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MULTIPLE ANISOTROPIC SCATTERING IN SLABS WITH AXIALLY SYMMETRIC FIELDS

Harriet Kagiwada and Robert Kalaba

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PREFACE

Much of the theory of multiple scattering is devoted to the cases of isotropic or mildly anisotropic scattering (such as the Rayleigh law). On the other hand, investigators such as Deirmendjian and van de Hulst have emphasized the practical need for the study of multiple scattering processes involving elementary acts of scattering with strongly peaked forward scattering. Such processes present formidable analytic and computational difficulties.

The aim of this Memorandum is to explore numerically certain axially symmetric fields for which the local phase function is approximated by a rational function of the cosine of the angle between the incident and scattered rays. The various analytic and numerical advantages are pointed out, and several sample numerical results are displayed.

SUMMARY

The invariant imbedding approach is applied to the problem of diffuse reflection from a slab which has a strongly elongated phase function. The diffuse radiation field is assumed axially symmetric. The analytic and computational advantages of using a rational function approximation to the phase function, rather than an expansion in Legendre polynomials, are examined. Numerical results for selected cases are presented, as are the FORTRAN programs.

CONTENTS

I. INTRODUCTION

Multiple scattering problems in which the local scattering diagram is highly anisotropic are difficult,[1-7] compared with the isotropic and mildly anisotropic cases.[8-13] However, these are important problems when one considers radiative transfer processes in the haze and clouds of the earth's atmosphere,[14-16] as well as in the sea.[17] In the mathematical treatment, if one expands the phase function in a series of Legendre polynomials,[8] one finds that on the order of a hundred terms are required to approximate the phase function moderately well.[1,2] The computational solution may then not be routine.

In this Memorandum, the possibilities of using a rational function to approximate a strongly peaked phase function are explored. The radiation fields are supposed to be axially symmetric. The plan is to derive the basic equations for the reflection and source functions, to compute the reflection function as the solution of an initial-value problem for ordinary differential equations, and to present some numerical results for cases of increasingly fiercer forward scattering. Interpolation methods are discussed. Appendix A contains listings of the FORTRAN program for the reflection function, and Appendix B contains a program for interpolation by expansion in orthogonal polynomials.

It is assumed that the reader is acquainted with the basic nomenclature of radiative transfer as presented in Ref. 8.

II. PRELIMINARIES

Consider a slab bounded by two horizontal parallel planes, and having a finite optical thickness x. At each point on the top plane a conical flux of energy is uniformly incident, and each individual ray forms an angle whose direction cosine is -u ($0 < u \leq 1$) when measured from the upward directed vertical. The energy passing through a unit horizontal area is πu per unit time. The local scattering is anisotropic, but because of the form of the incident energy and the phase function the internal and external fields are axially symmetric; i.e., they are not azimuth dependent. We assume for simplicity that the absorption and reemission characteristics of the slab do not depend on position. In passing through the optical distance Δ, the fraction Δ of the energy is temporarily absorbed by the medium. The albedo for single scattering, λ ($0 < \lambda \leq 1$), is the fraction of this absorbed energy that is reemitted in all directions. The phase function $p(\cos\theta)$ describes the angular distribution of the reemitted energy:

$$\lambda \frac{p(\cos\theta)d(\cos\theta)}{2} = \text{the fraction of energy that, after a local interaction with the medium, propagates in a direction forming an angle with the original direction, whose cosine lies between } \cos\theta \text{ and } \cos\theta + d(\cos\theta). \tag{1}$$

The phase function is normalized so that

$$2\pi \int_{-1}^{+1} \frac{p(\cos\theta)d(\cos\theta)}{4\pi} = 1. \tag{2}$$

If u and v are the cosines of the polar angles (measured from the upward directed vertical) before and after the interaction, and if φ is the change in the azimuth angle, then the well-known formula

$$\cos\theta = uv + \sqrt{1 - u^2}\sqrt{1 - v^2}\cos\varphi \tag{3}$$

relates the cosine of θ to u, v, and φ, and we may view p as $p(v,u,\varphi)$, a function of v, u, and φ.

It will be convenient to introduce the quantity $c(v,u)$,

$c(v,u)\,dv$ = the fraction of the energy that has direction cosine u, interacts with the medium, and then has a direction cosine between v and $v + dv$. (4)

From the above definitions,

$$c(v,u) = \lambda \int_0^{2\pi} \frac{p(v,u,\varphi)}{4\pi}\, d\varphi. \tag{5}$$

A frequently used measure of the degree of forward scattering is the fraction of the scattered energy which is scattered into the forward hemisphere:

$$f = 2\pi \int_0^1 \frac{p(\cos\theta)}{4\pi}\, d(\cos\theta). \tag{6}$$

It is assumed that the phase function is a rational function in $\cos\varphi$, so that the evaluation of $c(v,u)$ via Eq. (5) can, in principle, be made.

III. BASIC EQUATIONS

We shall derive equations for the reflection function,[10,11] and then for the source function. Let us define

$\rho(v,u,x)\,dv$ = the energy per unit horizontal area per unit time emerging from the top of the slab, in a solid angle with cosine of the polar angle between $(v, v + dv)$, due to multiple scattering in the slab of optical thickness x, the sources of energy being uniform conical flux of net flux unity per unit horizontal area in the top surface of the slab, and the cosine of the incident radiation being $-u$ $(0 < u \leq 1,\ 0 < v \leq 1)$. (7)

We first write a finite difference equation for this function using the invariant imbedding approach. We add a thin slab of thickness Δ to the top of the slab of thickness x, thus forming a slab of thickness $x + \Delta$. We write, to terms of order Δ,

$$\rho(v,u,x+\Delta)\,dv = \rho(v,u,x)\,dv\left[1 - \frac{\Delta}{u} - \frac{\Delta}{v}\right] + \frac{\Delta}{u}\,q(v,-u)\,dv + \int_0^1 \rho(v',u,x)\,dv'\,\frac{\Delta}{v'}\,q(v,v')\,dv, \tag{8}$$

where

$$q'(v,-u) = c(v,-u) + \int_0^1 c(-v',-u)\,dv'\,\rho(v,v',x)\,dv'. \tag{9}$$

This equation says that ρdv for the slab of thickness $x + \Delta$ is equal to ρdv for a slab of thickness x plus some changes. These changes are due to (1) absorption in the thin layer as the energy passes into the

slab and out of the slab; (2) absorption in the thin layer followed by ultimate emergence of the energy from the top of the slab, in the correct direction; and (3) the energy that would have emerged in the wrong direction but is then absorbed and ultimately emerges in the desired direction. The q function may be defined in words,

$q(v,u)\,dv$ = the fraction of the energy that interacts with the medium at the top while propagating with direction cosine u, which will emerge from the top of the slab with direction cosine between v and v + dv, either directly after one scattering, or after multiple scatterings ($-1 \le u \le +1$, $0 < v \le 1$). (10)

We assume that the lower boundary is a perfect absorber, so that any energy that impinges on it is absorbed. Now let the slab thickness approach zero, and we obtain the initial condition for ρ,

$$\rho(v,u,0) = 0. \tag{11}$$

Note that the ρ function depends on the phase function through Eqs. (5), (8), and (9).

Now let us introduce the reflection function,

$r(v,u,x)\,dv$ = the <u>intensity</u> of the energy emerging from the top of the slab with direction cosine v, due to multiple scattering in the slab of optical thickness x, the sources of energy being uniform conical flux of net flux πu per unit <u>horizontal</u> area, and the cosine of the incident direction being -u ($0 < u \le 1$, $0 < v \le 1$). (12)

From the definitions of r and ρ, it is clear that

$$r(v,u,x) = \pi u \cdot \frac{\rho(v,u,x)}{v} \cdot \frac{1}{2\pi} = \frac{u\rho(v,u,x)}{2v}. \tag{13}$$

We next introduce the S function, which is symmetric in u and v,

$$r(v,u,x) = \frac{S(v,u,x)}{4v}. \tag{14}$$

It follows that

$$\rho(v,u,x) = \frac{S(v,u,x)}{2u}. \tag{15}$$

We substitute Eq. (15) into Eq. (8), and we let $\Delta \to 0$. We thus obtain the differential-integral equation for S,

$$\begin{aligned} S_x(v,u,x) = &- \left(\frac{1}{u} + \frac{1}{v}\right) S \\ &+ 2\left\{c(v,-u) + \frac{1}{2}\int_0^1 c(-v',-u)S(v,v',x)\,\frac{dv'}{v'}\right\} \\ &+ \int_0^1 S(v',u,x)\left\{c(v,v') + \frac{1}{2}\int_0^1 c(-v'',v')S(v,v'',x)\,\frac{dv''}{v''}\right\}\frac{dv'}{v'}. \end{aligned} \tag{16}$$

In place of Eq. (16), we may use

$$\begin{aligned} S_x(v,u,x) = &-\left(\frac{1}{u} + \frac{1}{v}\right) S + 2q(v,-u) \\ &+ \int_0^1 S(v',u,x)q(v,v')\,\frac{dv'}{v'} \end{aligned} \tag{17}$$

and

$$q(v,w) = c(v,w) + \frac{1}{2}\int_0^1 c(-v',w)S(v,v',x)\,\frac{dv'}{v'}. \tag{18}$$

The initial condition is

$$S(v,u,0) = 0. \tag{19}$$

These are the desired basic equations for the S function.

We turn our attention now to the source function,[18]

$J(t,v;u,x)$ = energy scattered per unit volume per unit time into a unit solid angle with polar direction cosine v, at optical altitude t above the bottom; the input energy is conical flux of net flux πu per unit horizontal area, with direction cosine $-u$ $(0 < u \leq 1)$, and the slab thickness is x. (20)

We derive an equation for J in which the slab thickness is varied, while the internal point is held fixed at altitude t. Again, we write a finite difference equation correct to terms involving Δ, an incremental thickness,

$$J(t,v;u,x+\Delta) = J(t,v;u,x)\left(1 - \frac{\Delta}{u}\right) + \int_0^1 \frac{J(x,-v';u,x)}{\pi v'}\,\Delta \cdot J(t,v;v',x)2\pi\,dv'. \tag{21}$$

The source function for the slab of thickness $x + \Delta$ is expressed as the source function for the slab of thickness x modified by the absorption in the thin layer of thickness Δ (some of the incident energy never

gets to altitude t to cause any scattering there), and augmented by the scattering at altitude t due to energy emitted at the top in the thin layer. The differential-integral equation for J is

$$J_x(t,v;u,x) = -\frac{1}{u} J(t,v;u,x) + 2\int_0^1 W(-v',u,x)J(t,v;v',x)\,\frac{dv'}{v'}. \tag{22}$$

The source function at the top,

$$W(v,u,x) = J(x,v;u,x) \qquad (-1 \le v \le +1) \tag{23}$$

appears in this equation. In order to obtain an equation for Δt, we do the imbedding by adding a thin layer to the bottom of the slab. We derive the equation,

$$W_x(v,u,x) = 2\int_0^1 Z(-v',u,x)Z(v,v',x)\,\frac{dv'}{v'}, \tag{24}$$

which now involves the source function at the bottom,

$$Z(v,u,x) = J(0,v;u,x) \qquad (-1 \le v \le +1). \tag{25}$$

For the source function at the bottom, we obtain the equation

$$Z_x(v,u,x) = -\frac{1}{u} Z(v,u,x) + 2\int_0^1 W(-v',u,x)Z(v,v',x)\,\frac{dv'}{v'} \tag{26}$$

by putting t = 0 into Eq. (22).

