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# THE KERNEL AND BARGAINING SET FOR CONVEX GAMES

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# 1. INTRODUCTION

The basic papers dealing with the _kernel_ $\mathcal{K}$ of a cooperative game are [2], [5] and [6]. The basic papers dealing with the _bargaining set_ $\mathcal{M}_1^{(i)}$ are [3], [4] and [7]. For intuitive justification of the bargaining set as a solution concept, the reader is referred to [1].

Although the kernel of a game and the _core_ of a game are two different concepts, it has been proved in [5] that if a game has a nonempty core, its kernel for the grand coalition must intersect the core; however, it may contain points also outside the core. The bargaining set for the grand coalition always contains the core and, again, it may contain points outside the core.

A task which naturally arises is to sharpen these results for games whose core is in some sense "nice".

_Convex games_ were introduced in [9], where it was shown that these are precisely the games for which the core is _regular_.

Interpreting "nice" as regular, we shall accomplish the task by showing that, for the grand coalition, the kernel of a convex game lies in the relative interior of the core — i.e., its interior when regarded as a point set in the linear manifold of least dimension which contains it. We shall also show that the bargaining set $\mathcal{M}_1^{(i)}$ of a convex game coincides with the core, and hence, as shown in [9], with the unique von Neumann-Morgenstern solution.

Although all the definitions as well as the results taken from other papers will be fully stated, it is advisable that the reader make himself familiar at least with the relevant parts of [2], [4], [6], [8] and [9].

## 2. THE KERNEL OF A CONVEX GAME

An n-person cooperative game $(N;v)$, where $N = \{1,2,\dots,n\}$ is its set of players and $v$ is its characteristic function, is called convex if $v$ satisfies

(2.1) $\qquad v(\phi) = 0$ ,

(2.2) $\qquad v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ all $A, B \subseteq N$ .

Convex games were introduced and studied in [9], where it has been shown that they have non-empty cores. Moreover, it has been shown that these are precisely the games whose core is regular[1].

Convex games are superadditive but not necessarily monotonic[2].

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[1] The core of an n-person game $(N;v)$ is the set of all n-tuples $x = (x_1,\dots,x_n)$ such that $x(N) = v(N)$ and $x(S) \geq v(S)$ for all coalition $S$ . Here $x(S)$ is a short notation for $\Sigma_{i \in S} x_i$ . The notation $x(S)$ will be used throughout this paper. $x(\phi)$ is defined to be equal to zero.

A core of $C$ of an n-person game $(N,v)$ is called regular if it is not empty and if, in addition,

$$C_S \cap C_T \subset C_{S \cup T} \cap C_{S \cap T} .$$

Here $C_A = C \cap \{x = (x_1,\dots,x_n) \mid x(A) = v(A)\}$, $A = S,T$ . In particular $C_S \neq \phi$ for all $S$ if $C$ is regular. Thus, a regular core is quite "large" since it touches all the $(n-2)$-faces of the simplex of imputations. In [9] it is shown that a regular core always contains the (Shapley value), and is the unique von Neumann-Morgenstern solution of the game.

[2] A game $(N;v)$ is called monotonic if $v$ satisfies $v(S) \leq v(T)$ whenever $S \subset T$ .

However, if the characteristic function of a game satisfies

(2.3)      $v(\{i\}) = 0, \quad i = 1,2,\ldots,n$ ,

then monotonicity follows from superadditivity.

Note that the core, the bargaining set and the kernel of a game are relative invariants with respect to strategic equivalence[3], and convexity is invariant under strategic equivalence   For this reason we shall assume (2.3) in some of the proofs and this assumption will entail no loss of generality.

Let  $x = (x_1,x_2,\ldots,x_n)$  be an n-tuple of real numbers.  We define the excess of a coalition  S  with respect to  x  to be

(2.4)      $e(S,x) = v(S) - x(S)$ .

Lemma 2.1.      An n-person game  $(N;v)$  is convex if and only if for any arbitrarily chosen fixed n-tuple  x ,

(2.5)      $e(S,x) + e(T,x) \leq e(S \cup T,x) + e(S \cap T,x)$

for each pair of coalitions  S,T .

The proof is immediate.

Let  $\mathcal{C}$  be a given collection of coalitions in a convex n-person game  $(N;v)$ , and let  x  be an arbitrarily chosen n-tuple of real numbers.  Let  $\mathcal{D}(\mathcal{C},x)$  be a collection consisting of those coalitions in  $\mathcal{C}$  whose excess with respect to  x  is maximal; i.e.,

(2.6)      $\mathcal{D}(\mathcal{C},x) = \{S | S \in \mathcal{C} \text{ and } e(S) \geq e(T) \text{ all } T \in \mathcal{C}\}$ .

