (72)). Hence the comparison of these values of F with the corresponding m_{Fj} provides another check for the validity of the assumed relationship $F(\bar{x}, \bar{y}) = 0$. An investigation of the distribution of these values will help to detect systematic errors in the relationship assumed. Such errors will usually manifest in systematic deviations of $F(\bar{x}_j, \bar{y})$ from zero rather than the random deviations expected.

d. Accuracy of input data.

As stated in the previous Section c, a general check of the correctness of the standard errors e_{ji} is provided at the same time when the validity of the relationship $F(\bar{x}, \bar{y}) = 0$ is checked. If either the relationship is not valid or the accuracy is in general not correctly given, the quantity m_o (see (20)) will be different from one. Single blunders will be discovered by comparison of $F(\bar{x}_j, \bar{y})$ with its standard error m_{Fj} (see (72) and (24)). Thus erroneous sets can be detected, for instance by listing all such sets for which

$$|F(\bar{x}_j, \bar{y})| > 3m_{Fj} \qquad (j=1,2,\dots,r) \tag{77}$$

If the number of observation sets, r, is large and the observation errors are distributed normally, then 0.3% of the sets should satisfy (77).

The same kind of checking can be applied to single components of $\bar{x}$. First the (dimensionless) quantities

$$m_{xi} = \sqrt{\frac{\sum_{j=1}^{r} \xi_{ji}^2/e_{ji}^2}{r - m}} \qquad (i=1,2,\dots,n) \tag{78}$$

should be of the order of one. These quantities are the standard errors of an observation of unit weight of the i-th component of $\bar{x}$. If m_{xi} is not of the order of one, then most probably not adequate observation errors e_{ji} for the i-th component were given. (We assume now that the relationship $F(\bar{x}, \bar{y}) = 0$ describes correctly the physical properties of the phenomenon observed.)

A more detailed check, which corresponds to (77), is to compare the single ξ_{ji} with the corresponding standard error m_{Xji} from (20). Since the ξ_{ji} are supposed to be the observation errors, they should be distributed normally for any fixed i. Hence we may detect blunders by listing all such measurements for which

$$|\xi_{ji}| > 3 \frac{m_o}{\sqrt{p_{ji}}} \qquad \begin{matrix} (j=1,2,\ldots,r \\ i=1,2,\ldots,n) \end{matrix} \tag{79}$$

In (79) we have used m_o instead of m_{xi} (from (78)) because m_o will usually be less affected by blunders.

In the special cases where observations of a component of $\bar{x}$ are of the same accuracy for all observation sets (i.e., $e_{ji} = e_{ki}$ for a fixed i and j, k = 1,2,...,r), we can compute the standard error of one observation of that component. With (78) it is equal to

$$m_{Xi} = e_{(j)i}\, m_{xi} = \sqrt{\frac{\sum\limits_{j=1}^{n} \xi_{ji}^2}{r - m}} \qquad (i=1,2,\ldots,n) \tag{80}$$

(By writing the index j of e_{ji} in parenthesis we indicate, that the errors e_{ji} are independent of the index j.) The error m_{Xi} has the same dimension as the i-th component of $\bar{x}$ and it has the correct value if m_{xi} is of the order of one. Using the square root in (80) we can compute m_{Xi} also in cases where different sets j have different accuracies e_{ji}. In these cases, however, m_{Xi} has no physical meaning.

11. First Machine Program for Solving General Least Squares Problems.

The program described in this Section is designed for such cases where the arguments x_i of the function $F(x_i, y_\ell)$ (see (1)) are independent of each other. If they are correlated, a modified version of the program should be used. (The x_i may be correlated for instance in cases where they are not observed directly but are results of other adjustments.) The theory of such modifications and a corresponding machine program are described in Sections 12 through 18.

The least squares method will generally be used as a part of more extensive calculations, which check the validity of the observed data, compute initial values, fix the degree of approximation etc. Therefore, the least squares machine program described in this section is written in the form of a subroutine. This subroutine calculates the unknown values of the parameters and returns the control to the calling program before the final controls of Section 10 are carried out. Depending on the properties of the problem and the results of the calculations, the process can then be repeated with other initial values or the final controls of Section 10 carried out. Therefore the subroutine has two entries. Entering the first part a number of arguments are necessary. Entering the second part (for final controls) only an indicator may be used to specify the amount of printed output.

Entering the Program.

The least squares subroutine can be entered by the following **FORAST** statements:

1st part

```
ENTER (L·SQ·1) (N)(M)(F) ... (MO)(Q11)%
```

2nd part

```
ENTER (L·SQ·2) (PR)%
```

When entering the second part, the indicator (here denoted by PR) should be zero for reduced output and non-zero for full output. For particulars see the flow chart and the section "Printed Results."

When entering the first part (L·SQ·1), the addresses of the following arguments should be transmitted:

Addresses of arguments

- n = address of the number of x-variables, i.e.,"observations". ($n \leq 10$ and integer).
- m = address of the number of y-variables, i.e.,"parameters". ($m \leq 10$ and integer).
- F = address to enter the subroutine F.
- A1,...,AN addresses to enter the n subroutines A_i B1,...,BM and the m subroutines B_i.
- Y_1 = address of Y_1. It is assumed that the other initial approximations $Y_2,\ldots,Y_m$ of the parameters are stored in cells subsequent to that of Y_1.

Addresses for storing the results

- y_1 = address of y_1. Other y_i will be stored in subsequent cells.
- m_{y1} = address of the standard error of y_1. Other standard errors m_{yi} will be stored in subsequent cells.
- m_o = address of the standard error of measurements with weight one.
- Q_{11} = address of the first element of the matrix Q (inverse to the normal equations matrix). The elements of Q will be stored in the sequence $Q_{11},Q_{12},\ldots Q_{1m}, Q_{12},Q_{22},\ldots Q_{2m},Q_{13},\ldots Q_{3m},\ldots Q_{mm}$. For sake of simplicity the complete matrix is stored and storage space of m^2 words required.

The total number of arguments in the ENTER (L·SQ·1) - statement is hence 8 + m + n. These 8 + m + n addresses may be given in the same order as they are listed here.

About the subroutines F, A_i and B_i it is assumed that they can be called by statements of the form

ENTER (NAME)(RESULT)(INDIC)(ARGN)(ARGM)%

where the symbols in the statement stand for the following addresses:

NAME - address to enter the subroutine.

RESULT - address to store the result.

INDIC - address of an indicator. Its value is set zero if the result can be computed. If the arguments are such that no result can be computed, the indicator is non-zero.

ARGN - address of the argument x_1. It is assumed that $x_2, \ldots, x_n$ are stored in cells subsequent to x_1.

ARGM - address of the argument y_1. The arguments $y_2, \ldots, y_m$ are assumed to be stored in cells subsequent to y_1.

These subroutines may be programmed as shown by the following example for the subroutine named "NAME":

```
NAME        SET(7 = SELF + 1) GOTO(1,1)
            SETEA (RET = 1,1)RES = ,2)INDIC = ,3)XIND = ,4)YIND = ,5)%
```

Now the address of x_1 is in the index register XIND and the address of y_1 in the index register YIND. - After the calculations, the result may be stored and the control returned to the calling program by the following sequence of statements:

,RES = computed result %% This stores the result at the proper place.

,INDIC = 0%% This clears the indicator.

GOTO(,RET)%% This returns control to the calling program.

In case of not valid arguments when a proper result cannot be computed, the control may be returned to the calling program by the statements

,INDIC = 2%% This indicates invalid arguments.

GOTO(,RET)%% This returns control to the calling program.

Tape Units

The least squares program requires 3 tape units which are used as follows:

Tape unit 1 - Storage of input data

Tape unit 7 - Temporary storage

Tape unit 8 - Output tape. This tape is prepared for the printer with 132 characters per line.

Input Data

It is assumed that the input data are stored by the calling program on a magnetic tape on tape unit 1, using the Binary Tape-Output Routine. These data consist of r sets of observations, each containing 10 values X_{ji}, 10 corresponding observation errors e_{ji} and an identification of the set. The identification should consist of 20 characters stored in 2 computer words. It should be different for different sets because otherwise the error detecting part of the least squares program will not indicate the right sets.

The sequence of data on the tape within each set should be as follows

$$X_{j1},X_{j2},\ldots,X_{j10};\ e_{j1},e_{j2},\ldots,e_{j10};\ id_1,id_2.$$

(The last 2 words, id_1 and id_2, are the identification of the set. The quantities X_{j1} through e_{j10} are floating point numbers.)

The end of the data file should be indicated on the tape by an END FILE sentinel, consisting of one machine word with the alphanumerical contents (ENDbFILEbb).

The least squares subroutine will read these data from the tape on tape unit 1 starting at the position where the tape is at the instant of calling. It will stop the reading of data either after collecting 2000 valid sets or after sensing the END FILE sentinel. The tape will not be backspaced or rewound by the least squares subroutine. Thus, several data files can be prepared and stored on the tape for subsequent processing, separating them by the END FILE sentinel.

Example for preparing the input tape.

. . . .

The END FILE sentinel may be defined by the statement

EOF ALFNENDbFILE

The data may be written on the tape by the statements

```
ENTER(BT.WR)1)10)NOS.AT(X1)10)NOS.AT(E1)ID1)ID2)%
```

The END FILE sentinel may be written on the tape by the statement

```
ENTER(BT.WR)1)EOF)%
```

Results

The results of the least squares program are partly stored at places indicated by the calling program (cf. the Section "Entering the Program") and partly stored on tape unit 8 for printing. In the following, storage on tape unit 8 is referred to as "printing". The results are computed and presented in two portions:

1) Entered through the first entrance (L·SQ·1) the least squares routine computes and stores the values of the parameters y_ℓ with their corresponding standard errors $m_{y\ell}$, the standard error m_o of a measurement of weight one and the matrix Q (inverse to the normal equation matrix). At the same time all these results will be printed. (See the Section "Printed Results".) The control of the program is then returned to the calling program.

2) Entered through the second entrance (L·SQ·2) the least squares program carries out the controls which are described in Section 10 and prints the results of these controls. (Particulars about the controls see in the Section "Printed Results" and in the Flow Chart.) After that the control of the program is returned to the calling program.

Printed Results

The program prints (i.e., stores on tape unit 8 for printing) in normal cases the results in the form as given in this section. In case of trouble additional comments and error messages are produced (See the corresponding Sections.)

1. First part of the program.

Blank Line.
OUTPUT FROM THE SUBROUTINE (L·SQ·1)
Blank line.
THE TOTAL NUMBER OF SETS PROCESSED IS (r)
Blank line.
THE STANDARD ERROR OF AN OBSERVATION SET WITH THE WEIGHT ONE IS (m_0).
New line: THIS QUANTITY IS DIMENSIONLESS AND SHOULD BE APPROXIMATELY EQUAL TO ONE. DEVIATIONS FROM ONE INDICATE EITHER INCORRECT ESTIMATES OF OBSERVATION ERRORS OR NOT ADEQUATE DESCRIPTION OF THE PHENOMENON OBSERVED BY THE EQUATION F(X,Y) = 0.
2 blank lines
THE FINAL VALUES Y+ETA OF THE PARAMETERS Y ARE AS FOLLOWS
Blank line.

PARAMETER Y + ETA	STANDARD ERROR	INITIAL VALUE Y	CORRECTION ETA	DIFFERENCE OF CORRECTIONS
Blank line.				
(y)	(m_y)	(Y)	(η)	($\eta-\eta^*$)

In case of redundant parameters, only Y and y are printed in the corresponding line, the other values are then zero and at the end of the line the word REDUNDANT is printed.

2 blank lines.
THE INVERSE TO THE MATRIX OF NORMAL EQUATIONS (I.E. THE MATRIX OF COFACTORS) IS
Blank line.

$$\begin{array}{cccc} (Q_{11}) & (Q_{12}) & \cdots & (Q_{1m}) \\ (Q_{12}) & (Q_{22}) & \cdots & (Q_{2m}) \\ \cdots & \cdots & \cdots & \cdots \\ (Q_{1m}) & (Q_{2m}) & \cdots & (Q_{mm}) \end{array}$$

Blank line.