Equations (22), (24), and (26) form a complete system of equations for the source function at the top, at the bottom, and at altitude t. Equations (24) and (26) may be solved simultaneously, independently of Eq. (22). For these, the initial conditions are

$$W(v,u,0) = \frac{1}{2}\, c(v,-u),$$
$$Z(v,u,0) = \frac{1}{2}\, c(v,-u). \tag{27}$$

Equations (27) are obtained by considering a thin slab of thickness $x = \Delta$. It is so thin that the source function at the top is the same as that at the bottom, which we call $J(v,u,\Delta)$. The total rate of scattering in a portion of the slab with height Δ and unit base area is, to terms of order Δ,

$$\Delta \cdot J(v,u,\Delta) = \pi u \cdot \frac{\Delta}{u} \frac{1}{2\pi} \int_0^{2\pi} \lambda \frac{p(v,u,\varphi)}{4\pi}\, d\varphi. \tag{28}$$

The factors on the right-hand side are as follows: (πu) is the energy incident per unit horizontal area per unit time; (Δ/u) is the fraction of the energy that is absorbed in this thin slab; the remaining factor is the distribution of the reemitted energy. By letting $\Delta \to 0$ and using Eq. (5), we obtain the initial conditions of Eqs. (27).

Thus, the source function at the top and at the bottom of a slab may be evaluated as the slab thickness varies from zero to any desired finite value, by numerically solving Eqs. (24) and (26) subject to Eqs. (27). Let us suppose that this has been done for $x \geq t$. Then we

have the initial condition for the source function at altitude t,

$$J(t,v;u,x)\Big|_{x=t} = W(v,u,t). \tag{29}$$

Equation (22) with initial condition Eq. (29), when solved simultaneously with Eqs. (24) and (26), enables one to compute the source function at altitude t for $x \geq t$. Extensive calculations of source functions for the case of isotropic scattering in homogeneous slabs are reported in Ref. 10. For some basic equations in the case of anisotropic scattering in inhomogeneous slabs that do not assume axially symmetric fields, see Ref. 12.

IV. COMPUTATIONAL REMARKS

The computational scheme is based on the ability of current high speed computers to produce solutions of large systems of ordinary differential equations, subject to initial conditions, both rapidly and with high precision. In order to transform differential-integral equations into systems of ordinary differential equations, we employ the method of finite ordinates.[8,19] We make the approximation

$$\int_0^1 g(z)\,dz \cong \sum_{k=1}^{N} g(z_k)w_k \tag{30}$$

by utilizing a gaussian quadrature method. The abscissas z_k, $k = 1,2,\ldots,N$, are roots of the shifted Legendre polynomial of degree N, $P_N^*(z) = P_N(1 - 2z)$, taken in ascending order, and the w_k, $k = 1,2,\ldots,N$, are the Christoffel weights. The formula is exact for polynomials of degree $2N - 1$ or less. Tables of roots and weights for $N = 3,4,\ldots,15$ are available in Ref. 19.

Let us introduce the functions of one variable,

$$S_{ij}(x) = S(z_i,z_j,x), \tag{31}$$

which are S functions evaluated at $u = z_j$, $v = z_i$, where z_j and z_i are roots of $P_N^*(z)$, and $j = 1,2,\ldots,N$, $i = 1,2,\ldots,N$, the independent variable being the thickness x. In place of the original equations for S, Eqs. (17)-(19), we deal with the ordinary differential equations with initial values

$$\dot{S}_{ij}(x) = -\left(\frac{1}{z_i} + \frac{1}{z_j}\right)S_{ij} + 2q(z_i,-z_j)$$

$$+ \sum_{k=1}^{N} S_{kj}q(z_i,z_k)\frac{w_k}{z_k}, \tag{32}$$

$$q(z_i,\pm z_j) = c(z_i,\pm z_j) + \frac{1}{2}\sum_{k=1}^{N} c(z_k,\mp z_j)S_{ik}\frac{w_k}{z_k}, \tag{33}$$

$$S_{ij}(0) = 0. \tag{34}$$

The array S_{ij} contains N^2 elements, but because of the symmetry of S, i.e.,

$$S_{ij}(x) = S_{ji}(x), \tag{35}$$

we have to consider only $N(N + 1)/2$ differential equations.

Equations (32) are the equations that we solve, using Eq. (33) and initial conditions (34), by numerical integration on the IBM 7044. In some of the runs, $N = 7$ is used, with a grid size for numerical integration of $\Delta x = 0.01$. In other runs, we use $N = 5$, 9, 13, or 15, and step sizes of $\Delta x = 0.005$, 0.0025, and 0.001, as we shall explain later. Equations (32) have been found to be computationally stable.

Systems of ordinary differential equations could also be written for the J, W, and Z source functions, but we shall not do so here.

We would like to discuss now two ways in which one might interpolate to obtain S at arbitrary values of the arguments u and v,

given S evaluated at the roots of P_N^*. Both procedures are in no way restricted to dealing with the S function, but may be used for J, W, Z, and other appropriate functions. We need an interpolation procedure for comparing the results obtained with the use of quadrature formulas of various orders.

In one method, we add more differential equations for each new S function desired. Suppose we require S for $v = z_i$, $i = 1,2,\ldots,N$, and $u = a_\ell$, $\ell = 1,2,\ldots,M$. Let

$$A_{i\ell}(x) = S(z_i,a_\ell,x) \qquad (i = 1,2,\ldots,N;\ \ell = 1,2,\ldots,M). \tag{36}$$

Then the equation for $A_{i\ell}$ is

$$\dot{A}_{i\ell}(x) = -\left(\frac{1}{z_i} + \frac{1}{a_\ell}\right)A_{i\ell} + 2q(z_i,-a_\ell) + \sum_{k=1}^{N} A_{k\ell}q(z_i,z_k)\,\frac{w_k}{z_k}\,, \tag{37}$$

where

$$q(z_i,-a_\ell) = c(z_i,-a_\ell) + \frac{1}{2}\sum_{k=1}^{N} c(z_k,+a_\ell)A_{ki}\,\frac{w_k}{z_k}\,, \tag{38}$$

and $q(z_i,z_k)$ is given in terms of S_{ij} by Eq. (33). In Eq. (38) we have used the symmetry of A. The $N \cdot M$ Eqs. (37) are adjoined to the basic set in Eq. (32) and are integrated simultaneously with the initial conditions of Eq. (34) and

$$A_{i\ell}(0) = 0. \tag{39}$$

In the second method, we are given a function evaluated at the N points $z_1, z_2, \ldots, z_N$: $f(z_1), f(z_2), \ldots, f(z_N)$, and we wish to find $f(v)$ through approximating this function by a polynomial of degree $N - 1$. We make an expansion in the orthogonal polynomials $P_m^*(v)$,

$$f(v) = \sum_{m=0}^{N-1} \alpha_m P_m^*(v), \tag{40}$$

where the coefficients are

$$\alpha_m = (2m + 1) \int_0^1 f(v) P_m^*(v)\, dv. \tag{41}$$

Making the approximation

$$\alpha_m \cong (2m + 1) \sum_{i=1}^{N} f(z_i) P_m^*(z_i) w_i , \tag{42}$$

and substituting it into Eq. (40), we find the formula that gives $f(v)$ in terms of the given values $f(z_i)$,

$$f(v) = \sum_{m=0}^{N-1} (2m + 1) \sum_{i=1}^{N} f(z_i) P_m^*(z_i) w_i P_m^*(v). \tag{43}$$

We rewrite Eq. (43):

$$f(v) = \sum_{i=1}^{N} f(z_i) \beta_i(v), \tag{44}$$

where

$$\beta_i(v) = w_i \sum_{m=0}^{N-1} (2m + 1)P_m^*(z_i)P_m^*(v). \qquad (45)$$

We have evaluated these coefficients for N = 7 and v = 0,0.1, 0.2,...,1.0. These are listed in Table 1. These coefficients may be used, for example, in producing S(v,u,x) based on the N numerical values of $S(z_i,u,x)$, i = 1,2,...,N.

Table 1

SOME INTERPOLATION COEFFICIENTS, $\beta_i(v)$, FOR N = 7

V	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7
0.0000	1.574662	-0.970726	0.672107	-0.457142	0.284054	-0.144070	0.041115
0.1000	0.128800	1.028396	-0.242801	0.136943	-0.079364	0.039006	-0.010979
0.2000	-0.074052	0.571893	0.663517	-0.245789	0.128076	-0.060334	0.016688
0.3000	0.002129	-0.010718	0.996804	0.016675	-0.007230	0.003207	-0.000866
0.4000	0.031399	-0.135993	0.569418	0.670894	-0.193467	0.078217	-0.020469
0.5000	0.000000	0.000000	0.000000	1.000000	-0.000000	-0.000000	-0.000000
0.6000	-0.020470	0.078220	-0.193472	0.670899	0.569414	-0.135990	0.031399
0.7000	-0.000866	0.003207	-0.007230	0.016674	0.996804	-0.010718	0.002129
0.8000	0.016690	-0.060340	0.128084	-0.245796	0.663522	0.571890	-0.074051
0.9000	-0.010975	0.038992	-0.079345	0.136924	-0.242787	1.028387	0.128802
1.0000	0.041115	-0.144070	0.284054	-0.457142	0.672107	-0.970726	1.574662

V. COMPUTATIONAL RESULTS

Reflection functions are calculated for several strongly peaked forward scattering diagrams. A simple rational function that exhibits this property is

$$p(\cos\theta) = \frac{k}{b - \cos\theta}, \tag{46}$$

where b is a number slightly greater than 1, and k is a positive constant chosen to satisfy the normalization condition of the phase function. This phase function, as a function of two polar angles and an azimuth angle, is

$$p(v,u,\varphi) = \frac{k}{b - (uv + \sqrt{1 - u^2}\sqrt{1 - v^2}\cos\varphi)}. \tag{47}$$

The function c(v,u) is then, by analytically evaluating the integral in Eq. (5),

$$c(v,u) = \frac{\lambda k}{2\sqrt{(b - uv)^2 - (1 - u^2)(1 - v^2)}}, \tag{48}$$

and the constant k is

$$k = 2\left(\log\frac{b + 1}{b - 1}\right)^{-1}. \tag{49}$$

The forward scattered fraction of Eq. (6) is

$$f = \frac{k}{2}\log\frac{b}{b - 1}. \tag{50}$$

Polar diagrams of phase functions for b = 1.5, 1.1, and 1.01 may be seen in Fig. 1. The numerical values are given for the quantity $(b - \cos\theta)^{-1}$, rather than p itself. In Table 2 are found the fraction, f, of scattering into the forward hemisphere, and the ratio of forward to backward scattering, $p(\cos 0^{\circ})/p(\cos 180^{\circ})$, for each of six values of the parameter b.

