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F-B2294



# ON THE EIGENVALUES OF COUETTE FLOW IN A FULLY-FILLED CYLINDRICAL CONTAINER



THE FRANKLIN INSTITUTE RESEARCH LABORATORIES
BENJAMIN FRANKLIN PARKWAY • PHILADELPHIA, PENNA. 19103

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# ON THE EIGENVALUES OF COUETTE FLOW IN A FULLY-FILLED CYLINDRICAL CONTAINER


*by*

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January 1967

*Prepared for*

EXTERIOR BALLISTIC LABORATORY
Aberdeen Proving Ground, Maryland

Contract No. DA-30-069-AMC-686(R)

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## SUMMARY

For a stationary flow in a cylindrical container of the Couette type in an outer radial zone, and of zero velocity in an inner radial zone, the normal mode equations are derived. For negative wave numbers in the θ-direction, these equations are found to have a signularity.

The eigenvalues are calculated by initial value methods employing the Runge-Kutta-Gill integration procedure. Values of the dependent function and their derivatives at the singularity are calculated by linear extrapolation coupled with continuity requirements.

Tables of eigenvalues for various slenderness ratios of the cylinder and various radial nodes are given for θ-wave numbers of -1.

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## INTRODUCTION

The object of the following investigation is to determine the eigenfrequencies of a viscous fluid, contained in a rotating cylinder, during the initial spin-up period. In order to make the problem tractable, an inviscid fluid is assumed and the fluid is divided into two zones; an outer radial zone in which a Couette type flow prevails and an interior zone which is static. The initial flow distribution, where the interface between the two zones is assumed to decrease from the outer radius to zero in a quasi-static manner, is assumed to be time independent.

## FORMULATION OF THE EIGENVALUE PROBLEM

In the following we are interested in the characteristic frequencies of an inviscid liquid contained in a cylindrical container. The liquid is assumed to have an initial stable motion of the Couette type for an outer radial zone whereas the interior is assumed to be at zero velocity. Thus, let the container be of radius a and height 2c. Then the stationary flow in the outer zone is given by

$$\begin{aligned} u_o &= 0 & a \geq r \geq b \\ v_o &= a\omega[(r/a) - (e^2 a/r)]/(1 - e^2) & 0 \leq [e = b/a] \leq 1 \\ w_o &= 0 & \qquad (1) \end{aligned}$$

In the inner zone,

$$\begin{aligned} u_i &= 0 \\ v_i &= 0 \qquad b \geq r \geq 0 \\ w_i &= 0 \qquad (2) \end{aligned}$$

where b is the radius of the interface between the static interior and the moving exterior; u, v, and w are the radial, longitudinal and axial components of velocity in a cylindrical coordinate system with the z-axis aligned along the axis of the cylinder. The origin of the coordinate system is located at the bottom of the cylinder so that the z-coordinates of the end faces are given by 0 and 2c. The velocity components in the inner and outer zones are distinguished by the subscripts i and o respectively.

For convenience, we consider the two zones, namely, the exterior zone of Couette flow, and the interior zone of zero velocity, separately. These are then coupled by boundary conditions imposed at the interface, where the radial velocities and pressures corresponding to the two regions are required to be equal. Considering the exterior zone first, the continuity and momentum equations with the appropriate boundary conditions are:

$$\frac{\partial u_o}{\partial t} + \left(Q_o \cdot \nabla\right) u_o - v_o^2/r = -\frac{\partial}{\partial r}\left(\frac{p_o}{\rho}\right)$$

$$\frac{\partial v_o}{\partial t} + \left(Q_o \cdot \nabla\right) v_o + u_o v_o/r = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{p_o}{\rho}\right)$$

$$\frac{\partial w_o}{\partial t} + \left(Q_o \cdot \nabla\right) w_o = -\frac{\partial}{\partial z}\left(\frac{p_o}{\rho}\right)$$

$$Q_o \cdot \nabla \equiv u_o \frac{\partial}{\partial r} + \frac{v_o}{r}\frac{\partial}{\partial \theta} + w_o \frac{\partial}{\partial z}$$

$$\frac{u_o}{r} + \frac{\partial u_o}{\partial r} + \frac{1}{r}\frac{\partial v_o}{\partial \theta} + \frac{\partial w_o}{\partial t} = 0 \qquad (3)$$

The boundary conditions are

$$r = a;\ u_o = 0$$

$$z = 0,\ 2c;\ w_o = 0$$

Moreover, at the interface, we have

$$u_{of} = u_{if};\ p_{of} = p_{if} \qquad (4)$$

where subscript f refers to the interface.

Substituting the following,

$$u_o^* = u_o/a\omega;\ v_o^* = v_o/a\omega;\ w_o^* = w_o/a\omega$$

$$r^* = r/at;\ z^* = z/at;\ t^* = \omega t;\ \theta^* = \theta$$

$$p^*/\rho^* = p/\rho a^2\omega^2 \tag{5}$$

and dropping the asterisk, the stationary flow is given by

$$u_o = 0$$

$$v_o = [r - e^2/r]/[1 - e^2] \qquad 1 \geq r \geq e$$

$$w_o = 0$$

The boundary conditions now become

$$r = 1;\ u_o = 0$$

$$z = 0,\ 2c/a;\ w_o = 0 \tag{6}$$

At the interface, they are

$$u_{of} = u_{if};\ p_{of} = p_{if} \tag{7}$$

Here $u_{if}$ and $p_{if}$ are the appropriate non-dimensional radial velocity and pressure prevailing in the flow field of the inner zone at the interface. From the non-dimensional form of Equation 3,

$$\frac{\partial P_o}{\partial r} = \frac{v^2}{r}$$

$$\frac{\partial P_o}{\partial \theta} = 0$$

$$\frac{\partial P_o}{\partial z} = 0 \tag{8}$$

where

$$P_o = p_o/\rho$$

$$v = [r - e^2/r]/(1 = e^2) \tag{9}$$

Integrating,

$$P_o = \int_e^r (v^2/y)\ dy + P_1 \tag{10}$$

where $P_1$ is the equilibrium pressure at the interface.
We now assume perturbations due to a radial displacement, $\eta$, of the interface from its equilibrium position corresponding to r = e.

$$u_o = u_o'$$

$$v_o = V + v_o'$$

$$w_o = w_o'$$

$$P_o = \int_e^r \frac{V^2(y)}{y}\,dy + P_o' + P_1 \tag{11}$$

Substituting the perturbations into the dimensionless Euler's equations of motion, and neglecting second and higher order quantities,

$$\frac{\partial u_o'}{\partial t} + \frac{V}{r}\frac{\partial u_o'}{\partial \theta} - \frac{2V}{r} v_o' = - \frac{\partial P_o'}{\partial r}$$

$$\frac{\partial v_o'}{\partial t} + u_o' \frac{dV}{dr} + \frac{V}{r}\frac{\partial v_o'}{\partial \theta} + \frac{u_o' V}{r} = - \frac{1}{r}\frac{\partial P_o'}{\partial \theta}$$

$$\frac{\partial w_o'}{\partial t} + \frac{V}{r}\frac{\partial w_o'}{\partial \theta} = - \frac{\partial P_o'}{\partial z}$$

$$\frac{u_o'}{r} + \frac{\partial u_o'}{\partial r} + \frac{1}{r}\frac{\partial v_o'}{\partial \theta} + \frac{\partial w_o'}{\partial z} = 0 \tag{12}$$

The boundary conditions become

$$r = 1; \; u_o' = 0$$

$$z = 0,\ 2c/a; \; w_o' = 0 \tag{13}$$

The boundary conditions due to displacement of the interface are obtained in the following manner:
Let the interface be at

$$r = e + \eta \tag{14}$$

Substituting into the last of Equations 11 and neglecting second and higher order terms,

$$P_{of} = P_1 + P_{of}' \tag{15}$$

At the interface, we have from 14

$$u_o' = \frac{\partial\eta}{\partial t} \tag{16}$$

where second and higher order quantities in the perturbations have been ignored.

