TECHNICAL MEMORANDUM NO. 66-01

REVISED THEORY OF VORTEX RINGS - A SIMPLIFIED REVIEW OF THE STATE-OF-THE-ART

INTERIM REPORT ON TASK NO. 01-S-65

By
Serge J. Zaroodny
U. S. Army Ballistic Research Laboratories

April 1966

U. S. ARMY LIMITED WAR LABORATORY
Aberdeen Proving Ground, Maryland 21005

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ABSTRACT

The vortex ring is an essential and ubiquitous phenomenon that has been rather neglected in aerodynamics and technology, perhaps because it is usually construed, unfortunately, as a mere exercise in old-fashioned mathematics. The existing venerable theories of this phenomenon are at once little known, difficult, uncoordinated, insufficient and inconsistent. They are reviewed, modified and combined, and are thus made ready for the long-overdue experimental tests. An essential mathematical preliminary, the theory of straight vortices, is discussed in an appendix.

FOREWORD

The U. S. Army Limited War Laboratory has an interest in the potentialities of vortex rings in applications to problems of counter-insurgency. Mr. Serge Zaroodny of the U. S. Army Ballistics Research Laboratory was known to have done some work in this field. Because of this, BRL was asked on 29 January 1965 to prepare a "study of the state-of-the-art in the field of vortex rings with some attention to their generation and decay...". The paper that follows was prepared in response to the LWL request.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure

INTRODUCTION

The fascination of the vortex ring lies in the fact that it is nature's way of transporting a finite body of fluid through some ambient fluid by rolling rather than by sliding; so that the role of viscosity of the fluid is minimized. Hence in aerodynamics the vortex ring is ubiquitous: whenever a finite body of fluid is moving through some ambient fluid without external constraints, or is made to move so, it probably moves so because it has been formed as a vortex ring, or it is forming itself into a vortex ring. It can indeed be said that this way of transport obviates the basic difficulty of aerodynamics, the boundary layer; and that the transport is achieved practically without drag. For this reason the theory of the vortex ring is one of the relatively few hydrodynamical phenomena which in principle can be treated sufficiently well by the classical theory of potential flow; and this is how all theories of the vortex ring are indeed construed. In reality, of course, the phenomenon is not entirely free of this basic difficulty. Since in the potential flow there is some shear, and a consequent diffusion of vorticity, there survives some analog of drag: there is a weak wake, the ring grows, decelerates and dissipates. Yet this dissipation is distinctly secondary to the fundamental "drag-less", or steady, character of this phenomenon.

Sometimes the vortex ring is conspicuous, precisely because its steadiness, or perseverance, is of interest; sometimes it is there, but remains unsuspected; often it is rather overshadowed by other complexities and constitutes but a secondary aspect of some more complicated phenomenon; but often it is only a helpful abstraction, and is of interest not so much by virtue of its steadiness, or precisely because of its mortality; for the vortex ring is the basic constituent of any turbulence. Thus, the vortex ring is obvious in a lenticular cloud around a mountain, in squid's ink, in the disturbance left on the surface of the water by an oar stroke, in an explosion of a missile in flight, or in the mushroom of an explosion at the surface of the earth (e.g., sometimes in the mushroom of an atomic explosion the thin fiery thread of the core of the vortex ring is distinctly visible) and on some rare occasions - for reasons which are still not entirely understood - a large and long-lasting smoke ring forms out of the blast from a large cannon. A vortex ring is easily recognizable in the billowing of smoke from a large (and unsteady) fire, and in the nearly-spherical puffs of cumulous clouds; it is less clearly defined, but still unmistakable, in any muzzle blast, in the intermittent flow around the spike of a spike-nosed projectile, and in the early stages of any wake from a moving body - whether a rushing train or a projectile. It is less widely recognized, but is no less essential, as a constituent, or "fire structure", of a meteorological "thermal", as radar "angels" [12], in the paths of soaring birds [11], and in the structure of the formations of birds and fishes.

But in such phenomena as the laminar flow in a pipe, the rocket exhaust, the flow about a projectile, or in the general turbulence, the vortex ring becomes an abstraction, or an individual vortex line or vortex tube, not possessing all of the features of a well-defined toroidal ring. In such cases we have, rather, a combination of many distributed vortex rings. This "assembly" of rings often is similar to a cylindrical coil of wire, rather than to an individual circular wire loop; viz., it may be a tubular core of a "long" ring, continually being regenerated (or "re-wound") on one end, and continually flaring and breaking out into individual, not particularly well formed, rings on the other end. Yet even in such case the resort to the concept of vortex rings often allows us to discern a somewhat simpler system in the general complexity. For instance, attention is often enough drawn to the fact that cigarette smoke starts as a smooth and long jet, and then quite suddenly breaks into swirls. The terms "laminar" and "turbulent" here, of course, only describe rather than explain. The vortex ring here is invisible, consisting of the air surrounding the smoke jet. As the smoke is sucked into this ring along its axis, it accelerates, and the narrowing of the jet counteracts its spread due to the diffusion; thus, the steady, or laminar, character of flow is emphasized. As the smoke passes to the front of the rings, it rather suddenly deviates from the axis in one or another direction, rotating about this invisible core; thus the apparent sudden onset of the turbulence is basically the "steady" behaviour of the fluid in a vortex ring. In the meanwhile the core moves up, too, though at a lower velocity, and in a more complicated manner than in an idealized single vortex ring; and the more quiet the ambient air, the farther this smooth jet, in this tubular core, extends before the upper end of this core begins breaking up into individual rings. With a little concentration one can almost see the individual rings about which the smoke strand is "wrapping itself up". Similarly, a vortex ring is at least qualitatively recognizable in such phenomena as projectile wake, the profile of the turbulent boundary layer, the formation of the sub-laminar boundary layer, etc.

In the demise of a vortex ring one can discern two aspects, which may be termed decay and distortion; meaning by <u>decay</u> the effects of diffusion, with a hypothecated preservation of the circular symmetry of the ring; and by <u>distortion</u> the effects of the inherent instability of the circular shape of a vortex, with the neglect of the decay (the straight and the circular shapes of a vortex are but shapes of unstable equilibrium). By far the simpler, and the prior, form of the demise is the decay: the ring seems to decay first, and only then starts distorting, passing into the general large-scale turbulence. In air, particularly, the decay is quite strong, and the distortion is entirely too complicated for any detailed consideration: a ring which starts distorting practically "blows up", or disappears. Yet even in the swirling of tobacco smoke the vortex-ring nature of turbulence is strongly pronounced: each segment of a vortex moves as though it were a part of a vortex ring and the wonder is not so much that diffusion occurs, but how long the individual strands of smoke persist, and how weak is the diffusion. This mechanism of mixing is in fact much stronger than mere diffusion: vortices move so as to be always in contact with a fresh mass of air, thus speeding up the mixing. The distortion is more readily seen in water rather than in air, apparently because the decay there is less rapid, due to the lower kinematic viscosity. A drop of ink or dyed water gently let into still water presently "cascades", or breaks up into a complicated, and yet remarkably systematic, tangle of fine (that is, still not thoroughly diffused) thread [21, 22], Figs. 19-20; yet even there the decay distinctly precedes the distortion. The drop at once forms a ring which rapidly travels down, slowing down and expanding, viz., decaying. The subsequent distortion, however, is so remarkably systematic only because of the presence of another, little understood, vortex phenomenon: the axial movement of the fluid <u>along</u> the vortex, viz., the spilling of the dyed fluid down the vortex as if down a chimney; this effect of gravity is unmistakable even when the differences in density are very slight. As the imperfections of the circular shape of the ring make it approximate a polygon, the corners of this polygon act as rings of smaller radius, having higher velocity; they travel ahead of the straighter sections, and as the ink spills into them, form a horseshoe vortex which presently forms a "derivative" (viz., not "seamless") ring, to the center of which there runs a vanishing "umbilical" vortex pair (Fig. 19); this ring then breaks in two (Fig. 20), and so on. Thus, theoretically, a single continuous circuit could be traced through the resulting tangle; but the elementary concepts of the continuity of the circulation around such a filament in the "inviscid" fluid will require much modification. The stretched umbilical vortex pair vanishes because of viscosity.

Technological applications of the vortex ring as a purposely-made artifact - as distinguished from its "natural", or unintended, occurrence - seem to have been limited, so far, to toys and advertising, or tobacco and perfume. The latter use is particularly interesting

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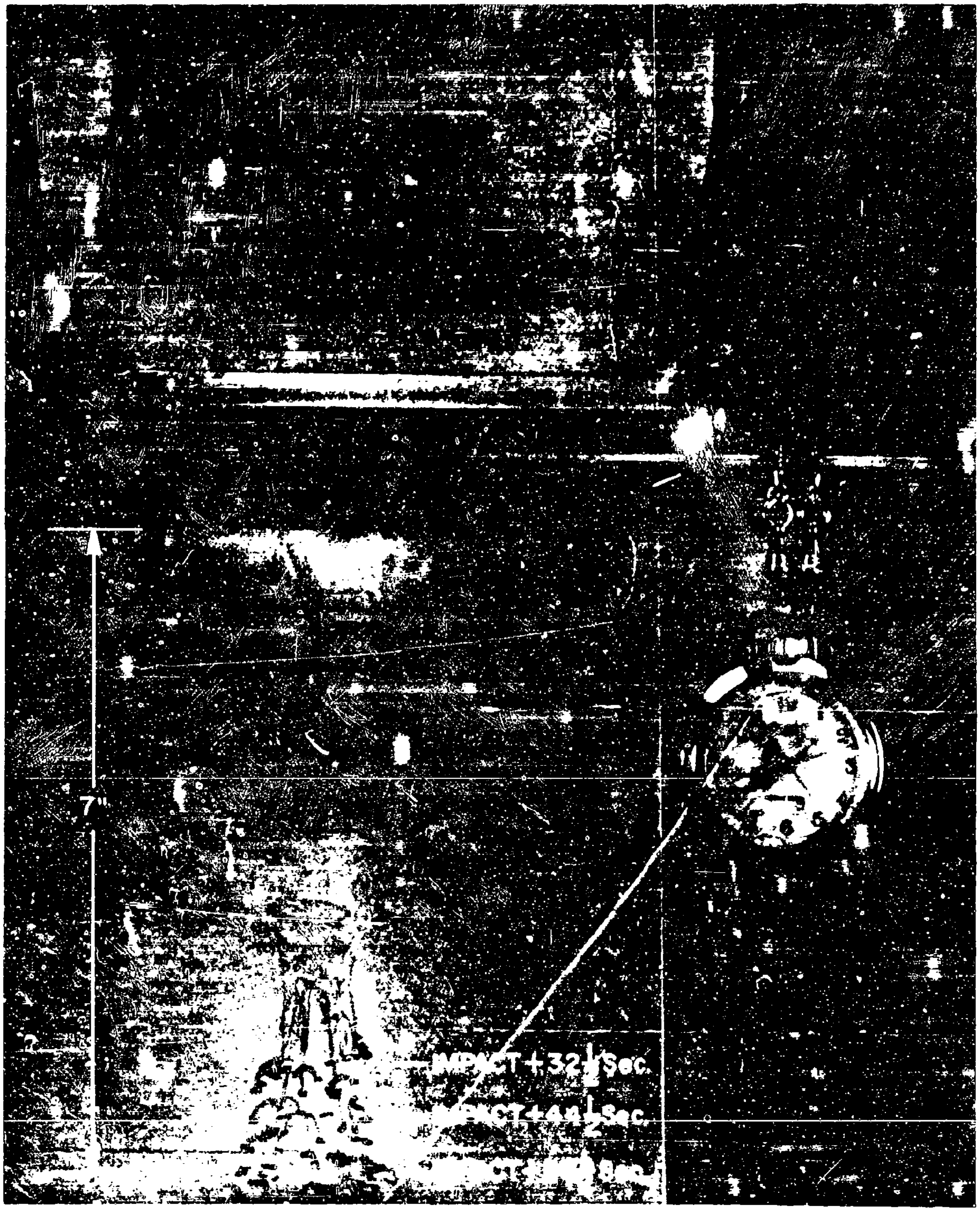

