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THE COST FUNCTION

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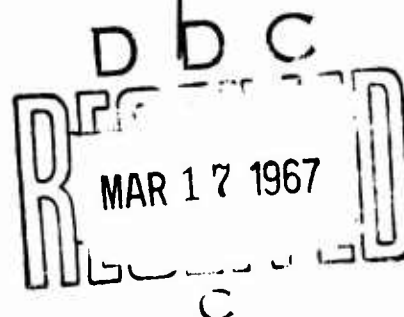
Ronald W. Shephard

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1. THE PRODUCTION STRUCTURE

For some technology, let $x = (x_1, x_2, \dots, x_n)$ denote a vector of nonnegative inputs per unit of time of the relevant factors of production. The technology prescribes a family of technical possibilities, which may or may not be realized in practice.

The technical possibilities of production are defined by a family of production possibility sets $L(u)$, which are subsets of the nonnegative domain $D = \{x \mid x \geq 0, x \in R^n\}$ specifying for each output rate $u \in [0, \infty)$ the input vectors which yield at least the output rate u . These sets are assumed to have the following properties:

P.1 $L(0) = 0$, $0 \notin L(u)$ for $u > 0$,

P.2 If $x \in L(u)$ and $x' \geq x$, then $x' \in L(u)$,

P.3 If $x > 0$, or $x \geq 0$ and $(\lambda x) \in L(u)$ for some $\lambda > 0$ and $u > 0$, the ray $\{\lambda x \mid \lambda \geq 0\}$ intersects all sets $L(u)$, $u \in [0, \infty)$,

P.4 $L(u_2) \subset L(u_1)$ if $u_2 \geq u_1$,

P.5 $\bigcap_{0 \leq u < u_0} L(u) = L(u_0)$ for any $u_0 > 0$,

P.6 $\bigcap_{u \in [0, \infty)} L(u) = \emptyset$, i.e., the empty set,

P.7 $L(u)$ is closed for any $u \in [0, \infty)$,

P.8 $L(u)$ is convex for any $u \in [0, \infty)$.

As shown in [4], the production function for the technology is a function

$\phi(x) = \text{Max}_{x \in L(u)} u$ defined on the nonnegative domain D of R^n with the properties:

A.1 $\phi(0) = 0$,

A.2 $\phi(x)$ is finite for x finite,

A.3 $\phi(x') \geq \phi(x)$ for $x' \geq x$,

A.4 If $x > 0$, or $x \geq 0$ and $\phi(\lambda x) > 0$ for some $\lambda > 0$, then

$$\phi(\lambda x) \rightarrow \infty \text{ as } \lambda \rightarrow \infty,$$

A.5 $\phi(x)$ is upper semi-continuous on D ,

A.6 $\phi(x)$ is quasi-concave on D .

The sign (\geq) implies (\geq) but not $(=)$.

To the properties P.1, ... P.8 we add that

P.9 $E(u)$ is bounded for all $u \in [0, \infty)$,

i.e., the set of efficient points $E(u) = \{x \mid x \in L(u), \phi(y) < u \forall y \leq x\}$

for each production possibility set is bounded, and (see [4], §4)

$$L(u) = \bar{E}(u) + D. \quad (1)$$

The assumption P.9 is no restriction on the generality of the production model, while it assures the existence of the minimum of a linear functional over the production set $L(u)$. $\bar{E}(u)$ is the closure of $E(u)$, and the sum of two sets $A + B = \{x \mid x = y + w, y \in A, w \in B\}$.

A nonnegative vector $p = (p_1, p_2, \dots, p_n)$ is used to denote the prices of the factors of production. When $p = 0$, all of the factors are free goods - a situation of trivial interest in economic theory, but this possibility is not excluded.

Some convention is required for the costing of mixed inputs per unit time of the factors. If x and y are two input vectors for the production structure,

the input $[(1 - \theta)x + \theta y]$ where $\theta \in [0,1]$ may be interpreted to mean that the input combination x is used a fraction $(1 - \theta)$ of the time interval and an input y is used the remaining fraction θ , taking the technology to be time divisible. The cost of this mixed input is calculated by $[(1 - \theta)p \cdot x + \theta p \cdot y]$ (where $p \cdot x$, $p \cdot y$ denote the innerproduct of two vectors), implying for capital services that the real capital is charged for only when used. If $[(1 - \theta)x + \theta y]$ is interpreted as an input rate of a combined input over the entire interval, the costing of the input is still calculated by $[(1 - \theta)p \cdot x + \theta p \cdot y]$. The physical interpretation, in any event, is made to conform to property A.6.

The factor input and factor price domains are regarded as superimposed on the set D , with each point of D representing an input vector or a price vector.

2. DEFINITION OF THE COST FUNCTION

For any price vector $p \in D$ and output rate u the cost structure of interest is that of the total cost $p \cdot x$, where the factor applications x are adjusted to yield the smallest total cost for the output rate $u \in [0, \infty)$, and the cost function is defined by

$$Q(u, p) = \min_{x \in L(u)} (p \cdot x) ; p \in D, u \in [0, \infty) \quad (2)$$

where $(p \cdot x)$ is the inner product of the two vectors p and x . Since $L(u) = \bar{E}(u) + D$, if $x \in L(u)$ then $x = y + w$ where $y \in \bar{E}(u)$, $w \in D$ and

$$\begin{aligned} \min_{x \in L(u)} (p \cdot x) &= \min_{y \in \bar{E}(u)} (p \cdot y) + \min_{w \in D} (p \cdot w) \\ &= \min_{y \in \bar{E}(u)} (p \cdot y) \end{aligned} \quad (3)$$

The compactness of $\bar{E}(u)$ (i.e., $\bar{E}(u)$ is bounded and closed) assures existence of minimum total cost. Moreover,

$$Q(u, p) = \min_{x \in \bar{E}(u)} (p \cdot x) . \quad (2.1)$$

Evidently, $Q(u, 0) = 0$ for all $u \in [0, \infty)$ and $Q(0, p) = 0$ for all $p \in D$. Neither of these two situations are of particular interest for economic theory; they are included for completeness of definition. For the minimum problem defined by (2), the components of the price vector p and the output rate u are arbitrary parameters, and the cost function $Q(u, p)$ gives the minimum total cost per unit time corresponding to any nonnegative output rate and nonnegative prices of the factors of production. In effect, it is assumed that the prices $p_i (i = 1, \dots, n)$ do not depend upon the amounts $x_i (i = 1, 2, \dots, n)$ of the factor inputs.

Under the assumptions made above, it is possible for $Q(u, p)$ to be zero when $p \neq 0$, $u \neq 0$. In order to see this, classify the price and factor input points of D by $D_1 = \{p \mid p > 0\} = \{x \mid x > 0\}$,
 $D_2 = \left\{ p \mid p \geq 0, \sum_{i=1}^n p_i = 0 \right\} = \left\{ x \mid x \geq 0, \sum_{i=1}^n x_i = 0 \right\}$. The set D_1 is the set of positive price or factor input vectors and the boundary of D is $\{0\} \cup D_2$.
 If $p \in D_2$ and $u > 0$ the cost minimizing input $x^*(u, p)$ may have zero components for factors with positive prices and positive components for factors with zero prices. Hence it is useful to pursue further a classification of the boundary points D_2 . Let

$$\mathcal{D}_2' = \{p \mid p \in D_2, Q(u, p) > 0 \text{ for all } u > 0\},$$

$$\mathcal{D}_2'' = \{p \mid p \in D_2, Q(u, p) = 0 \text{ for all } u > 0\}.$$

Proposition 1: $\mathcal{D}_2' \cap \mathcal{D}_2'' = \emptyset$ (the empty set) and $D_2 = \mathcal{D}_2' \cup \mathcal{D}_2''$.

Suppose for some $\bar{u} > 0$ and $p \in D_2$ that $Q(\bar{u}, p) = 0$. Then the cost minimizing input $x^*(\bar{u}, p)$ is such that $p \cdot x^*(\bar{u}, p) = 0$, which implies for each i that $x_i^*(\bar{u}, p) \cdot p_i = 0$ since both p and $x^*(\bar{u}, p)$ are nonnegative. Hence when $p_i > 0$, $x_i^*(\bar{u}, p) = 0$. Therefore $x^*(\bar{u}, p) \in D_2$. Moreover, the ray $\{\lambda x^*(\bar{u}, p) \mid \lambda \geq 0\}$ intersects all level sets $L(u)$ for $u > 0$ (see P.3, §1). Hence $Q(u, p) = 0$ for all $u > 0$, if $Q(\bar{u}, p) = 0$ for some $\bar{u} > 0$ and $p \in D_2$. Thus the sets \mathcal{D}_2' and \mathcal{D}_2'' are exclusive and $\mathcal{D}_2' \cap \mathcal{D}_2''$ is an empty set. Further $D_2 = \mathcal{D}_2' \cup \mathcal{D}_2''$, since no point of D_2 can yield both a positive and a zero value for $Q(u, p)$ for different positive output rates, because suppose $Q(\bar{u}, p) = 0$ for some $\bar{u} > 0$. Then $Q(\bar{u}, p) = p \cdot x^*(\bar{u}, p) = 0$, implying that $x^*(\bar{u}, p) \in D_2$. But the ray $\{\lambda x^*(\bar{u}, p) \mid \lambda \geq 0\}$ intersects all level sets and hence for any $u > 0$ there is some $\lambda > 0$ where $Q(u, p) = p \cdot \lambda x^*(\bar{u}, p) = \lambda p \cdot x^*(\bar{u}, p) = 0$.

If \mathcal{D}_2'' is nonvoid, all positive output rates may be obtained at zero cost for price vectors of this set, but the price vectors \mathcal{D}_2'' need not be available for any particular production regime. The cost function expresses potential minimum total cost-output-factor price combinations which need not be realized in practice.

It is interesting to inquire when \mathcal{D}_2'' is an empty set and zero cost for positive output is excluded. For this purpose we classify the boundary points D_2 relative to factor inputs by (see [5], §1)

$$D_2' = \{x \mid x \in D_2, (\lambda x) \in L(u) \text{ for some } \lambda > 0 \text{ and } u > 0\},$$

$$D_2'' = \{x \mid x \in D_2, (\lambda x) \notin L(u) \text{ for all } \lambda > 0, u > 0\}.$$

By the property P.3, the ray $\{\lambda x \mid \lambda \geq 0\}$ intersects all production possibility sets $L(u)$, $u > 0$ if $x \in D_2'$, and if $x \in D_2'$ then $x \notin D_2''$, and conversely. Hence $D_2' \cap D_2'' = \emptyset$ and $D_2 = D_2' \cup D_2''$.

Regarding whether \mathcal{D}_2'' is nonempty, the following proposition holds:

Proposition 2: \mathcal{D}_2'' is nonempty if and only if D_2' is nonempty.

Assume D_2' is nonempty and let $x \in D_2'$. Then there is a price point $p^0 \neq 0$ such that $p^0 \cdot x = 0$. Moreover, the ray $\{\lambda x \mid \lambda \geq 0\}$ intersects all sets $L(u)$ for $u > 0$, by virtue of property P.3, and for any $u > 0$ there is a point $(\lambda x) \in L(u)$ with $(\lambda x) \cdot p^0 = 0$. Hence $p^0 \in \mathcal{D}_2''$, and \mathcal{D}_2'' is nonempty. If D_2' is empty no point of D_2 yields positive output and hence for all $p \in D_2$ any cost minimizing output $x^*(u, p) \in D_1$ for $u > 0$. Consequently $Q(u, p) = p \cdot x^*(u, p) > 0$ if $p \in D_2$ and $u > 0$, which implies \mathcal{D}_2'' is empty.

Thus, in order to exclude zero minimum total cost for positive output we must require that positive output is possible only with positive input for all factors of production, i.e. none of the boundaries of the production possibility sets

coincide with the boundary of D for $u > 0$. This restriction is too strong, since some of the factors may be alternatives for others, i.e., not all the factors are assential by themselves. (See [4], §8.) We need not require the set D_2' to be empty, since the subset of prices \mathcal{D}_2'' is merely a formal possibility, which may or may not be realized in practice. On the other hand, we do not wish to require that p be positive. Some factors may be free goods, and it is not correct technologically to exclude them from the input vectors. What is free depends upon the exchange economy which may vary from place to place and from time to time.