We now consider the interior zone. The Euler's equations of motion and the equation of continuity are once again given by Equation 3 in a non-dimensional form. The boundary conditions are

$$z = 0,\ 2c/a;\ w_i = 0 \tag{17}$$

$$r = 0;\ u_i,\ v_i \text{ and } w_i \text{ are bounded.}$$

At the interface, they are

$$u_{if} = u_{of}$$

$$P_{if} = P_{of} \tag{18}$$

We once again assume perturbations due to radial displacement, $\eta$, of the interface from its equilibrium position corresponding to $r = e$.

$$u_i = u_i'$$

$$v_i = v_i'$$

$$w_i = w_i'$$

$$P_i = P_i' + P_1 \tag{19}$$

Substituting in Equation 3,

$$\frac{\partial u_i'}{\partial t} = -\frac{\partial P_i'}{\partial r}$$

$$\frac{\partial v_i'}{\partial t} = -\frac{1}{r}\frac{\partial P_i'}{\partial \theta}$$

$$\frac{\partial w_i'}{\partial t} = -\frac{\partial P_i'}{\partial z}$$

$$\frac{u_i'}{r} + \frac{\partial u_i'}{\partial r} + \frac{1}{r}\frac{\partial v_i'}{\partial \theta} + \frac{\partial w_i'}{\partial z} = 0 \tag{20}$$

The boundary conditions become

$$z = 0,\ 2c/a,\ w_i' = 0 \tag{21}$$

$$r = 0;\ u_i',\ v_i' \text{ and } w_i' \text{ are bounded.}$$

At the interface

$$u_i' = \frac{\partial \eta}{\partial t}$$

$$P_{if}' = P_{of} - P_1 \tag{22}$$

## NORMAL MODES

In accordance with the usual procedure of treating characteristic value problems, the perturbations are analyzed into normal modes. In view of the boundary conditions, it is natural to suppose that the perturbations are given by quantities which have a (r, θ, z, t) dependence given by

$$u_o' = U_o(r) \cos [h_o \pi az/2c] e^{i(K_o t + m_o \theta)}$$

$$v_o' = B_o(r) \cos [h_o \pi az/2c] e^{i(K_o t + m_o \theta)}$$

$$w_o' = W_o(r) \sin [h_o \pi az/2c] e^{i(K_o t + m_o \theta)}$$

$$P_o' = R_o(r) \cos [h_o \pi az/2c] e^{i(K_o t + m_o \theta)} \tag{23}$$

Where $K_o$ is a constant (which can be complex), $m_o$ is an integer (which can be positive, zero, or negative) and $h_o$ is the wave number in the z-direction (which can be 1, 2, 3, etc.). $U_o$, $B_o$, $W_o$ and $R_o$ are functions or r only.

In an analogous manner, for the interior zone, the perturbation quantities may be assumed to be of the form

$$u_i' = U_i(r)\cos(h_i\pi az/2c)e^{i(K_it + m_i\theta)}$$
$$v_i' = B_i(r)\cos(h_i\pi az/2c)e^{i(K_it + m_i\theta)}$$
$$w_i' = W_i(r)\sin(h_i\pi az/2c)e^{i(K_it + m_i\theta)}$$
$$P_i' = R_i(r)\cos(h_i\pi az/2c)e^{i(K_it + m_i\theta)} \tag{24}$$

where $K_i$, $m_i$, and $h_i$ are defined as before. $U_i$, $B_i$, $W_i$, and $R_i$ are once again assumed to be functions of the radius r only.

Substituting Equation 23 into Equation 12, we obtain

$$i\left(K_o + m_o \frac{V}{r}\right)U_o - \frac{2V}{r} B_o = -\frac{dR_o}{dr}$$
$$\left(\frac{dV}{dr} + \frac{V}{r}\right)U_o + i\left(K_o + m_o \frac{V}{r}\right)B_o = -i\, m_o \frac{R_o}{r}$$
$$\left(K_o + m_o \frac{V}{r}\right)W_o = -\frac{h_o\pi a}{2c}\, i\, R_o$$
$$\left(\frac{dU_o}{dr} + \frac{U_o}{r}\right) + i\, m_o \frac{B_o}{r} + \frac{h_o\pi a}{2c} W_o = 0 \tag{25}$$

In a similar manner, substituting Equation 24 into 20, we obtain

$$U_i = \frac{i}{K_i}\frac{dR_i}{dr}$$
$$B_i = -\frac{1}{r}\frac{m_i}{K_i} R_i$$
$$W_i = -\frac{i}{K_i}\frac{h_i\pi a}{2c} R_i$$
$$\frac{U_i}{r} + \frac{dU_i}{dr} + \frac{im_i}{r} B_i + \frac{h_i\pi a}{2c} W_i = 0 \tag{26}$$

Since

$$u_{of}' = u_{if}'$$

we have

$$U_i(e + \eta) \cos (h_i \pi az/2c) e^{i(K_i t + m_i \theta)}$$
$$= U_o(e + \eta) \cos (h_o \pi az/2c) e^{i(K_o t + M_o)} \tag{27}$$

Inasmuch as $\theta$, z and t are independent variables, we require that $K_i$, $m_i$ and $h_i$ be equal to $K_o$, $m_o$ and $h_o$ respectively. Thus, the subscripts for K, m and h will be omitted in what follows. Equation 25 can be simplified to the following

$$\sigma \left[ \frac{dU_o}{dr} + \frac{U_o}{r} \right] - \frac{m}{r^2} U_o \frac{d}{dr} \left(\Omega r^2\right) = \left[ \frac{m^2}{r^2} + \frac{\pi^2 h^2 a^2}{4c^2} \right] i \; R_o$$

$$\sigma^2 U_o - \frac{2\Omega}{r} U_o \frac{d}{dr} \left(\Omega r^2\right) = \left[ 2\Omega m \frac{R_o}{r} + \sigma \frac{dR_o}{dr} \right] i \tag{28}$$

where

$$\Omega = V/r \tag{29}$$

and

$$\sigma = K + m\Omega \tag{30}$$

In a similar manner, Equation 26 becomes

$$\frac{U_i}{r} + \frac{dU_i}{dr} = \frac{i}{K} \left[ \frac{m^2}{r^2} + \frac{\pi^2 h^2 a^2}{4c^2} \right] R_i$$

$$U_i = \frac{i}{K} \frac{dR_i}{dr} \tag{31}$$

The boundary conditions are

$$U_o(1) = 0 \tag{32}$$

$U_i(o)$ and $R_i(0)$ are bounded

The interface conditions are obtained as follows: Since $P_{of}$ is equal to $P_{if}$, from Equations 15 and 22,

$$P'_{if} = P'_{of} \tag{33}$$

Substituting for $P'_{of}$ and $P'_{if}$ from Equations 23 and 24 respectively,

$$R_i(e) \cos(h\pi az/2c)e^{i(Kt + m\theta)}$$
$$= R_o(e) \cos(h\pi az/2c)e^{i(Kt + m\theta)} \qquad (34)$$

where we consider $\eta << e$.

$$\left[K\left\{\frac{U_i}{r} + \frac{dU_i}{dr}\right\} = \sigma\left\{\frac{dU_o}{dr} + \frac{U_o}{r}\right\} - \frac{m}{r^2} U_o \frac{d}{dr}\left(\Omega r^2\right)\right]_{r=e} \qquad (35)$$

$$U_o(e) = U_i(e) \qquad (36)$$

Thus, the characteristic value problem consists of Equations 28 and 31 with boundary conditions 32, 35, and 36. Since we are interested only in those motions which produce a couple on the casing, interest centers around values of $h = (2j + 1)$, where $j$ is an integer.