Figure 19

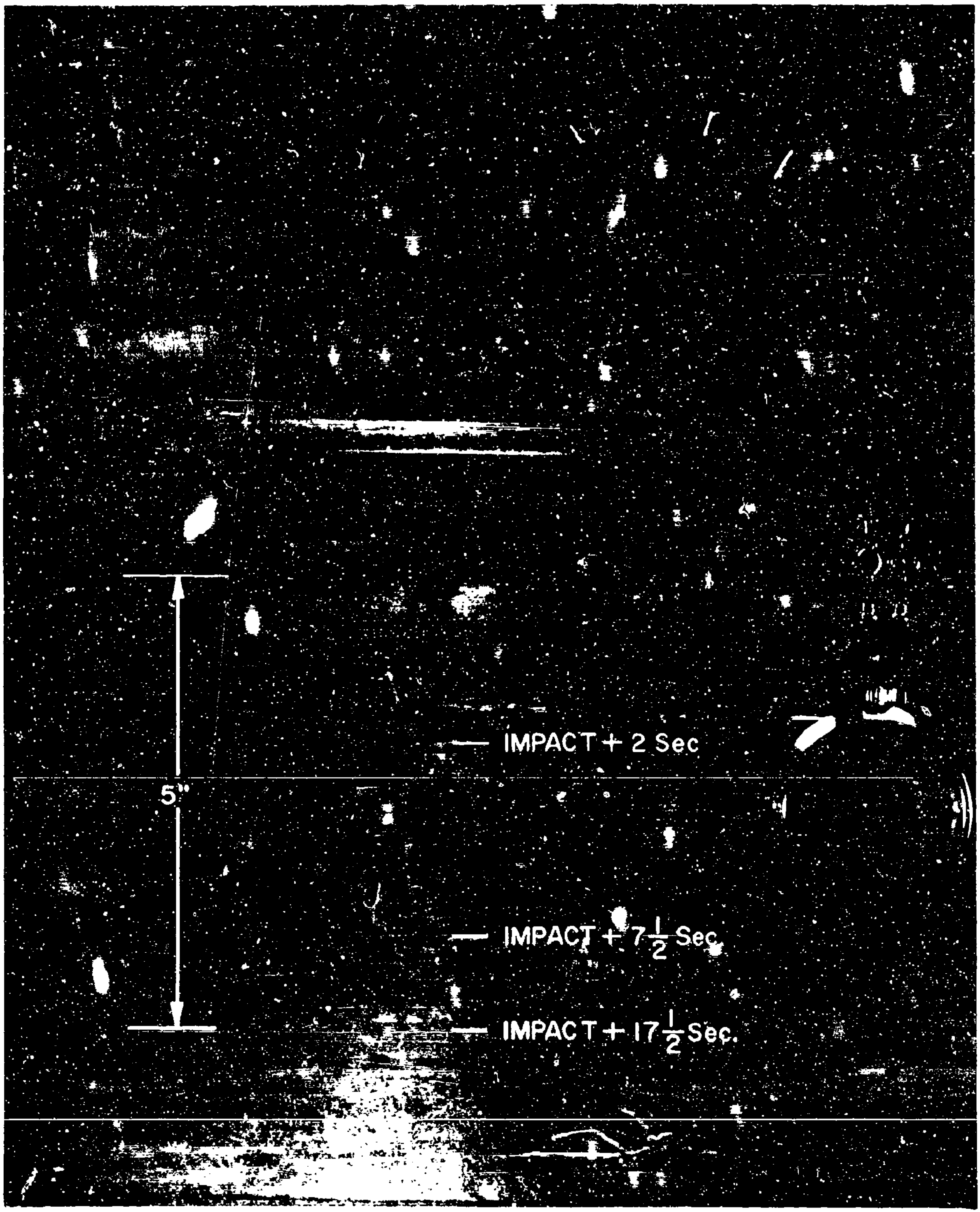

Figure 20

because of the remarkably weak concentrations of gas sufficient for olfactory sensation. Such occasional uses of the vortex ring (including, perhaps, theatrical technology) can be expected to recur, but are not likely to lead to much serious study. Less innocent applications suggest themselves readily. Most probably, the principal applications would lie simply in the unconstrained, but directed and selective, transport of gas, smoke, dust and aerosol. Used for such purpose, the vortex ring will always have difficulty in competing with a projectile; but it is easy to visualize many diverse hypothetical circumstances of human conflict in which a vortex ring may have, over a projectile, the advantages of stealth, silence, perhaps invisibility, and puzzlement: e.g., a bush on the wayside, or a particular individual in a mob, might not be worth the expenditure of a tear-gas grenade, but might be worth a few vortex-ring puffs; a quick dash through a field of fire might be only advertised by the launching and the burst of a smoke grenade, but may conceivably be disguised by smoke conveyed as vortex rings. In any case, man's reaction to smoke or gas conveyed in vortex rings is bound to be entirely different from his reaction to a grenade. One ought not suppose, though, that the effects of a vortex ring manifested in the transport of energy, and perhaps of momentum (say, by production of disturbances of pressure at a distance) must always be non-competitive with a projectile: there has not been much serious effort to produce particularly strong and efficient vortex rings, and we really do not have enough experience to conclude that a vortex ring must always be "impractical". In matters of unconstrained transport of fluid, the vortex ring might even echo the role of the laser in optics.

However, were we to attempt a serious feasibility study of technological applications of vortex rings, we could not do so with the present standard theory: for today we lack not only the experimental techniques for detecting, recognizing and evaluating a passing vortex ring, and the quantitative understanding of the decay (not to speak of the distortion) of a vortex ring: but we lack even a sufficiently clear understanding of the structure of the ring. The known theories of the vortex ring appear to have arisen mainly through the curiosity of mathematicians, as exercises in the general aerodynamic theory; one may suppose that the purely analytical approach has reached its point of diminishing returns, and that the further quantitative progress will ensue mainly through the use of computing machines and experiment.

EXISTING THEORIES OF VORTEX RINGS

With an exception that will presently be noted [15], there seem to exist but two theories of vortex rings, both heavily qualified by assumptions.

The classical, and by far the more popular one, is that of Helmholtz, dating to 1858; or to 1867, when it was expounded by Sir W. Thomson, who considered it in connection with his "vortex atom" theory. Today his theory may be viewed as one of the precursory gropings toward quantum mechanics, particularly interesting because coming from the proverbial exponent of "classical" physics. For our present purposes, however, it can be said simply that, with Helmholtz's setting aside the aspects of the demise of the vortex ring, Kelvin was particularly impressed with the aspects of the permanence of the ring [26]. This theory is concerned with thin-cored rings; heuristically, this is merely a direct mathematical analog of the magnetic field of a current-carrying wire, and is not particularly meant to apply to actual, reasonably diffused, vortex rings. The main accomplishment of this theory is a resolution of certain mathematical, and rather artificial, indeterminacies. This theory is particularly interesting because it describes quite a variety of possible structures of a vortex ring.

Less widely known, and perhaps unnecessarily modest, is M. J. M. Hill's theory, dating to 1894. This theory is concerned mainly with the flow inside the core, and assumes that vorticity has been thoroughly spread throughout the core in a certain manner. From the summary by Lamb [1], this theory is known by its one thoroughly-completed phase, the theory of the "spherical vortex": i.e., a vortex ring in which the body of transported fluid has a spherical shape. In principle, the theory describes an even greater variety of possible structures of a vortex ring than Helmholtz's theory does; but the flow outside the core is manageably simple only for the spherical-vortex case.

Both of these theories use three basic assumptions:

1. Inviscid fluid; viz., they consider only the steady vortices. This restriction, however, still might provide a good picture of an instantaneous structure of a vortex ring.

2. Incompressible fluid; viz., both theories restrict the problem to that of hydrodynamics. In comparison with the other assumptions, this restriction seems quite reasonable.

3. The field of flow is sharply divided in two regions: the field of outer, irrotational and potential flow, and the field of a core containing vorticity, the latter being assumed to be distributed uniformly in a certain sense. Mathematically, this feature is much like the "lumped parameters" approximation to any continuum problem, which is usually forced upon us whenever we have difficulty in specifying the detailed distribution of a physical quantity. This artificial division of the field into the rotational and irrotational flows should not be confused with another important, legitimate and sharp division: that into the fluid transported with the vortex ring, and the ambient fluid. The latter division occurs in the outer flow, the core forming distinctly a part of the transported fluid. In a real ring this division is somewhat blurred by the decay [15].

With assumptions 1 and 2 the real problem of the hydrodynamics of a vortex ring lies in the fact that fluid has both vorticity and "shear", of these two concepts vorticity being the simpler one (see Appendix). Assumption 3 divides the problem accordingly. In the outer flow vorticity is "assumed out of problem", and in Helmholtz's core shear is "assumed out of problem", viz., the core is viewed as a rigid body not having any shear deformation. In reality, of course, vorticity is negligible only in regions far from the core, while shear is negligible, in addition, only near a certain curved "axis" of the core. Everywhere else shear exists, and can be said - somewhat contrary to the famous Helmholtz's dictum (see Appendix) - to "generate" vorticity; or more precisely, to cause a redistribution of vorticity. Yet assumptions 1 and 3 taken together are eminently legitimate at this stage of the game; they purport merely to describe approximately the instantaneous structure of a real ring.

Helmholtz's and Hill's theories of the vortex ring are briefly reviewed in the encyclopaedic text by Lamb [1], where they are treated apparently merely as exercises in the introduction of the more advanced methods of hydrodynamics preparatory to the more popular problems of aerodynamics; but these theories seem progressively abridged, and even omitted entirely, in the more modern texts [2, 3, 4, 6] , perhaps for the reason that vortex rings as such have not had, so far, direct application to aeronautics. It has been said that a proper modern theory of vortex rings should start with Navier-Stokes equations; but such an approach is too laborious for our present purposes. It will have its place some day in a formulation of a theory of the decay of vortex rings. The classical introductory theories, reviewed here, constitute in effect a series of shortcuts to the Navier-Stokes approach, and deserve a careful consideration. Some of these shortcuts might be not especially fortunate, and if applied mechanistically lead to contradictions, infinities, indeterminacies, etc.; but something useful is to be learned even from such pitfalls, and most of these shortcuts are excellent idealized models of reality.

We feel that the shortcomings of the existing theories lie not so much in any lack of merit of assumptions 1, 2, 3, which are so natural at this stage of the game, but in the fact that these theories have traditionally been viewed as isolated mathematical exercises. The facts that Helmholtz's and Hill's theories have never (to our knowledge) been combined, nor subjected to experimental tests, are cases in point. It may even be said figuratively that these theories, viewed as primitive preliminaries to some omniscient, but in practice unachievable, general theory, have served to discourage engineers from organizing thorough experiments, or studies of technological applications, of vortex rings. In fact, of course, these theories were meant to help precisely this type of student; for, while he does need some theoretical background, he does not have to be a full-fledged aerodynamicist, or expert on partial differential equations. By the time he does become such an expert, he will probably be lured to subjects more glamorous than vortex rings.

MATHEMATICAL PRELIMINARIES

The following peculiarities distinguish the theories of vortex rings from those of straight vortices, discussed in the Appendix.

Because of the axial symmetry, the obvious coordinate system is the cylindrical, rather than rectilinear; e.g., a study of Hill's original paper [28] would show that treatment in the rectilinear system is quite feasible, but unnecessarily laborious. Traditionally, the axis of the ring is denoted by x, and the component of velocity of the fluid along that axis, by u. Two important and readily-interchangeable alternatives should not be confused: the coordinate system may be stationary, or it may move with some steady velocity U. In the former case the flow is distinctly not steady: e.g., the plane x = 0 is the central plane of the ring only at the instant when the ring is passing through that plane; u at infinity is zero. In the latter case u at infinity is - U, and the ring velocity is diminished by U. In particular, if U is in fact the steady velocity of the ring, the flow is steady, viz., the streamlines are also the pathlines; the plane x = 0 remains the central plane of the ring, and the pathlines passing through certain stagnation points delineate the volume of the fluid which is transported with the ring. This concept applies, of course, only by the virtue of assumption 1, for in reality the ring is decelerating.

The radial coordinate, viz., the distance from the axis, $\sqrt{y^2 + z^2}$ is often denoted by $\tilde{\omega}$; but little confusion, greater convenience, and a better parallel with the mathematics of that essential preliminary, the vortex pair, results if this coordinate is denoted simply by y. The corresponding component of fluid velocity is traditionally denoted by v, as in the rectilinear system. An occasional reversion to the rectilinear system will remain necessary, but it will be clear in context, and will not call for a change of notation.

The third, circumferential, component of fluid velocity is generally taken as zero; this may be reckoned as our fourth basic assumption, 4. There are some literally-nebulous indications that this may not necessarily be so with actual rings occurring in the atmosphere; but such refinements are outside our scope.

To describe the field of flow fully it is both necessary and sufficient to describe u and v in terms of x and y. Most of the consequent mathematical difficulties stem from this basic problem of display of two functions of two variables. Yet, here this problem of display is so much simpler than the basic problem of hydrodynamics (which may be that of displaying four functions, u, v, w, p as functions of x, y, z). In the more general and advanced hydrodynamics it is customary to resort to slightly more sophisticated concept; and in particular, it is often desirable to specify a single scalar function of the coordinates x and y from which both u and v are in principle determinable. Three such possible functions should be mentioned, though only the third seems used.

1. This function could be simply the velocity potential ϕ, such that the velocity $\underline{q}$ is $-\nabla\phi$. In our case this simplifies to

$$u = -\partial\phi/\partial x, \quad v = -\partial\phi/\partial y$$

However, with vortices this concept is not convenient. It is convenient only for the specification of those parts of u and v which are the result of the presence of definite sources and sinks in the flow; e.g., for the uniform flow of velocity $-U$, $\phi = Ux$. One limitation of this concept is that only the irrotational flow (the region where $\nabla \times \underline{q} = 0$) can be so described; e.g., this concept cannot describe so simple a phenomenon as the rotation of a rigid body. Another limitation is that when the streamlines are closed, i.e., link a core, ϕ is multiple-valued, and so requires an adoption of some convention. Still another limitation is that with vortex rings expressions for ϕ are quite involved [1]. However, if u and v are known in any way, one can always estimate ϕ as $-\int \underline{q} \cdot d\underline{s}$, where $d\underline{s}$ is the element of distance, and draw the equipotential surfaces; these surfaces can be of help in visualizing the flow.

2. This function could be the vector potential of velocity, say $\underline{P}$, a function such that $\underline{q} = \nabla \times \underline{P}$. In general, specification of $\underline{P}$ implies specifying three scalar functions of x,y,z, and so is no simpler than the direct specification of velocity components. But with vortex rings the components of $\underline{P}$ in the x,y plane can be put equal to zero, and the surviving circumferential component, P, is in fact a scalar function. There results

$$u = P/y + \partial P/\partial y, \quad v = -\partial P/\partial x,$$

which now holds in the presence of vorticity (when $\nabla \times \underline{q} \neq 0$) as well; and there are no difficulties with the multiple-valuedness of $\underline{P}$ when streamlines loop a vortex. The uniform flow $\underline{q} = -\underline{i}U$ can be represented by $P = -Uy/2$ in cylindrical coordinates (and alternatively, by $\underline{P} = \underline{j}Uz$ or by $\underline{P} = -\underline{k}Uy$ in rectilinear coordinates). There are, however, certain difficulties. The fact that only "divergenceless" (viz., "solenoidal", or incompressible) flow can be so represented is hardly a difficulty with vortex rings; but the fact that a simple rotation in the x,y plane turns out to be represented by a rather strange function $\underline{P}$, is indeed a difficulty. The physical significance of $\underline{P}$ (which is that of the "potential of the circulation") is far from obvious, and the lines of P = const can easily be confused with streamlines. So this concept is used only in derivations, rather than in the final specification of a flow field; it turns out that this concept is useful only as a peculiar sophisticated simplification of Ampere's law - which law could be used more directly.

