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STOCHASTIC RENTAL INVENTORY MODEL

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ABSTRACT

An inventory of rented equipment is studied. Equipment is withdrawn from the inventory by customers who use it for a length of time and then return it. Decisions about the amount of equipment to rent can be made at certain points in time. This paper describes a policy for making these decisions which minimizes expected costs.

A dynamic programming model is formulated to describe the problem. The model is different than the usual ones considered in the literature in three respects. First, the equipment which is withdrawn from the inventory is not consumed; it is only used for a certain length of time then it returns to the inventory. The amount of equipment in the inventory, consequently, can fluctuate <u>up</u> <u>or</u> <u>down</u>. Second, all the equipment in the inventory is rented. Thus, when a decision is made about how much equipment is needed, either <u>more</u> <u>or</u> <u>less</u> than currently is on hand can be rented. Third, convexity of the cost function is not important because simple optimal policies can be found when the cost function is nonconvex.

The exact form of the optimal policy depends on the specific assumptions made, however, all of the optimal policies have the following generic structure. At the time a decision is to be made two numbers t and u, with $t \leq u$, can be computed. If the amount of equipment currently on hand is greater than u (less than t) then it is optimal to rent u (t). If the amount of equipment is in between t and u then, assuming convexity, it is optimal to continue renting the same quantity or, assuming nonconvexity, no general statement can be made about what to do because for each number rented in between t and u there may be a different optimal number to rent.

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1. Introduction

1.1 Inventory Problems

An inventory is a stock of physical goods which is held or stored for use at a later time. Inventory problems deal with deciding what quantity of goods should be stocked in anticipation of a future demand. The usual inventory problems studied in the literature deal with the determination of how many goods should be purchased and when they should be purchased; for example see Dvoretzky, Kiefer, and Wolfowitz [1], Arrow, Karlin, and Scarf [2], and Hadley and Whiten [3]. A problem exists because the demand for the commodity being stocked is not known with certainty; only a probability distribution is assumed to be known. If all of the demands cannot be satisfied a loss (cost) is incurred. On the other hand, a loss (cost) also occurs if more goods are stocked then are demanded. An optimal solution to the problem is a policy for the operation of the inventory system which minimizes the sum of these two losses and strikes a balance between over stocking and under stocking goods.

1.2 The Inventory Problem for Rented Equipment

The usual inventory problems in the literature implicitly utilize two assumptions. First, when the commodity in stock is demanded it is consumed. Second, the commodities are owned by the management of the inventory system. This paper studies an inventory problem in which neither assumption is valid. When a commodity is demanded it will be assumed to be used for a length of time and then returned in good condition to the inventory for possible use in the future. In addition the commodities in the inventory will be assumed to be rented or leased so that periodically a rental or lease fee must be paid. The latter assumption allows the operation of the inventory system to be very flexible because the number of units stored can be decreased in addition to the usual alternative of increasing the number of units.

For example, the problem of storing spare parts for machines can be viewed as an inventory problem. When a machine breaks down a spare part is withdrawn from the inventory and used. Techniques are available in the literature to solve this type of inventory problem. The problem of an organization which rents or leases a group or pool of cars for the intermittent use of a large number of their employees does not satisfy the two implicit assumptions mentioned previously. Problems similar to this latter type will be called inventory problems for rented equipment and are the subject of this paper. Several related problems have been studied by Kirby [4], Mico [5], Phelphs [6], Fukuda [7], Tainiter [21], and Iglehart and Karlin [22].

1.3 Historical Origin

The inventory problem for rented equipment originally arose when the question of pooling the aircraft used in fire fighting was investigated by the United States Forest Service. Aircraft are used to carry personnel to fire areas, to drop smoke jumpers in a region, to drop borate bombs or other chemicals on the fires, etc. An alternative to having aircraft scattered throughout a region is to group or pool all aircraft at a central location. Aircraft can be dispatched to an area if a fire occurs. Then they are returned to the pool when the fire is extinguished.

When and where a forest fire will occur is not known with certainty. In addition, there is a seasonal variation in fires; most fires occur during dry summer seasons and only a few occur in the wet winter season. An uneconomical solution would be for the Forest Service to own enough aircraft to handle the peak number of fires because many of the aircraft would remain idle a large part of the time. An alternative to ownership is the renting or leasing of the aircraft. Periodically, say once a week, once a month, or at the beginning and end of a fire season, a decision could be made concerning the number of aircraft that would be rented in the ensuing period. In this manner the number of aircraft could be adjusted, either up or down, much more easily than if the aircraft were owned.

In this study it is assumed that all the goods in the inventory are rented. It is quite possible that an organization may already possess a certain number of units and the question may be to decide how many additional units are to be rented during peak seasons. Or it may turn out that there is an optimal number of units to be owned with the rental of additional units occurring during peak seasons. Questions such as these which are concerned with the quantity of goods to own in addition to the number to rent are not considered in this paper.

1.4 An Inventory Model for Rented Equipment

In this section we describe a model of the inventory system discussed in the previous section. Our model can be viewed as an abstract representation of the real inventory system. It is used primarily for the purposes of prediction and control. The chief function of the model is to explain the operation of the inventory system rather than just providing a description [8].

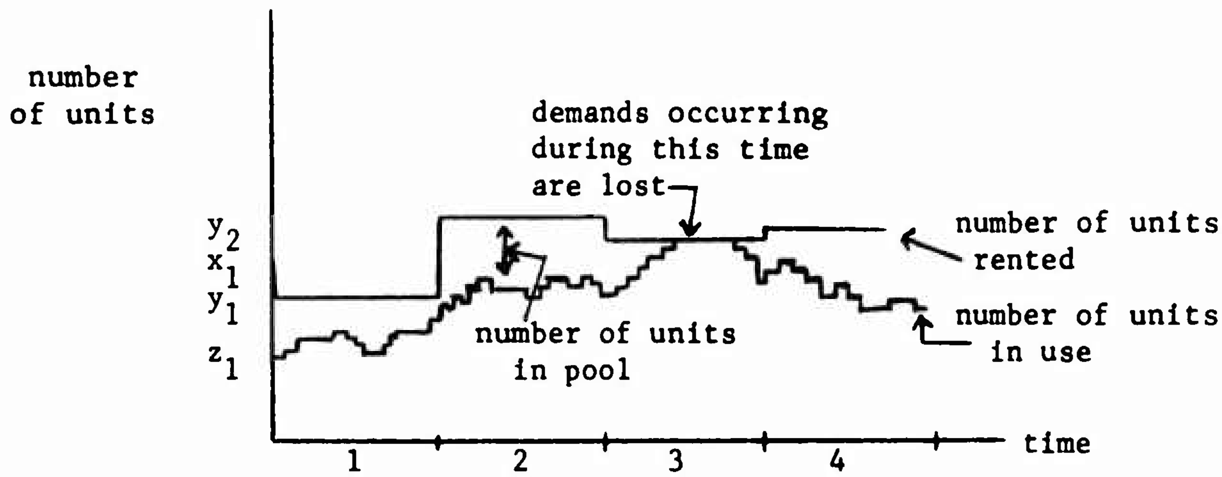
At this stage we will be content with just describing our model verbally. Chapter 2 will extend this discussion by formulating the mathematical model, i.e., the mathematical equations which describe the inventory system.

Before proceeding some terminology which will be used throughout the paper will be discussed. The goods in the inventory will be called equipment and the location of the goods when they are not being used will be called the pool. The entity which gives rise to a demand will be called a customer. When a demand occurs a unit of equipment leaves the pool and is used by the customer for a certain length of time. The equipment is said to be in the field when it is in use. Thus, demands for the equipment originate from customers in the field and equipment goes from the pool to the field to satisfy the demands. The length of time for which equipment is rented will be called the time period or the period.

The inventory model for rented equipment can be described as follows. At a specific location there is a pool of rented equipment. The amount of equipment in the pool can be changed only at specific points in time, i.e., at the beginning of any time period. For clarity the periods are numbered $1, 2, 3, \ldots$ in some manner. The amounts of equipment rented at the beginning of a period, before and after a decision is made about the quantity to rent during the period, are denoted by x_n and y_n, respectively, where n refers to the number of the period. Generally, let the integer valued, continuous parameter stochastic process $\{z(t), t \geq 0\}$ denote the number of units of equipment in use in the field at time t; specifically, let the integer valued, discrete parameter stochastic process $\{z_n, n = 1, 2, \ldots\}$ denote the number in use at the beginning of period n. Subscripts and arguments will be suppressed when no confusion will arise. For example, consider a fixed period, say the n^{th}. At the beginning of the period z_n units of equipment are in use and $x_n - z_n$ are in the pool. A decision is made to rent y_n units during the period. If y_n is larger than x_n then $y_n - x_n$ additional units are rented. If y_n is less than x_n then $x_n - y_n$ units are returned to the renter of the equipment. During the period the new number of units rented y_n remains constant but the number of units in use $z(t)$ changes with time as new demands occur and as units in the field are returned to the pool. We note that the variables x_n, y_n, and z_n are either zero or a positive integer.

We assume when equipment is in use in the field it cannot be returned to the renter; only equipment in the pool is eligible to be returned. Thus, y_n can never be less than $z(t)$, for all t in the n^{th} period. Also, we assume when a demand occurs and all the equipment is already in use that the customer either goes somewhere else and has his demand satisfied (at a higher cost) or goes away with his demand unfulfilled and does not return. Third, we assume that when equipment is returned to the pool from the field it is in the same physical condition as unused equipment. All the equipment in the pool can be considered identical, regardless of how many times it has been used.

Figure 1.1 illustrates how the number of rented units and the number of units in use might vary during several periods.



The Variation of the Amount of Equipment Rented and the Amount in Use During Several Periods.

Figure 1.1

During the operation of the inventory system the following sequence of events occurs. Suppose the current period is numbered 1, the next period 2, the next 3, etc. (We emphasize that other numberings of the periods are possible. This numbering method has been chosen for convenience. Later on it will be more convenient to number the periods backwards, see Section 3.1.) At the beginning of the first period (time = 0) there are x_1 units of equipment being rented. There are $z(0) = z_1$ units in use. At this time we decide to change the number of rented units to y_1. The new number of rented units y_1 may be larger or smaller than x_1 but it never can be less than z_1. Then during period one the number of units rented y_1 remains the same but the number of units in use changes. After a decision is made at the beginning of the period the number of units that will be in use at some time t during the period, $z(t)$, is not known with certainty; only its probability distribution is known. At the beginning of period two a decision is made to change the number of leased units from y_1 to another new level y_2. The sequence of events is repeated during the remainder of the second period, during period three, ad infinitum.

1.5 The Costs

To be able to decide how one method of operating the inventory system compares with another we must have a criterion. We assume the criterion is the minimization of cost, or if uncertainity is involved, expected cost. That method of operation, called a policy, which costs less or has a smaller expected cost than any other is the best method or the optimal policy.

First we discuss the costs associated with the obtaining or returning of equipment and the rental costs. At the beginning of every period a decision is made to either (1) continue renting the same number of units; or (2) to increase the number rented; or (3) to decrease the number rented for the next period. Making decision (1) incurs a cost of $c \cdot x$ where x is the number of items being rented or leased. The cost c is interpreted as the rental cost, or lease cost, per unit for the time period. If decision (2) is made a cost of $k \cdot (y - x) + c \cdot y$ is incurred, where $y - x$ is the additional number of units rented and where $k \geq 0$. The proportionality factor k can be interpreted as an order cost; it is the additional cost required if the items rented during a period were not rented in the preceeding period. When decision (3) is made a cost of $d \cdot (y - x) + c \cdot y$ is incurred, with $d \leq 0$. The proportionality constant $(-d)$ represents a return cost. Note that since $y - x \leq 0$, $d \cdot (y - x)$ is always non-negative and thus is really a cost. Included in the order and return costs are the costs of processing the paperwork for the order or return, any modification costs, and the transportation costs incurred in bringing the equipment to the pool or in returning it to the renter. In general the order or return cost may contain a fixed "set up" cost as a component in addition to the component proportional to the number ordered or returned. This possibility is not considered in this paper.

Second we discuss, briefly, the holding costs; the costs associated with holding the inventory. Complete discussions are given in the literature by Hadley and Whiten [3], Siegal [9], and Whiten [10]. Costs included under this heading are out-of-pocket costs such as the operating expenses of a warehouse in which the equipment is stored (heat, light, night watchmen), insurance, taxes, etc. An important cost which is always included as a component of the holding cost in the usual inventory models is the cost of the capital tied up in the inventory. In the model we are considering the entire inventory is rented so there is no capital invested in the inventory. Therefore, the cost of capital is not included in the holding cost. The holding cost will be assumed to be proportional to the number of units in the pool and to the length of time each unit is held in the pool.

Third, we mention, also briefly, the shortage costs; the costs associated with a demand occurring when all of the equipment is in use in the field. A thorough discussion is given in the literature by Hadley and Whiten [3] and Whiten [10]. If we suppose that when demands cannot be satisfied the customer goes away unsatisfied then the shortage cost - i.e., the cost of losing the customer - is often difficult to measure; still it is a very real cost. As an example, if the inventory pool contains aircraft used for fighting forest fires then the shortage cost can be identified as the cost of the loss due to the fire. When demands cannot be satisfied at the pool if the customer can fulfill his demand elsewhere then the additional cost of doing this is the shortage cost. Consider the example of the pool of aircraft again. It is possible that when the pool is empty and a demand occurs the customer may go directly to the renter (or to another supplier) to obtain the aircraft. The additional cost of doing this can be interpreted as a shorage cost. In any case we will say that the demands which occur when no equipment is in the pool are lost. The shortage cost is assumed proportional to the number of lost customers only. Assuming a shortage cost proportional to time has no meaning because customers do not wait; they are lost immediately and never return again.

When making a decision at the beginning of a period the number of units in the pool at any time t during the period, the length of time a unit is in the pool, and the number of lost customers up to t are not known precisely. Thus the expected values of holding and shortage costs are the relevant quantities to be considered. The expected holding and shortage cost for one period is assumed to be a function of the amount of equipment rented during a period, y, and the number of units in use at the beginning of the period, z. It will be denoted by $L_z(y)$.

A word must be said about extra costs which are incurred when the equipment is in use in the field, e.g., operating costs. We assume that the management of the pool has no authority and no influence over the customers who demand the equipment. The occurrence of demands is a phenomenon independent of the operation of the inventory system. Under this assumption operating costs are irrelevant to the managers of the inventory system and not included in the model. However, in the case of the Forest Service problem, to the decision maker who has to determine the number of aircraft to use in fighting a forest fire costs such as operating costs are relevant and must be considered.

Summing all of the costs mentioned in this section we obtain the total expected cost incurred during a period. (1) If the same amount of equipment is rented as in the previous period the total expected cost is $cx + L_z(x)$. (2) If the number rented is increased the total expected cost is $k(y - x) + cy + L_z(y)$. (3) If the number rented is decreased the total expected cost is $d(y - x) + cy + L_z(y)$.

1.6 Relation with Queueing Theory

An unusual aspect of this inventory problem is that there are two sources from which the inventory pool obtains equipment. At the beginning of a period additional equipment can be acquired, as in the usual inventory models. However, during a period equipment also becomes available as soon as it returns from the field. Because of the return of equipment, in addition to withdrawals, the inventory level in the pool can fluctuate up or down.

The problem we have been discussing up to this point, using inventory terminology, can be identified with the telephone trunking problem of queueing theory. Suppose y units of equipment are in the inventory pool. From the queueing point of view they may be considered as y parallel service channels in a service center. The demands which occur for the equipment can be interpreted as customers arriving at the service center. If there is a unit of equipment in the pool when a demand occurs then the equipment is sent to the field for a length of time which is a rancom varaible. Any other demand can only be satisfied by some other piece of equipment which is in the pool. This corresponds, from the queueing viewpoint, to an arriving customer going into an available service channel; the service time of the customer being identical with the length of time a piece of equipment is in use in the field. When a piece of equipment is through being used in the field it is returned to the pool. This corresponds to a customer completing his service and a service channel becoming available again. Since demands which occur when no equipment is in the pool are lost the queueing problem is a y channel telephone trunking model. We have a one to one correspondence between the stochastic behavior of the inventory model for rented equipment and the telephone trunking model.

By using the analogy between the stochastic behavior of the inventory level and queueing theory we can think of other versions of the rental inventory model.

As an example, consider y parallel service channels in a service center, as before. Only this time suppose customers who arrive when all channels are busy wait until equipment becomes available. The corresponding inventory problem is identical with the discussed in the preceding sections except that the customers who arrive when no inventory is in the pool wait until some becomes available. In inventory terminology we say that backlogging is or is not allowed depending on whether the customer waits or does not wait. Other possible versions of the rental inventory problem are: (1) customers are allowed to wait but an upper limit on the number waiting exists so that all subsequent demands are lost; (2) some customers who arrive wait and some do not wait; (3) priorities exist among the customers so that first preference for equipment is given to high priority customers. Clearly, for each one of the inventory problems there is a corresponding queueing model. This paper investigates only the lost customer or no backlogging case. The main reason is that the expressions obtained from queueing theory which describe the stochastic behavior of the inventory level are simpler than in any of the other cases.

1.7 Summary of Results

The results of this paper show that there is an optimal method of operating the inventory system described in the preceding sections. The exact form of the optimal policy depends on the assumptions made about the stochastic process $\{z(t), t \geq 0\}$, which represents the amount of equipment in use at time t. However, generally speaking, all the optimal policies have the following structure. At the beginning of each period two real numbers can be computed. They are called the <u>upper</u> and <u>lower</u> <u>critical</u> <u>numbers</u>; the former always being larger than the latter. If the quantity of equipment rented at the beginning of a period is larger than the upper critical number then too much equipment is currently on hand. The optimal amount of equipment to rent for the ensuing period is the upper critical number.

If the quantity of equipment rented at the beginning of the period is smaller than the lower critical number then too little equipment currently is on hand. The optimal amount to rent for the ensuing period is the lower critical number. Whenever the amount of equipment at the beginning of the period is in between the upper and lower critical numbers then in general it is not known whether too much or too little equipment is on hand. In this last case the optimal policy may call for either: (1) the renting of additional equipment, (2) the return of equipment, or (3) no action, i.e., continuing to rent the same number.

The various assumptions considered in this paper give rise to four variations of the generic optimal policy described above. The four cases considered are the various combinations of the assumptions concerning the dependence or independence of the relevant expected total cost function and the amount of equipment in use at the beginning of the period and whether or not this cost function is convex or non-convex. The importance of the independence and dependence cases stems from the following facts. When the stochastic process $\{z(t), t \geq 0\}$ is stationary then the relevant total cost function is independent of the quantity of equipment in use at the beginning of the period; however, when stationarity is not assumed then the total cost function does depend on the quantity of equipment in use at the beginning of the period. The simplest situation assumes independence and convexity. In this case the optimal policy described above is simplified because there is exactly one pair of critical numbers and when the quantity of equipment rented is in between these numbers at the beginning of the period the optimal policy is to continue renting the same number. When the convexity assumption is dropped the only change in the optimal policy occurs when the amount of rented equipment is in between the upper and lower critical numbers. In this case no general statements can be made about what to do; for each number rented at the beginning of the period which is in between the critical numbers there may be a different optimal number to rent during the ensuing period. When assuming convexity but not necessarily independence it is shown that there exists a sequence of pairs of critical numbers. There is one pair of critical numbers for each possible amount of equipment in use at the beginning of the period. Again in this case it is shown that when the amount of rented equipment at the beginning of a period is in between the upper and lower critical numbers the optimal policy is to continue renting the same number. Finally, when the convexity assumption is dropped the only change in the optimal policy is when the amount of equipment is in between the upper and lower critical numbers. As in the previous nonconvex case no general statements can be made about what to do in this situation. For each number rented which is in between the critical numbers there may be a different optimal number to rent during the ensuing period.