3. GEOMETRIC INTERPRETATION OF THE COST FUNCTION

Consider the hyperplane $p \cdot x = Q(u, p)$. Since $Q(u, p)$ is the minimum of $p \cdot x$ for all points $x \in L(u)$, it follows that

$$L(u) \subset \{x \mid p \cdot x \geq Q(u, p)\} \text{ for all } p \neq 0,$$

and the hyperplane is a supporting hyperplane of the production possibility set $L(u)$ (see [6], Part II, §B) and the cost function $Q(u, p)$ is a support functional of $L(u)$ (see [6], Part V) for any $u \geq 0$.

The relation of the hyperplane $p \cdot x = Q(u, p)$ to the production possibility set $L(u)$ is depicted in Figure 1, where $x^*(u, p)$ denotes an input at which the minimum of $p \cdot x$ is attained for $x \in L(u)$. Note that the contact point $x^*(u, p)$ is not necessarily unique.

Let r denote the intersection of the ray $\{\theta p \mid \theta \geq 0\}$ from the origin normal to the hyperplane $p \cdot x = Q(u, p)$, for $p \neq 0$. For some value of θ , say θ_0 , $r = \theta_0 \cdot p$, and, since r lies in the hyperplane $p \cdot x = Q(u, p)$,

$$p \cdot r = \theta_0 \|p\|^2 = Q(u, p)$$

and

$$\theta_0 = \frac{Q(u, p)}{\|p\|^2}, \quad p \neq 0.$$

Consequently

$$r = \frac{Q(u, p)}{\|p\|^2} \cdot p, \quad p \neq 0,$$

and

$$\|r\| = \frac{Q(u, p)}{\|p\|}, \quad p \neq 0.$$

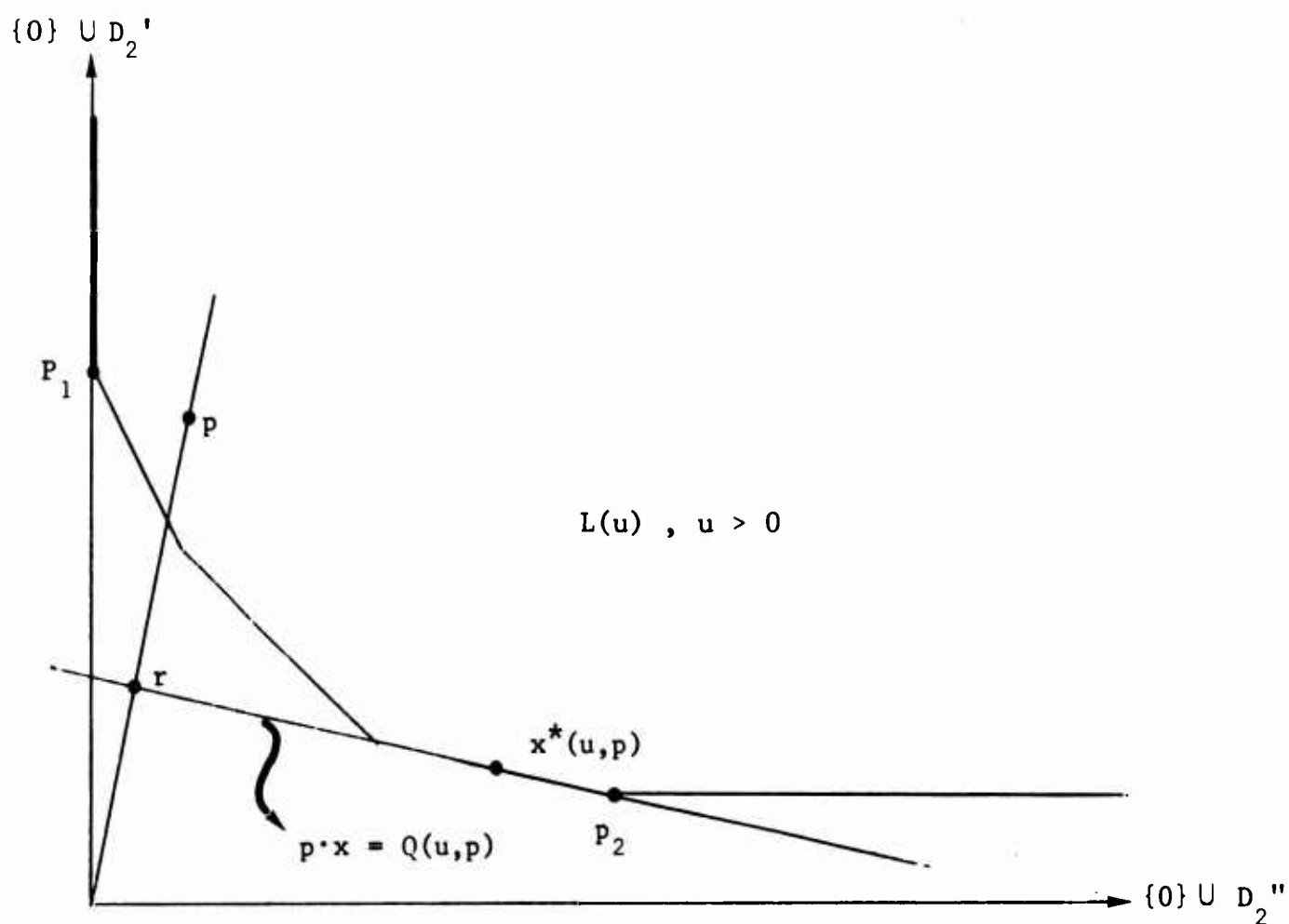


FIGURE 1: RELATION OF HYPERPLANE $p \cdot x = Q(u, p)$ TO $L(u)$

Thus

$$Q(u,p) = ||r|| \cdot ||p||, \quad p \neq 0. \quad (4)$$

If the price point p is normalized so that $||p|| = 1$, the minimum total cost $Q(u,p)$ is merely the normal distance of the supporting hyperplane $p \cdot x = Q(u,p)$ from the point 0 .

The closure of the efficient point set $\bar{E}(u)$ in Figure 1 is the boundary of $L(u)$ comprised between the points P_1 and P_2 . In general, the total minimum cost occurs for an input vector $x^*(u,p) \in \bar{E}(u)$, and $x^*(u,p)$ will be understood always to be such a point, unless otherwise specified.

4. PROPERTIES OF THE MINIMUM TOTAL COST FUNCTION $Q(u,p)$

For positive output rates u the following proposition states in detail the properties of the minimum total cost function $Q(u,p)$:

Proposition 3: If the production structure has production possibility sets

$L(u)$ satisfying P.1, ... P.9, then for any $u > 0$

- Q.1 $Q(u,p) = 0$ for all $p \in \{0\} \cup \mathcal{D}_2''$,
- Q.2 $Q(u,p)$ is finite for finite $p \in D$ and positive for all $p \in D_1 \cup \mathcal{D}_2'$,
- Q.3 $Q(u,\lambda p) = \lambda Q(u,p)$ for $\lambda \geq 0$ and all $p \in D$,
- Q.4 $Q(u,p+q) \geq Q(u,p) + Q(u,q)$ for all $p,q \in D$,
- Q.5 $Q(u,p') \geq Q(u,p)$ if $p' \geq p \in D$,
- Q.6 $Q(u,p)$ is a concave function of p on D ,
- Q.7 $Q(u,p)$ is a continuous function of p on D ,
- Q.8 For any $p \in D$, $Q(u_2,p) \geq Q(u_1,p)$ if $u_2 \geq u_1$,
- Q.9 For any $p \in D_1$, $Q(u,p) \rightarrow \infty$ as $u \rightarrow \infty$,
- Q.10 If there exists for $\delta > 0$ an open neighborhood $N(0) = \{x \mid ||x|| < \delta, x \in D\}$ such that $x \notin L(u)$ for any $u > 0$ when $x \in N(0)$, then as $u \rightarrow 0$
 $\inf Q(u,p) > 0$ for all $p \in D_1$, otherwise
 $\inf Q(u,p) = 0$ for all $p \in D_1$.
- Q.11 For any $p \in D$, $Q(u,p)$ is a lower semi-continuous function of u for all $u \in [0,\infty)$.

Property Q.1 is merely a restatement of the definition of \mathcal{D}_2'' and the evident fact that $Q(u,0) = 0$ for all $u > 0$.

Regarding property Q.2, $Q(u,p) = p \cdot x^*(u,p)$ where $x^*(u,p)$ belongs to the closure of a bounded efficient point set $\bar{E}(u)$ (see (3)) and hence $Q(u,p)$ is finite for finite p . When $p \in D_1$, $\|p\| > 0$, since otherwise the support plane $p \cdot x = Q(u,p)$ has contact with points of D only at 0 and $0 \notin L(u)$ for $u > 0$; from equation (4) it follows then that $Q(u,p) > 0$ since $\|p\| > 0$. If $p \in \mathcal{D}_2'$ it follows from the definition of \mathcal{D}_2' that $Q(u,p) > 0$ for all $u > 0$. Thus property Q.2 holds.

Property Q.3 applies because

$$Q(u, \lambda p) = \min_{x \in L(u)} (\lambda p) \cdot x = \lambda \cdot \min_{x \in L(u)} p \cdot x = \lambda Q(u, p)$$

for any $\lambda > 0$, $p \neq 0$ and when $\lambda = 0$ or $p = 0$ it is obvious that $Q(u, \lambda p) = \lambda Q(u, p)$.

In order to show that Q.4 holds, let $p, q \neq 0$ and let $x^*(u, p+q)$, $x^*(u, p)$, $x^*(u, q)$ denote input combinations minimizing $(p+q) \cdot x$, $p \cdot x$ and $q \cdot x$ respectively. Then

$$Q(u, p+q) = p \cdot x^*(u, p+q) + q \cdot x^*(u, p+q) .$$

Clearly

$$p \cdot x^*(u, p+q) \geq p \cdot x^*(u, p) = Q(u, p) ,$$

$$q \cdot x^*(u, p+q) \geq q \cdot x^*(u, q) = Q(u, q) ,$$

and

$$Q(u, p+q) \geq Q(u, p) + Q(u, q) .$$

If $p = q = 0$, then $Q(u, p+q) = Q(u, p) + Q(u, q)$ since

$Q(u, p+q) = Q(u, p) = Q(u, q) = 0$. When $p \neq 0$, $q = 0$, then

$Q(u, p+q) = Q(u, p) + Q(u, q)$ since $Q(u, p+q) = Q(u, p)$ and $Q(u, q) = 0$.

Similarly, for $p = 0$ and $q \neq 0$.