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[3] i.e., they undergo the transformation  $x \to ax + \alpha$  when  v(S)  is replaced by  $av(S) + \alpha(S)$  for each coalition  S .  Here  a  is a real positive constant,  $\alpha = (\alpha_1,\ldots,\alpha_n)$  is an n-tuple of real constants, and  $\alpha(S)$  is a short notation for  $\Sigma_{i \in S}\alpha_i$ .

<u>Lemma 2.2.</u>    <u>Under these conditions</u> $\mathcal{D}(\mathcal{C},x)$ <u>is "nearly" closed under</u>

<u>unions and intersections</u>: namely,

(2.7)        $S,T \in \mathcal{D}(\mathcal{C},x) \Rightarrow S \cup T \in \mathcal{D}(\mathcal{C},x)$ ,

(2.8)        $S,T \in \mathcal{D}(\mathcal{C},x) \Rightarrow S \cap T \in \mathcal{D}(\mathcal{C},x)$ ,

<u>provided that both</u> $S \cup T$ <u>and</u> $S \cap T$ <u>belong to</u> $\mathcal{C}$ .

The proof is an immediate consequence of Lemma 2.1 and the definitions.


<u>Notation:</u>        We shall write $\mathcal{D}(x)$  instead of $\mathcal{D}(\mathcal{C},x)$  if  $\mathcal{C}$  is the

set of all the coalitions of the game except  N  and  $\phi$ .  We shall

write $\mathcal{E}(x)$  instead of $\mathcal{D}(\mathcal{C},x)$  if  $\mathcal{C}$  is the set of all the coalitions

of the game.


<u>Corollary 2.3.</u>    $\mathcal{E}(x)$  <u>is closed under unions and intersections.</u>

An n-tuple  $x = (x_1,x_2,\ldots,x_n)$  is called an <u>imputation</u> in an n-

person game  $\Gamma = (N;v)$  if  $x(\{i\}) \geq v(\{i\})$  for all  $i$ ,  $i = 1,2,\ldots,n$

and  $x(N) = v(N)$ .  An imputation  x  belongs to the <u>core</u> of the game if

and only if  $e(S,x) \leq 0$  for every coalition  S .  The core of a game

will be denoted by  $C$  or by  $C(\Gamma)$ .  An imputation  x  is said to belong

to the <u>kernel</u> of the game  $\Gamma$  for the grand coalition  N  if for every

ordered pair of players  $(k,\ell)$ ,

(2.9)        $\underset{S:k\in S,\ell\notin S}{\text{Max}} e(S,x) \leq \underset{S:\ell\in S,k\notin S}{\text{Max}} e(S,x)$  or  $x_\ell = v(\{i\})$ .

The kernel for the grand coalition will be denoted in this paper by[4]

$\mathcal{K}$  or $\mathcal{K}(\Gamma)$ .  It has been shown in general that  $\mathcal{K} \neq \phi$  and that

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[4] If coalition-structures other than  $\{N\}$  are also being considered,
the usual notation is slightly different.

$\mathcal{K} \cap C \neq \phi$ if $C \neq \phi$ (see [2], [5]). In [9] it is shown that for convex games $C \neq \phi$. Both $C$ and $\mathcal{K}$ are closed sets.

Theorem 2.4. The kernel for the grand coalition of a convex game is contained in the core.

Proof: Let $\Gamma = (N;v)$ be a convex game. Without loss of generality we may assume that (2.3) holds, in which case $\Gamma$ is a monotonic game. For monotonic games satisfying (2.3) it has been proved in [6] that if $x \in \mathcal{K}$ then $\mathcal{D}(x)$ (see notation prior to Corollary 2.3) has the following property [5]:

If a coalition in $\mathcal{D}(x)$ contains a player $k$ and does not contain a player $\ell$ then another coalition exists in $\mathcal{D}(x)$ which contains player $\ell$ and does not contain player [6] $k$.

It follows from this property that [7]

$$(2.10) \qquad \cap_{S:S\in\mathcal{D}(x)} S = \phi$$

$$(2.11) \qquad \cup_{S:S\in\mathcal{D}(x)} S = N$$

for $n \geq 2$, whenever $x \in \mathcal{K}$, because $\mathcal{D}(x)$ is not empty and its members are proper non-empty subsets of $N$.