Axisymmetric Cases*

Setting $m = 0$ reduces Equations 28 and 31 to the following

$$\frac{d^2U_o}{dr^2} + \frac{1}{r}\frac{dU_o}{dr} - \left[\frac{1}{r^2} + N^2 - \frac{N^2}{K^2}\frac{2\Omega}{r}\frac{d}{dr}\left(\Omega r^2\right)\right]U_o = 0$$

$$\frac{d^2U_i}{dr^2} + \frac{1}{r}\frac{dU_i}{dr} - \left[\frac{1}{r^2} + N^2\right]U_i = 0 \qquad (37)$$

where $N^2 \equiv \pi^2h^2a^2/4c^2$ (38)

The boundary conditions become

$$U_o(1) = 0$$
$$U_o(e) = U_i(e); \quad \frac{dU_o(e)}{dr} = \frac{dU_i(e)}{dr}$$
$$U_i(o) = 0 \qquad (39)$$

*The axisymmetric case is of academic interest only.

Case i, The Narrow Gap Approximation for m = 0

In specifying $\Omega$, interest is in those velocity distributions which are realizable in a viscous fluid. Equations 9 and 29 determine the case of primary interest as

$$\Omega = \left(1 - e^2/r^2\right)/\left(1 - e^2\right) \tag{40}$$

However, as $e \to 1$, an important simplification in the characteristic value problem given by Equations 37, 39 and 40 is possible provided that $(1 - e)$ is small when compared with $(1 + e)/2$. In this case, Equation 40 may be expressed as

$$\Omega = s \tag{41}$$

$$s = (r - e)/(1 - e) \tag{42}$$

The first of Equation 37 may be expressed as

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - N^2\right] U_o = -\frac{N^2}{K^2}\left[2\Omega r \frac{d\Omega}{dr} + 4\Omega^2\right] U_o \tag{43}$$

Using the transformation defined by Equation 42, consistent with the "narrow gap" approximation of small $(1 - e)$, equation 43 becomes

$$\left[\frac{d^2}{ds^2} - a_1^2\right] U_o = -\frac{a_1^2}{K^2} \cdot \frac{2e}{1-e}\, s\, U_o \tag{44}$$

where

$$a_1^2 \equiv (1 - e)^2 N^2 \tag{45}$$

By means of the following transformation,

$$x = a_1^{2/3}\, b^{2/3}\left[\frac{1}{b^2} - s\right] \tag{46}$$

where

$$b^2 \equiv 2e/K^2(1 - e) \tag{47}$$

Equation 44 may be expressed as

$$\frac{d^2U_o}{dx^2} - x\,U_o = 0 \tag{48}$$

The general solution of Equation 48 may be given in terms of either Bessel Functions of order 1/3 or somewhat more conveniently in terms of Airy's functions AI(x) and BI(x); thus,

$$U_o = A_o\,AI(x) + B_o\,BI(x) \tag{49}$$

where $A_o$ and $B_o$ are the integration constants.

In order to determine $U_i$, the general solution of the second of the differential Equation 37 may be obtained from

$$\frac{d^2U_i}{dR^2} + \frac{1}{R}\frac{dU_i}{dR} - \left[\frac{1}{R^2} + 1\right] U_i = 0 \tag{50}$$

where $R^2 = N^2r^2$ (51)

The general solution of Equation 50 may be written as

$$U_i = A_iI_1(R) + B_iK_1(R) \tag{52}$$

The boundary conditions in terms of the new independent variable x defined by Equation 46 are

$$U_o(x_1) = 0 \tag{53a}$$

$$U_o(x_2) = U_i(Ne) \tag{53b}$$

$$-\frac{dU_o}{dx} = \frac{N(1 - e)}{a_1^{2/3}\,b^{2/3}}\frac{dU_i}{dR} \text{ at } x = x_2;\ R = Ne \tag{53c}$$

$$U_i(o) = 0 \tag{53d}$$

where

$$x_1 \equiv a_1^{2/3}\ b^{2/3}\left[\frac{1}{b^2} - s\right]$$

$$x_2 \equiv a_1^{2/3}\ b^{2/3}/b^2 \tag{54}$$

From boundary conditions 53d, we have

$$A_i I_1(o) + B_i K_1(o) = 0$$

Therefore,

$$B_i = 0 \tag{55}$$

Substituting the general solutions given by Equations 49 and 52 and the condition given by Equation 55 into the boundary conditions 53a, 53b and 53c, we obtain

$$A_o AI(x_1) + B_o BI(x_1) = 0 \tag{56}$$

$$A_o AI(x_2) + B_o BI(X_2) = A_i I_1(Ne) \tag{57}$$

$$-\frac{a_1^{2/3}b^{2/3}}{a_1}\left[A_o AI'(x_2) + B_o BI'(x_2)\right]$$

$$= A_i\left[I_o(Ne) - \frac{I_1(Ne)}{Ne}\right] \tag{58}$$

where the primes refer to differentation with respect to x. $A_o$, $B_o$ and $A_i$ may be eliminated from Equations 56, 57 and 58 to give the characteristic equation as

$$x_2^{-1/2}\left[\frac{AI'(x_2)\ BI(x_1) - BI'(x_2)\ AI(x_1)}{AI(x_1)\ BI(x_2) - AI(x_2)\ BI(x_1)}\right] = \frac{I_o(Ne)}{I_1(Ne)} - \frac{1}{Ne} \tag{59}$$

From the definition of $x_1$, and $x_2$ in Equation 54, we obtain

$$x_1 = x_2 - N(1 - e)/x_2^{1/2} \tag{60}$$

For given N and e, the values of $x_1$ and $x_2$ which satisfy Equations 59 and 60 simultaneously are the desired values. However, not all pairs of $x_1$ and $x_2$ lead to admissible values of $K^2$. From Equation 61 we observe that since

$$K^2 = \frac{2e}{1 - e} \cdot \frac{x_2}{x_2 - x_1} = \frac{2e}{N(1 - e)^2} x_2^{3/2} \tag{61}$$

and since $K^2$ is a real number, first of all $x_2$ must be a positive number. Secondly, $(x_2 - x_1)$ must also be positive. Thus, of all the values which satisfy Equations 59 and 60, we select only those pairs of $x_1$ and $x_2$ values which satisfy the conditions that $x_2 > 0$ and that $x_2 > x_1$.