Property 0.5 follows directly from the nonnegativity and super-additivity of $Q(u,p)$, since $p' = p + \Delta p$ where $\Delta p \geq 0$ and

$$Q(u,p') \geq Q(u,p) + Q(u,\Delta p) \geq Q(u,p) .$$

The concavity of $Q(u,p)$ on D is a simple consequence of the homogeneity and super-additivity of $Q(u,p)$, since for any $p,q \in D$, $(1-\theta)p$ and $\theta \cdot q$ belong to D for any $\theta \in [0,1]$ and

$$Q(u,(1-\theta)p + \theta q) \geq (1-\theta)Q(u,p) + \theta Q(u,q) .$$

The continuity of $Q(u,p)$ in p on D may be established as follows:
First, for any $u > 0$ the function $Q(u,p)$ is continuous on the interior of D , i.e., for $p \in D_1$, since $-Q(u,p)$ is a convex function and a convex function defined on a convex open set in R^n is continuous on this open set (see [2], p193).
Second, regarding the boundary of D , i.e., for $p \in \{0\} \cup D_2$, $Q(u,p)$ is lower semi-continuous (see Theorem, p31, [4]). But $Q(u,p)$ is also upper semi-continuous for $p \in \{0\} \cup D_2$. In order to show this last statement, we extend the definition of the cost function to all points $p \in R^n$ in the following way.
Let

$$L(u/R) = L(u) \cap S_R(0) ; u > 0 , R \in \{R \mid L(u/R) \neq \emptyset\}$$

where

$$S_R(0) = \left\{ x \mid \|x\| \leq R , x \in R^n \right\} .$$

The set $L(u/R)$ is compact, i.e., bounded and closed. Then let

$$Q(u,p/R) = \min_{x \in L(u/R)} (p \cdot x)$$

for any $u > 0$, $R > 0$ and $p \in R^n$. The extended cost function $Q(u, p/R)$ is a concave function defined on R^n , since the arguments used for $Q(u, p)$ apply also to $Q(u, p/R)$, and the function $Q(u, p/R)$ is continuous in p for all $p \in R^n$. Now suppose $p^0 \in \{0\} \cup D_2$ and let $\{R_n\}$ be a nondecreasing sequence of values of R tending to ∞ . The sequence $\{Q(u, p^0/R_n)\}$ is nonincreasing, since

$$Q(u, p^0/R_{n+1}) \leq Q(u, p^0/R_n)$$

for any n , because $L(u/R_{n+1}) \supset L(u/R_n)$ for any n . Also, since $p^0 \geq 0$ the sequence $\{Q(u, p^0/R_n)\}$ is uniformly bounded below by zero and $\lim_{n \rightarrow \infty} Q(u, p^0/R_n)$ exists. The greatest lower bound of the sequence is $Q(u, p^0)$, since $L(u/R_n) \rightarrow L(u)$ monotonically as $n \rightarrow \infty$ and $Q(u, p^0) = \min_{x \in L(u)} (p^0 \cdot x)$. In fact $\bar{E}(u/R) = \bar{E}(u)$ for R sufficiently large. Thus

$$\lim_{n \rightarrow \infty} Q(u, p^0/R_n) = Q(u, p^0).$$

Let α be positive. Then there is a positive integer N such that

$$Q(u, p^0/R_N) < Q(u, p^0) + \frac{\alpha}{2}$$

Further, since $Q(u, p^0/R_N)$ is continuous at p^0 for $n = N$, there is a $\delta > 0$ such that, for all $p \in S_\delta(p^0) = \left\{ p \mid \|p - p^0\| < \delta, p \in R^n \right\}$

$$Q(u, p/R_N) < Q(u, p^0/R_N) + \frac{\alpha}{2}.$$

Thus

$$Q(u, p/R_N) < Q(u, p^0) + \alpha$$

for all $p \in S_\delta(p^0)$ and therefore also for all $p \in D \cap S_\delta(p^0)$. Then, since

$Q(u, p) \leq Q(u, p/R_N)$ for all $p \in D \cap S_\delta(p^0)$, it follows that

$$Q(u, p) < Q(u, p^0) + \alpha$$

for all $p \in D \cap S_\delta(p^0)$, and $Q(u, p)$ is upper semi-continuous on the boundary of D . Therefore, $Q(u, p)$ is continuous for all $p \in D$.

We turn our attention now to property Q.8. Since $L(u_2) \subset L(u_1)$ if $u_2 \geq u_1$ (see property P.4, §1), it follows for all $u_2 \geq u_1$ that

$$Q(u_2, p) = \min_{x \in L(u_2)} (p \cdot x) \geq \min_{x \in L(u_1)} (p \cdot x) = Q(u_1, p).$$

Hence $Q(u, p)$ is nondecreasing in u for any $p \in D$.

Regarding property Q.9, let $x^*(u, p)$ denote a contact point of the hyperplane $px = Q(u, p)$ with $\bar{E}(u)$ for any $p \in D_1$, and suppose that $Q(u, p)$ is bounded for $u \rightarrow \infty$, i.e., $Q(u, p) \leq Q_0$ for all $u \in [0, \infty)$. Then let $\{u_n\}$ be an arbitrary nondecreasing sequence of values tending to ∞ . Due to property Q.8, the corresponding sequence $\{p \cdot x^*(u_n, p)\}$ is then monotone nondecreasing and bounded above by Q_0 . Hence

$$\lim_{n \rightarrow \infty} (p \cdot x^*(u_n, p)) = p \cdot x^0(p) \text{ finite,}$$

and since $p > 0$ by supposition it follows that $x^0(p)$ is finite and by property P.4 it belongs to all $L(u)$ for $u \in [0, \infty)$ contrary to property P.6. Thus $Q(u, p) \rightarrow \infty$ as $u \rightarrow \infty$ for any $p \in D_1$.

When $p \in \mathcal{D}_2''$, $Q(u, p) = 0$ for all $u \in [0, \infty)$ and when $p \in \mathcal{D}_2'$, it is possible for $Q(u, p)$ to be positive and finite for all $u \in (0, \infty)$ as shown by the example in Figure 2 where $p = (0, p_2^0) \in \mathcal{D}_2'$ and $Q(u, p) = Q_0$ for all $u \in [0, \infty)$.

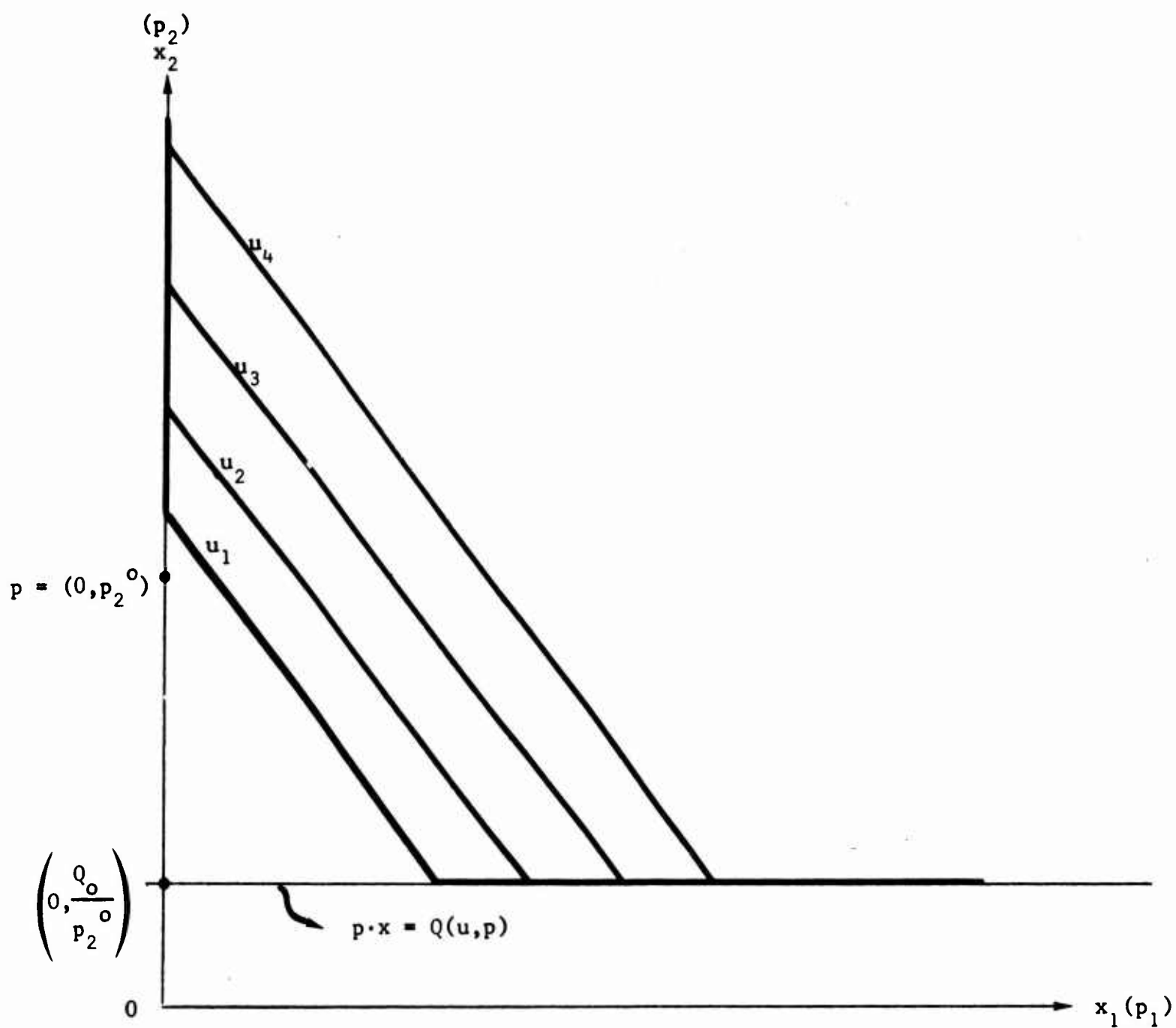


FIGURE 2: EXAMPLE OF BOUNDED $Q(u, p)$ FOR $p \in \mathcal{D}_2'$

Property Q.10 holds, because let $\{u_n\}$ be an arbitrary nonincreasing sequence tending to zero. The corresponding sequence $\{p \cdot x^*(u_n, p)\}$ is nonincreasing and bounded below by zero, and hence the

$$\liminf_{n \rightarrow \infty} \{p \cdot x^*(u_n, p)\} = p \cdot x^*(0, p)$$

exists. Suppose $p \cdot x^*(0, p) = 0$. Then, since $p > 0$ it follows that $x^*(0, p) = 0$. But this is impossible, because $\phi(x) = 0$ for any $x \in N(0)$ and $p \cdot x^*(0, p)$ would not be the greatest lower bound of the sequence $\{p \cdot x^*(u_n, p)\}$. If $p \in \mathcal{D}_2'$, it is possible for $\inf_{n \rightarrow \infty} Q(u_n, p) = 0$ when the neighborhood $N(0)$ exists such that $x \notin L(u)$ for any $u > 0$ when $x \in N(0)$, as illustrated in Figure 3 where the boundaries of the level sets converge to the axes as $n \rightarrow \infty$.

Finally, regarding the continuity of $Q(u, p)$ in $u \in [0, \infty)$ we note first that the cost function is generally not upper semi-continuous in u . The example shown in Figures 4 and 5 illustrates this fact. The step function of Figure 4 satisfies the properties A.1, ... A.6 (see §1). At $u = 0$, $Q(0, p) = 0$ and $\frac{1}{p}Q(0, p) = 0$. For any $u \in (i, i+1]$, clearly $x^*(u, p) = (i+1)$, $p \cdot x^*(u, p) = p(i+1)$ and $\frac{1}{p}Q(u, p) = (i+1)$, and the cost function is evidently not upper semi-continuous in u ; because suppose $u = 3$, then $\frac{1}{p}Q(u, p) = Q(u, 1) > Q(3, 1) + \alpha$ for $0 < \alpha < 1$ and $u > 3$.