Table 2

FORWARD SCATTERING PARAMETERS

b	f	$p(\cos 0^{\circ})/p(\cos 180^{\circ})$
1000.0	0.500	1.00
2.0	0.631	3.59
1.5	0.683	5.00
1.1	0.788	21.0
1.01	0.870	201.
1.001	0.909	2008.

A FORTRAN IV program is written for the IBM 7044 to compute the reflection function as described in Section IV. The interpolation method of adjoining additional differential equations is incorporated into the code so that the quantities $S_{ij} = S(z_i, z_j, x)$, $A_{i\ell} = S(z_i, a_\ell, x)$, and $B_{k\ell} = S(a_k, a_\ell, x)$ are evaluated. The A and B arrays (or the corresponding reflection matrices) are convenient for the purpose of comparing the effects of the order of the quadrature formula N, and the grid size Δx. They can also be compared with interpolated values obtained by the method of expansion in orthogonal polynomials.

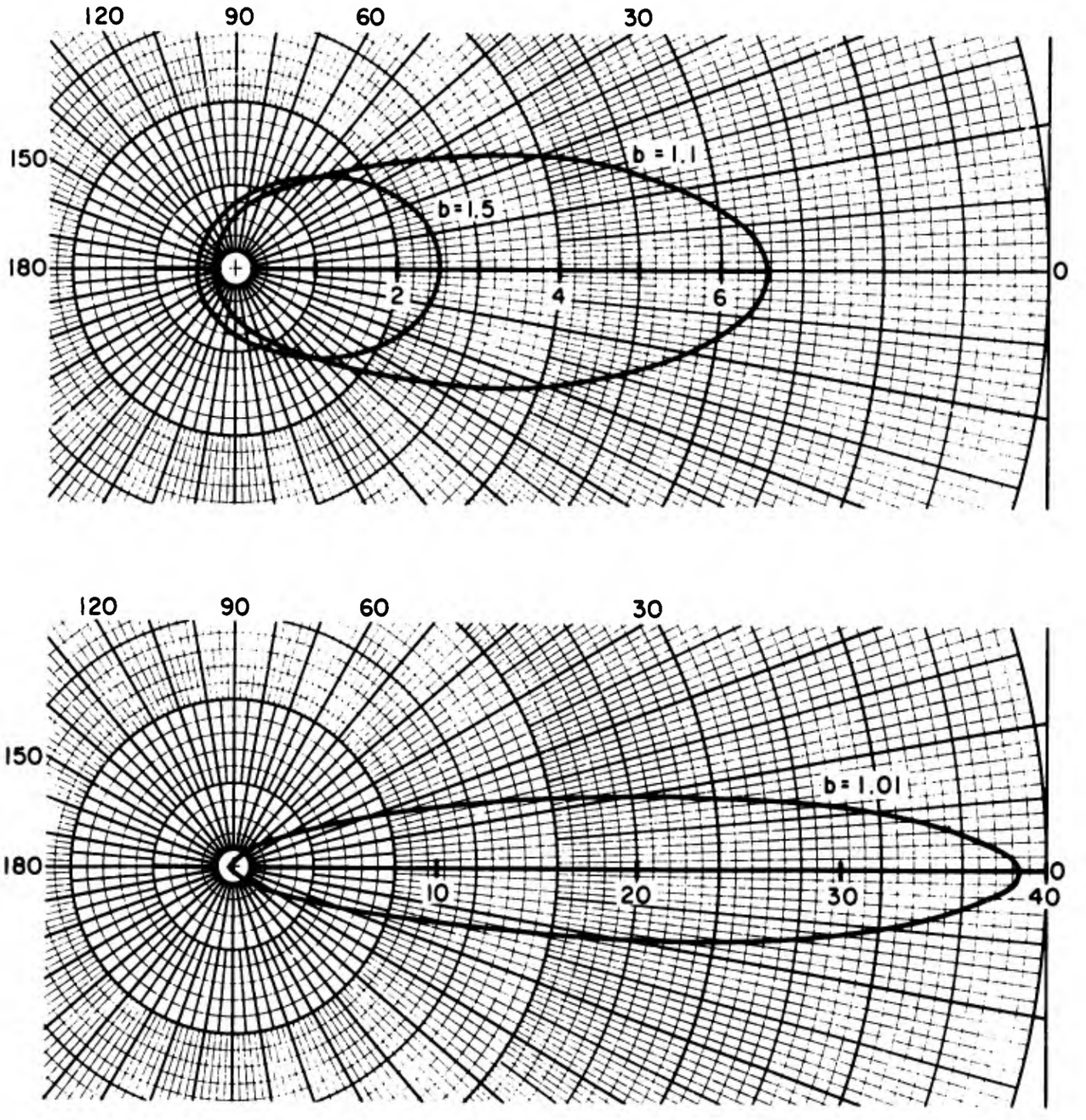

Fig. 1—Phase diagrams for b=1.5, 1.1 and 1.01

In Figs. 2-5, we see reflection functions plotted against v, the cosine of the output angle, for incident cosines u = 1.0, 0.5, and 0.1. The values of b are 1000 (essentially isotropic), 1.5, 1.1, and 1.01 in Figs. 2-5, respectively. The albedo for single scattering is $\lambda = 1$, and the thickness is x = 1.0, in each case. There is a remarkably smooth transition in the reflection pattern as one departs more and more from the spherical scattering diagram (bear in mind Fig. 1). The intensities drop in value, generally, except for grazing outgoing directions with grazing incidence. This increase at grazing angles may be attributed to single scatterings, and the decrease at other angles may be due to the greater forward scattering with less backward scattering. In this connection, it would be of interest to examine the diffuse transmission functions, which should exhibit behavior reverse to that of the reflection functions.

From the same computer runs that produced numerical values for Figs. 2 and 4, we are able to obtain reflection patterns for slabs of thicknesses less than one. Figures 6 and 7 are graphs of reflection functions for b = 1000 and b = 1.1, for slab thickness 0.1 and albedo $\lambda = 1$. Very little difference is found between the two figures. In inverse problems for the determination of the phase diagram based on reflection measurements, it would seem that thicker slabs and consequently a greater number of multiple scatterings would provide more information about the phase function. A class of inverse problems for estimating the coefficients in the expansion of the phase function in Legendre polynomials has been considered in Ref. 20.