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[5] $\mathcal{D}(x)$ is denoted in [6] as $\mathcal{D}(N,x)$.

[6] In the terminology of [6], no player is "separated out" by $\mathcal{D}(x)$.

[7] (2.10) holds for a much larger class of games (see [6]).

The theorem certainly holds for 1-person games. Assuming $n \geq 2$ and applying Lemma 2.2 repeatedly to unions and intersections of members of $\mathcal{D}(x)$ , one concludes that either there exist two coalitions $S_1$ and $T_1$ in $\mathcal{D}(x)$ such that $S_1 \cap T_1 = \phi$ , or there exist two coalitions $S_2$ and $T_2$ in $\mathcal{D}(x)$ such that $S_2 \cup T_2 = N$ . In view of the fact that $e(N,x) = e(\phi,x) = 0$ , it follows from Lemma 2.1 and from the meaning of $\mathcal{D}(x)$ that $e(S,x) \leq 0$ for every coalition of the game. Consequently $x$ belongs to the core of the game. This concludes the proof.

Let us now ask under what conditions the kernel of a convex game can intersect the boundary of the core, when the core is being regarded as a point-set in the $n - 1$ dimensional hyperplane $x(N) = v(N)$ . Assume, therefore, that $x \in \mathcal{K}$ and $e(S,x) = 0$ for a coalition $S$ other than $N$ and $\phi$ . By Theorem 2.4 it follows that the coalitions in $\mathcal{D}(x)$ have excess equal to $0$ and therefore, in this case,

(2.12)  $\qquad \mathcal{E}(x) = \mathcal{D}(x) \cup \{N\} \cup \{\phi\}$ .

Definition 2.5.  The partition $\{T_1, T_2, \ldots, T_u\}$ of $N$ <u>induced</u> by a given collection of coalitions $\mathcal{D}$ is the set of equivalence classes $T_1, T_2, \ldots, T_u$ such that two players $k$ and $\ell$ belong to the same equivalence class if and only if they appear simultaneously in the coalitions of $\mathcal{D}$ ; i.e., if and only if

(2.13)  $\qquad k \in S \in \mathcal{D} \iff \ell \in S \in \mathcal{D}$ .

Clearly, one obtains the same partition if $\mathcal{D}$ is replaced by $\mathcal{D} \cup \{N\} \cup \{\phi\}$ .

We shall now show that the equivalence classes $T_1, T_2, \ldots, T_u$ of the partition induced by $\mathcal{D}(x)$ (or by $\mathcal{E}(x)$ , see (2.12)) are themselves elements of $\mathcal{D}(x)$ , provided that $n \geq 2$ .

Since $\mathcal{D}(x)$ and therefore $\{T_1, T_2, \ldots, T_u\}$ are invariant under strategic equivalence[8], we may assume that $(N;v)$ satisfies (2.3) and is, therefore, a monotonic game.

Consider the intersection $I(\ell)$ of all the coalitions in $\mathcal{E}(x)$ which contain a player $\ell$ in an equivalence class $T_j$ . Clearly $I(\ell) \supset T_j$ . By Corollary 2.3, $I(\ell) \in \mathcal{E}(x)$ . By (2.11) and because $n \geq 2$ , $I(\ell) \neq N$ and $I(\ell) \neq \phi$ ; therefore

(2.14)  $\qquad I(\ell) \in \mathcal{D}(x)$ .

If a player $k$ existed in $I(\ell) - T_j$ then there would exist a coalition in $\mathcal{D}(x)$ containing $k$ and not $\ell$ , whereas every coalition in $\mathcal{D}(x)$ containing player $\ell$ would also contain player $k$ . This is impossible by the property of $\mathcal{D}(x)$ mentioned prior to (2.10). We conclude that $I(\ell) = T_j$ , and, by (2.14), $T_j \in \mathcal{D}(x)$ . Thus, $e(T_1, x) = e(T_2, x) = \ldots = e(T_u, x) = 0$ , and since $\{T_1, T_2, \ldots, T_u\}$ is a partition of $N$ it follows from (2.4) that

(2.15)  $\qquad v(T_1) + v(T_2) + \ldots + v(T_u) = v(N)$ .