3. The customarily used function is the Stokes stream function ψ. In our axisymmetric case, in particular, the lines of ψ (x,y) = const are the streamlines; and the value of ψ is $1/2\pi$ times the <u>rate of flux</u> of the fluid through the circle which is centered on the axis x and passes through the point x,y (lying in the plane normal to $\underline{x}$). In fact, streamlines are much more convenient for visualizing the flow than the lines of ϕ = const. The relation between ψ and $\underline{P}$ is $\psi = -yP$. This is derived by noting the flux of the fluid through that circle is the area integral of $\nabla \times \underline{P}$, and hence can be computed, by Stoke's theorem, as the line integral of $\underline{P}$ around the circumference of that circle. Thus the computation of ψ amounts to a sophisticated application of Ampere's law, which for a while may remain unrecognized by the student. Fluid velocity is determined from ψ by

$$u = -(\partial\psi/\partial y)/y, \qquad v = +(\partial\psi/\partial x)/y$$

[illegible]ysical significance of which expressions becomes more clear if [illegible]flects that $\partial\psi/\partial y$ is the rate of flow through an elementary unit [illegible]s, while $\partial\psi/\partial x$ is the rate of flow through an elementary unit l[illegible] of a cylindrical surface, except for the factor $1/2\pi$; the divi[illegible]on by y represents the division of the rate of flux by the area $2\pi y$. The concept of ψ is particularly useful when computation of u and v is easy; this is in fact the case with Hill's vortices, but less so with Helmholtz's flow. The dimensions of ψ are L^3/T, while those of ϕ and P are L^2/T. For uniform flow of velocity u = - U, we have $\psi = Uy^2/2$, and the addition of this latter function to ψ is equivalent to a transfer into the moving coordinate system.

The concept of an analogous, but different, function also named "stream function" is particularly useful in the theory of straight vortices (see Appendix), where it dovetails with ϕ to form a complex potential. With vortex rings, however, those mathematical tricks are denied us. In brief, the condition of the irrotationality of the flow, $\nabla \times \mathbf{q} = 0$, which is the necessary and sufficient condition for existence of ϕ, has in the cylindrical coordinate system the form $\partial u/\partial y = \partial v/\partial x$, just as with the straight vortices; and it does provide one of the two Cauchy-Riemann equations necessary for the existence of a complex potential. But the condition of incompressibility, or of "divergencelessness", of the flow, $\nabla \cdot \mathbf{q} = 0$, now has the more complicated form $\partial(yv)/\partial y = - \partial(yu)/\partial x$, or $\partial u/\partial x = - \partial v/\partial y - v/y$; this is the necessary and sufficient condition for existence of ψ, but because of the presence of the addend $- v/y$, it is no longer the second of the Cauchy-Riemann equations. The similarity of the latter expression to the relations between $\underline{P}$, u and v leads to Clebsch's transformation, but that is entirely too advanced for our present purposes.

The differential equation for ϕ, holding for the irrotational region only, is the Laplace equation $\nabla^2\phi = 0$, which in our coordinates becomes $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + (\partial\phi/\partial y)/y = 0$. The differential equation for ψ, holding everywhere, is $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 - (\partial\psi/\partial y)/y = y\sigma$, where $\sigma = \partial v/\partial x - \partial u/\partial y$ is the vorticity; the equation arises from the definition of σ, with u and v expressed in terms of ψ. Both equations hold in coordinate systems moving with any velocity U; the differences in the patterns as affected by U are a matter of the boundary values. These equations are sometimes referred to as the principal forms of the Stokes-Belhami equation [20].

HELMHOLTZ'S VORTEX RING: GENERAL DESCRIPTION

The essential assumption of Helmholtz's theory is simply this: the field of the irrotational flow is that of an "effective line vortex", viz., a core of zero thickness.

The assumption is eminently reasonable at fair distances from this idealized core; but in the vicinity of this core some difficulties naturally arise. Carried all the way, this assumption leads to absurdities - infinite energy and infinite velocity of the ring - and requires an obvious modification: the assumption of a finite, though small, thickness of the core. The subsequent further assumption that the cross-section of the core is circular, with the center of the circle on the line vortex, and that the vorticity in the core is uniform (viz., the assumption of the "rigidity" of the core) is a carry-over from the theory of straight vortices; usually it is not spelled out specifically, but is rather inherent in the basic assumption. Both the finiteness and the rigidity of the core are contradictory to the basic assumption; but it will be shown that with a few refinements both of these auxiliary assumptions can be made rather rigorous.

The field of Helmholtz's flow is simpler in the stationary coordinate system. Although this field is that of the very simple and essential concept in electromagnetics, the magnetic field of a circular current-carrying wire, it is not entirely simple from the computational viewpoint; and for this reason, perhaps, is not discussed in most standard texts on electromagnetics. If the radius of the ring and the strength of the vortex are taken as units, the stream function is the dimensionless quantity

$$\psi(x,y) = - y \int_0^{\pi} \frac{\cos \theta}{r} d\theta$$

where θ is the angle in the plane of the ring from the axis y to the element $d\theta$ of the unit circle, and $r = r(x,y,\theta)$ is the distance to that element from the point in the x,y plane. If the radius is a, and strength is S, naturally ψ carries also the dimensional factor Sa, and velocities derived from it carry the factor S/a. This field is illustrated in Figs. 1 a, b. Fig. 1a, taken from [1], is properly drawn for equal increments of ψ . Fig. 1b is drawn, as an exercise necessary for drawing the more complicated patterns, for equal increments of $y < 1$ in the plane $x = 0$; this form of exhibit of ψ is convenient and sufficient for our purposes, for with few refinements the values of ψ on the streamlines can be determined, whenever needed.

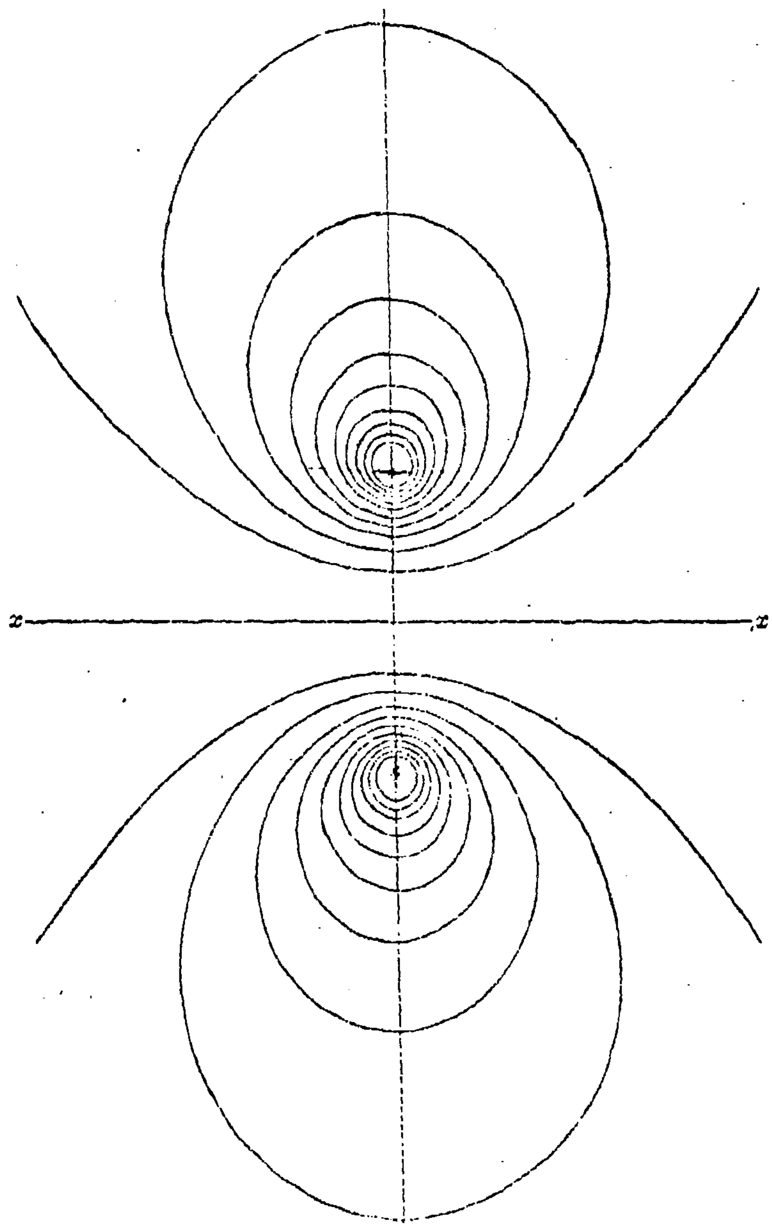

Figure 1a

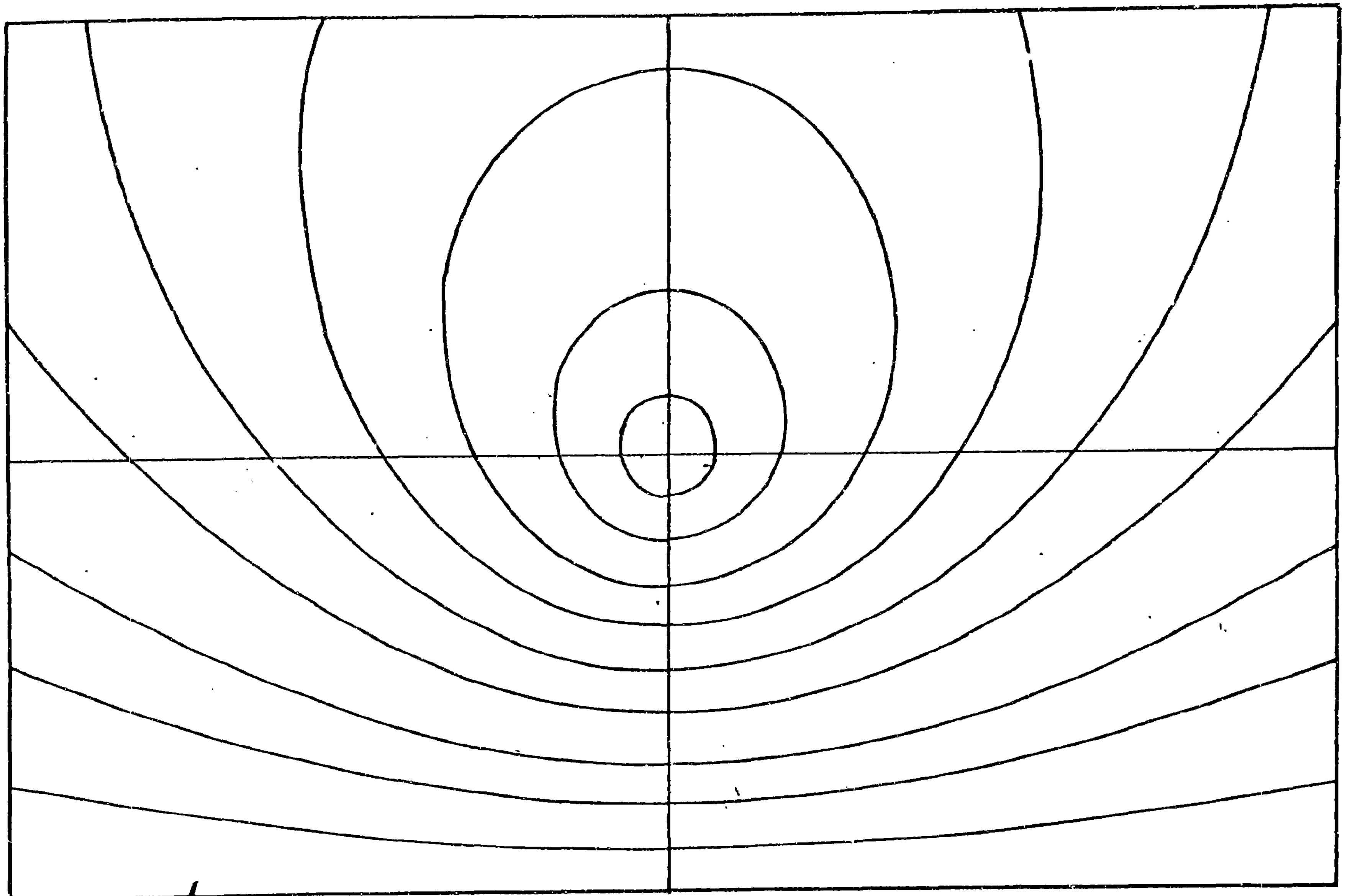

Figure 1b. Streamlines of an idealized vortex ring in a stationary coordinate system at the instant when the ring passes through the plane x = 0.

It is now necessary to note that the integral is "improper"; also, at the vortex the stream function, all fluid velocities, and the kinetic energy of the fluid are infinite. Hence the determination of the velocity of Helmholtz's ring requires a resolution of certain mathematical indeterminacies. At the center of the ring $u = \pi$.