The usual inventory models considered in the literature give optimal policies only under strict conditions such as convexity. For the rental inventory model convexity is not as important because simple optimal policies can be found when the relevant expected total cost functions are nonconvex. However, when convexity is assumed the optimal policies have a simpler form.

Finally, examples are given which show that in both the cases of stationary and nonstationary $\{z(t) , t \geq 0\}$ convex and nonconvex cost functions are obtained under certain conditions.

2. The One Period Model

2.1 Introduction

In this chapter the inventory model for rented equipment, described verbally in Chapter 1, will be presented more precisely. The mathematical model for a one period problem will be formulated, described, and solved. Then we discuss a special case in which the occurrence of demands for equipment follows a Poisson process.

2.2 The Mathematical Model

When dealing with a one period model we must consider what happens to the rented equipment at the end of the period. We will suppose that all the equipment is returned to the renter. The cost to return a unit of equipment, in general, will be different if it is in use than if it is in the pool. However, in what follows we make the assumption that the cost of returning each unit is the same so that the cost can be considered as a component of the rental cost c. Chapter 6 considers the more general case.

Let $C_1(x , z)$ be the minimum expected cost incurred during one period if x and z are the initial numbers of equipment rented and in use, respectively, at the beginning of the period (i.e., before a decision is made to rent y units for the ensuing period). We note that this function is only defined for $x \geq z$ since the amount of equipment in use can never be larger than the amount rented. Combining the order and return costs mentioned in Section 1.5 let

$$a(y - x) = \begin{cases} k \cdot (y - x) & \text{if } y \geq x \\ \\ d \cdot (y - x) & \text{if } y \leq x \end{cases} \qquad (2.1)$$

where k is the unit order cost and $(-d) \geq 0$ is the unit return cost. Similarly, following the notation of Section 1.5 again we denote the rental cost by $c \cdot y$

and the one period expected holding and shortage cost by $L_z(y)$ where y is the amount of equipment rented during the period. The total expected cost incurred during one period is

$$a(y - x) + cy + L_z(y) \qquad (2.2)$$

The following assumptions are made, in addition to those listed in Section 1.4:

(1) $L_z(y) \geq 0$ and is a function of the amount of equipment rented during the period, y, only; i.e., $L_z(y) = L(y)$.

(2) $\Delta^2 L(y) \geq 0$ for $y = 0, 1, 2, \ldots$ where Δ is the usual difference (forward) operator.

(3) $k \geq 0$, $d \leq 0$, $c \geq 0$, and $k + c > 0$.

The motivation for the second part of assumption (1), i.e., $L_z(y) = L(y)$, is that this case of independence naturally arises when the stochastic process $\{z(t), t \geq 0\}$ is assumed stationary.

We want to find that value of y for which (2.2) is the smallest, thus

$$C_1(x, z) = \min_{y \geq z} \{a(y - x) + cy + L(y)\}. \qquad (2.3)$$

The minimum exists by virtue of the fact that $a(y - x) + cy + L(y) \to \infty$ as $y \to \infty$. The variables x, y, and z take nonnegative integral values only since each represents a certain quantity of equipment.

When a real valued function has nonnegative second differences we will say that it is <u>discrete convex</u>. Consequently, assumptions (1) and (2) state that the one period expected holding and shortage cost is a nonnegative, discrete convex function which does not depend on the initial amount of equipment in use. The third assumption says that the order and return cost function $a(y - x)$ and the unit rental cost c are nonnegative, i.e., they are true costs. Chapter 5 considers a more general rental inventory model where the one period expected holding and shortage cost $L_z(y)$ does not have a special structure. That is, it may depend on the initial amount in use, z, and it is not necessarily discrete convex.

Define two nonnegative real numbers t_1 and u_1 by

$$\begin{aligned} k(t_1 - x) + ct_1 + L(t_1) &= \min_{y \geq 0} \{k(y - x) + cy + L(y)\} \\ d(u_1 - x) + cu_1 + L(u_1) &= \inf_{y \geq 0} \{d(y - x) + cy + L(y)\} . \end{aligned} \tag{2.4}$$

The minimum can be used in the first equation of (2.4) because $k(y - x) + cy + L(y) \to \infty$ as $y \to \infty$. By contrast, the infimum must be used in the second equation of (2.4) because $d(y - x) \to -\infty$ as $y \to +\infty$. The subscript 1 will be suppressed whenever no confusion will arise, i.e., $t_1 = t$ and $u_1 = u$. For convenience assume t and u are unique. Chapter 5 discussed the case when this is not true.

Because $L(y)$ is discrete convex $k + c + \Delta L(0) \geq 0$ if and only if $t = 0$. Clearly $k + c + \Delta L(0) < 0$ if and only if $0 < t < \infty$, in which case t is determined by

$$k + c + \Delta L(t - 1) < 0 \leq k + c + \Delta L(t) . \tag{2.5}$$

Similarly, $d + c + \Delta L(0) \geq 0$ if and only if $u = 0$. Also $d + c + \Delta L(y) < 0$ for $y = 0, 1, 2, \ldots$ if and only if $u = \infty$. Finally, $d + c + \Delta L(0) < 0$ and $d + c + \Delta L(y) \geq 0$ for some finite nonnegative integral y when and only when $0 < u < \infty$, in which case u is calculated from

$$d + c + \Delta L(u - 1) < 0 \leq d + c + \Delta L(u) . \tag{2.6}$$

The <u>upper</u> and <u>lower</u> <u>critical</u> <u>numbers</u> u and t defined in (2.6) and (2.5), respectively, obviously exist and are uniquely determined. We will show, now, that $t \leq u$.

Theorem 2.1

If assumptions (1), (2), and (3) hold, i.e., if $L(y)$ is nonnegative, discrete convex, independent of the initial amount of equipment in use z, and if $k \geq 0$, $c \geq 0$ and $d \leq 0$ then $t \leq u$.

Proof: If $0 < t < \infty$ and $0 < u < \infty$ then from (2.5) and (2.6) and by the hypothesis $\Delta L(u) \geq -d - c \geq -k - c > \Delta L(t - 1)$. By discrete convexity $\Delta L(u) > \Delta L(t - 1)$ implies that $u > t - 1$. But t and u assume integral values only therefore $t \leq u$. Clearly, the theorem is true if $t = 0$ and $u = 0$ or $u = \infty$. The only other alternative $t > 0$ and $u = 0$ cannot occur because then $k + c + \Delta L(0) \geq d + c + \Delta L(0) \geq 0$, a contradiction since $t > 0$ implies $k + c + \Delta L(0) < 0$.

2.3 Solution of the One Period Model

The solution to the one period model is presented in this section, i.e., the optimal policy is determined. Initially we assume that the upper and lower critical numbers, u and t, are calculated from (2.6) and (2.5), i.e., $0 < t \leq u < \infty$. From these results the optimal policy can be obtained for any values of the critical numbers $0 \leq t \leq u \leq \infty$.

Under the assumption in the above paragraph three cases must be distinguished: the initial amount of rented equipment x can be larger than u, less than t, or in between t and u.

Case (1): $x > u$. Let

$$h_x(y) = a(y - x) + cy + L(y) . \tag{2.7}$$

That is, $h_x(y)$ is the total expected cost incurred during one period if x is the initial amount of rented equipment and y is the amount rented during the period. From (2.3) we have

$$C_1(x\ ,\ z) = \min_{y \geq z} \{h_x(y)\}\ .$$

For each fixed x , $a(y - x)$ is discrete convex in y therefore so is $h_x(y)$ because the sum of discrete convex functions is discrete convex. Fix $x > u$. When $y < x$

$$h_x(y) = d(y - x) + cy + L(y)$$

and when $y \geq x$

$$h_x(y) = k(y - x) + cy + L(y) \geq d(y - x) + cy + L(y)\ .$$

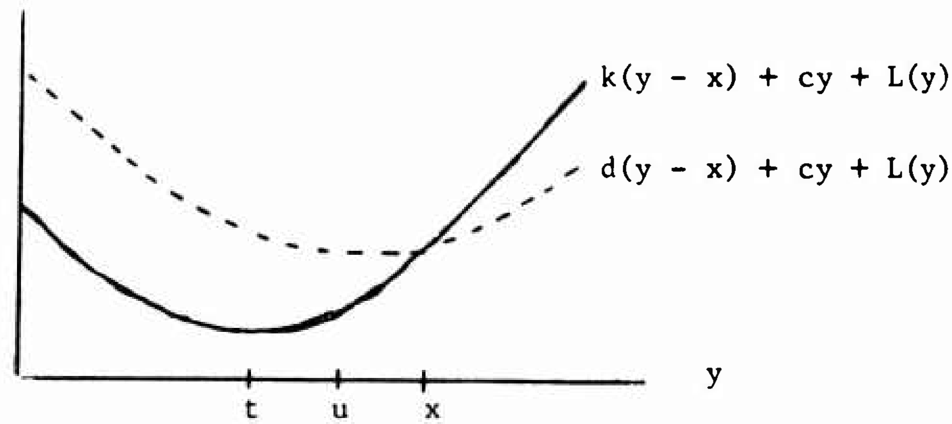
Thus using the definition of u , the discrete convexity of $h_x(y)$, and the fact that $u < x$ implies

$$\min_{y \geq 0} \{h_x(y)\} = h_x(u)$$

so that

$$C_1(x\ ,\ z) = \min_{y \geq z} \{h_x(y)\} = h_x(\max(u\ ,\ z))\ .$$

Pictorially, the situation can be depicted as in Figure 2.1.



The Cost Function when $x > u$.

Figure 2.1

Clearly $h_x(y) = \max\ [k(y - x) + cy + L(y);\ d(y - x) + cy + L(y)]$. Strictly speaking the lines in Figure 2.1 are not continuous but they are defined only at nonnegative integers. For purposes of clarity from this point on, unless otherwise specifically noted, all discrete valued functions will be drawn in illustrations as if they are continuous functions. It is clear also from Figure 2.1 that $h_x(y)$ attains its minimum at $y = u$.

At the beginning of a period if the initial number of units of rented equipment x is larger than the upper critical number u then the optimal amount of equipment to rent during the ensuing period is u if u is larger than the number of units in use z , otherwise it is z . In other words whenever $x > u$ then $y = \max(u , z)$. The difference $x - \max(u , z)$ is the number of units returned.

<u>Case (2):</u> $x < t$. For a fixed $x < t$ we have, when $y \leq x$,

$$h_x(y) = d(y - x) + cy + L(y) \geq k(y - x) + cy + L(y)$$

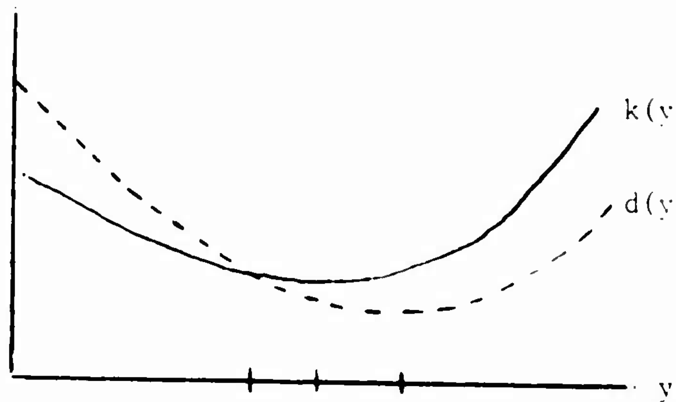
and when $y > x$

$$h_x(y) = k(y - x) + cy + L(y) .$$

Using the definition of t and the discrete convexity of $h_x(y)$, since $y > x \geq z$, we conclude

$$C_1(x, z) = \min_{y \geq z} h_x(y) = h_x(t) .$$

Pictorially, the situation can be depicted as in Figure 2.2.



The Cost Function when $x < t$.

Figure 2.2

Figure 2.2 also clearly indicates that in this case $h_x(y) = \max [k(y - x + cy + L(y) ; d(y - x) + cy + L(y)]$ and it attains its minimum at $y = t$.

If the initial amount of equipment being rented at the beginning of a period, x , is less than the lower critical number t then the optimal policy is to rent t units during the coming period. The difference $t - x$ is the additional number of units rented.

<u>Case (3)</u>: $t \leq x \leq u$. The definition of $h_x(y)$ can be rewritten as

$$h_x(y) = \begin{cases} d(y - x) + cy + L(y) & y < x \\ k(y - x) + cy + L(y) & y \geq x . \end{cases}$$

Differencing with respect to y gives

$$h_x(y) = \begin{cases} d + c + L(y) & y < x \\ k + c + L(y) & y \geq x . \end{cases}$$

Equations (2.5) and (2.6) and the discrete convexity of $L(y)$ imply

$$h_x(y) = \begin{cases} < 0 & y < x \\ \geq 0 & y \geq x . \end{cases}$$

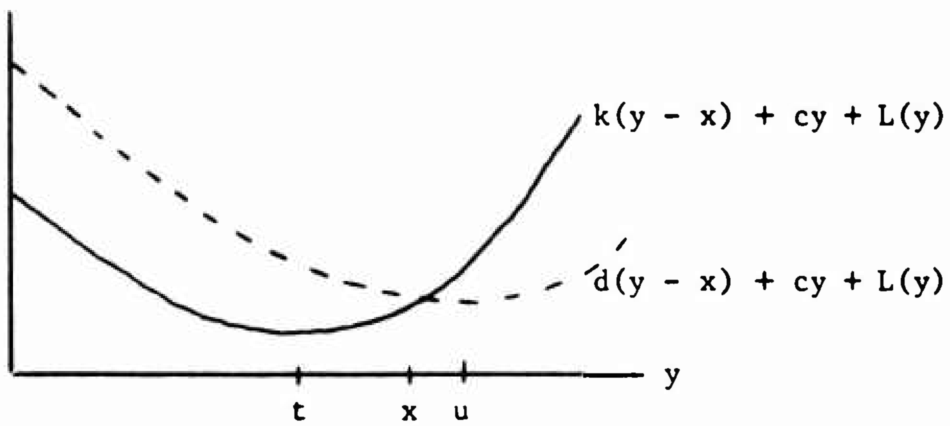
In particular we have the conditions for a minimum at x

$$h_x(x - 1) < 0 \leq h_x(x)$$

so that, because $x \geq z$,

$$C_1(x , z) = \min_{y \geq z} \{h_x(y)\} = h_x(x) .$$

Pictorially, the situation can be depicted as in Figure 2.3.



The Cost Function when $t \leq x \leq u$.

Figure 2.3

Figure 2.3 also clearly indicates that in this case $h_x(y) = \max[k(y - x) + cy + L(y);$

$d(y - x) + cy + L(y)]$ and it attains its minimum at $y = x$.

If the initial amount of equipment rented at the beginning of a period is in between the upper and lower critical numbers then the optimal policy is to rent the same amount during the period.

The optimal policy has been determined whenever $0 < t \leq u < \infty$. Other situations possible are: (i) $t = 0$, $u = 0$; (ii) $t = 0$, $0 < u < \infty$; (iii) $t = 0$, $u = \infty$; and (iv) $t > 0$, $u = \infty$. When (i) occurs only case (1), $x > u$, can happen therefore the optimal policy is $y = \max(u , z) = z$. For (ii) cases (1) , $x > u$, and (3), $t \leq x \leq u$, can occur so that the optimal policy is $y = \max(u , z)$ when $x > u$ and $y = x$ if $t \leq x \leq u$. If (iii) occurs then only case (3), $t \leq x \leq u$, is possible, thus $y = x$ is the optimal policy. Finally, whenever (iv) occurs only cases (2), $x < t$, and (3), $t \leq x \leq u$, can happen therefore the optimal policy is $y = t$ if $x < t$ and $y = x$ if $x > t$.

Remember that when the critical numbers are calculated by (2.5) and (2.6) $0 < t \leq u < \infty$. This is the normal situtation. Other situations which occur are not normal in the sense that $t = 0$ and either $u = 0$ or $u = \infty$. The optimal method of operating the inventory system in the first case is to have no inventory while in the latter case it is never to change the inventory no matter what the initial level is. Such situtations are considered degenerate.

We make a remark about the return cost. It seems reasonable to suppose that if the return cost $(-d)$ is very large it might never be economical to return equipment. That it is indeed possible that there is a value of $(-d)$ which is too large follows from (2.6). As $(-d)$ increases d decreases. If there is a value of d , say d_0 , such that $d + c + \Delta L(y) < 0$ for all nonnegative integral y then $u = \infty$ for all values of $(-d) \geq (-d_0)$. A finite inventory level is always less than this value of the upper critical number, thus, equipment never is returned.

The next theorem shows that the one period minimum expected cost function $C_1(x , z)$ is discrete convex in $x \geq z$ for each z, under certain conditions.

<u>Theorem 2.2</u>

If $L(y)$ if discrete convex then $C_1(x , z)$ is discrete convex for all $x \geq z$ for each value of z.

Proof: The optimal policy given in the above sections implies

$$C_1(x , z) = \begin{cases} k(t - x) + ct + L(t) & x < t \\ cx + L(x) & t \leq x \leq u \\ \left.\begin{array}{ll} d(u - x) + cu + L(u) & \text{if } z \leq u \\ d(z - x) + cz + L(z) & \text{if } z > u \end{array}\right\} & x > u \end{cases}$$

for $x \geq z$. It is to be understood that when $z < t$ then x may be in any of the three ranges: $x < t$, $t \leq x \leq u$, or $x > u$. However, whenever $t \leq z \leq u$ then $C_1(x , z)$ is defined only for $t \leq z \leq x \leq u$ and $x > u$ and when $z > u$ then $C_1(x , z)$ is defined only for $u < z \leq x$. Two cases must be considered, $z \leq u$ and $z > u$.

Case (i): $z \leq u$. In this case the expression for $C_1(x , z)$ simplifies to

$$C_1(x , z) = \begin{cases} k(t - x) + ct + L(t) & x < t \\ cx + L(x) & t \leq x \leq u \\ d(u - x) + cu + L(u) & x > u \end{cases}$$

for $x \geq z$. In the regions $x < t$ and $x > u$, $C_1(x , z)$ is "linear" (in the sense that $\Delta C_1(x , z)$ = constant) and in the region $t \leq x \leq u$ it is discrete convex. By the definitions of t and u

$$\Delta C_1(t - 1 , z) = -k \leq c + \Delta L(t) = \Delta C_1(t , z)$$

$$\Delta C_1(u - 1 , z) = c + \Delta L(u - 1) < -d = \Delta C_1(u , z)$$

so that $C_1(x, z)$ is discrete convex in $x \geq z$ for each value of $z \leq u$.

Case (ii): $z > u$. In this case the expression for $C_1(x, z)$ simplifies to

$$C_1(x, z) = d(z - x) + cz + L(z)$$

which is "linear" and hence discrete convex in $x \geq z$ for each z.