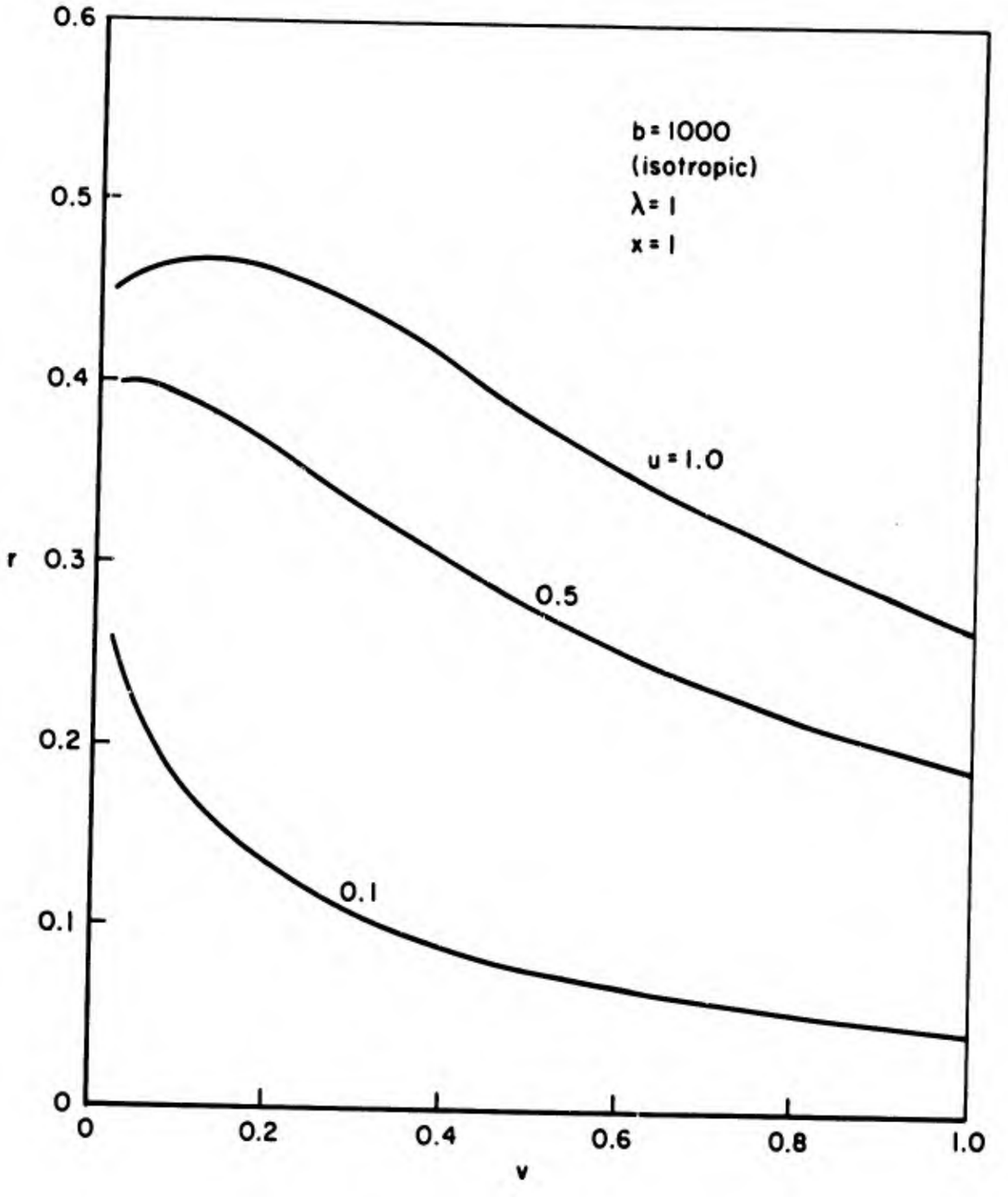

Fig. 2 — Some reflection curves

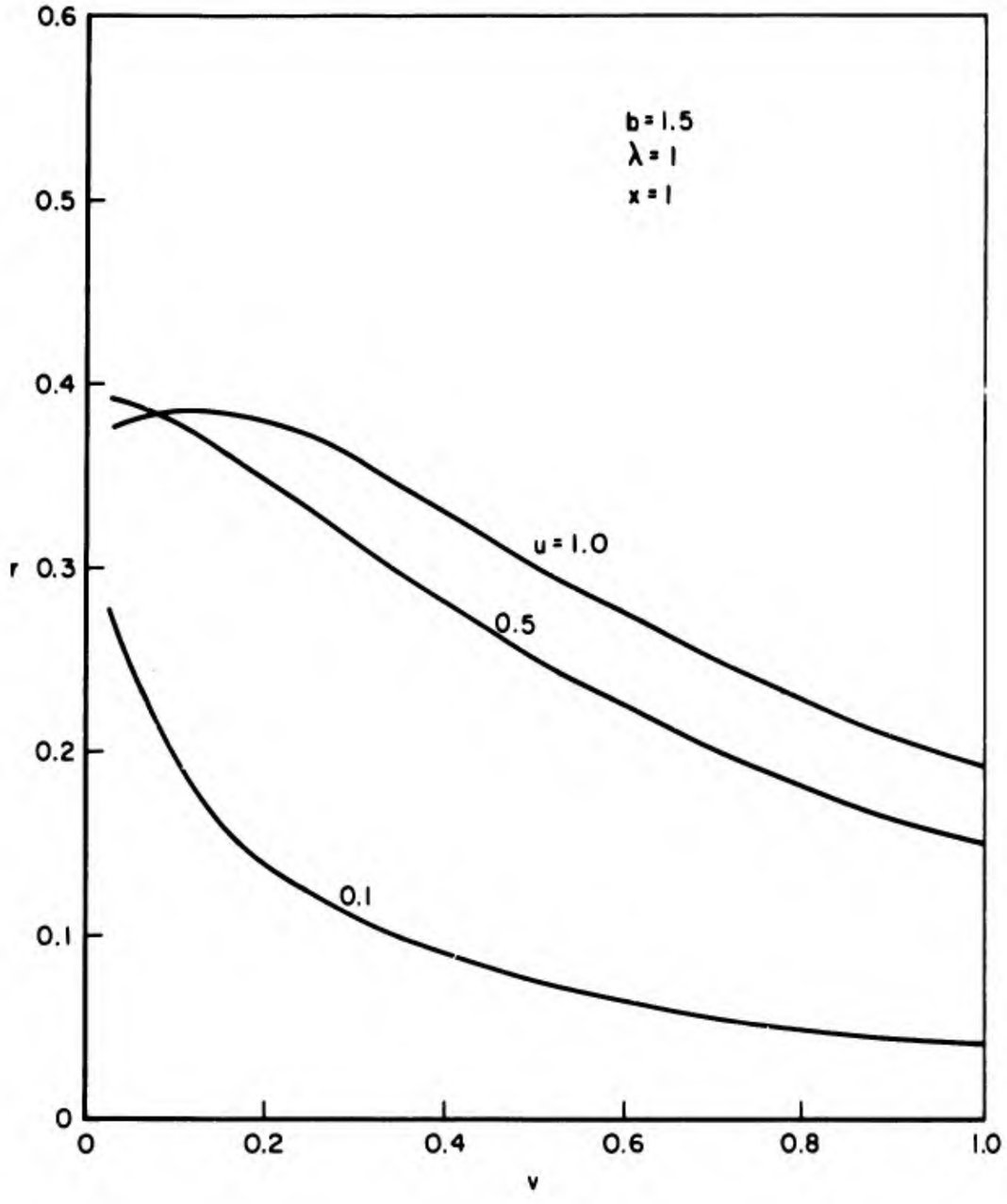

Fig. 3 — Some reflection curves

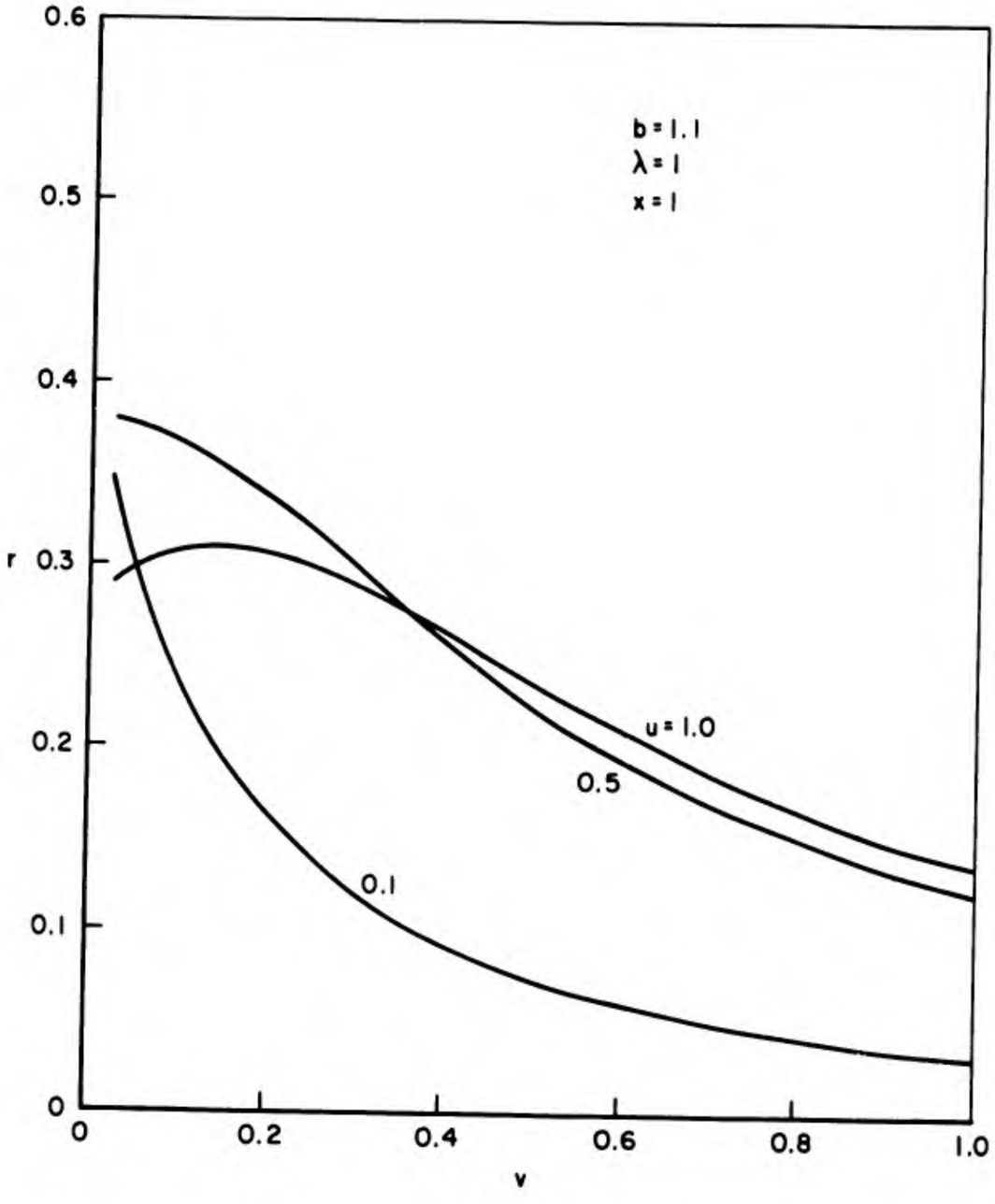

Fig. 4 — Some reflection curves

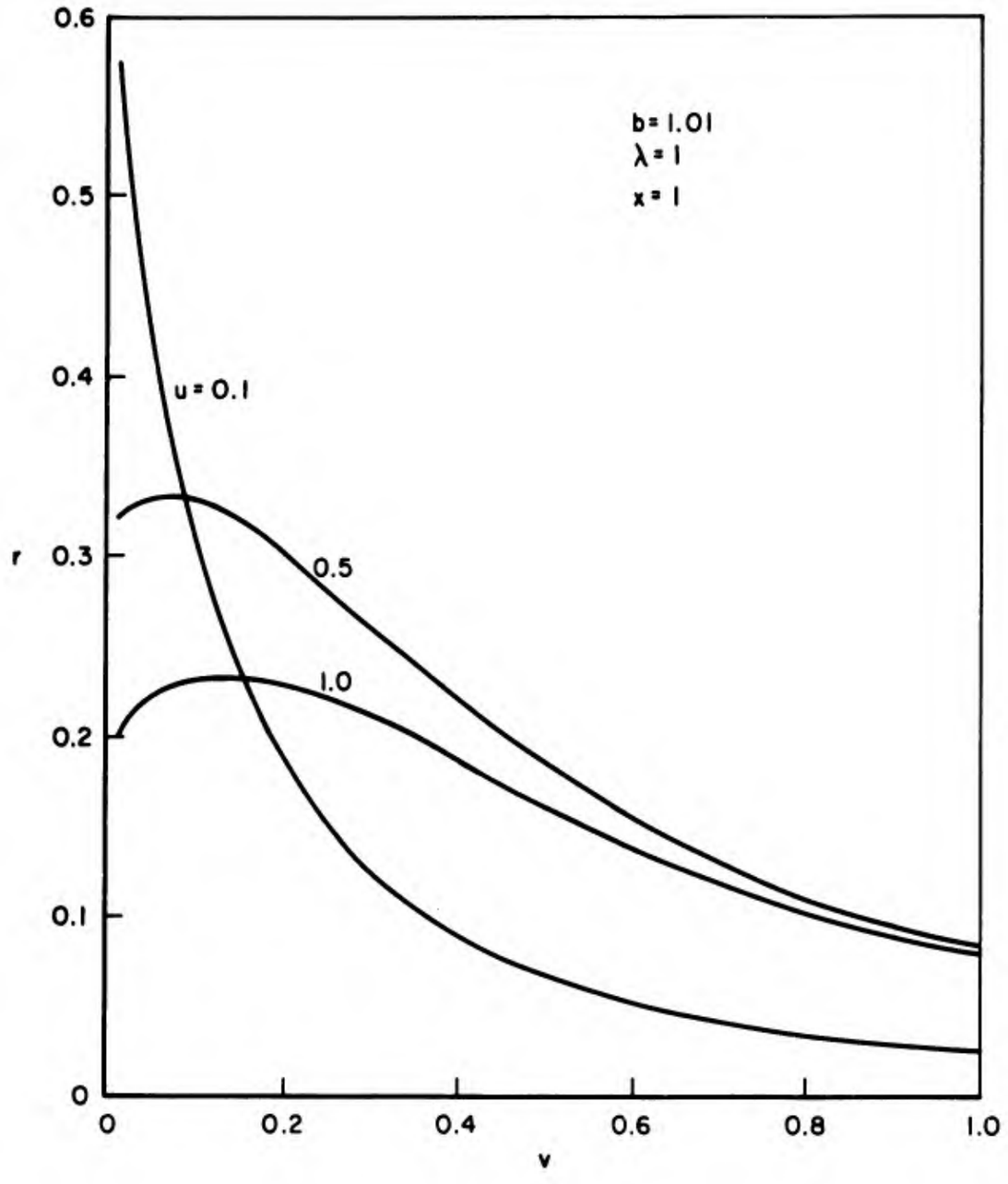

Fig. 5 — Some reflection curves

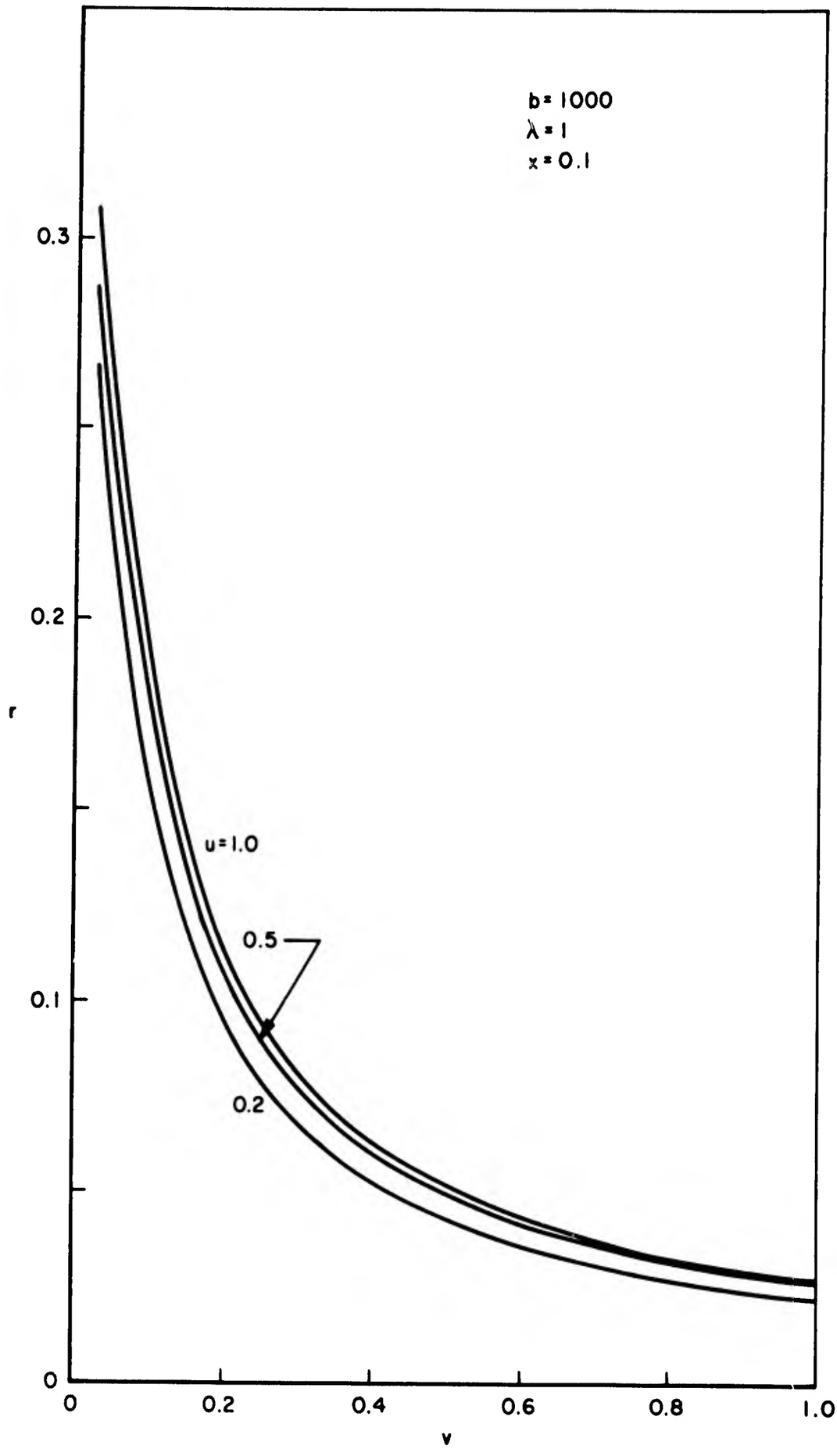

Fig. 6 — Some reflection curves for a thin slab

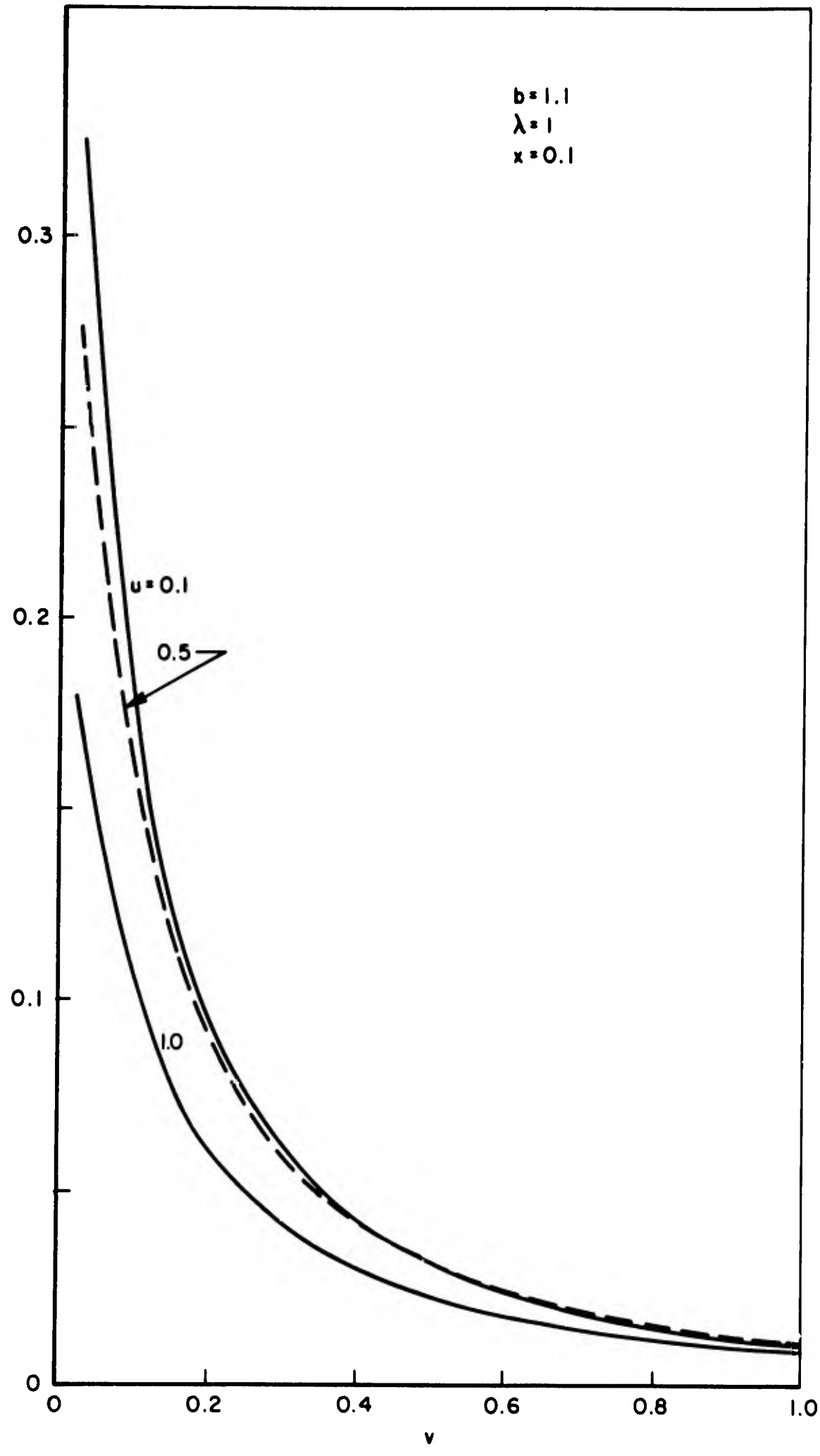

Fig. 7 — Some reflection curves for a thin slab

The value of N and the integration step sizes used to calculate the reflection functions drawn in Figs. 2-5 are given in Table 3. We also list the approximate number of significant figures obtained. These estimates are made by varying N and Δx, and comparing the numerical values of r. Attempts are made to calculate the reflection function for the extremely peaked phase function for b = 1.001. As can be seen from this table, highly accurate results are not obtained for this case, even with a large number of quadrature points and a tiny integration step size. The computing times are 1 min for the results of Figs. 2-4 and 5 min for Fig. 5.

Table 3

ACCURACY OF EVALUATION OF r BY INTEGRATION

Figure	b	N	Δx	Approximate Number of Significant Figures
2	1000.	7	0.01	6
3	1.5	7	0.01	4-5
4	1.1	7	0.01	4-5
5	1.01	9	0.005	2-3
-	1.001	15	0.001	1

We apply the second interpolation scheme, expansion in orthogonal polynomials, to obtain values of r at evenly spaced input cosine values, first when the output cosines are v = 0.025, 0.5, and 0.975, and again for evenly spaced output cosines when the input cosines are u = 0.1 and 1.0. These evaluations are compared with those obtained by integrating additional equations for r, in Table 4 for the case in which b = 1.1, λ = 1, and x = 1. In the expansion calculation, the coefficients