If this field is now viewed in a coordinate system moving with some velocity U, the pattern of the streamlines changes as it is shown in Figs. 2-6, drawn for $U = \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4$. With $U < 0$ the pattern is that of the magnetic field of a coil reinforcing a uniform field [17](or, of a flow through an orifice); with $U > 0$, that of a coil bucking the uniform field. With $0 < U < \pi$ there are two "stagnation" points on the axis (Figs. 2-4); for $U \to 0$ these points recede to infinity (Fig. 1); for $U = \pi$ the two points coalesce (Fig. 5); with $U > \pi$ the "stagnation" region becomes a ring within the vortex ring (Fig. 6). If U is actually the velocity of the ring, as mentioned, the streamlines passing through the stagnation points delineate the volume of fluid <u>transported</u> with the ring. For small U this volume first appears as a very large sphere; then it generally resembles an oblate ellipsoid of revolution; then a balloon pinched in the center; and finally it becomes a toroid (and the vortex ring becomes a "true ring", rather than a "blob", since some of the ambient fluid now is flowing through this ring).

In comparison with the theory of the vortex pair (see Appendix) the curious thing is that in principle all of these patterns may indeed exist - although the "true" rings may be difficult to produce. The key feature of Helmholtz's theory is that the velocity U of the ring depends upon the size of the core, becoming logarithmically infinite as core thickness approaches zero. Most of the difficulties of the theory stem from the questions: just what is the relation between core thickness and ring velocity and, in particular, to how thick a core can this theory be extended. The traditional theory, developed in the pre-computer era, tackles this question by way of resolution of the mathematical indeterminacies. Today it is practicable to approach this question in a different, and more natural, manner. We may postulate a velocity U; determine its pattern of streamlines; and inspect whether some of these streamlines may not be such as might represent a rotation of a rigid body. If there is one, and only one, such pathline, it represents the "profile" of a hypothetical rigid core corresponding to the postulated ring velocity U. This was the original object of our study. Our conclusion is that the plausible ring size for a given U is generally considerably smaller than computed on the basis of the commonly-used formula; and

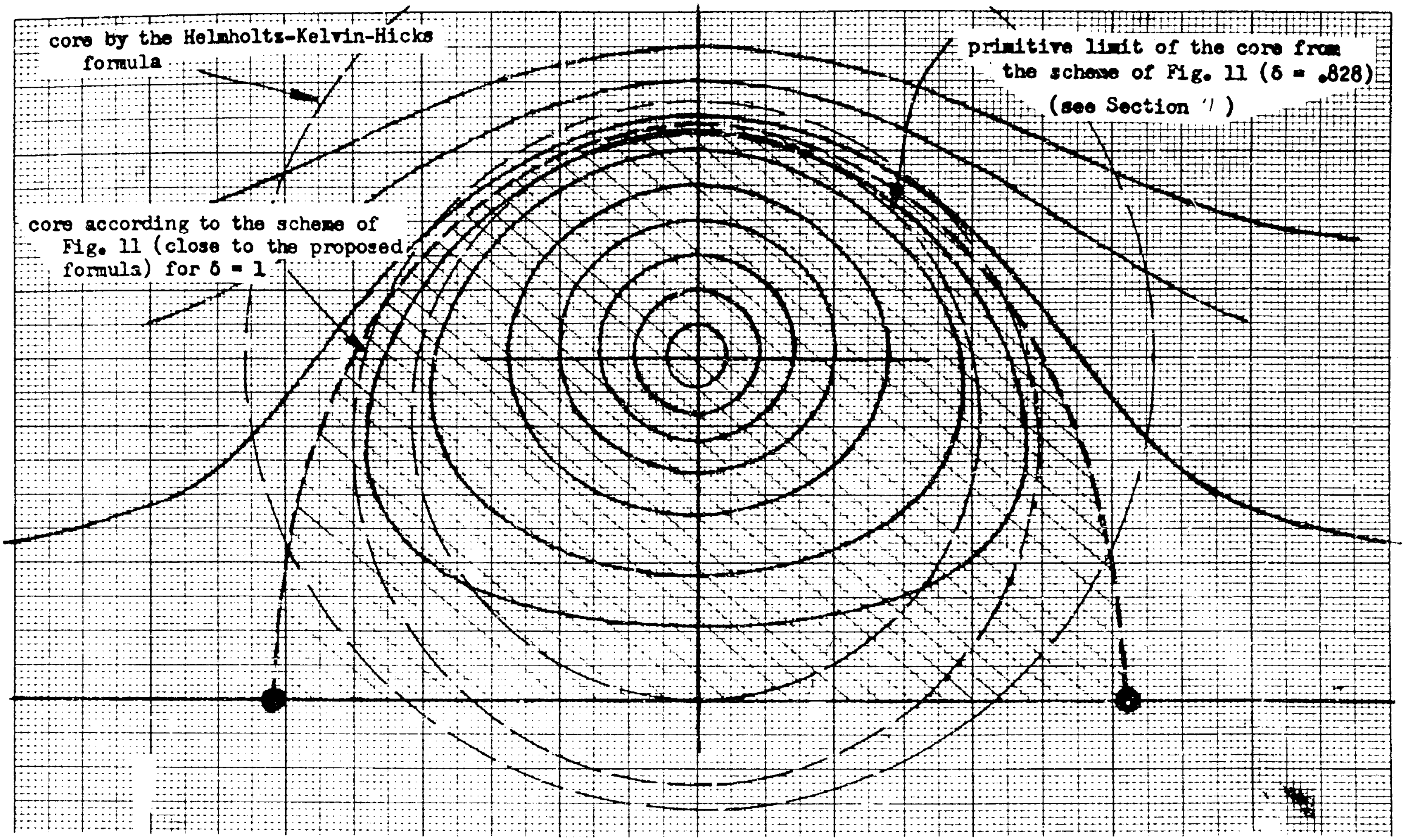

Figure 2. Helmholtz-flow streamlines as they appear in a coordinate system which moves with the velocity $U = \pi/4$. If this were also the velocity of the vortex ring, the pattern would have been stationary (the streamlines would be also the pathlines), the black dots would have been the stagnation points, and the shaded area (the dotted streamline) would have outlined the fluid moving with the ring. However, it appears doubtful that a real ring of so low a velocity can exist.

The thin dotted circles indicate the core of this ring as extrapolated from formulas discussed in the section on "Helmholtz's Vortex Ring: General Description". It can be seen that these theories are not really applicable to so slow a ring.

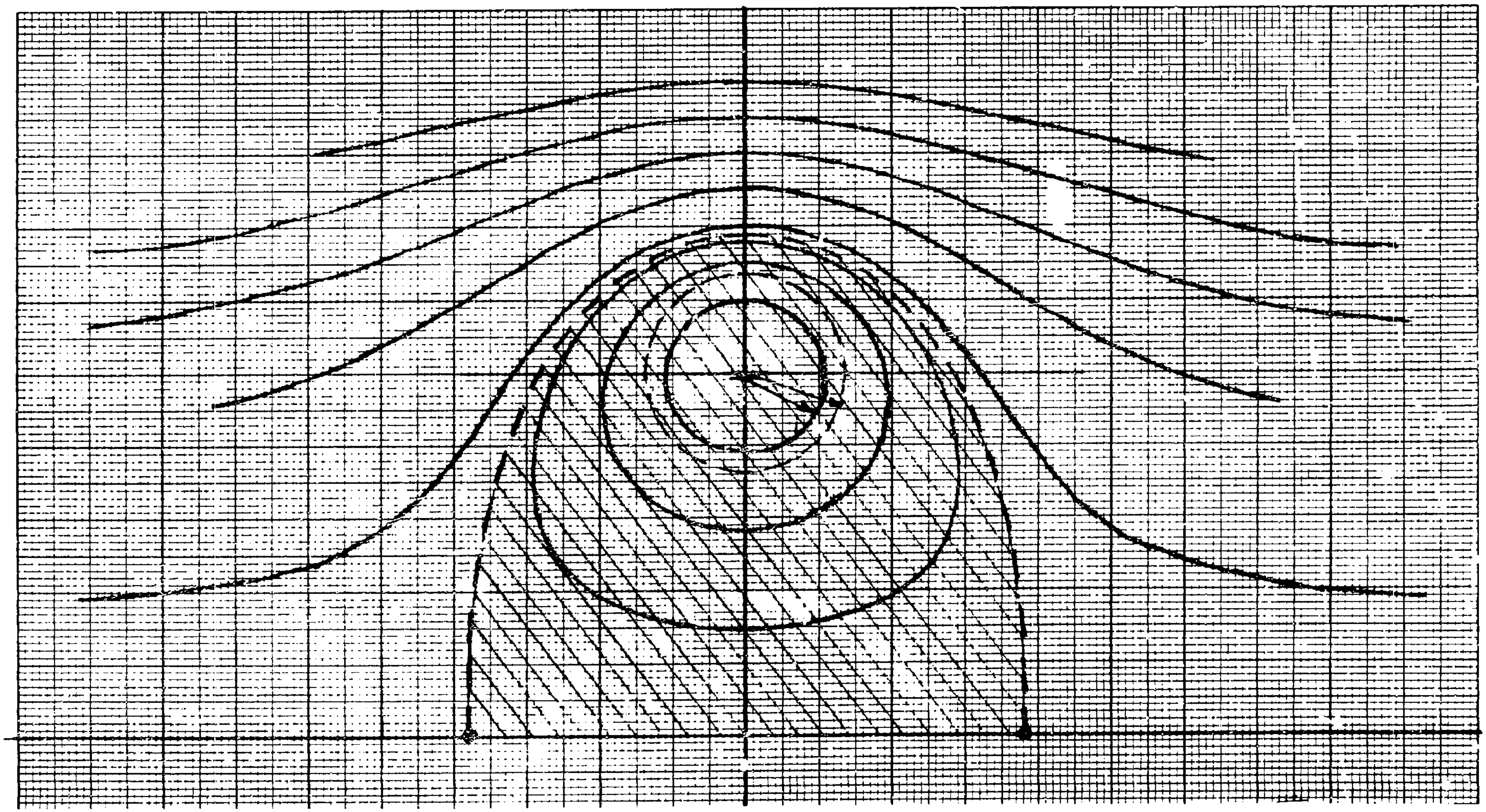

Figure 3. Helmholtz's flow streamlines for $U = \pi/2$. The inner streamline shown is practically a circle that moves as a rigid body. The dotted circle represents the core according to Helmholtz-Kelvin-Hicks formula (see the section on "Helmholtz's Vortex Ring: General Description").

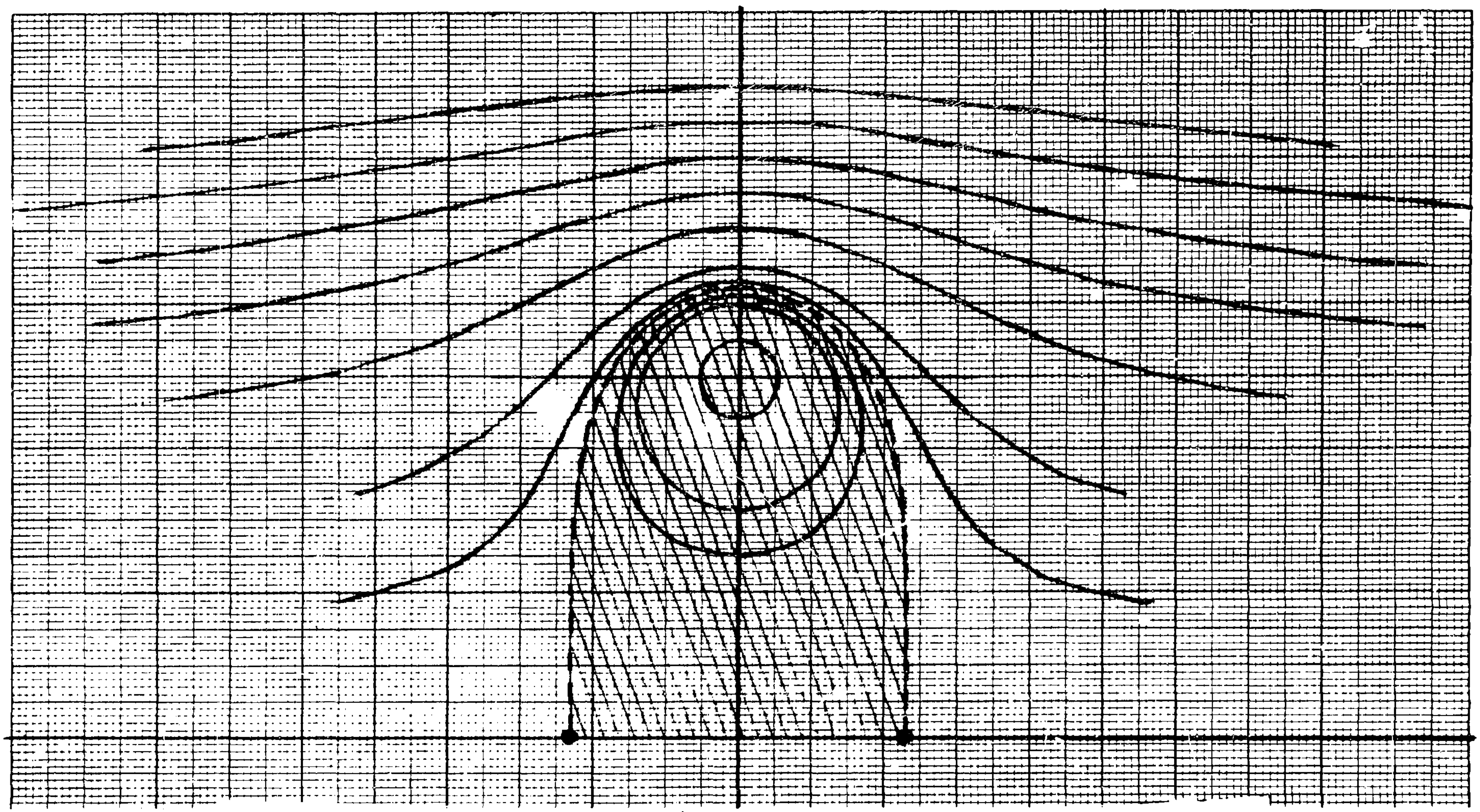

Figure 4. Helmholtz's flow for $U = 3\pi/4$. The streamline which moves as the rigid body constitutes a circle so small it cannot very well be shown on this plot.

