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## Comparison of Two Methods of Obtaining Confidence Intervals for System Reliability

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COMPARISON OF TWO METHODS OF OBTAINING  
CONFIDENCE INTERVALS FOR SYSTEM RELIABILITY

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## SUMMARY

Some specific comparisons are made in this note between the use of the asymptotic Chi-square distribution of the likelihood ratio and the asymptotic normality of the maximum likelihood estimates to obtain confidence interval for reliabilities of arbitrary systems when only failure data on the components is known. In all the comparisons made, using moderate samples and systems of average complexity, the asymptotic Chi-square appears to give much more accurate confidence intervals. Although the asymptotic Chi-square method requires machine computation for all but the simplest systems while the asymptotic normal method can be done easily by desk calculator, these examples would indicate the Chi-square method would be superior in most practical instances.

## 1. INTRODUCTION

In a recent publication [2], Madansky has shown how to use the asymptotic distribution of the logarithm of the likelihood ratio to obtain approximate confidence intervals for series, parallel and series-parallel systems. That this method could be extended to the class of systems which are monotone was shown by Saunders and Myhre [6]. In this study we make some comparisons between this likelihood ratio method and the only other method presently known which can be used to obtain approximate confidence limits for the reliability of an arbitrary system. This alternate method was proposed by Rosenblatt in [5] as a special case of the U-statistic which was discussed in that paper. But this method also depends upon the asymptotic distribution of the statistic used and hence is also approximate in the same sense for finite sample size.

We can help to answer the question of which approximation is best by finding a situation in which exact confidence intervals can be found by other means and then making a comparison between the two. We also might ask which statistic has its distribution, for certain moderate sample sizes likely to arise in practice, closer to its asymptotic distribution. We do both of these things believing that these answers, for finite samples, help resolve, much more effectively, the question of which method is better than the answer to the question of which test, the inversion of which gives the approximate confidence interval, has the highest asymptotic relative efficiency.

Let  $y_i$ , a Bernoulli random variable, be the indicator of performance for the  $i^{\text{th}}$  component among  $m$ . The state of the components then is the vector  $y = (y_1, \dots, y_m)$ . By a monotone structure we follow [1] to mean there exists a non-decreasing Boolean function  $\phi$  of the state of the components,  $\phi(y)$ , which is the indicator of the state of the structure. If  $Ey_i = p_i$  for  $i = 1, \dots, m$  is the reliability of the  $i^{\text{th}}$  component, then the components reliability is the vector  $p = (p_1, \dots, p_m)$  and  $E\phi(y) = h(p)$  is the reliability of the structure.

Our data consists of the vectors  $x = (x_1, \dots, x_m)$  and  $n = (n_1, \dots, n_m)$  where we have observed  $x_i$  successes in  $n_i$  trials of the  $i^{\text{th}}$  component. The number of successes  $x_i$  has a binomial distribution.

## 2. THE LIKELIHOOD RATIO METHOD

Let us summarize the likelihood ratio (LR) method presented in [6]. Let a monotone system reliability function  $h$  of order  $m$  and data  $(x, n)$  be given. Now we define a function  $A$  from the  $m$ -dimensional unit hypercube into the same space for each given real  $\delta$  by the equation for its  $j^{\text{th}}$  coordinate

$$(2.1) \quad A_j(p, \delta) = \frac{x_j - \delta p_j}{n_j - \delta p_j} \frac{\delta_j h(p)}{\delta_j h(p)} \quad j = 1, \dots, m.$$

where  $\delta_j h(p)$  represents the partial derivative of  $h$  with respect to the  $j^{\text{th}}$  coordinate.

For given  $p^0$  in the domain of  $A$  we set

$$p^i(\delta) = A(p^{i-1}(\delta), \delta) \quad i = 1, 2, \dots, .$$

Call the limit which exists  $p^*(\delta)$ , which is a fixed point of  $A$ . Then one computes  $hp^*(\delta)$  where we make here and throughout the convention that juxtaposition of functions denotes composition.

Now let the maximum likelihood estimate of component reliability be given by  $\tilde{p}_i = x_i/n_i$  for  $i = 1, \dots, m$  and set

$$(2.2) \quad N(p) = \sum_{i=1}^m n_i [\tilde{p}_i \ln p_i + (1-\tilde{p}_i) \ln(1-p_i)].$$

Then we also must compute  $-\Lambda(\delta)$  where

$$(2.3) \quad \Lambda(\delta) = Np^*(\delta) - N(\tilde{p}).$$

A confidence interval of level  $\gamma$  can be obtained from the set

$$(2.4) \quad \Delta_\gamma = \{\delta: -\Lambda(\delta) \leq \frac{1}{2} \chi_{1-\gamma}^2(1)\},$$

where  $\chi_{1-\gamma}^2(1)$  is the  $\gamma^{\text{th}}$  quantile of the Chi-square with one degree of freedom distribution, by taking

$$(2.5) \quad \{hp^*(\delta): \delta \in \Delta_\gamma\}.$$

That this method gives an approximate confidence interval is the substance of [6].

A program which performs these calculations automatically and which requires only the inputs

- 1<sup>o</sup> the reliability function  $h$
- 2<sup>o</sup> the data on component performance  $(x, n)$
- 3<sup>o</sup> the desired confidence level  $\gamma$

to obtain the boundary points of the interval given in (2.4) is available by request from the authors.

To make the ideas clear we illustrate with a simple example. Let us consider the "2 out of 3" quorum structure reliability function

$$h(p) = p_3[1 - (1-p_1)(1-p_2)] + (1-p_3)p_1p_2.$$

If

$$(2.6) \quad x_1 = 7, \quad x_2 = 8, \quad x_3 = 9 \quad \text{and} \quad n_1 = n_2 = n_3 = 10$$

then  $h(\tilde{p}) = .902$ .

We give in Figure 1 the graphs of  $hp(\delta)$  and  $-\Lambda(\delta)$  as functions of  $\delta$ .

