AD647926

ARL 66-0198
OCTOBER 1966

# Aerospace Research Laboratories

## POINT AND INTERVAL ESTIMATORS, BASED ON m ORDER STATISTICS, FOR THE SCALE PARAMETER OF A WEIBULL POPULATION WITH KNOWN SHAPE PARAMETER

(Reprint from Technometrics, Vol. 7, No. 3, August 1965)

H. LEON HARTER

APPLIED MATHEMATICS RESEARCH LABORATORY

ALBERT H. MOORE

AIR FORCE INSTITUTE OF TECHNOLOGY



Distribution of this document is unlimited

**OFFICE OF AEROSPACE RESEARCH**

**United States Air Force**

## NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

. . . . . . . . . .

Qualified requesters may obtain copies of this report from the Defense Documentation Center, (DDC), Cameron Station, Alexandria, Virginia.

(Reproduction in whole or in part is permitted for any purpose of the U.S. Gov't)

Distribution of this document is unlimited



Copies of ARL Technical Documentary Reports should not be returned to Aerospace Research Laboratories unless return is required by security considerations, contractual obligations or notices on a specified document.

TECHNOMETRICS VOL. 7, NO. 3 AUGUST, 1965

# Point and Interval Estimators, Based on m Order Statistics, for the Scale Parameter of a Weibull Population with Known Shape Parameter

H. LEON HARTER AND ALBERT H. MOORE

*Aerospace Research Laboratories and Air Force Institute of Technology Wright-Patterson Air Force Base*

A derivation is given of the maximum likelihood estimator $\hat{\theta}$, based on the first $m$ out of $n$ ordered observations, of the scale parameter $\theta$ of a Weibull population with known shape parameter $K$. It is shown that $2m\,\hat{\theta}^K/\theta^K$ has a chi-square distribution with $2m$ degrees of freedom (independent of $n$). Use is made of this fact to set upper confidence bounds with confidence level $1 - P$ (lower confidence bounds with confidence level $P$) on the scale parameter $\theta$. Formulas are given for the mean squared deviations of the upper and lower confidence bounds from the true value of the parameter. From these one can obtain expressions for the efficiency of confidence bounds and confidence intervals. The expected value of $\hat{\theta}$ is also determined, and from it the unbiasing factor $\tilde{\theta}/\hat{\theta}$ by which $\hat{\theta}$ must be multiplied to obtain an unbiased estimator $\tilde{\theta}$. An expression for the variance of the unbiased estimator $\tilde{\theta}$ is found. Values of the unbiasing factor and of the variance of the unbiased estimator, both of which are independent of $n$, are tabled for $m = 1(1)100$ and $K = 0.5(0.5)4.0(1.0)8.0$. A section on use of the table and a numerical example are included.

## 1. INTRODUCTION

Epstein and Sobel (1953) have pointed out the advantages of the use of ordered data from truncated tests to estimate the parameters of parent populations, and have worked out details for the exponential distribution. In particular, they have derived an estimator $\hat{\sigma}$ (which is maximum likelihood, unbiased, and minimum variance), based on the first $m$ out of $n$ ordered observations, of the parameter $\sigma$ of an exponential population and have shown that $2m\hat{\sigma}/\sigma$ has a chi-square distribution with $2m$ degrees of freedom (independent of $n$). They have also given without derivation the maximum likelihood estimator $\hat{\theta}^K$, based on the first $m$ out of $n$ ordered observations, of $\theta^K$, where $\theta$ is the scale parameter of a Weibull population with known shape parameter $K$. N. R. Mann (1963, p. 39) has stated without proof that $2m\hat{\theta}^K/\theta^K$ has a chi-square distribution with $2m$ degrees of freedom. The missing derivation and proof are supplied in the present paper. Expressions are given for upper and lower confidence bounds, $\bar{\theta}$ and $\underline{\theta}$, and for the efficiencies, as defined by Harter (1964b, c), of $\bar{\theta}$ and the central confidence interval $(\underline{\theta}, \bar{\theta})$. Brief discussions of the method of computation of the table and of its use are given, as well as a numerical example which illustrates the computation of both point and interval estimates and the efficiencies of both point and interval estimators.

* Received Feb. 1964; final revision Dec. 1964.

## 2. Maximum Likelihood Estimator for the Scale Parameter

The probability density function of the random variable $Y$ having a Weibull distribution with location parameter 0, scale parameter $\theta$, and shape parameter $K$ is given by

$$(1) \qquad f(y) = \begin{cases} (K/\theta)(y/\theta)^{K-1} \exp[-(y/\theta)^K], & y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now if we define the random variable $X$ by $X = Y^K$ and make the change of variable $x = y^K$ in (1), we find the probability density function of the random variable $X$ to be

$$(2) \qquad g(x) = \begin{cases} \exp(-x/\theta^K)/\theta^K, & x > 0 \\ 0 & \text{elsewhere,} \end{cases}$$

which is the familiar exponential density function with parameter $\sigma = \theta^K$. Therefore if $Y$ has a Weibull distribution with scale parameter $\theta$ and shape parameter $K$ and if $X = Y^K$, then $X$ is exponentially distributed with parameter $\sigma = \theta^K$. Hence a maximum likelihood $m$-order-statistic estimator for $\theta$ can be obtained from the "best" $m$-order-statistic estimator for $\sigma = \theta^k$ derived by Epstein and Sobel (1953), which is given by

$$(3) \qquad \hat{\sigma}_{mn} = [x_{1n} + x_{2n} + \cdots + x_{mn} + (n - m)x_{mn}]/m,$$

where $x_{in}$ $(i = 1, 2, \cdots, m)$ are the first $m$ order statistics of a sample of size $n$ from an exponential population. Now, taking the $K$-th root of both sides of (3), we obtain

$$(4) \qquad \hat{\sigma}_{mn}^{1/K} = \{[x_{1n} + x_{2n} + \cdots + x_{mn} + (n - m)x_{mn}]/m\}^{1/K}.$$

Since $x_{in} = y_{in}^K$ we can write

$$(5) \qquad \hat{\theta}_{mn} = \hat{\sigma}_{mn}^{1/K} = \{[y_{1n}^K + y_{2n}^K + \cdots + y_{mn}^K + (n - m)y_{mn}^K]/m\}^{1/K}.$$

Now, since $\hat{\sigma}_{mn}$ is a maximum likelihood estimator of $\sigma$, $\hat{\theta}_{mn} = \hat{\sigma}_{mn}^{1/K}$ is a maximum likelihood estimator of $\theta = \sigma^{1/K}$.

The probability density function of the random variable $X_1 = \hat{\sigma}_{mn}$ is given by Epstein and Sobel (1953) as

$$(6) \qquad f_m(x_1) = \begin{cases} [1/\Gamma(m)](m/\sigma)^m x_1^{m-1} \exp(-mx_1/\sigma), & x_1 > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now we let $\hat{\theta}_{mn} = \hat{\sigma}_{mn}^{1/K}$ or $Y_1 = X_1^{1/K}$ and we find the probability density function of $Y_1 = \hat{\theta}_{mn}$ to be

$$(7) \qquad g_m(y_1) = \begin{cases} [K/\Gamma(m)](m/\sigma)^m y_1^{Km-1} \exp(-my_1^K/\sigma), & y_1 > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now, making the substitution $\sigma = \theta^K$ in (7), we obtain

$$g_m(y_1) = \begin{cases} [K/\Gamma(m)](m/\theta^K)^m y_1^{Km-1} \exp(-my_1^K/\theta^K), & y_1 > 0 \\ 0 \quad \text{elsewhere}. \end{cases} \tag{8}$$

Hereafter, for simplicity, we will denote $\hat{\theta}_{mn}$ by $\hat{\theta}$.

## 3. Confidence Bounds for Scale Parameter $\theta$

From (8) it can be easily seen that $2m\hat{\theta}^K/\theta^K$ has a chi-square distribution with $2m$ degrees of freedom:

$$2m\hat{\theta}^K/\theta^K = \chi^2_{2m}. \tag{9}$$

Solving for $\theta$, we obtain

$$\theta = (2m/\chi^2_{2m})^{1/K}\hat{\theta} \tag{10}$$

Then an upper confidence bound with confidence level $1 - P$ (lower confidence bound with confidence level $P$) on $\theta$ is given by

$$\bar{\theta}_{1-P} = \underline{\theta}_P = (2m/\chi^2_{2m,P})^{1/K}\hat{\theta}, \tag{11}$$

where the first subscript on $\chi^2$ is the number of degrees of freedom and the second one is the cumulative probability. The interval between lower and upper confidence bounds, each with confidence level $1 - P$, will be call a central confidence interval with confidence level $1 - 2P$. Equations (9)-(11) remain valid when $m = n$, in which case (11) is an expression for the conventional confidence bound based on all $n$ observations.