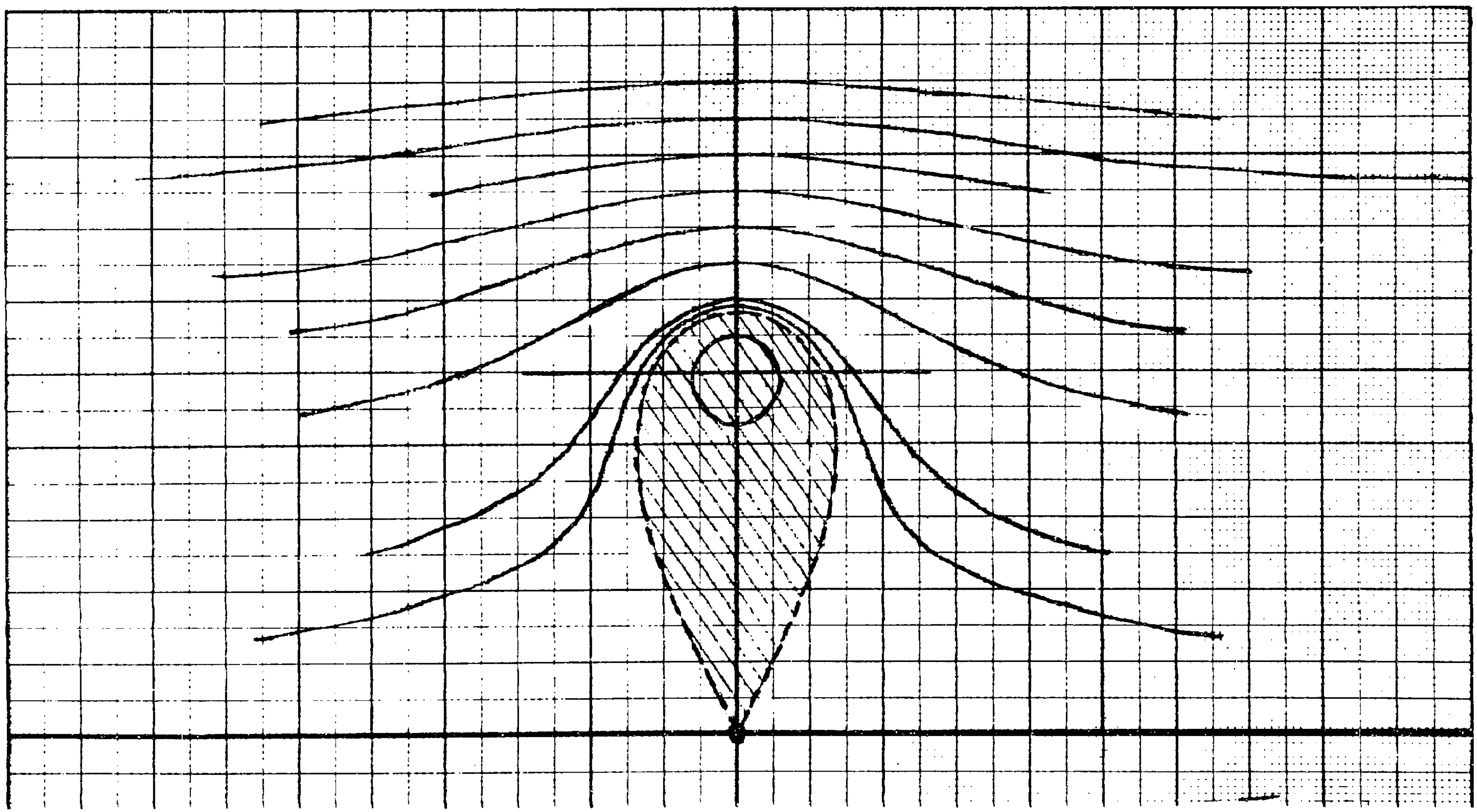

Figure 5. Helmholtz's flow for $U = \pi$, which is the dividing line between the "blobs" and "true rings". Both stagnation points (the forward and the rear) coincide at the center of the ring.

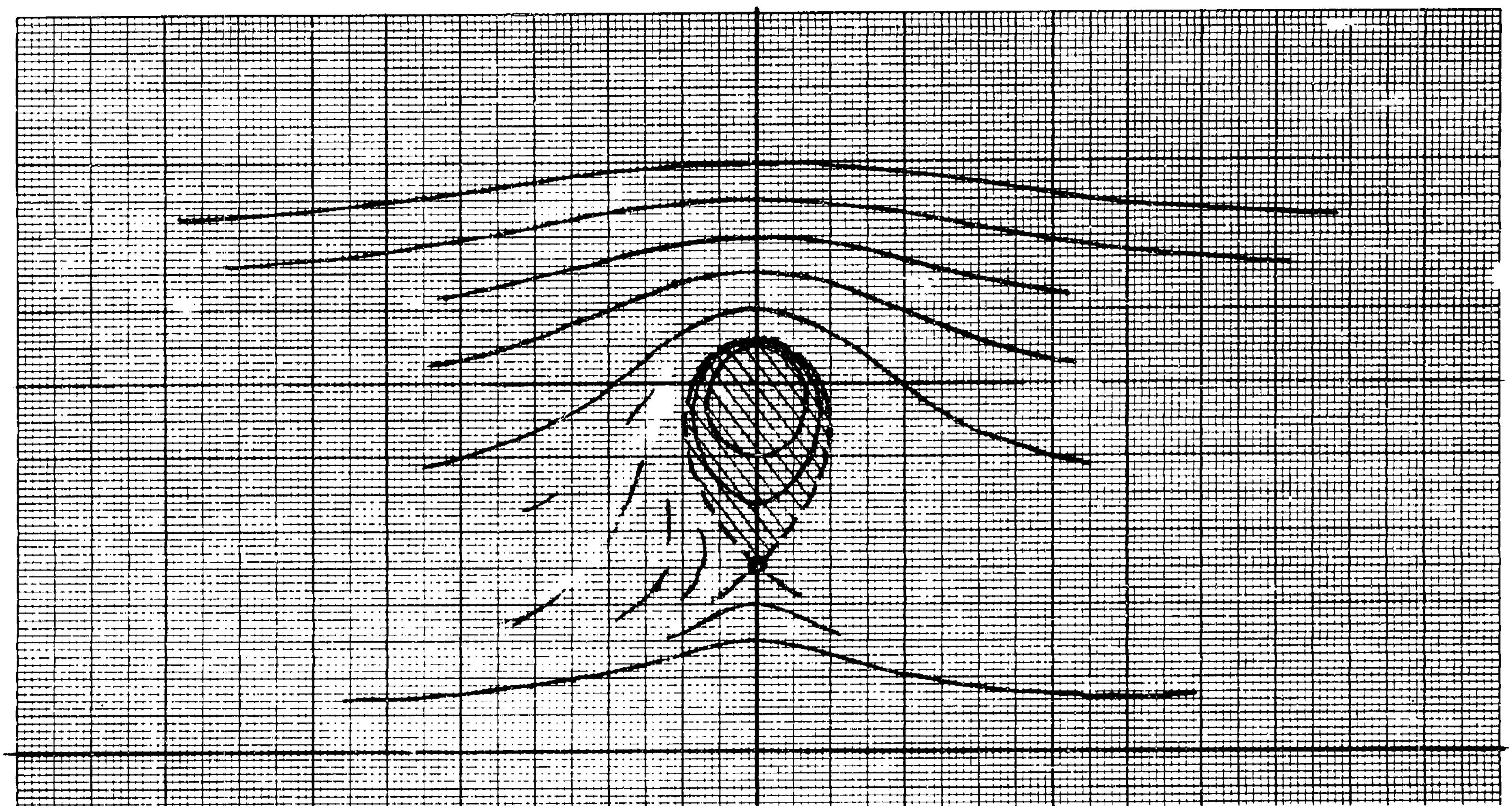

Figure 6. Helmholtz's flow for $U = 5\pi/4$, representative of "true" rings. The stagnation points became a ring within the ring. The size of the core is entirely too small to be represented here. It is doubtful that such rings exist.

conversely, that the plausible ring velocity for a given core thickness is considerably less than is conventionally considered. This is discussed in the section entitled "Velocity of Helmholtz's Vortex Ring"; but here we are not concerned with the thickness of the core as yet.

The components of fluid velocity are given by the integrals

$$u = \int_0^{\pi} \frac{1 - \cos\theta}{r^3}\, d\theta - U, \qquad v = x \int_0^{\pi} \frac{\cos\theta}{r^3}\, d\theta$$

which can be derived from ψ (u requires some integration by parts); however, these integrals are more easily derived directly by Ampere's law,

$$\underline{q} = \frac{1}{2} \int \frac{\underline{r} \times d\underline{s}}{r^3}$$

where the integral is taken over the circumference 2π of the ring; the factor 1/2 is the result of a rather unfortunate hydrodynamic terminology, whereby the strength S of a vortex (circulation/2π) corresponds not to a current i, or to the quantity (magnetomotive force)/4π, but to 2i. Reverting briefly to the rectilinear coordinates x,y,z we have the components

of $\underline{r}$, as $\quad -x; \quad \cos\theta - y; \quad \sin\theta;$

and of $d\underline{s}$, as $\quad 0; \quad -\sin\theta\, d\theta; \cos\theta\, d\theta.$

Thus the x-component of the product $\underline{r} \times d\underline{s}$ is

$$r_y\, ds_z - r_z\, ds_y = (\cos\theta - y)\cos\theta\, d\theta - \sin\theta\,(-\sin\theta\, d\theta)$$
$$= (1 - y\cos\theta)\, d\theta$$

and the y-component is $r_z ds_x - r_x ds_z = x \cos\theta\, d\theta$. The z-component cancels out upon integration around the circle. The factor 1/2 drops out when the domain of integration $(0, 2\pi)$ is changed to $(0, \pi)$. The integrals for u and v then follow.

The computation of the integrals for ψ , u and v, unfortunately, is rather laborious. While it can be done in a number of ways, the customary method is to express these integrals in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\pi/2} d\alpha/\Delta\alpha \quad \text{and} \quad E(k) = \int_0^{\pi/2} \Delta\alpha \, d\alpha \ ,$$

where $\Delta\alpha$ (read "delta-amplitude of α") is defined as the elliptic function [16]

$$\Delta\alpha = \sqrt{1 - k^2 \sin^2\alpha}$$

These integrals can be viewed as known functions of some "independent variable" such as the modulus k, or the parameter k^2, or the modular angle $\sin^{-1}k$, or the complementary modulus $k' = \sqrt{1 - k^2}$, etc.; except that in this application this "independent" variable is just another function of x,y, that can again be defined in a number of ways. The business of expressing the functions ψ, u and v of x and y with the assistance of the auxiliary functions k, k', K and E of x and y can become quite involved, since in the theory of elliptic integrals there are many transformations that can be resorted to as a matter of convenience of computations. But in principle the relevance of elliptic integrals to our problem arises as follows. Since

$$\begin{aligned} r^2 &= x^2 + (\cos\theta - y)^2 + \sin^2\theta = x^2 + y^2 + 1 - 2y\cos\theta \\ &= x^2 + y^2 + 2y + 1 - 2y(1 - \cos\theta) \\ &= [x^2 + (y+1)^2] - 4y\sin^2(\theta/2) \end{aligned}$$

we can write $r = R'\Delta\alpha$ if we put

$$R'^2 = x^2 + (y+1)^2, \quad \alpha = \theta/2 \ , \quad k^2 = 4y/R'^2$$

If we also use the natural transformations such as

$$\cos\theta = \frac{(x^2 + y^2 + 1) - (x^2 + y^2 + 1 - 2y\cos\theta)}{2y}$$

(which of course introduces a basic "weakness" of computations, viz., the expression of a quantity as a difference between two large numbers), it readily follows, for instance, that

$$\psi = -y \int_0^{\pi} \frac{\cos\theta}{r} d\theta = -y \int_0^{\pi/2} \frac{(x^2 + y^2 + 1) - r^2}{2yr} (2d\alpha)$$

$$= - (x^2 + y^2 + 1) \int_0^{\pi/2} d\alpha/r + \int_0^{\pi/2} r \, d\alpha$$

$$= - \frac{x^2 + y^2 + 1}{R'} K + R'E$$

In evaluating the integrals for u and v, the terms which contain the cancelling factor r^2 lead to K; while the other term, with r^3 in the denominator, requires the use of the formula*

$$\int_0^{\pi/2} \frac{d\alpha}{\Delta^3\alpha} = E(k)/k'^2$$

There results

$$u = K/R' - (x^2 + y^2 - 1)E/R'^3k'^2$$

$$v = - (x/y) \; [K/R' - (x^2 + y^2 + 1)E/R'^3k'^2]$$

Since

$$R'^2k'^2 = R'^2(1 - 4y/R'^2) = R'^2 - 4y = x^2 + (y - 1)^2 \equiv R''^2, \text{ say,}$$

the above can be written as

$$u = [K - (x^2 + y^2 - 1)E/R''^2]/R'$$

$$v = - (x/y)[K - (x^2 + y^2 + 1)E/R''^2]/R'$$

whereby the role of R" is exhibited more directly (R" and R' are the shortest and the greatest distances from the point x,y to the vortex). At great distances from the vortex, when $k \to 0$, these formulas describe the dipole field; at $x = 0$ and $y = \pm 1$, we have $k = 1$, and ψ , u and v become infinite. The transfer to a system moving with velocity U is accomplished simply by subtracting U from u, or by adding $Uy^2/2$ to ψ.