In case of redundant parameters, the rows and columns of the matrix Q, which correspond to such parameters, contain zeros only. In such case the following comment is printed
THE SUBROUTINE AT THE SEXADECIMAL ADDRESS ...... (DECIMAL = ...)
FURNISHES ZERO RESULTS FOR ALL INPUT SETS. THE CORRESPONDING PARAMETER Y...(here the number of the component follows) IS REDUNDANT AND THE CALCULATIONS WERE CARRIED OUT WITH THE REMAINING PARAMETERS ONLY.

2. <u>Second part of the program</u>.

In case of full output complete information about every observation and observation set is printed. This printout can become rather large (n + 2 lines per observation set). Therefore the reduced output should be used normally. This output provides informations about error distributions and lists such sets and observations where systematic observation errors or blunders are suspected.

In case of <u>full output</u> the following is printed:

Skip to a new page.

OUTPUT FROM THE SUBROUTINE (L·SQ·2)

Blank line

SET IDENTIFICATION	OBSERVATIONS X	STANDARD ERROR OF X	CORRECTIONS KSI	ADJUSTED OBSERV. X + KSI
Blank line.				
(20 characters)	(X_{ji})	(e_{ji})	(ξ_{ji})	(x_{ji})
Blank line at the end of the set.				

STANDARD ERROR OF X+KSI	F(X,Y)	F(X+KSI,Y+ETA)	STANDARD ERROR OF F	COMMENT
Blank line.				
(m_{xji}; cf.(52))	(F(X,Y))	(F(x,y))	$(m_o/\sqrt{g_j})$	

The values of F and their standard error as well as the identification of the set are printed on the first line for each set only. The last column contains a comment about the value of F(x,y). If this value is within the range $(-3\ m_{Fj},\ +3\ m_{Fj})$, the comment "NORMAL F" is printed. If F(x,y) is outside that range, the comment "F LARGER THAN 3*M" is printed. If the arguments $(\overline{x}_j,\overline{y})$ are such that F cannot be computed, then in the column F(X + KSI, Y + ETA) the word "FAILURE" is printed instead of the corresponding value. In this case no further comment in the last column is printed.

The headline is repeated on the top of every new page.

After completion of this list the following output is printed (this output is the "reduced output" of (L·SQ·2)):

Skip to a new page.

DISTRIBUTION OF THE VALUES OF F(X + KSI, Y + ETA).

Blank line.

LIMITS	NUMBER OF SETS	% OF SETS	NUMBER OF WEIGHT UNITS	% OF UNITS	NORMAL PERCENTAGES
Blank line.					
BELOW -3M	(+1234)	(+123.1)	(Σg_i)	(+123.1)	0.1
(-3M,-2M)					2.2
(-2M,- M)					16.6
(- M, 0)					34.1
(0, M)					34.1
(M, 2M)					16.6
(2M, 3M)					2.2
OVER 3M					0.1
Blank line.					
TOTAL	(...)	100.0	(...)	100.0	100.0

2 blank lines.

The following <u>List A</u> is not printed if it is empty.

THE FOLLOWING SETS ARE OUTSIDE THE 3M-LIMITS.

Blank line.

SET IDENTIFICATION	WEIGHT UNITS	F(X+KSI,Y+ETA)	STANDARD ERROR	ARGUMENTS	X + KSI
Blank line.					
(20 characters)	(g_j)	($F(x_j,y)$)	($m_o/\sqrt{g_j}$)	($x_{j1},x_{j2},\dots x_{jn}$)	

2 blank lines at the end of the <u>List A.</u>

The following list is printed in any case.

Start new page if less than 20 lines are left on the previous page.

STANDARD OBSERVATION ERRORS OF SINGLE COMPONENTS OF THE ARGUMENTS X.

Blank line.

THE STANDARD ERRORS OF A UNIT WEIGHT OBSERVATION ARE DIMENSIONLESS AND OF THE ORDER ONE IF THE OBSERVATION ERRORS ARE GIVEN CORRECTLY BY THE INPUT AND THE RELATION F(X,Y) = 0 IS CORRECT. THE AVERAGES IN BOTH LAST COLUMNS REPRESENT STANDARD ERRORS OF OBSERVATIONS ONLY IN CASE ALL OBSERVATIONS OF A FIXED COMPONENT HAVE THE SAME ACCURACY.

COMPONENT OF X	STANDARD ERROR OF UNIT WEIGHT OBSERV.	AVERAGE STANDARD ERROR FROM INPUT	AVERAGE STANDARD ERROR FROM ADJUSTMENTS
(i)	(m_{xi})(cf.(78))	(see flow chart)	(m_{Xi}(cf.(80))

4 blank lines at the end of the list.

The following List B is not printed if it is empty.

Start new page if less than 20 lines are left on the previous page.

THE FOLLOWING SETS CONTAIN OBSERVATIONS WITH LARGE ADJUSTMENTS.

Blank line.

ASTERISKS INDICATE COMPONENTS WITH KSI LARGER THAN 3 TIMES THE STANDARD

ERROR OF THE OBSERVATION (EQUATION (22)).

Blank line.

SET IDENTIFICATION	OLD ARGUMENTS X	(X_{j1})	(X_{j2}) *	(X_{j3})	....
(20 characters)	NEW ARGUMENTS X+KSI	(x_{j1})	(x_{j2})	(x_{j3})	....

Blank line.

Next set, etc.

4 blank lines at the end of the List. B

The following text is printed in any case.

THE WEIGHTED SUM OF CORRECTION SQUARES IS

COMPUTED FROM NORMAL EQUATIONS (LAST EXPRESSION IN (21)) ($[p\xi]_1$)
COMPUTED USING CORRELATES (SECOND EXPRESSION IN (21)) ($[p\xi]_2$)
COMPUTED USING ORIGINAL RELATIONSHIP (EQUATION (75)) ($[p\xi]_3$)

Blank line.

THE FIRST TWO VALUES ARE EQUAL IF THE CALCULATIONS WERE CARRIED OUT WITH A

SUFFICIENT NUMBER OF DIGITS. THE THIRD VALUE IS NOT EQUAL TO THE FIRST TWO IF

LINEARIZATION WITH RESPECT TO THE PARAMETERS Y IS NOT PERMISSIBLE.

Blank line.
The output is now complete and the following comments and List C are ommitted if the List C is empty.

DEVIATIONS OF THE THIRD VALUE ARE POSSIBLY CAUSED BY FAILURES OF THE X-VALUES.

(SEE THE FOLLOWING LIST.)

2 blank lines

LIST OF SETS FOR WHICH F(X+KSI, Y+ETA) OR F(X,Y+ETA) CANNOT BE COMPUTED BECAUSE THE ARGUMENTS ARE NOT VALID.

Blank line.

SET IDENTIFICATION (20 characters)	WEIGHT UNITS (g_j)	OLD ARGUMENTS X	(X_{j1}) ...
		NEW ARGUMENTS X+KSI	(X_{j1}) ...

Blank line.

Next set etc.

4 blank lines at the end of the list.

THE TOTAL NUMBER OF SETS WITH NOT VALID ARGUMENTS IS

(+1234) SETS OR (+123.1)% OF THE VALID SETS CORRESPONDING TO

(...) WEIGHT UNITS OR (+123.1)% OF THE TOTAL WEIGHT OF VALID SETS.

Printed Comments

The following comments are printed in case of troubles with input data. The data sets which caused the troubles are not used for the calculations (see the flow chart).

Comment 1.

Blank line.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE IT CONTAINS OBSERVATIONS WITH ZERO ERROR.

Comment 2.

Blank line.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE THE CORRESPONDING FUNCTION F(X,Y) OR ITS DERIVATIVES CANNOT BE COMPUTED.

Comment 3.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE ALL CORRESPONDING DERIVATIVES AI OF F(X,Y) WITH RESPECT TO X ARE ZERO.

Error Messages.

In cases where the computation of the parameters is not possible, the least squares program prints a message which should help to detect the error in the calling program. After printing the message the computation is interrupted, and the control returned to the monitor for processing the next program (by GOTO(N.PROB)).

In the Flow Chart 3 such error exits are indicated. The corresponding error messages are given on the following pages. They differ only by their first sentences, the other information being the same for all messages.
The first sentences are:

Error message 1

EXIT FROM (L.SQ.1) BECAUSE A NUMBER OF VARIABLES EXCEEDS 10.

Error message 2

EXIT FROM (L.SQ.1) BECAUSE THE NUMBER OF OBSERVED SETS IS NOT BIGGER THAN THE NUMBER OF PARAMETERS TO BE COMPUTED

Error message 3

EXIT FROM (L.SQ.1) BECAUSE ALL PARTIAL DERIVATIVES BI OF THE FUNCTION F(X,Y) WITH RESPECT TO THE PARAMETERS Y VANISH FOR ALL OBSERVATION SETS.

Blank line.

The part which is common to all error messages is as follows:

THE PROGRAM WAS ENTERED FROM THE LOCATION (sexadecimal address)

(DECIMAL = (decimal address)) WITH THE FOLLOWING ARGUMENT VALUES

Blank line.

NUMBER OF X-VARIABLES N = (integer n)

NUMBER OF PARAMETERS Y M = (integer m)

ADDRESS OF THE FUNCTION F(X,Y) (sexadecimal address) (DECIMAL = (dec.addr.)))

ADDRESSES OF THE FUNCTIONS AI	SEXADECIMAL	DECIMAL
	(sexadec. addr.)	(decimal addr.)

There will be printed n addresses if $n \leq 10$ and 10 addresses if $n > 10$.

ADDRESSES OF THE FUNCTIONS BI	SEXADECIMAL (sexadec. addr.)	DECIMAL (decimal addr.)

There will be printed m addresses if $m \leq 0$ and 10 addresses if $m > 10$.

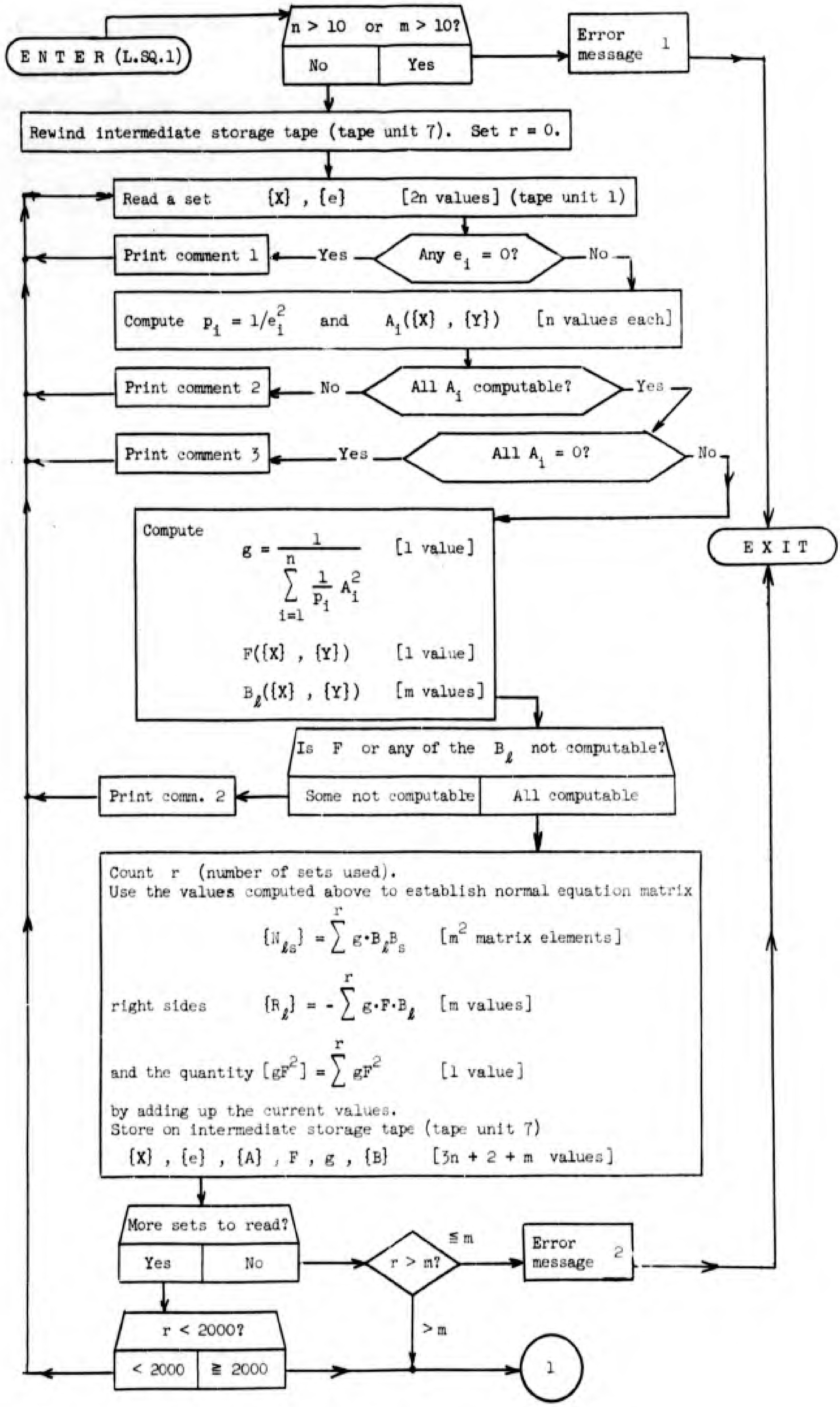
INITIAL VALUES OF THE PARAMETERS Y (...) (...) ...