Now in order to establish the lower semi-continuity of the cost function, we need only consider $p \in D_1 \cup \mathcal{D}_2'$ because if $p \in \{0\} \cup \mathcal{D}_2''$ then $Q(u, p) = 0$ for all $u \in [0, \infty)$ and the cost function is continuous and therefore lower semi-continuous in this case. Hence let $p \in D_1 \cup \mathcal{D}_2'$ and consider an arbitrary output rate $u_0 \in [0, \infty)$. Then, by the property Q.8, $Q(u, p) \geq Q(u_0, p)$ for all $u \geq u_0$ and $Q(u, p) > Q(u_0, p) - \alpha$ for any $\alpha > 0$ and $u \geq u_0$. For $u \leq u_0$,

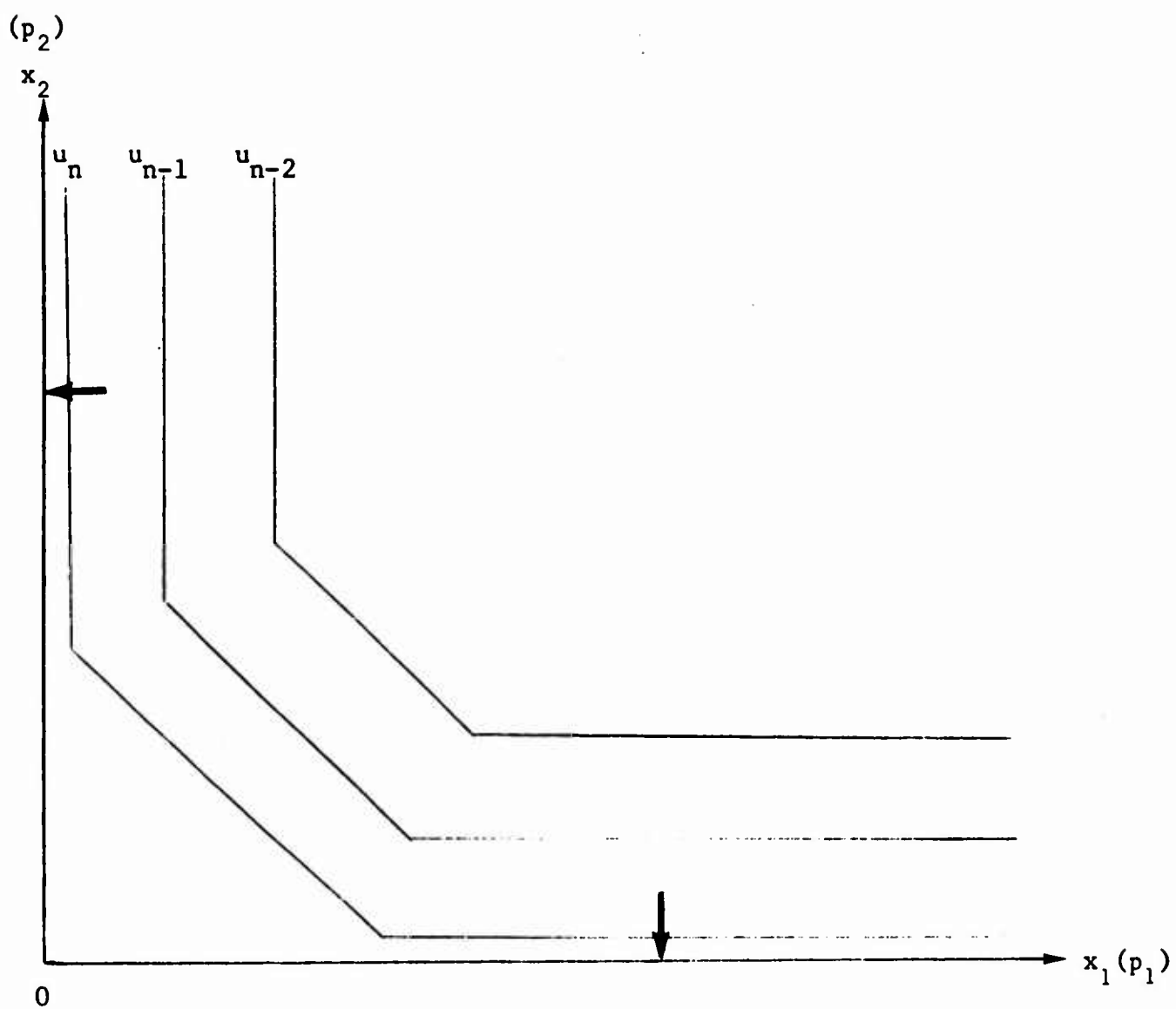


FIGURE 3: EXAMPLE OF $\inf_{n \rightarrow \infty} Q(u_n, p) = 0$ FOR $\phi(x) = 0$ IF $x \in N(0)$

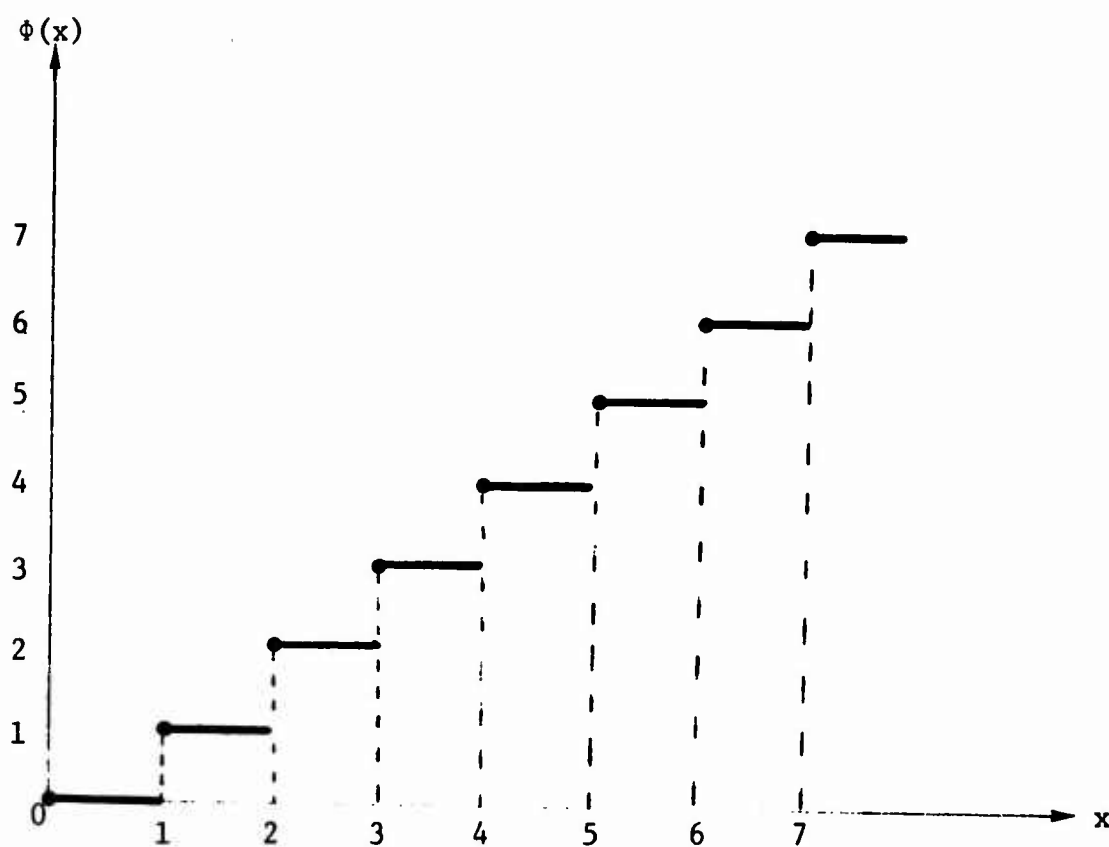


FIGURE 4: UPPER SEMI-CONTINUOUS PRODUCTION STEP FUNCTION

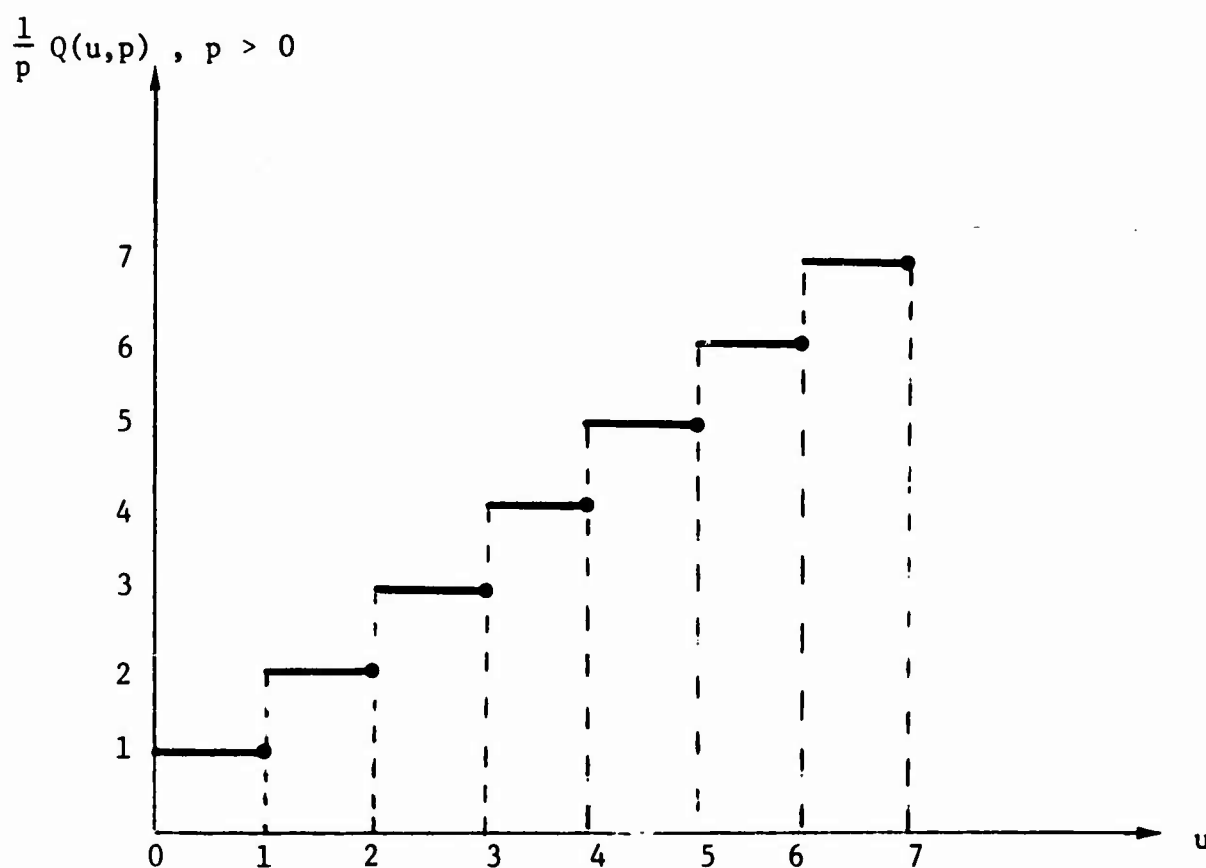


FIGURE 5: COST FUNCTION FOR THE PRODUCTION STEP FUNCTION

let $\{u_n\}$ be an arbitrary nondecreasing sequence of output rates less than u_0 and converging to u_0 , assuming that $u_0 > 0$, otherwise the lower semi-continuity has been established. The corresponding sequence $\{Q(u_n, p)\} = \{p \cdot x^*(u_n, p)\}$, where $x^*(u_n, p)$ is a contact point of the hyperplane $p \cdot x = Q(u_n, p)$ with the set $\bar{E}(u_n)$, is nondecreasing and bounded above by $Q(u_0, p)$, due to property Q.8.

Hence $\lim_{n \rightarrow \infty} Q(u_n, p) = p \cdot \bar{x}(u_0, p)$ where $\bar{x}(u_0, p) \in L(u_0)$, because

$$\bar{x}(u_0, p) \in \bigcap_{n=1}^{\infty} L(u_n) = L(u_0) \text{ by P.4 and P.5. Moreover,}$$

$$\lim_{n \rightarrow \infty} Q(u_n, p) = p \cdot \bar{x}(u_0, p) = p \cdot x^*(u_0, p) = Q(u_0, p),$$

since $\lim_{n \rightarrow \infty} Q(u_n, p) < Q(u_0, p)$ implies a point $\bar{x}(u_0, p) \in L(u_0)$ such that

$p \cdot \bar{x}(u_0, p) < Q(u_0, p)$ contradicting the fact that $Q(u_0, p) = \min_{x \in L(u_0)} (p \cdot x)$. Then,

since the sequence $\{Q(u_n, p)\}$ converges monotonically to $Q(u_0, p)$, there exists for any $\alpha > 0$ on N such that

$$Q(u_0, p) - Q(u_N, p) < \alpha,$$

or

$$Q(u_N, p) > Q(u_0, p) - \alpha$$

for all u satisfying $(u_N < u \leq u_0)$. Thus the cost function is lower semi-continuous in u for any $p \in D$.