*Perhaps the simplest way of deriving this formula (not prominently exhibited in a number of handbooks) is the following. Note that $(d/d\alpha)(\cos \alpha/\Delta\alpha) = (k^2 - 1) \sin \alpha/\Delta^3\alpha$, and $(d/d\alpha)(\sin \alpha/\Delta\alpha) = \cos \alpha/\Delta^3\alpha$. Multiply the integrand by $\cos^2\alpha + \sin^2\alpha$, regroup, integrate by parts, note the cancellations of the integrated terms between 0 and $\pi/2$, add the remaining integrals, recognize $\Delta^2\alpha$ and cancel, recognizing E(k).

It may be noted that with small k, elliptic integrals do not constitute a "strong" method of computation, since certain small quantities have to be determined as differences between large numbers.

In practice the expression $\psi(x,y) = \text{const}$ is not a convenient algebraic equation in x and y. Streamlines can be drawn much more easily by noting that (at the given instant, and in a coordinate system moving with any velocity U) we have

$$dx/dt = u \quad \text{and} \quad dy/dt = v$$

in the Lagrangian sense (where t is time, and x and y refer to a particular particle); viz., by noting that the differential equation for the streamlines is $dy/dx = v/u$, and solving it numerically.

In the economy of a computing machine several refinements are necessary in order to make the computations "stronger". We found that one convenient transformation of these equations is

$$-F_2R'^3u \equiv x' = k^2y - F_1 + F_2R'^3U$$

$$-F_2R'^3v \equiv y' = -k^2x .$$

where

$$F_1 \equiv E/F_3$$

$$F_2 \equiv (1 - k^2)/2F_3$$

$$F_3 \equiv [E - 2(1 - k^2)(K - E)/k^2]/k^2,$$

where k, R', K and E are functions of x and y as before, and where the primes on x and y denote differentiation with respect to a new independent variable T ("machine time") related to t by $dt/dT = -F_2R'^3$. The resultant system is well-behaved in the sense that no infinities occur: velocities x' and y' are zero at the vortex (the situation thereby is made somewhat analogous to that with Hill's vortex, q.v.). Functions $F_1(k^2)$ and $F_2(k^2)$ are particularly convenient in that they are nearly linear and are easy to represent in the machine.

Besides giving a general qualitative description of the irrotational flow field, the part of Helmholtz's theory described by the above mathematics and exhibited in Figs. 2-6 yields a number of substantive results. It outlines the profile of the fluid transported with the ring. It shows that the pathlines are indeed substantially circles, though for larger circles their centers become displaced toward the axis, and their shape becomes distorted so as to approximate eventually the half-profile of the transported fluid. In particular it shows that for the most-probable, low, values of U the existing theory is not well applicable (see the section entitled "Velocity of Helmholtz's Vortex Ring"). Finally, of course, from these patterns one can in principle work out all other information on the irrotational flow, such as the distribution of pressure and shear in the fluid. The pressure, of course, has a "hump" in the stagnation regions and a "drop" at the "equator" of the transported fluid shape, this drop increasing rapidly as one approaches the vortex itself (see Appendix). While the inviscid fluid is supposed to be without shear stress, the distribution of "shear" in the sense of rate of shear deformation is of interest; it indicates, at least qualitatively, the ways in which a real vortex ring will depart from this idealized picture. The shear most directly indicated by these patterns is the one in the planes normal to the axial (x,y) planes and at 45° to the streamlines; this shear is much like that in the irrotational flow about straight vortices (see Appendix), and is greater near the vortex itself, and on the "equator" of the transported fluid. Less directly indicated is the shear in the planes which are at 45° to the normal to the x,y plane; this shear results from the fact that each elementary torus on the pathline is alternately shrunk and stretched as it approaches, or recedes from, the axis. Its magnitude reflects essentially the radial velocity component v, vanishes in the plane $x = 0$ and at infinity, and reaches some maximum at some $|x|$ near $y = 1$.

Further information on the size and the shape of the body of transported fluid is given in Fig. 7.

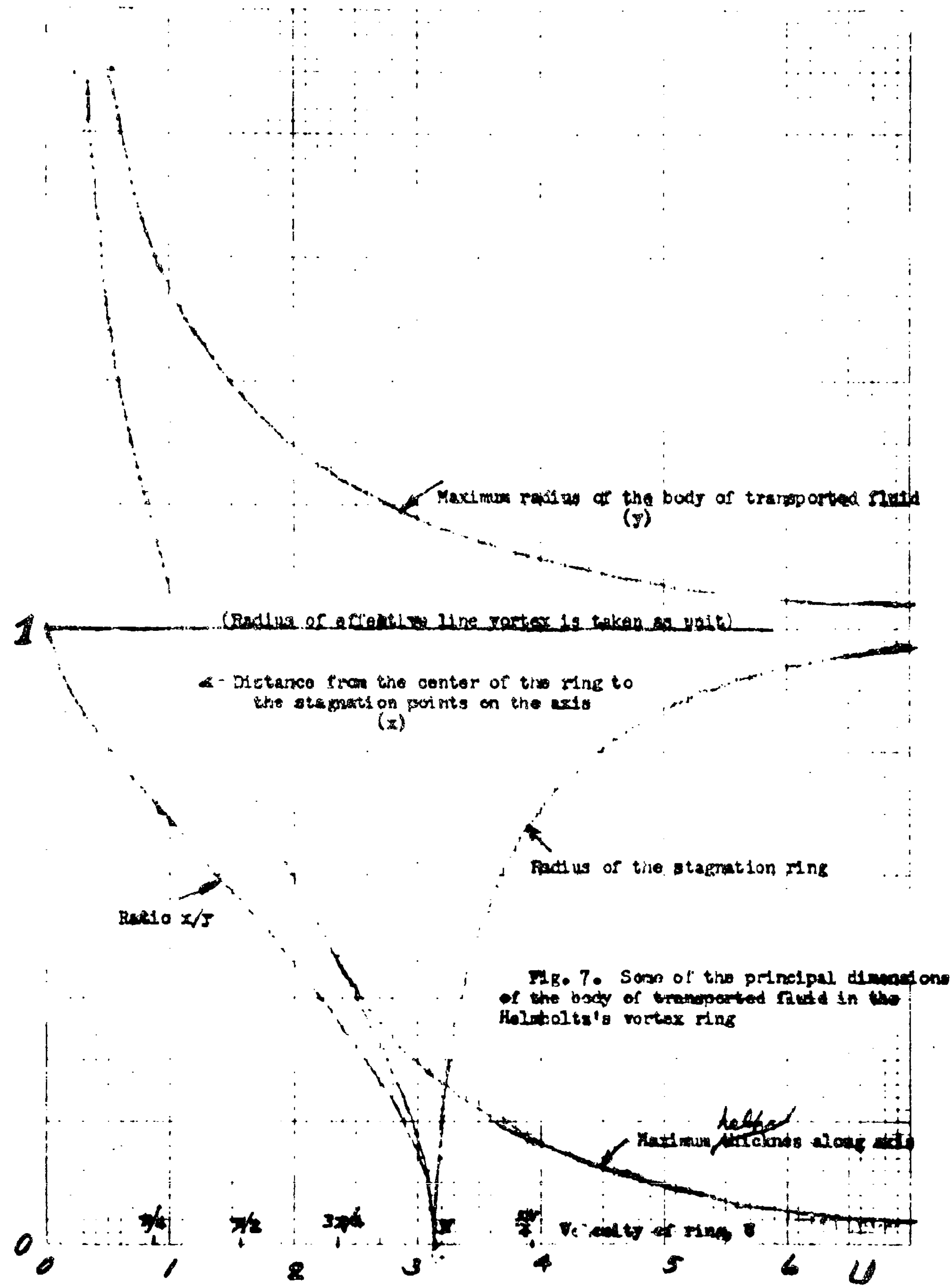

Figure 7. Some of the principal dimensions of the body of transported fluid in the Helmholtz's vortex ring.

HILL'S SPHERICAL VORTEX

Hill's theory is concerned mainly with flow within the core of a vortex ring. Indeed, the rigid core postulated by Helmholtz is an excellent concept for straight vortices, but it cannot properly be "bent" into a torus; that is, vorticity in the core cannot be uniform. As regards the variation of vorticity on each pathline in the core, it can readily be seen that vorticity must be proportional to the distance y from the axis. Indeed, consider an elementary torus on this pathline. As it "spins" about some "center of circulation" in the core, both its volume and its circulation must remain constant; its length is proportional to y, hence its cross-sectional area is proportional to $1/y$; so the vorticity σ, which is circulation per unit area, comes out as $\sigma = \beta y$, with $\beta = \beta(\psi)$ being a constant for each pathline. The pathline can be viewed as a projection on the x,y plane of a section of the surface $\sigma(x,y)$, representing the vorticity distribution, by a plane passing through the x-axis at a slope β to the y-axis. Generally one could expect that the inner pathlines have a greater slope β, this slope becoming zero gradually.

The very reasonable assumption of Hill's theory is that $\beta = 0$ in the ambient fluid, but that the vorticity has spread throughout the transported fluid so that β is the same for all pathlines in that fluid. Thus as regards the presence of a sharp boundary between the rotational and the irrotational fluid, this theory is much like that of a rigid straight vortex. Yet there is now a good reason for this boundary (the ambient fluid is being continually replaced), and the difference in vorticity is great only in a small part of the circumference of the "core", viz., near the "equator" of the vortex. At the forward "pole" where particles of ambient and transported fluid "meet", both are irrotational; but as they proceed side by side along the meridian (so to say without forming a boundary layer), the former remains irrotational, while the inner acquires vorticity through some Coriolis forces which are too complicated for our present purposes. In a real fluid there would then commence a diffusion of vorticity (its form being simply that the inner particle will not gain as much, while the outer will gain some), but here this is neglected. Thus the assumption of inviscidity of the fluid here appears quite reasonable.

With $\sigma = \beta y$ the differential equation for the stream function within the core becomes

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 - (\partial\psi/\partial y)/y = y^2\beta(\psi)$$

With Hill's assumption that β is constant this equation is satisfied by the simple expression

$$\psi = y^2(Ax^2 + By^2 + C),$$

provided $2A + 8B = \beta$. Here A,B,C are constants, and we revert temporarily to standard units. The value of β is readily seen, and those of A and B can be shown, to be invariant with respect to our choice of the velocity U of the coordinate system; but the value of C is obviously affected by the transfer term $Uy^2/2$. In the following the coordinate system moving with the ring is assumed.

The most important streamline, of course, is the separatrix, $\psi = 0$. It consists of the "evancescent cylinder" on axis x (the line $y = 0$) and the conic $Ax^2 + By^2 + C = 0$, which meet at the stagnation points. Since near $y = 0$ we have $u > 0$, it follows that $\partial\psi/\partial y < 0$, and so $\psi < 0$ throughout the core; it can easily be shown, also, that $C<0$, $A>0$, $B>0$, and the conic is an ellipse. Letting $A = BQ^2$, $C = -BR^2$, we can write

$$\psi = By^2\ (Q^2x^2 + y^2 - R^2),$$

whence the semi-axes of the ellipse are R/Q on x-axis, and R on y-axis. The fluid velocities are $u = -2B(Q^2x^2 + 2y^2 - R^2)$ and $v = 2BQ^2xy$. At $x = 0$, $y = R/\sqrt{2}$, both u and v vanish; this will be called the "center of circulation".

The particularly simple case, of course, is that of the spherical vortex, when $Q = 1$, and ψ can be written as $By^2(r^2 - R^2)$. Then $\beta = 10B$. The total circulation in the half-circle $y> 0$, $r<R$ is easily seen to be

$$2\pi S = \int_0^{\pi/2} (\beta R \sin\lambda)(2R\cos\theta)d(R\sin\lambda) = 2R^3\beta/3 = (20/3)BR^3$$

whence $B = .3\pi S/R^3$. At $x = 0$ and $y = R$ we have $u = -2BR^2 = -.6\pi S/R$.

The irrotational flow <u>outside the core</u> is given by another well-known and particularly simple function

$$\psi = (1/2)Uy^2(1 - R^3/r^3)$$

which at $x = 0$ and $y = R$ has the velocity $= -3U/2$. Identifying

$$-.6\pi S/R = -3U/2,$$

we have the velocity of Hill's spherical vortex as

$$U = .4\pi S/R$$

in standard units. To relate that to the velocity of Helmholtz's ring (see the section on "Helmholtz's Vortex Ring: General Description" and also the section on the "Velocity of Helmholtz's Ring"), where S and the radius a of the effective vortex were taken as units, we need only note that the radius most nearly corresponding to a here is $R/\sqrt{2}$. In these analogous units

$$U = .4\pi/\sqrt{2} = .889$$

The concept of the spherical vortex is particularly attractive because with it the outer flow matches the inner flow everywhere on the meridian $r = R$. The fact that the irrotational flow of this vortex does not differ much from Helmholtz's flow can be demonstrated by showing that at great distances the latter tends to become the dipole flow, too (in doing that one should remember that with $k \to 0$ both K and E tend to $\pi/2$ with the relation $(K - E)/k^2 \to \pi/4$). The spherical vortex is illustrated in Fig. 8.