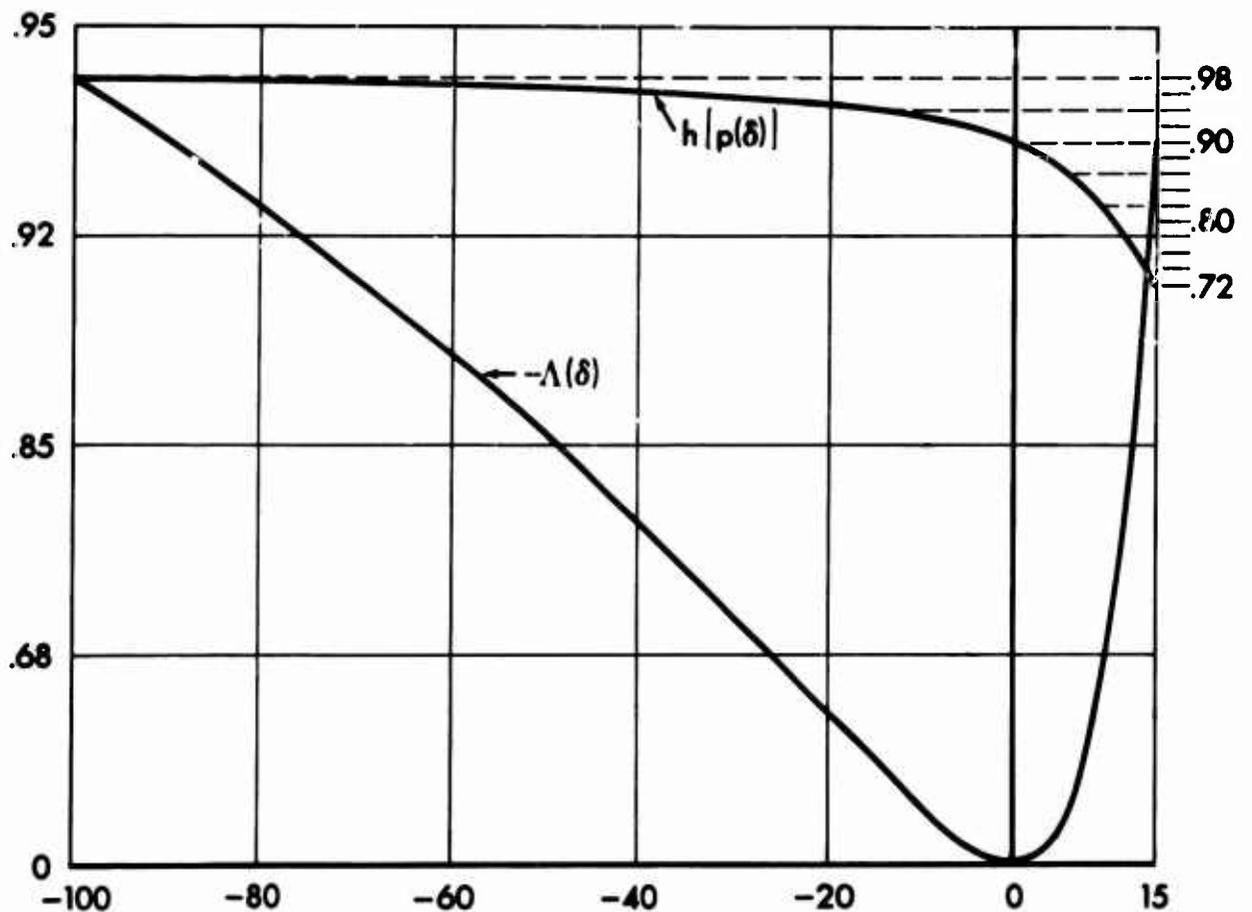


Figure 1

The abscissa contains values of  $\delta$ . On the left ordinate are plotted values of  $\gamma$  such that  $\chi_{1-\gamma}^2(1) = -2\Lambda(\delta)$  and on the right ordinate are the values of  $hp(\delta)$ .

The data summarized in this graph was obtained by a program (which is also available by request from the authors). This program is the basis of the one mentioned previously and uses only the inputs  $h$  and data  $(x,n)$  in the following manner: Take a sequence of values, say,  $\delta_i = i$  for  $i = 0, \pm 1, \pm 2, \dots$  over a sufficiently large range, here taken as  $i = -100, \dots, 20$ . Beginning with  $\delta = 0$ ,  $p^*(0) = \tilde{p}$ , we solve for  $p^*(\delta_i)$  by using as the initial estimate the solution of the previous iteration. Let  $p^0(\delta_{i+1}) = p^*(\delta_i)$  and have the computer obtain the iterates  $p^j(\delta_{i+1}) \rightarrow p^*(\delta_{i+1})$ . We can then compute  $hp^*(\delta_i)$ ,  $-\Lambda(\delta_i)$  at each value of  $i$ . For a given  $\delta$  we plot the values of  $hp^*(\delta)$  on the right ordinate and the Chi-square value corresponding to  $-\Lambda(\delta)$  on the left ordinate.

To obtain a confidence interval we follow the analogous procedure for the function which we now give for the graphs. We select a value of the confidence level  $\gamma$  on the left ordinate, then we find the two values of  $\delta$  on the abscissa for which  $-\Lambda(\delta)$  assumes that value. We then read the two corresponding values of  $hp(\delta)$  on the right ordinate and these values are the confidence intervals.

We see by examining (2.1) at the unique fixed point  $p^*(\delta)$  of  $A(\cdot, \delta)$ , which is the function of the sample data  $(x,n)$ , that this point satisfies for any positive integer  $\omega$  the equation

$$(2.7) \quad p^*(\delta: \omega x, \omega n) = p^*(\delta/\omega: x, n).$$

Accordingly, from (2.2) we have

$$N(p: \omega x, \omega n) = \omega N(p: x, n).$$

Hence from (2.3) it follows that

$$(2.8) \quad \Lambda(\delta: \omega x, \omega n) = \omega \Lambda(\delta/\omega: x, n).$$

These two relations make possible an instructive comparison of the decreasing length of the confidence interval with increasing sample size.

If we take the same data as given in (2.6) and set  $\omega = 10$  and use the relation (2.7), (2.8), we have Figure 2 as the graphical solution.