2.4 The Optimal Policy

We summarize the optimal policy derived in the preceeding section. If the amount of equipment being rented at the beginning of the period is x and if the number of units in use at the beginning of the period is $z \leq x$ then the optimal policy is:

(1) If $x > u$ then $y = \max(u, z)$, i.e., $x - \max(u, z)$ units are returned;

(2) If $t \leq x \leq u$ then $y = x$, i.e., the same number is rented;

(3) If $x < t$ then $y = t$, i.e., $t - x$ additional units are rented;

where t and u are the critical number determined in Section 2.2 and y is the quantity of equipment to be rented during the period. It always is true that $t \leq u$ and it usually is true that $t > 0$ and $u < \infty$. The optimal policy intuitively says that whenever the inventory is too high $(x > u)$ equipment should be returned. If the inventory level is too low $(x < t)$ then additional equipment should be rented. However, because of the cost of renting additional equipment and the cost of returning equipment, whenever the inventory level is just about right $(t \leq x \leq u)$ no action should be taken.

2.5 $L_z(y)$: A Stationary Approximation

Before proceeding to the multi-period formulation of the model the structure of $L_z(y)$, the one period expected holding and shortage cost, is investigated. To determine this cost we must know the probability law of the number of units of equipment in the pool at every time t. Using this probability law $L_z(y)$ is calculated in a special case. For this case it is shown that in fact $L_z(y)$ does not depend on the initial amount of equipment in use, z, and that it is nonnegative and discrete convex. Thus the results of Sections 2.1 through 2.4 apply for this special case. At the end of this section more explicit expressions for the critical numbers t and u are given for this case and a numerical example is presented.

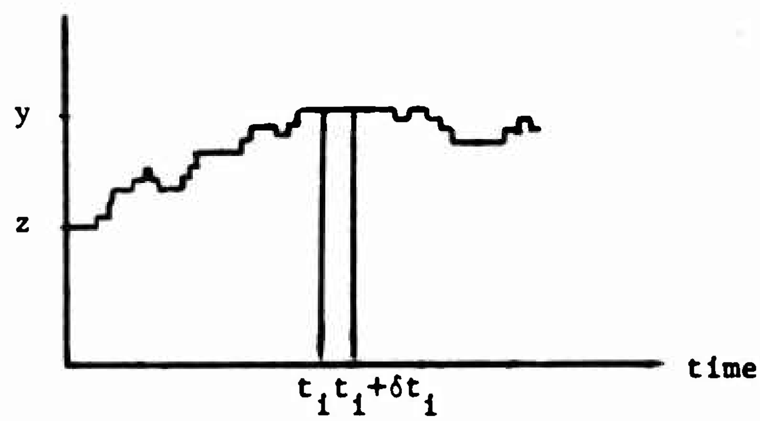
Let $p_{z,k}(t, y)$ be the required probability law, i.e., it represents the probability that, given z units of equipment in use at time 0 and y units rented during the period, k units are in use at time t. Recall the one to one correspondence between the stochastic behavior of the inventory model and the telephone trunking problem mentioned in Section 1.6. The transition probabilities for a y channel trunking problem - i.e., the p. d. f.'s which give the probabilities that there are k channels occupied at time t given that z are occupied at time 0 - clearly are identical with $p_{z,k}(t, y)$. In general, given an arbitrary demand process and an arbitrary usage distribution the transition probabilities $p_{z,k}(t, y)$ are very difficult to determine. When Poisson demands and negative exponential usage times are assumed the transition probabilities can be calculated (or their Laplace transforms) but the expressions are so complex that they are of little practical use; see Riordan [11], pages 81-7, and Takács [12], pages 174-87. To obtain an expression which can be efficiently manipulated the transition probabilities will be approximated by their stationary distribution $\pi(k, y)$, where

$$\pi(k, y) = \lim_{t\to\infty} p_{z,k}(t, y) .$$

The stationary distribution $\pi(k , y)$ can be interpreted in two ways. It can represent the probability that, at a random point in time, k units will be in use given that y are being rented (and that t is large). Also it can be thought of as the proportion of time, over some long period of time, that k units are in use when y units are being rented. Notice that $\pi(k , y)$ is independent of both z , the initial number in use, and time; this can be shown rigorously for the systems we consider (and it also can be shown that $\pi(k , y)$ exists), see Takács [12], page 183.

In the following the one period expected holding and shortage cost $L_z(y)$ is calculated. The p. d. f. of the equipment in use is approximated by its stationary distribution. We consider the holding cost first and then the shortage cost.

A holding cost is charged for all of the items stored in the pool. It is proportional to the number of units at a given time and it is proportional to the length of time each unit is held in the pool.



The Expected Holding Cost in a Small Interval of Time $(t , t + \delta t)$.

Figure 2.4

The expected holding cost during a small interval of time $(t_1 , t_1 + \delta t_1)$ is,

$$h' \sum_{k=0}^{y} (y - k)\pi(k , y)\delta t_i$$

where h' is the constant of proportionality, in dollars per item per unit time. Summing over all intervals δt_i in $[0 , T]$ gives an expression for the total holding cost during $[0 , T]$,

$$\sum_i h'[\sum_{k=0}^{y} (y - k)\pi(k , y)]\delta t_i . \tag{2.8}$$

Letting $\delta t_i \to 0$ gives

$$h' \int_0^T \sum_{k=0}^{y} (y - k)\pi(k , y)dt \tag{2.9}$$

since (2.8) is just a Riemann sum. Thus we obtain

$$\text{Expected holding cost} = h'T \sum_{k=0}^{y} (y - k)\pi(k , y) \tag{2.10}$$

$$= h \sum_{k=0}^{y} (y - k)\pi(k , y) . \tag{2.11}$$

Since h' is in units of dollars per item per unit time h is in dollars per item per period. Note that h is identical with h' when one time period is taken as the unit of time, i.e., when $T = 1$. Notice also that (2.10) could have been obtained directly from (2.8) in this simple case. In more complicated cases where $p_{z,k}(t , y)$ is used or an approximation is used which depends on t the dependence on t makes such a short cut impossible; in order to arrive at (2.10) we must go through (2.9).

A shortage cost is charged for each unit demanded when there are none in the pool. It is assumed proportional to the number of such lost customers only. Since a customer does not wait when there is no inventory charging a shortage cost proportional to time has no meaning. It can be shown under our assumptions that the expected number of lost customers during a period of time is $\lambda \cdot \Pr\{\text{all equipment is in use}\} = \lambda \cdot \pi(y, y)$; see Hadley and Whiten [13], pages 177-80, and Takács [12], page 183. Here λ represents the average demand during the period. Denoting the shortage cost, in units of dollars per lost customer, by s implies that the one period expected shortage cost is $s \cdot \lambda \cdot \pi(y, y)$. The one period expected holding and shortage cost is the sum of the expected holding cost and the expected shortage cost,

$$L(y) = h \sum_{k=0}^{y} (y - k)\pi(k, y) + s\lambda\pi(y, y). \qquad (2.12)$$

Assuming h and s are nonnegative we immediately have $L(y)$ nonnegative also. It is important to note that in this case when $p_{z,k}(t, y)$ is approximated by $\pi(k, y)$ $L_z(y)$ does not depend on the initial amount of equipment in use z , i.e., $L_z(y) = L(y)$.

The last part of this section is devoted to showing that under certain conditions $L(y)$ is discrete convex. We calculate an explicit expression for $L(y)$ and show that $\Delta^2 L(y) \geq 0$ if $\lambda/\mu \leq 1$ where λ is the average demand per period and μ^{-1} is the average usage time.

From queueing theory it is well known that for a y channel queueing system with Poisson arrivals and arbitrary independent holding times that,

$$\pi(k, y) = \begin{cases} \dfrac{(\lambda/\mu)^k}{k!}\left[\displaystyle\sum_{j=0}^{y} \dfrac{(\lambda/\mu)^j}{j!}\right]^{-1} & 0 \leq k \leq y \\ 0 & \text{otherwise,} \end{cases}$$

Takács [12], page 186. The only condition necessary for the existence of $\pi(k\ ,\ y)$ is that $\mu \neq 0$. Define

$$P(k) = \sum_{j=0}^{k} \frac{(\lambda/\mu)^j e^{-(\lambda/\mu)}}{j!}$$

$$p(k) = \frac{(\lambda/\mu)^k e^{-(\lambda/\mu)}}{k!}$$

then the stationary probabilities are

$$\pi(k\ ,\ y) = \begin{cases} \dfrac{p(k)}{P(y)} & 0 \leq k \leq y \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

Before proceeding an important fact must be noted. First, we have assumed the demands for equipment follow a Poisson process. In reality this assumption is not too restrictive because of the Central Limit Theorem of Reliability, see Barlow and Proschan [13], pages 13-19. This theorem states that if demands are originating from n sources independently of each other then the superimposed demand of these n sources tends to a Poisson process regardless of the distribution of demand at each individual source as the number of sources, n , gets very large, if certain weak (from a practical point of view) conditions are satisfied. Consider the aircraft pooling problem discussed previously. The demands are generated by the occurrence of forest fires. The number of possible starting locations for fires is very very large. Assuming the probability of a fire starting at a location is independent of what happens at other locations implies that the Central Limit Theorem of Reliability can be applied. In some cases this assumption is valid although it is obviously not universally true, e.g., when a fire is in existence at a location it may be quite likely to spread to several other surrounding locations.

The shortage cost per period is now very easy to calculate. By (2.12) we have

$$\text{shortage cost} = s\lambda\frac{p(y)}{P(y)} \ . \tag{2.14}$$

The expected holding cost per period is also easy to calculate when using (2.11)

$$\text{holding cost} = h\sum_{k=0}^{y}(y - k)\frac{p(k)}{P(y)}$$

$$\text{holding cost} = h\left[y - (\lambda/\mu)\frac{P(y - 1)}{P(y)}\right] \tag{2.15}$$

where $P(-1) = 0$ by definition. Equation (2.12) gives the one period expected holding and shortage cost to be the sum of (2.14) and (2.15)

$$L(y) = h[y - (\lambda/\mu)\frac{P(y - 1)}{P(y)}] + s\lambda\frac{p(y)}{P(y)} \tag{2.16}$$

for $y = 0, 1, 2, \ldots$ with $P(-1) = 0$. Finally, we are able to show that under certain conditions $L(y)$ is discrete convex. This is done in the following theorem.

<u>Theorem 2.3</u>

If $0 \leq (\lambda/\mu) \leq 1$ then (i) $L(y) \to \infty$ as $y \to \infty$, (ii) $\Delta L(0) = (h - s\lambda)/[1 + (\lambda/\mu)]$, and (iii) $\Delta^2 L(y) \geq 0$ for $y = 0, 1, 2, \ldots$

Proof: Note first that $L(0) = s\lambda > 0$. The proof of (i) is immediate because $p(y)/P(y) \to 0$ as $y \to \infty$ and $0 \leq P(y - 1)/P(y) \leq 1$ for all y. To show (ii) we calculate the first difference $\Delta L(y) = L(y + 1) - L(y)$,

$$\Delta L(y) = h + h\frac{\lambda}{\mu}\left[\frac{P(y-1)}{P(y)} - \frac{P(y)}{P(y+1)}\right] + s\lambda\left[\frac{p(y+1)}{P(y+1)} - \frac{p(y)}{P(y)}\right]$$

$$\Delta L(0) = h - h\frac{\lambda}{\mu}\frac{p(0)}{P(1)} + s\lambda\frac{p(1)}{P(1)} - s\lambda \; .$$

Using the definition of $p(y)$ and $P(y)$ gives

$$L(0) = \frac{h - s\lambda}{1 + \frac{\lambda}{\mu}}$$

proving (ii). To show (iii) we calculate the second difference $\Delta^2 L(y) = \Delta L(y+1) - \Delta L(y)$.

$$\Delta^2 L(y) = h\frac{\lambda}{\mu}\left[\frac{2P(y)}{P(y+1)} - \frac{P(y+1)}{P(y+2)} - \frac{P(y-1)}{P(y)}\right] + s\lambda\left[\frac{p(y+2)}{P(y+2)} + \frac{p(y)}{P(y)} - \frac{2p(y+1)}{P(y+1)}\right]$$

$$= h\frac{\lambda}{\mu}\left[\frac{2[P(y)]^2P(y+2) - [P(y+1)]^2P(y) - P(y-1)P(y+1)P(y+2)}{P(y)P(y+1)P(y+2)}\right]$$

$$+ s\frac{\lambda}{\mu}\left[\frac{p(y+2)P(y)P(y+1) + p(y)P(y+2)P(y+1)}{P(y)P(y+1)P(y+2)} - 2\frac{p(y+1)P(y+2)P(y)}{P(y)P(y+1)P(y+2)}\right] .$$

Concentrating attention on just the numerator of the coefficient of $h\frac{\lambda}{\mu}$ let $P(y+2) = P(y) + p(y+1) + p(y+2)$ and $P(y+1) = P(y) + p(y+1)$. Upon collecting terms this numerator is

$$[P(y)]^2[p(y+2) - 2p(y+1) + p(y)] + P(y)[p(y) - p(y+1)][2p(y+1) + p(y+2)] + p(y)p(y+1)[p(y+1) + p(y+2)] . \qquad (2.17)$$

When $y = 0$ (2.17) equals $e^{-(\lambda/\mu)}$ which is strictly positive. When $y = 1, 2, 3, \ldots$ we note that since $p(y) \geq p(y+1)$ for $0 \leq (\lambda/\mu) \leq 1$ the last two terms of (2.17) are positive. But a closer examination of the first term, using the definition of $p(y)$, shows that $p(y+2) - 2p(y+1) + p(y) \geq 0$ for $0 \leq (\lambda/\mu) \leq 1$. Now we look at the numerator of the coefficient of $s\lambda$,

$$p(y+2)P(y)P(y+1) + p(y)P(y+2)P(y+1) - 2p(y+1)P(y+2)P(y) .$$

Because $p(y) \geq 2p(y+1)$ for $y = 1 , 2 , 3 , . . .$ and $0 \leq (\lambda/\mu) \leq 1$ it is evident that this numerator is nonnegative. Checking it at $y = 0$ it is easy to see that it is nonnegative also. The positivity of both terms of $\Delta^2 L(y)$ for $y = 0 , 1 , 2 , . . .$ completes the proof of the theorem.

The results of this section tell us that this special case of the rental inventory model satisfies the assumptions of Section 2.2 so that all of the results of this chapter apply. In particular, the optimal policy has the form given in Section 2.4. We gather together here all of the assumptions for this special case.

(i) Arrivals are a Poisson process with parameter λ .

(ii) Usage times are independent, identically distributed according to an arbitrary distribution with mean μ^{-1} .

(iii) $0 \leq (\lambda/\mu) \leq 1$.

(iv) $k \geq 0$, $c \geq 0$, $d \leq 0$, and $k + c > 0$.

Making these assumptions the conditions determining the two critical numbers t and u can be made more explicit than the representation in Section 2.2. Theorem 2.4 deals with the lower critical number t and Theorem 2.5 is concerned with the upper critical number u .

Theorem 2.4

Let assumptions (i) through (iv) hold and suppose that $k + c + h > 0$. (1) If $k + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) \geq 0$ then $t = 0$. (2) If $k + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) < 0$ then $t > 0$ and t is determined from

$$k + c + h + h\frac{\lambda}{\mu}\left[\frac{P(t-2)}{P(t-1)} - \frac{P(t-1)}{P(t)}\right] + s\lambda\left[\frac{p(t)}{P(t)} - \frac{p(t-1)}{P(t-1)}\right] < 0$$

$$k + c + h + h\frac{\lambda}{\mu}\left[\frac{P(t-1)}{P(t)} - \frac{P(t)}{P(t+1)}\right] + s\lambda\left[\frac{p(t+1)}{P(t+1)} - \frac{p(t)}{P(t)}\right] \geq 0 \qquad (2.18)$$

where $P(-1) = 0$.

Proof: From the proof of Theorem 2.2

$$\Delta L(y) = h + h\frac{\lambda}{\mu}\left[\frac{P(y-1)}{P(y)} - \frac{P(y)}{P(y+1)}\right] + s\lambda\left[\frac{p(y+1)}{P(y+1)} - \frac{p(y)}{P(y)}\right]$$

thus

$$k + c + \Delta L(0) = k + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) \geq 0$$

proving (1). Part (2) follows immediately from the hypothesis and the fact that

$$\lim_{y\to\infty} k + c + \Delta L(y) = k + c + h > 0 .$$

Thus, the theorem is proven.

<u>Theorem 2.5</u>

Let assumptions (i) through (iv) hold. (1) If $d + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) \geq 0$ then $u = 0$. (2) If $d + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) < 0$ and $d + c + h + h\frac{\lambda}{\mu}\left[\frac{P(y-1)}{P(y)} - \frac{P(y)}{P(y+1)}\right] + s\lambda\left[\frac{p(y+1)}{P(y+1)} - \frac{p(y)}{P(y)}\right] \geq 0$ for some finite y then $0 < u < \infty$ and u is determined from

$$d + c + h + h\frac{\lambda}{\mu}\left[\frac{P(u-2)}{P(u-1)} - \frac{P(u-1)}{P(u)}\right] + s\lambda\left[\frac{p(u)}{P(u)} - \frac{p(u-1)}{P(u-1)}\right] < 0$$

$$d + c + h + h\frac{\lambda}{\mu}\left[\frac{P(u-1)}{P(u)} - \frac{P(u)}{P(u+1)}\right] + s\lambda\left[\frac{p(u+1)}{P(u+1)} - \frac{p(u)}{P(u)}\right] \geq 0 . \qquad (2.19)$$

(3) If $d + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) < 0$ and $d + c + h + h\frac{\lambda}{\mu}\left[\frac{P(y-1)}{P(y)} - \frac{P(y)}{P(y+1)}\right] + s\lambda\left[\frac{p(y+1)}{P(y+1)} - \frac{p(y)}{P(y)}\right] < 0$ for all finite y then $u = \infty$.

Proof: The difference $\Delta L(y)$ is computed in the proof of Theorem 2.4. Therefore

$$d + c + \Delta L(0) = d + c + (h - s\lambda)/(1 + \frac{\lambda}{\mu}) \geq 0 .$$

Using the definition of u in Section 2.2 proves all three parts of the theorem immediately.

Note that these theorems show that it is impossible to have $t > 0$ and $u = 0$. Also notice that the upper critical number is finite only if $-d \leq c + h$. Similarly, if $-d < c + h$ then the upper critical number is finite. It seems reasonable to conjecture that ordinarily in real problems the return cost will be less than the rental cost, $-d < c$.

2.6 An Example

Suppose an inventory system is described by the following values of parameters:

$\lambda = 10$	$d = -5$	$z = 2$
$\mu = 20$	$h = 1$	
$c = 7$	$s = 5$	
$k = 5$	$x = 4$	

Using part (2) of Theorem 2.4

$$k + c + \frac{h - s\lambda}{1 + \frac{\lambda}{\mu}} = -20.7$$

therefore $t > 0$ and t is determined by (2.18). The results of Table 2.1 facilitate the computation of the critical numbers.