It is a remarkable feature of any Hill's vortex (whether spherical or ellipsoidal) that any one of its pathlines may be a candidate for the profile of the core; that is, if the remainder of the flow outside of this pathline is replaced by a special irrotational flow that matches this pathline, the flow within this pathline will not be affected. This feature well deserves being called the "vortex analog of Faraday's cage", the analog of the absence of gravity within a shell-like planet. Its proof amounts to a sophisticated re-inspection of Ampere's law, which is discussed in the Appendix.

The difficulty with the spherical vortex, however, is that its inner pathlines are not circles, and so are not suitable candidates for the core of Helmholtz's ring - for which, Figs. 2-6 show, the cross-section of the core must indeed be circular. The fact that the inner pathlines of the spherical vortex are ellipses elongated along the x-axis in ratio 2: 1 can easily be seen by inspecting a pathline $y^2(x^2 + y^2 - 2) = \text{const}$ and putting $y = 1 + Y$, $Y \ll 1$. There results

$$x^2 + 4Y^2 + (\text{terms of higher order in } x \text{ and } Y) = \text{const.}$$

The attractiveness of the concept of core cross-section rotating as a rigid body lies, of course, in the absence of shear in such a motion: one naturally supposes that as the processes in a real vortex ring go on, the situation would eventually become such that at least the central filaments of the core have no internal shear, viz., rotate as a rigid body. The importance of this concept, however, should not be overemphasized. Even though the core be small, and rotate as a rigid

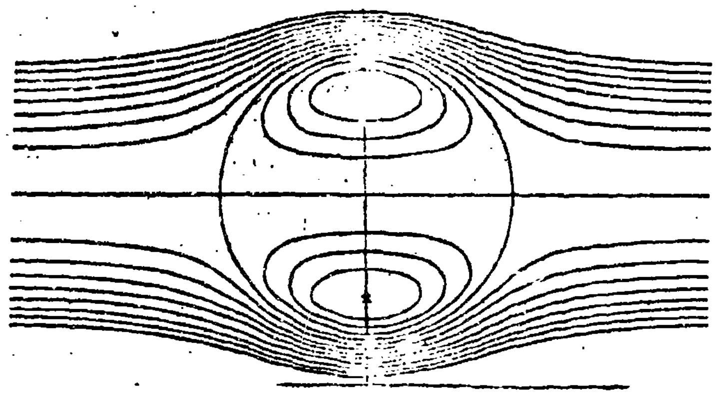

Figure 8

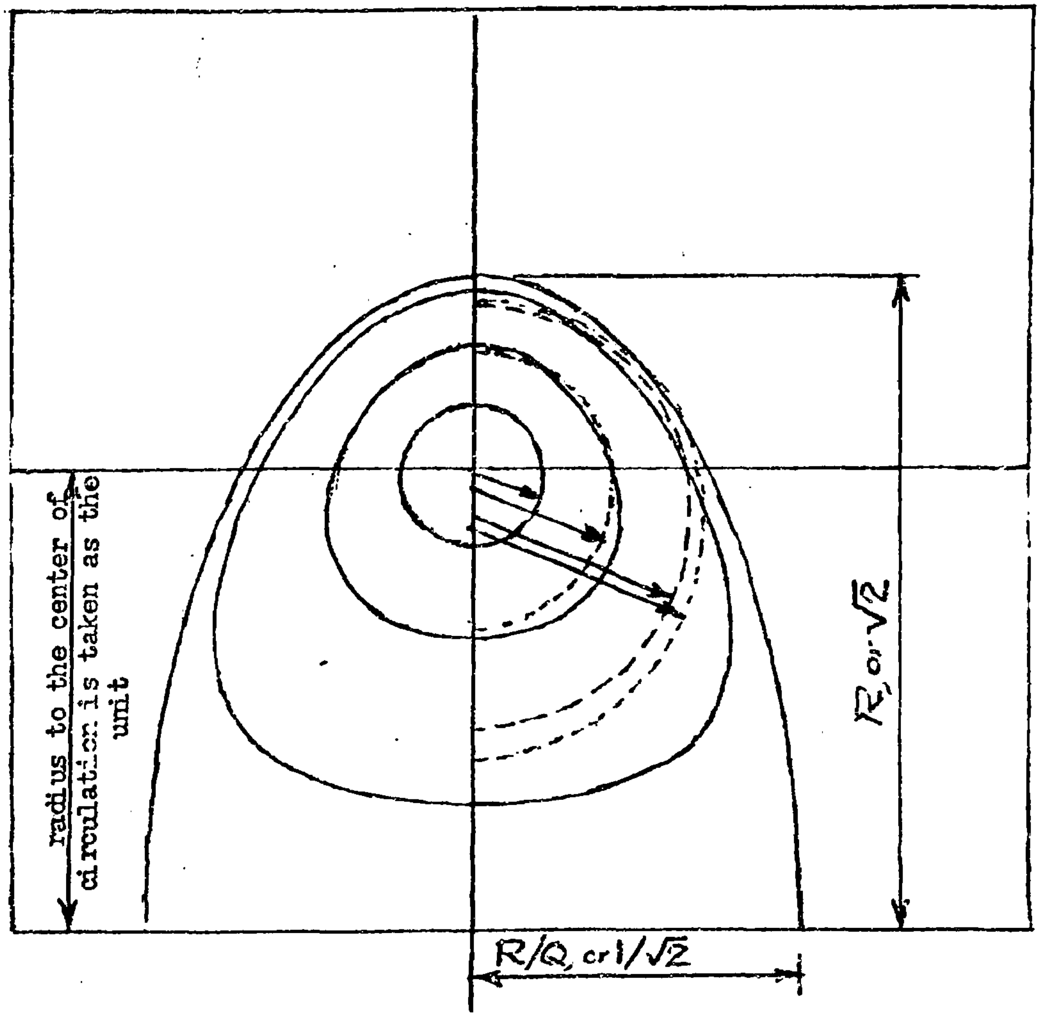

Figure 9. Pathlines of Hill's flow for Q = 2. The outer irrotational flow is simple only in the case of Q = 1. For other Q the pattern can be stretched or shrunk along the axis x. The dotted circles are those with radius , offset by $\lambda^2/2$, with the values of λ = .15, .3, .4608 and .5 corresponding to the four pathlines shown.

VELOCITY OF HELMHOLTZ'S VORTEX RING

With the above preliminaries we may attempt to tie together the theory of the vortex ring by inspecting the relation between the velocity of the Helmholtz ring and the thickness of the core. Usually this is given by the much-quoted formula, which in our units reduces to

$$U = [\ln(8/\delta) - 1/4]/2 \qquad (1)$$

where δ is the ratio of core and ring radii. In the standard units, of course, velocity U has also the dimensional factor S/a. Our object here is to show certain inconsistencies implied by this formula, and to propose a modification for it. The seemingly more direct argument, via a criticism of the derivation of (1), unfortunately, is out of our scope. It may be noted, though, that (1) is meant only as approximation heavily relying upon the assumption $\delta \ll 1$ and utilizing a number of other simplifications in the manipulation of the elliptic integrals (which, as mentioned, do not constitute a particularly "strong" method of computation). In his two-page outline of the derivation, Lamb [1] mentions that this formula was stated by Kelvin in 1867 without proof, and was formally derived by Hicks (in fifty-five pages) only in 1885 [27]. In particular, Sommerfeld [6] specifically points out that the derivation is quite complicated, that the subtrahend 1/4 is not quite exact, and that the whole concept of expressing the relation between U and δ is "einigermassen illusorisch".

Among the simplifications consequent on the assumption $\delta \ll 1$ utilized in (1) there are the relations

$$k' \cong \delta/2, \quad R \cong 2, \quad K \cong \ln(4/k'), \quad E \cong 1 \qquad (2)$$

Using substantially the same simplifications, we find that the mathematical model of Helmholtz's vortex ring can be made internally more consistent, and can be extended to larger δ than is implied by (1), if the velocity is reckoned as

$$U = [\ln(8/\delta) - 1/2]/2 \qquad (3)$$

with certain qualifications as to how the radius of the ring (here taken as 1) is to be measured (see Figs. 10 and 11). The inconsistencies in question arise from the ease of confusion between the three distinct concepts:

(i) The outer Helmholtz's flow of the section entitled "Helmholtz's Vortex Ring: General Description", both in the ambient and in the transported fluid, with its "center" at the effective line vortex;

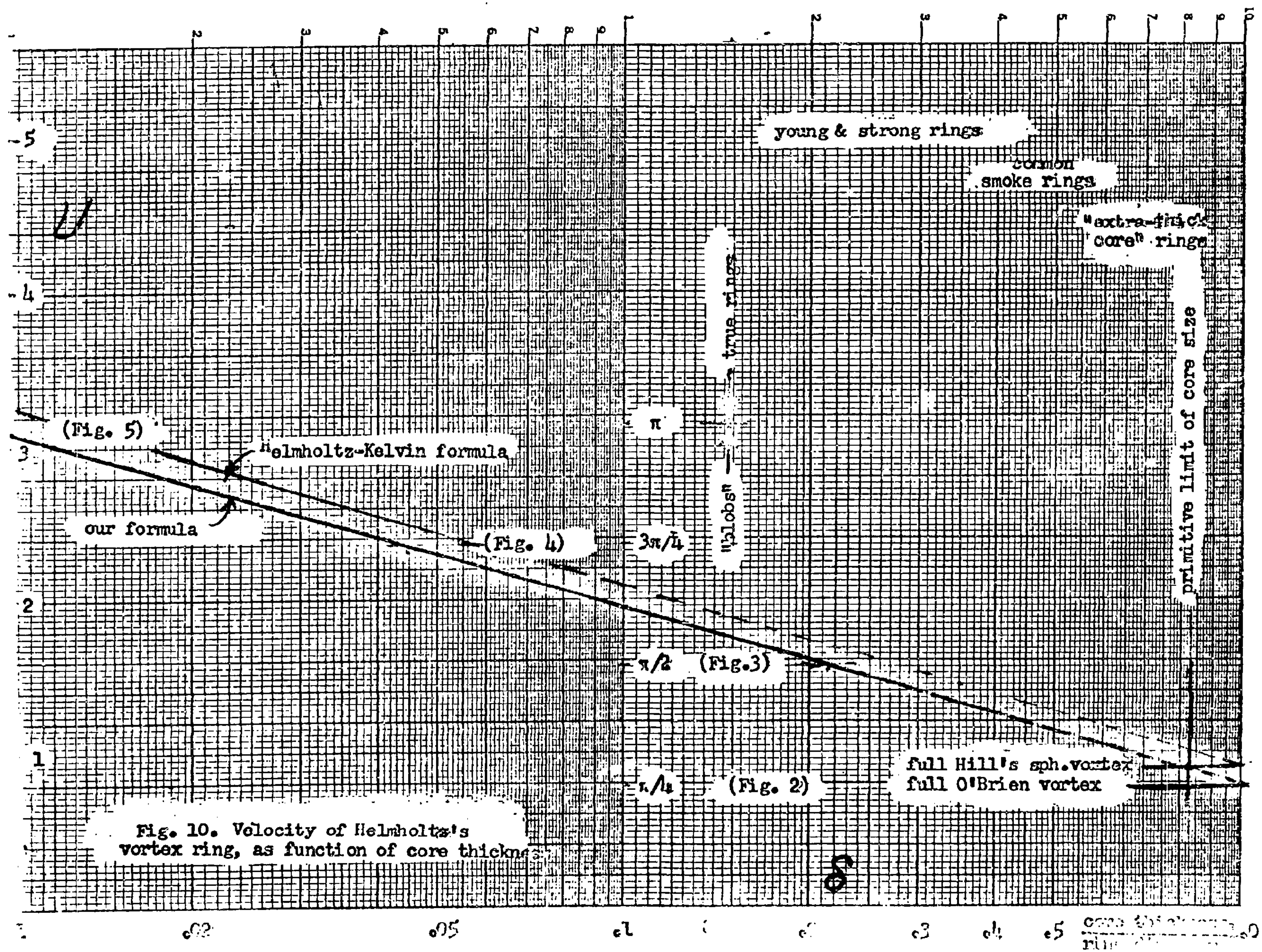

Figure 10. Velocity of Helmholtz's vortex ring, as function of core thickness.

(ii) The inner, Hill-O'Brien, flow of the section entitled "O'Brien's Oblate Vortices", with its "center of circulation" * and

(iii) The concept of the center of the core interpreted as the center of the circular shape of the core. The "rigidity of the core" concept of 1858 is clearly a carry-over from the discoveries of about 1820 concerning the magnetic field around a straight wire (since heuristically the vortex core of fluid mechanics is nothing but a current-carrying wire). While this rigid core cannot be bent - viz., the core of a vortex ring cannot rotate as a rigid body - if we interpret the assumption of the incompressibility of the fluid literally, there survives this much of this concept: the circumference of a fairly-thin core indeed may move as a rigid body. This concept is not essential, but is very convenient in tying up the concepts (i) and (ii).

Our whole point is that these three centers are not coincident. Formula (3) results from the fact that as long as the pathlines in Figs. 2-6 and 9 can be approximated by circles, these three concepts can be made coherent if the three centers are arranged as is shown in Fig. 11: the radius of the effective line vortex is taken as the unit; the center of circulation is displaced outward from that radius by $\delta^2/4$; and the center of the rigid circumference is displaced toward the axis by $\delta^2/4$. The rigid circle then constitutes, approximately, a pathline in both Helmholtz's and Hill-O'Brien flow, and so forms a transition between these two regimes.

Thus the qualifications for (3) are that the profile of the core be small enough merely to be reckoned as circular; that the radius of the ring be measured not to that circle but to the effective line vortex of the outer flow; that the radius to the center of that circle be $1 - \delta^2/4$; and that the radius to the center of circulation of that core be $1 + \delta^2/4$. The formula may be viewed as a primitive form of $U = (K - E/2)/(2 + \dots)$, but formula (3) happens to be particularly fortunate in this connection: for the quantity $K - E/2$ is approximated by $\ln(4/k') - 1/2$ much more closely than either K or E are approximated by (2).