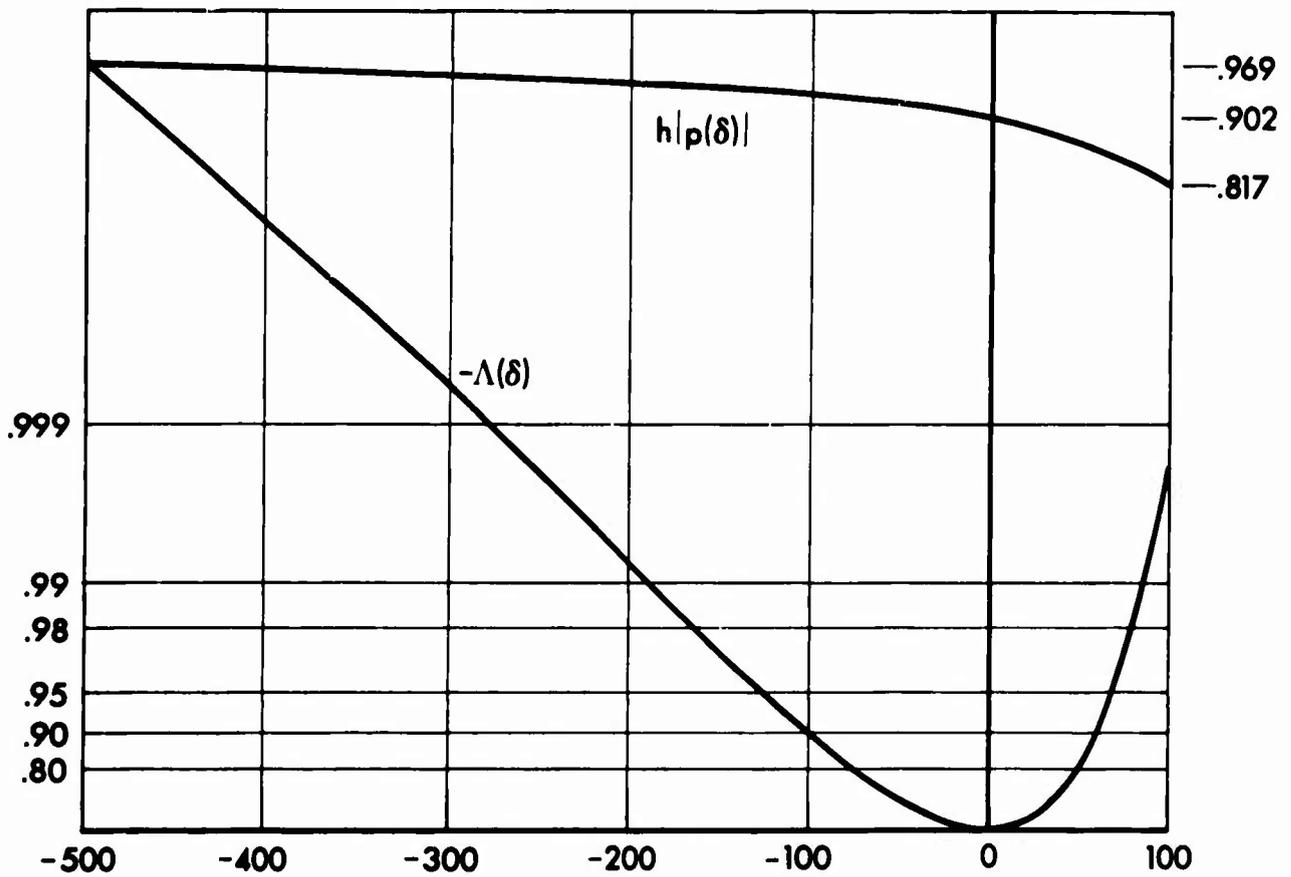


Figure 2

The abscissa contains values of  $\delta$ . On the left ordinate are plotted values of  $\gamma$  such that  $\chi^2_{1-\gamma}(1) = -2\Lambda(\delta)$  and on the right ordinate are the values of  $h p(\delta)$ .

If we take  $\omega = 100$ , we obtain Figure 3.