TABLE 2.1

THE CALCULATION OF THE CRITICAL NUMBERS

y	p(y)	P(y)	P(y)/P(y+1)	p(y)/P(y)
0	0.61	0.61	0.67	1.00
1	0.30	0.91	0.92	0.33
2	0.08	0.99	0.99	0.08
3	0.01	1.00	1.00	0.01
4	0.00	1.00	1.00	0.00

It easily can be checked using Table 2.1 and (2.19) that $t = 1$. Before calculating u we note that $-d < c + h$ therefore u is finite. Again it can be checked easily using the above table and (2.19) that $u = 3$. The optimal policy described in Section 2.4 gives the number to rent during the period, y , to be 3 because $z \leq u$. That is, too many units currently are in inventory so one unit is returned.

The following table gives the value of the one period expected holding and shortage cost, $L(y)$, as calculated from (2.16).

TABLE 2.2

THE ONE PERIOD EXPECTED
HOLDING AND SHORTAGE COST

y	L(y)
0	50
1	17
2	5.5
3	2
4	3.5
5	4.5
6	5.5

The two critical numbers can be calculated directly from their definition in (2.4) by using Table 2.2. This method actually is simpler for this example and, as it must, it gives identical results.

The one period total expected cost when following the optimal policy can be computed from the first expression for $C_1(x , z)$ given in the proof of Theorem 2.2. It can be checked easily that this expected cost is 28. Using the same expression $C_1(x , z)$ can be calculated for any $x \geq z$ since in this example it reduces to

$$C_1(x , z) = \begin{cases} k(1 - x) + c + L(1) & x < 1 \\ cx + L(x) & 1 \leq x \leq 3 \\ \left.\begin{array}{ll} d(3 - x) + 3c + L(3) & \text{if } z \leq 3 \\ d(z - x) + zc + L(z) & \text{if } z > 3 \end{array}\right\} & x > 3 \end{cases}$$

for $x \geq z$. This expression in turn yields the entries in the following table, for $x \geq z$.

TABLE 2.3

THE ONE PERIOD COST FUNCTION

	$C_1(x,z)$		
x	$z \leq 3$,	$z = 4$	$z = 5$
0	29	-	-
1	24	-	-
2	19.5	-	-
3	23	-	-
4	28	31.5	-
5	33	36.5	39.5
6	38	41.5	44.5
7	43	46.5	49.5
8	48	51.5	54.5

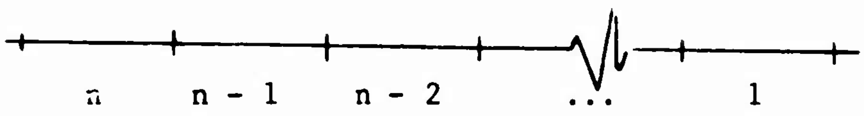
Consider two variations of this example. Suppose all of the parameters describing the inventory system are the same as above except that the initial numbers rented, x , and in use, z , are different. First, let $z = 0$ and $x = 2$. The optimal policy is to continue renting the same number of units because x is between the upper and lower critical numbers. The expected cost in this case is, from Table 2.3, 19.5. Second, let $z = 5$ and $x = 8$. Now the optimal policy is to return 3 units and to rent only 5 during the period. The expected cost is, in this case, 54.5.

3. The Multi-Period Model

3.1 Introduction

This chapter is devoted to extending the results of Chapter 2 to cover multi-period rental inventory problems. We consider an inventory system which operates for a number of period. Demands for equipment occur according to a Poisson process and the usage times of equipment in the field follow independent arbitrary distributions. The length of each period is assumed large enough so that the p. d. f. of the quantity of equipment in use at the end of the period is approximated very closely by the stationary distribution $\pi(k, y)$. Decisions must be made at the beginning of each period about the amount of inventory to rent. What is the optimal policy? It is shown in this chapter that the optimal policy is the logical extension of the one period optimal policy. In each period n there exist two critical numbers t_n and u_n $(t_n \leq u_n)$ such that if the amount of equipment rented at the beginning of the period is (1) larger than u_n then the new quantity rented is $\max(u_n, z)$, (2) smaller than t_n then the new quantity rented is t_n, (3) in between t_n and u_n then the same amount is rented.

A two period problem is considered first. When its solution is obtained the solution to a general n period problem becomes obvious. In everything that follows the standard dynamic programming convention of numbering time periods backwards is used. That is when considering an n period problem the current time period is called n, the next $n - 1$, and so on until the last which is called 1. Refer to Figure 3.1.



The Convention for Numbering Time Periods.

Figure 3.1

3.2 The Two Period Mathematical Model

The Mathematical model is developed for a two period problem in this section. Let $C_2(x , z)$ be the minimum expected cost incurred during two periods if x and z are the initial numbers of equipment rented and in use, respectively, at the beginning of period 2 (i.e., the present). This function is defined for $x \geq z$ only. The stochastic behavior of the inventory level at the end of the period is assumed to be described by the stationary p. d. f. $\pi(k , y)$ discussed in Section 2.5. The one period expected holding and shortage cost $L(y)$ is assumed to be discrete convex but not necessarily as given by (2.16). For example, the p. d. f. of the equipment in use at any time during the period may not be approximated very closely by the stationary p. d. f. $\pi(k , y)$. Or the holding and shortage costs may not be linear functions of the amount of equipment in the pool or the number of lost customers as assumed in the example of Section 2.5. By analogy with the one period model in Section 2.2 the total expected cost incurred during both periods is

$$a(y - x) + cy + L(y) + \alpha \sum_{j=0}^{y} C_1(y , j)\pi(j , y) \qquad (3.1)$$

where α is the discount factor, $0 \leq \alpha \leq 1$. We wish to determine the value of y for which (3.1) is the smallest, hence,

$$C_2(x , z) = \min_{y \geq z}\{a(y - x) + cy + L(y) + \alpha \sum_{j=0}^{y} C_1(y , j)\pi(j , y)\} \qquad (3.2)$$

The minimum is attained since $C_1(y , j) \geq 0$ for $y \geq j$.

Defining

$$M_1(y) = \sum_{j=0}^{y} C_1(y\ ,\ j)\pi(j\ ,\ y) \tag{3.3}$$

we can write (3.2) as

$$C_2(x,\ z) = \min_{y \geq z}\{a(y - x) + cy + L(y) + \alpha M_1(y)\}\ . \tag{3.4}$$

The subscript 1 will be suppressed when ever no confusion will arise, i.e., $M_1(y) = M(y)$.

Define two nonnegative real numbers t_2 and u_2 by

$$k(t_2 - x) + ct_2 + L(t_2) + \alpha M(t_2) = \min_{y \geq 0}\{k(y - x) + cy + L(y) + \alpha M(y)\}$$

$$\tag{3.5}$$

$$d(u_2 - x) + cu_2 + L(u_2) + \alpha M(u_2) = \inf_{y \geq 0}\{d(y - x) + cy + L(y) + \alpha M(y)\}\ .$$

The minimum can be used in the first equation of (3.5) because $k(y - x) + cy + L(y) + \alpha M(y) \to \infty$ as $y \to \infty$. However, the infimum must be used in the second equation of (3.5) because $d(y - x) \to -\infty$ as $y \to +\infty$. For convenience assume t_2 and u_2 are unique. Chapter 5 discusses the case when this is not true.

If $M(y)$ is discrete convex then the two critical numbers can be defined more explicitly, analogous with Section 2.2, by the conditions on the first differences of (3.1). In the standard inventory models considered in the literature once convexity is shown for the one period model it is shown easily for the multi-period model. We cannot show discrete convexity as easily because the p. d. f. $\pi(k\ ,\ y)$ depends on y (the standard models assume the p. d. f. independent of y). Therefore even if the one period cost $C_1(y\ ,\ k)$ is discrete convex in y we have no guarantee that $C_1(y\ ,\ k)\pi(k\ ,\ y)$ is discrete convex; and even if it is we cannot conclude that $M(y)$ is discrete convex because the limits of summation depend on y . We will show that in fact $M(y)$ is not necessarily discrete convex. Theorem 3.1 will show that in some cases it is and in others it is not.

Theorem 3.1

Suppose $\pi(k, y) = p(k)/P(y)$ for $0 \le k \le y$ and 0 otherwise. Let assumptions (i) through (iv) of Section 2.5 hold.

(1) If $1 < t < u + 1$ then

$$\Delta^2 M(y) = \begin{cases} 0 & 0 \le y < t - 1 \\ k + c + \Delta L(t) & y = t - 1 \\ \Delta^2 L(y) & t \le y < u - 1 \end{cases}$$

(2) If $t = u - 1$ then

$$\Delta^2 M(t) = [d + c + \Delta L(u)][\pi(u + 1, u + 1) - 1]$$

Proof: Calculating the first and second differences

$$\begin{aligned} \Delta M(y) &= M(y + 1) - M(y) \\ &= \sum_{j=0}^{y} [C_1(y + 1, j)\pi(j, y + 1) - C_1(y, j)\pi(j, y)] \\ &\quad + C_1(y + 1, y + 1)\pi(y + 1, y + 1) \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Delta^2 M(y) &= \Delta M(y + 1) - \Delta M(y) \\ &= \sum_{j=0}^{y} [C_1(y + 2, j)\pi(j, y + 2) + C_1(y, j)\pi(j, y) \\ &\quad - 2C_1(y + 1, j)\pi(j, y + 1)] \\ &\quad + C_1(y + 2, y + 2)\pi(y + 2, y + 2) + C_1(y + 2, y + 1)\pi \\ &\quad (y + 1, y + 2) - 2C_1(y + 1, y + 1)\pi(y + 1, y + 1) . \end{aligned} \tag{3.7}$$

The optimal policy given in Section 2.4 implies that if $j < t$ then

$$C_1(y\ ,\ j) = \begin{cases} k(t-y) + ct + L(t) & y < t \\ cy + L(y) & t \le y \le u \\ d(u-y) + cu + L(u) & y > u \end{cases} \tag{3.8}$$

and if $t \le j \le u$

$$C_1(y\ ,\ j) = \begin{cases} cy + L(y) & t \le j \le y \le u \\ d(u-y) + cu + L(u) & y > u \end{cases} \tag{3.9}$$

and if $j > u$

$$C_1(y\ ,\ j) = d(j-y) + cj + L(j) \qquad u < j \le y\ . \tag{3.10}$$

Using (3.8) through (3.10) we obtain, after simplification, for $0 < t < u$,

$$\Delta M(y) = \begin{cases} -k & 0 \le y < t \\ c + \Delta L(y) & t \le y < u \end{cases} \tag{3.11}$$

from which the first part of the theorem easily follows. Or, proceeding directly, by using (3.8) through (3.10) in (3.7) we obtain the same result. The last part of the theorem follows in a straight forward manner by using (3.8) through (3.10) in (3.7), after the cancellation of a few terms. Thus the theorem is proven.

It follows from this theorem that $M(y)$ is not necessarily discrete convex because $\Delta^2 M(t) < 0$ when $t = u - 1$. However, it is also true that in some cases $M(y)$ is discrete convex. For example, if $d + c + \Delta L(y) < 0$ for $y \ge 0$ then $u = \infty$. Thus $\Delta^2 M(y) \ge 0$ if $L(y)$ is discrete convex since $k + c + \Delta L(t) \ge 0$ by the definition of t .

Theorem 3.1 illustrates the fact that in some cases $M(y)$ definitely is or is not discrete convex. If $\Delta^2 M(y)$ is calculated for $y \geq u$ when $u < \infty$ expressions are obtained that are much more complex. It is difficult to determine the sign of $\Delta^2 M(y)$ in this case because of the complexity of the expressions. Points can be found, however, where the second difference is positive or negative depending on the shape of the expected holding and shortage cost function $L(y)$ even though $L(y)$ still is assumed discrete convex.

3.3 Solution of the Two Period Model

The solution to the two period model when $M(y)$ is discrete convex is presented in this section. The case when $M(y)$ is not discrete convex is discussed and solved in Chapter 5. For the remainder of this chapter $M(y)$ is assumed to be discrete convex. First we show that the two critical numbers t_2 and u_2 exist and that $t_2 \leq u_2$. Then the optimal policy is determined.

We follow Section 2.2. If $M(y)$ is discrete convex so is $a(y - x) + cy + L(y) + \alpha M(y)$ because $\alpha \geq 0$. Thus $k + c + \Delta L(0) + \alpha \Delta M(0) \geq 0$ if and only if $t_2 = 0$. Clearly $k + c + \Delta L(0) + \alpha \Delta M(0) < 0$ if and only if $0 < t < \infty$ in which case t_2 is determined by

$$k + c + \Delta L(t_2 - 1) + \alpha \Delta M(t_2 - 1) < 0 \leq k + c + \Delta L(t_2) + \alpha \Delta M(t_2) . \tag{3.12}$$

Similarly $d + c + \Delta L(0) + \alpha \Delta M(0) \geq 0$ if and only if $u_2 = 0$. Also $d + c + \Delta L(y) + \alpha \Delta M(y) < 0$ for $y = 0 , 1 , 2 , \ldots$ if and only if $u_2 = \infty$. Finally, $d + c + \Delta L(0) + \alpha \Delta M(0) < 0$ and $d + c + \Delta L(y) + \alpha \Delta M(y) \geq 0$ for some finite non-negative value of y when and only when $0 < u_2 < \infty$ in which case u_2 is determined from

$$d + c + \Delta L(u_2 - 1) + \alpha\Delta M(u_2 - 1) < 0 \leq d + c + \Delta L(u_2) + \alpha\Delta M(u_2) \ . \tag{3.13}$$

The two critical numbers t_2 and u_2 defined in (3.12) and (3.13) obviously exist and are uniquely determined. It is easy to see that $t_2 \leq u_2$.

Theorem 3.2

If assumptions (1), (2), and (3) of Section 2.2 are true and if $M(y)$ is discrete convex then $t_2 \leq u_2$.

Proof: Applying the method of proof used in Theorem 2.1 gives this theorem immediately.

Now we show that whenever the lower critical number from the one period model $t = t_1 > 0$ then the lower critical number from the two period model $t_2 > 0$.

Theorem 3.3

If $t = t_1 > 0$ then $t_2 > 0$.

Proof: From the proof of Theorem 3.1 we have $\Delta M(y) = -k$ for $0 \leq y < t$. By hypothesis $t > 0$ so that $k + c + \Delta L(0) < 0$ thus we have $k + c + \Delta L(0) + \alpha\Delta M(0) < 0$ proving the theorem.

When applying the analysis of Section 2.3 it is very easy to see that the optimal policy for the two period model has the same form as the one period optimal policy given in Section 2.4. That is,

(1) if $x > u_2$ then $y = \max(u_2 , z)$,

(2) if $t_2 \leq x \leq u_2$ then $y = x$,

(3) if $x < t_2$ then $y = t_2$,

where x and z are as defined at the beginning of this section and y is the quantity rented during the current period.

Theorem 3.4

Suppose $\pi(k , y) = p(k)/P(y)$ for $0 \le k \le y$ and 0 otherwise. Let assumptions (i) through (iv) of Section 2.5 hold. Let $L_2(y) = L(y) + \alpha M(y)$ and

$$M_2(y) = \sum_{j=0}^{y} C_2(y , j)\pi(j , y) .$$

(1) If $0 < t_2 < u_2 + 1$ then

$$\Delta^2 M_2(y) \begin{cases} 0 & 0 \le y < t_2 - 1 \\ k + c + \Delta L_2(t) & y = t_2 - 1 \\ \Delta^2 L_2(y) & t_2 \le y < u_2 - 1 \end{cases}$$

(2) If $t_2 = u_2 + 1$ then

$$\Delta^2 M_2(t_2) = [d + c + \Delta L_2(u_2)][\pi(u_2 + 1 , u_2 + 1) - 1] .$$

Proof: It is clear that the proof of Theorem 3.1 can be followed step by step changing only $C_1(x , z)$ to $C_2(x , z)$, $L(y)$ to $L_2(y)$, and $M(y)$ to $M_2(y)$.

3.4 The N Period Model and Its Solution

Now we discuss the general multi-period model and its solution. In a manner analogous with Section 3.2 let $C_n(x , z)$ be the minimum expected cost incurred during n periods if x and z are the initial numbers of equipment rented and in use, respectively (i.e., at the beginning of period n). This function is defined only for $x \ge z \ge 0$. The stochastic behavior of the inventory level at the end of the period is assumed to be described by the stationary p. d. f.

$\pi(k, y)$. However, as noted in the beginning of Section 3.2 $L(y)$ is assumed discrete convex but not necessarily as given by (2.16). The functional equations for and N period model are,

$$C_n(x, z) = \min_{y \geq z}\{a(y - x) + cy + L(y) + \alpha \sum_{j=0}^{y} C_{n-1}(y, j)\pi(j, y)\} \tag{3.14}$$

for $n = 1, 2, \ldots, N$ with $C_0(x, z) = 0$. The minimum in (3.14) exists because $C_{n-1}(y, j) \geq 0$ for $y \geq j$. Let

$$M_{n-1}(y) = \sum_{j=0}^{y} C_{n-1}(y, j)\pi(j, y)$$

and $L_n(y) = L(y) + \alpha M_{n-1}(y)$ for $n = 1, 2, \ldots$ with $M_0(y) = 0$.

Define two nonnegative real numbers t_n and u_n by

$$k(t_n - x) + ct_n + L(t_n) + \alpha M_{n-1}(t_n) = \min_{y \geq 0} \{k(y - x) + cy + L(y) + \alpha M_{n-1}(y)\} \tag{3.15}$$

$$d(u_n - x) + cu_n + L(u_n) + \alpha M_{n-1}(u_n) = \inf_{y \geq 0} \{d(y - x) + cy + L(y) + \alpha M_{n-1}(y)\} . \tag{3.16}$$

By this time it should be clear that the minimum can be used in (3.15) because $k(y - x) \to \infty$ as $y \to \infty$ but the infimum must be used in (3.16) because $d(y - x) \to -\infty$ as $y \to \infty$ For convenience assume t_n and u_n are unique. Chapter 5 discusses the case when this is not true. When $M_{n-1}(y)$ is discrete convex then the two critical numbers t_n and u_n can be defined more explicitly, as done previously for the one and two period problems.

Theorem 3.5

If all subscripts in Theorem 3.4 are changed from 2 to n then the theorem is still true for $n = 1, 2, 3, \ldots$ with $M_1(y) = M(y)$.

Proof: Identical with the proof of Theorem 3.1 except that $C_1(x, z)$, $L(y)$, and $M(y)$ must be changed to $C_{n-1}(x, z)$, $L_{n-1}(y)$, and $M_{n-1}(y)$.

This theorem indicates that a necessary condition for the discrete convexity of $M_n(y)$ is that $L_n(y) = L(y) + \alpha M_{n-1}(y)$ is discrete convex. Thus $M_{n-1}(y)$, $M_{n-2}(y), \ldots, M_1(y)$ must necessarily be discrete convex. The case when $u_n = \infty$ is a trivial example of a discrete convex $M_n(y)$. From Section 3.2 we know that there exist examples for which $M(y)$ is not discrete convex and hence there are cases when $M_n(y)$ is not discrete convex. We will consider only the case when $M_n(y)$ is discrete convex in the remainder of this chapter. Chapter 5 examines the problem when $M_n(y)$ is not discrete convex.

When $M_n(y)$ is discrete convex then the critical numbers can be defined in a manner identical with the definitions of t_2 and u_2 in Section 3.3, changing only the subscripts 2 to n. Theorems 3.2 and 3.3 can be extended.

Theorem 3.6

If assumptions (1), (2), and (3) of Section 2.2 are true and if $M_{n-1}(y)$ is discrete convex then $t_n \leq u_n$.

Proof: The method of proof used in Theorem 2.1 gives the theorem immediately.