Definition 2.6.    A game $(N;v)$ is said to be <u>decomposable</u> into the games $(T_1; v/T_1)$, $(T_2; v/T_2)$, $\ldots$, $(T_u; v/T_u)$ , where $\{T_1, T_2, \ldots, T_u\}$ , $u \geq 2$ , is a partition of $N$ and $v/T_j$, $j = 1, 2, \ldots, u$ , is the restriction of the characteristic function $v$ to the subsets of $T_j$ , if for each coalition $S, S \subset N$ ,

(2.16)  $\qquad v(S) = v(S \cap T_1) + v(S \cap T_2) + \ldots + v(S \cap T_u)$ .

The notion of decomposition was introduced by J. von Neumann and O. Morgenstern in [10]. We shall refer to the games $(T_j; v/T_j)$ as

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[8] When $x$ undergoes the transformation mentioned in footnote (3).

components of (N;v) . It is easily seen from (2.2) that a decomposable

game is convex if and only if all its components are convex. Thus, many

non-trivial decomposable games exist for n ≥ 4 ; none of them, however,

is "strictly convex" in the sense of [9].

Lemma 2.7     A convex game (N;v) is decomposable into

$(T_1;v/T_1)$, $(T_2;v/T_2)$,..., $(T_u;v/T_u)$ , where $\{T_1,T_2,...,T_u\}$ is a

partition of N , if and only if

(2.17)        $v(N) = v(T_1) + v(T_2) + ... + v(T_u)$ .

Proof[9].        Clearly, (2.16) implies (2.17). Suppose (2.17) holds.
By convexity

$$v(S) + v(T_1) \leq (S \cap T_1) + v(S \cup T_1)$$

$$v(S \cup T_1) + v(T_2) \leq v(S \cap T_2) + v(S \cup T_1 \cup T_2)$$

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$$v(S \cup T_1 \cup ... \cup T_{u-1}) + v(T_u) \leq v(S \cap T_u) + v(S \cup T_1 \cup ... \cup T_u).$$

Adding these inequalities to the equation (2.17), we obtain

$v(S) \leq v(T_1 \cap S) + v(T_2 \cap S) + ... + v(T_u \cap S)$ . But equality must hold

because (N;v) is a superadditive game. This concludes the proof.

A consequence of Theorem 2.4, Lemma 2.7 and the preceding discussion

leading to (2.15) is:

Theorem 2.8.    The kernel for the grand coalition of an indecomposable

n-person convex game lies strictly in the interior of the core, regarded

as a point-set in the n - 1 dimensional hyperplane defined by x(N) = v(N) .

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[9] Several proofs can be given. The proof presented here is, perhaps,
the most elegant one. It was pointed out to one of the authors by
Y. Kannai.

This has been proved for $n \geq 2$. It certainly holds if $n = 1$.

The converse statement is also true in view of

Lemma 2.9.     The core of a decomposable game is the cartesian product

of the cores of its components. Therefore it is not of full dimension,

and each of its points is a boundary point if the core is being regarded

as a point-set in $x(N) = v(N)$.

Proof.     Let $x \in C(\Gamma)$, where $\Gamma = (N;v)$ is decomposable into

$\Gamma_1 \equiv (T_1;v/T_1)$, $\Gamma_2 \equiv (T_2;v/T_2)$, ..., $\Gamma_u \equiv (T_u;v/T_u)$, where

$\{T_1, T_2, \ldots, T_u\}$ is a partition of $N$, $u \geq 2$. By (2.17), $0 = v(N) - x(N) =$

$[v(T_1) - x(T_1)] + [v(T_2) - x(T_2)] + \ldots + [v(T_u) - x(T_u)] =$

$e(T_1,x) + e(T_2,x) + \ldots + e(T_u,x)$. Therefore, $e(T_1,x) = e(T_2,x) = \ldots$

$= e(T_u,x) = 0$ and $x/T_j$ - the restriction of $x$ into the coordinates

which belong to $T_j$ - belongs to $C(\Gamma_j)$, $j = 1,\ldots,u$. And conversely: If

$x/T_j \in C(\Gamma_j)$, $j = 1,2,\ldots,u$, then, by (2.16) and (2.4), for each sub-

set $S$ of $N$, $e(S,x) = e(S \cap T_1,x) + e(S \cap T_2,x) + \ldots + e(S \cap T_u,x) \leq 0$

and therefore $x \in C(\Gamma)$.

It has been proved in [7] that, in general, the kernel for the

grand coalition of a decomposable game is not equal to the cartesian

product of the kernels for the component games, because a transfer (see

[8], [10]) may take place. For convex games, however, no transfer is

possible.

Lemma 2.10.     The kernel for the grand coalition of a decomposable

convex game is the cartesian product of the kernels for the grand

coalitions of the component games.