Table 4

REFLECTION FUNCTIONS BY TWO INTERPOLATION METHODS
FOR THE CASE b = 1.1, λ = 1, x = 1

	v = 0.025		v = 0.500		v = 0.975	
u	Integ.	Expans.	Integ.	Expans.	Integ.	Expans.
0.1	.3498	.344	.0733	.0734	.03178	.03176
0.2	.3985	.403	.1335	.1334	.06330	.06329
0.3	.4054	.40_	.1774	.1774	.0894	.0894
0.4	.3957	.394	.2069	.2070	.1087	.1087
0.5	.3791	.379	.2253	.2253	.1220	.1220
0.6	.3601	.362	.2357	.2356	.1306	.1306
0.7	.3408	.341	.2406	.2406	.1358	.1358
0.8	.3222	.321	.2416	.2417	.1384	.1385
0.9	.3049	.306	.2401	.2400	.1392	.1392
1.0	.2887	.286	.2368	.2370	.1388	.1389

	u = 0.1		u = 1.0	
v	Integ.	Expans.	Integ.	Expans.
0.1	.2360	.238	.3063	.3062
0.2	.1612	.160	.3059	.3057
0.3	.1192	.119	.2885	.2885
0.4	.0921	.0930	.2634	.2636
0.5	.0733	.0731	.2368	.2368
0.6	.0597	.0591	.2114	.2113
0.7	.0495	.0495	.1885	.1885
0.8	.0417	.0422	.1683	.1683
0.9	.0355	.0352	.1505	.1504
1.0	.0306	.0320	.1350	.1353

$\beta_i(v)$ of Eq. (45) were computed in double precision arithmetic (about 15 to 16 decimal figures), for v(or u) = 0.1,0.2,...,1.0, and i = 1,2,...,7. We first compute the r functions using about eight figures of accuracy in $\beta_i(v)$. Then we repeat the calculations after truncating $\beta_i(v)$ after the sixth decimal place. The r functions using six and eight figures are in agreement to about six places. Thus, these coefficients to six decimals, listed in Table 1, are suitable for hand computation on a desk calculator.