## 4. Efficiency of Confidence Bounds and Intervals

Harter (1964b, c) has defined the efficiency of a substitute upper confidence bound as the ratio, expressed as a percentage, of the mean squared deviation of the conventional upper confidence bound from the true parameter value to the mean squared deviation of the substitute upper confidence bound from the true parameter value. This definition may be expressed symbolically in the form

$$E_u = 100E[(\bar{\theta} - \theta)^2]/E[(\bar{\theta}' - \theta)^2], \tag{12}$$

where $E_u$ is the efficiency (in percent) of the substitute upper confidence bound, $E[(\bar{\theta} - \theta)^2]$ is the mean squared deviation of the conventional upper confidence bound $\bar{\theta}$ from the true value $\theta$ of the parameter, and $E[(\bar{\theta}' - \theta)^2]$ is the mean squared deviation of the substitute upper confidence bound $\bar{\theta}'$ from the true value $\theta$ of the parameter. Further, the efficiency of a substitute central confidence interval is defined as the ratio, expressed as a percentage, of the sum of the mean squared deviations of the conventional upper and lower confidence bounds from the true parameter value to the sum of the mean squared deviations of the substitute upper and lower confidence bounds from the true parameter value. This definition in symbolic form is given by

$$E_c = 100\{E[(\bar{\theta} - \theta)^2] + E[(\underline{\theta} - \theta)^2]\}/\{E[(\bar{\theta}' - \theta)^2] + E[(\underline{\theta}' - \theta)^2]\}, \tag{13}$$

where $E_c$ is the efficiency (in percent) of the substitute central confidence interval, $E[(\bar{\theta} - \theta)^2]$ and $E[(\bar{\theta}' - \theta)^2]$ are defined as above, and $E[(\underline{\theta} - \theta)^2]$

and $E[(\underline{\theta}' - \theta)^2]$ are respectively the mean squared deviations of the conventional and substitute lower confidence bounds $\underline{\theta}$ and $\underline{\theta}'$ from the true value $\theta$ of the parameter.

Quayle (1963) has shown that the mean squared deviation $E[(\bar{\theta} - \theta)^2]$ of the conventional upper confidence bound with confidence level $1 - P$, based on all $n$ observations, from the true value $\theta$ of the scale parameter of a Weibull population with known shape parameter $K$ is given by

$$(14) \qquad E[(\bar{\theta} - \theta)^2] = 1 - 2^{1+1/K}[\Gamma(n + 1/K)/\Gamma(n)][1/\chi^2_{2n,P}]^{1/K} + 2^{2/K}[\Gamma(n + 2/K)/\Gamma(n)] \times [1/\chi^2_{2n,P}]^{2/K}$$

and that the mean squared deviation $E[(\underline{\theta} - \theta)^2]$ of the corresponding conventional lower confidence bound is found by replacing $\bar{\theta}$ by $\underline{\theta}$ and $P$ by $1 - P$ in (14). Since $2n\hat{\theta}^K_{nn}/\theta^K$ is distributed as $\chi^2$ with $2n$ degrees of freedom and $2m\hat{\theta}^K/\theta^K$ as $\chi^2$ with $2m$ degrees of freedom, the mean squared deviations of the substitute confidence bounds based on the first $m$ order statistics are found as follows: $E[(\bar{\theta}' - \theta)^2]$ by replacing $\bar{\theta}$ by $\bar{\theta}'$ and $n$ by $m$ in (14); $E([\underline{\theta}' - \theta)^2]$ by replacing $\bar{\theta}$ by $\underline{\theta}'$, $n$ by $m$, and $P$ by $1 - P$ in (14). Substitution of the results in (12) and (13) then yields the efficiencies of the substitute upper confidence bound and the substitute central confidence interval, respectively, as compared with the conventional bound and interval based on all $n$ observations.

## 5. Unbiased Estimator for the Scale Parameter

The expected value of $\hat{\theta}$ is found by using (8) to be

$$(15) \qquad E(\hat{\theta}) = \{\theta\Gamma(m + 1/K)\}\{m^{1/K}\Gamma(m)\}.$$

Hence an unbiased estimator of $\theta$ is given by

$$(16) \qquad \tilde{\theta} = [m^{1/K}\Gamma(m)/\Gamma(m + 1/K)]\hat{\theta}.$$

The variance of the unbiased estimator $\tilde{\theta}$ is found to be

$$(17) \qquad \mathrm{Var}\ \tilde{\theta} = \theta^2\{[\Gamma(m)\Gamma(m + 2/K)/\Gamma^2(m + 1/K)] - 1\}.$$

Values of the unbiasing factor $\tilde{\theta}/\hat{\theta}$ for a maximum likelihood estimator and of the ratio $\mathrm{Var}\ \tilde{\theta}/\theta^2$ of the variance of an unbiased estimator to $\theta^2$, expressions for which can be obtained by dividing both sides or (16) by $\hat{\theta}$ and both sides of (17) by $\theta^2$, were computed for $m = 1(1)100$ and $K = 0.5(0.5)4.0(1.0)8.0$. The computations were performed on the IBM 1620 computer with FORTRAN programming, use being made of Stirling's approximation to the Gamma function. Twelve decimal digits were carried in the computations, but the values of the unbiasing factor were rounded to 6 decimal places (6 or 7 significant digits) and those of the variance to 8 decimal places (5 to 9 significant digits). The results are shown in Table 1.

## 6. Use of Table

In life-testing situations, one may wish to terminate the test without waiting for all $n$ of the items placed on test to fail. If the life distribution is Weibull with known shape parameter $K$, where $K$ is one of the values included in Table 1,

TABLE 1

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 0.5$

| $m$ | $\hat{\theta}/\hat{\theta}$ | Var $\hat{\theta}/\theta^2$ | $m$ | $\hat{\theta}/\hat{\theta}$ | Var $\hat{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | .500000 | 5.0000000 | 51 | .980769 | .07918552 |
| 2 | .666667 | 2.3333333 | 52 | .981132 | .07764877 |
| 3 | .750000 | 1.5000000 | 53 | .981481 | .07617051 |
| 4 | .800000 | 1.1000000 | 54 | .981818 | .07474747 |
| 5 | .833333 | .86666667 | 55 | .982143 | .07337662 |
| 6 | .857143 | .71428571 | 56 | .982456 | .07205[illegible] |
| 7 | .875000 | .60714286 | 57 | .982758 | .07078040 |
| 8 | .888889 | .52777778 | 58 | .983051 | .06954997 |
| 9 | .900000 | .46666667 | 59 | .983333 | .06836158 |
| 10 | .909091 | .41818182 | 60 | .983607 | .06721311 |
| 11 | .916667 | .37878788 | 61 | .983871 | .06610259 |
| 12 | .923077 | .34615385 | 62 | .984127 | .06502816 |
| 13 | .928571 | .31868132 | 63 | .984375 | .06398810 |
| 14 | .933333 | .29523810 | 64 | .984615 | .06298077 |
| 15 | .937500 | .27500000 | 65 | .984848 | .06200466 |
| 16 | .941176 | .25735294 | 66 | .985075 | .06105834 |
| 17 | .944444 | .24183007 | 67 | .985294 | .06014047 |
| 18 | .947368 | .22807018 | 68 | .985507 | .05924979 |
| 19 | .950000 | .21578947 | 69 | .985714 | .05838509 |
| 20 | .952381 | .20476190 | 70 | .985915 | .05754527 |
| 21 | .954545 | .19480519 | 71 | .986111 | .05672926 |
| 22 | .956522 | .18577075 | 72 | .986301 | .05593607 |
| 23 | .958333 | .17753623 | 73 | .986486 | .05516475 |
| 24 | .960000 | .17000000 | 74 | .986667 | .05441441 |
| 25 | .961538 | .16307692 | 75 | .986842 | .05368421 |
| 26 | .962963 | .15669516 | 76 | .987013 | .05297334 |
| 27 | .964286 | .15079365 | 77 | .987180 | .05228105 |
| 28 | .965517 | .14532020 | 78 | .987342 | .05160662 |
| 29 | .966667 | .14022989 | 79 | .987500 | .05094937 |
| 30 | .967742 | .13548387 | 80 | .987654 | .05030864 |
| 31 | .968750 | .13104839 | 81 | .987805 | .04968383 |
| 32 | .969697 | .12689304 | 82 | .987952 | .04907435 |
| 33 | .970588 | .12299465 | 83 | .988095 | .04847963 |
| 34 | .971429 | .11932773 | 84 | .988235 | .04789916 |
| 35 | .972222 | .11587302 | 85 | .988372 | .04733242 |
| 36 | .972973 | .11261261 | 86 | .988506 | .04677894 |
| 37 | .973684 | .10953058 | 87 | .988636 | .04623824 |
| 38 | .974359 | .10661269 | 88 | .988764 | .04570991 |
| 39 | .975000 | .10384615 | 89 | .988889 | .04519351 |
| 40 | .975610 | .10121951 | 90 | .989011 | .04468864 |
| 41 | .976190 | .09872242 | 91 | .989131 | .04419404 |
| 42 | .976744 | .09634551 | 92 | .989247 | .04371201 |
| 43 | .977273 | .09408034 | 93 | .989362 | .04323953 |
| 44 | .977778 | .09191919 | 94 | .989474 | .04277716 |
| 45 | .978261 | .08985507 | 95 | .989583 | .04232456 |
| 46 | .978723 | .08788159 | 96 | .989691 | .04188114 |
| 47 | .979167 | .08599201 | 97 | .989796 | .04144751 |
| 48 | .979592 | .08418367 | 98 | .989899 | .04102247 |
| 49 | .980000 | .08244898 | 99 | .990000 | .04060606 |
| 50 | .980392 | .08078431 | 100 | .990099 | .04019802 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 1.0$