Case ii, The Formal Solution for m = 0

In case we wish to consider the complete range of e in $1 \geq e \geq 0$, we must consider Equation 37 without making any approximation. We then have

$$\frac{d^2U_o}{dr^2} + \frac{1}{r}\frac{dU_o}{dr} - \left[\frac{\nu^2}{r^2} + \alpha^2\right] U_o = 0 \tag{62}$$

where

$$\nu^2 \equiv 1 + \frac{4N^2e^2}{K^2(1 - e^2)^2} \tag{63}$$

$$\alpha^2 \equiv N^2 - \frac{4N^2}{K^2(1 - e^2)^2} \tag{64}$$

Equation 62 would need to be considered with the second of Equation 37 and boundary conditions 39. The general solution of Equation 62 is

$$U_o = \ell_\nu(\alpha r) \tag{65}$$

where $\ell$ is a general cylinder function of order $\nu$. The boundary conditions require $\ell_\nu$ to vanish at r = 1. The required solution may be expressed as

$$U_o = M\,[J_{-\nu}(\alpha)\; J_\nu(\alpha r) - J_\nu(x)\; J_{-\nu}(\alpha r)] \tag{66}$$

where M is a constant.

Defined in this manner, $U_o$ clearly vanishes at r = 1. We have further at r = e.

$$M[J_{-\nu}(\alpha)\; J_\nu(\alpha c) - J_\nu(\alpha)\; J_{-\nu}(\alpha c)] = A_i I_1(Ne) \tag{67}$$

Moreover,

$$\left.\frac{dU_o}{dr}\right|_{r=c} = A_i N \left[I_o(Ne) - \frac{I_1(Ne)}{Ne}\right] \tag{68}$$

The integration constants M and $A_i$; may be eliminated from Equations 67 and 68 to yield the characteristic equation. For arbitrarily assigned $\nu$, one may compute $K^2$ from the characteristic equation and Equations 63 and 64 for given N and e.

## THE GENERAL CASE

Consistent with boundedness of the solutions of $U_i$ and $R_i$ at the origin Equation 31 has the following solution

$$R_i = AI_m(Nr) \tag{69}$$

$$U_i = \frac{i}{K} R_i' = A\left[\frac{m}{r} I_m + NI_{m+1}\right] \tag{70}$$

where A is any arbitrary constant. Equation 28 may be rewritten as

$$U_o' = \frac{1}{\sigma}\left[\left(\frac{m^2}{r^2} + N^2\right) R_o + \frac{2m}{1 - e^2} \cdot \frac{1}{r}\, U_o\right] - \frac{U_o}{r} \tag{71}$$

$$R_o' = -\frac{2}{\sigma}\left[\frac{2}{1 - e^2}\, U_o + \frac{m}{r}\, R_o\right] + \sigma U_o \tag{72}$$

The boundary conditions become

$$U_o(e) = \frac{A}{K}\left[\frac{m}{e} I_m(Ne) + NI_{m+1}(Ne)\right] \tag{73}$$

$$R_o(e) = AI_m(Ne) \tag{74}$$

$$U_o(1) = 0 \tag{75}$$

where $R_o$ is redefined as $-iR_o$ of equations prior to Equation 69. The eignevalue problem now is the following: find those values of K for which the Equations 71 and 72, and the boundary conditions 73, 74, and 75 are satisfied.

Since Equations 71 and 72 are homogeneous, any constant times any solution that satisfies the boundary conditions will also be a solution. Thus, in the eigenvalue problem, we can arbitrarily assign any value to the constant A. For convenience, we set it equal to unity.

## NUMERICAL SOLUTION

Since Equations 71 and 72 are too cumbersome for closed solutions, we employ numerical integration methods. The numerical procedure for solving the eigenvalue problem is as follows: for an assumed value of K, we can compute the values of $U_o(e)$ and $R_o(e)$ for a given set of values of m, e, and N. Using these as the initial values, we can integrate Equations 71 and 72 by numerical methods to yield $U_o(1)$. If this value is zero, then our choice of K is indeed the right one. If not, we change our assumed value of k by a selected increment and recompute. In practice, since it is uneconomical, computer time-wise, to calculate a precise zero value for $U_o(1)$, we arbitrarily say that if the absolute value of $U_o(1)$ is less than a prescribed positive small number, then our choice of the k value is the right one.

For negative values of m, Equations 71 and 72 have a singularity depending on whether or not the quantity $1 + (1 - \varepsilon^2)\, K/m$ is real or complex. If the quantity is real, the singularity is at a raduis given by

$$r_{singular} = e \Big/ \sqrt{1 + (1 - e^2)\ K/m} \qquad (76)$$

At this radius, the derivates of $U_o$ and $R_o$ cannot be computed from Equations 71 and 72. However, from physical considerations, we require that $U_o$ and $R_o$ be continuous at this point.

When a singularity exists, we employ the following procedure: We divide the two zones, $e \leq r \leq r_{sing}$ and $r_{sing} \leq r \leq 1$ into $N_1$ and $N_2$ intervals, where $N_1$ and $N_2$ are selected to be such that the increments in radius for an interval is approximately equal in both zones. Starting from $r = e$, we integrate the Equations 71 and 72 till we reach the last interval of $e \leq r \leq r_{sing}$. We subdivide this into four smaller intervals and integrate the equations to obtain the values of $U_o$, $R_o$ and their derivatives at the three intermediate points shown in Figure 1. Using these values, by linear extrapolation, we estimate the values of $U_o$, $R_o$ and their derivatives at the singularity. Since we cannot use Equations 71 and 72 at the singularity, using the estimated derivatives, we integrate numerically to the one-fourth point of the next interval. From here on, we revert to the conventional integration procedure using Equations 71 and 72.

The numerical integration procedure used in the computer program for calculating the eigenvalues is the familiar Runge-Kutta-Gill method.

## DESCRIPTION OF THE COMPUTER PROGRAM

For computing an eigenvalue, the input data to the program is supplied by two cards. The input map for the program, describing the breakdown of the entries in the cards are shown in Table 1. The third I-field on the first card of each set was originally intended to supply the value of n. Since, this was subsequently generalized to be a