There will be printed m floating point numbers starting with Y_1 if

$m \leq 10$ and 10 numbers if $m > 10$.

Blank line.

TIME OF INTERRUPTION (12) HRS. (12.12) MIN.



Flow chart of L.SQ.1. Part 1.

1

All diagonal elements of normal equation matrix vanish?

No	Yes

Yes → Error message 3 → EXIT

No ↓

Check whether any diagonal elements of the normal equation matrix vanish. If so, reduce the matrix accordingly and solve for the remaining parameters. The components of $\{\eta_\ell\}$ which corresponds to zero diagonal elements set equal to zero.

Normal equations $\{N_{\ell s}\}\ \{\eta_\ell\} = \{R_\ell\}$ [Solution: m values]

Compute the inverse matrix to $\{N_{\ell s}\}$. Store the m^2 elements of the inverse $\{Q_{\ell s}\}$ at places specified by the calling program. Set rows and columns corresponding to redundant parameters zero.

Compute $\{y_\ell\} = \{Y_\ell\} + \{\eta_\ell\}$ [m values]

Store these values at places specified by the calling program.

Compute $[p\xi^2]_1 = [gF^2] - \{R_\ell\}\ \{\eta_\ell\}$ [1 value]

$m_o = \sqrt{[p\xi^2]_1/(r-m)}$ [1 value]

$\{m_{y\ell}\} = m_o\sqrt{Q_{\ell\ell}}$ [m values]

Store $\{m_{y\ell}\}$ and m_o at places specified by the calling program.

Compute $\{\eta^*_\ell\} = \{Q_{\ell s}\}\ \{R_\ell\}$ [m values]

↓

Print r (total number of sets used) and m_o.
Print the results

$|y|m_y|Y|\eta|\eta^*-\eta|$ [m lines]

Print the matrix $\{Q_{\ell s}\}$. [m^2 elements in m lines]

↓

Any redundant parameters?

Yes	No

Yes → Print comment 4 → RETURN

No → RETURN

Flow chart of L.SQ.1. Part 2.

ENTER(L.SQ.2)ARG)$

Rewind temporary storage tape (tape unit 7). Print headline.

Read from tape unit 7 one set consisting of

$\{X\}$, $\{e\}$, $\{A\}$, F , g , $\{B\}$ [$3\cdot n+2+m$ values]

Compute

$k = -(F + \{B\}\cdot\{\eta\})\cdot g$ [1 value]

$\xi_i = A_i\, e^2\cdot k$ [n values]

$x_i = X_i + \xi_i$ [n values]

$F^* = F(\{x_i\}\ ,\ \{y_\ell\})$ [1 value]

Compute distribution of F^* by adding g and 1 in the respective subtotals. Store set identification if $|F^*| > 3\cdot m_F = 3\cdot m_o/\sqrt{g}$ (List A) or if F^* cannot be computed (List C).

Argument of the ENTER statement

$\neq 0$	$= 0$

Compute $\{m_{xji}\}$ and print identification of the set, F, F^*, m_F and the vectors $\{X\}$, $\{e\}$, $\{\xi\}$, $\{x\}$, $\{m_{xji}\}$

Store set identification and number of component if $|\xi_{ji}| > 3\cdot m_o\cdot e_{ji}$ (List B).
Add the current values to the $3\cdot n$ sums

$$\sum_j^r (\xi_{ji}^2/e_{ji}^2),\ \sum_j^r e_{ji}^2,\ \sum_j^r \xi_{ji}^2$$

Compute

$$\tilde{F} = F(\{X_i\}\ ,\ \{y_\ell\})$$

and store set identification number if $\tilde{F}$ cannot be computed (List C)
Add the current values to the sums

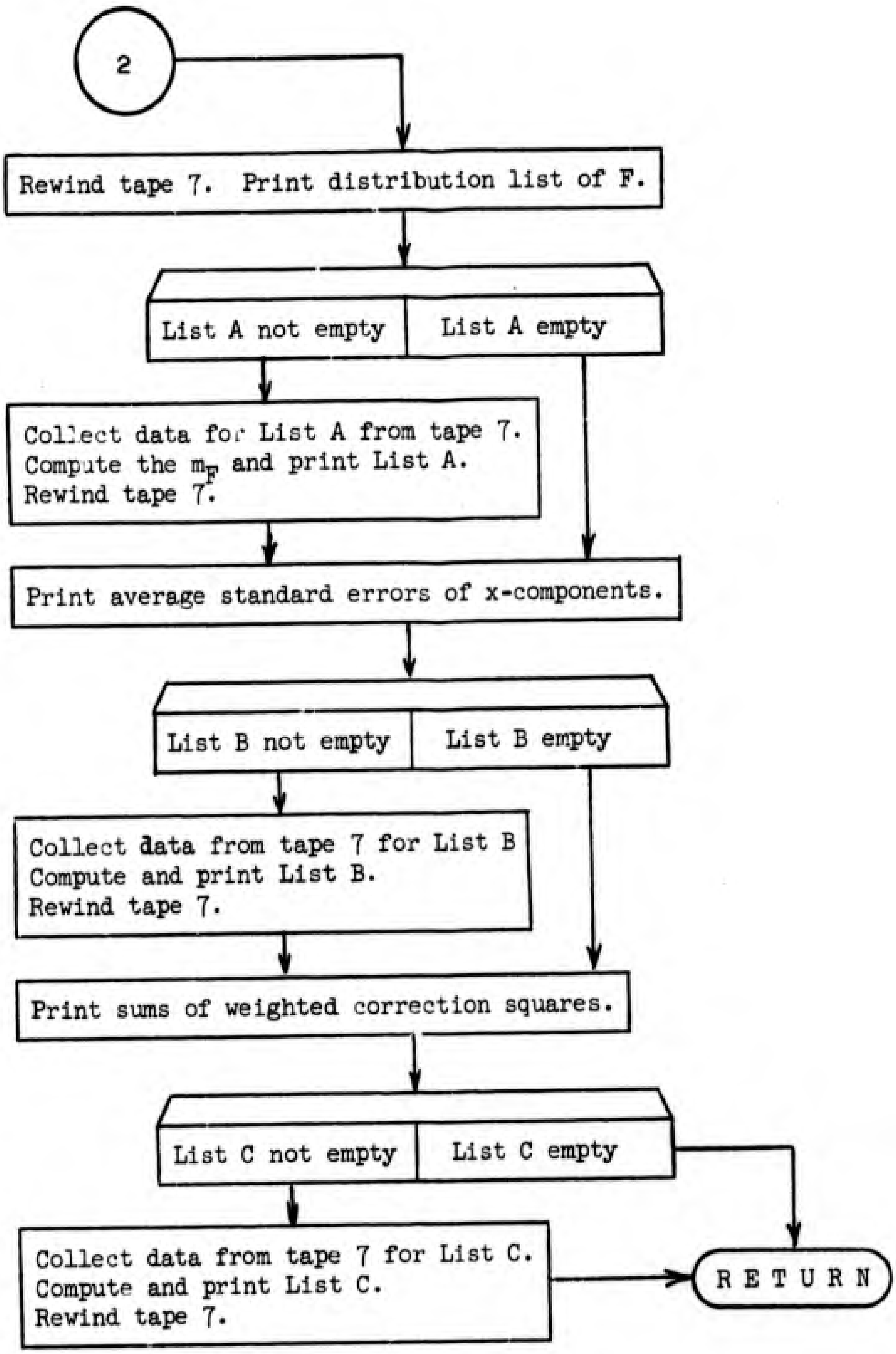
$$[p\xi^2]_2 = -\sum k\,F \quad \text{and} \quad [p\xi^2]_3 = \sum \tilde{F}^2\, g$$

All sets read?

No	Yes

2

Flow chart of L.SQ.2. Part 1.



<u>Flow chart of L.SQ.2.</u> Part 2.

12. Least Squares Problem with Correlated Data.

Suppose we are given, as in Section 1, a functional relationship

$$F(x_1,\dots,x_n;\ y_1,\dots y_m) = 0 \tag{81}$$

between n + m variables. We assume that we know the values of $r > m$ sets of the variables x_i, each set consisting of a vector $\bar{x}'_j = (x_{j1},\dots,x_{jn})$. The problem is to determine the corresponding values of the remaining m variables (parameters) y_ℓ. This problem was solved in Sections 1 through 11 under the assumption that each component x_{ji} has been observed independently of the other components within the set. In the following we will generalize the problem assuming some correlation between the components of each observation set. A correlation between separate sets will not be assumed.

13. Co-factors.

We assume that the correlation between the components x_{ji} of the vector $\bar{x}_j$ is expressed by a matrix of co-factors $\bar{R}_j$. In addition to this matrix, also a corresponding "standard error of an observation of weight one" m_{oj} may be given for each set.

The co-factor matrix $\bar{R}_j$ as well as m_{oj} are known if the vector $\bar{x}_j$ is the result of previous adjustments of other observations. (In Section 6 the co-factor matrix $\bar{Q}$ of the parameters was determined and (20) furnished the corresponding value of m_o. The values of $\bar{Q}$ and m_o are both furnished by the machine program described in Section 11.) If the vector $\bar{x}_j$ is computed from some observations $u_k^{(j)}$ directly, then the co-factors and m_{oj} can be computed from these observations and their standard errors $E_k^{(j)}$ as will be shown below.

The co-factors are determined in such manner that a formula of the same type as (43) can be used to compute the standard errors of a function F of the x_{ji}. We consider here the function (81) and compute the standard error of F in terms of the standard observation errors of the $u_k^{(j)}$. Using the symbols A_{ji} for the partial derivatives of F (see Section 2), the differential dF_j of F_j can be expressed by

$$dF_j = \sum_{i=1}^{n} A_{ji} dx_{ji}. \tag{82}$$

(Since we are interested in the error propagation through the x_{ji}, we consider the parameters y_ℓ as constants for the present purpose.)

For convenience we introduce the vectors

$$\bar{A}_j = \begin{pmatrix} A_{j1} \\ \cdots \\ A_{jn} \end{pmatrix} \qquad d\bar{x}_j = \begin{pmatrix} dx_{j1} \\ \cdots\cdot \\ dx_{jn} \end{pmatrix} \tag{83}$$

and write (81) in the form

$$dF_j = \bar{A}_j'(d\bar{x}_j). \tag{84}$$

In order to express the differentials dx_{ji} in terms of the differentials of the observations we need the partial derivatives of the x_{ji} with respect to the variables $u_k^{(j)}$. We denote the matrix of these derivatives (taken at the places $u_j^{(j)}$) by

$$\bar{X}_u^{(j)} = \begin{pmatrix} \partial x_{j1}/\partial u_1^{(j)}, \ldots, \partial x_{ji}/\partial u_s^{(j)} \\ \cdots\cdots\cdots\cdots \\ \partial x_{jn}/\partial u_1^{(j)}, \ldots, \partial x_{jn}/\partial u_s^{(j)} \end{pmatrix}. \tag{85}$$

With this expression

$$dF_j = \bar{A}_j' \, \bar{X}_u^{(j)}(d\bar{u}^{(j)}). \tag{86}$$

Since the $u_k^{(j)}$ are independent observations, the law of error propagation can be applied to (86). We denote by

$$\bar{E}^{(j)} = \begin{pmatrix} E_s^{(j)} & 0 & \dots & 0 \\ 0 & E_2^{(j)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & E_s^{(j)} \end{pmatrix} \tag{87}$$

the matrix of the standard errors of the observations $\bar{u}^{(j)}$. Then the standard error m_{Fj} of F_j is

$$m_{Fj} = \sqrt{\bar{A}_j' \; \bar{X}_u^{(j)} \; \overline{E^{(j)}}^2 \; \bar{X}_u^{(j)'} \; \bar{A}_j} \tag{88}$$

Comparing (88) with (43) we see that the co-factor matrix can be defined by

$$\bar{R}_j = \bar{X}_u^{(j)} \; \bar{E}^{(j)^2} \; \bar{X}_u^{(j)'} \tag{89}$$

The (dimensionless) standard error of weight one m_{oj} is in this case equal to 1.