| $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ | $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.000000 | 1.00000000 | 51 | 1.000000 | .01960784 |
| 2 | 1.000000 | .50000000 | 52 | 1.000000 | .01923077 |
| 3 | 1.000000 | .33333333 | 53 | 1.000000 | .01886792 |
| 4 | 1.000000 | .25000000 | 54 | 1.000000 | .01851852 |
| 5 | 1.000000 | .20000000 | 55 | 1.000000 | .01818182 |
| 6 | 1.000000 | .16666667 | 56 | 1.000000 | .01785714 |
| 7 | 1.000000 | .14285714 | 57 | 1.000000 | .01754386 |
| 8 | 1.000000 | .12500000 | 58 | 1.000000 | .01724138 |
| 9 | 1.000000 | .11111111 | 59 | 1.000000 | .01694915 |
| 10 | 1.000000 | .10000000 | 60 | 1.000000 | .01666667 |
| 11 | 1.000000 | .09090909 | 61 | 1.000000 | .01639344 |
| 12 | 1.000000 | .08333333 | 62 | 1.000000 | .01612903 |
| 13 | 1.000000 | .07692308 | 63 | 1.000000 | .01587302 |
| 14 | 1.000000 | .07142857 | 64 | 1.000000 | .01562500 |
| 15 | 1.000000 | .06666667 | 65 | 1.000000 | .01538462 |
| 16 | 1.000000 | .06250000 | 66 | 1.000000 | .01515152 |
| 17 | 1.000000 | .05882353 | 67 | 1.000000 | .01492537 |
| 18 | 1.000000 | .05555556 | 68 | 1.000000 | .01470588 |
| 19 | 1.000000 | .05263158 | 69 | 1.000000 | .01449275 |
| 20 | 1.000000 | .05000000 | 70 | 1.000000 | .01428571 |
| 21 | 1.000000 | .04761905 | 71 | 1.000000 | .01408451 |
| 22 | 1.000000 | .04545455 | 72 | 1.000000 | .01388889 |
| 23 | 1.000000 | .04347826 | 73 | 1.000000 | .01369863 |
| 24 | 1.000000 | .04166667 | 74 | 1.000000 | .01351351 |
| 25 | 1.000000 | .04000000 | 75 | 1.000000 | .01333333 |
| 26 | 1.000000 | .03846154 | 76 | 1.000000 | .01315789 |
| 27 | 1.000000 | .03703704 | 77 | 1.000000 | .01298701 |
| 28 | 1.000000 | .03571429 | 78 | 1.000000 | .01282051 |
| 29 | 1.000000 | .03448276 | 79 | 1.000000 | .01265823 |
| 30 | 1.000000 | .03333333 | 80 | 1.000000 | .01250000 |
| 31 | 1.000000 | .03225806 | 81 | 1.000000 | .01234568 |
| 32 | 1.000000 | .03125000 | 82 | 1.000000 | .01219512 |
| 33 | 1.000000 | .03030303 | 83 | 1.000000 | .01204819 |
| 34 | 1.000000 | .02941176 | 84 | 1.000000 | .01190476 |
| 35 | 1.000000 | .02857143 | 85 | 1.000000 | .01176471 |
| 36 | 1.000000 | .02777778 | 86 | 1.000000 | .01162791 |
| 37 | 1.000000 | .02702703 | 87 | 1.000000 | .01149425 |
| 38 | 1.000000 | .02631579 | 88 | 1.000000 | .01136364 |
| 39 | 1.000000 | .02564103 | 89 | 1.000000 | .01123596 |
| 40 | 1.000000 | .02500000 | 90 | 1.000000 | .01111111 |
| 41 | 1.000000 | .02439024 | 91 | 1.000000 | .01098901 |
| 42 | 1.000000 | .02380952 | 92 | 1.000000 | .01086957 |
| 43 | 1.000000 | .02325581 | 93 | 1.000000 | .01075269 |
| 44 | 1.000000 | .02272727 | 94 | 1.000000 | .01063830 |
| 45 | 1.000000 | .02222222 | 95 | 1.000000 | .01052632 |
| 46 | 1.000000 | .02173913 | 96 | 1.000000 | .01041667 |
| 47 | 1.000000 | .02127660 | 97 | 1.000000 | .01030928 |
| 48 | 1.000000 | .02083333 | 98 | 1.000000 | .01020408 |
| 49 | 1.000000 | .02040816 | 99 | 1.000000 | .01010101 |
| 50 | 1.000000 | .02000000 | 100 | 1.000000 | .01000000 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 1.5$

| $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ | $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.107732 | .40099849 | 51 | 1.002179 | .00872401 |
| 2 | 1.055040 | .22723873 | 52 | 1.002137 | .00855607 |
| 3 | 1.036879 | .15053631 | 53 | 1.002096 | .00839447 |
| 4 | 1.027710 | .11250205 | 54 | 1.002058 | .00823886 |
| 5 | 1.022187 | .08979793 | 55 | 1.002020 | .00808891 |
| 6 | 1.018498 | .07471422 | 56 | 1.001984 | .00794432 |
| 7 | 1.015800 | .06396708 | 57 | 1.001949 | .00780482 |
| 8 | 1.013880 | .05592196 | 58 | 1.001916 | .00767012 |
| 9 | 1.012340 | .04967390 | 59 | 1.00188[illegible] | .00754000 |
| 10 | 1.011107 | .04468139 | 60 | 1.001852 | .00741422 |
| 11 | 1.010098 | .04060061 | 61 | 1.001821 | .00729257 |
| 12 | 1.009257 | .03720273 | 62 | 1.001792 | .00717484 |
| 13 | 1.008545 | .03432959 | 63 | 1.001764 | .00706085 |
| 14 | 1.007935 | .03186847 | 64 | 1.001736 | .00695043 |
| 15 | 1.007406 | .02973941 | 65 | 1.001709 | .00684341 |
| 16 | 1.006943 | .02787179 | 66 | 1.001683 | .00673964 |
| 17 | 1.006535 | .02622719 | 67 | 1.001658 | .00663897 |
| 18 | 1.006172 | .02476585 | 68 | 1.001634 | .00654126 |
| 19 | 1.005847 | .02345875 | 69 | 1.001610 | .00644638 |
| 20 | 1.005555 | .02228270 | 70 | 1.001587 | .00635422 |
| 21 | 1.005291 | .02121803 | 71 | 1.001565 | .00626465 |
| 22 | 1.005050 | .02025205 | 72 | 1.001543 | .00617758 |
| 23 | 1.004831 | .01936952 | 73 | 1.001522 | .00609289 |
| 24 | 1.004629 | .01856066 | 74 | 1.001501 | .00601040 |
| 25 | 1.004444 | .01781064 | 75 | 1.001481 | .00593029 |
| 26 | 1.004273 | .01712997 | 76 | 1.001462 | .00585221 |
| 27 | 1.004115 | .01649427 | 77 | 1.001443 | .00577615 |
| 28 | 1.003968 | .01590405 | 78 | 1.001424 | .00570204 |
| 29 | 1.003831 | .01535462 | 79 | 1.001406 | .00562981 |
| 30 | 1.003704 | .01484188 | 80 | 1.001389 | .00555939 |
| 31 | 1.003584 | .01436227 | 81 | 1.001372 | .00549071 |
| 32 | 1.003472 | .01391270 | 82 | 1.001355 | .00542371 |
| 33 | 1.003367 | .01349041 | 83 | 1.001339 | .00535832 |
| 34 | 1.003268 | .01309300 | 84 | 1.001323 | .00529449 |
| 35 | 1.003174 | .01271833 | 85 | 1.001307 | .00523216 |
| 36 | 1.003086 | .01236452 | 86 | 1.001292 | .00517128 |
| 37 | 1.003003 | .01202985 | 87 | 1.001277 | .00511180 |
| 38 | 1.002924 | .01171282 | 88 | 1.001263 | .00505368 |
| 39 | 1.002849 | .01141208 | 89 | 1.001248 | .00499686 |
| 40 | 1.002778 | .01112639 | 90 | 1.001235 | .00494131 |
| 41 | 1.002710 | .01085465 | 91 | 1.001221 | .00488697 |
| 42 | 1.002645 | .01059587 | 92 | 1.001208 | .00483382 |
| 43 | 1.002584 | .01034914 | 93 | 1.001195 | .00478181 |
| 44 | 1.002525 | .01011365 | 94 | 1.001182 | .00473091 |
| 45 | 1.002469 | .00988863 | 95 | 1.001170 | .00468109 |
| 46 | 1.002415 | .00967340 | 96 | 1.001157 | .00463230 |
| 47 | 1.002364 | .00946735 | 97 | 1.001145 | .00458451 |
| 48 | 1.002315 | .00926989 | 98 | 1.001134 | .00453771 |
| 49 | 1.002268 | .00908049 | 99 | 1.001122 | .00449185 |
| 50 | 1.002222 | .00889869 | 100 | 1.001111 | .00444690 |