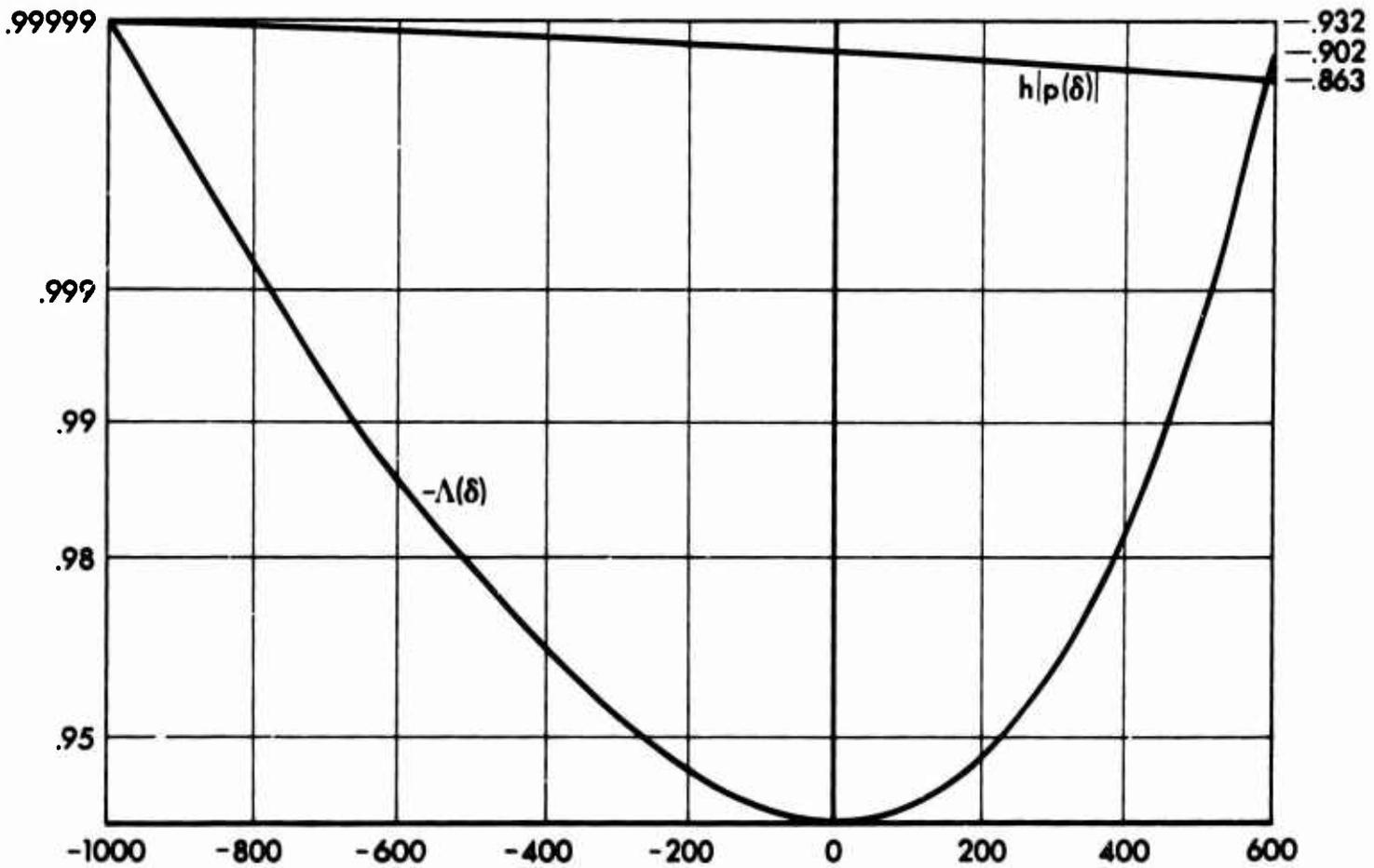


Figure 3

The abscissa contains values of  $\delta$ . On the left ordinate are plotted values of  $\gamma$  such that  $\chi^2_{1-\gamma}(1) = -2\Lambda(\delta)$  and on the right ordinate are the values of  $hp(\delta)$ .

Let us now consider a more realistic case where we have a "fail safe" reliability system (an  $m-1$  out of  $m$  quorum structure), letting  $q_j = 1-p_j$  we write

$$(2.9) \quad h(p) = \prod_{i=1}^m p_i + \sum_{j=1}^m q_j \prod_{i \neq j} p_i = \prod_{i=1}^m p_i \left[ 1 + \sum_{j=1}^m \frac{q_j}{p_j} \right].$$

Consider the data for  $m = 10$ ,  $n_1 = \dots = n_m = 100$

$$(2.10) \quad \begin{aligned} x_1 &= 87, & x_2 &= 86, & x_3 &= 82, & x_4 &= 89, & x_5 &= 98 \\ x_6 &= 95, & x_7 &= 97, & x_8 &= 96, & x_9 &= 91, & x_{10} &= 96. \end{aligned}$$

Now  $h(\bar{p}) = .803$ . Using the iteration method just described we obtain for  $\delta = -500, (15), 100$  the values of  $h(p(\delta))$  and  $-\Lambda(\delta)$  given in Figure 4. In particular, for  $\gamma = .95$  we have the interval  $(.72, .88)$  for the true reliability  $h(\pi)$ .

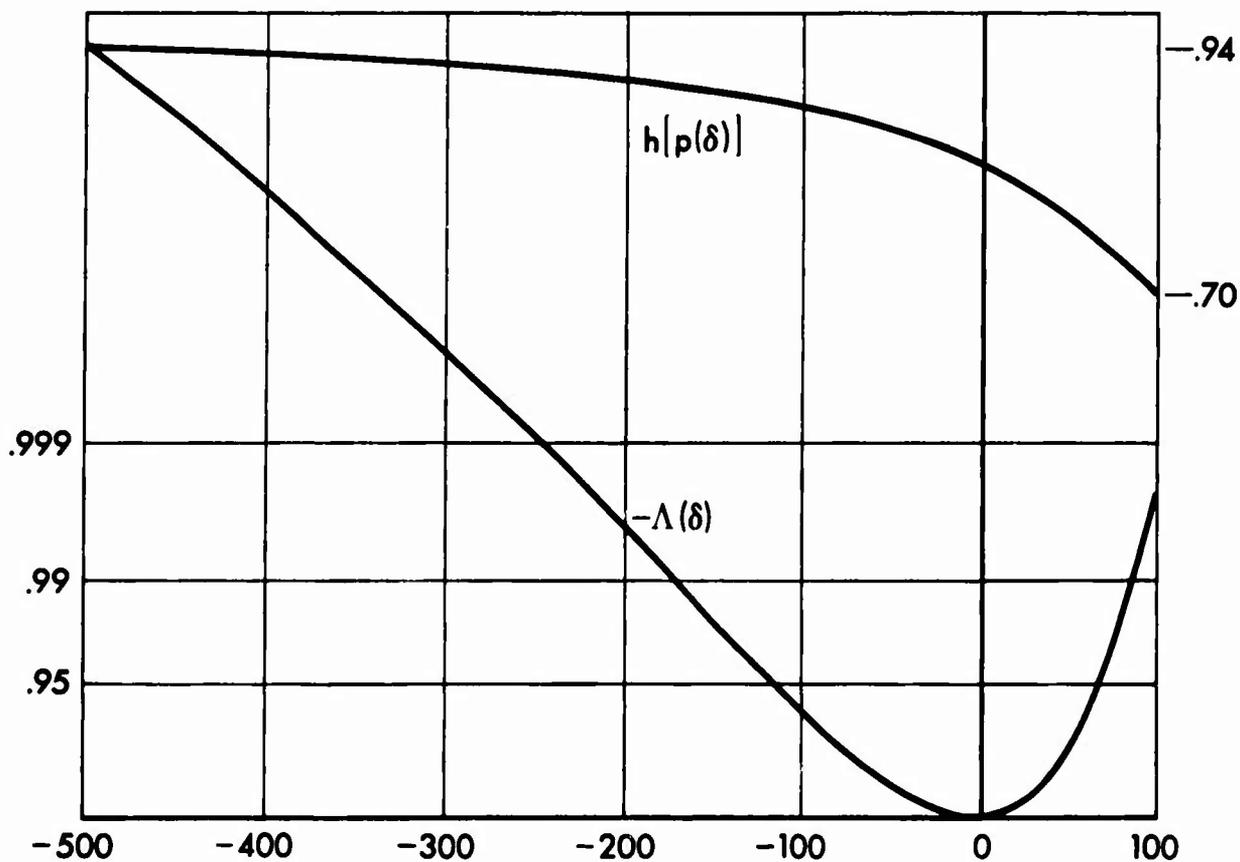


Figure 4

The abscissa contains values of  $\delta$ . On the left ordinate are plotted values of  $\gamma$  such that  $\chi_{1-\gamma}^2(1) = -2\Lambda(\delta)$  and on the right ordinate are the values of  $h(p(\delta))$ .