<u>Proof.</u>   We use the notation of the previous proof.  It has been shown in [8] that $\mathcal{K}(\Gamma) \supset \mathcal{K}(\Gamma_1) \times \mathcal{K}(\Gamma_2) \times \ldots \times \mathcal{K}(\Gamma_u)$ .  If $x \in \mathcal{K}(\Gamma)$  then $x \in \mathcal{C}(\Gamma)$  (Theorem 2.4) and therefore  $x/T_j$  is an imputation in  $\Gamma_j$ , $j = 1,2,\ldots,u$ .  Thus there is no transfer and, as it has been shown in [8],  $x/T_j \in \mathcal{K}(\Gamma_j)$ .  This concludes the proof.

We can now sharpen the result stated in Theorems 2.4 and 2.8:

<u>Theorem 2.11.</u>          <u>The kernel for the grand coalition of a convex game lies strictly in the relative interior of the core.</u>

<u>Proof</u>.     Suppose  $\Gamma$  is decomposable into  $\Gamma_1,\Gamma_2,\ldots,\Gamma_u$  and none of the  $\Gamma_j$'s  is further decomposable.  If  $x \in \mathcal{K}(\Gamma)$   then  $x/T_j \in \mathcal{K}(\Gamma_j)$  and, by Theorem 2.8,  $x/T_j$  lies in the interior of  $\mathcal{C}(\Gamma_j)$ , where  $\mathcal{C}(\Gamma_j)$  is being regarded as a point-set in the linear manifold spanned by the imputations of  $\Gamma_j$ ,  $j = 1,2,\ldots,u$ .  The rest of the proof now follows easily.

## 3.   <u>THE</u> <u>BARGAINING</u> <u>SET</u> $\mathcal{M}_1^{(i)}$  <u>FOR A CONVEX GAME</u>

Let  $x$  be an imputation in a game  $(N;v)$ .  An <u>objection</u> of a player  $k$  against a player  $\ell$ , with respect to  $x$  is a pair  $(\hat{y};C)$ , where  $C$  is a coalition containing player  $k$  and not containing player  $\ell$ ,  $\hat{y}$  is a vector whose indices are the members of  $C$ ,  $\hat{y}(C) = v(C)$  and  $\hat{y}_i > x_i$  for each  $i$  in  $C$ .  A <u>counter objection</u> to the above objection is a pair  $(\hat{z};D)$ , where  $D$  is a coalition containing player  $\ell$  and not containing player  $k$  and  $\hat{z}$  is a vector whose indices are members of  $D$ ,  $\hat{z}(D) = v(D)$ ,  $z_i \geq y_i$  for  $i \in D \cap C$ , and  $z_i \geq x_i$  for  $i \in D - C$ .

An imputation  x  is said to belong to the <u>bargaining set</u> $\mathcal{M}_1^{(i)}$

for the grand coalition if for any objection of one player against

another there exists a counter objection to this objection[10].  Clearly

$\mathcal{M}_1^{(i)}$  contains the core, because if  $x \in \mathcal{C}$  no objections are possible.

In this section we shall show that for convex games  $\mathcal{M}_1^{(i)} = \mathcal{C}$.  Since

$\mathcal{M}_1^{(i)} \supset \mathcal{K}$  (see [2]) , this result will furnish another proof of Theorem 2.4.

<u>Theorem 3.1.</u>   <u>The bargaining set</u> $\mathcal{M}_1^{(i)}$  <u>for the grand coalition of a</u>

<u>convex game coincides with the core of the game.</u>

<u>Proof.</u>    All we have to show is that if  x  is an imputation not in the

core then  $x \notin \mathcal{M}_1^{(i)}$ .

Let  s(x)  be the excess of the coalitions in  $\mathcal{E}(x)$   (see Notation

prior to Corollary 2.3).  Since  $x \notin \mathcal{C}$ ,

(3.1)            $s(x) > 0$ .

Consequently

(3.2)          $N \notin \mathcal{E}(x)$    $\phi \notin \mathcal{E}(x)$ .

By Corollary 2.3,

(3.3)          $\mathcal{E}(x) \ni Q_1 \equiv \underset{S:S\in\mathcal{E}(x)}{\cup} S \neq N$ ,

(3.4)          $\mathcal{E}(x) \ni Q_2 \equiv \underset{S:S\in (x)}{\cap} S \neq \phi$ .

Thus, there exists a player  k  in  $Q_2$   and a player  $\ell$  in  $N - Q_1$ .