TABLE 1 (*continued*)
*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*
Shape Parameter $K = 2.0$

| $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ | $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.128379 | .27323954 | 51 | 1.002454 | .00491392 |
| 2 | 1.063846 | .13176848 | 52 | 1.002407 | .00481919 |
| 3 | 1.042352 | .08649774 | 53 | 1.002361 | .00472805 |
| 4 | 1.031661 | .06432432 | 54 | 1.002317 | .00464030 |
| 5 | 1.025273 | .05118452 | 55 | 1.002275 | .00455574 |
| 6 | 1.021027 | .04249704 | 56 | 1.002235 | .00447421 |
| 7 | 1.018002 | .03632842 | 57 | 1.002195 | .00439554 |
| 8 | 1.015737 | .03172251 | 58 | 1.002157 | .00431959 |
| 9 | 1.013979 | .02815254 | 59 | 1.002121 | .00424623 |
| 10 | 1.012573 | .02530447 | 60 | 1.002085 | .00417531 |
| 11 | 1.011424 | .02297952 | 61 | 1.002051 | .00410672 |
| 12 | 1.010468 | .02104572 | 62 | 1.002018 | .00404035 |
| 13 | 1.009659 | .01941205 | 63 | 1.001986 | .00397610 |
| 14 | 1.008967 | .01801368 | 64 | 1.001955 | .00391385 |
| 15 | 1.008367 | .01680320 | 65 | 1.001925 | .00385352 |
| 16 | 1.007842 | .01574513 | 66 | 1.001896 | .00379503 |
| 17 | 1.007379 | .01481240 | 67 | 1.001867 | .00373828 |
| 18 | 1.006968 | .01398398 | 68 | 1.001840 | .00368320 |
| 19 | 1.006600 | .01324330 | 69 | 1.001813 | .00362973 |
| 20 | 1.006269 | .01257713 | 70 | 1.001787 | .00357778 |
| 21 | 1.005970 | .01197477 | 71 | 1.001762 | .00352730 |
| 22 | 1.005697 | .01142746 | 72 | 1.001738 | .00347823 |
| 23 | 1.005449 | .01092799 | 73 | 1.001714 | .00343050 |
| 24 | 1.005222 | .01047035 | 74 | 1.001691 | .00338407 |
| 25 | 1.005012 | .01004949 | 75 | 1.001668 | 00333887 |
| 26 | 1.004819 | .00966116 | 76 | 1.001646 | 00329487 |
| 27 | 1.004640 | .00930172 | 77 | 1.001625 | .00325201 |
| 28 | 1.004474 | .00896807 | 78 | 1.001604 | .00321025 |
| 29 | 1.004319 | .00865752 | 79 | 1.001584 | .00316955 |
| 30 | 1.004175 | .00836776 | 80 | 1.001564 | .00312987 |
| 31 | 1.004040 | .00809677 | 81 | 1.001544 | .00309117 |
| 32 | 1.003914 | .00784278 | 82 | 1.001526 | .00305341 |
| 33 | 1.003795 | .00760423 | 83 | 1.001507 | .00301657 |
| 34 | 1.003683 | .00737977 | 84 | 1.001489 | .00298061 |
| 35 | 1.003578 | .00716818 | 85 | 1.001472 | .00294549 |
| 36 | 1.003478 | .00696839 | 86 | 1.001455 | .00291119 |
| 37 | 1.003384 | .00677943 | 87 | 1.001438 | .00287768 |
| 38 | 1.003295 | .00660045 | 88 | 1.001421 | .00284493 |
| 39 | 1.003210 | .00643067 | 89 | 1.001405 | .00281292 |
| 40 | 1.003130 | .00626941 | 90 | 1.001390 | .00278163 |
| 41 | 1.003053 | .00611604 | 91 | 1.001375 | .00275102 |
| 42 | 1.002981 | .00596999 | 92 | 1.001360 | .00272107 |
| 43 | 1.002911 | .00583076 | 93 | 1.001345 | .00269178 |
| 44 | 1.002845 | .00569787 | 94 | 1.001331 | .00266310 |
| 45 | 1.002782 | .00557090 | 95 | 1.001317 | .10263503 |
| 46 | 1.002721 | .00544947 | 96 | 1.001303 | .00260755 |
| 47 | 1.002663 | .00533322 | 97 | 1.001289 | .00258063 |
| 48 | 1.002608 | .00522183 | 98 | 1.001276 | .00255427 |
| 49 | 1.002554 | .00511499 | 99 | 1.001263 | .00252843 |
| 50 | 1.002503 | .00501244 | 100 | 1.001251 | .00250312 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter $\theta$ of Weibull Population*

Shape Parameter $K = 2.5$

| $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ | $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.127060 | .18310455 | 51 | 1.002357 | .00314830 |
| 2 | 1.062261 | .08652458 | 52 | 1.002312 | .00308754 |
| 3 | 1.041086 | .05634335 | 53 | 1.002268 | .00302909 |
| 4 | 1.030634 | .04172268 | 54 | 1.002226 | .00297281 |
| 5 | 1.024414 | .03311340 | 55 | 1.002185 | .00291859 |
| 6 | 1.020292 | .02744474 | 56 | 1.002146 | .00286630 |
| 7 | 1.017359 | .02343128 | 57 | 1.002109 | .00281586 |
| 8 | 1.015167 | .02044098 | 58 | 1.002072 | .00276716 |
| 9 | 1.013466 | .01812705 | 59 | 1.002037 | .00272012 |
| 10 | 1.012108 | .01628345 | 60 | 1.002003 | .00267465 |
| 11 | 1.010999 | .01478008 | 61 | 1.001970 | .00263067 |
| 12 | 1.010075 | .01353073 | 62 | 1.001938 | .00258812 |
| 13 | 1.009295 | .01247607 | 63 | 1.001908 | .00254692 |
| 14 | 1.008627 | .01157389 | 64 | 1.001878 | .00250701 |
| 15 | 1.008049 | .01079335 | 65 | 1.001849 | .00246834 |
| 16 | 1.007543 | .01011142 | 66 | 1.001821 | .00243084 |
| 17 | 1.007097 | .00951052 | 67 | 1.001794 | .00239446 |
| 18 | 1.006701 | .00897702 | 68 | 1.001767 | .00235916 |
| 19 | 1.006346 | .00850019 | 69 | 1.001741 | .00232488 |
| 20 | 1.006027 | .00807145 | 70 | 1.001717 | .00229158 |
| 21 | 1.005739 | .00768388 | 71 | 1.001692 | .00225922 |
| 22 | 1.005477 | .00733182 | 72 | 1.001669 | .00222777 |
| 23 | 1.005238 | .00701061 | 73 | 1.001646 | .00219717 |
| 24 | 1.005019 | .00671635 | 74 | 1.001624 | .00216741 |
| 25 | 1.004818 | .00644580 | 75 | 1.001602 | .00213844 |
| 26 | 1.004632 | .00619820 | 76 | 1.001581 | .00211024 |
| 27 | 1.004460 | .00596521 | 77 | 1.001560 | .00208277 |
| 28 | 1.004300 | .00575082 | 78 | 1.001540 | .00205601 |
| 29 | 1.004151 | .00555131 | 79 | 1.001521 | .00202992 |
| 30 | 1.004012 | .00536517 | 80 | 1.001502 | .00200449 |
| 31 | 1.003882 | .00519111 | 81 | 1.001483 | .00197969 |
| 32 | 1.003761 | .00502799 | 82 | 1.001465 | .00195549 |
| 33 | 1.003647 | .00487481 | 83 | 1.001447 | .00193188 |
| 34 | 1.003539 | .00473069 | 84 | 1.001430 | .00190884 |
| 35 | 1.003438 | .00459484 | 85 | 1.001413 | .00188633 |
| 36 | 1.003342 | .00446657 | 86 | 1.001397 | .00186435 |
| 37 | 1.003251 | .00434528 | 87 | 1.001381 | .00184288 |
| 38 | 1.003166 | .00423039 | 88 | 1.001365 | .00182189 |
| 39 | 1.003084 | .00412143 | 89 | 1.001350 | .00180138 |
| 40 | 1.003007 | .00401793 | 90 | 1.001335 | .00178133 |
| 41 | 1.002933 | .00391951 | 91 | 1.001320 | .00176171 |
| 42 | 1.002863 | .00382579 | 92 | 1.001306 | .00174253 |
| 43 | 1.002797 | .00373645 | 93 | 1.001292 | .00172375 |
| 44 | 1.002733 | .00365119 | 94 | 1.001278 | .00170538 |
| 45 | 1.002672 | .00356973 | 95 | 1.001264 | .00168740 |
| 46 | 1.002614 | .00349183 | 96 | 1.001251 | .00166979 |
| 47 | 1.002558 | .00341725 | 97 | 1.001238 | .00165254 |
| 48 | 1.002505 | .00334579 | 98 | 1.001226 | .00163565 |
| 49 | 1.002454 | .00327726 | 99 | 1.001213 | .00161910 |
| 50 | 1.002404 | .00321149 | 100 | 1.001201 | .00160288 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter $\theta$ of Weibull Population*