Summarizing the results we note that the standard error of the function F_j (see (6)) is given by

$$m_{Fj} = m_{oj} \sqrt{\bar{A}_j' \, \bar{R}_j \, \bar{A}_j} \tag{90}$$

where $\bar{R}_j$ is the matrix of co-factors of the variables $\bar{x}_j$ and m_{oj} is the corresponding standard error of weight one. The co-factor matrix $\bar{R}_j$ is either obtained from (89) in which case $m_{oj} = 1$, or from (32) in which case m_{oj} is given by (20). $\bar{R}_j$ is in any case a symmetric and positive definite matrix.

A different notation of the formula (90) is

$$m_{Fj} = m_{oj} \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} R_{jik} \, A_{ji} \, A_{jk}} \, , \tag{91}$$

where R_{jik} are the elements of the matrix $\bar{R}_j$.

As an example for the computation of co-factors by (89) consider the transformation of cylindrical to Cartesian coordinates. Assume that the distances $u_1^{(j)}$, angles $u_2^{(j)}$ and elevations $u_3^{(j)}$ have been measured directly. The standard errors of the observations may be $E_k^{(j)}$. If we use the corresponding Cartesian coordinates x_{ji} for the calculations of the function values F_j, we need co-factor matrices $\bar{R}_j$ in order to compute the standard errors of F_j. In this case we have the following relationships between the $\bar{x}_j$ and $\bar{u}^{(j)}$:

$$x_{j1} = u_1^{(j)} \cos u_2^{(j)}$$

$$x_{j2} = u_1^{(j)} \sin u_2^{(j)}$$

$$x_{j3} = u_3^{(j)}$$

The matrix of the partial derivatives is

$$\bar{X}_u = \begin{pmatrix} \cos u_2, & \sin u_2, & 0 \\ -u_1 \sin u_2, & u_1 \cos u_2, & 0 \\ 0 & , \quad 0 & , 1 \end{pmatrix}$$

The co-factor matrix $\bar{R}_j$ for the Cartesian coordinates $\bar{x}_j$ can then be obtained from the following formula by substituting the corresponding values of $u_k^{(j)}$ and $E_k^{(j)}$.

$$\bar{R} = \begin{pmatrix} E_1^2 \cos^2 u_2 + E_2^2 u_1^2 \sin^2 u_2, & (E_1^2 - E_2^2 u_1^2) \sin u_2 \cos u_2, & 0 \\ (E_1^2 - E_2^2 u_1^2) \sin u_2 \cos u_2, & E_1^2 \sin^2 u_2 + E_2^2 u_1^2 \cos^2 u_2, & 0 \\ 0 & 0 & E_3^2 \end{pmatrix}$$

In case we use the direct observations $u_k^{(j)}$ for the computation of F_j, the corresponding co-factor matrix of the $u_k^{(j)}$ is

$$\bar{R}_j = \begin{pmatrix} E_1^{(j)^2} & 0 & 0 \\ 0 & E_2^{(j)^2} & 0 \\ 0 & 0 & E_3^{(j)^2} \end{pmatrix}$$

This is the special case considered in Sections 1 through 11. It can be easily verified that with this matrix $\bar{R}_j$ the general formula (91) is identical with (23), which was used in those Sections.

14. Normal Equations in Case of Correlated Data.

The normal equations for the unknown parameters y_ℓ can be established in this case in the same manner as in Section 6 where independent observations were assumed. In the next Section 15 we will then show that the parameters determined by these normal equations indeed minimize the weighted sum of correction squares of the original observations $u_k^{(j)}$.

As in Section 6 we consider the values F_j as independent observations and assign to F_j a weight inversely proportional to the square of its standard error, namely (with (90) and (91))

$$g_j = \frac{1}{m_{oj}^2 \, \bar{A}_j' \, \bar{R}_j \, \bar{A}_j} =$$

$$= \frac{1}{m_{oj}^2 \sum_{i=1}^{n} \sum_{k=1}^{n} R_{jik} \, A_{ji} \, A_{jk}} \quad . \qquad (92)$$

We define v_j by

$$v_j = - \sum_{i=1}^{n} A_{ji} \xi_{ji} \qquad (93)$$

and write the constraints (81) (see (9)) in the form of error equations for the η_ℓ:

$$F_j + \sum_{\ell=1}^{m} B_{j\ell}\eta_\ell = v_j. \quad (j=1,2,\ldots,r) \tag{94}$$

We determine then the η_ℓ such that the sum $\sum_{j=1}^{r} g_j v_j^2 = \min.$ This requirement furnishes in the usual way the normal equations (31) of Section 6, namely

$$\bar{B}' \; \bar{G} \; \bar{B} \; \bar{\eta} = - \bar{B}' \; \bar{G} \; \bar{F}$$

or, in the notation of Section 4, (19)

$$\sum_{\ell=1}^{m} \left(\sum_{j=1}^{r} g_j B_{j\ell} B_{js} \right) \eta_\ell + \sum_{j=1}^{r} g_j F_j B_{js} = 0 \qquad (s=1,2,\ldots,m)$$

The only difference between the present process and that for independently observed x_{ji} is the more general definition of the weights g_j by (92). Consequently the (dimensionless) error of unit weight m_o can be computed using the last expression in (21), namely

$$m_o = \sqrt{\frac{1}{r-m} \left(\sum_{j=1}^{r} g_j F_j^2 + \sum_{s=1}^{m} \left(\sum_{j=1}^{r} g_j F_j B_{js} \right) \eta_s \right)} \, . \tag{95}$$

The standard error of F_j is then

$$m_{Fj} = \frac{m_o}{\sqrt{g_j}} = m_o \cdot m_{oj} \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} R_{jik} A_{ji} A_{jk}} \, . \tag{96}$$

The standard errors of the parameters y_ℓ are furnished by (36), namely

$$m_{y\ell} = m_o \sqrt{Q_{\ell\ell}} \qquad (\ell=1,2,\ldots,m) \tag{97}$$

where $\bar{Q} = (\bar{B}' \; \bar{G} \; \bar{B})^{-1}$, and the standard errors of functions of the y_ℓ by (44) (Section 7).

15. The Principal Property of the Method of Least Squares.

We will show in this Section that the parameters $y_\ell = Y_\ell + \eta_\ell$, obtained from the normal equations of the previous Section 14, minimize the weighted sum of correction squares of the original observations $u_k^{(j)}$. This property of the adjustment problem treated here is a special case of the following Principal Property as formulated by Tienstra (Ref. 1)

> Every problem of adjustment may be divided into a number of phases, provided that in each following phase the cofactors resulting from preceding phase(s) are used.

With the notations of Sections 13 and 14 we formulate the least squares problem as follows:

Suppose we are given the constraints

$$F(x_1,\ldots,x_n;\ y_1,\ldots,y_m) = 0 \tag{98}$$

which have to be satisfied by the unknown parameters y_ℓ at r points $\bar{x}_j$ in the n-dimensional x-space. Further, each of these points $\bar{x}_j$ has to satisfy the constraints

$$T_j(u_1^{(j)},\ldots,u_{nj}^{(j)};\ x_{j1},\ldots,x_{jn}) = 0 \tag{99}$$

at r_j points $\bar{u}^{(j)}$ in a n_j-dimensional $u^{(j)}$ space. The components $u_{sk}^{(j)}$ of $\bar{u}_s^{(j)}$ may be observed independently of each other and the standard errors $e_{sk}^{(j)}$ of $u_{sk}^{(j)}$ may be known. We assume as before that $r > m$ and $r_j > n$ $(j=1,2,\ldots,r)$, so that the equation system is overdetermined with respect to the $r\cdot n$ unknown values of x_{jk} and the m unknown values of y_ℓ. We will satisfy the r equations (98) and $r_1 + r_2 + \ldots + r_r$ equations (99) by introducing corrections to the observations $u_{sk}^{(j)}$ as additional unknowns. (This adds $r_1n_1 + r_2n_2 + \ldots + r_rn_r$ more unknowns to the problem.) These corrections and the x_{jk} and y_ℓ will then be determined such that the weighted sum of the correction squares assumes a minimum and (98) as well as (99) are satisfied.

We will linearize the constraint functions and introduce for this purpose approximations of the solutions. In case of the $u_{sk}^{(j)}$ the approximations are the uncorrected observations $U_{sk}^{(j)}$. The approximations we denote by capital letters, the corrections by greek letters and the final values by small letters. Thus we have

$$\bar{y} = \bar{Y} + \bar{\eta} \tag{100}$$

($\bar{y}$ is a m-dimensional vector),

$$\bar{x} = \bar{X} + \bar{\chi} + \bar{\xi} \tag{101}$$

($\bar{x}$ is a $(n \cdot r)$-dimensional vector),

$$\bar{u} = U + \bar{\omega} + \bar{\zeta} \tag{102}$$

(the dimension of the vector $\bar{u}$ is $r_1 \cdot n_1 + r_2 n_2 + \ldots + r_r n_r$).

The standard errors of the components of $\bar{u}$ may be given in form of a matrix

$$\bar{e} = \begin{pmatrix} e_{11}^{(1)} & & 0 \\ & \ddots & \\ 0 & & e_{r_r n_r}^{(r)} \end{pmatrix} \tag{103}$$

If we linearize the constraints (98) by retaining the linear terms of Taylor series only, we obtain instead of (98) the constraints (see (9)!)

$$\bar{F} + \bar{A}\,(\bar{X}+\bar{\xi}) + \bar{B}\,\bar{\eta} = 0. \tag{104}$$

$\bar{F}$ is a r-dimensional vector and $\bar{F}$, $\bar{B}$ and $\bar{\eta}$ are defined by (28), (30), (6) and (8). The matrix $\bar{A}$ has in the present case the form

$$\overline{A} = \begin{pmatrix} \overline{A}_1' & 0 & \cdots & 0 \\ 0 & \overline{A}_2' & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \overline{A}_r' \end{pmatrix} \tag{105}$$

where $\overline{A}_j$ are the n-dimensional vectors defined by (83) and (7).

In the same manner we may linearize the constraints (99) and obtain

$$\overline{T} + \overline{C}\,(\overline{\omega}+\overline{\zeta}) + \overline{D}(\overline{\chi}+\overline{\xi}) = 0. \tag{106}$$

The vector $\overline{T}$ has $r_1 + r_2 + \ldots + r_r$ components. The matrix $\overline{C}$ has in the present case a structure similar to $\overline{A}$, namely

$$\overline{C} = \begin{pmatrix} \overline{C}_1' & 0 & \cdots & 0 \\ 0 & \overline{C}_2' & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \overline{C}_r' \end{pmatrix} \tag{107}$$

where the matrix $\overline{C}_j'$ $(j=1,2,\ldots,r)$ is composed by r_j vectors of the dimension n_j as $\overline{A}$ is composed by r vectors with n components each. The matrix $\overline{D}$ has the form

$$\overline{D} = \begin{pmatrix} \overline{D}_1' & 0 & \cdots & 0 \\ 0 & \overline{D}_2' & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \overline{D}_r' \end{pmatrix} \tag{108}$$

where $\overline{D}_j'$ $(j=1,2,\ldots,r)$ are matrices with n columns and r_j rows.

As in Section 3 we introduce as many correlates as there are constraint functions by defining the correlate vectors

$$\bar{k}' = (k_1, k_2, \ldots, k_r) \tag{109}$$

and

$$\bar{c}' = (c_1, \ldots, c_t). \tag{109a}$$

(The vector $\bar{c}$ has the same dimension as $\bar{T}$, namely $r_1 + r_2 + \ldots + r_r$.)