5. COST FUNCTION OF A HOMOTHETIC PRODUCTION STRUCTURE

We consider now the special form of the cost function when the production structure has a production function of the form $F(\phi(x))$, where $\phi(x)$ is a homogeneous function satisfying A.1, ... A.6 and $F(\cdot)$ is any nonnegative, continuous, strictly increasing function with $F(0) = 0$ and $F(v) \rightarrow \infty$ as $v \rightarrow \infty$. As shown in [4], §7, the function $\phi(x)$ is continuous for $x \in D$ and also strictly increasing along rays $\{\lambda x \mid \lambda \geq 0\}$ where $x \in D_1 \cup D_2'$.

The production possibility sets (level sets) of the homothetic production structure are

$$\begin{aligned} L_F(u) &= \{x \mid F(\phi(x)) \geq u, x \in D\} \\ &= \{x \mid \phi(x) \geq f(u), x \in D\} = L_\phi(f(u)), \end{aligned}$$

where $f(\cdot)$ is the inverse function of $F(\cdot)$. But for $u > 0$, let $\hat{x} = \frac{x}{f(u)}$ and define $L_\phi(1) = \{\hat{x} \mid \phi(\hat{x}) \geq 1, \hat{x} \in D\}$. Then

$$\begin{aligned} L_\phi(f(u)) &= \left\{x \mid \phi\left(\frac{x}{f(u)}\right) \geq 1, x \in D\right\} \\ &= \{x \mid \phi(\hat{x}) \geq 1, x \in D\}, \end{aligned}$$

since $\phi(\cdot)$ is homogeneous. Thus for any $u > 0$, $x \in L_F(u)$ if and only if $\hat{x} \in L_\phi(1)$ independently of u .

Consequently, for homothetic production structures and $u > 0$

$$Q(u, p) = \min_{x \in L_F(u)} (p \cdot x) = \min_{x \in L_F(u)} f(u) \left(p \cdot \frac{x}{f(u)} \right) = \min_{\hat{x} \in L_\phi(1)} f(u) (p \cdot \hat{x}),$$

and

$$\min_{\hat{x} \in L_\phi(1)} (p \cdot \hat{x}) = P(p)$$

where $P(p)$ is independent of u and homogeneous of degree one in the price vector p . If $u = 0$, clearly $L_F(u) = D$ and $Q(0, p) = 0$. Therefore, for and $p \in D$ and $u \in [0, \infty)$ the cost function $Q(u, p)$ is given by

$$Q(u, p) = f(u) \cdot P(p). \quad (5)$$

The properties of the homogeneous factor price function $P(p)$ follow from those for the cost function given above. First, if $p \in \{0\} \cup \mathcal{D}_2''$, then by Q.1 we have $P(p) = 0$, and by Q.2 it follows that $P(p) > 0$ for $p \in D_1 \cup \mathcal{D}_2'$.

Thus

$$P(p) \text{ is } \begin{cases} 0 & \forall p \in \{0\} \cup \mathcal{D}_2'' \\ > 0 & \forall p \in D_1 \cup \mathcal{D}_2' \end{cases}. \quad (6)$$

It is interesting to note that if the factor price vector p belongs to \mathcal{D}_2'' , then the scalar measure of this price vector is zero.

Property Q.3 is consistent with the homogeneity of the factor price function $P(p)$ and adds nothing new, while the properties Q.4, Q.5, Q.6, and Q.7 imply that the price function $P(p)$ is super-additive, nondecreasing, concave and continuous for $p \in D$.

Thus, the following proposition holds for the cost function of homothetic production structures:

Proposition 4: The cost function of a homothetic production structure is

$$Q(u, p) = f(u) \cdot P(p) \text{ where the factor price function } P(p)$$

has the following properties:

$$\text{HQ.1 } P(p) = 0 \text{ for all } p \in \{0\} \cup \mathcal{D}_2'',$$

- HQ.2 $P(p)$ is finite for finite $p \in D$ and $P(p) > 0$ for all $p \in D_1 \cup \mathcal{D}_2'$,
- HQ.3 $P(\lambda p) = \lambda P(p)$ for $\lambda \geq 0$ and all $p \in D$.
- HQ.4 $P(p + q) \geq P(p) + P(q)$ for all $p, q \in D$,
- HQ.5 $P(p') \geq P(p)$ if $p' \geq p \in D$,
- HQ.6 $P(p)$ is a concave function of p on D ,
- HQ.7 $P(p)$ is a continuous function of p on D .

The property Q.8 is consistent with the strictly increasing character of $F(u)$. But property Q.9 is strengthened to $Q(u, p) \rightarrow \infty$ as $u \rightarrow \infty$ for all $p \in D_1 \cup \mathcal{D}_2'$, since $P(p) > 0$ and $f(u) \rightarrow \infty$ as $u \rightarrow \infty$. The example of Figure 2 does not apply, since along any ray $\{\lambda x \mid \lambda \geq 0\}$ where $x \in D_1 \cup D_2'$ the homogeneous function $\phi(\cdot)$ is positive for $\lambda > 0$ and the level sets $L(u)$ cannot be supported by a hyperplane $p \cdot x = Q(u, p)$ for all $u > 0$. Similarly, the neighborhood $N(0)$ of property Q.10 cannot exist and $\inf Q(u, p) = 0$ for $p \in D_1 \cup \mathcal{D}_2'$ as $u \rightarrow 0$, since $f(u) \rightarrow 0$ as $u \rightarrow 0$. Finally property Q.11 is strengthened to: $Q(u, p)$ is a continuous function of u for all $p \in D$, since $f(u)$ is continuous.

Hence, regarding the properties of the cost function $Q(u, p)$ in respect to output rate for homothetic production structures, the following proposition holds:

Proposition 5: If the production structure is homothetic:

- HQ.8 For any $p \in D$, $Q(u_1, p) \geq Q(u_2, p)$ if $u_2 \geq u_1$,
- HQ.9 For any $p \in D_1 \cup \mathcal{D}_2'$, $Q(u, p) \rightarrow \infty$ as $u \rightarrow \infty$,
- HQ.10 For any $p \in D_1 \cup \mathcal{D}_2'$, $\inf Q(u, p) = 0$ as $u \rightarrow 0$,
- HQ.11 For any $p \in D$, $Q(u, p)$ is a continuous function of u for all $u \in [0, \infty)$.

The special form (5) of the cost function is of some interest for the study of changing returns to scale, because for any $p \in D_1 \cup \mathcal{D}_2'$ it implies

$$f(u) = \frac{Q(u,p)}{P(p)} \quad \text{or} \quad u = F\left(\frac{Q(u,p)}{P(p)}\right)$$

and, if cost data reflects minimum cost operation for the output rates and factor prices encountered, then $f(u)$ and hence $F(\cdot)$ may be investigated by studying the relation between output rate and factor price deflated costs. The function $F(\cdot)$ has direct meaning for changing returns to scale, since $\phi(x)$ has the properties of a scalar measure of input. It will be shown later that homotheticity of production structure is an if and only if condition for the factorization of the cost function given by equation (5), and thus for the use of factor price deflated costs to estimate changing returns to scale.

6. COST STRUCTURE

We note that the cost function $Q(u, p)$ has the properties of a distance function (see [5]) for a family of convex subsets in the factor price domain $D = \{p \mid p \geq 0\} \subset \mathbb{R}^n$, bounded by the unit cost surfaces $Q(u, p) = 1$, $u \in (0, \infty)$. These subsets of D define the cost structure for any output rate $u > 0$, which is not surprising, because for any $u > 0$ the locus $Q(u, p) = 1$ states all relevant cost information, since $Q(u, p)$ is linear homogeneous in p and the minimum total cost $Q(u, q)$ for any $q \in D_1 \cup \mathcal{D}_2'$ is derivable from $Q(u, p^0) = 1$ where $p^0 = \theta_0 \cdot q$ is the intersection of the ray $\{\theta q \mid \theta \geq 0\}$ with the unit cost locus $Q(u, p) = 1$. In fact $Q(u, p^0) = \theta_0 Q(u, q) = 1$ and $Q(u, q) = \frac{1}{\theta_0}$.

Hence in order to proceed carefully along these lines, we define the cost structure by the subsets

$$\mathcal{L}(u) = \{p \mid Q(u, p) \geq 1, p \in D\}, u \geq 0 \quad (7)$$

of the linear space \mathbb{R}^n . Corresponding to $u = 0$,

$$\mathcal{L}(0) = \{p \mid Q(0, p) \geq 1, p \in D\} = \emptyset,$$

the empty set. The set $\mathcal{L}(u)$ of the cost structure for any $u \in [0, \infty)$ is the set of price vectors which yield a minimum total cost equal to or greater than unity.

Before demonstrating that the cost function $Q(u, p)$ is a distance function for the sets $\mathcal{L}(u)$, $u > 0$, consider first the properties of the price sets of the cost structure which are summarized in the following proposition:

Proposition 6: The price sets of a cost structure $\mathcal{L}(u)$, $u \in [0, \infty)$ corresponding to a cost function $Q(u, p)$ have the following properties:

$\pi.1 \quad \mathcal{L}(0) = \emptyset$, the empty set, and $0 \notin \mathcal{L}(u)$ for any $u > 0$.

$$\mathcal{L}(\infty) = D_1 \cup \{p \mid p \in \mathcal{D}_2', Q(u,p) \geq 1 \text{ for some } u > 0\},$$

$\pi.2 \quad \text{If } p \in \mathcal{L}(u) \text{ and } p' \geq p, \text{ then } p' \in \mathcal{L}(u),$

$\pi.3 \quad \text{If } p > 0, \text{ or } p \geq 0 \text{ and } p \in \mathcal{L}(u) \text{ for some } u > 0,$
 then the ray $\{\theta p \mid \theta \geq 0\}$ intersects all price sets

$$\mathcal{L}(u), u > 0,$$

$\pi.4 \quad \mathcal{L}(u_2) \supset \mathcal{L}(u_1) \text{ if } u_2 \geq u_1,$

$\pi.5 \quad \bigcap_{u > u_0} \mathcal{L}(u) = \mathcal{L}(u_0),$

$\pi.6 \quad \bigcap_{u \in [0, \infty)} \mathcal{L}(u) \text{ is empty},$

$\pi.7 \quad \mathcal{L}(u) \text{ is closed for } u \in [0, \infty),$

$\pi.8 \quad \mathcal{L}(u) \text{ is convex for } u \in [0, \infty).$

We shall verify these properties in turn.

First, since $Q(0,p) = 0$ for all $p \in D$, it is evident that $\mathcal{L}(0)$ is empty; and the price vector 0 does not belong to any $\mathcal{L}(u)$ for $u > 0$ because $Q(u,0) = 0$ for all $u \geq 0$. For the second part of property $\pi.1$, suppose first that $p \in D_1$. Then by property Q.9 of the cost function, $Q(u,p) \rightarrow \infty$ as $u \rightarrow \infty$, and for any $p \in D_1$ we have $p \in \mathcal{L}(\infty)$. If $p \in \mathcal{D}_2'$, then according to the example of Figure 2, §4, it is possible to have $Q(u,p)$ fixed and bounded as $u \rightarrow \infty$, and hence $p \in \mathcal{L}(\infty)$ only if $Q(u,p) \geq 1$ for some $u > 0$; thus only if $p \in \{p \mid p \in \mathcal{D}_2', Q(u,p) \geq 1 \text{ for some } u > 0\}$ does $p \in \mathcal{L}(\infty)$. If $p \in \mathcal{D}_2''$, $Q(u,p) = 0$ for all $u > 0$ (see property Q.1 of the cost function) and $p \notin \mathcal{L}(\infty)$.

Property $\pi.2$ follows directly from property Q.5 of the cost function, since $p \in \mathcal{L}(u)$ implies $Q(u,p) \geq 1$ and $Q(u,p') \geq Q(u,p) \geq 1$, implying $p' \in \mathcal{L}(u)$.