VI. DISCUSSION

The theoretical and numerical treatment of multiple scattering involving highly anisotropic elementary acts of scattering is in a rudimentary state. We have chosen to attack some problems for which there is no azimuth dependence of the diffuse radiation field though the local scattering is anisotropic. This is half way between the desired case and the simple isotropic case. To simulate the great forward scattering lobes suggested by Deirmendjian,[14] we suggest using a simple rational function, for both its analytical and its computational convenience. More realistic approximations will require the use of higher polynomials in both the numerator and denominator. In these cases it may be necessary to evaluate the function $c(v,u)$ in Eq. (5) numerically. The function $c(v,u)$ would have a sharp peak along $v = u$ that could cause difficulty in evaluating the integrals in Eqs. (17) and (18).

The treatment of a variety of inverse problems for anisotropic scattering, i.e., the estimation of the local phase function based on multiple scattering measurements, now appears feasible. A start is made in Ref. 20. An early work is represented by Ref. 21. It is of interest that the function $c(v,u)$ (an integral of the phase function) appears in the initial conditions on the source functions W and Z, Eqs. (27). It does not appear in the differential-integral Eqs. (24) and (26). An interesting inverse problem is the determination of the missing initial conditions on W and Z.

Extensions to the case of isotropic sources,[22] reflecting surfaces, and so forth are readily carried out.

Appendix A

FORTRAN PROGRAM FOR S AND r FUNCTIONS, WITH AXIALLY SYMMETRIC FIELDS, AND PHASE FUNCTION $p(\cos\theta) = k/(b - \cos\theta)$

This program is written to compute the functions $S(v,u,x)$ and $r(v,u,x)$ first for $u = z_1,\ldots,z_N$ and $v = z_1,\ldots,z_N$; second for $u = a_1,\ldots,a_M$ and $v = z_1,\ldots,z_N$; and third for $u = a_1,\ldots,a_M$ and $v = a_1,\ldots,a_M$. The phase function is $p(\cos\theta) = k/(b - \cos\theta)$. The phase function and the matrices $c(v,u)$ are evaluated in subroutine CHANGE, and printed.

There are $NEQ = N(N+1)/2 + M(M+1)/2 + N \cdot M$ differential equations that are simultaneously integrated. The value of the independent variable x is kept in cell T(2), the increment Δx in T(3). The dependent variables, the S functions, are stored in T(4) through T(NEQ+3), and their derivatives in T(NEQ+4) through T(2 · NEQ+3). The derivatives are evaluated in subroutine DAUX, which is called on by subroutines INTS and INTM. The latter two subroutines are for the numerical integration of a system of ordinary differential equations using a fourth-order formula. They are called in the MAIN program.

Subroutine EMERG, called upon by DAUX, calculates the $q(v,u)$ functions. Subroutine OUTPUT computes the r functions and prints all of the S and r matrices for each thickness between 0 and the final thickness, at evenly spaced values of x.