Shape Parameter $K = 3.0$

| $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ | $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.119847 | .13209336 | 51 | 1.002183 | .00218813 |
| 2 | 1.058189 | .06133753 | 52 | 1.002141 | .00214587 |
| 3 | 1.038277 | .03967758 | 53 | 1.002101 | .00210521 |
| 4 | 1.028495 | .02928080 | 54 | 1.002062 | .00206607 |
| 5 | 1.022689 | .02319038 | 55 | 1.002024 | .00202835 |
| 6 | 1.018846 | .01919354 | 56 | 1.001988 | .00199199 |
| 7 | 1.016115 | .01637029 | 57 | 1.001953 | .00195691 |
| 8 | 1.014075 | .01427035 | 58 | 1.001919 | .00192304 |
| 9 | 1.012494 | .01264752 | 59 | 1.001887 | .00189032 |
| 10 | 1.011231 | .01135588 | 60 | 1.001855 | .00185870 |
| 11 | 1.010201 | .01030348 | 61 | 1.001825 | .00182812 |
| 12 | 1.009343 | .00942952 | 62 | 1.001795 | .00179853 |
| 13 | 1.008619 | .00869217 | 63 | 1.001767 | .00176988 |
| 14 | 1.007998 | .00806173 | 64 | 1.001739 | .00174213 |
| 15 | 1.007461 | .00751654 | 65 | 1.001712 | .00171524 |
| 16 | 1.006992 | .00704040 | 66 | 1.001686 | .00168916 |
| 17 | 1.006578 | .00662097 | 67 | 1.001661 | .00166387 |
| 18 | 1.006210 | .00624870 | 68 | 1.001637 | .00163932 |
| 19 | 1.005882 | .00591606 | 69 | 1.001613 | .00161549 |
| 20 | 1.005586 | .00561704 | 70 | 1.001590 | .00159234 |
| 21 | 1.005319 | .00534678 | 71 | 1.001567 | .00156984 |
| 22 | 1.005076 | .00510133 | 72 | 1.001546 | .00154797 |
| 23 | 1.004854 | .00487743 | 73 | 1.001524 | .00152670 |
| 24 | 1.004651 | .00467235 | 74 | 1.001504 | .00150601 |
| 25 | 1.004464 | .00448383 | 75 | 1.001484 | .00148587 |
| 26 | 1.004292 | .00430992 | 76 | 1.001464 | .00146626 |
| 27 | 1.004132 | .00414900 | 77 | 1.001445 | .00144716 |
| 28 | 1.003984 | .00399966 | 78 | 1.001427 | .00142856 |
| 29 | 1.003846 | .00386070 | 79 | 1.001408 | .00141042 |
| 30 | 1.003717 | .00373107 | 80 | 1.001391 | .00139274 |
| 31 | 1.003597 | .00360986 | 81 | 1.001374 | .00137550 |
| 32 | 1.003484 | .00349628 | 82 | 1.001357 | .00135868 |
| 33 | 1.003378 | .00338962 | 83 | 1.001340 | .00134227 |
| 34 | 1.003279 | .00328928 | 84 | 1.001324 | .00132625 |
| 35 | 1.003185 | .00319471 | 85 | 1.001309 | .00131060 |
| 36 | 1.003096 | .00310543 | 86 | 1.001294 | .00129533 |
| 37 | 1.003012 | .00302100 | 87 | 1.001279 | .00128040 |
| 38 | 1.002932 | .00294104 | 88 | 1.001264 | .00126581 |
| 39 | 1.002857 | .00286520 | 89 | 1.001250 | .00125155 |
| 40 | 1.002785 | .00279318 | 90 | 1.001236 | .00123761 |
| 41 | 1.002717 | .00272469 | 91 | 1.001222 | .00122398 |
| 42 | 1.002652 | .00265947 | 92 | 1.001209 | .00121064 |
| 43 | 1.002591 | .00259731 | 93 | 1.001196 | .00119760 |
| 44 | 1.002532 | .00253798 | 94 | 1.001183 | .00118483 |
| 45 | 1.002475 | .00248131 | 95 | 1.001171 | .00117232 |
| 46 | 1.002421 | .00242711 | 96 | 1.001159 | .00116008 |
| 47 | 1.002370 | .00237523 | 97 | 1.001147 | .00114810 |
| 48 | 1.002320 | .00232551 | 98 | 1.001135 | .00113636 |
| 49 | 1.002273 | .00227784 | 99 | 1.001124 | .00112485 |
| 50 | 1.002227 | .00223208 | 100 | 1.001112 | .00111358 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 3.5$

| $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ | $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.111423 | .10014607 | 51 | 1.002006 | .00160864 |
| 2 | 1.053765 | .04581787 | 52 | 1.001967 | .00157756 |
| 3 | 1.035293 | .02947697 | 53 | 1.001930 | .00154765 |
| 4 | 1.026247 | .02169264 | 54 | 1.001894 | .00151885 |
| 5 | 1.020886 | .01715179 | 55 | 1.001859 | .00149111 |
| 6 | 1.017341 | .01417983 | 56 | 1.001826 | .00146436 |
| 7 | 1.014824 | .01208442 | 57 | 1.001794 | .00143856 |
| 8 | 1.012945 | .01052796 | 58 | 1.001763 | .00141365 |
| 9 | 1.011488 | .00932639 | 59 | 1.001733 | .00138958 |
| 10 | 1.010326 | .00837081 | 60 | 1.001704 | .00136633 |
| 11 | 1.009378 | .00759275 | 61 | 1.001676 | .00134383 |
| 12 | 1.008589 | .00694696 | 62 | 1.001649 | .00132207 |
| 13 | 1.007922 | .00640237 | 63 | 1.001623 | .00130100 |
| 14 | 1.007351 | .00593692 | 64 | 1.001597 | .00128059 |
| 15 | 1.006857 | .00553455 | 65 | 1.001573 | .00126081 |
| 16 | 1.006426 | .00518324 | 66 | 1.001549 | .00124164 |
| 17 | 1.006045 | .00487386 | 67 | 1.001526 | .00122304 |
| 18 | 1.005707 | .00459932 | 68 | 1.001503 | .00120498 |
| 19 | 1.005405 | .00435406 | 69 | 1.001481 | .00118745 |
| 20 | 1.005133 | .00413362 | 70 | 1.001460 | .00117043 |
| 21 | 1.004887 | .00393443 | 71 | 1.001440 | .00115389 |
| 22 | 1.004664 | .00375355 | 72 | 1.001420 | .00113780 |
| 23 | 1.004460 | .00358857 | 73 | 1.001400 | .00112216 |
| 24 | 1.004273 | .00343748 | 74 | 1.001381 | .00110695 |
| 25 | 1.004101 | .00329859 | 75 | 1.001363 | .00109214 |
| 26 | 1.003943 | .00317049 | 76 | 1.001345 | .00107772 |
| 27 | 1.003796 | .00305197 | 77 | 1.001327 | .00106368 |
| 28 | 1.003660 | .00294199 | 78 | 1.001310 | .00104999 |
| 29 | 1.003533 | .00283966 | 79 | 1.001294 | .00103666 |
| 30 | 1.003415 | .00274421 | 80 | 1.001277 | .00102366 |
| 31 | 1.003305 | .00265496 | 81 | 1.001262 | .00101098 |
| 32 | 1.003201 | .00257134 | 82 | 1.001246 | .00099862 |
| 33 | 1.003104 | .00249283 | 83 | 1.001231 | .00098655 |
| 34 | 1.003012 | .00241896 | 84 | 1.001217 | .00097477 |
| 35 | 1.002926 | .00234935 | 85 | 1.001202 | .00096327 |
| 36 | 1.002844 | .00228363 | 86 | 1.001188 | .00095203 |
| 37 | 1.002767 | .00222149 | 87 | 1.001175 | .00094106 |
| 38 | 1.002694 | .00216264 | 88 | 1.001161 | .00093033 |
| 39 | 1.002625 | .00210683 | 89 | 1.001148 | .00091985 |
| 40 | 1.002559 | .00205382 | 90 | 1.001135 | .00090960 |
| 41 | 1.002496 | .00200342 | 91 | 1.001123 | .00089958 |
| 42 | 1.002437 | .00195543 | 92 | 1.001111 | .00088977 |
| 43 | 1.002380 | .00190969 | 93 | 1.001099 | .00088018 |
| 44 | 1.002326 | .00186604 | 94 | 1.001087 | .00087079 |
| 45 | 1.002274 | .00182434 | 95 | 1.001075 | .00086160 |
| 46 | 1.002224 | .00178446 | 96 | 1.001064 | .00085260 |
| 47 | 1.002177 | .00174629 | 97 | 1.001053 | .00084379 |
| 48 | 1.002131 | .00170971 | 98 | 1.001043 | .00083515 |
| 49 | 1.002088 | .00167464 | 99 | 1.001032 | .00082670 |
| 50 | 1.002046 | .00164098 | 100 | 1.001022 | .00081841 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter $\theta$ of Weibull Population*

Shape Parameter $K = 4.0$

| $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ | $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.103263 | .07870520 | 51 | 1.001843 | .00123225 |
| 2 | 1.049606 | .03555099 | 52 | 1.001807 | .00120843 |
| 3 | 1.032516 | .02277234 | 53 | 1.001773 | .00118551 |
| 4 | 1.024163 | .01672043 | 54 | 1.001740 | .00116344 |
| 5 | 1.019220 | .01320237 | 55 | 1.001709 | .00114218 |
| 6 | 1.015954 | .01090486 | 56 | 1.001678 | .00112168 |
| 7 | 1.013635 | .00928741 | 57 | 1.001648 | .00110190 |
| 8 | 1.011905 | .00808731 | 58 | 1.001620 | .00108281 |
| 9 | 1.010564 | .00716161 | 59 | 1.001592 | .00106437 |
| 10 | 1.009495 | .00642591 | 60 | 1.001566 | .00104655 |
| 11 | 1.008622 | .00582721 | 61 | 1.001540 | .00102932 |
| 12 | 1.007896 | .00533050 | 62 | 1.001515 | .00101264 |
| 13 | 1.007283 | .00491179 | 63 | 1.001491 | .00099649 |
| 14 | 1.006758 | .00455404 | 64 | 1.001468 | .00098086 |
| 15 | 1.006304 | .00424485 | 65 | 1.001445 | .00096570 |
| 16 | 1.005907 | .00397497 | 66 | 1.001423 | .00095101 |
| 17 | 1.005557 | .00373734 | 67 | 1.001402 | .00093675 |
| 18 | 1.005246 | .00352652 | 68 | 1.001381 | .00092292 |
| 19 | 1.004968 | .00333820 | 69 | 1.001361 | .00090949 |
| 20 | 1.004718 | .00316898 | 70 | 1.001342 | .00089645 |
| 21 | 1.004492 | .00301608 | 71 | 1.001323 | .00088377 |
| 22 | 1.004286 | .00287725 | 72 | 1.001304 | .00087145 |
| 23 | 1.004099 | .00275064 | 73 | 1.001287 | .00085946 |
| 24 | 1.003927 | .00263470 | 74 | 1.001269 | .00084781 |
| 25 | 1.003769 | .00252814 | 75 | 1.001252 | .00083646 |
| 26 | 1.003624 | .00242987 | 76 | 1.001236 | .00082541 |
| 27 | 1.003489 | .00233894 | 77 | 1.001220 | .00081465 |
| 28 | 1.003364 | .00225458 | 78 | 1.001204 | .00080417 |
| 29 | 1.003247 | .00217609 | 79 | 1.001189 | .00079396 |
| 30 | 1.003138 | .00210288 | 80 | 1.001174 | .00078400 |
| 31 | 1.003037 | .00203443 | 81 | 1.001159 | .00077428 |
| 32 | 1.002942 | .00197030 | 82 | 1.001145 | .00076481 |
| 33 | 1.002852 | .00191009 | 83 | 1.001131 | .00075556 |
| 34 | 1.002768 | .00185345 | 84 | 1.001118 | .00074654 |
| 35 | 1.002688 | .00180007 | 85 | 1.001105 | .00073773 |
| 36 | 1.002614 | .00174968 | 86 | 1.001092 | .00072912 |
| 37 | 1.002543 | .00170204 | 87 | 1.001079 | .00072071 |
| 38 | 1.002476 | .00165692 | 88 | 1.001067 | .00071250 |
| 39 | 1.002412 | .00161413 | 89 | 1.001055 | .00070447 |
| 40 | 1.002351 | .00157349 | 90 | 1.001043 | .00069662 |
| 41 | 1.002294 | .00153485 | 91 | 1.001032 | .00068894 |
| 42 | 1.002239 | .00149806 | 92 | 1.001020 | .00068143 |
| 43 | 1.002187 | .00146300 | 93 | 1.001009 | .00067408 |
| 44 | 1.002137 | .00142954 | 94 | 1.000999 | .00066688 |
| 45 | 1.002089 | .00139757 | 95 | 1.000988 | .00065984 |
| 46 | 1.002044 | .00136701 | 96 | 1.000978 | .00065295 |
| 47 | 1.002000 | .00133775 | 97 | 1.000968 | .00064620 |
| 48 | 1.001958 | .00130972 | 98 | 1.000958 | .00063959 |
| 49 | 1.001918 | .00128283 | 99 | 1.000948 | .00063311 |
| 50 | 1.001880 | .00125703 | 100 | 1.000939 | .00062676 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 5.0$