Using these correlates we minimize the expression

$$W = (\bar{\omega}+\bar{\zeta})' \, \bar{e}^{-2} \, (\bar{\omega}+\bar{\zeta}) - 2\bar{c}' \, (\bar{T} + \bar{C}(\bar{\omega}+\bar{\zeta}) + \bar{D}(\bar{X}+\bar{\xi})) - 2\bar{k}' (\bar{F} + \bar{A}(\bar{X}+\bar{\xi}) + \bar{B}\,\bar{\eta}). \tag{110}$$

Setting the derivatives of W with respect to the corrections $\bar{\omega}+\bar{\zeta}$ of the original observations $\bar{U}$ equal to zero we obtain the equations

$$(\bar{\omega} + \bar{\zeta})' \, \bar{e}^{-2} - \bar{c}'\bar{C} = 0$$

or

$$\bar{\omega} + \bar{\zeta} = \bar{e}^{2} \, \bar{C}'\bar{c}. \tag{111}$$

Substitution of (111) into the constraints (106) yields

$$\bar{T} + \bar{C}\,\bar{e}^{2}\,\bar{C}'\bar{c} + \bar{D}(\bar{X} + \bar{\xi}) = 0. \tag{112}$$

We define the matrix

$$\bar{S} = (\bar{C}\,\bar{e}^{2}\,\bar{C}')^{-1}. \tag{113}$$

Because of the structure of the matrix $\bar{C}$ and $\bar{e}$, the matrix $\bar{S}$ is a diagonal matrix with $r_1 + r_2 + \ldots + r_r$ rows:

$$\bar{S} = \begin{pmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_k \end{pmatrix} \tag{114}$$

With (113) the equation (112) becomes

$$\bar{S}^{-1}\,\bar{c} + \bar{D}(\overline{X+\xi}) + \bar{T} = 0 \tag{115}$$

or

$$\bar{c} + \bar{S}\,\bar{D}(\overline{X+\xi}) + \bar{S}\,\bar{T} = 0. \tag{116}$$

We now set the derivatives of W with respect to the corrections $\overline{X+\xi}$ of $\bar{X}$ equal to zero and obtain the equation

$$\bar{c}'\bar{D} + \bar{k}'\bar{A} = 0$$

or

$$\bar{D}'\bar{c} + \bar{A}'\bar{k} = 0 \tag{117}$$

With (117) we can eliminate $\bar{c}$ from (116) obtaining the equation

$$-\bar{A}'\bar{k} + \bar{D}'\bar{S}\,\bar{D}(\overline{X+\xi}) + \bar{D}'\bar{S}\,\bar{T} = 0. \tag{118}$$

We define now a matrix

$$\bar{R} = (\bar{D}'\bar{S}\,\bar{D})^{-1}. \tag{119}$$

$\bar{R}$ has the structure

$$\bar{R} = \begin{pmatrix} \bar{R}_1 & 0 & \cdots & 0 \\ 0 & \bar{R}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \bar{R}_r \end{pmatrix} \tag{120}$$

where the $\bar{R}_j$ are positive definite symmetric matrices with n columns.

With (119) the equation (118) becomes

$$\overline{X+\xi} = -\,\bar{R}\,\bar{D}'\bar{S}\,\bar{T} + \bar{R}\,\bar{A}'\bar{k}. \tag{121}$$

Substituting (121) in the constraint equations (104) we obtain

$$\bar{F} - \bar{A}\,\bar{R}\,\bar{D}'\bar{S}\,\bar{T} + \bar{A}\,\bar{R}\,\bar{A}'\bar{k} + \bar{B}\,\bar{\eta} = 0. \tag{122}$$

We define

$$\overline{G} = (\overline{A}\;\overline{R}\;\overline{A}')^{-1}, \tag{123}$$

which is again a diagonal matrix with the diagonal elements g_j $(j=1,\ldots,r)$ and write (122) in the form

$$\overline{G}^{-1}\;\overline{k} + \overline{B}\;\overline{\eta} + \overline{F} - \overline{A}\;\overline{R}\;\overline{D}'\overline{S}\;\overline{T} = 0 \tag{124}$$

or

$$\overline{k} + \overline{G}\;\overline{B}\;\overline{\eta} + \overline{G}\;\overline{F} - \overline{G}\;\overline{A}\;\overline{R}\;\overline{D}'\overline{S}\;\overline{T} = 0. \tag{125}$$

Finally we compute the derivatives of W with respect to the corrections $\overline{\eta}$ of the parameters $\overline{Y}$. Setting these derivatives equal to zero yields

$$\overline{k}'\overline{B} = 0$$

or

$$\overline{B}\;\overline{k}' = 0. \tag{126}$$

With (126) we can eliminate the correlates from (125). The result is the following equation for the unknown parameters $\overline{\eta}$

$$\overline{B}'\overline{G}\;\overline{B}\;\overline{\eta} + \overline{B}'\overline{G}\;\overline{F} - \overline{B}'\overline{G}\;\overline{A}\;\overline{R}\;\overline{D}'\overline{S}\;\overline{T} = 0$$

or

$$\overline{B}'\overline{G}\;\overline{B}\;\overline{\eta} + \overline{B}'\overline{G}(\overline{F} - \overline{A}\;\overline{R}\;\overline{D}'\overline{S}\;\overline{T}) = 0. \tag{127}$$

We now assume that a first adjustment has been done at the r points $\overline{X}_j$ in the x-space. This means that at each of these points the constraints (99) (or (106)) have been satisfied and the corresponding sum of weighted correction squares minimized. Assume that the corrections of the observations $\overline{U}$ due to this adjustment are $\overline{\omega}$, and the corresponding corrections of the approximations $\overline{X}$ are $\overline{\chi}$. The adjustment problem at each of the r points $\overline{X}_j$ is of the type handled in the Sections 1 through 11. From Section 4 or 6 we know the corresponding normal equations for the parameters $\overline{\chi}_j$, namely

$$\overline{D}_j' \, \overline{S}_j \, \overline{D}_j \, \overline{X}_j + \overline{D}_j' \, \overline{S}_j \, \overline{T}_j = 0 \qquad (j=1,2,\ldots,r). \tag{128}$$

Using the complete matrices $\overline{D}$ and $\overline{S}$ (128) can be written in the form

$$\overline{D}' \overline{S} \, \overline{D} \, \overline{X} + \overline{D}' \overline{S} \, \overline{T} = 0 \tag{129}$$

and, with (119)

$$\overline{X} = - \overline{R} \, \overline{D}' \overline{S} \, \overline{T}. \tag{130}$$

Substitution of (130) in (127) furnishes

$$\overline{B}' \overline{G} \, \overline{B} \, \overline{\eta} + \overline{B}' \overline{G} (\overline{F} + \overline{A} \, \overline{X}) = 0. \tag{131}$$

By the definition of $\overline{A}$ we have the relation

$$\overline{F}^{(2)} = \overline{F}(\overline{X} + \overline{X}, \overline{Y}) = \overline{F} + \overline{A} \, \overline{X}, \tag{132}$$

i.e.,the expressions in the parenthesis in (131) are values of the function F computed at the points $\overline{X}_j + \overline{X}_j$ which are the results of the first adjustment phase. The weights $\overline{G}$ are defined by (123) and $\overline{R}_j$ are the cofactor matrices corresponding to the first adjustment phases. (They are the inverse matrices of the normal equation matrices (128).)

Hence the equations (127) for the unknown parameters $\overline{\eta}$ which are obtained by minimizing the correction squares are identical with the normal equations for $\overline{\eta}$ of Section 14. Starting a new adjustment phase it is, therefore, not necessary to know the original constraints $T_k = 0$. The knowledge of the cofactor matrices $\overline{R}_j$ (and the corresponding m_{oj}) is sufficient to compute the new adjustments. It is evident that the same is true if further adjustments are applied to the $\overline{\eta}$. Such adjustments will be correct if the corresponding cofactor matrix of the $\overline{\eta}$, namely $\overline{Q} = (\overline{B}' \overline{G} \, \overline{B})^{-1}$, will be taken into account.

A formula for the standard error of unit weight m_o was given by (95), Section 14. A comparison with the formula (21), Section 5, for independent observations shows that (95) corresponds to the last expression in (21). For controls of the calculations, other expressions for m_o, corresponding to the first two expressions in (21) and the equation (75) of Section 10 are desirable. In order to obtain these formulas we denote the numerator in (95) by W_2 and write it using the notations of this Section as follows:

$$W_2 = \overline{F}^{(2)'} \; \overline{G} \; \overline{F}^{(2)'} + \overline{F}^{(2)'} \; \overline{G} \; \overline{B} \; \overline{\eta} \tag{133}$$

From (131) we obtain the zero expression

$$\overline{\eta}' \; \overline{B}' \; \overline{G} \; \overline{B} \; \overline{\eta} + \overline{\eta}' \; \overline{B}' \; \overline{G} \; \overline{F}^{(2)} = 0 \tag{134}$$

and add it to (133). This completes the square and we obtain

$$W_2 = (\overline{F}^{(2)} + \overline{B} \; \overline{\eta})' \; \overline{G} \; (\overline{F}^{(2)} + \overline{B} \; \overline{\eta}). \tag{135}$$

The equation (125) together with (130) yield

$$\overline{k} + \overline{G} \; (\overline{B} \; \overline{\eta} + \overline{F}^{(2)}) = 0 \tag{136}$$

and therefore

$$W_2 = - \overline{k}' (\overline{F}^{(2)} + \overline{B} \; \overline{\eta}) \tag{137}$$

or, with (126)

$$W_2 = - \overline{k}' \; \overline{F}^{(2)}. \tag{138}$$

The last equation corresponds to the second expression in (21). A formula corresponding to the first expression in (21) can be obtained from (137). First we note that the constraints of the second adjustment phase yield

$$-\overline{A} \; \overline{\xi} = \overline{F}^{(2)} + \overline{B} \; \overline{\eta} \tag{139}$$

(see (104) and (132)). From (118) and (129) we obtain

$$\overline{A}' \; \overline{k} = \overline{D}' \; \overline{S} \; \overline{D} \; \overline{\xi} = \overline{R}^{-1} \; \overline{\xi}. \tag{140}$$

If we now substitute in (137) first the expression (139) and subsequently (140) we obtain

$$W_2 = \bar{\xi}' \bar{R}^{-1} \bar{\xi}. \tag{141}$$

The expressions (135), (138) and (141) can be used for calculation controls. By the definition of $\bar{F}^{(2)}$ and $\bar{B}$ we have the relation

$$\begin{aligned} \bar{F}^{(2)} + \bar{B}\,\bar{\eta} &= \bar{F}(\bar{X} + \bar{\chi},\ \bar{Y} + \bar{\eta}) = \\ &= \bar{F}(\bar{X} + \bar{\chi},\ \bar{y}). \end{aligned} \tag{142}$$

Substituting (142) in (135) we obtain another formula for calculation controls.

$$\begin{aligned} W_2 &= \bar{F}'(\bar{X} + \bar{\chi},\ \bar{y})\ \bar{G}\ \bar{F}\ (\bar{X} + \bar{\chi},\ \bar{y}) = \\ &= \sum_{j=1}^{r} F_j^2\ (\bar{X} + \bar{\chi},\ \bar{y})\ g_j \end{aligned} \tag{143}$$

The latest formula corresponds to (75) of Section 10 in case of independent observations. Note that the arguments $\bar{X} + \bar{\chi}$ of F in (143) are those x-values which are used as "observations" for the second adjustment phase. The arguments $\bar{y}$ are the final values of the parameters.