### 3. THE MAXIMUM LIKELIHOOD METHOD

If  $y_i(j)$  for  $j = 1, \dots, n_i$  are the Bernoullian random variables which represent the data on the behavior of the  $i^{\text{th}}$  component then in terms of our previous notation  $x_i = y_i(1) + \dots + y_i(n_i)$  is the total number of successes for the  $n_i$  trials of the  $i^{\text{th}}$  component. In [5] it is noted that

$$U = \sum_{i=1}^{n_1} \dots \sum_{j=1}^{n_m} \frac{\phi[y_1(i), \dots, y_m(j)]}{\prod_{k=1}^m n_k} = h(\tilde{p}).$$

Now an alternate method of obtaining approximate confidence intervals can be based on either the well known asymptotic normality of maximum likelihood estimates or equivalently in this case the asymptotic normality of U-statistics. This method was called "linearization" by Madansky in the comparisons which he made in his paper loc. cit.

We now state the appropriate

**THEOREM:** *If  $\tilde{p}$  is the maximum likelihood estimator of the true component reliability vector  $\pi$ , then  $h(\tilde{p})$  is asymptotically normal with mean  $h(\pi)$  and variance*

$$\sigma^2 = \sum_{j=1}^m \left[ \partial_j h(\pi) \right]^2 \text{var}(\tilde{p}_j).$$

**Proof:** See Rao [p.207, 4].

Using this result, approximate confidence limits of level  $\gamma$  for  $h(\pi)$  can be obtained from  $h(\tilde{p}) \pm z_{(1-\gamma)/2} \cdot \tilde{\sigma}$  where  $z_\gamma$  is the  $\gamma^{\text{th}}$  percentile of the standard normal distribution. For the structure given in (2.9) one

calculates

$$\partial_j h(p) = \prod_{i \neq j} p_i \left( \sum_{i \neq j} \frac{q_i}{p_i} \right) \quad j=1, \dots, m.$$

So

$$(3.1) \quad \tilde{\sigma}^2 = \prod_{i=1}^m (\tilde{p}_i)^2 \sum_{j=1}^m \left( \sum_{i \neq j} \frac{\tilde{q}_i}{\tilde{p}_i} \right)^2 \left( \frac{\tilde{q}_j}{n_j \tilde{p}_j} \right).$$

Calculation with a desk computer for the data in (2.10) using (3.1) with  $m = 10$  yields

$$h(\tilde{p}) = .803 \quad \tilde{\sigma}^2 = .001047.$$

By choosing  $z_{.025} = 1.960$  the 95% confidence interval becomes (.740, .866) which is noticeably different from the one obtained previously. The latter interval is shorter on each side by about .02.

To make another comparison, take the data given in (2.6) and we calculate by desk computer using (3.1) with  $m = 3$ ,  $h(\tilde{p}) = .902$ ,  $\tilde{\sigma} = \sqrt{.00449}$ . Now we take  $\gamma = .90$  and the confidence interval becomes (.79, 1.01) which can be compared to the interval (.74, .97) generated by the likelihood ratio method. Again the intervals differ significantly.

As before we ask what happens to the ML confidence interval for the data  $(\omega x, \omega n)$  and we notice that  $h(\tilde{p})$  remains unchanged but, by (3.1) that  $\tilde{\sigma}(\omega x, \omega n) = \tilde{\sigma}(x, n) / \sqrt{\omega}$ .

Using the data (2.6) in the 2 out 3 system and the numerical results of which graph 1 is a summary we obtain for a 90% confidence interval

	LR	ML
$\omega = 1$	(.74,.97)	(.79,1.01)
$\omega = 10$	(.862,.932)	(.867,.937)

Of course, the above calculations show the comparative simplicity of the ML method and it also shows that in cases when sample sizes are of the order of 100 observations for each component, (impractically high in most situations), and the reliability estimates not too close to one, that both the ML and LR methods will yield essentially the same confidence intervals *for simple enough structures.*

As the structure becomes more complex, however, the number of observations for each component must increase to obtain the same limits by the two methods as we saw with the structure (2.9).

To see how this works consider the series system  $h(p) = \prod_{i=1}^m p_i$ . Since  $\tilde{p}_i = x_i/n_i$ , if  $n_i = n_0$  and  $\pi_i = \pi_0$  for  $i=1, \dots, m$  then for  $n = 5$  and  $m = 10$ , by the rapidity of convergence in the central limit theorem when the summands are identically distributed, the distribution of  $h(\tilde{p})$  is very close to log-normality. To make the distribution close to normal, we have to take  $n_0$  at least 100. While if  $m = 2$ , then  $n_0 = 25$  may be enough to have  $h(\tilde{p})$  close to normality. If  $m = 25$ , no sample size  $n_0$  of practically attainable magnitude might be large enough to do the trick.

A weakness of the ML method is that the confidence limits are always symmetric about the estimate of the reliability and when this estimate is near one and the sample sizes moderate, both of which occur frequently in

practice, it may give intervals in which the upper limit exceeds one. Of course this objection is partially vitiated in case interest is confined to lower confidence intervals.

#### 4. COMPARISON WITH EXACT CONFIDENCE INTERVALS

Actually comparisons between the actual confidence intervals given by the ML and LR methods are not too meaningful when they do not agree since in general there is not a unique exact confidence interval to which they can be compared. However, there do exist special cases for which we can obtain exact confidence intervals.

If all component reliabilities were known to be equal in a  $k$  out of  $m$  quorum structure, the system reliability would be [p.20, Ref. 1], (here letting  $p_1$  be the common value of the vector  $(p_1, \dots, p_1)$  and disregarding our previous notation that  $h$  map the unit hypercube into the unit interval)

$$(4.1) \quad h(p_1) = \sum_{i=k}^m \binom{m}{i} p_1^i q_1^{m-i}.$$

Using the notation for the incomplete beta function found in the tabulation by Karl Pearson [3]

$$I(s; \alpha, \beta) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^s t^{\alpha-1} (1-t)^{\beta-1} dt$$

we have

$$(4.2) \quad h(p_1) = I(p_1; k, m-k+1).$$

I: The exact method.