Sample data are given for the case $\lambda = 0.1$, $b = 1.5$, $x = 0.1, 0.2, \ldots, 1.0$; $N = 7$, $M = 10$, $a_1 = 0.1$, $a_2 = 0.2, \ldots, a_{10} = 1.0$.

```
$JOB            2890,SAXSYM,K0160,45,00,100,P       PAL 77
$IBJOB          MAP
$IBFTC MAIN
      COMMON N,M,NPRNT,MPRNT,DELTA,B,ZK,ALAM,ZKD2,NEQ,MX,FORE,PEAK,
     1 Z1(15),Z2(11),A1(15,15),A2(11,11),A3(15,11),WZ(15),C11P(15,15),
     2 C11M(15,15),C22M(11,11),C12M(15,11),C21P(11,15),S1(15,15),
     3 S2(11,11),S3(15,11),Q11P(15,15),Q11M(15,15),Q22M(11,11),
     4 Q12M(15,11),Q21P(11,15),R(15,15),T(4213),WT(15),C12P(15,11)
C
C             S FUNCTION FOR AXIALLY SYMMETRIC FIELD
C             ANISOTROPIC PHASE FUNCTION  P = K/(B-Z), Z=COSINE ALPHA
C             N=NO. OF ROOTS, Z1(I)
C             M=NO. OF COSINES, Z2(I)
C             S1(I,J) = S(Z1(I),Z1(J))
C             S2(I,J) = S(Z2(I),Z2(J))
C             S3(I,J) = S(Z1(I),Z2(J))
C             C MATRICES--PROB. OF SCATTERING FROM ONE D.C. TO ANOTHER
C             Q MATRICES--PROB. OF EMERGENCE WITH D.C. V, IF PARTICLE
C                         INTERACTS AT TOP WHILE IN STATE WITH D.C. U
C             R(I,J)  REFLECTED INTENSITY
C
C             INPUT
C
    1   READ(5,100)N,M,NPRNT,MPRNT
        WRITE(6,90)N,M,NPRNT,MPRNT
        READ(5,101)DELTA,B,ALAM
        WRITE(6,91)DELTA,B,ALAM
        READ(5,101)(Z1(I),I=1,N)
        WRITE(6,92)(Z1(I),I=1,N)
        READ(5,101)(WT(I),I=1,N)
        WRITE(6,93)(WT(I),I=1,N)
        READ(5,101)(Z2(I),I=1,M)
        WRITE(6,94)(Z2(I),I=1,M)
C
        DO 2 I=1,N
        WZ(I)=WT(I)/Z1(I)
        ZZ1=1.0/Z1(I)
        DO 3 J=1,N
    3   A1(I,J)=ZZ1 + 1.0/Z1(J)
        DO 2 J=1,M
    2   A3(I,J)=ZZ1 + 1.0/Z2(J)
        DO 4 I=1,M
        ZZ2=1.0/Z2(I)
        DO 4 J=1,M
    4   A2(I,J)=ZZ2 + 1.0/Z2(J)
C
C             LOCAL SCATTERING PROPERTIES
C
        CALL CHANGE
C
C             INITIAL CONDITIONS FOR S
C
        DO 5 I=1,4213
    5   T(I)=0.0
        T(3)=DELTA
        NEQ = N*(N+1)/2 + M*(M+1)/2 + N*M
        CALL INTS(T,NEQ,2,0,0,0,0,0,0)
        MX=1
C
```

```
C            INTEGRATE
C
      DO 6 M1=1,MPRNT
      DO 7 M2=1,NPRNT
      MX=MX+1
    7 CALL INTM
    6 CALL OUTPUT
C
      GO TO 1
C
C
  100 FORMAT(6I12)
   90 FORMAT(1H1,2HN=,I4,3X2HM=,I4,3X6HNPRNT=,I4,3X6HMPRNT=,I4)
  101 FORMAT(6E12.8)
   91 FORMAT(1HO, 6HDELTA=,F 6.3,3X2HB=,F8.3,3X7HLAMBDA=,F6.3)
   92 FORMAT(1HO,14HN ROOTS, Z1(I)/(1X6E20.8))
   93 FORMAT(1HO, 9HN WEIGHTS/     (1X6E20.8))
   94 FORMAT(1HO,16HM COSINES, Z2(I)/(1X6E20.8))
      END
$IBFTC DAUX
      SUBROUTINE DAUX
      COMMON N,M,NPRNT,MPRNT,DELTA,B,ZK,ALAM,ZKD2,NEQ,MX,FORE,PEAK,
     1 Z1(15),Z2(11),A1(15,15),A2(11,11),A3(15,11),WZ(15),C11P(15,15),
     2 C11M(15,15),C22M(11,11),C12M(15,11),C21P(11,15),S1(15,15),
     3 S2(11,11),S3(15,11),Q11P(15,15),Q11M(15,15),Q22M(11,11),
     4 Q12M(15,11),Q21P(11,15),R(15,15),T(4213),WT(15),C12P(15,11)
C
      L=3
      DO 1 I=1,N
      DO 1 J=1,I
      L=L+1
      S1(I,J)=T(L)
    1 S1(J,I)=T(L)
      DO 2 I=1,M
      DO 2 J=1,I
      L=L+1
      S2(I,J)=T(L)
    2 S2(J,I)=T(L)
      DO 3 I=1,N
      DO 3 J=1,M
      L=L+1
      S3(I,J)=T(L)
    3 CONTINUE
C
      CALL EMERG
C
      DO 5 I=1,N
      DO 5 J=1,I
      SUM=0.0
      DO 6 K=1,N
    6 SUM=SUM + S1(K,J)*Q11P(I,K)*WZ(K)
      L=L+1
    5 T(L)=-A1(I,J)*S1(I,J) + 2.0*Q11M(I,J) + SUM
C
      DO 7 I=1,M
      DO 7 J=1,I
      SUM=0.0
      DO 8 K=1,N
    8 SUM=SUM + S3(K,J)*Q21P(I,K)*WZ(K)
```

```
      L=L+1
    7 T(L)=-A2(I,J)*S2(I,J) + 2.0*Q22M(I,J) + SUM
C
      DO 9 I=1,N
      DO 9 J=1,M
      SUM=0.0
      DO 10 K=1,N
   10 SUM=SUM + S3(K,J)*Q11P(I,K)*WZ(K)
      L=L+1
    9 T(L)=-A3(I,J)*S3(I,J) + 2.0*Q12M(I,J) + SUM
      RETURN
      END
$IBFTC CHANGE
      SUBROUTINE CHANGE
      COMMON N,M,NPRNT,MPRNT,DELTA,B,ZK,ALAM,ZKD2,NEQ,MX,FORE,PEAK,
     1 Z1(15),Z2(11),A1(15,15),A2(11,11),A3(15,11),WZ(15),C11P(15,15),
     2 C11M(15,15),C22M(11,11),C12M(15,11),C21P(11,15),S1(15,15),
     3 S2(11,11),S3(15,11),Q11P(15,15),Q11M(15,15),Q22M(11,11),
     4 Q12M(15,11),Q21P(11,15),R(15,15),T(4213),WT(15),C12P(15,11)
C
C           NORMALIZATION CONSTANT, ZK
C
      ZLOG=ALOG((B+1.0)/(B-1.0))
      ZK=2.0*ALAM/ZLOG
      ZKD2=0.5*ZK
      WRITE(6,90)ZK
   90 FORMAT(1HO,3HZK=,E16.6)
C
C           PHASE FUNCTION VS. SCATTERING ANGLE (COSINE OF)
      WRITE(6,91)
      DEL=2.0/40.0
      DO 1 I=1,41
      FI=I-1
      Z=FI*DEL - 1.0
      PHASE = ZK/(B-Z)
    1 WRITE(6,92) Z, PHASE
C
   91 FORMAT(///20X17HPHASE FUNCTION, P /14X 6HCOSINE,19X 1HP)
   92 FORMAT(F20.3, E20.6)
C
C           FORWARD FRACTION AND PEAKEDNESS
C
      ZLOG=ALOG(B/(B-1.0))
      FORE=ZKD2*ZLOG
      WRITE(6,93)FORE
C
   93 FORMAT(1HO,18HFORWARD FRACTION =,F6.3)
C
C           C(V,U) MATRICES
C       C11P
      WRITE(6,94)(Z1(I),I=1,N)
   94 FORMAT(///20X16HC MATRIX  (C11P) /(1HO11X,8F14.3)/8X4HINC.//)
      DO 3 J=1,N
      DO 2 I=1,N
      SQ1=(B - Z1(I)*Z1(J))**2
      SQ2=(1.0 - Z1(I)**2) * (1.0 - Z1(J)**2)
    2 C11P(I,J) = ZKD2 / SQRT(SQ1-SQ2)
    3 WRITE(6,95) Z1(J), (C11P(I,J),I=1,N)
   95 FORMAT(F12.3,8E14.6/12X8E14.6)
```

```
C
C       C11M
C
      WRITE(6,96)(Z1(I),I=1,N)
   96 FORMAT(///20X16HC MATRIX  (C11M) /(1H011X,8F14.3)/8X4HINC.//)
      DO 5 J=1,N
      DO 4 I=1,N
      SQ1=(B + Z1(I)*Z1(J))**2
      SQ2=(1.0 - Z1(I)**2) * (1.0 - Z1(J)**2)
    4 C11M(I,J)=ZKD2 / SQRT(SQ1-SQ2)
    5 WRITE(6,95) Z1(J), (C11M(I,J),I=1,N)
C
C            C22M
C
      WRITE(6,97)(Z2(I),I=1,M)
   97 FORMAT(///20X16HC MATRIX  (C22M) /(1H011X,8F14.3)/8X4HINC.//)
      DO 7 J=1,M
      DO 6 I=1,M
      SQ1=(B + Z2(I)*Z2(J))**2
      SQ2=(1.0 - Z2(I)**2) * (1.0 - Z2(J)**2)
    6 C22M(I,J)=ZKD2 / SQRT(SQ1-SQ2)
    7 WRITE(6,95) Z2(J),(C22M(I,J),I=1,M)
C
C            C12P
C
      WRITE(6,89)(Z1(I),I=1,N)
   89 FORMAT(///20X16HC MATRIX  (C12P) /(1H011X,8F14.3)/8X4HINC.//)
      DO 13 J=1,M
      DO 12 I=1,N
      SQ1=(B - Z1(I)*Z2(J))**2
      SQ2=(1.0 - Z1(I)**2) * (1.0 - Z2(J)**2)
   12 C12P(I,J)=ZKD2 / SQRT(SQ1-SQ2)
   13 WRITE(6,95)Z2(J),(C12P(I,J),I=1,N)
C
C            C12M
C
      WRITE(6,98)(Z1(I),I=1,N)
   98 FORMAT(///20X16HC MATRIX  (C12M) /(1H011X,8F14.3)/8X4HINC.//)
      DO 9 J=1,M
      DO 8 I=1,N
      SQ1=(B + Z1(I)*Z2(J))**2
      SQ2=(1.0 - Z1(I)**2) * (1.0 - Z2(J)**2)
    8 C12M(I,J)=ZKD2 / SQRT(SQ1-SQ2)
    9 WRITE(6,95) Z2(J),(C12M(I,J),I=1,N)
C
C            C21P
C
      WRITE(6,99)(Z2(I),I=1,M)
   99 FORMAT(///20X16H C MATRIX (C21P) /(1H011X,8F14.3)/8X4HINC.//)
      DO 11 J=1,N
      DO 10 I=1,M
      SQ1=(B - Z2(I)*Z1(J))**2
      SQ2=(1.0 - Z2(I)**2) * (1.0 - Z1(J)**2)
   10 C21P(I,J)=ZKD2 / SQRT(SQ1-SQ2)
   11 WRITE(6,95) Z1(J),(C21P(I,J),I=1,M)
C
      RETURN
      END
$IBFTC OUTPUT
```

```
      SUBROUTINE OUTPUT
      COMMON N,M,NPRNT,MPRNT,DELTA,D,ZK,ALAM,ZKD2,NEQ,MX,FORE,PEAK,
     1 Z1(15),Z2(11),A1(15,15),A2(11,11),A3(15,11),WZ(15),C11P(15,15),
     2 C11M(15,15),C22M(11,11),C12M(15,11),C21P(11,15),S1(15,15),
     3 S2(11,11),S3(15,11),Q11P(15,15),Q11M(15,15),Q22M(11,11),
     4 Q12M(15,11),Q21P(11,15),R(15,15),T(4213),WT(15),C12P(15,11)
C
      L=3
      DO 11 I=1,N
      DO 11 J=1,I
      L=L+1
      S1(I,J)=T(L)
   11 S1(J,I)=T(L)
      DO 12 I=1,M
      DO 12 J=1,I
      L=L+1
      S2(I,J)=T(L)
   12 S2(J,I)=T(L)
      DO 13 I=1,N
      DO 13 J=1,M
      L=L+1
   13 S3(I,J)=T(L)
C
      WRITE(6,50)
C
      WRITE(6,51)
   51 FORMAT(50X9HS1 MATRIX)
      WRITE(6,59)T(2)
   59 FORMAT(47X9HTHICKNESS, F6.2)
      WRITE(6,52)(I,I=1,N)
   52 FORMAT(1H0,11X,8I14)
      WRITE(6,53)(Z1(I),I=1,N)
   53 FORMAT(12X8F14.3)
      WRITE(6,54)
   54 FORMAT(8X4HINC. )
C
      DO 1 J=1,N
    1 WRITE(6,55) J,Z1(J),(S1(I,J),I=1,N)
   55 FORMAT(I5,F7.3,8E14.6/12X8E14.6)
C
      DO 2 I=1,N
      DO 2 J=1,N
    2 R(I,J)=0.25*S1(I,J)/Z1(I)
C
      WRITE(6,56)
   56 FORMAT(///50X9HR  MATRIX)
      WRITE(6,52)(I,I=1,N)
      WRITE(6,53)(Z1(I),I=1,N)
      WRITE(6,54)
      DO 3 J=1,N
    3 WRITE(6,55) J,Z1(J),( R(I,J),I=1,N)
C
C
      WRITE(6,57)
   57 FORMAT(////50X9HS2 MATRIX)
      WRITE(6,59)T(2)
      WRITE(6,52)(I,I=1,M)
      WRITE(6,53)(Z2(I),I=1,M)
      WRITE(6,54)
```

```
      DO 4 J=1,M
    4 WRITE(6,55) J,Z2(J),(S2(I,J),I=1,M)
C
      DO 5 I=1,M
      DO 5 J=1,M
    5 R(I,J)=0.25*S2(I,J)/Z2(I)
      WRITE(6,56)
      WRITE(6,52)(I,I=1,M)
      WRITE(6,53)(Z2(I),I=1,M)
      WRITE(6,54)
      DO 6 J=1,M
    6 WRITE(6,55) J,Z2(J), (R(I,J),I=1,M)
C
C
      WRITE(6,58)
   58 FORMAT(////50X9HS3· MATRIX)
      WRITE(6,59)T(2)
      WRITE(6,52)(I,I=1,N)
      WRITE(6,53)(Z1(I),I=1,N)
      WRITE(6,54)
      DO 7 J=1,M
    7 WRITE(6,55) J,Z2(J),(S3(I,J),I=1,N)
C
      DO 8 I=1,N
      DO 8 J=1,M
    8 R(I,J)=0.25*S3(I,J)/Z1(I)
      WRITE(6,56)
      WRITE(6,52)(I,I=1,N)
      WRITE(6,53)(Z1(I),I=1,N)
      WRITE(6,54)
      DO 9 J=1,M
    9 WRITE(6,55) J,Z2(J),(R(I,J),I=1,N)
C
   50 FORMAT(1H1)
      RETURN
      END
$IBFTC EMERG
      SUBROUTINE EMERG
      COMMON N,M,NPRNT,MPRNT,DELTA,B,ZK,ALAM,ZKD2,NEQ,MX,FORE,PEAK,
     1 Z1(15),Z2(11),A1(15,15),A2(11,11),A3(15,11),WZ(15),C11P(15,15),
     2 C11M(15,15),C22M(11,11),C12M(15,11),C21P(11,15),S1(15,15),
     3 S2(11,11),S3(15,11),Q11P(15,15),Q11M(15,15),Q22M(11,11),
     4 Q12M(15,11),Q21P(11,15),R(15,15),T(4213),WT(15),C12P(15,11)
C
C           Q MATRICES, PROBABILITIES OF EMERGENCE
C
      DO 2  I=1,N
      DO 2  J=1,N
      U11P=0.0
      U11M=0.0
      DO 1 K=1,N
      U11P=U11P + C11P(K,J)*S1(I,K)*WZ(K)
    1 U11M=U11M + C11M(K,J)*S1(I,K)*WZ(K)
      Q11P(I,J)=C11P(I,J) + 0.5*U11M
    2 Q11M(I,J)=C11M(I,J) + 0.5*U11P
C
      DO 4  I=1,M
      DO 4  J=1,M
      U22P=0.0
```

```
      DO 3 K=1,N
    3 U22P=U22P + C12P(K,J)*S3(K,I)*WZ(K)
    4 Q22M(I,J)=C22M(I,J) + 0.5*U22P
C
      DO 6  I=1,M
      DO 6  J=1,N
      U21M=0.0
      DO 5 K=1,N
    5 U21M=U21M + C11M(K,J)*S3(K,I)*WZ(K)
    6 Q21P(I,J)=C21P(I,J) + 0.5*U21M
C
      DO 7 I=1,N
      DO 7 J=1,M
      U12P=0.0
      DO 8 K=1,N
    8 U12P=U12P + C12P(K,J)*S1(I,K)*WZ(K)
    7 Q12M(I,J)=C12M(I,J) + 0.5*U12P
C
      RETURN
      END
$ENTRY          MAIN
           7          10          10          10
        0.01         1.5         0.1
25446046E-0112923441E-0029707742E-0050000000E 0070292258E 0087076559E 00     03
97455396E 00
64742484E-0113985269E-0019091502E-0020897958E-0019091502E-0013985269E-00    00
64742484E-01
         0.1         0.2         0.3         0.4         0.5         0.6
         0.7         0.8         0.9         1.0
```

Appendix B

FORTRAN PROGRAM FOR THE INTERPOLATION OF FUNCTIONS BY EXPANSION IN SHIFTED LEGENDRE POLYNOMIALS

In this program, the roots and weights for $N \leq 15$ are input in double precision. The M values of V(J), and the N given values of the function FZ(I), are read in, in single precision. The coefficients are calculated in subroutine LCOF and returned in matrix C(I,J), I = 1, N; J = 1, M. The interpolated function is F(J) $= \sum$ FZ(I) · C(I,J). The coefficients C(I,J) may be truncated after MDEC decimal places.