Theorem 3.7

If $t = t_1 > 0$ then $t_n > 0$.

Proof: The proof is by induction. By Theorem 3.3 $t_2 > 0$. Assuming $t_{n-1} > 0$ we will show that $t_n > 0$. By hypothesis $t > 0$ so that $k + c + \Delta L(0) < 0$. But $\Delta M_{n-1}(0) = -k$ by Theorem 3.5 since $t_{n-1} > 0$. Thus $k + c +$

$\Delta L(0) + \alpha\Delta M(0) < 0$ which proves the theorem.

The two period optimal policy extends to the n period case. If at the beginning of the nth period there are x units of equipment rented and z units in use then the new amount of rented equipment y is

(1) if $x > u_n$ then $y = \max(u_n , z)$,

(2) if $t_n \leq x \leq u_n$ then $y = x$,

(3) if $x < t_n$ then $y = t_n$.

Theorem 3.8

If $L(y)$ and $M_n(y)$ are discrete convex then $C_{n+1}(x , z)$ is discrete convex for all $x \geq z$ for each value of z .

Proof: The optimal policy given above implies, for $x \geq z$,

$$C_{n+1}(x , z) = \begin{cases} k(t_{n+1} - x) + ct_{n+1} + L_{n+1}(t_{n+1}) & x < t_{n+1} \\ cx + L_{n+1}(x) & t_{n+1} \leq x \leq u_{n+1} \\ d(u_{n+1} - x) + cu_{n+1} + L_{n+1}(u_{n+1}) \quad \text{if } z \leq u_{n+1} & x > u_{n+1} \\ d(z - x) + cz + L_{n+1}(z) \quad \text{if } z > u_{n+1} & x > u_{n+1} \end{cases}$$

where $L_{n+1}(y) = L(y) + \alpha M_n(y)$ which is discrete convex. The remainder of the proof is almost identical with the proof of Theorem 2.2, the only difference being that $L(y)$ must be replaced by $L_{n+1}(y)$.

3.5 An Example

Suppose an inventory system operates for two periods and in each period the parameters have the same values given in the example of Section 2.6. Assume a discount factor of unity. The initial numbers rented and in use are 4 and 2 therefore $x = x_2 = 4$ and $z = z_2 = 2$. The first step in solving the two period problem is to solve the one period problem. This was done in Section 2.6. Thus $t_1 = 1$ and $u_1 = 3$. To determine the critical numbers t_2 and u_2 it is simplest computationally to use their definitions given in (3.5) rather than the conditions on the first differences. Using the definition of $M(y)$ in (3.3) and $\pi(j , y)$ in (2.13) we can make the following table:

TABLE 3.1

THE CALCULATION OF THE CRITICAL NUMBERS

	(1)	(2)	(3)	(4)	(5)	(6)
y	(k+c)y	(d+c)y	M(y)	L(y)+M(y)	(1)+(4)	(2)+(4)
0	0	0	29	79	79	79
1	12	2	24	41	53	43
2	24	4	19.5	25	49	29
3	36	6	23	25	61	31
4	48	8	28	31.5	79.5	39.5
5	60	10	33	37.7	97.5	47.5

Columns (5) and (6) of Table 3.1 give $t_2 = 2$ and $u_2 = 2$. Consequently the optimal policy for the two period problem is as follows. At the beginning of the current period $(x_2 = 4 , z_2 = 2)$ return 2 units and continue renting 2 during the period because $x_2 > u_2$ implies $y_2 = \max(u_2 , z_2) = 2$. Then at the beginning of the next period $x_1 = y_2 = 2$ is in between the critical numbers of the period $(t_1 = 1 , u_1 = 3)$ so that the same number will be rented during the period.

The two period total expected cost when this policy is followed, $C_2(x , z)$, can be computed by using the expression in the proof of Theorem 3.4. It can be checked that this expected cost is 49 . Using the same expression $C_2(x , z)$ can be calculated for any $x \geq z$ since in this example it reduces to

$$
C_2(x\ ,\ z) = \begin{cases} k(2 - x) + 2c + L(2) + M(2) & x \le 2 \\ \left.\begin{array}{ll} d(2 - x) + 2c + L(2) + M(2) & \text{if } z \le 2 \\ d(2 - x) + zc + L(z) + M(z) & \text{if } z > 2 \end{array}\right\} & x > 2\ . \end{cases}
$$

This expression in turn yields the entries in the following table, for $x \ge z$.

TABLE 3.2

THE ONE PERIOD COST FUNCTION

	$C_2(x\ ,\ z)$		
x	$z \le 2$	$z = 3$	$z = 4$
0	49	-	-
1	44	-	-
2	39	-	-
3	44	46	-
4	49	51	59.5
5	54	56	64.5
6	59	61	69.5
7	64	66	74.5
8	69	71	79.5

A variation of this example is considered. Assume all parameters of the inventory system are the same except the initial numbers rented and in use x_2 and z_2 are different. Suppose $x_2 = 8$ and $z_2 = 4$. The optimal policy is to return 4 units and to continue renting 4 units since $x_2 > u_2$ implies $y_2 = \max(u_2\ ,\ z_2) = z_2$. The expected cost associated with doing this is, from Table 3.2, 79.5 . At the beginning of the next period $x_1 = y_2 = 4 > u_1$ therefore $y_1 = \max(u_1\ ,\ z_1) = 4$ or 3 depending on whether or not $z_1 = x_1$ or $z_1 < x_1$, respectively.

4. The Infinite Period Model

4.1 Introduction

In this chapter an inventory system which operates for an infinite number of periods is considered. The mathematical model is formulated first. Then the solution is presented. It turns out to be similar to the solution for the single and multi-period models. The relationships between the multi-period and infinite period models are discussed next. We will see that in a sense the infinite period model is a simple approximation of an n period model for large n .

4.2 The Mathematical Model

We have a system which operates forever; there is never a last period. The problem which is faced at the beginning of one time period is identical with that faced at the beginning of any other time period. Let $C(x , z)$ be the minimum expected cost incurred during the operation of the inventory system if x and z are the initial quantities of equipment rented and in use, respectively. This function is defined for $x \geq z$ only. Demands for equipment occur according to a Poisson process and the usage times of equipment in the field follow independent arbitrary distributions. The length of each period is assumed large enough so that the p. d. f. of the quantity of equipment in use at the end of the period is approximated very closely by the stationary distribution. As noted in the beginning of Section 3.2 $L(y)$ is assumed discrete convex but not necessarily as given by (2.16). Expenses occur during each of an infinite number of time periods. In order to keep the sum of these expenses finite the discount factor α is assumed strictly less than one, i.e., $0 \leq \alpha < 1$.

When writing down the functional equation of the infinite period model the key fact to notice is that the future situation faced at the beginning of the next period is identical with the situation at the current time (except possibly the current state of the system will be different). Thus,

$$C(x , z) = \min_{y \geq z} \{a(y - x) + cy + L(y) + \alpha \sum_{j=0}^{y} C(y , j)\pi(j , y)\} \quad (4.1)$$

where $0 \leq \alpha < 1$. The single equation (4.1) describes the infinite period problem. A system of n equations, see (3.14), is needed to describe the n period problem. Thus the infinite period model is simpler than an n period model in the sense that only one equation is needed rather than a system of n equations. Because of this simplicity it would be very nice if by solving the infinite period model we could infer something about the solution of an n period model. A desirable relationship would be the knowledge that the solution to (4.1) approximates, in some sense, the solution to (3.14).

The solution of the infinite period model is discussed in the next section and then in the sections after that its relation to the solution of a multi-period model is analyzed.

4.3 Solution of the Infinite Period Model

The solution of the infinite period model is considered in this section. Define

$$M_\infty(y) = \sum_{j=0}^{y} C(y , j)\pi(j , y) .$$

Then rewrite (4.1) as

$$C(x , z) = \min_{y \geq z} \{a(y - x) + cy + L(y) + \alpha M_\infty(y)\} .$$

It will be shown in the next section that $M_\infty(y) = \lim M_n(y)$ as $n \to \infty$. Thus if

$M_n(y)$ is discrete convex for each n then $M_\infty(y)$ is discrete convex. Only this case is considered in this section; the more general case is discussed in the next chapter.

Define two nonnegative real nonnegative real numbers t_∞ and u_∞ by

$$k(t_\infty - x) + ct_\infty + L(t_\infty) + \alpha M_\infty(t_\infty) = \min_{y\geq 0} \{k(y - x) + cy + L(y) + \alpha M_\infty(y)\}$$

$$d(u_\infty - x) + cu_\infty + L(u_\infty) + \alpha M_\infty(u_\infty) = \inf_{y\geq 0} \{d(y - x) + cy + L(y) + \alpha M_\infty(y)\} .$$

The reasons for the usage of the minimum and the infimum are similar to the reasons for their use in the multi-period model in Section 3.2. As in that section, also, we assume t_∞ and u_∞ are unique. Chapter 5 discusses the case when this is not true. Since $M_\infty(y)$ is supposed to be discrete convex these two critical numbers can be defined more explicitly by the conditions on the first differences as done for the one and two period models. Theorem 3.6 can be extended easily to show that $t_\infty \leq u_\infty$. Because of the discrete convexity of $M_\infty(y)$ the optimal policy clearly is

(1) if $x > u_\infty$ then $y = \max(u_\infty , z)$,

(2) if $t_\infty \leq x \leq u_\infty$ then $y = x$,

(3) if $x < t_\infty$ then $y = t_\infty$,

where x and z are the amount of equipment rented and in use, respectively, at the beginning of the current period and y is the new amount rented.

4.4 Convergence, Existence, and Uniqueness Properties

The object of this section is threefold. First it is to show that the n period minimum cost function $C_n(x , z)$ converges as $n \to \infty$ monotonically and uniformly in x for each z , for all x in any finite interval. Second we show that the limit $C^*(x, z) = \lim C_n(x, z)$, as $n \to \infty$, satisfies the functional equation of the infinite period model (4.1). Third, any other solution to (4.1) which is bounded in finite intervals is identical with $C^*(x, z)$, i.e., the solution of (4.1) is unique among this class of bounded solutions. Thus the limit $C^*(x, z)$ is the minimum expected cost for the infinite period model, i.e., $C^*(x, z) = C(x, z)$. Similar results have been obtained in the literature for standard inventory models; see Dvoretzky, Kiefer, and Wolfowitz [1], Karlin [14], Bellman, Glicksberg, and Gross [15], Bellman [16], Whisler and Parikh [17], Igelhart [18], and Abrams [19]. The following theorems are based on these results.

Theorem 4.1

For all x in any finite interval $\lim C_n(x, z) = C^*(x, z)$, as $n \to \infty$, exists for each z . Moreover, the convergence is monotone and uniform in x for each z and if $L(y)$ and $M_n(y)$ are discrete convex then $C^*(x, z)$ is discrete convex.

Proof: The proof proceeds in three parts. We first show that $C_n(x, z) \geq C_{n-1}(x, z)$. Then by showing that $C_n(x, z)$ is bounded from above we conclude that $\lim C_n(x, z)$ exists. The final step is showing that the convergence is uniform.

(i) Monotonicity of $C_n(x, z)$. The proof is by induction. Let $A(y, x) = a(y - x) + cy + L(y)$. Then

$$C_n(x, z) = \min_{y \geq z} \{A(y, x) + \alpha \sum_{j=0}^{y} C_{n-1}(y, j)\pi(j, y)\}$$

for $n = 1, 2, \ldots$ with $C_0(x, a) = 0$ by definition. Let

$$T(y , x , C_{n-1}) = A(y , x) + \alpha \sum_{j=0}^{y} C_{n-1}(y , j)\pi(j , y) \tag{4.2}$$

$$T(y_n^* , x , C_{n-1}) = \min_{y \geq z} T(y , x , C_{n-1}) \tag{4.3}$$

where y_n^* is the optimal policy in period n . Therefore we have $C_n(x , a) = T(y_n^* , x , C_{n-1})$ for $n = 1 , 2 , \ldots$. By definition $C_1(x , z) \geq C_0(x , z) = 0$ for all $x \geq z$. Assuming $C_n(x , z) \geq C_{n-1}(x , z)$ for $x \geq z$ we will show that $C_{n+1}(x , z) \geq C_n(x , z)$. We have

$$\begin{aligned} C_{n+1}(x , z) - C_n(x , z) &= T(y_{n+1}^* , x , C_n) - T(y_n^* , x , C_{n-1}) \\ &\geq T(y_{n+1}^* , x , C_n) - T(y_{n+1}^* , x , C_{n-1}) \\ &= \alpha \sum_{j=0}^{y_{n+1}^*} [C_n(y_{n+1}^* , j) - \\ &\quad C_{n-1}(y_{n+1}^* , j)]\pi(j , y_{n+1}^*) \\ &\geq 0 \end{aligned}$$

by the inductive hypothesis. Thus we have proven that $C_n(x , z)$ is monotone increasing.

(ii) Boundedness of $C_n(x , z)$. We show that the $C_n(x , z)$ are bounded from above by a finite number for all n .

$$\begin{aligned} C_n(x , z) = T(y_n^* , x , C_{n-1}) &\leq T(x , x , C_{n-1}) \\ &= cx + L(x) + \alpha \sum_{j=0}^{y} C_{n-1}(x , j)\pi(j , x) . \end{aligned} \tag{4.4}$$

Similarly

$$C_{n-1}(x , z) \leq cx + L(x) + \alpha \sum_{j=0}^{x} C_{n-2}(x , j)\pi(j , x) . \tag{4.5}$$

Substituting (4.5) into (4.4) and continuing the iterative process yields

$$C_n(x , z) = cx + L(x) + \alpha[cx + L(x)] + \dots + \alpha^{n-1}[cx + L(x)] . \tag{4.6}$$

From (4.6) we conclude that, for $n = 0 , 1 , 2 , \dots ,$

$$C_n(x , z) \leq \frac{cx + L(x)}{1 - \alpha} \leq \frac{M}{1 - \alpha} < \infty$$

for $0 \leq \alpha < 1$ and $0 \leq z \leq x \leq X < \infty$ where X is a very large fixed real number. It is so large that the number of rented units never exceeds it (i.e., $y_n < X$ for all n); in reality such a number must exist. Thus there exists a limit to which $C_n(x , z)$ converges. Define $C^*(x , z) = \lim C_n(x , z)$ for $x \geq z$. By Theorem 3.8 each $C_n(x , z)$ is discrete convex therefore $C^*(x , z)$ is also because it is just the limit of discrete convex functions.

(iii) Uniformity of Convergence. First it is shown that

$$C_n(x , z) - C_{n-1}(x , z) \leq \alpha^{n-1}M \tag{4.7}$$

for $n = 1 , 2 , \dots$ where M is as defined in (ii). The proof of this fact is by induction. We have

$$C_1(x , z) - C_0(x , z) = T(y_1^* , x , C_0) \leq T(x , x , C_0) \leq M < \infty$$

for x in the interval $[z , X]$. Assuming (4.7) true we look at

$$\begin{aligned} C_{n+1}(x , z) - C_n(x , z) &= T(y_{n+1}^* , x , C_n) - T(y_n^* , x , C_{n-1}) \\ &\leq T(y_n^* , x , C_n) - T(y_n^* , x , C_{n-1}) \\ &= \alpha \sum_{j=0}^{y_n^*} [C_n(y_n^* , j) - C_{n-1}(y_n^* , j)\pi(j , y_n^*) . \end{aligned}$$

Using the inductive hypothesis we obtain

$$C_{n+1}(x , z) - C_n(x , z) \leq \alpha^n M .$$

Thus (4.7) holds for $n = 1 , 2 , \ldots$ as claimed. But if $0 \leq \alpha < 1$ then

$$\sum_{n=0}^{\infty} \alpha^n M = \frac{M}{1 - \alpha} < \infty . \tag{4.8}$$

To show that the convergence is uniform we look at the series

$$\sum_{n=0}^{\infty} [C_{n+1}(x , z) - C_n(x , z)] \tag{4.9}$$

and note that by (4.8) each bracketed term is bounded from above by a real number such that the sum of all of the real numbers is finite. We conclude that the series (4.9) converges uniformly. Thus by definition $C_n(x , z)$ converges uniformly. This completes the proof of the theorem.

From this theorem we see that $|C_{n+1}(x , z) - C_n(x , z)| \to 0$ as $n \to \infty$ for each $z \leq X$ and all x in $[z , X]$ where X is the large finite upper bound on the inventory level defined in the theorem.

Theorem 4.2

$C^*(x , z)$ satisfies (4.1).

Proof: The proof proceeds in two steps. First it will be shown that

$$\lim_{n\to\infty} C_n(x , z) = C^*(x , z) \leq \min_{y\geq z} T(y , x , C^*) \tag{4.10}$$

$$\lim_{n\to\infty} C_n(x , z) = C^*(x , z) \geq \min_{y\geq z} T(y , x , C^*) . \tag{4.11}$$

To show (4.10) we note that

$$C_n(x\ ,\ z) = \min_{y\geq z} T(y\ ,\ x\ ,\ C_{n-1}) \leq \min_{y\geq z} T(y\ ,\ x\ ,\ C^*)$$

because $C_n(x\ ,\ z) \leq C^*(x\ ,\ z)$ by Theorem 4.1. Letting $n \to \infty$ gives (4.10).
To show (4.11) we note

$$C^*(x\ ,\ z) \geq C_n(x\ ,\ z) = \min_{y\geq z} T(y\ ,\ x\ ,\ C_{n-1}) = T(y^*\ ,\ x\ ,\ C_{n-1})$$

where y^* is the minimizing value of y. Letting $n \to \infty$ gives

$$C^*(x\ ,\ z) \geq \lim_{n\to\infty} T(y^*\ ,\ x\ ,\ C_{n-1}) \geq \min_{y\geq z} \lim_{n\to\infty} T(y\ ,\ x\ ,\ C_{n-1})$$

$$= \min_{y\geq z} T(y\ ,\ x\ ,\ C^*)$$

by Theorem 4.1, thus proving (4.11).

Theorem 4.3

Any solution $g(x\ ,\ z)$ to (4.1) which is bounded in finite intervals is identical with $C^*(x\ ,\ z)$, i.e., $g(x\ ,\ z) = C^*(x\ ,\ z)$.