16. Corrections and Standard Errors of Data.

The formulae of Section 14 permit the computation of the unknown parameters $\bar{\eta}$ and their standard errors. The corresponding adjustments $\bar{\xi}$ of the data can be most easily computed using the correlates $\bar{k}$. Solving (140) for the corrections $\bar{\xi}$ we obtain

$$\bar{\xi} = \bar{R}\ \bar{A}'\bar{k}. \tag{144}$$

Comparing (144) with the corresponding equation (14) for not correlated observations, we see that (14) is a special case of (144), namely the case where the cofactor matrix is a diagonal matrix with the elements $1/p_{ji}$. From (144) we find for each element of $\bar{\xi}$ the formula

$$\xi_{ji} = m_{oj}^2 \cdot k_j \cdot \sum_{\ell=1}^{n} A_{j\ell} R_{j\ell i}. \tag{145}$$

(In the notation of (144) the factor m_{oj}^2 is included in the $R_{j\ell i}$. In case of correct adjustment and error assumptions this factor is equal to 1.) The standard errors of the corrected x_{ji} can be computed in the same manner as in Section 8 from the errors of the correlates. With g_j defined by (92) we obtain for the standard errors m_{kj} of the correlates k_j (see (51))

$$m_{kj} = m_o \sqrt{g_j} \sqrt{1 - g_j \sum_{\ell=1}^{m} \sum_{t=1}^{m} Q_{\ell t} \, B_{j\ell} \, B_{jt}} \tag{146}$$

where $Q_{\ell t}$ are elements of the cofactor matrix $\bar{Q}$ of the parameters $\bar{y}$. The standard error m_{xji} of the corrected component x_{ji} is with (145) and (146)

$$m_{xji} = m_{oj}^2 \, m_{kj} \left| \sum_{\ell=1}^{n} A_{j\ell} \, R_{j\ell i} \right| \tag{147}$$

Again the corresponding formula (52) for independent data is a special case of (147).

The standard errors of the input data $\bar{\bar{X}}^{(2)} = \bar{X} + \bar{x}$ before the adjustment of the second phase follow from (91). Considering $\bar{x}^{(2)}$ as a special function of $\bar{\bar{X}}^{(2)}$ we find

$$m_{xji(2)} = m_{oj} \sqrt{R_{jii}} \tag{148}$$

17. Corrections of the Original Observations.

The considerations of Section 15 permit us to compute the corrections $\bar{\zeta}$ of the original observations $\bar{U}$ which are caused by the adjustment $\bar{\xi}$ of the x-values. These corrections can be computed most easily using the corresponding correlates $\bar{c}$. First we obtain from (116)

$$\bar{c} = -\bar{S}\,(\bar{T} + \bar{D}\,\bar{\chi} + \bar{D}\,\bar{\xi}). \tag{149}$$

The $\bar{\chi}$ are the results of the first adjustment phase. The corresponding values of the correlates are

$$\bar{c}_{old} = -\bar{S}\,(\bar{T} + \bar{D}\,\bar{\chi}) \tag{150}$$

(see for instance the analogous formula (136) for the correlates $\bar{k}$). Hence the corrections of the correlates $\bar{c}$ due to the new adjustments is

$$\bar{c}_{corr} = -\bar{S}\,\bar{D}\,\bar{\xi}. \tag{151}$$

The corrections of the original observations is with (111)

$$\bar{\varpi} + \bar{\zeta} = \bar{e}^{2}\,\bar{C}'\,\bar{c}$$

Here we have denoted by $\bar{\varpi}$ the corrections by the first adjustment phase, corresponding to the correlates $\bar{c}_{old}$. Hence the additional corrections $\bar{\zeta}$ are

$$\bar{\zeta} = \bar{e}^{2}\,\bar{C}'\bar{c}_{corr}. \tag{152}$$

18. Machine Program for Least Squares Problems with Correlated Data.

The machine program described in this Section has essentially the same structure as the program described in Section 11. However, there is a significant difference between the applicability of the two programs. The program described in this Section handles correctly input data with finite correlations between them, including the special case of zero correlation. The program of Section 11 is designed for cases with zero correlation only. Consequently, more input is needed for the present problem, namely the cofactor matrices of the observations in addition to the values observed.

The least squares problem is formulated in Section 12. The user of the least squares program should read also Sections 1, 2 and 13, where some symbols are defined which will be used in the following program description.

The least squares method will generally be used as a part of more extensive calculations, which check the validity of the observed data, compute initial values, fix the degree of approximation etc. Therefore, the least squares machine program described in this section is written in the form of a subroutine. This subroutine calculates the unknown values of the parameters and returns the control to the calling program before the final controls of Section 10 are carried out. Depending on the properties of the problem and the results of the calculations, the process can then be repeated with other initial values or the final controls of Section 10 carried out. Therefore the subroutine has two entries. Entering the first part a number of arguments are necessary. Entering the second part (for final controls) only an indicator may be used to specify the amount of printed output.

Entering the Program.

The least squares subroutine can be entered by the following FORAST statements:

1st part

```
ENTER (COLS 1) (N)(M)(F) ... (MO)(Q11)%
```

2nd part

```
ENTER (COLS 2) (ARG)%
```

When entering the second part, the indicator (here denoted by ARC) should be zero for reduced output and non-zero for full output. For particulars see the flow chart and the section "Printed Results."

When entering the first part (COLS 1), the addresses of the following arguments should be transmitted:

Addresses of arguments

n = address of the number of x-variables, i.e.,"observations". ($n \le 10$ and integer).

m = address of the number of y-variables, i.e.,"parameters". ($m \le 10$ and integer).

F = address to enter the subroutine F.

A1,...,AN addresses to enter the n subroutines A_i
B1,...,BM and the m subroutines B_i.

Y_1 = address of Y_1. It is assumed that the other initial approximations $Y_2,\ldots,Y_m$ of the parameters are stored in cells subsequent to that of Y_1.

Addresses for storing the results

y_1 = address of y_1. Other y_i will be stored in subsequent cells.

m_{y1} = address of the standard error of y_1. Other standard errors m_{yi} will be stored in subsequent cells.

m_o = address of the standard error of measurements with weight one.

Q_{11} = address of the first element of the matrix Q (inverse to the normal equations matrix). The elements of Q will be stored in the sequence $Q_{11}, Q_{12}, \ldots Q_{1m}, Q_{12}, Q_{22}, \ldots Q_{2m}, Q_{13}, \ldots Q_{3m}, \ldots Q_{mm}$. For sake of simplicity the complete matrix is stored and storage space of m^2 words required.

The total number of arguments in the ENTER (COLS 1) - statement is hence $8 + m + n$. These $8 + m + n$ addresses may be given in the same order as they are listed here.

About the subroutines F, A_i and B_i it is assumed that they can be called by statements of the form

ENTER (NAME)(RESULT)(INDIC)(ARGN)(ARGM)%

where the symbols in the statement stand for the following addresses:

NAME - address to enter the subroutine.

RESULT - address to store the result.

INDIC - address of an indicator. Its value is set zero if the result can be computed. If the arguments are such that no result can be computed, the indicator is non-zero.

ARGN - address of the argument x_1. It is assumed that $x_2,\ldots,x_n$ are stored in cells subsequent to x_1.

ARGM - address of the argument y_1. The arguments $y_2,\ldots,y_m$ are assumed to be stored in cells subsequent to y_1.

These subroutines may be programmed as shown by the following example for the subroutine named "NAME":

NAME SET(7 = SELF + 1) GOTO(1,1)

SETEA (RET = 1,1)RES = ,2)INDIC = ,3)XIND = ,4)YIND = ,5)%

Now the address of x_1 is in the index register XIND and the address of y_1 in the index register YIND. - After the calculations, the result may be stored and the control returned to the calling program by the following sequence of statements:

,RES = computed result %% This stores the result at the proper place.

,INDIC = 0%% This clears the indicator.

GOTO(,RET)%% This returns control to the calling program.

In case of not valid arguments when a proper result cannot be computed, the control may be returned to the calling program by the statements

,INDIC = 2%% This indicates invalid arguments.

GOTO(,RET)%% This returns control to the calling program.

Tape Units

The least squares program requires 3 tape units which are used as follows:

Tape unit 1 - Storage of input data

Tape unit 7 - Temporary storage

Tape unit 8 - Output tape. This tape is prepared for the printer with 132 characters per line.

Input Data

It is assumed that the input data are stored by the calling program on a magnetic tape on tape unit 1, using the Binary Tape-Output Routine. These data consist of r sets of observations, each containing 10 values X_{ji}, the corresponding standard error m_{jo} of weight one, the cofactor matrix R_{jis} and an identification of the set. The identification should consist of 20 characters stored in 2 computer words. It should be different for different sets because otherwise the error detecting part of the least squares program will not indicate the right sets.

The sequence of data on the tape within each set should be as follows

$$X_{j1}, X_{j2}, \ldots, X_{j10};\ m_{jo};\ R_{j11}, \ldots, R_{j,10,10};\ id_1,\ id_2$$

(The last 2 words, id_1 and id_2, are the identification of the set. The quantities X_{j1} through $R_{j,10,10}$ are floating point numbers.) In order to simplify the standards this sequence of 113 computer words for each set is used also if there are less than 10 values X_{ji}.

The end of the data file should be indicated on the tape by an END FILE sentinel, consisting of one machine word with the alphanumerical contents (ENDbFILEbb).

The least squares subroutine will read these data from the tape on tape unit 1 starting at the position where the tape is at the instant of calling.

It will stop the reading of data either after collecting 2000 valid sets or after sensing the END FILE sentinel whichever comes first. The tape will not be backspaced or rewound by the least squares subroutine. Thus, several data files can be prepared and stored on the tape for subsequent processing, separating them by the END FILE sentinel.

Example for preparing the input tape.

. . . .

The END FILE sentinel may be defined by the statement

EOF ALFNENDbFILE

The data may be written on the tape by the statements

```
ENTER(BT.WR)1)10)NOS.AT(X1)MO)100)NOS.AT(R11)ID1)ID2)%
```

The END FILE sentinel may be written on the tape by the statement

```
ENTER(BT.WR)1)EOF)%
```

Results

The results of the least squares program are partly stored at places indicated by the calling program (cf. the Section "Entering the Program") and partly stored on tape unit 8 for printing. In the following, storage on tape unit 8 is referred to as "printing". The results are computed and presented in two portions:

1) Entered through the first entrance (COLS 1) the least squares routine computes and stores the values of the parameters y_ℓ with their corresponding standard errors $m_{y\ell}$, the standard error m_o of a measurement of weight one and the matrix Q (inverse to the normal equation matrix). At the same time all these results will be printed. (See the Section "Printed Results".) The control of the program is then returned to the calling program.

2) Entered through the second entrance (COLS 2) the least squares program carries out the controls of the type described in Section 10 and prints the results of these controls. (Particulars about the controls see in the Section "Printed Results" and in the Flow Chart.) After that the control of the program is returned to the calling program.

Printed Results

The program prints (i.e., stores on tape unit 8 for printing) in normal cases the results in the form as given in this section. In case of trouble additional comments and error messages are produced. (See the corresponding Sections.)

1. First part of the program.

Blank Line.
OUTPUT FROM THE SUBROUTINE (COLS 1) FOR CORRELATED DATA
Blank Line.
THE TOTAL NUMBER OF SETS PROCESSED IS (r)
Blank Line.
THE STANDARD ERROR OF AN OBSERVATION SET WITH THE WEIGHT ONE IS (m_o).
New Line: THE QUANTITY IS DIMENSIONLESS AND SHOULD BE APPROXIMATELY EQUAL TO ONE. DEVIATIONS FROM ONE INDICATE EITHER INCORRECT ESTIMATES OF OBSERVATION ERRORS OR NOT ADEQUATE DESCRIPTION OF THE PHENOMENON OBSERVED BY THE EQUATION F(X,Y) = 0.
2 Blank Lines
THE FINAL VALUES Y+ETA OF THE PARAMETERS Y ARE AS FOLLOWS
Blank Line.

PARAMETER Y + ETA	STANDARD ERROR	INITIAL VALUE Y	CORRECTION ETA	DIFFERENCE OF CORRECTIONS
Blank Line.				
(y)	(m_y)	(Y)	(η)	($\eta-\eta^*$)

In case of redundant parameters, only Y and y are printed in the corresponding line, the other values are then zero and at the end of the line the word REDUNDANT is printed.

2 Blank Lines.
THE INVERSE TO THE MATRIX OF NORMAL EQUATIONS (I.E., THE MATRIX OF COFACTORS) IS
Blank Line.

$$\begin{array}{cccc} (Q_{11}) & (Q_{12}) & \cdots & (Q_{1m}) \\ (Q_{12}) & (Q_{22}) & \cdots & (Q_{2m}) \\ \cdots & \cdots & \cdots & \cdots \\ (Q_{1m}) & (Q_{2m}) & \cdots & (Q_{mm}) \end{array}$$

Blank Line.