| $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ | $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.089124 | .05246525 | 51 | 1.001573 | .00078924 |
| 2 | 1.042563 | .02323010 | 52 | 1.001543 | .00077397 |
| 3 | 1.027845 | .01477365 | 53 | 1.001513 | .00075928 |
| 4 | 1.020674 | .01080969 | 54 | 1.001485 | .00074514 |
| 5 | 1.016435 | .00851761 | 55 | 1.001458 | .00073151 |
| 6 | 1.013637 | .00702572 | 56 | 1.001432 | .00071837 |
| 7 | 1.011653 | .00597782 | 57 | 1.001407 | .00070570 |
| 8 | 1.010172 | .00520161 | 58 | 1.001383 | .00069347 |
| 9 | 1.009025 | .00460363 | 59 | 1.001359 | .00068165 |
| 10 | 1.008110 | .00412886 | 60 | 1.001336 | .00067023 |
| 11 | 1.007304 | .00374281 | 61 | 1.001314 | .00065918 |
| 12 | 1.006744 | .00342274 | 62 | 1.001293 | .00064850 |
| 13 | 1.006219 | .00315307 | 63 | 1.001273 | .00063815 |
| 14 | 1.005771 | .00292278 | 64 | 1.001253 | .00062813 |
| 15 | 1.005383 | .002[illegible]383 | 65 | 1.001233 | .00061842 |
| 16 | 1.005043 | .002[illegible]023 | 66 | 1.001215 | .00060900 |
| 17 | 1.004744 | .00239742 | 67 | 1.001197 | .00059987 |
| 18 | 1.004479 | .00226189 | 68 | 1.001179 | .00059101 |
| 19 | 1.004241 | .00214086 | 69 | 1.001162 | .00058240 |
| 20 | 1.004028 | .00203212 | 70 | 1.001145 | .00057404 |
| 21 | 1.003835 | .00193389 | 71 | 1.001129 | .00056592 |
| 22 | 1.003659 | .00184472 | 72 | 1.001113 | .00055803 |
| 23 | 1.003499 | .00176341 | 73 | 1.001098 | .00055035 |
| 24 | 1.003353 | .00168896 | 74 | 1.001083 | .00054288 |
| 25 | 1.003218 | .00162054 | 75 | 1.001069 | .00053561 |
| 26 | 1.003093 | .00155745 | 76 | 1.001055 | .00052853 |
| 27 | 1.002978 | .00149909 | 77 | 1.001041 | .00052164 |
| 28 | 1.002871 | .00144494 | 78 | 1.001027 | .00051493 |
| 29 | 1.002772 | .00139457 | 79 | 1.001014 | .00050838 |
| 30 | 1.002679 | .00134759 | 80 | 1.001002 | .00050200 |
| 31 | 1.002592 | .00130367 | 81 | 1.000989 | .00049578 |
| 32 | 1.002511 | .00126253 | 82 | 1.000977 | .00048971 |
| 33 | 1.002434 | .00122390 | 83 | 1.000965 | .00048379 |
| 34 | 1.002363 | .00118757 | 84 | 1.000954 | .00047801 |
| 35 | 1.002295 | .00115333 | 85 | 1.000943 | .00047236 |
| 36 | 1.002231 | .00112101 | 86 | 1.000932 | .00046685 |
| 37 | 1.002170 | .00109045 | 87 | 1.000921 | .00046146 |
| 38 | 1.002113 | .00106151 | 88 | 1.000911 | .00045620 |
| 39 | 1.002059 | .00103407 | 89 | 1.000900 | .00045106 |
| 40 | 1.002007 | .00100802 | 90 | 1.000890 | .00044603 |
| 41 | 1.001958 | .00098324 | 91 | 1.000880 | .00044111 |
| 42 | 1.001911 | .00095965 | 92 | 1.000871 | .00043630 |
| 43 | 1.001867 | .00093717 | 93 | 1.000862 | .00043159 |
| 44 | 1.001824 | .00091571 | 94 | 1.000852 | .00042698 |
| 45 | 1.001783 | .00089522 | 95 | 1.000843 | .00042247 |
| 46 | 1.001744 | .00087562 | 96 | 1.000835 | .00041806 |
| 47 | 1.001707 | .00085687 | 97 | 1.000826 | .00041373 |
| 48 | 1.001672 | .00083890 | 98 | 1.000817 | .00040950 |
| 49 | 1.001637 | .00082167 | 99 | 1.000809 | .00040535 |
| 50 | 1.001604 | .00080513 | 100 | 1.000801 | .00040128 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 6.0$

| $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ | $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.077912 | .03754820 | 51 | 1.001366 | .00054838 |
| 2 | 1.037070 | .01637374 | 52 | 1.001339 | .00053776 |
| 3 | 1.024223 | .01035970 | 53 | 1.001314 | .00052755 |
| 4 | 1.017974 | .00756092 | 54 | 1.001289 | .00051772 |
| 5 | 1.014284 | .00594882 | 55 | 1.001266 | .00050825 |
| 6 | 1.011850 | .00490205 | 56 | 1.001243 | .00049911 |
| 7 | 1.010124 | .00416801 | 57 | 1.001221 | .00049030 |
| 8 | 1.008837 | .00362492 | 58 | 1.001200 | .00048180 |
| 9 | 1.007840 | .00320692 | 59 | 1.001180 | .00047359 |
| 10 | 1.007045 | .00287528 | 60 | 1.001160 | .00046565 |
| 11 | 1.006396 | .00260576 | 61 | 1.001141 | .00045797 |
| 12 | 1.005857 | .00238241 | 62 | 1.001123 | .00045054 |
| 13 | 1.005401 | .00219431 | 63 | 1.001105 | .00044335 |
| 14 | 1.005012 | .00203373 | 64 | 1.001088 | .00043639 |
| 15 | 1.004674 | .00189504 | 65 | 1.001071 | .00042964 |
| 16 | 1.004380 | .00177405 | 66 | 1.001055 | .00042309 |
| 17 | 1.004120 | .00166758 | 67 | 1.001039 | .00041675 |
| 18 | 1.003889 | .00157317 | 68 | 1.001023 | .00041059 |
| 19 | 1.003683 | .00148887 | 69 | 1.001009 | .00040461 |
| 20 | 1.003497 | .00141314 | 70 | 1.000994 | .00039880 |
| 21 | 1.003330 | .00134474 | 71 | 1.000980 | 00039315 |
| 22 | 1.003177 | .00128266 | 72 | 1.000966 | .00038767 |
| 23 | 1.003038 | .00122605 | 73 | 1.000953 | .00038233 |
| 24 | 1.002914 | .00117423 | 74 | 1.000940 | .00037714 |
| 25 | 1.002794 | .00112661 | 75 | 1.000928 | .00037209 |
| 26 | 1.002686 | .00108271 | 76 | 1.000915 | .00036717 |
| 27 | 1.002586 | .00104269 | 77 | 1.000904 | .00036238 |
| 28 | 1.002493 | .00100442 | 78 | 1.000892 | .00035771 |
| 29 | 1.002407 | .00096937 | 79 | 1.000881 | .00035317 |
| 30 | 1.002326 | .00093668 | 80 | 1.000870 | .00034873 |
| 31 | 1.002251 | .00090613 | 81 | 1.000859 | .00034441 |
| 32 | 1.002180 | .00087751 | 82 | 1.000848 | .00034019 |
| 33 | 1.002114 | .00085064 | 83 | 1.000838 | .00033607 |
| 34 | 1.002051 | .00082537 | 84 | 1.000828 | .00033206 |
| 35 | 1.001992 | .00080155 | 85 | 1.000818 | .00032813 |
| 36 | 1.001937 | .00077907 | 86 | 1.000809 | .00032430 |
| 37 | 1.001884 | .00075782 | 87 | 1.000800 | .00032056 |
| 38 | 1.001834 | .00073769 | 88 | 1.000790 | .00031690 |
| 39 | 1.001787 | .00071861 | 89 | 1.000782 | .00031333 |
| 40 | 1.001742 | .00070049 | 90 | 1.000773 | .00030983 |
| 41 | 1.001700 | .00068326 | 91 | 1.000764 | .00030642 |
| 42 | 1.001659 | .00066686 | 92 | 1.000756 | .00030307 |
| 43 | 1.001620 | .00065123 | 93 | 1.000748 | .00029980 |
| 44 | 1.001584 | .00063631 | 94 | 1.000740 | .00029660 |
| 45 | 1.001548 | .00062206 | 95 | 1.000732 | .00029347 |
| 46 | 1.001514 | .00060843 | 96 | 1.000724 | .00029040 |
| 47 | 1.001482 | .00059539 | 97 | 1.000717 | .00028740 |
| 48 | 1.001451 | .00058290 | 98 | 1.000710 | .00028445 |
| 49 | 1.001421 | .00057092 | 99 | 1.000702 | .00028157 |
| 50 | 1.001393 | .00055942 | 100 | 1.000695 | .00027874 |