If we denote  $p_1 = x_1/n_1$ , then we obtain a  $100\gamma\%$  upper limit  $\bar{p}_1$  as the solution for the real variable  $p_1$  such that

$$\sum_{j=0}^{x_1} \binom{n_1}{j} p_1^j q_1^{n_1-j} = 1 - I(p_1; x_1+1, n_1-x_1+1) = \frac{1-\gamma}{2}$$

and a lower limit  $\underline{p}_1$  as the solution for the real variable  $p_1$  such that

$$\sum_{j=x_1}^{n_1} \binom{n_1}{j} p_1^j q_1^{n_1-j} = I(p_1; x_1, n_1-x_1+1) = \frac{1-\gamma}{2}$$

which can be solved using Pearson's tables loc.cit. So we have with confidence  $\gamma$

$$\underline{p}_1 < \pi_1 < \bar{p}_1,$$

and hence

$$h(\underline{p}_1) < h(\pi_1) < h(\bar{p}_1)$$

is an exact confidence interval of level  $\gamma$  for the true system reliability.

Assuming  $\pi = (\pi_1, \dots, \pi_1)$  for the structure given in (4.1), and that  $x_1 = 23$ ,  $n_1 = 25$  and interpolating linearly in Pearson's tables loc. cit. we obtain the 95% confidence limits for  $\pi_1$ :

$$\bar{p}_1 = .976 \quad \underline{p}_1 = .740$$

Fixing  $m = 10$ , and transforming this interval by the function defined in (4.2) we have exact upper and lower tolerance limits as given in I table 1, (page 15).

## II. The ML Method

By the theorem we have for a  $k$  out of  $m$  structure with  $p_1 = (p_1, \dots, p_1)$  that

$$\sigma_k = k \binom{m}{k} p_1^{k-1} q_1^{m-k} \sqrt{\frac{p_1 q_1}{n_1}}$$

and substituting  $\tilde{p}_1 = x_1/n_1$  for  $p_1$  we obtain an approximate 100% confidence interval of the form

$$I(\tilde{p}_1: k, m-k+1) \pm \tilde{\sigma}_k \cdot z_{(1-\gamma)/2}$$

Using  $n_1 = 25$ ,  $x_1 = 23$ ,  $m = 10$  and  $\gamma = .95$  with the fact that  $\sigma_{k+1}/\sigma_k = (m-k)p_1/kq$  we obtain the following 95% confidence intervals for  $h(\pi_1)$  as given in II table 1, (page 15)

## III. The LR Method

In this one dimensional case the LR method yields a simple procedure for obtaining an approximate confidence interval for  $h(\pi_1)$  of level  $\gamma$ . Now the transformation  $A$  is from the unit interval into the unit interval and thus

$$N(p_1) = x_1 \ln p_1 + (n_1 - x_1) \ln(1-p_1)$$

so that instead of using the parameter  $\delta$  we can as well use  $p_1^*(\delta)$ . We then have

$$\Delta_\gamma = \{p_1: N(\tilde{p}_1) - \frac{1}{2} \chi_{1-\gamma}^2(1) \leq N(p_1)\}$$

and the confidence interval becomes

$$\{h(p_1): p_1 \in \Delta_\gamma\}.$$

Using the data  $n_1 = 25$ ,  $x_1 = 23$ ,  $m = 10$ ,  $\gamma = .95$  we have

$$\Delta_\gamma = (.773, .986)$$

Again by transforming this interval by the function  $h(p_1)$  as defined in (4.2) we obtain the LR confidence intervals as given in III Table 1.

TABLE I

k	Exact	ML	LR
	I	II	III
10	(.049, .787)	(- .070, .938)	(.076, .871)
9	(.221, .977)	(.418, 1.21)	(.2997, .9912)
8	(.495, .9984)	(.823, 1.097)	(.595, .9996)
7	(.7511, .99993)	(.9664, 1.022)	(.8266, .99994)

Comparison of 95% confidence intervals for k out of 10 quorum systems with identical component reliabilities.

Of course no one in his right mind would use such a procedure as II or III in this case but in real situations these conditions might be approximated.

This evidence would lead one to suspect that for all structures in which the true reliabilities are nearly the same and near one, and the number of observations for each component small but the structure not simple (in short, those cases which are usually encountered in practice) that the LR method would yield results which are superior.

We now make some comparisons with a few selected cases in the tables of exact confidence limits for systems which are available. These are for series systems of order two and three with the same numbers of observations taken for each component.

We shall use the tables of Lipow and Riley [2] which we shall denote by RL. These tables provide directly upper bounds on the unreliability but by taking the entries and subtracting for one unity we obtain lower bounds on the reliability which we use for our comparisons.

Based on the LR method, utilizing our previous notation, a lower confidence bound on the true reliability of level  $(1 + \gamma)/2$  is  $h_p(\delta_\gamma^+)$  where  $\delta_\gamma^+ = \sup \Delta_\gamma > 0$ . (The true confidence level used for the comparisons of one sided confidence bounds in [3] for the LR method was not correct. The three upper limits were compared for  $n_1 = n_2 = 100$  when the number of failures was 3 and 5 respectively. At a 90% confidence level the upper limits are

Buehler's value	= .00412	
LR	= .00349	(not .00518 as given)
ML	= .00164	

the .00518 value corresponds to a 97.5% confidence level.) Similarly a lower confidence bound on the true reliability of level  $\gamma$  is  $h(\bar{p}) + z_{1-\gamma} \cdot \bar{\sigma}$  when based on the ML method. Again here  $z_\gamma$  is the  $\gamma^{\text{th}}$  percentile of the standard normal distribution.

Selecting typical entries which might arise in practice from volume I of reference [2] we have the comparisons of table II.