Regarding property $\pi.3$, note that if $p > 0$ then $Q(u,p) > 0$ for all $u > 0$ (see property Q.2 of the cost function). Hence if $p > 0$, then for any $u > 0$

there is a positive scalar θ such that $Q(u, \theta p) = \theta Q(u, p) \geq 1$, and the ray $\{\theta p \mid \theta \geq 1\}$ intersects all price sets $\mathcal{L}(u)$ for $u > 0$. On the other hand, if $p \geq 0$ and $Q(u, p) \geq 1$ for some $u > 0$, then $p \in \mathcal{L}_2'$ (see Proposition 1) and, by property Q.2 of the cost function, $Q(u, p) > 0$ for $u > 0$, so that again the ray $\{\theta p \mid \theta \geq 1\}$ intersects all price sets $\mathcal{L}(u)$ for $u > 0$.

Property $\pi.4$ follows directly from the property Q.8 of the cost function, since if $p \in \mathcal{L}(u_1)$ then $Q(u_1, p) \geq 1$ and for $u_2 \geq u_1$ we have $Q(u_2, p) \geq Q(u_1, p) \geq 1$, whence $p \in \mathcal{L}(u_2)$. Thus $\mathcal{L}(u_1) \subset \mathcal{L}(u_2)$.

Property $\pi.5$ may be established as follows. First, if $p \in \mathcal{L}(u_0)$ then by $\pi.4$ we have $p \in \mathcal{L}(u)$ for all $u > u_0$ and $p \in \bigcap_{u > u_0} \mathcal{L}(u)$. Contrary wise, if $p \in \bigcap_{u > u_0} \mathcal{L}(u)$ then $p \in \mathcal{L}(u_0)$, because if $p \notin \mathcal{L}(u_0)$ there is a $\bar{u} > u_0$ such that $Q(\bar{u}, p) < 1$ and $p \notin \mathcal{L}(\bar{u})$, a contradiction.

Property $\pi.6$ is obvious, since $\mathcal{L}(0)$ is empty. We note that $\bigcap_{u > 0} \mathcal{L}(u)$ is not necessarily empty due to the possibility illustrated in Figure 2 above.

Property $\pi.7$ holds because for any $u > 0$ the cost function $Q(u, p)$ is a continuous function of p on D and therefore upper semi-continuous on D which implies that

$$\{p \mid Q(u, p) \geq Q_0, p \in D\},$$

is closed for all numbers Q_0 , because this property is an if and only if condition for the upper semi-continuity of $Q(u, p)$ in p on D . (See [2], p76.) Hence, for $Q_0 = 1$

$$\mathcal{L}(u) = \{p \mid Q(u, p) \geq 1, p \in D\}$$

is closed for any $u > 0$. For $u = 0$, $\mathcal{L}(0)$ is empty.

Finally, property $\pi.8$ follows directly from the concavity in p on D of the cost function, i.e., property Q.6. Let $p \in \mathcal{L}(u)$, $q \in \mathcal{L}(u)$, then for any scalar $\theta \in [0,1]$ and any $u > 0$

$$Q(u, (1 - \theta)p + \theta q) \geq (1 - \theta)Q(u, p) + \theta Q(u, q),$$

recalling that $Q(u, p)$ is homogeneous in the price vector p , and since $p \in \mathcal{L}(u) \Rightarrow Q(u, p) \geq 1$, $q \in \mathcal{L}(u) \Rightarrow Q(u, q) \geq 1$ it follows that $Q(u, (1 - \theta)p + \theta q) \geq 1$. Hence the point $[(1 - \theta)p + \theta q] \in \mathcal{L}(u)$. If $u = 0$, $\mathcal{L}(0)$ is empty.

Now we may verify that the cost function $Q(u, p)$ is a distance function for the price sets $\mathcal{L}(u)$ of the cost structure. Consider any price vector $p \in D$. If $p \in \{0\} \cup \mathcal{D}_2''$ we note that $Q(u, p) = 0$ (property 0.1) and due to the homogeneity of the cost function $Q(u, \theta p) = 0$ for all scalars $\theta \geq 0$. Hence, the ray $\{\theta p \mid \theta \geq 0\}$ does not intersect any price set $\mathcal{L}(u)$ and for the reasons explained in [5] the distance function may be taken zero, i.e., $Q(u, p) = 0$ if $p \in \{0\} \cup \mathcal{D}_2''$. Now suppose $p \in D_1 \cup \mathcal{D}_2'$ (see Figure 6), then $Q(u, p) > 0$ (property Q.2) and by property $\pi.3$ of the price sets $\mathcal{L}(u)$ it follows that the ray $\{\theta p \mid \theta \geq 0\}$ intersects all sets $\mathcal{L}(u)$ for $u > 0$. For any $u > 0$, we may define

$$\theta_o = \min_{\theta p \in \mathcal{L}(u)} \theta,$$

since the price set $\mathcal{L}(u)$ is closed. Let $\xi = \theta_o p$. The distance ratio

$$\frac{\|p\|}{\|\xi\|} = \frac{1}{\theta_o}$$

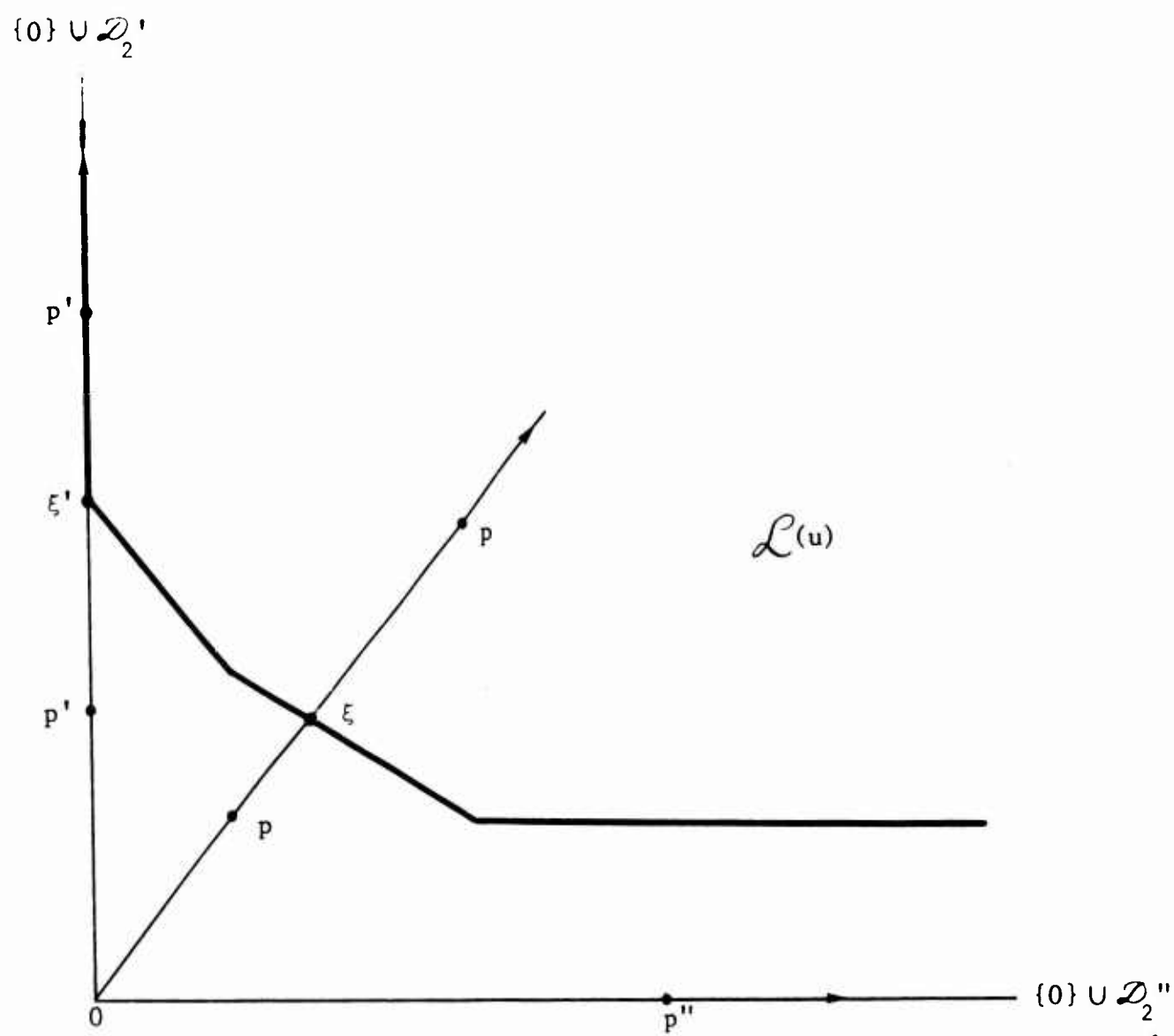


FIGURE 6: INTERSECTIONS OF PRICE RAYS WITH $\mathcal{L}(u)$

and

$$Q(u, p) = Q\left(u, \frac{\xi}{\theta_0}\right) = \frac{1}{\theta_0} Q(u, \xi) .$$

But, by the definition of the price point ξ it lies on the boundary of the closed set $\mathcal{L}(u)$ and $Q(u, \xi) = 1$. Thus

$$Q(u, p) = \begin{cases} \frac{||p||}{||\xi||} & \text{for } p \in D_1 \cup \mathcal{D}_2' \\ 0 & \text{for } p \in \{0\} \cup \mathcal{D}_2'' \end{cases} \quad (8)$$

and the cost function is a distance function for the price sets of the cost structure.

7. EFFICIENT PRICE VECTORS OF THE COST STRUCTURE

Following the definition of the efficient points of a production possibility set $L(u)$, see §4 of [4], we use the following definition of an efficient price vector p of the price set $\mathcal{L}(u)$:

Definition: A price vector $p \in \mathcal{L}(u)$ is efficient relative to the price set $\mathcal{L}(u)$ if and only if $0(u, q) < 1$ for all price vectors $q \leq p$.[†]

Hence, for any output rate u a price vector is efficient if and only if the minimum total cost is less than unity for all price vectors which are equal to or less than but not equal to the given price vector.

Definition: The efficient subset $\mathcal{E}(u)$ of a price set $\mathcal{L}(u)$ of the cost structure is defined by

$$\mathcal{E}(u) = \{p \mid p \in \mathcal{L}(u), Q(u, q) < 1 \ \forall \ q \leq p\}$$

From a cost-factor price standpoint, the efficient price vectors are those which for the given output rate cannot be decreased without making the minimum total cost less than unity.

Now, in all essential respects so far as efficiency is concerned, the price sets $\mathcal{L}(u)$ have the same properties in regards to the price vectors p as the production possibility sets $L(u)$ have in terms of the input vectors x - compare $\pi.2, \pi.3, \pi.4, \pi.7, \pi.8$ with $P.2, P.3, P.4, P.7$, and $P.8$.

In particular, the argument given in §4 of [4] to show that $\mathcal{E}(u)$ is nonempty may be used here to verify that:

[†] $q \leq p \Rightarrow q_1 \leq p_1$, but $q \neq p$.

Proposition 7: The efficient point set $\mathcal{E}(u)$ of a price set

$\mathcal{L}(u)$ is nonempty.

The counter example referred to in [4], see [1], shows that $\mathcal{E}(u)$ need not be closed. However, for our purposes it will be sufficient to work with the closure $\bar{\mathcal{E}}(u)$ of $\mathcal{E}(u)$, and $\bar{\mathcal{E}}(u) \subset \mathcal{L}(u)$ since $\mathcal{L}(u)$ is closed.

For reasons explained in [4] it is suitable to assume that the efficient point set $E(u)$ of a production possibility set $L(u)$ is bounded. But the question remains whether boundedness of $E(u)$ implies that $\mathcal{E}(u)$ is bounded.