In case of redundant parameters, the rows and columns of the matrix Q, which correspond to such parameters, contain zeros only. In such case the following comment is printed
THE SUBROUTINE AT THE SEXADECIMAL ADDRESS ...... (DECIMAL = ...)
FURNISHES ZERO RESULTS FOR ALL INPUT SETS. THE CORRESPONDING PARAMETER Y...(here the number of the component follows) IS REDUNDANT AND THE CALCULATIONS WERE CARRIED OUT WITH THE REMAINING PARAMETERS ONLY.

2. Second part of the program.

In case of full output complete information about every observation and observation set is printed. This printout can become rather large (n + 2 lines per observation set). Therefore the reduced output should be used normally. This output provides information about error distributions and lists such sets and observations where systematic observation errors or blunders are suspected.

In case of full output the following is printed
Skip to a new page.

OUTPUT FROM THE SUBROUTINE (COLS 2)

Blank Line

SET IDENTIFICATION	OBSERVATIONS X	STANDARD ERROR OF X	CORRECTIONS KSI	ADJUSTED OBSERV. X + KSI
Blank Line.				
(20 characters)	(X_{ji})	$(m_{jo}\sqrt{R_{jii}})$	(ξ_{ji})	(x_{ji})
Blank line at the end of each set.				

STANDARD ERROR OF X+KSI	F(X,Y)	F(X+KSI,Y+ETA)	STANDARD ERROR OF F	COMMENT
Blank Line.				
$(m_{xji}$; cf.(147)	(F(X,Y))	(F(x,y))	$(m_o/\sqrt{g_j})$	

The values of F and their standard error as well as the identification of the set are printed on the first line for each set only. The last column contains a comment about the value of F(x,y). If this value is within the range $(-3\ m_{Fj}, +3\ m_{Fj})$, the comment "NORMAL F" is printed. If F(x,y) is outside that range, the comment "F LARGER THAN 3*M" is printed. If the arguments $(\overline{x}_j,\overline{y})$ are such that F cannot be computed, then in the column F(X + KSI, Y + ETA) the word "FAILURE" is printed instead of the corresponding value. In this case no further comment in the last column is printed.

The headline is repeated on the top of every new page.

After completion of this list the following output is printed (this output is the "reduced output" of (COLS 2)):

Skip to a new page.

DISTRIBUTION OF THE VALUES OF F(X + KSI, Y + ETA).

Blank Line.

LIMITS	NUMBER OF SETS	% OF SETS	NUMBER OF WEIGHT UNITS	% OF UNITS	NORMAL PERCENTAGES
Blank Line.					
BELOW -3M	(+1234)	(+123.1)	(Σ g_i)	(+123.1)	0.1
(-3M,-2M)					2.2
(-2M,- M)					16.6
(- M, 0)					34.1
(0, M)					34.1
(M, 2M)					16.6
(2M, 3M)					2.2
OVER 3M					0.1
Blank Line.					
TOTAL	(...)	100.0	(...)	100.0	100.0

2 blank lines.

The following List A is not printed if it is empty.

THE FOLLOWING SETS ARE OUTSIDE THE 3M-LIMITS.

Blank Line.

SET IDENTIFICATION	WEIGHT UNITS	F(X+KSI,Y+ETA)	STANDARD ERROR	ARGUMENTS	X + KSI
Blank Line.					
(20 characters)	(g_j)	($F(x_j,y)$)	($m_o/\sqrt{g_j}$)	($x_{j1},x_{j2},\dots x_{jn}$)	

2 blank lines at the end of the List A.

The following list is printed in any case.

Start new page if less than 20 lines are left on the previous page.

STANDARD OBSERVATION ERRORS OF SINGLE COMPONENTS OF THE ARGUMENTS X.

Blank line.

THE STANDARD ERRORS OF A UNIT WEIGHT OBSERVATION ARE DIMENSIONLESS AND OF THE ORDER ONE IF THE OBSERVATION ERRORS ARE GIVEN CORRECTLY BY THE INPUT AND THE RELATION F(X,Y) = 0 IS CORRECT. THE AVERAGES IN BOTH LAST COLUMNS REPRESENT STANDARD ERRORS OF OBSERVATIONS ONLY IN CASE ALL OBSERVATIONS OF A FIXED COMPONENT HAVE THE SAME ACCURACY.

COMPONENT OF X	STANDARD ERROR OF UNIT WEIGHT OBSERV.	AVERAGE STANDARD ERROR FROM INPUT	AVERAGE STANDARD ERROR FROM ADJUSTMENTS
(i)	(see flow chart)	(see flow chart)	(see flow chart)

4 blank lines at the end of the list.

The following List B is not printed if it is empty.

Start new page if less than 20 lines are left on the previous page.

THE FOLLOWING SETS CONTAIN OBSERVATIONS WITH LARGE ADJUSTMENTS.

Blank Line.

ASTERISKS INDICATE COMPONENTS WITH KSI LARGER THAN 3 TIMES THE STANDARD ERROR OF THE OBSERVATION (EQUATION (22)).

Blank Line.

SET IDENTIFICATION (20 characters)	OLD ARGUMENTS X	(X_{j1})	(X_{j2}) *	(X_{j3}) ....
	NEW ARGUMENTS X+KSI	(x_{j1})	(x_{j2})	(x_{j3}) ....

Blank Line.

Next set, etc.

4 blank lines at the end of the List. B

The following text is printed in any case.

THE WEIGHTED SUM OF CORRECTION SQUARES IS

COMPUTED FROM NORMAL EQUATIONS (EQUATION (95)) $([p\xi]_1)$
COMPUTED USING CORRELATES (EQUATION (138)) $([p\xi]_2)$
COMPUTED USING ORIGINAL RELATIONSHIP (EQUATION (143)) $([p\xi]_3)$

Blank Line.

THE FIRST TWO VALUES ARE EQUAL IF THE CALCULATIONS WERE CARRIED OUT WITH A SUFFICIENT NUMBER OF DIGITS. THE THIRD VALUE IS NOT EQUAL TO THE FIRST TWO IF LINEARIZATION WITH RESPECT TO THE PARAMETERS Y IS NOT PERMISSIBLE.

Blank Line.
The output is now complete and the following text and List C are ommitted if the List C is empty.

DEVIATIONS OF THE THIRD VALUE ARE POSSIBLY CAUSED BY FAILURES OF THE X-VALUES.

(SEE THE FOLLOWING LIST.)

2 blank lines

LIST OF SETS FOR WHICH F(X+KSI, Y+ETA) OR F(X,Y+ETA) CANNOT BE COMPUTED BECAUSE THE ARGUMENTS ARE NOT VALID.

Blank Line.

SET IDENTIFICATION (20 characters)	WEIGHT UNITS (g_j)	OLD ARGUMENTS X	(X_{jl}) ...
		NEW ARGUMENTS X+KSI	(X_{jl}) ...

Blank Line.

Next set etc.

4 blank lines at the end of the list.

THE TOTAL NUMBER OF SETS WITH NOT VALID ARGUMENTS IS

(+1234) SETS OR (+123.1)% OF THE VALID SETS CORRESPONDING TO

(...) WEIGHT UNITS OR (+123.1)% OF THE TOTAL WEIGHT OF VALID SETS.

Printed Comments

The following comments are printed in case of troubles with input data. The data sets which caused the troubles are not used for the calculations (see the flow chart).

Comment 1.

Blank Line.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE IT CONTAINS OBSERVATIONS WITH ZERO ERROR.

Comment 2.

Blank Line.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE THE CORRESPONDING FUNCTION F(X,Y) OR ITS DERIVATIVES CANNOT BE COMPUTED.

Comment 3.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE ALL CORRESPONDING DERIVATIVES AI OF F(X,Y) WITH RESPECT TO X ARE ZERO.

Comment 4.

SET WITH THE IDENTIFICATION (20 characters) IS NOT USED BECAUSE ITS WEIGHT COMPUTED WITH (92) IS EITHER NEGATIVE OR CAUSES OVERFLOW.

Error Messages.

In cases where the computation of the parameters is not possible, the least squares program prints a message which should help to detect the error in the calling program. After printing the message the computation is interrupted, and the control returned to the monitor for processing the next program (by GOTO(N.PROB)).

In the Flow Chart 3 such error exists are indicated. The corresponding error messages are given on the following pages. They differ only by their first sentences, the other information being the same for all messages. The first sentences are:

Error message 1

EXIT FROM (COLS 1) BECAUSE A NUMBER OF VARIABLES EXCEEDS 10.

Error message 2

EXIT FROM (COLS 1) BECAUSE THE NUMBER OF OBSERVED SETS IS NOT BIGGER THAN THE NUMBER OF PARAMETERS TO BE COMPUTED

Error message 3

EXIT FROM (COLS 1) BECAUSE ALL PARTIAL DERIVATIVES BI OF THE FUNCTION F(X,Y) WITH RESPECT TO THE PARAMETERS Y VANISH FOR ALL OBSERVATION SETS.

Blank Line.

The part which is common to all error messages is as follows:

THE PROGRAM WAS ENTERED FROM THE LOCATION (sexadecimal address)

(DECIMAL = (decimal address)) WITH THE FOLLOWING ARGUMENT VALUES

Blank Line.

NUMBER OF X-VARIABLES N = (integer n)

NUMBER OF PARAMETERS Y M = (integer m)

ADDRESS OF THE FUNCTION F(X,Y) (sexadecimal address) (DECIMAL = (dec.addr.)))

ADDRESSES OF THE FUNCTIONS AI	SEXADECIMAL	DECIMAL
	(sexadec, addr.)	(decimal addr.)

There will be printed n addresses if $n \leqq 10$ and 10 addresses if $n > 10$.

ADDRESSES OF THE FUNCTIONS BI	SEXADECIMAL (sexadec. addr.)	DECIMAL (decimal addr.)

There will be printed m addresses if $m \leqq 0$ and 10 addresses if $m > 10$.

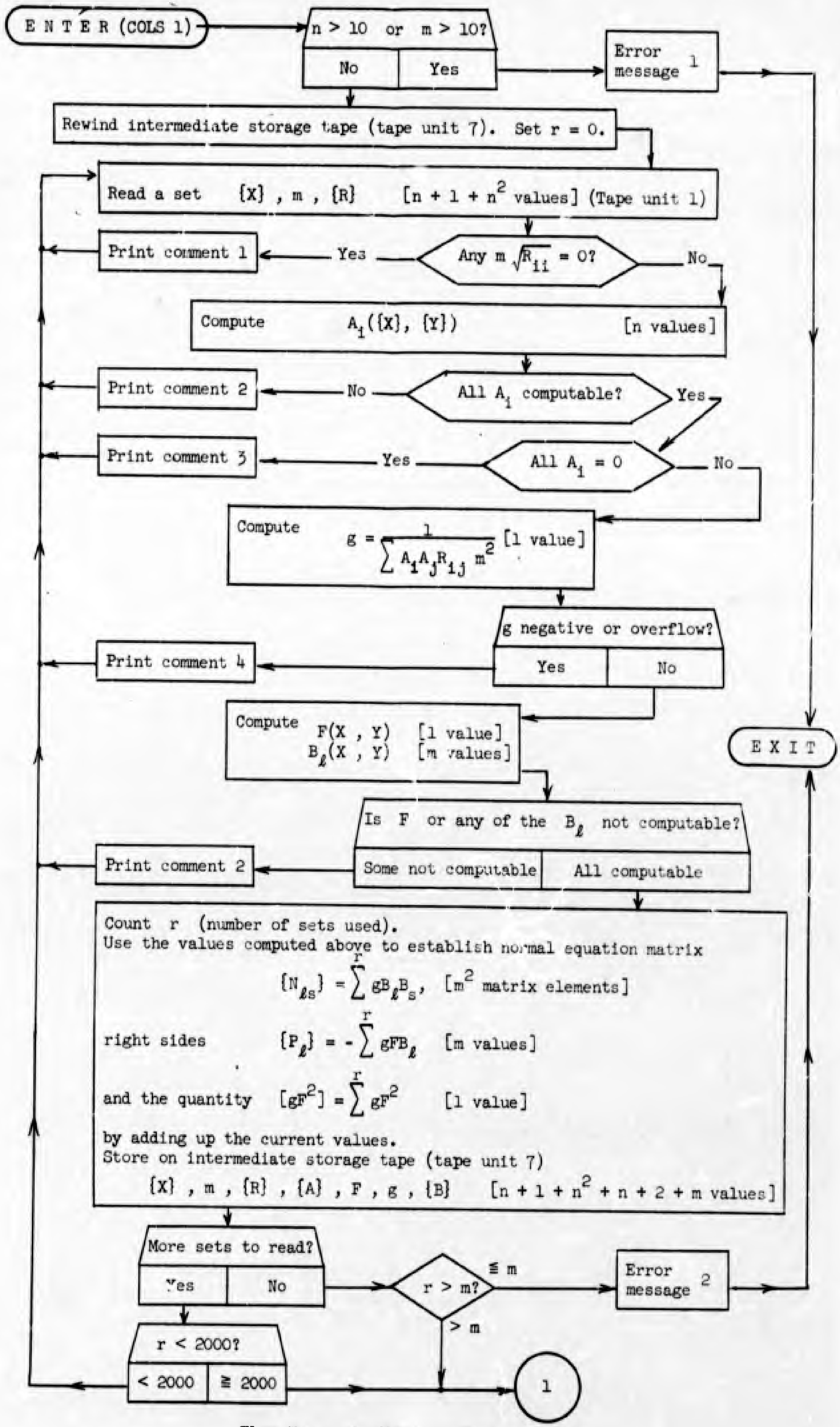
INITIAL VALUES OF THE PARAMETERS Y (...) (...) ...