Proof: Let $g(x\ ,\ z)$ be any other solution to (4.1) that is bounded in $[z\ ,\ X]$. We proceed by induction in a method similar to the proof of Theorem 4.1. By the hypothesis

$$|g(x\ ,\ z) - C_0(x\ ,\ z)| \leq M' < \infty$$

for x in $[z\ ,\ X]$. Now we assume that

$$|g(x\ ,\ z) - C_{n-1}(x\ ,\ z)| \leq \alpha^{n-1} M' .$$

Then

$$g(x\ ,\ z) - C_n(x\ ,\ z) = T(\bar{y}\ ,\ x\ ,\ g) - T(y_n\ ,\ x\ ,\ C_{n-1})$$
$$\leq T(y_n, x\ ,\ g) - T(y_n\ ,\ x\ ,\ C_{n-1})$$

and

$$g(x\ ,\ z) - C_n(x\ ,\ z) \geq T(\bar{y}\ ,\ x\ ,\ g) - T(\bar{y}\ ,\ x\ ,\ C_{n-1})$$

where $\bar{y}$ is the minimizing value of y . Hence

$$|g(x\ ,\ z) - C_n(x\ ,\ z)| \leq \max \{|T(y_n\ ,\ x\ ,\ g) - T(y_n\ ,\ x\ ,\ C_{n-1})|\ ;\ |T(\bar{y}\ ,\ x\ ,\ g) - T(\bar{y}\ ,\ x\ ,\ C_{n-1})|\}$$

$$\leq \max \{\alpha \sum_{j=0}^{y_n} |g(y_n\ ,\ j) - C_{n-1}(y_n\ ,\ j)|\pi(j\ ,\ y_n)\ ;\ \alpha \sum_{j=0}^{\bar{y}} |g(\bar{y}\ ,\ j) - C_{n-1}(\bar{y}\ ,\ j)|\pi(j\ ,\ \bar{y})\}$$

which is by the inductive assumption

$$|g(x\ ,\ z) - C_n(x\ ,\ z)| \leq \alpha^n M' \ .$$

Since $0 \leq \alpha < 1$ we conclude that the infinite series

$$\sum_{n=0}^{\infty} [g(x\ ,\ z) - C_n(x\ ,\ z)]$$

converges absolutely and uniformly, therefore $C_n(x\ ,\ z)$ converges uniformly to $g(x\ ,\ z)$ which implies that $C^*(x\ ,\ z)$ is identical with $g(x\ ,\ z)$ on the finite interval $z \leq x \leq X$, i.e., $C^*(x\ ,\ z) = g(x\ ,\ z)$.

The conclusions of the above three theorems have interesting implications.

In practical situations the results of the infinite period model can serve as an approximate solution to multi-period problems when the number of periods is sufficiently large. Such an approximation is useful because for infinite period problems only one functional equation has to be solved whereas for multi-period models many functional equations must be solved (n for an n period problem). Any reduction in effort by solving only one rather than many functional equations is an attractive feature of the infinite period formulation. Also, the structure of the optimal policy for an infinite period model can be considered simpler than the structure of the optimal policy of a multi-period model in the following sense. The critical numbers in an infinite period problem remain the same from period to period whereas in a multi-period problem the critical numbers may be different each period. Using the solution of the infinite period model in a multi-period problem naturally is not optimal. However, the theorems of this section show that such a policy is approximately optimal for large n. The method of proof in Theorem 4.3 provides a bound on the difference between the costs of the two policies.

4.5 Limiting Behavior of the Critical Numbers

The critical numbers t_n and u_n are discussed in Chapter 3. Their relationship to the critical numbers t_∞ and u_∞ defined in Section 4.3 are analyzed in this section. We will show that under certain conditions the sequence $\{t_n\}$ is monotone increasing and converges to t_∞. However, only the weaker conclusion that u_n converges to u_∞, under certain conditions can be shown.

Theorem 4.4

The sequence $\{t_n\}$ is monotone increasing and bounded from above by t_∞ if

$M_n(y)$ is discrete convex for $n = 1 , 2 , \ldots$.

Proof: From (3.11) $\Delta M_n(y) = -k$ for $y < t_n$ therefore using discrete convexity gives,

$$\Delta M_n(y) = \begin{cases} -k & y < t_n \\ \geq -k & y \geq t_n . \end{cases} \tag{4.12}$$

The definitions of t_n and t_{n+1} are, for $n = 1 , 2 , \ldots$,

$$k + c + \Delta L(t_n - 1) + \alpha \Delta M_{n-1}(t_n - 1) < 0 \leq k + c + \Delta L(t_n) + \alpha \Delta M_{n-1}(t_n) \tag{4.13}$$

$$k + c + \Delta L(t_{n+1} - 1) + \alpha \Delta M_n(t_{n+1} - 1) < 0 \leq k + c + \Delta L(t_{n+1}) + \alpha \Delta M_n(t_{n+1}) . \tag{4.14}$$

Now we show that $t_{n+1} \geq t_n$ for any positive integral value of n . Clearly this is true if $t_n = 0$ so assume $t_n > 0$. Using (4.12) in (4.13) and (4.14) yields,

$$k + c + \Delta L(t_n - 1) - \alpha k \leq k + c + \Delta L(t_n - 1) + \alpha \Delta M_{n-1}(t_n - 1)$$
$$< 0 \leq k + c + \Delta L(t_{n+1}) + \alpha \Delta M_n(t_{n+1}) .$$

If $t_{n+1} < t_n$ then by (4.12) $\Delta M_n(t_{n+1}) = -k$ so that

$$k + c + \Delta L(t_n - 1) - \alpha k < k + c + \Delta L(t_{n+1}) - \alpha k$$

which implies $t_n \leq t_{n+1}$ by the discrete convexity of $L(y)$ which is a contradiction. Thus for $n = 1 , 2 , \ldots$ $t_n \leq t_{n+1}$.

The definition of t_∞ is

$$k + c + \Delta L(t_\infty - 1) + \alpha \Delta M_\infty(t_\infty - 1) < 0 \leq k + c + \Delta L(t_\infty) + \alpha \Delta M_\infty(t_\infty) \quad (4.15)$$

Using the same method of proof as above shows that $t_n \leq t_\infty$ since (4.12) holds for $n = \infty$. The proof is complete.

Theorem 4.4 implies that $\lim t_n = t'$, as $n \to \infty$, exists since $t_n < \infty$. We will show that in fact $t' = t_\infty$.

Theorem 4.5

If $M_n(y)$ is discrete convex for $n = 1, 2, \ldots$ then $\lim t_n = t' = t_\infty$, as $n \to \infty$.

Proof: Let

$$h_x(y, h) = k(y - x) + cy + L(y) + \alpha M_{n-1}(y)$$

then

$$h_x(t_n, n) \leq h_x(y, n)$$
$$h_x(t_\infty, \infty) \leq h_x(y, \infty)$$

for all $y \geq 0$. But $C_n(y, z) \nearrow C(y, z)$ by Theorem 4.1 therefore $M_n(y) \nearrow M(y)$ which implies that $h_x(y, n) \nearrow h_x(y, \infty)$ as $n \to \infty$. Thus

$$h_x(t_n, n) \leq h_x(y, n) \leq h_x(y, \infty) .$$

Letting $n \to \infty$ gives

$$h_x(t', \infty) \leq h_x(y, \infty)$$

for all $y \geq 0$ which implies that $t' = t_\infty$, proving the theorem since t_∞ is unique.

Theorem 4.6

If $M_n(y)$ is discrete convex for $n = 1, 2, \ldots$ and all u_n are finite then $\lim u_n = u' = u_\infty$, as $n \to \infty$.

Proof: By hypothesis all u_n are bounded thus there exists a convergent subsequence of the sequence $\{u_n\}$. An argument similar to the one used in Theorem 4.5 shows

$$g_x(u_n , n) \leq g_x(y , \infty)$$

where

$$g_x(y , n) = d(y - x) + cy + L(y) + \alpha M_{n-1}(y) .$$

Hence any limit point of the sequence $\{u_n\}$ minimizes $g_x(y , \infty)$. But $g_x(y , \infty)$ has a unique minimum by assumption, therefore the sequence $\{u_n\}$ has exactly one limit point. Thus $\lim u_n = u'$ exists and $u' = u_\infty$ proving the theorem, Graves [20], page 49.

5. A Model for Discrete Non-Convex Cost Functions

5.1 Introduction

All of the problems studied up to this time assume that the relevant cost functions are discrete convex. Although such cases do occur in reality, a simple example is given in Section 3.2 in which this is not true. The present chapter considers the rental inventory problem when no assumptions are made about the one period expected holding and shortage cost $L_z(y)$ other than it being nonnegative. If a function is not discrete convex it will be called discrete non-convex. Note that in addition to $L_z(y)$ being arbitrary it is now permitted to depend on the initial number of units in use, z. In Section 5.8 we will see that such a case can occur if the nonstationary transition probabilities $p_{z,k}(t, y)$ are used to calculate $L_z(y)$ rather than the stationary probabilities $\pi(k, y)$.

As in the previous chapters the one period model and its solution forms the basis for the multi and infinite period models. Thus it is discussed first and in the greatest detail. The variations of this solution which occur when $L_z(y)$ is assumed discrete convex (but still a function of z) and when $L(y)$ is assumed discrete non-convex and independent of z are discussed. Also, we explore the relationships between the optimal policy previously obtained for the discrete convex case and the optimal policy of this section for the discrete non-convex case. The multi and infinite period models are discussed next. Finally, we discuss the calculation of $L_z(y)$ when using $p_{z,k}(t, y)$ in the special case of Poisson arrivals and negative exponential usage times.

5.2 The One Period Model

In this section the one period model is formulated first. Then it is shown that there exist critical numbers t_z and u_z, $z = 0, 1, 2, \ldots$, such that

$t_z \leq u_z$ where z is the initial quantity of equipment in use.

Following the method of Section 2.2 the one period functional equation is

$$C_1(x , z) = \min_{y \geq z} \{a(y - x) + cy + L_z(y)\} \tag{5.1}$$

for $x \geq z \geq 0$. The only assumptions made are that $k \geq 0$, $d \leq 0$, $c \geq 0$, $k + c > 0$ and $L_z(y) \geq 0$. Consequently, the minimum in (5.1) exists.

Define two sequences of real numbers $\{t_z\}$ and $\{u_z\}$, $z = 0 , 1 , 2 , \ldots$ by

$$k(t_z - x) + ct_z + L_z(t_z) = \min_{y \geq z}\{k(y - x) + cy + L_z(y)\} \tag{5.2}$$

$$d(u_z - x) + cu_z + L_z(u_z) = \inf_{y \geq z}\{d(y - x) + cy + L_z(y)\} . \tag{5.3}$$

The reasons for the usage of the minimum and the infimum are similar to the reasons for their use in the multi-period model in Section 3.2. For convenience assume that t_z and u_z are uniquely defined. If not some ancillary rule may be used for choosing one unique value, for each z . One such rule, for example, is

$$t_z = \min_{1 \leq i \leq I_z} \{t_z^{(i)}\}$$

$$u_z = \min_{1 \leq j \leq J_z} \{u_z^{(j)}\}$$

where $\{t_z^{(i)} : i = 1 , 2 , \ldots , I_z\}$ and $\{u_z^{(j)} : j = 1 , 2 , \ldots , J_z\}$ are, for each z , all values which satisfy (5.2) and (5.3), respectively. Both critical numbers exist for each z but $t_z < \infty$ while u_z may be infinite since $k \geq 0$ and $d \leq 0$. In the follwoing theorem we show that $t_z \leq u_z$.

Theorem 5.1

If $L_z(y) \geq 0$, $k \geq 0$, and $d \leq 0$ then $t_z \leq u_z$.

Proof: It can be assumed that u_z is finite otherwise the theorem obviously is true. In this case the greatest lower bound in (5.3) is attined and inf may be replaced by min . Define y_z^* by

$$cy_z^* + L_z(y_z^*) = \min_{y \geq z} [cy + L_z(y)] \ . \tag{5.4}$$

The proof proceeds in two parts. First it is shown that $t_z \leq y_z^*$ and then it is shown that $y_z^* \leq u_z$; the theorem then follows.

(i) Proof that $t_z \leq y_z^*$. For all $y \geq y_z^*$

$$k(y - x) + cy + L_z(y) \geq k(y_z^* - x) + cy_z^* + L_z(y_z^*) \tag{5.5}$$

because $k \geq 0$. Comparison of (5.5) with (5.2) gives that $t_z \leq y_z^*$. That is, if $t_z > y_z^*$ then by (5.5)

$$k(t_z - x) + ct_z + L_z(t_z) \geq k(y_z^* - x) + cy_z^* + L_z(y_z^*)$$

and by (5.2) the opposite inequality holds so that

$$k(t_z - x) + ct_z + L_z(t_z) = k(y_z^* - x) + cy_z^* + L_z(y_z^*) \ .$$

But $t_z > y_z^*$ so that if $k \neq 0$

$$ct_z + L_z(t_z) < cy_z^* + L_z(y_z^*)$$

contradicting the definition of y_z^* . If $k = 0$ then from (5.2) and (5.4) $t_z = y_z^*$, a contradiction.

(ii) Proof that $y_z^* \leq u_z$. For all $y \leq y_z^*$

$$d(y_z^* - x) + cy_z^* + L_z(y_z^*) \leq d(y - x) + cy + L_z(y) \qquad (5.6)$$

because $d \leq 0$. Comparing (5.6) with (5.3) implies that $y_z^* \leq u_z$. That is, if $u_z < y_z^*$ then by (5.6)

$$d(y_z^* - x) + cy_z^* + L_z(y_z^*) \leq d(u_z - x) + cu_z + L_z(u_z) .$$

Equation (5.3) implies the opposite inequality so that

$$d(y_z^* - x) + cy_z^* + L_z(y_z^*) = d(u_z - x) + cu_z + L_z(u_z) .$$

But $u_z < y_z^*$ so that if $d \neq 0$

$$cy_z^* + L_z(y_z^*) > cu_z + L_z(u_z)$$

contradicting the definition of y_z^* . If $d = 0$ then from (5.3) and (5.4) $u_z = y_z^*$ furnishing the desired contradicition. This completes the proof.

5.3 Solution of the One Period Model

The solution of the one period model is derived in this section. At the beginning of a period for each value of z there are three possible situations. The initial number of rented units x is larger than u_z , less than t_z , or in between t_z and u_z . In what follows all critical numbers are assumed positive and finite. When u_z is allowed to be infinite or t_z is zero the required modifications will become clear.

Case (1): $x > u_z$. Let

$$h_x(y , z) = a(y - x) + cy + L_z(y) .$$

Then

$$C_1(x , z) = \min_{y \geq z} \{h_x(y , z)\} .$$

Fix $x > u_z$. When $y < x$

$$h_x(y , z) = d(y - x) + cy + L_z(y)$$

and when $y \geq x$

$$h_x(y , z) = k(y - x) + cy + L_z(y) \geq d(y - x) + cy + L_z(y) .$$

Thus

$$\min_{z \leq y \leq x} \{h_x(y , z)\} = \min_{z \leq y \leq x} \{d(y - x) + cy + L_z(y)\}$$

and since $x > u_z$ the minimum occurs at $y = u_z$ by the definition of u_z . Also

$$\min_{y \geq x} \{h_z(y , z)\} = \min_{y \geq x} \{k(y - x) + cy + L_z(y)\}$$

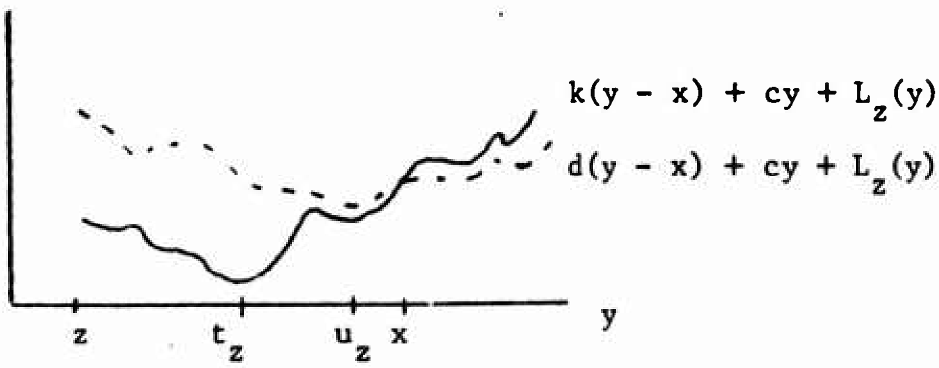
$$\geq \min_{y \geq x} \{d(y - x) + cy + L_z(y)\}$$

$$\geq d(u_z - x) + cu_z + L_z(u_z)$$

again by the definition of u_z . Thus we have

$$\min_{y \geq z} \{h_x(y , z)\} = h_x(u_z , z) .$$

The optimal policy is, whenever $x > u_z$ set $y = u_z$. We note that if $u_z = \infty$ then this case cannot occur. This case is illustrated in Figure 5.1.



The Cost Function when $x > u_z$.

Figure 5.1

<u>Case (2)</u>: $x < t_z$. For a fixed $x < t_z$ we have when $y \leq x$

$$h_x(y\ ,\ z) = d(y - x) + cy + L_z(y) \geq k(y - x) + cy + L_z(y)$$

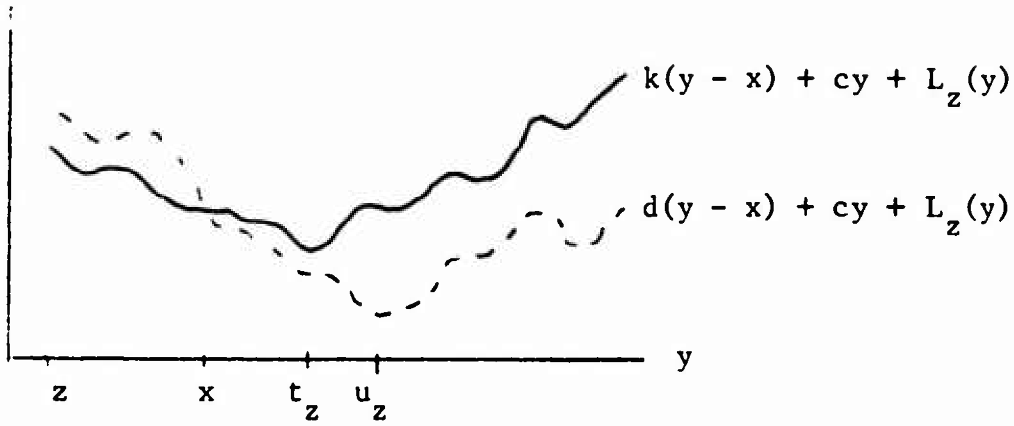
and when $y > x$

$$h_x(y\ ,\ z) = k(y - x) + cy + L_z(y)\ .$$

Using the same method as in case (1) and the definition of t_z implies

$$C_1(x\ ,\ z) = \min_{y \geq z} \{h_x(y\ ,\ z)\} = h_x(t_z\ ,\ z)\ .$$

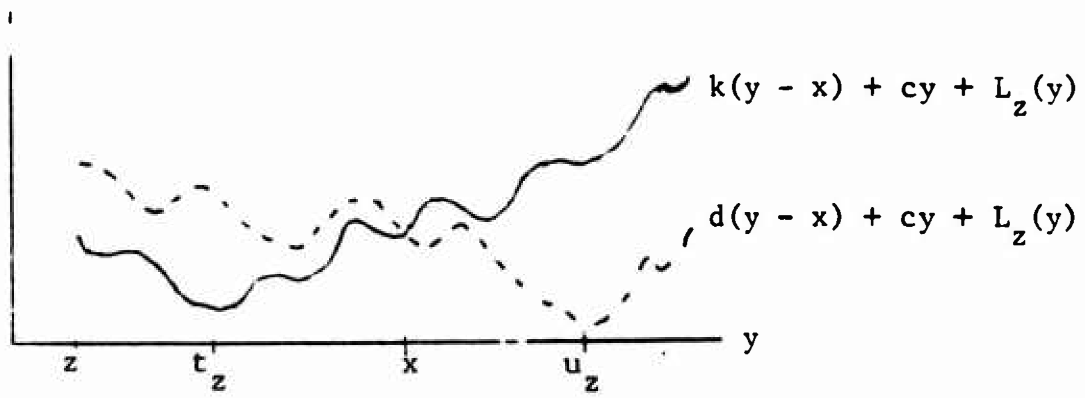
Thus the optimal policy is to set $y = t_z$ whenever $x < t_z$. We note that this case cannot occur if $t_z = 0$. This case is illustrated in Figure 5.2.