We note first that if $p \in \mathcal{E}(u)$ then p belongs to the boundary of $\mathcal{L}(u)$. Now let $p \in \mathcal{E}(u)$ and suppose that p is unbounded, i.e., for at least one factor of production, say the i^{th} , p_i is unbounded. First, if $p \in D_1$, then since $Q(u, p)$ is a distance function for the price set $\mathcal{L}(u)$ and $p \in \text{Boundary } \mathcal{L}(u)$ it follows that $Q(u, p) = 1$. But $Q(u, p) = p \cdot x^*(u, p)$ where $x^*(u, p)$ belongs to $\bar{E}(u)$ (see (3), §1 above) and $x^*(u, p)$ is positive and bounded in all components, if the efficient set $E(u)$ of $L(u)$ is bounded. Hence, if $p \in D_1$, $p \in \mathcal{E}(u)$ and $\bar{E}(u)$ is bounded, then p is bounded. The only uncertainty arises when $p \in \mathcal{D}_2'$. But here too, $Q(u, p) = 1$ if $p \in \mathcal{E}(u)$, because otherwise if $Q(u, p) = Q_0 > 1$ then $Q\left(u, \frac{p}{Q_0}\right) = 1$ due to the homogeneity of the cost function and for $p/Q_0 < p$ we have $Q(u, p/Q_0) = 1$, a contradiction to the efficiency of the point p . But again $p \cdot x^*(u, p) = 1$, where $x^*(u, p)$ is bounded but not necessarily positive in all components. Here we may have a component p_i unbounded only if the corresponding component $x_i^*(u, p)$ is zero. However, such a price point p cannot be efficient, because any bounded non-negative value of p_i yields $Q(u, p) = 1$. Thus, if $p \in \mathcal{D}_2'$, $p \in \mathcal{E}(u)$ and $\bar{E}(u)$ is bounded, then p is bounded. Hence the following proposition holds:

Proposition 8: If the efficient set $E(u)$ of $L(u)$ is bounded,
 then the efficient set $\mathcal{E}(u)$ of $\mathcal{L}(u)$ is bounded.

Finally, by a proof which parallels that given in [4] for the property
 $L(u) = E(u) + D$, one may verify the following proposition and corollary:

$$\pi.9 \quad \mathcal{L}(u) = \mathcal{E}(u) + D ,$$

$$\pi.10 \quad \mathcal{L}(u) = \bar{\mathcal{E}}(u) + D .$$

For the study of the cost structure we shall assume that $E(u)$ is bounded for
 each $u \in [0, \infty)$ and correspondingly $\mathcal{E}(u)$ is bounded for each $u \in (0, \infty)$.

8. MINIMUM OUTPUT FUNCTION

Recall that the production function $\phi(x)$ is definable as the maximum output corresponding to a given input vector x , relative to the production structure (production possibility sets $L(u)$, $u \in [0, \infty)$). In an analogous way we may define a minimum output function $\Gamma(p)$ relative to the cost structure as the minimum output corresponding to a given factor price vector p , i.e., the minimum output for the price vector p to belong to a price set $\mathcal{L}(u)$ or to yield at least unit minimum total cost.

For any given cost structure $\mathcal{L}(u)$, $u \in (0, \infty)$, having the properties $\pi.1$, ... $\pi.8$ consider the function $\Gamma(p)$ defined on the price sets $\mathcal{L}(u)$ by

$$\Gamma(p) = \inf_{p \in \mathcal{L}(u)} u \quad (9)$$

We need to determine on which subset of D the function $\Gamma(p)$ is defined.

Clearly when $p \in \{0\} \cup \mathcal{D}_2''$, $Q(u, p) = 0$ for all $u > 0$ (see property Q.1 of the cost function) and $p \notin \mathcal{L}(u)$ for any $u > 0$. Hence the function $\Gamma(p)$ is not defined for $p \in \{0\} \cup \mathcal{D}_2''$.

Now consider price vectors p belonging to $D_1 \cup \mathcal{D}_2'$. We note first that if $p \in D_1$ then $\inf_{u \rightarrow 0} Q(u, p)$ is greater than zero or equal to zero according as there

exists or does not exist for $\delta > 0$ an open neighborhood

$$N(0) = \left\{ x \mid \|x\| < \delta, x \in D \right\} \text{ such that } x \notin L(u) \text{ for any } u > 0 \text{ when}$$

$x \in N(0)$ (see property Q.10). Also, by the property $\pi.4$ the sets $\mathcal{L}(u)$ are contained in their predecessors for decreasing u . Hence, if $p \in D_1$,

$\inf_{u \rightarrow 0} Q(u, p) \geq 0$. Also, by property Q.8, $Q(u, p) \rightarrow \infty$ monotonically as $u \rightarrow \infty$

for all $p \in D_1$. Thus, if $\inf_{u \rightarrow 0} Q(u, p) \geq 1$ when the neighborhood $N(0)$ exists,

then $p \in \mathcal{L}(u)$ for all $u > 0$ and $\inf_{p \in \mathcal{L}(u)} u = 0$; otherwise, there exists a

finite, positive value of output rate, say \bar{u} , such that $p \in \mathcal{L}(u)$ for all $u > \bar{u}$ and $u_0 = \inf_{p \in \mathcal{L}(u)} u$ is positive and finite. Since u_0 is the infimum of u it follows that $p \in \mathcal{L}(u)$ for all $u > u_0$ and, by the property $\pi.5$, $\mathcal{L}(u_0) = \bigcap_{u > u_0} \mathcal{L}(u)$ so that also $p \in \mathcal{L}(u_0)$. Then $p \in \mathcal{L}(u)$ for all u in the closed interval $[u_0, \infty)$ and $u_0 = \min_{p \in \mathcal{L}(u)}$. Thus, if $p \in D_1$, and $\inf_{u \rightarrow 0} Q(u, p) \geq 1$, $\Gamma(p) = 0$, otherwise $\Gamma(p) > 0$ and \inf may be replaced by \min in the definition of $\Gamma(p)$. Next, if $p \in \mathcal{D}_2'$, it may happen that $Q(u, p) = Q_0 > 0$ and finite for all $u \in (0, \infty)$ (see discussion of property Q.9), and if $Q_0 < 1$ the function $\Gamma(p)$ is not defined, since $Q(u, p) < 1$ for all $u \in (0, \infty)$. However, if $Q_0 \geq 1$, then $Q(u, p) \geq 1$ for all $u \in (0, \infty)$, in which case $\Gamma(p) = 0$.

Thus the Minimum Output Function (9) is defined for $p \in D_1$ and $p \in \{p \mid p \in \mathcal{D}_2', Q(u, p) \geq 1 \text{ for some } u > 0\}$. If $p \in D_1$ and $\inf_{u \rightarrow 0} Q(u, p) < 1$, then \inf may be replaced by \min in the definition of $\Gamma(p)$ and $\Gamma(p) > 0$, otherwise $\Gamma(p) = 0$. If $p \in \mathcal{D}_2'$ and for some $u > 0$, $Q(u, p) \geq 1$, the function $\Gamma(p)$ is defined, but in this case $Q(u, p)$ may be equal to or greater than one for all $u > 0$ with $\Gamma(p) = 0$ and otherwise $\Gamma(p) > 0$ and \inf may be replaced by \min in the definition of $\Gamma(p)$. The possibility that $\Gamma(p)$ is not defined on the boundary of D is a natural consequence of the general properties of the cost function.

The minimum output function $\Gamma(p)$ gives the minimum output corresponding to any $p \in D_1 \cup \{p \mid p \in \mathcal{D}_2', Q(u, p) \geq 1 \text{ for some } u > 0\}$ for minimum total cost to be at least unity. It is, in a sense, a dual of the production function and it will be shown later (see [3]) that the production function can in fact be interpreted as the maximum output corresponding to any $x \in D_1 \cup D_2'$ for minimum

total cost to be at least unity, where cost is minimized with respect to factor prices instead of factor inputs.

The properties of the Minimum Output Function are summarized in the following proposition:

Proposition 9: The minimum output function $\Gamma(p)$ for a cost structure $\mathcal{L}(u)$, $u \in (0, \infty)$ satisfying the properties $\pi.1, \dots, \pi.8$ has the following properties:

- $\alpha.1$ $\Gamma(p)$ is not defined for $p \in \{0\} \cup \mathcal{D}_2''$, and $\Gamma(p) = 0$ if $p \in D_1 \cup \mathcal{D}_2'$ and $\inf_{u \rightarrow 0} Q(u, p) \geq 1$,
- $\alpha.2$ $\Gamma(p)$ is finite for finite $p \in D_1 \cup \{p \mid p \in \mathcal{D}_2', Q(u, p) \geq 1 \text{ for some } u > 0\}$,
- $\alpha.3$ $\Gamma(p') \leq \Gamma(p)$ for $p' \geq p \in D_1 \cup \mathcal{D}_2'$,
- $\alpha.4$ If $p > 0$ or $p \geq 0$ and $\Gamma(\theta p) > 0$ for some scalar $\theta > 0$, then $\Gamma(\theta p) \rightarrow 0$ as $\theta \rightarrow \infty$,
- $\alpha.5$ $\Gamma(p)$ is lower semi-continuous on $D_1 \cup \mathcal{D}_2'$,
- $\alpha.6$ $\Gamma(p)$ is quasi-convex on $D_1 \cup \mathcal{D}_2'$.

The property $\alpha.1$ is merely a restatement of the properties of $\Gamma(p)$ described above following the definition of the minimum output function.

Regarding property $\alpha.2$, we need not concern ourselves with the cases where $\Gamma(p) = 0$. Hence, consider $p \in D_1$ where $\Gamma(p) > 0$, and, by the property Q.9 of the cost function $Q(u, p)$, $Q(u, p) \rightarrow \infty$ monotonically as $u \rightarrow \infty$ and for some finite $u > 0$ we have $Q(u, p) \geq 1$. If $p \in \mathcal{D}_2'$, and $\Gamma(p)$ is defined and $\Gamma(p) > 0$, the same argument applies.

Property $\alpha.3$ follows directly from the property $\pi.2$. Suppose

$u_0 = \inf_{p \in \mathcal{L}(u)} u$. Then $p \in \mathcal{L}(u_0)$, and by $\pi.2$ $p' \in \mathcal{L}(u_0)$ if $p' \geq p$, whence

$$\inf_{p' \in \mathcal{L}(u)} u \leq u_0.$$

For property $\alpha.4$, it follows from the property $\pi.4$ of the price sets $\mathcal{L}(u)$ that the ray $\{\theta p \mid \theta \geq 0\}$ intersects all price sets $\mathcal{L}(u)$, $u > 0$, and $Q(u, \theta p) = \theta Q(u, p)$ (homogeneity of the cost function). As $\theta \rightarrow \infty$ monotonically, the sequence $\{\Gamma(\theta p)\}$ is monotone nonincreasing (property $\alpha.3$) bounded below by zero and $\lim_{\theta \rightarrow \infty} \Gamma(\theta p)$ equals $u_0 \geq 0$. Suppose $u_0 > 0$ and let $0 < \bar{u} < u_0$. Then $Q(\bar{u}, \theta p) = \theta Q(\bar{u}, p)$, where $Q(\bar{u}, p) > 0$ since the ray $\{\theta p \mid \theta \geq 0\}$ intersects all $\mathcal{L}(u)$ for $u > 0$, and by taking θ large enough, say $\bar{\theta}$, $Q(\bar{u}, \bar{\theta} p) \geq 1$. Hence, a contradiction, since $u_0 > 0$ implies $\Gamma(\theta p) \geq u_0$ for all $\theta > 0$, but $\Gamma(\bar{\theta} p) < u_0$.