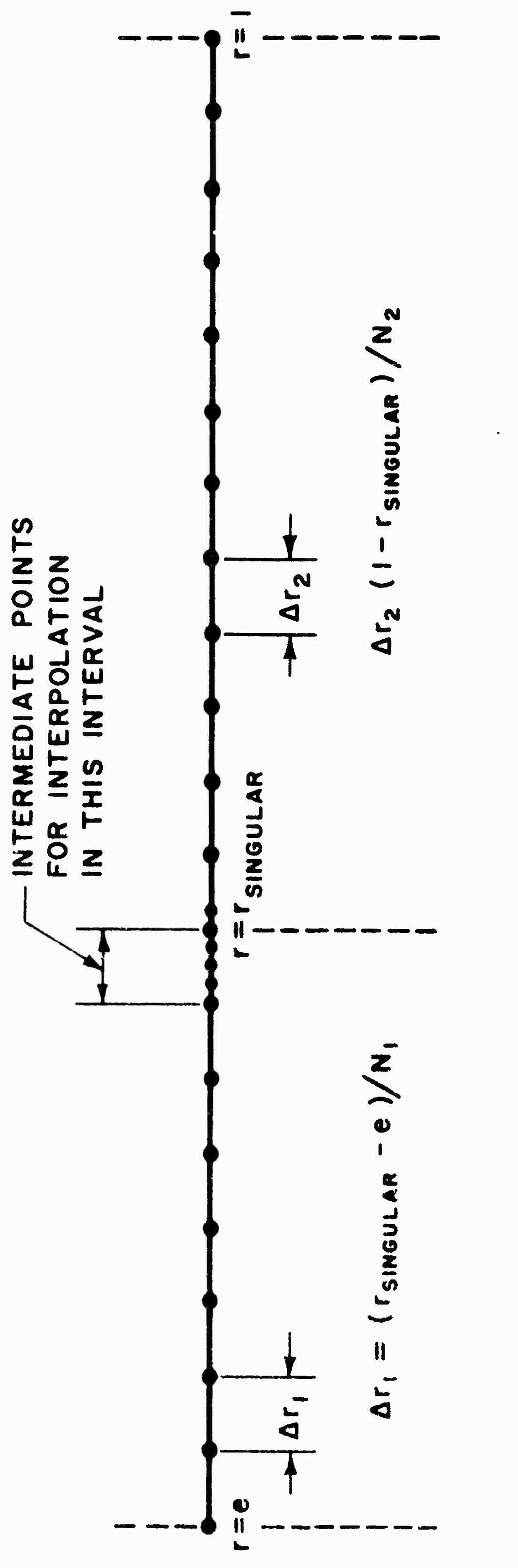


*FIG. 1. INTEGRATION INTERVALS*

Table 1

INPUT MAP

| | Variable | Definition | Format |
|---|---|---|---|
| Card No. 1 | M | Run Number | I6 |
| | MM | -m | I6 |
| | NN | Enter zeros | I6 |
| | NR | Number of integration intervals $\leq$ 500 | I6 |
| | IGEN | Number of Eigenvalues required | I6 |
| | NSTOP | If last set of data being read, should be entered non-zero. Otherwise, enter zero. | I6 |
| | LC | 10 Ne $\leq$ 50 | I6 |
| Card No. 2 | PLAO | Initial trial value of Eigen-value | E12.0 |
| | DPLA | Increment of Eigenvalue | E12.0 |
| | TRUNC | Permissible truncation difference | E12.0 |
| | CRAD | e | F12.6 |

factor in N (CAPN in the program), it may be filled in with a zero or any other number. This number will appear on the printout as the value of n (N in the program). Since it has no other effect on the program, it may be ignored for all other purposes.

The computer program, written in Fortran IV, is given in the Appendix: the first subroutine, NING(NR1) provides the Runge-Kutta-Gill integration method. It also calls a subroutine, DER(Y, FM, CØN, CRAD, FK, AK) which provides the derivatives of $U_o$ and $R_o$ as given by Equations 71 and 72. Subroutine NUESTI, called by the main program, provides a method for incrementing the eigenvalue on the basis of current derivation in $U_o(1)$ from the specified truncation error. Subroutine RKGCØN (AIN, BIN, CIN) supplies the RKG constants to the main program. Subroutine BESCØN(AI, BI), called by the main program supplies the values of the modified Bessel functions. If the program is to be run for any values of m other than -1, this subroutine will need to be changed to supply the modified Bessel functions of the appropriate order.

## RESULTS

Table 2 shows the effect of various step sizes on the computed eigenvalues. Truncation is of $10^{-6}$, initial eigenvalue is 0.01, eigenvalue increment is .05.

Table 2

EFFECT OF STEP SIZE ON COMPUTED EIGENVALUES

| Total No. of Steps | Lowest eigenvalue K for Ne = .2, b/a = e = 0.45, m = -1 | Lowest eigenvalue K for Ne = .2, b/a = e = .1, m = -1 |
|---|---|---|
| 100 | .76079984 | .35125716 |
| 200 | .75407226 | .35046438 |
| 300 | .75447285 | .35032046 |
| 400 | .75445469 | .35025621 |
| 500 | .75479767 | .35021697 |

The results show that if 200 or more total number of steps are employed, the variation is in the fourth significant digit.

Table 3 lists the results for various Ne and e values computed with 200 integration steps, an initial eigenvalue of 0.01, and an eigenvalue increment of 0.05. With this sweep procedure, the listed eigenvalues were found to be the lowest. It is possible that if the eigenvalue increment is lowered from 0.05 to some smaller value, other, ever lower, eigenvalues may be discovered; but this is considered to be extremely unlikely.

In Stewartson's* notation, we have

$$\frac{c}{a(2j+1)} = \frac{\pi}{2N}$$

$$b/a = e$$

$$\tau = K$$

*Stewartson, K., *On The Stability of a Spinning Top Containing Liquid*, J. Fluid Mech., 5, 1959, pp. 577-592.

Table 3

LOWEST EIGENVALUES

| 10XNe | $\frac{\pi}{2}\frac{a(2j+1)}{c}$ | b/a | K = τ | n | 10XNe | $\frac{\pi}{2}\frac{a(2j+1)}{c}$ | b/a | K = τ | n |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | .05 | .35294417 | 2 | 3 | 6 | .05 | 0.10161652 | 4 |
| | 1 | .1 | .36637878 | 1 | | 3 | .1 | 0.079672609 | 2 |
| | .5 | .2 | .68914656 | 1 | | 1.5 | 0.2 | 0.077654683 | 1 |
| | .333 | .3 | .78440604 | 1 | | 1 | 0.3 | 0.38693088 | 1 |
| | .25 | .4 | .95493352 | 1 | | .75 | 0.4 | 0.53024257 | 1 |
| | .2 | .5 | .98246679 | 1 | | .6 | 0.5 | 0.65401489 | 1 |
| | .166 | .6 | .99240964 | 1 | | .5 | 0.6 | 0.93675748 | 1 |
| | .142857 | .7 | .99677785 | 1 | | .42871 | 0.7 | 0.96985291 | 1 |
| | .125 | .8 | .99793433 | 1 | | .375 | 0.8 | 0.98589012 | 1 |
| | .1111 | .9 | .99749419 | 1 | | .333 | 0.9 | 0.99401352 | 1 |
| 2 | 4 | .05 | .17450405 | 3 | 4 | 8 | .05 | 0.062517198 | 5 |
| | 2 | .1 | .35046438 | 2 | | 4 | .1 | 0.1748773 | 3 |
| | 1 | .2 | .37567224 | 1 | | 2 | .2 | 0.33798441 | 2 |
| | .666 | .3 | .57999146 | 1 | | 1.33 | .3 | 0.19826175 | 1 |
| | .5 | .4 | .68582012 | 1 | | 1 | 0.4 | 0.39889403 | 1 |
| | .4 | .5 | .93226799 | 1 | | 0.8 | 0.5 | 0.5155985 | 1 |
| | .333 | .6 | .97111366 | 1 | | 0.6667 | 0.6 | 0.69132881 | 1 |
| | .285714 | .7 | .98598000 | 1 | | 0.571429 | 0.7 | 0.94881139 | 1 |
| | .25 | .8 | .99387097 | 1 | | 0.5 | 0.8 | 0.97572948 | 1 |
| | .222 | .9 | .99698953 | 1 | | 0.444 | 0.9 | 0.98943233 | 1 |

n is the number of radial nodes in the range 0<r<1

Table 3 (Concl.)