The Cost Function when $x < t_z$.

Figure 5.2

Case (3): $t_z \leq x \leq u_z$. This case is a little different. A typical example of $h_x(y , z)$ is shown in Figure 5.3.



The Cost Function when $t_z \leq x \leq u_z$.

Figure 5.3

Note that

$$h_x(y , z) = \max\{k(y - x) + cy + L_z(y) ; d(y - x) + cy + L_z(y)\} .$$

If $h_x(y , z)$ is discrete convex then from Figure 5.3 the optimal policy clearly is to set $y = x$. But when $h_x(y , z)$ is not discrete convex its minimum does not necessarily occur at x . Let $v_z(x)$ represent the value of y which minimizes $h_x(y, z)$. Then the optimal policy is $y = v_z(x)$. It is evident that $v_z(x)$ exists for each z and x. If we define $T_z^{(j)}$ by

$$k(T_z^{(j)} - x) + cT_z^{(j)}) + L_z(T_z^{(j)}) = c_x + L_z(x)$$

and

$$T_z^{(j)} \geq x$$

then $T_z^{(j)}$, $j = 1, 2, \ldots, r$, are all of the y values larger than x for which $k(y - x) + cy + L_z(y) = cx + L_z(x)$. By assumption $1 \leq r < \infty$. Let

$$T_z = \max_{1 \leq j \leq r} \{T_z^{(j)}\} .$$

In the same manner define $U_z^{(j)}$ and U_z by

$$d(U_z^{(j)} - x) + cU_z^{(j)} + L_z(U_z^{(j)}) = cx + L_z(x)$$

and

$$z \leq U_z^{(j)} \leq x \qquad j = 1, 2, \ldots, s < \infty$$

$$U_z = \min_{1 \leq j \leq s} \{U_z^{(j)}\} .$$

That is $U_z^{(j)}$, $j = 1, 2, \ldots, s$, are all of the y values less than x for which $d(y - x) + cy + L_z(y) = cx + L_z(x)$, and U_z is the smallest of these values.

Then $v_z(x)$ may be redefined by

$$h_x(v_z(x), z) = \min_{u_z \leq y \leq T_z} \{h_x(y, z)\} .$$

5.4 The Optimal Policy and Some Variations

In this section we summarize the optimal policy obtained above. Then we discuss variations of this policy which occur when various assumptions are made about the one period expected holding and shortage cost function, $L_z(y)$. The assumptions considered are: (1) $L_z(y)$ is discrete convex, (2) $L_z(y)$ is not discrete convex, (3) $L_z(y)$ is independent of z , i.e., $L_z(y) = L(y)$, and (4) $L_z(y)$ is a function of z . Suppose $L_z(y)$ is not discrete convex and it is a function of z . If the number of units of equipment being rented at the beginning of a period is x and if the number of units in use at the beginning of the period is $z \leq x$ then the optimal policy is, whenever $0 < t_z \leq u_z < \infty$,

(1) if $x > u_z$ then $y = u_z$,

(2) if $t_z \leq x \leq u_z$ then $y = v_z(x)$,

(3) if $x < t_z$ then $y = t_z$.

If $h_x(y , z) - h_x(v_z(x) , z) \geq 0$ is small for all y in the interval $[U_z , T_z]$ then there is a simpler policy which is nearly optimal:

(1) if $x > u_z$ then $y = u_z$,

(2) if $t_z \leq x \leq u_z$ then $y = x$,

(3) if $x < t_z$ then $y = t_z$.

If $h_x(y , z)$ is a fairly smooth function this policy will give approximately the least cost. If $h_x(y , z)$ is discrete convex in $y \geq z$ then this policy is the exact optimal policy. A sufficient condition for $h_x(y , z)$ to be discrete convex is that $L_z(y)$ is discrete convex in $y \geq z$ for each z .

If t_z is allowed to assume the value 0 and u_z is permitted to take the values 0 or ∞ then the modification of the optimal policy clearly is analogous with the results of Section 2.3.

The optimal policy obtained in this chapter should simplify to the optimal policy of Chapter 2 when $L_z(y)$ is assumed to be independent of z (i.e., $L_z(y) = L(y)$) and discrete convex. That this is the case is now shown.

Assume $L_z(y)$ is independent of z and discrete convex. Then

$$\min_{y \geq z} \{k(y - x) + cy + L(y)\} = k[\max(t , z) - x] + c \cdot \max(t , z) + L(\max(t , z))$$

$$\inf_{y \geq z} \{d(y - x) + cy + L(y)\} = d[\max(u , z) - x] + c \cdot \max(u , z) + L(\max(u , z))$$

where t and u are defined by

$$k(t - x) + ct + L(t) = \min_{y \geq 0} \{k(y - x) + cy + L(y)\}$$

$$d(u - x) + cu + L(u) = \inf_{y \geq 0} \{d(y - x) + cy + L(y)\} .$$

Consequently from (5.2) and (5.3) we have

$$t_z = \max (t , z)$$

$$u_z = \max (u , z) .$$

The definitions of t and u given above are identical with the definitions given in Section 2.2. Because $L(y)$ is discrete convex the approximate optimal policy given in this section is the exact optimal policy:

(1) if $x > u_z$ then $y = \max(u , z)$,

(2) if $t_z \leq x \leq u_z$ then $y = x$,

(3) if $x < t_z$ then $y = \max(t , z)$.

This optimal policy can be simplified to the optimal policy of Section 2.4 by utilizing the following three theorems:

Theorem 5.2

$$x < t_z = \max(t , z) \text{ if and only if } x < t .$$

Theorem 5.3

$$x \geq u_z = \max(u , z) \text{ if and only if } x \geq u .$$

Theorem 5.4

$$t_z = \max(t , z) \leq x < u_z = \max(u , z) \text{ if and only if } t \leq x < u .$$

Proof of Theorem 5.2: If $x < t_z$ then $z \leq x < t_z = \max(t , z)$ and we must have $t > z$ (otherwise $z < z$) therefore $\max(t , z) = t$. Conversely, if $x < t$ then $x < t \leq \max(t , z) = t_z$.

Proof of Theorem 5.3: Suppose $x \geq u_z$. Then $x \geq \max(u , z) \geq u$. Conversely, if $x \geq u$ then $x \geq \max(u , z)$.

Proof of Theorem 5.4: If $t_z = \max(t , z) \leq x < u_z = \max(u , z)$ then $x \geq t$. Now is $z > u$ then $x < u_z \leq z$, a contradiction. Conversely, if $t \leq x < u$ then $x \geq \max(t , z) = t_z$ and $x < u = \max(u , z)$ because $z \leq x < u$.

Finally, we consider the special case when $L_z(y)$ is independent of z but not necessarily discrete convex. As an example, this case can arise when the stochastic behavior of the inventory system is described by the stationary p. d. f. $\pi(k , y)$ with $\lambda/\mu > 1$ or when the holding and shortage costs may not be linear functions of the amount of equipment in the pool or the number of lost customers. The optimal policy under this assumption is

(1) if $x > u$ then $y = u_z$,

(2) if $t \leq x \leq u_z$ then $y = v_z(x)$,

(3) if $x < t$ then $y = t$,

where t and u_z are defined in (2.4) and (5.3) and $v_z(x)$ is defined by

$$a(v_z(x) - x) + c \cdot v_z(x) + L(v_z(x)) = \min_{y \geq z} \{a(y - x) + cy + L(y)\} ,$$

for $t \leq x \leq u_z$.

5.5 The Multi-Period Model

The extension of the optimal policy of the one period model to multi-period models is discussed in this section. The demands for equipment occur according to an arbitrary distribution and the usage times follow independent arbitrary distributions. Let T be the length of the period under consideration so that $p_{z,k}(T,y)$ represents the probability of k units being in use at the end of the period given z were in use at the beginning of the period and y units of equipment were rented during the period. By now it is clear that the N period functional equations are

$$C_n(x,z) = \min_{y \geq z} \{a(y - x) + cy + L_z(y) + \alpha \sum_{j=0}^{y} C_{n-1}(y,j)p_{z,j}(T,y)\} \qquad (5.7)$$

for $n = 1, 2, \ldots, N$ with $C_0(x,z) = 0$ and $0 \leq \alpha \leq 1$. Define two sequences of real numbers $\{t_n(z)\}$ and $\{u_n(z)\}$, $z = 0, 1, 2, \ldots,$ by

$$k(t_n(z) - x) + ct_n(z) + L_z(t_z(z)) + \alpha M_{n-1}(t_n(z),z)$$

$$= \min_{y \geq z} \{k(y - x) + cy + L_z(y) + \alpha M_{n-1}(y,z)\} \qquad (5.8)$$

$$d(u_n(z) - x) + cu_n(z) + L_z(u_n(z)) + \alpha M_{n-1}(u_n(z)\ ,\ z) \tag{5.9}$$

$$= \inf_{y \geq z} \{d(y - x) + cy + L_z(y) + \alpha M_{n-1}(y\ ,\ z)\}\ ,$$

where

$$M_n(y\ ,\ z) = \sum_{j=0}^{y} C_n(y\ ,\ j) p_{z\ ,\ j}(T\ ,\ y)\ .$$

The reasons for the usage of the minimum and the infimum are similar to the reasons for their use in the multi-period model in Section 3.2. Assume $t_n(z)$ and $u_n(z)$ are unique for convenience. If they are not an auxiliary rule may be used to choose unique values, as in Section 5.2.

Theorem 5.5

If $L_z(y) \geq 0$, $k \geq 0$, and $d \leq 0$ then $t_n(z) \leq u_n(z)$.

Proof: The proof is identical with the proof of Theorem 5.1 if $L_z(y)$ is replaced by $L_n(y\ ,\ z) = L_z(y) + \alpha M_{n-1}(y\ ,\ z)$.

To derive the optimal policy we follow the method of Section 5.3 replacing $L_z(y)$ by $L_n(y\ ,\ z)$, which is defined in the proof of the above theorem. Hence, the optimal policy is

(1) if $x > u_n(z)$ then $y = u_n(z)$,

(2) if $t_n(z) \leq x \leq u_n(z)$ then $y = v_n(x\ ,\ z)$,

(3) if $x < t_n(z)$ then $y = t_n(z)$,

where $v_n(x\ ,\ z)$ is defined by

$$h_x(v_n(x\ ,\ z)\ ,\ z\ ,\ n) = \min_{U_n(z) \leq y \leq T_n(z)} \{h_x(y\ ,\ z\ ,\ n)\}$$

where

$$h_x(y , z , n) = a(y - x) + cy + L_z(y) + \alpha M_{n-1}(y , z)$$

$$U_n(z) = \min_j \{U_n^{(j)}(z)\}$$

$$T_n(z) = \max_j \{T_n^{(j)}(z)\}$$

$$h_x(x , z , n) = h_x(T_n^{(j)}(z) , z , n) = h_x(U_n^{(j)}(z) , z , n)$$

$$T_n^{(j)}(z) \geq x \qquad U_n^{(j)}(z) \leq x .$$

This optimal policy may be simplified by putting $v_n(x , z) = x$. When $h_x(y , z , n)$ is discrete convex in y then doing this gives the exact optimal policy. Whereas if $h_x(y , z , n)$ is not discrete convex but yet fairly regular then a policy is obtained which approximately is optimal. Finally, if $h_x(y , z , n)$ is not discrete convex and does not depend on z the optimal policy for each period of the multi-period model has the same form as the optimal pclicy of a single period model (see the discussion at the end of Section 5.4).

5.6 The Infinite Period Model

In this section we consider the infinite period model and show that its solution is, in a sense, an approximation to the solution of a multi-period model. The functional equation is

$$C(x , z) = \min_{y \geq z} \{a(y - x) + cy + L_z(y) + \alpha \sum_{j=0}^{y} C(y , j)p_{z , j}(T , y)\} \tag{5.10}$$

where $0 \leq \alpha < 1$. When examining the proofs of Theorems 4.1, 4.2, and 4.3 it will be noticed that in no case was discrete convexity needed except in the last part of Theorem 4.1 when proving the discrete convexity of $C^*(x, z)$. With this one exception noted, all the remaining parts of these theorems are true for our generalized model. The n period minimum expected cost function $C_n(x, z)$ converges monotonically and uniformly, as $n \to \infty$, to a limit which is the unique solution of (5.10) (among the class of solutions bounded in finite intervals) .

Define two sequences of real numbers $\{t_\infty(z)\}$ and $\{u_\infty(z)\}$, $z = 0, 1, 2, \ldots$, by (5.8) and (5.9) with $n = \infty$. As in Section 5.5 we may assume they are unique. Using the proof of Theorem 5.5 we see that $t_\infty(z) \leq u_\infty(z)$. The optimal policy is identical with that given in Section 5.5 when $n = \infty$.

5.7 Limiting Behavior of the Critical Numbers

The limiting behavior of the critical numbers $t_n(z)$, $u_n(z)$, and $v_n(x, z)$ is discussed in this section. In certain cases it will be shown that the sequences of critical numbers contain convergent subsequences.

Since $t_n(z)$ and $v_n(x, z)$ are always finite for all n , for each z , and for each x in $[U_n(z), T_n(z)]$, the sequences $\{t_n(z)\}$ and $\{v_n(x, z)\}$ contain convergent subsequences for each z . If $u_n(z)$ is finite for all n and for each z then the sequence $\{u_n(z)\}$ contains at least one convergent subsequence. In a manner similar to the proof used in Theorem 4.5 and Theorem 4.6 we can show that limit points of the above sequences are solutions to the infinite period model cost equations.

Theorem 5.6

If $u_n(z)$ is finite for all n and for each value of z then convergent subsequences of $\{t_n(z)\}$, $\{u_n(z)\}$, and $\{v_n(x, z)\}$ exist for each z and every limit point of the sequence minimizes the corresponding infinite period cost equation.

Proof: The existence of the convergent subsequences follows immediately from the boundedness of the sequences. Following the proof of Theorem 4.5 we find

$$f_x(t_n(z) , z , n) \leq f_x(y , z , \infty)$$
$$g_x(u_n(z) , z , n) \leq g_x(y , z , \infty)$$
$$h_x(v_n(x , z) , z , n) \leq h_x(y , z , \infty)$$

for $y \geq z$, where

$$f_x(y , z , n) = k(y - x) + cy + L_z(y) + \alpha M_{n-1}(y)$$

$$g_x(y , z , n) = d(y - x) + cy + L_z(y) + \alpha M_{n-1}(y) .$$

Thus any limit point of the sequence $\{t_n(z)\}$ minimizes $f_x(y , z , \infty)$; any limit point of the sequence $\{u_n(z)\}$ minimizes $g_x(y , z , \infty)$; any limit point of the sequence $\{v_n(x , z)\}$ minimizes $h_x(y , z , \infty)$; proving the theorem.

Naturally if all of these minima are unique then the sequences of critical numbers all converge: $t_n(z) \to t_\infty(z)$, $u_n(z) \to u_\infty(z)$, and $v_n(x , z) \to v_\infty(x , z)$ as $n \to \infty$.

5.8 $L_z(y)$: A Non-Stationary Approximation

The one period expected holding and shortage cost $L_z(y)$ is calculated in this section using the transition probabilities $p_{z , k}(t , y)$ and assuming Poisson demands and negative exponential usage times. Then an approximation to $p_{z , k}(t , y)$ which is better than the stationary approximation of Section 2.5 is discussed. It is shown that under certain conditions use of this approximation in place of $p_{z , k}(t , y)$ leads to a discrete convex $L_z(y)$. The holding cost is considered first then the shortage cost.

Following the method in Section 2.5 the holding cost can be calculated from (2.9) if $p_{z,k}(t, y)$ is substituted for $\pi(k, y)$. The holding cost is

$$h' \int_0^T \sum_{k=0}^{y} (y - k) p_{z,k}(t, y) dt . \tag{5.11}$$

This simplifies to

$$h[y - \frac{1}{T} \int_0^T A_z(t, y) dt] \tag{5.12}$$

where

$$A_z(t, y) = \sum_{k=0}^{y} k p_{z,k}(t, y)$$

represents the expected number of units of equipment in use at time t if z are in use at time 0 and y units are rented during the period.

In order to calculate the holding cost $A_z(t, y)$ is calculated first. A Poisson arrival, negative exponential usage, y channel queueing system with no queue permitted satisfies the following system of birth death differential equations.

$$p'_{z,0}(t) = -\lambda p_{z,0}(t) + \mu p_{z,1}(t)$$

$$p'_{z,j}(t) = \lambda p_{z,j-1}(t) - (\lambda + j\mu) p_{z,j}(t) + \mu(j + 1) p_{z,j+1}(t) \quad 1 \leq j \leq y - 1 \tag{5.13}$$

$$p'_{z,y}(t) = \lambda p_{z,y-1}(t) - \mu y p_{z,y}(t)$$

where $p_{z,j}(t)$ represents the probability that j channels are occupied at time t if z are at time 0 and λ and μ are the parameters of the Poisson and negative exponential distributions, respectively. For convenience, the dependence of the transition probabilities on y will be suppressed when no confusion is likely to arise, i.e., $p_{z,j}(t) = p_{z,k}(t , y)$. Also, the prime notation represents diff[illegible]ation with respect to t . Multiplying the jth equation by j and summing gives

$$A_z'(t , y) + \mu A_z(t , y) = \lambda[1 - p_{z,y}(t)] . \quad (5.14)$$

Equation (5.14) is a linear differential equation of the first order and can be solved by making the substitution $A_z(t , y) = u(t)\cdot v(t)$. Thus

$$A_z(t , y) = \frac{\lambda}{\mu} - \lambda e^{-\mu t}\int p_{z,y}(t)e^{\mu t}dt + ce^{-\mu t}$$

where c is a constant of integration. Clearly this solution satisfies (5.14). To evaluate c we note that $A_z(0 , y) = z$. Therefore

$$c = z - \frac{\lambda}{\mu} + \lambda\left[\int p_{z,y}(t)e^{\mu t}dt\right]_{t=0}$$

so that

$$A_z(t , y) = \frac{\lambda}{\mu}(1 - e^{-\mu t}) + ze^{-\mu t} - \lambda e^{-\mu t}\Big\{\int p_{z,y}(t)e^{\mu t}dt - \left[\int p_{z,y}(t)e^{\mu t}dt\right]_{t=0}\Big\} . \quad (5.15)$$

Riordan [11], page 85, gives the following expression for $p_{z,y}(t)$.

$$p_{z,y}(t) = \frac{p(y)}{P(y)} + \left(\frac{\lambda}{\mu}\right)^{y-z} \sum_{k=1}^{y} \frac{D_z(r_k(y))}{r_k(y)D_y'(r_k(y) + 1)} e^{r_k(y)\mu t} \quad (5.16)$$

where p(y) and P(y) are as defined previously, and $r_k(y)$, k = 1 , 2 , ... , y , are the y eigenvalues of the matrix of the coefficients of the system of birth death equations (5.13) (their dependence on y , the order of the matrix, is explicitly noted). The polynomials $D_n(s)$ are a sequence of Sturm functions defined by

$$D_n(s) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda}{\mu}\right)^{n-k} s(s+1) \ldots (s+k-1) \tag{5.17}$$

where $s(s+1) \ldots (s+k-1) = 0$ for $k = 0$. The prime on D in the denominator indicates the derivative with respect to s . It can be shown that the y eigenvalues $r_k(y)$ are all distinct, real, negative, and that they are they y roots of $D_y(s+1) = 0$. Let

$$a_k(y\ ,\ z) = \frac{D_z(r_k(y))}{r_k(y)D'_y(r_k(y)+1)}$$

then (5.16) becomes

$$p_{z,k}(t) = \pi(y\ ,\ y) + \left(\frac{\lambda}{\mu}\right)^{y-z} \sum_{k=1}^{y} a_k(y\ ,\ z)e^{r_k(y)\mu t} \ . \tag{5.18}$$