TABLE II

$$h(p) = \prod_{i=1}^k p_i \quad n_i - x_i = y_i, \quad n_i = n_0 \quad \text{for } i = 1, \dots, k$$

			Confidence Level						
			.90			.95			
k = 2	y <sub>1</sub>	y <sub>2</sub>	ML	LR	RL	ML	LR	RL	
n <sub>0</sub> =10	1	1	.655	.629	.607	.611	.571	.548	
	1	2	.545	.529	.497	.495	.473	.443	
	2	2	.456	.451	.445	.405	.397	.392	
	1	4	.347	.350	.344	.292	.301	.298	
	2	3	.373	.375	.364	.320	.326	.304	
n <sub>0</sub> =20	1,	2	.756	.739	.716	.728	.700	.677	
	2,	2	.701	.687	.683	.670	.647	.643	
	1,	3	.697	.683	.660	.665	.643	.620	
	2,	3	.647	.636	.622	.614	.597	.582	
	3,	3	.599	.591	.585	.565	.551	.544	
k = 3	y <sub>1</sub> , y <sub>2</sub> , y <sub>3</sub>								
n <sub>0</sub> =20	1,	1,	1	.760	.743	.747	.732	.705	.709
	1,	1,	2	.704	.690	.693	.673	.651	.644
	1,	2,	2	.654	.643	.639	.621	.604	.598
	1,	2,	3	.605	.596	.595	.571	.557	.544
n <sub>0</sub> =30	1,	2,	3	.723	.714	.705	.698	.683	.674
	1,	1,	1	.835	.822	.825	.816	.794	.796
	2,	2,	2	.725	.715	.712	.700	.685	.681
n <sub>0</sub> =50	1	2	4	.805	.798	.789	.788	.776	.767
	1	1	2	.874	.865	.861	.860	.845	.841
n <sub>0</sub> =100	1,	1,	2	.936	.931	.929	.929	.920	.918
	2,	3,	5	.866	.861	.858	.855	.848	.844

The values for the lower confidence bound obtained by the LR method for  $N > 200$  which we computed did not differ by more than .002 from those tabled in volume II ref [2], while the ML bounds differed by about twice this amount. This small discrepancy was not thought sufficient to display. We point out that at every instance in table II, the LR method lies closer to the RL value than does the ML bound.

##### 5. THE EMPIRIC DISTRIBUTION OF THE TWO STATISTICS

The question about which of the LR or ML methods is the better in a given instance can only be answered by knowing whether the distribution of  $h(\tilde{p})$  is closer to normality, than  $-2\Lambda(\delta_\pi)$  is closer to a Chi-square distribution where  $\delta_\pi$  is that argument of  $p^*(\delta)$  such that  $hp^*(\delta) = h(\pi)$ . More precisely, for a given  $h$ ,  $\pi$  and  $n$ , is the distribution law of  $[(h(\tilde{p})-h(\pi))/\sigma]^2$  or  $-2\Lambda(\delta_\pi)$  closer to a Chi-square with one degree of freedom. In this form, the question can be answered by simulation with a computer.

Select a monotone reliability function  $h$  of certain order  $m$ , the vector of true reliabilities  $\pi = (\pi_1, \dots, \pi_m)$  and the set of sample sizes  $n = (n_1, \dots, n_m)$ . We now generate the binomial observations  $x = (x_1, \dots, x_m)$  where each  $x_j$  is binomial with parameters  $\pi_j, n_j$ . We now compute  $\tilde{p}$ ,  $h(\pi)$ ,  $h(\tilde{p})$  and

$$\bar{\sigma}^2 = \sum_{j=1}^m [\partial_j h(\tilde{p})]^2 \tilde{p}_j \tilde{q}_j / n_j$$

and then calculate

$$X = [(h(\tilde{p}) - h(\pi) / \tilde{\sigma})]^2.$$

We also use the operator  $A$  to calculate  $p^*(\delta_\pi)$  with  $\delta_\pi$  such that  $hp^*(\delta_\pi) = h(\pi)$ , and then

$$Y = -2\Lambda(\delta_\pi) = 2N(\tilde{p}) - 2Np^*(\delta_\pi)$$

where  $N$  was defined in (2.2).

All of the above steps are repeated to obtain a set of observations

$$(X_i, Y_i) \quad \text{for } i=1, \dots, K$$

where  $K$  is in the thousands, and the frequency histograms and the empiric cumulative are compared with  $\chi^2(1)$  distribution.

The results of two comparisons are now given: Figure 5 shows how much closer the true distribution of  $Y$  is to Chi-square with one degree of freedom than  $X$  is when only 10 observations for each component are used. Figure 6 shows that for 20 observations both distributions lie closer but, especially in the tail, the distribution of  $Y$  is closer to its asymptotic distribution.