We shall conclude the proof by showing that  k  has an objection against

$\ell$  with respect to  x  which cannot be countered.

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[10] The notation is slightly different if coalition-structures other
     than  $\{N\}$  are also being considered.

Let C be any coalition in $\mathcal{E}(x)$ . Then $k \in C$ and $\ell \notin C$ . It has been shown in [4] that an objection $(\hat{y};C)$ which cannot be countered exists with respect to x if (and only if) the following conditions hold:

(i)     $e(R,x) < 0$ whenever $\ell \in R$ , $R \cap C = \phi$ ,

(ii)    $e(C,x) > e(R,x)$ whenever $\ell \in R$ and $R \cap C = C - \{k\}$ ,

(iii)   The game $(C - \{k\}; v^*_C)$ has a full dimensional core. Here,

(3.5)   $$v^*_C(S) = \begin{cases} e(C,x) & \text{if } S = C - \{k\} \\ \text{Max}(0, \underset{\substack{R:R\cap C=S \\ \ell\in R}}{\text{Max}} e(R,x)) & \text{if } S \subset C - \{k\}, S \neq C - \{k\} \end{cases}.$$

We shall show that these conditions hold. Condition (ii) holds because $R \notin \mathcal{E}(x)$ due to the fact that $\ell \in R$ .

If $R \cap C = \phi$ and $\ell \in R$ , then, by Lemma 2.1,

(3.6)   $s(x) + e(R,x) = e(C,x) + e(R,x) \leq e(C \cup R) < s(x)$ .

This implies $e(R,x) < 0$ as required by condition (i).

In order to prove (iii), we shall construct a <u>convex</u> <u>game</u> $(C - \{k\}, v^{**}_C)$ whose characteristic function $v^{**}_C$ satisfies

(3.7)   $v^{**}_C(C - \{k\}) < v^*_C(C - \{k\})$

(3.8)   $v^{**}_C(S) \geq v^*_C(S)$ whenever $S \subset C - \{k\}$ , $S \neq C - \{k\}$ .

Being convex, this new game would have a non-empty core, and because of (3.7) and (3.8), $(C - \{k\}, v^*_C)$ would have a full dimensional core.

To this end, let us define

(3.9)   $$v^{**}_C(S) = \text{Max}(0, \underset{\substack{R:R\cap C=S \\ \text{and } \ell\in R}}{\text{Max}} e(R,x))$$

for all S , $S \subset C - \{k\}$ .

Relation (3.7) now follows from (3.1), (3.5), the fact that $C \in \mathcal{E}(x)$ and the fact that $R \notin \mathcal{E}(x)$ if $\ell \in R$ .

Relation (3.8) is established by comparing (3.5) with (3.9).

Note that $v_C^{**}(S) \geq 0$ and that the game $(C - \{k\}, v_C^{**}(S))$ is monotonic (see footnote (2)). Therefore, for $A, B \subseteq C - \{k\}$,

(3.10)     $v_C^{**}(A) + v_C^{**}(B) \leq v_C^{**}(A \cup B) + v_C^{**}(A \cap B)$

holds whenever either $v_C^{**}(A) = 0$ or $v_C^{**}(B) = 0$. If $v_C^{**}(A) > 0$ and $v_C^{**}(B) > 0$ then, by (3.9), coalitions $R_A$ and $R_B$ exist in the original game such that $\ell \in R_A$, $\ell \in R_B$, $R_A \cap C \subseteq A$, $R_B \cap C \subseteq B$, $v_C^{**}(A) = e(R_A, x)$ and $v_C^{**}(B) = e(R_B, x)$. It now follows from Lemma 2.1 and (3.9) that $v_C^{**}(A) + v_C^{**}(B) = e(R_A, x) + e(R_B, x) \leq e(R_A \cup R_B, x) + e(R_A \cap R_B, x) \leq v_C^{**}(A \cup B) + v_C^{**}(A \cap B)$. We have therefore showed that (3.10) holds also in this case. Thus, $(C - \{k\}, v_C^{**}(S))$ is, indeed, a convex game. This concludes the proof.

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Many solution concepts for cooperative games agree or partially agree if the game happens to be convex. For example, convex games have a unique von-Neumann Morgenstern solution which coincides with the core. Also, the (Shapley) value is a center of gravity of the extreme points of the core of a convex game; [namely, the center of gravity when the extreme points are assigned appropriate multiplicities].

It is proved in this paper that the kernel (for the grand coalition) of a convex game lies in the **relative interior of its core and that the bargaining** set (for the grand coalition) coincides with the core.

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