Using (5.18) in (5.15) gives

$$A_z(t\ ,\ y) = \frac{\lambda}{\mu}(1 - e^{-\mu t}) + ze^{-\mu t} - (\lambda/\mu)\frac{p(y)}{P(y)}(1 - e^{-\mu t})$$

$$- \frac{\lambda}{\mu} \sum_{k=1}^{y} \frac{a_k(y\ ,\ z)}{r_k(y)+1}(e^{r_k(y)\mu t} - e^{-\mu t}) \ . \tag{5.19}$$

As a check on the correctness of (5.19) if we let $t \to \infty$ in $A_z(t\ ,\ y)$ we should obtain the steady state value given in (2.16). Since all the eigenvalues $r_k(y)$ are negative letting t approach infinity yields

$$A_z(t\ ,\ y) \to (\lambda/\mu) - (\lambda/\mu)\frac{p(y)}{P(y)} = (\lambda/\mu)\frac{P(u-1)}{P(y)} \ .$$

Comparing (5.12) with (2.16) shows that in steady state $A_z(t , y)$ does reduce to the value given in (2.16). Rewriting (5.19) and using the definition of $a_k(y , z)$,

$$A_z(t , y) = (\lambda/\mu)\frac{P(y - 1)}{P(y)} (1 - e^{-\mu t}) + ze^{-\mu t}$$

$$- (\frac{\lambda}{\mu})^y \sum_{k=1}^{y} \frac{D_z(r_k(y))}{r_k(y)[r_k(y) + 1]D'_y(r_k(y) + 1)} (e^{r_k(y)\mu t} - e^{-\mu t}) . \quad (5.20)$$

A simplified approximation of $A_z(y , t)$ can be obtained by neglecting all the exponential terms in (5.20) which have eigenvalues in the exponent except the one which has the largest negative eigenvalue. Let

$$r(y) = \max_{1 \le k \le y} \{r_k(y)\}$$

then (5.20) reduces to

$$A_z(t , y) \approx (\lambda/\mu)\frac{P(y - 1)}{P(y)} (1 - e^{-\mu t}) + ze^{-\mu t}$$

$$- (\frac{\lambda}{\mu})^y \frac{D_z(r(y))}{r(y)[r(y) + 1]D'_y(r(y) + 1)} (e^{r(y)\mu t} - e^{-\mu t}) \quad (5.21)$$

where $\approx$ denotes approximate equality. The objective in calculating $A_z(y , t)$ is to determine whether or not the expected holding cost (5.12) is discrete convex in y . If (5.21) is used in (5.12) to give an approximate expression for the expected holding cost it will be very difficult to determine the first and second differences of $A_z(t , y)$ with respect to y because r depends on y . That is, the relationship between the largest eigenvalue of a matrix and the order of the matrix must be known. We circumvent this difficulty by making another approximation. Riordan [11], pages 86-7, shows that as $(\lambda/\mu) \to 0$

$r(y) \to -1$. Thus assuming (λ/μ) is small and approximating $r(y)$ by -1 gives an expression even simpler than (5.21),

$$A_z(t\ ,\ y) \approx (\lambda/\mu)\frac{P(y-1)}{P(y)}(1 - e^{-\mu t}) + ze^{-\mu t}\ . \qquad (5.22)$$

Consequently, an approximate expression for the expected holding cost can be obtained by substituting (5.22) into (5.12). Doing this gives,

$$h[y - (\lambda/\mu)\frac{P(y-1)}{P(y)} - \frac{1}{T\mu}(1 - e^{-\mu t})(z - (\lambda/\mu)\frac{p(y)}{P(y)})]\ . \qquad (5.23)$$

The first two terms withing the bracket are the stationary component of the expected holding cost and by Theorem 2.3 we know that they are discrete convex if $0 \le (\lambda/\mu) \le 1$. Concentrating on the last term, if $\Delta^2[p(y)/P(y)] \ge 0$ then the expected holding cost is discrete convex.

Theorem 5.7

For small (λ/μ) $\Delta^2[p(y)/P(y)] \ge 0$.

Proof: The numerator of $\Delta^2[p(y)/P(y))]$ is

$$\begin{aligned}&p(y+2)[P(y)]^2 - 2p(y+1)[P(y)]^2 - 2[p(y+1)]^2P(y)\\&- p(y+1)p(y+2)P(y) + p(y)[P(y)]^2 + 2p(y)p(y+1)P(y)\\&+ p(y)p(y+2)P(y) + p(y)[p(y+1)]^2 + p(y)p(y+1)p(y+2)\\&\qquad\qquad \ge [P(y)]^2[p(y+2) + p(y) - 2p(y+1)]\end{aligned}$$

but the terms in the second bracket are nonnegative for small (λ/μ) for all $y \ge 0$. The proof of the theorem is complete.

We have shown up to this point that for small values of (λ/μ) the approximate expression (5.23) for the expected holding cost is discrete convex. The next term to be determined is the expected shortage cost. As in Section 2.5 it will be s times the expected number of lost customers $M(t)$ in some time interval t. The symbol M will have this meaning in this section only. In pervious sections it was used for something else. It is used in this section for the expected number of lost customers because it will subsequently be identified as the renewal function which is denoted in the literature by $M(t)$. First we calculate $M(t)$.

Let $G(t)$ be the distribution function of the time between successive lost customers and let $N(t)$ be the number of lost customers in $(0, t)$. Clearly the successive lost customers forms a renewal process so that

$$P[N(t) = n] = G^{(n)}(t) - G^{(n+1)}(t)$$

where $G^{(n)}(t)$ represents the n fold convolution of the distribution $G(t)$ with itself. Also,

$$M(t) = E[N(t)] = \sum_{j=1}^{\infty} G^{(j)}(t)$$

where E denotes expectation. Denoting Laplace-Stieltjes transforms by $*$ we have

$$M^*(s) = \frac{G^*(s)}{1 - G^*(s)}$$

where

$$M^*(s) = \int_0^{\infty} e^{-st} dM(t)$$

$$G^*(s) = \int_0^{\infty} e^{-st} dG(t) .$$

Takács [12], page 185, calculates an explicit expression for $M^*(s)$ assuming the input process is a recurrent process and the holding times are negative exponential.

We could follow his method but since his expression for $M^*(s)$ is very complex from a computational point of view another, and simpler, method is used. Divide a time interval $(0 , T)$ into n small equal subintervals of length δt . That is, $\delta t = T/n$. In order to have a lost customer during one of these intervals of length δt all of the equipment must be in use at the beginning of the interval and a demand must occur, together with no completions of service. Thus,

$$P\left[\begin{matrix}1 \text{ lost customer} \\ \text{in } (t , t + \delta t)\end{matrix}\right] = p_{z,y}(t , y)\lambda'\delta t(1 - \mu'\delta t)^y + o(\delta t)$$

where $(t , t + \delta t)$ is the interval under consideration, $\lambda' = \lambda/T$, and $\mu' = \mu/T$. The symbol $o(\delta t)$ represents terms of order $(\delta t)^2$ or higher. Mathematically, terms are of order $(\delta t)^2$ or higher if and only if

$$\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$$

or equivalently

$$o(\delta t) < \epsilon\delta t$$

for $\epsilon > 0$ and arbitrary and δt small. Consequently,

$$P\left[\begin{matrix}n \text{ lost customers} \\ \text{in } (t , t + \delta t)\end{matrix}\right] = p_{z,y}(t , y)(\lambda'\delta t)^n(1 - \mu'\delta t)^y + o[(\delta t)^n] .$$

Hence the expected number of lost customers in $(t , t + \delta t)$ is, for small δt ,

$$\sum_{n=0}^{\infty} n\{P_{z,y}(t , y)(\lambda'\delta t)^n(1 - \mu'\delta t)^y + o[(\delta t^n]\}$$

which reduces to

$$p_{z,y}(t , y)(\lambda'\delta t) + \sum_{n=0}^{\infty} no[(\delta t)^n] .$$

However this equals

$$p_{z,y}(t\ ,\ y)(\lambda'\delta t) + o(\delta t)$$

because

$$\sum_{n=0}^{\infty} no[(\delta t)^n] < \varepsilon\sum_{n=0}^{\infty} n(\delta t)^n = \varepsilon \frac{\delta t}{(1-\delta t)^2} = \varepsilon'\delta t$$

where $\varepsilon > 0$ is arbitrary and δt is sufficiently small. The expected number of lost customers in $(0\ ,\ T)$ is by definition $M(T)$ and is equal to

$$M(T) = \lim_{\delta t\to 0} \sum_{\substack{\text{all inter-}\\ \text{vals } \delta t}} [p_{z,y}(t\ ,\ y)(\lambda'\delta t) + o(\delta t)] \ .$$

Since $p_{z,y}(t\ ,\ y)$ is continuous in t it is Riemann integrable, thus

$$M(T) = \lambda' \int_0^T p_{z,y}(t\ ,\ y)dt \tag{5.24}$$

because

$$\lim_{\delta t\to 0} \Sigma\, o(\delta t) < \lim_{\delta t\to 0} \Sigma\, \varepsilon\delta t = \varepsilon T = \varepsilon'$$

where $\varepsilon > 0$ is arbitrary so that $\varepsilon' > 0$ is arbitrary. Note that the ε and ε' used here are not related to those used previously.

The shortage cost can be calculated now as $s\cdot M(T)$ or, indicating the dependence on z and y explicitly, as $s\cdot M_z(T\ ,\ y)$. The p. d. f. $p_{z,y}(t\ ,\ y)$ is given by (5.16). Again note that if the first and second differences of $M_z(T\ ,\ y)$ with respect to y are taken that the dependence of the eigenvalues on y creates very complex computational difficulties. Thus we will approximate

$p_{z,y}(t\ ,\ y)$ by

$$p_{z,y}(t\ ,\ y) \approx \frac{p(y)}{P(y)} + (\lambda/\mu)^{y-z}\frac{D_z(r(y))}{r(y)D_y'(r(y)+1)}\ e^{r(y)\mu t}$$

where, as before, $r(y)$ is the largest negative eigenvalue. As $(\lambda/\mu) \to 0$ we know that $r(y) \to -1$, thus, we make the simpler approximation,

$$p_{z,y}(t\ ,\ y) \approx \frac{p(y)}{P(y)} - (\lambda/\mu)^{y-z}\frac{D_z(-1)}{D_y'(0)}\ e^{-\mu t}\ .$$

By (5.17)

$$D_z(-1) = (\frac{\lambda}{\mu})^{z-1}(\frac{\lambda}{\mu} - z)$$

$$D_y'(0) = y\sum_{j=0}^{y-1}\frac{(\lambda/\mu)^j}{j!}\ \frac{1}{y-j}$$

so that

$$p_{z,y}(t\ ,\ y) \approx \frac{p(y)}{P(y)} - \frac{p(y-1)(z-(\lambda/\mu))}{y\sum_{j=0}^{y-1}\frac{p(j)}{y-j}}\ e^{-\mu t} \qquad (5.25)$$

if $y \geq 1$ and $p_{z,y}(t\ ,\ y) = 1$ if $y = 0$ since $z \leq y$ for small values of (λ/μ). Using (5.24) the expected shortage cost becomes

$$s\lambda'\int_0^T p_{z,y}(t\ ,\ y)dt$$

which is, when substituting (5.25), for $y \geq 1$,

$$s\lambda'\frac{p(y)}{P(y)} + \frac{p(y-1)}{y\sum_{j=0}^{y-1}\frac{p(j)}{y-j}}\left[\frac{(z-\frac{\lambda}{\mu})s\lambda'}{\mu}\right](1-e^{-\mu t})\ .$$

In order to show that the expected shortage cost is discrete convex for $y \geq 1$ and small (λ/μ) it suffices to show that

$$\Delta^2\left[\frac{p(y-1)}{y \sum_{j=0}^{y-1} \frac{p(j)}{y-j}}\right] \geq 0$$

for $z \geq (\lambda/\mu)$ because by Theorem 5.7 $p(y)/P(y)$ is discrete convex.

<u>Theorem 5.8</u>

For small (λ/μ) $\Delta^2\left[\frac{p(y-1)}{y \sum_{j=0}^{y-1} \frac{p(j)}{y-j}}\right] \geq 0$ for $1 \leq y \leq X$, where X is as defined in the proof of Theorem 4.1.

Proof: Let

$$f(y) = \frac{p(y-1)}{y}$$

$$g(y) = \sum_{j=0}^{y-1} \frac{p(j)}{y-j}$$

then using the standard formula for the difference of the quotient of two functions gives,

$$\Delta^2\left[\frac{p(y-1)}{y \sum_{j=0}^{y-1} \frac{p(j)}{y-j}}\right] = \Delta^2\left[\frac{f(y)}{g(y)}\right]$$

$$= \{g(y)g(y+1)[g(y+1)\Delta^2 f(y) - f(y+1)\Delta^2 g(y)]$$

$$- [g(y)\Delta f(y) - f(y)\Delta g(y)][g(y+1)\Delta g(y+1)$$

$$+ g(y+1)\Delta g(y)]\}/\{g(y)[g(y+1)]^2 g(y+2)\}$$

for $y \geq 1$. But $\Delta f(y) \leq 0$, $\Delta^2 f(y) \geq 0$, $\Delta g(y) \leq 0$, and $\Delta^2 g(y) \geq 0$ for

$1 \le y \le X$ and small (λ/μ) . Consequently it is sufficient to show that

$$g(y+1)\Delta^2 f(y) - f(y+1)\Delta^2 g(y) \geq 0 \tag{5.26}$$

$$g(y)\Delta f(y) - f(y)\Delta g(y) \geq 0 \, . \tag{5.27}$$

Starting with (5.26) after some elementary computation it is equal to

$$\frac{y(\lambda/\mu)^{y-1} e^{-2(\lambda/\mu)}}{(y+1)!}[y^2 + (3 - 2\frac{\lambda}{\mu})y + 2 + (\frac{\lambda}{\mu})^2 - 4(\frac{\lambda}{\mu})] - \frac{2(\lambda/\mu)^y e^{-2(\lambda/\mu)}}{(y+2)!y(y+1)}$$

$$+ \frac{(\lambda/\mu)^{y-1} e^{-2(\lambda/\mu)}}{y-1} o(\frac{\lambda}{\mu})(y^2 + (3 - 2(\frac{\lambda}{\mu})y + 2 + (\frac{\lambda}{\mu})^2 - 4\frac{\lambda}{\mu})$$

$$- \frac{(\lambda/\mu)^y e^{-2(\lambda/\mu)}}{(y+2)!y(y+1)} o(\frac{\lambda}{\mu}) \, .$$

The first and third terms are of the order $(\frac{\lambda}{\mu})^{y-1}$ but the second and fourth are of the order $(\frac{\lambda}{\mu})^y$ therfore for sufficiently small (λ/μ) the expression (5.26) is nonnegative.

Similarly for (5.27) we obtain an expression, for small (λ/μ) ,

$$\frac{1}{y}p(y-1)[\frac{y-1}{y(y+1)} o(\frac{\lambda}{\mu})]$$

which is nonnegative. The proof of the theorem is complete.

The theorem implies that the one period expected shortage cost is discrete convex for y in $[1 , X]$. Examining the special case at $y = 0$ will show that the second difference is nonnegative also.

In summary, it has been shown that the one period expected holding and shortage cost

$$L_z(y) = h' \int_0^T \sum_{k=0}^{y} (y-k)p_{z,k}(t , y)dt + s\lambda' \int_0^T p_{z,y}(t , y)dt$$

is discrete convex in y for each $z \geq 1$ when $p_{z,k}(t , y)$ is approximated by (5.25) and (λ/μ) is small.

6. Extensions

6.1 Introduction

Two important extensions or variations of the inventory model for rented equipment are considered in this Chapter. In a one or multi-period model the question of what happens to the rented equipment at the end of the last period is examined more closely. The constant d representing the return cost was restricted to nonpositive values in Chapters 1 through 5. Now we examine the case where d can be positive.

6.2 The Equipment in Use at the End of a Period

Focus attention on a one period problem or on the last period of a multiperiod problem. The models considered in the previous chapters implicitly assume that at the end of the period all of the rented equipment is returned; the cost of returning each unit being the same. The cost of returning the equipment at the end of the period is a component of the rental cost c . In reality this may or may not be the case. It is possible that a rental agreement may specify that at the end of the period the equipment not in use is returned immediately but the equipment in use can be returned as they finish their work in the field at no extra cost. The rental cost c would be increased, most likely, to allow for such a situation. Another possibility is that an extra cost is incurred at the end of the period for all equipment still in use. The equipment in use may have to be rented for another period or maybe just as long as it is in use. As an example, if this cost is proportional to the number in use and if the stochastic behavior of the inventory system is described approximately by the staionary transition probabilities then it is

$$r \sum_{k=0}^{y} k \frac{p(k)}{P(y)} = r(\frac{\lambda}{\mu}) \frac{P(y-1)}{P(y)}$$

where r is the constant of proportionality. A similar expression can be derived is the nonstationary transition probabilities are used. Adding this to (2.2) gives the total expected cost incurred during the period. If the cost equation is discrete convex then the analysis of Chapter 2 is applicable. If it is not discrete convex the analysis of Chapter 5 applies. It is clear from (2.16) that if $r < h$ or $r = s\lambda$ then the cost equation is discrete convex.

6.3 An Inventory Model for Owned Equipment

If $0 \leq d < k$ then all of the results in the previous chapters remain valid. The only conclusions which do not extend to this immediately are Theorems 5.1 and 5.5. The following theorem shows that Theorem 5.1 (and hence 5.5) is true for $0 \leq d < k$.

Theorem 6.1

If $L_z(y) \geq 0$, for each z , and $0 \leq d < k$ then $t_z \leq u_z$.

Proof: Assume the contrary that $t_z \not\leq u_z$. By the definition of t_z

$$k(t_z - x) + ct_z + L_z(t_z) \leq k(u_z - x) + cu_z + L_z(u_z) .$$

Thus

$$(k - d)(t_z - x) + d(t_z - x) + ct_z + L_z(t_z) \leq (k - d)(u_z - x) + d(u_z - x) + cu_z + L_z(u_z) .$$

Since $k - d > 0$ and $t_z > u_z$ it must be true that

$$d(t_z - x) + ct_z + L_z(t_z) < d(u_z - x) + cu_z + L_z(u_z)$$

contradicting the definition of u_z . The theorem is proven.

In addition if the rental cost c is zero then the rental inventory model can be given another interpretation. All the equipment can be considered as owned rather than rented and at the beginning of each period additional units can be bought at a cost of k per unit or excess items can be disposed of for a revenue (e.g., salvage value) of d per unit. In the light of this interpretation it makes sense that the gain from used equipment should be less, ordinarily, than the cost to buy new equipment (i.e., $0 \leq d < k$).

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