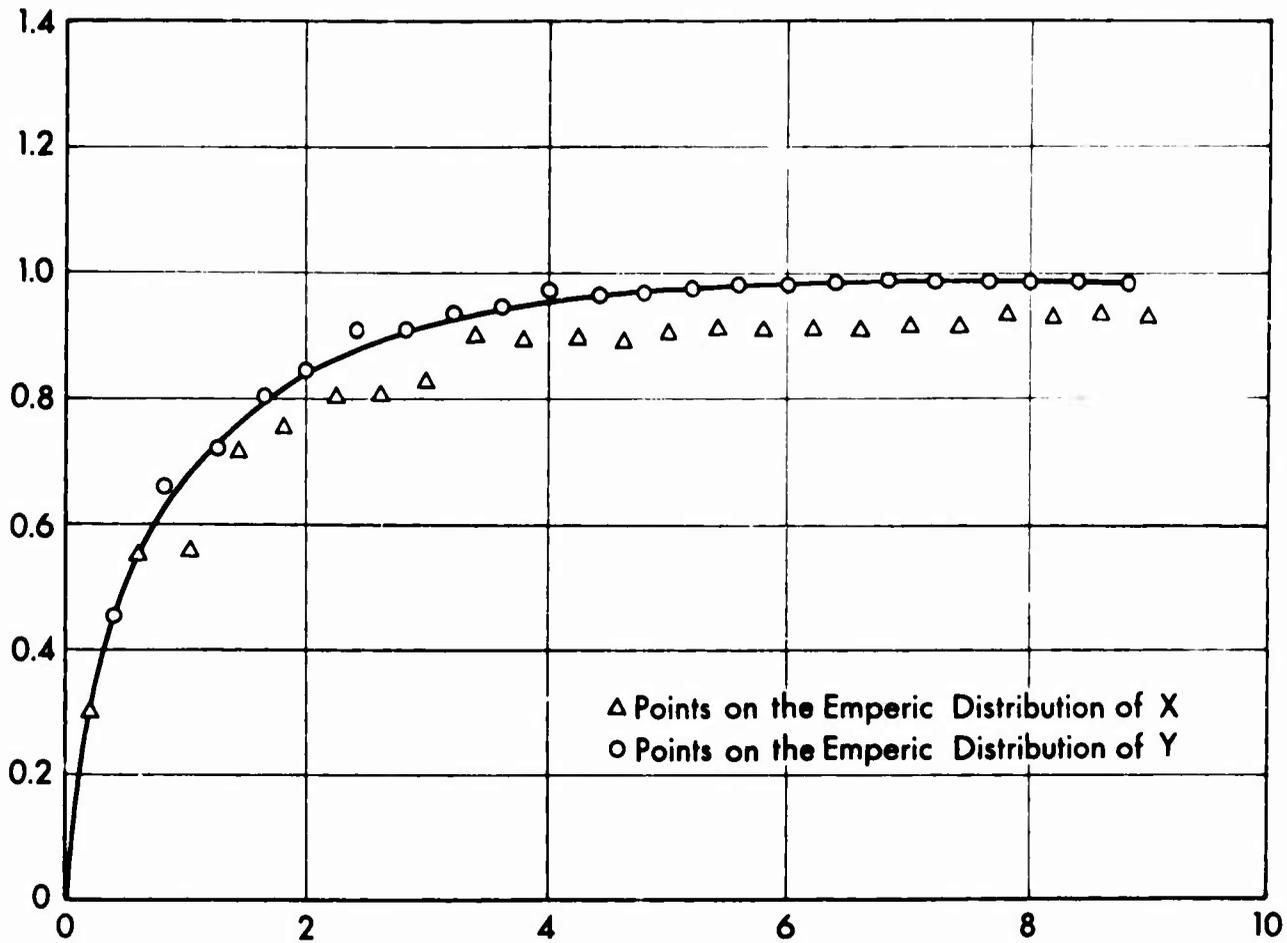


Figure 5

The empiric distributions of X, using the ML method, and of Y using the LR method based on 2000 observations and compared with their asymptotic distribution for

$$n_1 = n_2 = n_3 = 10, \quad \pi_1 = \pi_2 = \pi_3 = .7$$

$$h(p) = p_3[1-(1-p_1)(1-p_2)] + (1-p_3)p_1p_2$$

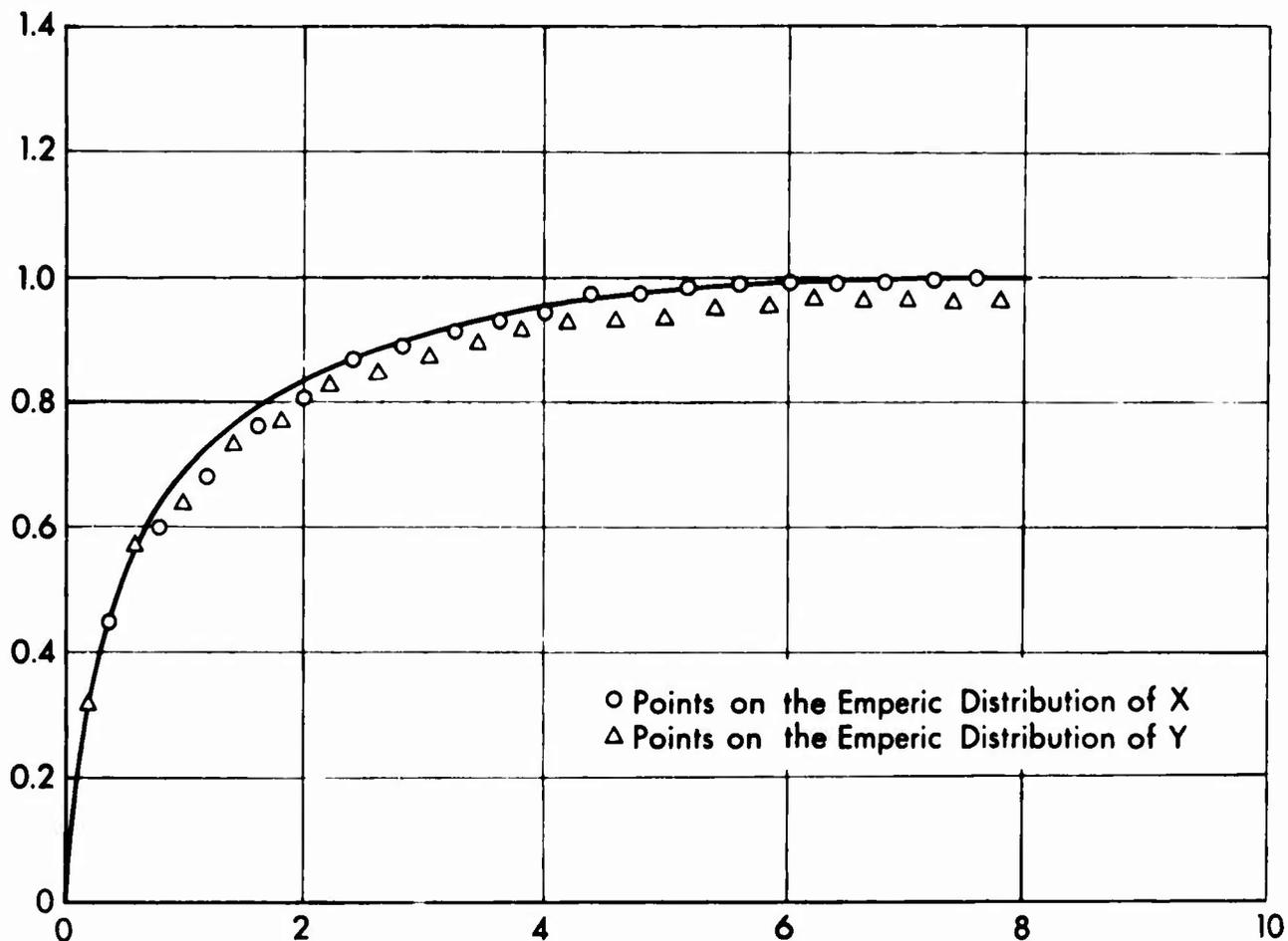


Figure 6

The empiric distributions of X, using the ML method, and of Y using the LR method based on 2000 observations and compared with their asymptotic distribution for

$$n_1 = n_2 = n_3 = 20, \quad \pi_1 = \pi_2 = \pi_3 = .7$$

$$h(p) = p_3[1-(1-p_1)(1-p_2)] + (1-p_3)p_1p_2$$

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