| 10XNe | $\frac{\pi}{2}\frac{a(2j+1)}{c}$ | b/a | k = τ | n | 10XNe | $\frac{\pi}{2}\frac{a(2j+1)}{c}$ | b/a | k = τ | n |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 10 | .05 | 0.03821307 | 6 | 7 | 14 | .05 | .094461123 | 9 |
| | 5 | .1 | 0.021487953 | 3 | | 7 | .1 | .15244942 | 5 |
| | 2.5 | 0.2 | 0.21049394 | 2 | | 3.5 | .2 | .24648402 | 3 |
| | 1.666 | 0.3 | 0.028242248 | 1 | | 2.333 | .3 | .24666304 | 2 |
| | 1.25 | 0.4 | 0.27032753 | 1 | | 1.75 | .4 | .036690115 | 1 |
| | 1 | 0.5 | 0.41369377 | 1 | | 1.4 | .5 | .23270392 | 1 |
| | .833 | 0.6 | 0.54600539 | 1 | | 1.166 | .6 | .36577586 | 1 |
| | .714268 | 0.7 | 0.9210945 | 1 | | 1 | .7 | .53633700 | 1 |
| | .625 | 0.8 | 0.96316761 | 1 | | .875 | .8 | .93412168 | 1 |
| | .555 | 0.9 | 0.98512214 | 1 | | .777 | .9 | .97391060 | 1 |
| 6 | 12 | .05 | 0.021654819 | 7 | 8 | 16 | .05 | .076405596 | 10 |
| | 6 | .1 | 0.1034529 | 4 | | 8 | .1 | .065089666 | 5 |
| | 3 | .2 | 0.094563088 | 2 | | 4 | .2 | .17042725 | 3 |
| | 2 | .3 | 0.32489160 | 2 | | 2.666 | .3 | .17456094 | 2 |
| | 1.5 | .4 | 0.14853735 | 1 | | 2 | .4 | .32350274 | 2 |
| | 1.2 | .5 | 0.32126868 | 1 | | 1.6 | .5 | .14869081 | 1 |
| | 1 | .6 | 0.44177624 | 1 | | 1.333 | .6 | .29887522 | 1 |
| | .857143 | .7 | 0.69269504 | 1 | | 1.142857 | .7 | .44032151 | 1 |
| | .75 | .8 | 0.94963371 | 1 | | 1 | .8 | .91714270 | 1 |
| | .666 | .9 | 0.97912326 | 1 | | .888 | .9 | .96724922 | 1 |