There will be printed m floating point numbers starting with Y_1 if

$m \leqq 10$ and 10 numbers if $m > 10$.

Blank Line.

TIME OF INTERRUPTION (12) HRS. (12.12) MIN.



Flow Chart of COLS 1 Part 1.

1

All diagonal elements of normal equation matrix vanish?

Yes → Error message 5 → EXIT

No ↓

Check whether any diagonal elements of the normal equation matrix vanish. If so, reduce the matrix accordingly and solve for the remaining parameters. The components of $\{\eta_\ell\}$ which corresponds to zero diagonal elements set equal to zero.

Normal equations $\{N_{\ell s}\}\{\eta_\ell\} = \{P_\ell\}$ [Solution: m values]

Compute the inverse matrix to $\{N_{\ell s}\}$. Store the m^2 elements of the inverse $\{Q_{\ell s}\}$ at places specified by the calling program. Set rows and columns corresponding to redundant parameters zero.

Compute $\{y_\ell\} = \{Y_\ell\} + \{\eta_\ell\}$ [m values]

Store these values at places specified by the calling program.

Compute $[p\xi^2]_1 = [gF^2] - \{P_\ell\}\{\eta_\ell\}$ [1 value]

$m_o = \sqrt{[p\xi^2]_1/(r-m)}$ [1 value]

$\{m_{y\ell}\} = m_o\sqrt{Q_{\ell\ell}}$ [m values]

Store $\{m_{y\ell}\}$ and m_o at places specified by the calling program.

Compute $\{\eta^*_\ell\} = \{Q_{\ell s}\}\{P_\ell\}$ [m values]

↓

Print r (total number of sets used) and m_o.
Print the results

| y | m_y | Y | η | $\eta^* - \eta$ | [m lines]

Print the matrix $\{Q_{\ell s}\}$ [m^2 elements in m lines]

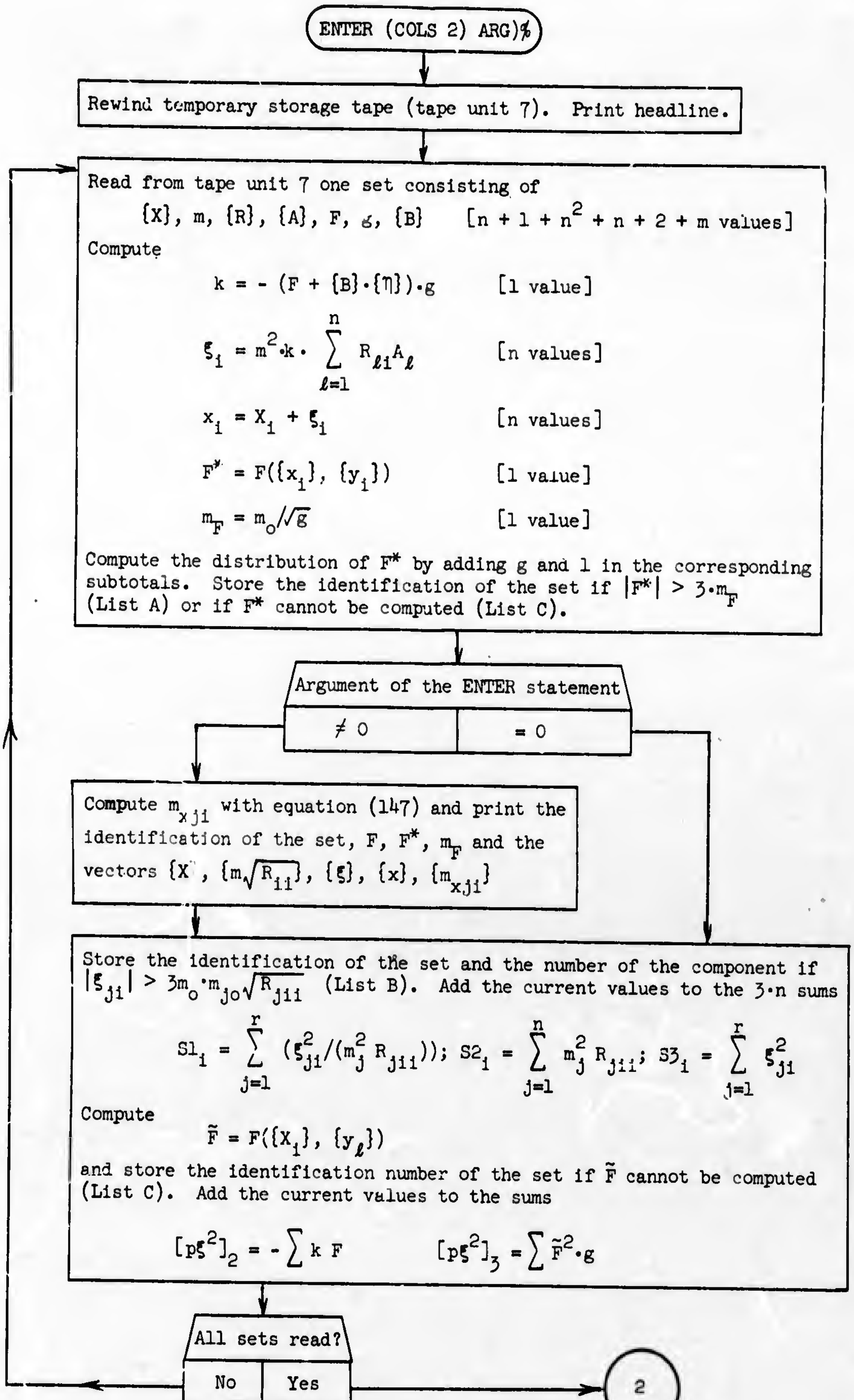
↓

Any redundant parameters?

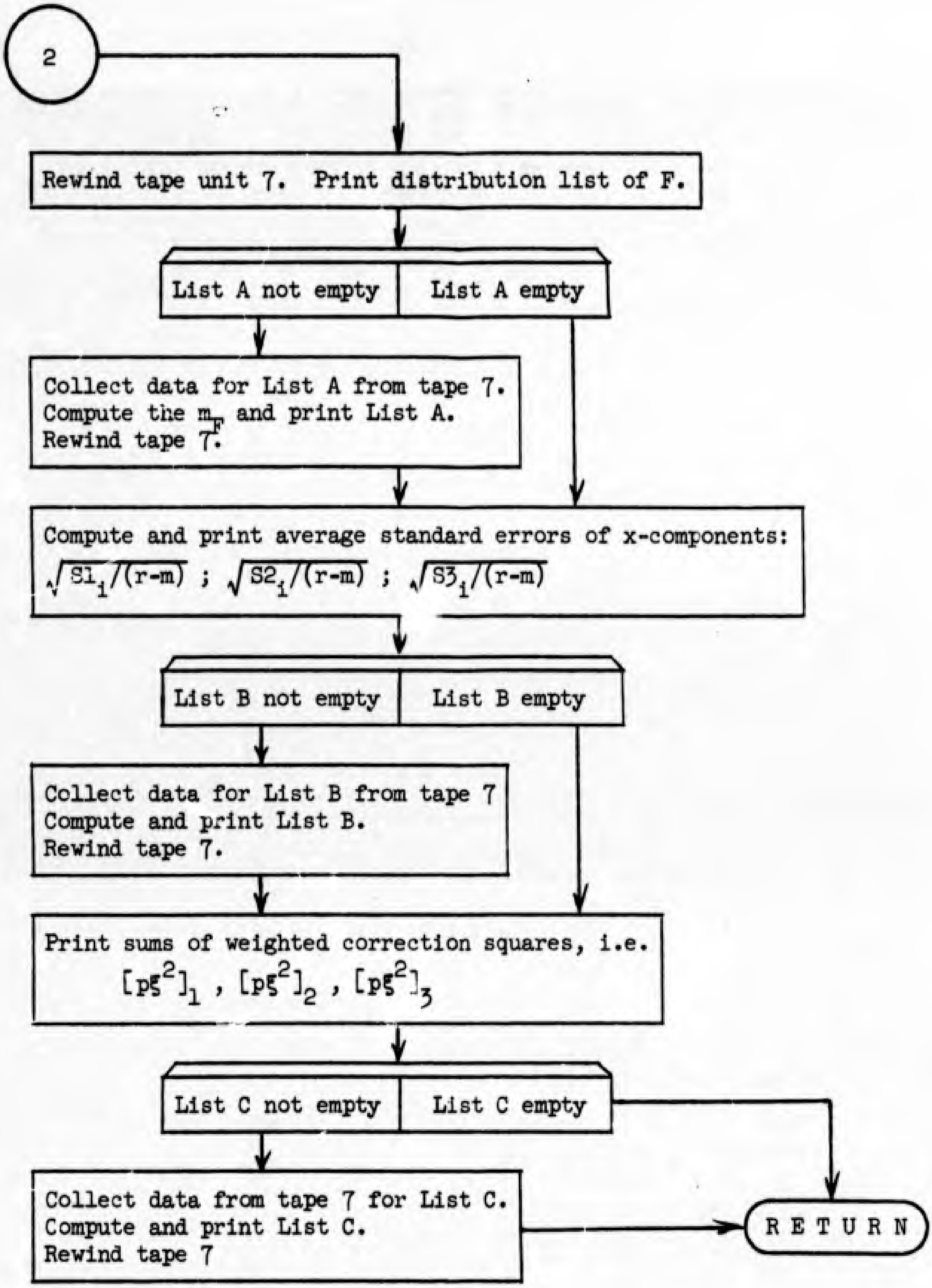
Yes → Print comment 4 → RETURN

No → RETURN

Flow chart of COLS 1. Part 2.



Flow Chart of COLS 2 Part 1.



<u>Flow Chart of COLS 2</u> Part 2.

REFERENCES

The ideas of the least squares processes used here are outlined in a number of thorough textbooks about adjustment of observations. For the preparation of this Appendix the following books were used.

1. G. M. Tienstra, Theory of the Adjustment of Normally Distributed Observations, Argus, Amsterdam, 1956.

2. W. Grossmann, Grundzuege der Ausgleichsrechnung, 2nd edition, Springer, Berlin, 1961.

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(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*
U.S. Army Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3. REPORT TITLE
DATA REDUCTION FROM PHOTOGRAPHS OF BLASTS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(Last name, first name, initial)*

Celmiņš, Aivars K. R.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO OF REFS
November 1966	150	2

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO RDT&E 1P014501A14B	Memorandum Report No. 1804
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. AVAILABILITY/LIMITATION NOTICES
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	U.S. Army Materiel Command Washington, D.C.

13. ABSTRACT

The ~~This~~ report gives a detailed description of a code for reduction of data from blast photographs. The code furnishes data about the orientation of the camera, computes scale factors of the frames, expresses the refractive distortion of the Sun's image in proper angles and determines the parameters of a function $R_o(f)$, which gives the radius of the blast bubble as a function of the frame number f (i.e., time). The distortion data are corrected by the code for a constant movement of the camera and can be used for the calculation of the air density within the blast bubble by other codes already in use.

DD FORM 1 JAN 64 1473

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Air Density in Blast Bubbles Optical Deformation Method High Altitude Explosions Least Squares						

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