In order to show the property $\alpha.5$, consider the level sets of the function $\Gamma(p)$ defined by

$$\mathcal{L}'(u) = \{p \mid \Gamma(p) \leq \bar{u}, p \in D\}, \bar{u} > 0.$$

If $\Gamma(p) \leq \bar{u}$, then $\inf_{p \in \mathcal{L}(u)} u \leq \bar{u}$ which implies $Q(\bar{u}, p) \geq 1$, since $Q(\bar{u}, p) \geq Q(u, p)$ for $\bar{u} \geq u$ by property Q.8; thus $p \in \mathcal{L}(\bar{u})$. Hence if $p \in \mathcal{L}'(\bar{u})$ then $p \in \mathcal{L}(\bar{u})$ and $\mathcal{L}'(\bar{u}) \subset \mathcal{L}(\bar{u})$. Contrarywise, if $p \in \mathcal{L}(\bar{u})$ then $Q(\bar{u}, p) \geq 1$ and $\Gamma(p) = \inf_{p \in \mathcal{L}(u)} u \leq \bar{u}$ so that $p \in \mathcal{L}'(\bar{u})$. Hence also $\mathcal{L}'(\bar{u}) \subset \mathcal{L}(\bar{u})$ and therefore $\mathcal{L}'(\bar{u}) = \mathcal{L}(\bar{u})$. Now, since the level sets $\mathcal{L}(\bar{u})$ are closed (property $\pi.7$) it follows that the level sets $\mathcal{L}'(\bar{u})$ are closed. Therefore, the function $\Gamma(p)$ is lower semi-continuous, because the closure of the level sets $\mathcal{L}'(\bar{u})$ is an if and only if property for the lower semi-continuity of $\Gamma(p)$ ([2], p76).

Finally, regarding property $\alpha.6$, it follows from the fact that the level sets

$\mathcal{L}'(u)$ of $\Gamma(p)$ are identical to the price sets $\mathcal{L}(u)$ and the convexity of the sets $\mathcal{L}(u)$ for any $u > 0$, that the level sets $\mathcal{L}'(u)$ are convex. Hence the function $\Gamma(p)$ is quasi-convex, i.e., $\Gamma((1 - \theta)p + \theta q) \leq \text{Max} [\Gamma(p), \Gamma(q)]$, for all $\theta \in [0, 1]$, because let $\bar{u} = \text{Max} [\Gamma(p), \Gamma(q)]$, then $p \in \mathcal{L}'(\bar{u})$, $q \in \mathcal{L}'(\bar{u})$ and $[(1 - \theta)p + \theta q] \in \mathcal{L}'(\bar{u})$ for all $\theta \in [0, 1]$, due to the convexity of $\mathcal{L}'(\bar{u})$, and

$$\Gamma((1 - \theta)p + \theta q) \leq \text{Max} [\Gamma(p), \Gamma(q)]$$

for all $\theta \in [0, 1]$ and any $p, q \in D_1 \cup \mathcal{D}_2'$.

9. COST STRUCTURE OF HOMOTHETIC PRODUCTION STRUCTURE

For a homothetic production structure the cost function has the special form

$$Q(u, p) = f(u) \cdot P(p) ,$$

where $P(p)$ is a homogeneous function having the properties HQ.1, ... HQ.11 (see §5 above), and $f(u)$ is a nonnegative, strictly increasing, continuous function of output rate with $f(0) = 0$.

The corresponding price sets $\mathcal{L}(u)$ of the cost structure are homothetic, as are the production possibility sets $L(u)$ of the production structure. Let $\{\theta p \mid \theta \geq 0\}$ be an arbitrary ray in the price domain D for $p \in D_1 \cup \mathcal{D}_2'$. Denote by ξ and η the intersections of this ray with the boundaries of the price sets $\mathcal{L}(u)$ and $\mathcal{L}(1)$ respectively, i.e., the price vectors $\xi = \theta_1 p$, $\eta = \theta_2 p$ for which $Q(u, \theta_1 p) = 1$ and $Q(1, \theta_2 p) = 1$, as illustrated in Figure 7. Since $P(p)$ is homogeneous (property HQ.3), it follows that

$$Q(u, \theta_1 p) = f(u)P(\theta_1 p) = \theta_1 f(u)P(p)$$

$$Q(1, \theta_2 p) = f(1)P(\theta_2 p) = \theta_2 f(1)P(p)$$

and

$$\theta_1 = \frac{f(1)}{f(u)} \cdot \theta_2 .$$

Hence

$$\xi = \frac{f(1)}{f(u)} \cdot \eta$$

and the price point ξ on the price set $\mathcal{L}(u)$ is obtained from the point η for $\mathcal{L}(1)$ by radial contraction with a scalar $f(1)/f(u)$ independently of the price direction $p \in D_1 \cup \mathcal{D}_2'$.

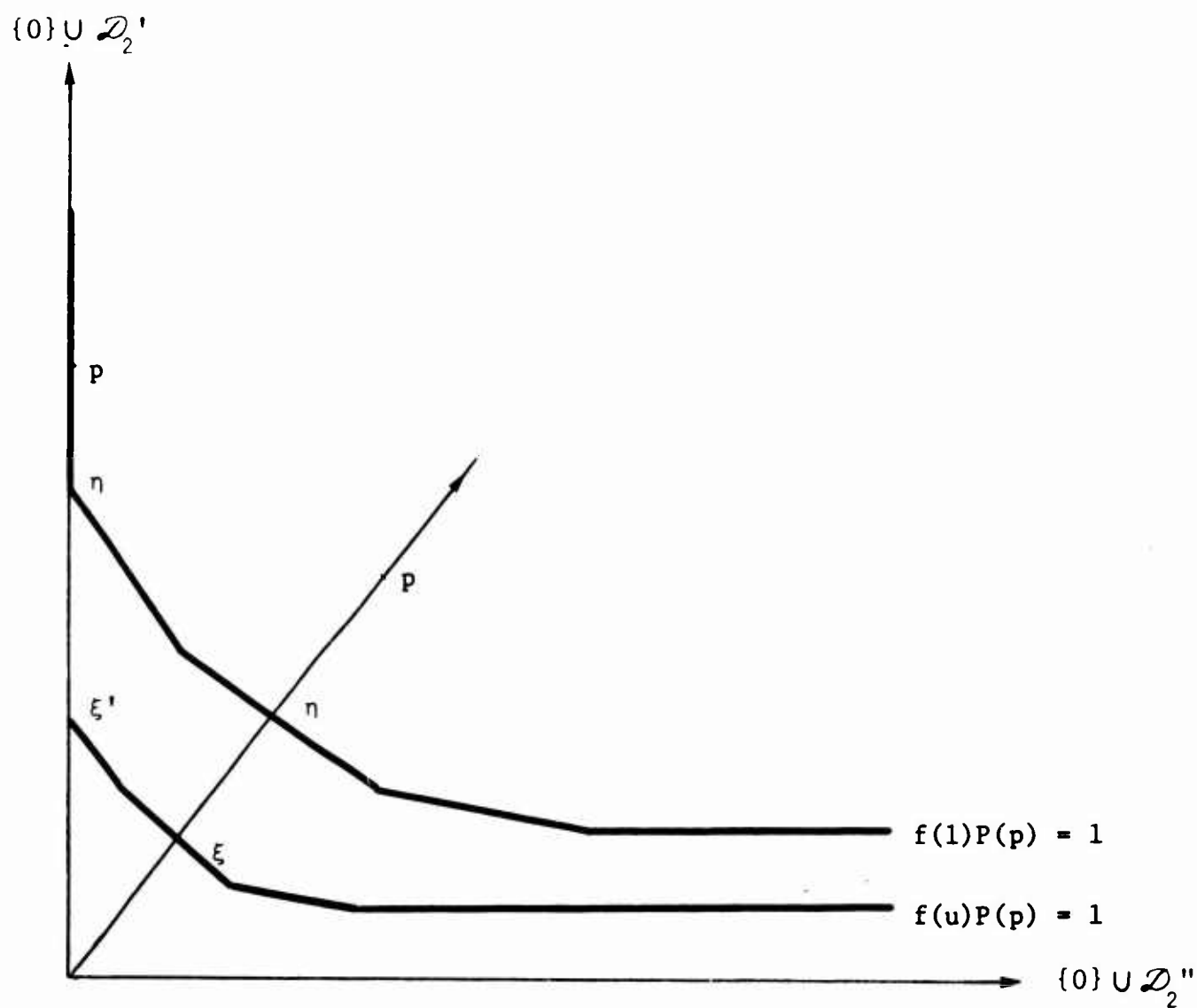


FIGURE 7: INTERSECTIONS OF PRICE RAYS WITH PRICE SETS $\mathcal{L}(u)$ FOR A HOMOTHETIC PRODUCTION STRUCTURE

The properties $\pi.1, \dots \pi.8$ (see Proposition 6) hold for the price set $\mathcal{L}(u)$, but they may be strengthened somewhat. First, since $f(u)$ is strictly increasing in u , there is for any $p \in \mathcal{D}_2'$ an output rate u such that $Q(u, p) = f(u)P(p) \geq 1$, since $P(p) > 0$ (HQ.2). Thus $\mathcal{L}(\infty) = D_1 \cup \mathcal{D}_2'$ in property $\pi.1$. Second, property $\pi.6$ may be strengthened to $\bigcap_{u>0} \mathcal{L}(u)$ is empty, because suppose a finite $p < \bar{p}$ belongs to $\bigcap_{u>0} \mathcal{L}(u)$. Then $P(p)$ is positive and finite (HQ.2) and there exists an output rate $\bar{u} > 0$ such that $f(\bar{u})P(p) < 1$, since $f(u) \rightarrow 0$ monotonically as $u \rightarrow 0$, and $p \notin \mathcal{L}(\bar{u})$, hence, $p \notin \bigcap_{u>0} \mathcal{L}(u)$, a contradiction.

In summary:

Proposition 10: If the production structure is homothetic, the cost structure is homothetic and the price sets $\mathcal{L}(u)$ satisfy:

H $\pi.1$ $\mathcal{L}(0)$ is empty, $0 \notin \mathcal{L}(u)$ for any $u > 0$

and $\mathcal{L}(\infty) = D_1 \cup \mathcal{D}_2'$,

H $\pi.6$ $\bigcap_{u>0} \mathcal{L}(u)$ is empty.

and the remaining properties of Proposition 6 apply as stated.

The minimum output function $\Gamma(p)$ for homothetic production structures is given by

$$\Gamma(p) = \min_{p \in \mathcal{L}(u)} u, \quad p \in D_1 \cup \mathcal{D}_2', \quad (9.1)$$

because by property HQ.10 $\inf Q(u, p) = 0$ as $u \rightarrow 0$ for any $p \in D_1 \cup \mathcal{D}_2'$ and the minimal output rate exists. Moreover, the first two properties stated in Proposition 9 are modified to:

H α .1 $\Gamma(p)$ is not defined for $p \in \{0\} \cup \mathcal{D}_2''$,

H α .2 $\Gamma(p)$ is finite for finite $p \in D_1 \cup \mathcal{D}_2'$,

while the fifth property is strengthened to

H α .5 $\Gamma(p)$ is a continuous function on $D_1 \cup \mathcal{D}_2'$.

From the properties HC 8 and HQ.9 we have $Q(u,p) \rightarrow \infty$ monotonically as $u \rightarrow \infty$ and for finite $u > 0$, $Q(u,p) \geq 1$ for any $p \in D_1 \cup \mathcal{D}_2'$, whence property H α .2 holds.

The strengthening of the fifth property follows from the special form

$$\Gamma(p) = F\left(\frac{1}{P(p)}\right) \quad (9.2)$$

of the minimum output function.

In order to verify that equation (9.2) is valid, consider an arbitrary point $p \in D_1 \cup \mathcal{D}_2'$ where $P(p) > 0$. Clearly $\min_{p \in \mathcal{L}(u)} u$ satisfies $Q(u,p) = 1$ or

$$f(\Gamma(p)) \cdot P(p) = 1,$$

and, since $f(\cdot)$ is the inverse function of $F(\cdot)$,

$$\Gamma(p) = F\left(\frac{1}{P(p)}\right).$$

Then, since F is a continuous function and $P(p)$ is continuous on $D_1 \cup \mathcal{D}_2'$ (property HQ.7), it follows that $\Gamma(p)$ is continuous on $D_1 \cup \mathcal{D}_2'$.

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