The C(I,J) array may be calculated by using the recurrence relations for the coefficients in the expansion of shifted Legendre polynomials by a power series,[18]

$$P_n^*(x) = \sum_{k=o}^{n} \gamma_{nk} x^k,$$

$$\gamma_{no} = 1,$$

$$\gamma_{nk} = \frac{n + k}{n - k} \gamma_{n-1,k}, \qquad k \neq n,$$

$$\gamma_{nk} = -\frac{2}{n} \gamma_{n,n-1}, \qquad k = n,$$

as done in the first subroutine LCOF that is listed. In the second version of LCOF, the C(I,J) array is calculated by using recurrence relations for the shifted Legendre polynomials themselves,

$$nP_n^*(x) = -(n - 1)P_{n-2}^*(x) + (2n - 1)(1 - 2x)P_{n-1}^*(x),$$

$$P_o^*(x) = 1,$$

$$P_1^*(x) = 1 - 2x.$$

Both subroutines give virtually the same set of coefficients.

The data shown are for the case N = 7, M = 11, MDEC = 6, V(1) = 0.0, V(2) = 0.1,..., V(11) = 1.0. The functions to be interpolated are nine different reflection functions (for nine input or output directions), with x = 1.0, b = 1.1, and λ = 1.0.

```
$JOB            2890,LCOF,K0160,1,50,10,C       PAL 77
$IBJOB          MAP
$IBFTC MAIN
      DIMENSION F(21),C(15,21),W(15),V(21),Z(15),FZ(15)
      DOUBLE PRECISION Z,W
C
C           INTERPOLATION
C           F(J)=SUM(FZ(I)*C(I,J))
C
      READ (5,1) N
      WRITE (6,2) N
      READ (5,1) M
      WRITE (6,3) M
      READ(5,1)MDEC
      WRITE(6,28)MDEC
C
C           MDEC = 0      C(I,J) IN FULL PRECISION
C           MDEC.GT. 0      C(I,J) TRUNCATED TO MDEC DECIMAL PLACES
C
      READ (5,4) (Z(I),I=1,N)
      WRITE (6,5)
      WRITE (6,6) (Z(I),I=1,N)
      READ (5,4) (W(I),I=1,N)
      WRITE (6,7)
      WRITE (6,6) (W(I),I=1,N)
C
      READ (5,8) (V(I),I=1,M)
      WRITE (6,9)
      WRITE (6,13) (V(I),I=1,M)
C
      CALL LCOF(N,Z,W,M,V,C)
C
      IF(MDEC.EQ.0) GO TO 25
C
      DO 22 I=1,N
      DO 22 J=1,M
      DEC=10.0**MDEC
      CD=C(I,J)*DEC
      CD=IFIX(CD)
   22 C(I,J)=CD/DEC
C
      WRITE (6,18)
      WRITE(6,21) (I,I=1,N)
      DO 23 J=1,M
   23 WRITE(6,24) V(J),(C(I,J),I=1,N)
      GO TO 20
C
C
   25 WRITE (6,18)
      WRITE(6,21) (I,I=1,N)
      DO 10 J=1,M
   10 WRITE(6,19) V(J),(C(I,J),I=1,N)
```

```
C
C
   20  READ (5,11) (FZ(I),I=1,N)
       WRITE (6,12)
       WRITE (6,13) (FZ(I),I=1,N)
C
       DO 15 J=1,M
       SUM=0.0
       DO 14 I=1,N
   14  SUM=SUM + FZ(I)*C(I,J)
    15 F(J)=SUM
       WRITE (6,16)
       WRITE (6,17) (J,F(J),J=1,M)
C
       GO TO 20
     1 FORMAT (I2)
     2 FORMAT (3H1N=I2)
     3 FORMAT (3H M=I2)
    4  FORMAT(D24.16)
     5 FORMAT (5H0Z(I) )
    6  FORMAT(D30.16)
     7 FORMAT (5H0W(I))
     8 FORMAT (6E12.8)
     9 FORMAT (5H0V(J))
    11 FORMAT (7F10.6)
    12 FORMAT (6H0F(ZI))
    13 FORMAT (E20.8)
    16 FORMAT (15H0 J          F(J) )
    17 FORMAT (1H I2,E23.8)
   18  FORMAT(9H0C MATRIX)
   19  FORMAT(F10.4,7E17.8)
   21  FORMAT(9X1HV,7I17)
   24  FORMAT(F10.4,7F17.6)
   28  FORMAT(1X5HMDFC=,I2)
       END
$IBFTC LCOF
       SUBROUTINE LCOF(N,Z,W,M,V,C)
       DIMENSION Z(15),W(15),V(21),C(15,21),PZ(15),PV(21),A(16,17)
       DOUBLE PRECISION Z,W,  PZ,PV,A,X1,X2,X,FK,SUM
C
C            COEFFICIENTS FOR P*
C
       NPLUS=N+1
       A(1,1)=1.0
       DO 4 I=1,NPLUS
       IP=I+1
       DO 3 N=IP,NPLUS
       X1=N + I - 2
       X2=N - I
    3  A(N,I)=A(N-1,I)*X1/X2
       N=IP
       X=I
```

```
    4   A(N,N)=-A(N,N-1)*2.0/X
 C
        N=NPLUS-1
 C
 C             COEFFICIENTS FOR INTERPOLATION
 C
        DO 2 I=1,N
        DO 2 J=1,M
        SUM=0.0
        DO 1 K1=1,N
        K=K1-1
        PZ(I)=A(K1,1)
        PV(J)=A(K1,1)
        IF(K1.EQ.1) GO TO 7
        DO 6 L1=2,K1
        L=L1-1
        PZ(I)=PZ(I) + A(K1,L1)*Z(I)**L
        VJ=DBLE(V(J))
        VJL=0.0
        IF(V(J).LT.0.00001) GO TO 6
        VJL=VJ**L
    6   PV(J)=PV(J) + A(K1,L1)*VJL
    7   FK=2*K+1
    1   SUM=SUM + FK*PZ(I)*PV(J)
    2   C(I,J)=W(I)*SUM
C
        DO 8 K1=1,N
    8   WRITE(6,90)K1,(A(K1,L1),L1=1,K1)
   90   FORMAT(I4,6E20.8/(4X6E20.8))
        RETURN
        END
$ENTRY          MAIN
 7
11                                                                    N
 6
  2.54460438286208660 -2                                              R 1/7
  1.29234407200302820 -1                                              R 2/7
  2.97077424311301450 -1                                              R 3/7
  5.00000000000000000 -1                                              R 4/7
  7.02922575688698530 -1                                              R 5/7
  8.70765592799697060 -1                                              R 6/7
  9.74553956171379090 -1                                              R 7/7
  6.47424830844348160 -2                                              W 1/7
  1.39852695744638280 -1                                              W 2/7
  1.90915025252559380 -1                                              W 3/7
  2.08979591836734660 -1                                              W 4/7
  1.90915025252559380 -1                                              W 5/7
  1.39852695744638280 -1                                              W 6/7
  6.47424830844348160 -2                                              W 7/7
         0.0          0.1          0.2          0.3          0.4          0.5
         0.6          0.7          0.8          0.9          1.0
 0.203020   0.371563   0.405522   0.379093   0.340232   0.309816   0.292717   V=.025
```

```
 0.0731602  0.242239  0.338482  0.357989  0.343726  0.324392  0.311595  V=.1
 0.0347348  0.147246  0.249669  0.296789  0.304510  0.297928  0.291086  V=.2
 0.0192928 0.0925291  0.176338  0.225259  0.240680  0.240762  0.237743  V=.5
 0.0123165 0.0631951  0.128695  0.171199  0.187150  0.189609  0.188390  V=.7
0.00905363 0.0481446 0.101643  0.138247  0.153061  0.156193  0.155742  V=.8
0.00764297 0.0413203 0.0887331 0.121975  0.135881  0.139000  0.139000  V=.9
  0.349797  0.208559  0.120187 0.0733123 0.0492430 0.0371810 0.0317777 U=0.1
  0.288726  0.308438  0.289201  0.236751  0.187859  0.155426  0.138773 U=1.0
$IBSYS          ENDJOB          TOTAL NUMBER OF CARDS IN YOUR INPUT DECK
```

```
$IBFTC LCOF
      SUBROUTINE LCOF(N,Z,W,M,V,C)
.129  DIMENSION Z(15),W(15),V(21),C(15,21),PZ(15),PV(21)
.297  DOUBLE PRECISION PZ,PV,SUM,PL1,Z,W,FL
.5    PZ(1)=1.0
.703  PV(1)=1.0
.871  N1=N-1
.975  DO 1 I=1,N
0.1   DO 1 J=1,M
1.0   PZ(2)=1.0-2.0*(Z(I))
      PV2  =1.0-2.0*(V(J))
      PV(2)=DBLE(PV2)
      SUM=1.0+PZ(2)*PV(2)*3.0
      DO 2 L=2,N1
      L1=L+1
      L2=L-1
      PL1=DBLE(FLOAT(L2))
      FL=DBLE(FLOAT(L))
      PZ(L1)=(-PL1*PZ(L2)+(2.0*FL-1.0)*PZ(2)*PZ(L))/FL
      PV(L1)=(-PL1*PV(L2)+(2.0*FL-1.0)*PV(2)*PV(L))/FL
    2 SUM=SUM+(2.0*FL+1.0)*PZ(L1)*PV(L1)
      C(I,J)=W(I)*SUM
    1 CONTINUE
      RETURN
      END
```

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10. ABSTRACT

The invariant imbedding approach is applied to the problem of diffuse reflection from a slab having a strongly elongated phase function. The diffuse radiation field is assumed to be axially symmetric. For both analytical and computational convenience, the study suggests using a simple rational function to approximate a strongly peaked phase function. Basic equations are derived for the reflection and source functions, and the reflection function is computed as the solution of an initial-value problem for ordinary differential equations. Numerical results are tabulated for increasingly fierce forward scattering. Two FORTRAN IV programs are included: one for the reflection function and the other for interpolation by expansion in orthogonal polynomials.

11. KEY WORDS

Radiative transfer
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Invariant imbedding