TABLE 1 (*continued*)
*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*
Shape Parameter $K = 7.0$

| $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ | $m$ | $\bar{\theta}/\hat{\theta}$ | Var $\bar{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.069018 | .02823145 | 51 | 1.001204 | .00040305 |
| 2 | 1.032756 | .01216533 | 52 | 1.001181 | .00039525 |
| 3 | 1.024387 | .00766682 | 53 | 1.001158 | .00038774 |
| 4 | 1.015863 | .00558486 | 54 | 1.001137 | .00038051 |
| 5 | 1.012604 | .00438916 | 55 | 1.001116 | .00037354 |
| 6 | 1.010454 | .00361417 | 56 | 1.001096 | .00036683 |
| 7 | 1.008931 | .00307138 | 57 | 1.001077 | .00036035 |
| 8 | 1.007794 | .00267015 | 58 | 1.001058 | .00035410 |
| 9 | 1.006915 | .00236154 | 59 | 1.001040 | .00034806 |
| 10 | 1.006213 | .00211683 | 60 | 1.001023 | .00034222 |
| 11 | 1.005641 | .00191803 | 61 | 1.001006 | .00033658 |
| 12 | 1.005135 | .00175335 | 62 | 1.000990 | .00033112 |
| 13 | 1.004763 | .00161470 | 63 | 1.000974 | .00032583 |
| 14 | 1.004420 | .00149636 | 64 | 1.000959 | .00032071 |
| 15 | 1.004122 | .00139418 | 65 | 1.000944 | .00031575 |
| 16 | 1.003862 | .00130506 | 66 | 1.000930 | .00031094 |
| 17 | 1.003633 | .00122664 | 67 | 1.000916 | .00030627 |
| 18 | 1.003430 | .00115711 | 68 | 1.000902 | .00030174 |
| 19 | 1.003248 | .00109504 | 69 | 1.000889 | .00029735 |
| 20 | 1.003084 | .00103929 | 70 | 1.000877 | .00029308 |
| 21 | 1.002936 | .00098893 | 71 | 1.000864 | .00028893 |
| 22 | 1.002802 | .00094324 | 72 | 1.000852 | .00028490 |
| 23 | 1.002679 | .00090157 | 73 | 1.000840 | .00028097 |
| 24 | 1.002567 | .00086343 | 74 | 1.000829 | .00027716 |
| 25 | 1.002464 | .00082839 | 75 | 1.000818 | .00027344 |
| 26 | 1.002368 | .00079608 | 76 | 1.000807 | .00026983 |
| 27 | 1.002280 | .00076620 | 77 | 1.000797 | .00026631 |
| 28 | 1.002198 | .00073848 | 78 | 1.000786 | .00026288 |
| 29 | 1.002122 | .00071269 | 79 | 1.000776 | .00025953 |
| 30 | 1.002051 | .00068864 | 80 | 1.000767 | .00025628 |
| 31 | 1.001984 | .00066617 | 81 | 1.000757 | .00025310 |
| 32 | 1.001922 | .00064511 | 82 | 1.000748 | .00025000 |
| 33 | 1.001864 | .00062534 | 83 | 1.000739 | .00024697 |
| 34 | 1.001809 | .00060675 | 84 | 1.000730 | .00024402 |
| 35 | 1.001757 | .00058924 | 85 | 1.000722 | .00024114 |
| 36 | 1.001708 | .00057270 | 86 | 1.000713 | .00023832 |
| 37 | 1.001661 | .00055707 | 87 | 1.000705 | .00023557 |
| 38 | 1.001618 | .00054227 | 88 | 1.000697 | .00023288 |
| 39 | 1.001576 | .00052823 | 89 | 1.000689 | .00023025 |
| 40 | 1.001536 | .00051491 | 90 | 1.000681 | .00022768 |
| 41 | 1.001499 | .00050224 | 91 | 1.000674 | .00022517 |
| 42 | 1.001463 | .00049017 | 92 | 1.000667 | .00022272 |
| 43 | 1.001429 | .00047868 | 93 | 1.000659 | .00022031 |
| 44 | 1.001396 | .00046771 | 94 | 1.000652 | .00021796 |
| 45 | 1.001365 | .00045723 | 95 | 1.000645 | .00021565 |
| 46 | 1.001335 | .00044721 | 96 | 1.000639 | .00021340 |
| 47 | 1.001307 | .00043762 | 97 | 1.000632 | .00021119 |
| 48 | 1.001279 | .00042843 | 98 | 1.000626 | .00020903 |
| 49 | 1.001253 | .00041962 | 99 | 1.000619 | .00020691 |
| 50 | 1.001228 | .00041117 | 100 | 1.000613 | .00020483 |

TABLE 1 (*continued*)

*Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population*

Shape Parameter $K = 8.0$

| $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ | $m$ | $\tilde{\theta}/\hat{\theta}$ | Var $\tilde{\theta}/\theta^2$ |
|---|---|---|---|---|---|
| 1 | 1.061801 | .02201333 | 51 | 1.001076 | .00030868 |
| 2 | 1.029305 | .00930588 | 52 | 1.001055 | .00030270 |
| 3 | 1.019123 | .00590316 | 53 | 1.001035 | .00029695 |
| 4 | 1.014180 | .00429371 | 54 | 1.001016 | .00029141 |
| 5 | 1.011265 | .00337150 | 55 | 1.000997 | .00028607 |
| 6 | 1.009343 | .00277461 | 56 | 1.000979 | .00028093 |
| 7 | 1.007980 | .00235606 | 57 | 1.000962 | .00027597 |
| 8 | 1.006965 | .00204845 | 58 | 1.000945 | .00027118 |
| 9 | 1.006178 | .00181128 | 59 | 1.000929 | .00026656 |
| 10 | 1.005551 | .00162328 | 60 | 1.000914 | .00026208 |
| 11 | 1.005040 | .00147062 | 61 | 1.000899 | .00025776 |
| 12 | 1.004615 | .00134419 | 62 | 1.000884 | .00025358 |
| 13 | 1.004250 | .00123776 | 63 | 1.000870 | .00024953 |
| 14 | 1.003949 | .00114695 | 64 | 1.000857 | .00024560 |
| 15 | 1.003683 | .00106854 | 65 | 1.000843 | .00024180 |
| 16 | 1.003450 | .00100017 | 66 | 1.000831 | .00023812 |
| 17 | 1.003246 | .00094002 | 67 | 1.000818 | .00023454 |
| 18 | 1.003064 | .00088669 | 68 | 1.000806 | .00023108 |
| 19 | 1.002901 | .00083908 | 69 | 1.000794 | .00022771 |
| 20 | 1.002755 | .00079633 | 70 | 1.000783 | .00022444 |
| 21 | 1.002623 | .00075772 | 71 | 1.000772 | .00022126 |
| 22 | 1.002503 | .00072268 | 72 | 1.000761 | .00021817 |
| 23 | 1.002393 | .00069074 | 73 | 1.000751 | .00021517 |
| 24 | 1.002293 | .00066150 | 74 | 1.000741 | .00021224 |
| 25 | 1.002201 | .00063463 | 75 | 1.000731 | .00020940 |
| 26 | 1.002116 | .00060987 | 76 | 1.000721 | .00020663 |
| 27 | 1.002037 | .00058696 | 77 | 1.000712 | .00020393 |
| 28 | 1.001964 | .00056571 | 78 | 1.000702 | .00020131 |
| 29 | 1.001896 | .00054595 | 79 | 1.000694 | .00019875 |
| 30 | 1.001832 | .00052752 | 80 | 1.000685 | .00019625 |
| 31 | 1.001773 | .00051025 | 81 | 1.000676 | .00019381 |
| 32 | 1.001717 | .00049415 | 82 | 1.000668 | .00019144 |
| 33 | 1.001665 | .00047901 | 83 | 1.000660 | .00018912 |
| 34 | 1.001616 | .00046476 | 84 | 1.000652 | .00018686 |
| 35 | 1.001569 | .00045133 | 85 | 1.000645 | .00018465 |
| 36 | 1.001526 | .00043866 | 86 | 1.000637 | .00018250 |
| 37 | 1.001484 | .00042669 | 87 | 1.000630 | .00018039 |
| 38 | 1.001445 | .00041534 | 88 | 1.000623 | .00017833 |
| 39 | 1.001408 | .00040459 | 89 | 1.000616 | .00017632 |
| 40 | 1.001372 | .00039438 | 90 | 1.000609 | .00017435 |
| 41 | 1.001339 | .00038467 | 91 | 1.000602 | .00017243 |
| 42 | 1.001307 | .00037543 | 92 | 1.000595 | .00017054 |
| 43 | 1.001276 | .00036662 | 93 | 1.000589 | .00016870 |
| 44 | 1.001247 | .00035822 | 94 | 1.000583 | .00016690 |
| 45 | 1.001219 | .00035019 | 95 | 1.000577 | .00016514 |
| 46 | 1.001193 | .00034251 | 96 | 1.000571 | .00016341 |
| 47 | 1.001167 | .00033516 | 97 | 1.000565 | .00016172 |
| 48 | 1.001143 | .00032813 | 98 | 1.000559 | .00016006 |
| 49 | 1.001120 | .00032138 | 99 | 1.000553 | .00015844 |
| 50 | 1.001097 | .00031490 | 100 | 1.000548 | .00015685 |