n is the number of radial nodes in the range $0<r<1$

# APPENDIX

# A

```
C     MAIN PROGRAM FOR THE DETERMINATION OF EIGENVALUES 31Q-B2294-01
C
     0COMMON FM,FN,CON,DEL,NR,X(1,501),Z(1,501),FREQU(3),DET(3),AIN(4),
     1BIN(4),CIN(4),CRAD,DPLA,IDET,ILU,KLUE,UCO,RCO,RADIUS(501)
      DIMENSION AI(50),BI(50)
      NSTOP=0
      CALL RKGCON(AIN,BIN,CIN)
      CALL BESCON(AI,BI)
   20 IF(NSTOP.NE.0)STOP
      READ(5,10)M,MM,NN,NR,IGEN,NSTOP,LC,PLAO,DPLA,TRUNC,CRAD
   10 FORMAT(7I6/3E12.0,F12.6)
      MM=-MM
      FM=MM
      FN=NN
      FNR=NR
      FREQU(1)=PLAO                                                     00000013
      BNC=LC
      FNC=BNC/10.
      CAPN=FNC/CRAD
      CON=CAPN**2
      AIM1=AI(LC)*EXP(FNC)
      AIM2=BI(LC)*EXP(FNC)
      IDET=1                                                            00000014
      KKK=1                                                             00000015
      WRITE(6,85)
   85 FORMAT(1H1 35HCURRENT VALUES OF INTEGRATION DATA //)
      WRITE(6,86)CAPN,LC
   86 FORMAT(5X7HCAPN =   F10.4,5X5HLC = I2)
      DO 30 IG=1,IGEN                                                   00000016
      ICOUNT=1                                                          00000017
   70 UCO=(FM*AIM1/CRAD+CAPN*AIM2)/FREQU(1)
      RCO=AIM1
      RADIUS(1)=CRAD
      X(1,1)=UCO
      Z(1,1)=RCO
      CALL NING(NR1)
      DET(1)=X(1,NR+1)
      WRITE(6,90)M,IG,FREQU(1),X(1,NR+1)
   90 FORMAT(                                   // 10X 13HRUN NUM
     1BER = I3, 5X  5HIG = I2, 5X 16HTRIAL EIGENV. = E15.8, 5X 9H  U(1) 00000023
     2= E15.8)                                                          00000024
      IF(IDET-2)170,170,180                                             00000025
  170 DL = DET(1)                                                       00000026
      AL = FREQU(1)                                                     00000027
      GO TO 190                                                         00000028
  180 IF(AL-FREQU(1))170,170,190                                        00000029
  190 ADET = ABS (DET(1))                                               00000030
      IF(ADET-TRUNC)80,80,100                                           00000031
  100 CALL NUESTI                                                       00000032
      GO TO (80,110, 20),KLUE                                           00000035
  110 IDET = IDET+1                                                     00000036
      ICOUNT=ICOUNT+1                                                   00000037
      IF(ICOUNT-50)70,70, 20
   80 WRITE(6,1)
    1 FORMAT(1H1 41X 32H EIGENVALUES AND EIGENFUNCTIONS   )             00000040
```

```
      WRITE(6,2)M,MM,NN,IG,FREQU(1),CAPN,CRAD
    2 FORMAT(//48X 10HRIN NO. = I3/// 7X 6H M =    I3,  5X 8H     N = I3,
     1     5X I2, 2X 13HEIGENVALUE = E15.8,9X6HCAPN = F12.6,5X4HC = F6.2)
      WRITE(6,3)
    3 FORMAT(///18X 2H R 4X 14H U MODE SHAPE 4X 14H R MODE SHAPE 18X 2H
     3R 4X 14H U MODE SHAPE 3X 15H  R MODE SHAPE  //)
      KPA=1
      NRR=NR/2
      DO 40 I=1,NRR
      IF (I-51*KPA) 50,60,50
   60 KPA = KPA+1
      WRITE(6,1)
      WRITE(6,2)M,MM,NN,IG,FREQU(1),CAPN,CRAD
      WRITE(6,3)
   50 K=NRR+I
      WRITE(6,130) RADIUS(I),X(1,I),Z(1,I),RADIUS(K),X(1,K),Z(1,K)
  130 FORMAT(F20.4,  3X E15.8, 3X E15.8,F20.4,   3X E15.8, 3X E15.8)
      IF(I-NRR) 40,140,40
  140 K = K+1
      WRITE(6,150) RADIUS(K),X(1,K),Z(1,K)
  150 FORMAT(40X F20.4,3X E15.8, 3X E15.8)
   40 CONTINUE
      FREQU(2)=AL
      DFT(2)  =DL
      FREQU(1)=AL+DPLA
   30 IDET=2
      GO TO 20
      END
```

```
      SUBROUTINE NING(NR1)
     0COMMON FM,FN,CON,DEL,NR,X(1,501),Z(1,501),FREQU(3),DET(3),AIN(4),
     1BIN(4),CIN(4),CRAD,DPLA,IDET,ILU,KLUE,UCO,RCO,RADIUS(501)
      DIMENSION AK(3),Y(3),Q(3),AM(3),BM(3)
      FK=FREQU(1)
      Y(1)=CRAD
      Y(2)=UCO
      Y(3)=RCO
      Q(1)=0.
      Q(2)=0.
      Q(3)=0.
      AK(1)=1.
      BING=1.+(1.-(CRAD**2))*FK/FM
      IF (BING)3,3,1
    1 RSING=CRAD/SQRT(BING)
      WRITE(6,2) RSING
    2 FORMAT(//20X 14HR(SINGULAR) = F9.5)
      GO TO 4
    3 WRITE(6,5)
    5 FORMAT(10X38HSINGULARITY OUTSIDE INTEGRATION DOMAIN        )
    4 BNR=NR
      NR1=BNR*(RSING-CRAD)/(1.-CRAD)
      NRM=NR-50
      IF(NR1.LT.50)NR1=50
      IF(NR1.GT.NRM)NR1=NRM
      NR2=NR-NR1
      FNR1=NR1
      FNR2=NR2
      DEL=(RSING-CRAD)/FNR1
      DEL1=(1.-RSING)/FNR2
      DELTA=DEL/4.
      NAB=NR1+2
      DO 1000 L=1,NAB
      IF(L.GE.NR1)DEL=DELTA
      DO 100 JJ=1,4
      CALL DER(Y,FM,CON,CRAD,FK,AK)
      DO 50 I=1,3
      AIKN =AIN(JJ)*(AK(I)-BIN(JJ)*Q(I))
      Y(I) = Y(I)+DEL*AIKN
   50 Q(I) = Q(I)+3.*AIKN -CIN(JJ)*AK(I)
  100 CONTINUE
      RADIUS(L+1)=Y(1)
      X(1,L+1)=Y(2)
 1000 Z(1,L+1)=Y(3)
      CALL DER(Y,FM,CON,CRAD,FK,AK)
      BU=AK(2)
      BR=AK(3)
      Y(1)=RADIUS(NAB)
      CALL DER(Y,FM,CON,CRAD,FK,AK)
      AU=AK(2)
      AR=AK(3)
      AK(2)=2.*BU-AU
      AK(3)=2.*BR-AR
      Y(1)=RSING
      Y(2)=2.*X(1,NAB+1)-X(1,NAB)
```

```
      Y(3)=2.*Z(1,NAB+1)-Z(1,NAB)
      RADIUS(NR1+1)=Y(1)
      X(1,NR1+1)=Y(2)
      Z(1,NR1+1)=Y(3)
      DEL=DEL1/4.
      DO 80 L=1,4
      DO 80 JJ=1,4
      IF(L.EQ.1)GO TO 81
      CALL DER(Y,FM,CON,CRAD,FK,AK)
   81 DO 82 I=1,3
      AIKN =AIN(JJ)*(AK(I)-BIN(JJ)*Q(I))
      Y(I) = Y(I)+DEL*AIKN
   82 Q(I) = Q(I)+3.*AIKN -CIN(JJ)*AK(I)
   80 CONTINUE
      RADIUS(NR1+2)=Y(1)
      X(1,NR1+2)=Y(2)
      Z(1,NR1+2)=Y(3)
      DEL=DEL1
      DO 900 L=NAB,NR
      IF(L.EQ.NR)DEL=1.-Y(1)
      DO 90 JJ=1,4
      CALL DER(Y,FM,CON,CRAD,FK,AK)
      DO 60 I=1,3
      AIKN =AIN(JJ)*(AK(I)-BIN(JJ)*Q(I))
      Y(I) = Y(I)+DEL*AIKN
   60 Q(I) = Q(I)+3.*AIKN -CIN(JJ)*AK(I)
   90 CONTINUE
      RADIUS(L+1)=Y(1)
      X(1,L+1)=Y(2)
  900 Z(1,L+1)=Y(3)
      RETURN
      END
```

```
SUBROUTINE DER(Y,FM,CON,CRAD,FK,AK)
DIMENSION Y(3),AK(3)
R2=1./(Y(1)*Y(1))
ALPHA=FM*FM*R2+CON
BETA=2./(1.-CRAD*CRAD)
OMEGA=.5*(1.-CRAD*CRAD*R2)*BETA
SIGMA=FK+FM*OMEGA
AK(2)=(ALPHA*Y(3)+FM*BETA*Y(2)/Y(1))/SIGMA-Y(2)/Y(1)
AK(3)=-2.*OMEGA*(BETA*Y(2)+FM*Y(3)/Y(1))/SIGMA+SIGMA*Y(2)
RETURN
END
```

```
      SUBROUTINE NUESTI
     0COMMON FM,FN,CON,DEL,NR,X(1,501),Z(1,501),FREQU(3),DET(3),AIN(4),
     1BIN(4),CIN(4),CRAD,DPLA,IDET,ILU,KLUE,UCO,RCO,RADIUS(501)
     0DOUBLE PRECISION F321,F213,FD,AF1,F23,F31,F12,D1F1,D2F2,D3F3,AF2,
     1F132,AF3,QD,CC,SR1,FF1,FF2
      IF(IDET-2)  10,  20,  30
   10 FREQU(2)=FREQU(1)
      FREQU(1)=FREQU(2)+DPLA
      DET(2)=DET(1)
      NCLUE= 3
      GO TO 40
   20 DDET = DET(1)* DET(2)
      NCLUE = 3
      IF(DDET) 50, 50, 60
   60 ILU = 2
      GO TO 70
   50 ILU = 1
   70 FREQU(3)=FREQU(2)
      DET(3) = DET(2)
      FREQU(2) =FREQU(1)
      DET(2) = DET(1)
   95 GO TO (90,100),ILU
   90 FREQU(1)=.5*((FREQU(3)+FREQU(2))-((FREQU(3)-FREQU(2))/(DET(3)-
     X  DET(2)))*(DET(3)+DET(2)))
      GO TO 40
  100 FREQU(1)=FREQU(2)+DPLA
      GO TO 40
   30 GO TO(110,120),ILU
  110 DDET = DET(1)*DET(2)
      IF(DDET)130,130,140
  130 NCLUE =1
      GO TO 150
  140 DDET=DET(1)*DET(3)
      NCLUE = 2
      IF(DDET)150,150,70
  150 IF(FREQU(3))165,165,300
  165 GO TO(175,185),NCLUE
  175 FREQU(3)=FREQU(1)
      DET(3)=DET(1)
      GO TO 95
  185 FREQU(2)=FREQU(1)
      DET(2)=DET(1)
      GO TO 95
  300 DF12=ABS ((FREQU(1)-FREQU(2))/FREQU(1))
      DF13=ABS ((FREQU(1)-FREQU(3))/FREQU(1))
      DF23=ABS ((FREQU(2)-FREQU(3))/FREQU(2))
      IF(DF12-.1E-04)312,312,313
  313 IF(DF13-.1E-04)312,312,314
  314 IF(DF23-.1E-04)312,312,311
  312 GO TO(165,50),ILU
  311 D1 = DET(1)
      D2 = DET(2)
      D3 = DET(3)
      F1 = FREQU(1)
      F2 = FREQU(2)
```

```
      F3 = FREQU(3)
      F321 =(F3-F2)/F1
      F132 =(F1-F3)/F2
      F213 =(F2-F1)/F3
      FD   = F321 + F132 + F213
      AF1  =(D1*F321 + D2*F132 + D3*F213)/FD
      F23  = F2/F3
      F31  = F3/F1
      F12  = F1/F2
      D1F1 = D1/F1
      D2F2 = D2/F2
      D3F3 = D3/F3
      AF2  = (D1F1*(F23-1.0/F23)+D2F2*(F31-1.0/F31)+D3F3*(F12-1.0/F12))/
     1FD
      AF3  = (D1F1*(1.0/F2-1.0/F3)+D2F2*(1.0/F3-1.0/F1)+D3F3*(1.0/F1-1.0
     1/F2))/FD
      QD   =  AF1/AF3
      CC   =  0.5*AF2/AF3
      CCD=CC**2-QD
      IF(CCD)312,312,397
  397 SR1 = SQRT (CCD)
      FF1  = -CC+SR1
      FF2  = -CC -SR1
      GO TO(160,400),ILU
  400 ILU = 1
      FREQU(3) = FREQ
      DE (3)    = DET(2)
      FREQU(2) = FREQU(1)
      DET(2)    = DET(1)
C
   80 IF(FF1-FREQU(3))500,500,600
  500 FREQU(1)=FF2
      GO TO 40
  600 IF(FF1-FREQU(2))700,700,500
  700 FREQU(1)=FF1
      GO TO 40
  160 GO TO(170,180),NCLUF
  170 FREQU(3)=FREQU(1)
      DET(3)   =DET(1)
      GO TO 80
  180 FREQU(2)=FREQU(1)
      DET(2)   =DET(1)
      GO TO 80
  120 DDET = DET(1)*DET(2)
      IF(DDET)190,190,200
  190 NCLUF = 2
      IF(FREQU(3))50,50,300
  200 NCLUF = 3
      GO TO 70
   40 GO TO(210,220,230),NCLUF
  210 DFRA = ABS (FREQU(1)-FREQU(3))
      GO TO 240
  220 DFRA = ABS (FREQU(1)-FREQU(2))
    0 IF(DFRA-.1E-07)250,250,230
  250 KLUF = 1
      GO TO(260,270,280),NCLUF
```

```
260 FREQU(1) = FREQU(3)
    GO TO 280
[illegible] FREQU(1) = FREQU(2)
    GO TO 280
230 KLUE = 2
280 CONTINUE
    RETURN
    END
```

```
SUBROUTINE RKGCON(AIN,BIN,CIN)
DIMENSION AIN(4),BIN(4),CIN(4)
AIN(1)=1./2.
SRT=SQRT(AIN(1))
AIN(2)=1.-SRT
AIN(3)=1.+SRT
AIN(4)=1./6.
BIN(1)=2.
BIN(2)=1.
BIN(3)=1.
BIN(4)=2.
CIN(1)=AIN(1)
CIN(2)=AIN(2)
CIN(3)=AIN(3)
CIN(4)=AIN(1)
RETURN
END
```

```
      SUBROUTINE BESCON(AI,BI)
      DIMENSION AI(50),BI(50)
      AI(1) =0.0452984                                         ISUBM
      AI(2) =0.0822831
      AI(3) =0.1123775                                         ACTUALLY
      AI(4) =0.1367632                                         ISUB1
      AI(5) =0.1564208
      AI(6) =0.1721644
      AI(7) =0.1846699
      AI(8) =0.1944987
      AI(9) =0.2021165
      AI(10)=0.2079104
      AI(11)=0.2122016
      AI(12)=0.2152568
      AI(13)=0.2172976
      AI(14)=0.2185076
      AI(15)=0.2190694
      AI(16)=0.2190195
      AI(17)=0.2185528
      AI(18)=0.2177263
      AI(19)=0.2166120
      AI(20)=0.2152693
      AI(21)=0.2137478
      AI(22)=0.2120877
      AI(23)=0.2103230
      AI(24)=0.2084811
      AI(25)=0.2065846
      AI(26)=0.2046523
      AI(27)=0.2026990
      AI(28)=0.2007374
      AI(29)=0.1987773
      AI(30)=0.1968267
      AI(31)=0.1948921
      AI(32)=0.1929786
      AI(33)=0.1910902
      AI(34)=0.1892299
      AI(35)=0.1873999
      AI(36)=0.1856022
      AI(37)=0.1838379
      AI(38)=0.1821076
      AI(39)=0.1804119
      AI(40)=0.1787508
      AI(41)=0.1771244
      AI(42)=0.1755325
      AI(43)=0.1739746
      AI(44)=0.1724502
      AI(45)=0.1709588
      AI(46)=0.1694997
      AI(47)=0.1680723
      AI(48)=0.1666757
      AI(49)=0.1653093
      AI(50)=0.1639723
      BI(1) =.9071009
      BI(2) =.8269385                                          ISUBM+1
      BI(3) =.7575806
                                                               ACTUALLY
```

```
      BI(4) =.6974022                                          ISUBZERO
      BI(5) =.6450353
      BI(6) =.5993272
      BI(7) =.5593055
      BI(8) =.5241489
      BI(9) =.4931630
      BI(10)=.4657596
      BI(11)=.4414404
      BI(12)=.4197821
      BI(13)=.4004249
      BI(14)=.3830625
      BI(15)=.3674336
      BI(16)=.3533150
      BI(17)=.3405157
      BI(18)=.3288719
      BI(19)=.3182432
      BI(20)=.3085083
      BI(21)=.2995631
      BI(22)=.2913173
      BI(23)=.2836930
      BI(24)=.2766223
      BI(25)=.2700464
      BI(26)=.2639140
      BI(27)=.2581801
      BI(28)=.2528055
      BI(29)=.2477557
      BI(30)=.2430003
      BI(31)=.2385126
      BI(32)=.2342688
      BI(33)=.2302480
      BI(34)=.2264314
      BI(35)=.2228024
      BI(36)=.2193462
      BI(37)=.2160494
      BI(38)=.2129001
      BI(39)=.2098875
      BI(40)=.2070019
      BI(41)=.2042345
      BI(42)=.2015774
      BI(43)=.1990232
      BI(44)=.1965656
      BI(45)=.1941983
      BI(46)=.1919160
      BI(47)=.1897134
      BI(48)=.1875862
      BI(49)=.1855300
      BI(50)=.1835408
      RETURN
      END
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ON THE EIGENVALUES OF COUETTE FLOW IN A FULLY-FILLED CYLINDRICAL CONTAINER

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(Last name, first name, initial)*

Reddi, M. M.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| January 1967 | | |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO.<br>DA-30-069-AMC-686(R)<br>b. PROJECT NO.<br>c.<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>Technical Report No. F-B2294<br>9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* |

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For a stationary flow in a cylindrical container of the Couette type in an outer radial zone, and of zero velocity in an inner radial zone, the normal mode equations are derived. For negative wave numbers in the $\theta$-direction, these equations are found to have a singularity.

The eigenvalues are calculated by initial value methods employing the Runge-Kutta-Gill integration procedure. Values of the dependent function and their derivatives at the singularity are calculated by linear extrapolation coupled with continuity requirements.

Tables of eigenvalues for various slenderness ratios of the cylinder and various radial nodes are given for $\theta$-wave numbers of -1.

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| 14. KEY WORDS | LINK A ROLE | LINK A WT | LINK B ROLE | LINK B WT | LINK C ROLE | LINK C WT |
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