and if the test is terminated at the time of the $m$-th failure ($m \leq 100$), one can compute a maximum likelihood estimator $\hat{\theta}$ of the scale parameter $\theta$ from (5) and then multiply $\hat{\theta}$ by the unbiasing factor $\tilde{\theta}/\hat{\theta}$ given in Table 1 to obtain an unbiased estimator $\tilde{\theta}$. The ratio, $\operatorname{Var} \tilde{\theta}/\theta^2$, of the variance of the unbiased estimator to $\theta^2$ is also given in the table. The efficiency $E_p$ of the unbiased estimator based on the first $m$ order statistics as compared with the one based on all $n$ order statistics ($m<n\leq 100$) can be found by taking the ratio of two entries in the $\operatorname{Var} \tilde{\theta}/\theta^2$ columns of Table 1. It can be seen that the percentage efficiency is approximately $100m/n$.

## 7. Numerical Example

As an illustration of the use of the above results, consider a simulated life test on forty components. Suppose the observed failure times in hours are as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 33 | 55 | 65 | 82 | 102 | 114 | 142 |
| 10 | 34 | 58 | 65 | 85 | 103 | 116 | 143 |
| 17 | 36 | 58 | 66 | 90 | 106 | 117 | 151 |
| 32 | 54 | 61 | 67 | 92 | 107 | 124 | 158 |
| 32 | 55 | 64 | 68 | 92 | 114 | 139 | 195 |

Suppose the experimenter knows that these times are from a Weibull population with shape parameter $K = 2.0$ and wishes to obtain a point estimate and set 80% upper and lower confidence bounds on the scale parameter $\theta$. The conventional confidence bounds are those based on all 40 observations, but the experimenter might not want to wait for all the components to fail and might therefore terminate the test at the time of the $m$th failure ($m < 40$). We can simulate this occurrence by censoring upper portions of the above ordered data. The values of the maximum likelihood estimator $\hat{\theta}$ were calculated from (5) for $m = 8(8)40$, and $\tilde{\theta}$ was obtained by multiplying by the unbiasing factor $\tilde{\theta}/\hat{\theta}$ given in Table 1. Then the lower and upper 80% confidence bounds, $\underline{\theta}_{.80}$ and $\bar{\theta}_{.80}$, were calculated from (11) with the aid of a table of percentage points of the chi-square distribution given by Harter (1964a). The intervals between paired values of $\underline{\theta}_{.80}$ and $\bar{\theta}_{.80}$ are central 60% confidence intervals for $\theta$. The efficiencies, $E_u$ and $E_c$, of upper confidence bounds and central confidence intervals, with confidence levels 80% and 60%, respectively, based on the first $m$ out of $n$ ordered observations, were calculated by substituting, in (12) and (13), values of $E[(\bar{\theta}' - \theta)^2]$ obtained from (11) and of $E[(\bar{\theta}' - \theta)^2]$, $E[(\underline{\theta} - \theta)^2]$, and $E[(\underline{\theta}' - \theta)^2]$ obtained from (11) modified as indicated in Section 4. The efficiency $E_p$ of the unbiased point estimator $\tilde{\theta}$ was computed as indicated in Section 7. The results are as follows:

| $m$ | $\hat{\theta}$ | $\tilde{\theta}$ | $\underline{\theta}_{.80}$ | $\bar{\theta}_{.80}$ | $E_u(\%)$ | $E_c(\%)$ | $E_p(\%)$ |
|---|---|---|---|---|---|---|---|
| 8 | 77.0 | 78.2 | 68.1 | 92.2 | 16.2 | 18.2 | 19.8 |
| 16 | 91.9 | 92.6 | 83.5 | 103.3 | 34.4 | 38.9 | 39.8 |
| 24 | 95.2 | 95.7 | 88.2 | 104.8 | 57.3 | 59.3 | 59.9 |
| 32 | 93.7 | 94.1 | 87.6 | 101.7 | 78.5 | 79.6 | 74.9 |
| 40 | 93.3 | 93.6 | 87.8 | 100.3 | 100.0 | 100.0 | 100.0 |

Note that $E_u \leq E_s \leq E_p \leq 100m/n$ and that $E_u \to E_s \to E_p \to 100m/n \to 100\%$ as $m \to n$.

## References


1. Epstein, Benjamin and Sobel, Milton, 1953. Life testing. *Journal of the American Statistical Association*, *48*, 486–502.
2. Harter, H. Leon, 1964a. *New Tables of the Incomplete Gamma-Function Ratio and of Percentage Points of the Chi-Square and Beta Distributions*, U. S. Government Printing Office, Washington.
3. Harter, H. Leon, 1964b. Exact confidence bounds, based on one order statistic, for the parameter of an exponential population. *Technometrics*, *6*, 301–317.
4. Harter, H. Leon, 1964c. Criteria for best substitute interval estimators with an application to the normal distribution. *Journal of the American Statistical Association*, *59*, 1133–1140.
5. Mann, N. R., 1963. Optimum Estimates of Parameters of Continuous Distributions, Research Report No. 63-41, Rocketdyne Division, North American Aviation, Inc., Canoga Park, California.
6. Quayle, Ronald J., 1963. Estimation of the Scale Parameter of the Weibull Probability Density Function by Use of One Order Statistic (unpublished thesis), Air Force Institute of Technology, Wright-Patterson Air Force Base.

Unclassified

Security Classification

**DOCUMENT CONTROL DATA - R&D**

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY (Corporate author)<br>Applied Mathematics Research Laboratory<br>Aerospace Research Laboratories<br>Wright-Patterson AFB, Ohio | 2a. REPORT SECURITY CLASSIFICATION<br>Unclassified<br>2b. GROUP |

3. REPORT TITLE

Point and Interval Estimators, Based on m Order Statistics, for the Scale Parameter of a Weibull Population with Known Shape Parameter

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific. Interim

5. AUTHOR(S) (Last name, first name, initial)

Harter, H. Leon
Moore, Albert H.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| October 1966 | 18 | 6 |

| | |
|---|---|
| 8a. ~~XXXXXXXXXXXXXX~~ In-House Researc | 9a. ORIGINATOR'S REPORT NUMBER(S) |
| b. PROJECT NO. 7071-00 11 | |
| c. 61445014 | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |
| d. 681304 | ARL 66-0198 |

10. AVAILABILITY/LIMITATION NOTICES

1. Distribution of this document is unlimited.

| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY |
|---|---|
| JOURNAL<br>(Technometrics, Vol. 7, No. 3, pp. 405-422, August 1965) | Aerospace Research Laboratories (ARM)<br>Wright-Patterson AFB, Ohio |

13. ABSTRACT


A derivation is given of the maximum likelihood estimator $\hat{\theta}$, based on the first m out of n ordered observations, of the scale parameter $\theta$ of a Weibull population with known shape parameter K. It is shown that $2m\,\hat{\theta}^K/\theta^K$ has a chi-square distribution with 2m degrees of freedom (independent of n). Use is made of this fact to set upper confidence bounds with confidence level 1−P (lower confidence bounds with confidence level P) on the scale parameter $\theta$. Formulas are given for the mean squared deviations of the upper and lower confidence bounds from the true value of the parameter. From these one can obtain expressions for the efficiency of confidence bounds and confidence intervals. The expected value of $\hat{\theta}$ is also determined, and from it the unbiasing factor $\tilde{\theta}/\hat{\theta}$ by which $\hat{\theta}$ must be multiplied to obtain an unbiased estimator $\tilde{\theta}$. An expression for the variance of the unbiased estimator $\tilde{\theta}$ is found. Values of the unbiasing factor and of the variance of the unbiased estimator, both of which are independent of n, are tabled for m = 1(1)100 and K = 0.5(0.5)4.0(1.0)8.0. A section on use of the table and a numerical example are included.


DD FORM 1 JAN 64 1473

Unclassified

Security Classification

| 14. KEY WORDS | LINK A ROLE | LINK A WT | LINK B ROLE | LINK B WT | LINK C ROLE | LINK C WT |
|---|---|---|---|---|---|---|
| maximum likelihood estimator | | | | | | |
| Weibull population | | | | | | |
| confidence bounds | | | | | | |
| confidence intervals | | | | | | |

## INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________ ."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________ ."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________ ."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.