One way of showing the correctness of (3) and of the arrangement of Fig. 11 is to work out, for both flows, the velocities u and v for two points on each diameter of that circle, using the condition $\delta \ll 1$;

*It may be noted that the location of Hill's "center of circulation" does depend upon the velocity U of the coordinate system in which the flow is viewed (the location of the effective line vortex, which implies infinite velocities, is invariant with respect to U).

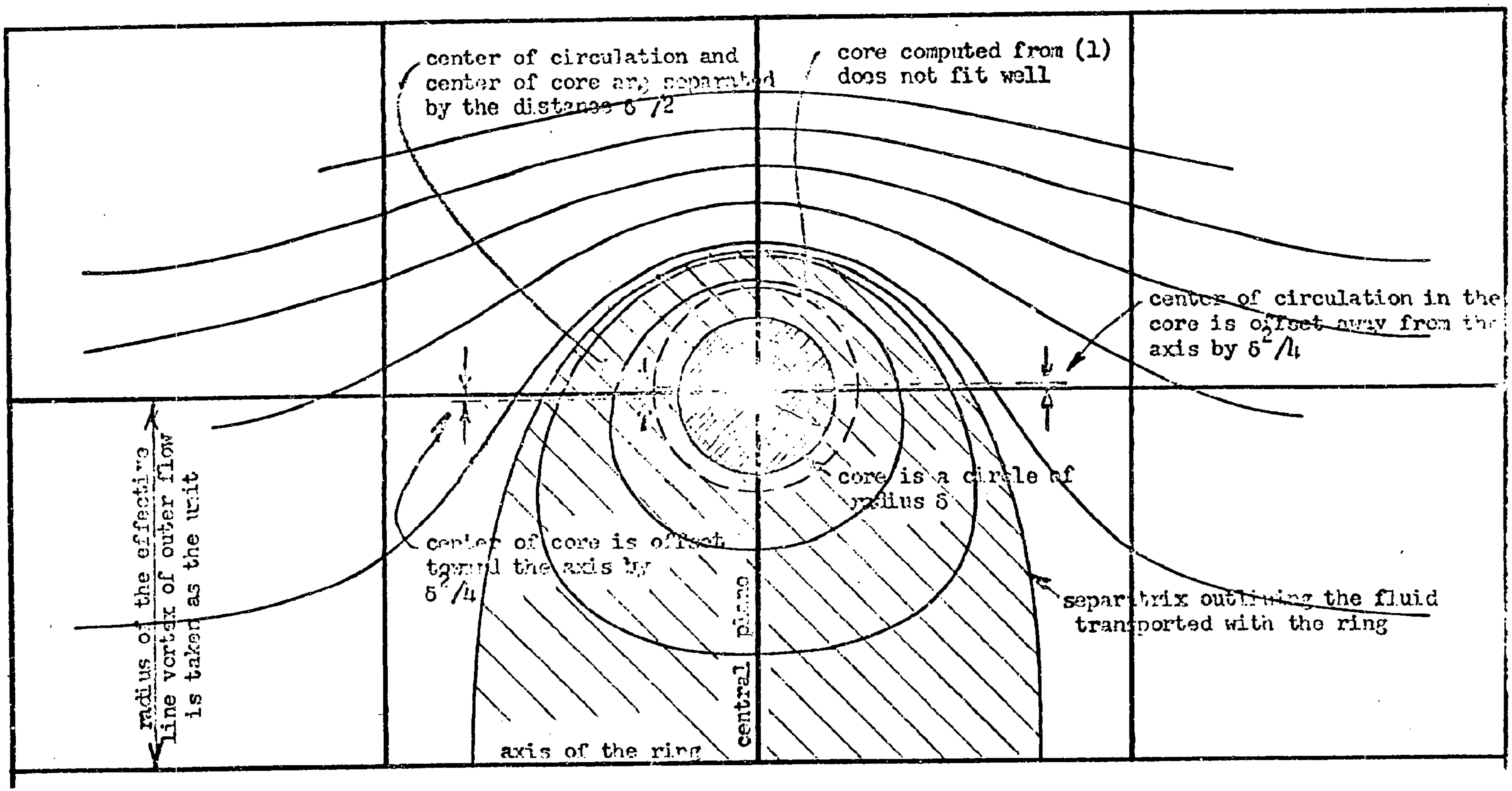

Figure 11. Structure of "consistent classical vortex ring", shown for $U = \pi/2$, $\delta = .21$, $\delta^2/4 = .01$. Showing that for "fairly-thin-cored" rings the circular contour of the core, offset from the effective line vortex by $\delta^2/4$, and from the center of circulation by $\delta^2/2$, matches both the outer Helmholtz's and the inner Hill's flows.

O'BRIEN'S OBLATE VORTICES

Strictly, the facts that the meridian of the body of transported fluid may be an ellipse as well as a circle, and that the core need not extend throughout this ellipse but may be defined by any one inner pathline, have been mentioned by Hill. Yet, since this is not mentioned in Lamb or in other introductory texts, and since ellipsoidal vortices have been studied in greater detail by O'Brien [18, 19] it is convenient to use the latter name to distinguish the ellipsoidal vortices from the spherical ones.

Theoretically, the ellipse may be either prolate or oblate. But with vortices - rather curiously, and contrary to our experience with rigid bodies - the oblate shape is the more efficient. A prolate ellipsoid would have both a much more elongated ellipse for its inner pathlines, and a much longer arc of its meridian near its equator for the diffusion of vorticity. As noted, Helmholtz's flow has no prolate spheroids.

Particularly interesting is the case of $Q = 2$ (Fig. 9); with these proportions of the ellipse the inner pathlines can easily be shown to be indeed circles, and hence suitable candidates for the core of Helmholtz's ring.

The velocity of a "full" O'Brien vortex of these proportions in our units differs remarkably little from that of the spherical vortex. The following preliminaries ensue simply from a re-inspection of the derivation of the velocity of the spherical vortex: $A = 4B$; $\theta = 16B$; $2\pi S = R^3\theta/3 = (16/3)BR^3$; $B = 3\pi S/8R^3$; at $x = 0$ and $y = R$ (in standard units), $u = -BR^2$ as before $= -3\pi S/4R$. Unfortunately, the irrotational flow about an oblate ellipse, although presumably an old problem of potential theory, is not entirely simple; nor does it match the O'Brien vortex as well as the flow around a sphere matched the spherical vortex. We shall therefore resort to certain approximations, again determining U from matching of u at that point in both flows. Dr. O'Brien [23] has kindly shown us that in the outer flow at that point $u = -1.42(3U/2)$. Identifying as before, we have

$$3\pi S/4R = 1.42(3U/2), \text{ or } U = \pi S/(2 \times 1.42)R,$$

or in the units of S and $R/\sqrt{2}$, $U = .782$. It will presently be seen (cf. Fig. 10) that this, curiously, is practically the theoretical velocity of the Helmholtz vortex ring extrapolated to the "formal" core thickness of $\delta = 1$.

and to show that components v average to 0, components u average to U of (3), and that the vectorial differences of velocities at these two points, divided by 2δ, are indeed $1/\delta^2$ - which is the angular velocity corresponding to the average vorticity $2/\delta^2$ of a circle having an area $\pi\delta^2$ and a circulation 2π (viz., strength S = 1), as assumed. Here we shall only sketch primitively how this arrangement can be arrived at.

Consider, in Helmholtz's flow, points 1 and 2 at $x = 0$, $y = 1 + \delta_1$ and $1 - \delta_2$; and points 3_+, 3_- at $y = 1$, $x = \pm\,\delta$. It can readily be shown that

$$u_1 = K_1/(2 + \delta_1) - E_1/\delta_1, \text{ with } k_1' = \delta_1/(2 + \delta_1)$$

$$u_2 = K_2/(2 - \delta_2) + E_2/\delta_2, \text{ with } k_2' = \delta_2/(2 - \delta_2)$$

$$u_3 = (K_3 - E_3)/\sqrt{4 + \delta_3^2}\,, \text{ with } k_3'^2 = \delta_3^2/(4 + \delta_3^2)$$

If we assume for a moment that the circular core is centered on the effective line vortex, we have $\delta_1 = \delta_2 = \delta_3 = \delta$, say. Using (2) we find that $(u_1+u_2)/2 = K/2$, while $u_3 = (K - 1)/2$. The principal inconsistency of this commonly-implied construction lies not in the fact that these expressions differ from (1), but simply in the fact that they are not the same: viz., the circle does not move as a rigid body.

Consider now points 1 and 2 in the Hill-O'Brien flow, the center of circulation of which coincides, for a moment, with the effective line vortex. The requirement that both of these points lie on the same streamline, viz., have the same stream function, is

$$y^2(y^2 - 2) = \text{constant, or}$$

$$(1 + \delta_1)^2[(1 + \delta_1)^2 - 2] = (1 - \delta_2)^2[(1 - \delta_2)^2 - 2]$$

which leads to

$$\delta_1(1 + \delta_1/2) = \delta_2(1 - \delta_2/2) = \delta, \text{ say.}$$

Utilizing the requirement $\delta \ll 1$, we obtain

$$\delta_1 = \delta - \delta^2/2, \quad \delta_2 = \delta + \delta^2/2,$$

that is, the center of the circular core must be displaced from the center of circulation by the distance $\delta^2/2$. Now, the mean of u_1 and u_2 computed for these δ_1 and δ_2 in Helmholtz's flow turns out to be $(K - 1)/2$, just what we had for u_3. Since points 3_+, 3_- now represent a chord, and no longer a diameter, of that circle, we compute u for points 4_+, 4_- at $x = \pm\,\delta$, $y = 1 - \delta^2/2$, obtaining $u_4 = K/2$. The inconsistency is as great as before, but now, curiously, it is reversed. Obviously, some averaging is in order: the values of δ_1 and δ_2 must be $\delta \mp \delta^2/4$; for this will yield $(u_1 + u_2)/2 = u_5$, say $= (K - 1/2)/2$. To preserve the distance $\delta^2/2$ between the center of the circle and the center of circulation, we need merely to place the latter at the distance $\delta^2/4$ above the effective vortex; hence the arrangement of Fig. 11. The fact that points 1 and 2 belong to the same pathline in Helmholtz's flow as well can be seen by inspecting the stream function of that flow,

$$\psi = -\,(x^2 + y^2 + 1)K/R' - R'E + Uy^2/2,$$

various refinements of these arguments calling for mere algebra.

Our modification of (1) into (3), and the arrangement of Fig. 11 go a long way toward removing the limitation $\delta \ll 1$; but some of this limitation remains, since the really-thick cores depart strongly from the circularity of their cross-section. Primitively, δ is limited by $1 - \delta^2/4 = \delta$, or $\delta = 2(\sqrt{2}-1) = .828$; for this δ we get from (3) formally $U = .885$, practically that of the idealized Hill's spherical vortex. As Fig. 2 shows, the pathlines which are conceivable candidates for the profile of the core at such low U are strongly non-circular; it seems rather clear, in fact, that one should not expect to meet rings of such low U, or such large δ, in reality. Our theory would surely apply to δ as great as .2; Fig. 3 shows that with $\delta = .21$, $U = 1.57$, and $\delta^2/4 = .011$ the theory applies excellently. The theory might conceivably be crudely applicable to rings with δ up to, say, 1/2 (Fig. 2); but with such rings it appears rather doubtful if our basic assumption (of β = constant within the core, and $\beta = 0$ outside the core) is at all a useful approximation.

Yet one possibility, perhaps merely academic, might be mentioned. With thick-cored rings we must first give up the circularity of the core, together with the idea that the pathline which forms both the border and the tie between the two regimes of flow must be rigid. Its rigidity, and its consequent circularity, is for our purposes merely a convenience. The basic property we seek for such a pathline is merely steadiness; and this ensues as soon as this pathline is found to belong to both steady flow regimes simultaneously. In both flow regimes the shape of pathlines departs from circularity in much the same way, circles changing into ovoids which eventually become half of an oval. Thus in the relatively narrow range of "extra-thick-cored rings", viz., in the range of δ, say, from .5 to .8, or very low U from 1.5 to .9, it should be possible to find, for each U, a pair of pathlines - one from each regime - that match most nearly; this ovoid can then be taken for the core corresponding to the assumed U. Our belief in the existence, and uniqueness, of such a determination rests on no more than physical intuition, and on the defense than merely an approximate agreement is sought. As guidelines for such a search, we have, so to say, three "degrees of freedom": change of scale between the two regimes, viz., a choice of the separation between the effective line vortex and the center of circulation; choice of proportions Q of Hill's ellipse; and the choice of the individual pathline in each flow. In this way one could arrive at an "atlas of the possible structures of a thick-cored ring of a certain type"; so that with an actual ring the few possible experimental observations would enable one to "fit" this ring among this series of theoretical structures.

Fig. 10 exhibits formulas (1) and (3), and shows how well they seem to "blend" with the values of U = .889 and .782 for the "full" spherical and O'Brien vortices (Q = 1 or 2). It is not clear at which values of δ these value of U should be marked at this plot (i.e., whether at the values of δ = .828, 1 or whatever); this would depend upon the manner of the definition of the quantity δ for the "extra-thick-cored" rings, and the detailed definition might just as well be made so as to fit (3). Fig. 10 also indicates our present idea about the probable range of the applicability of this theory.

REMARKS

The biggest gap in our understanding of vortex rings is the poverty of our understanding of the mechanism of the dissipation of the ring. A proper approach to this subject will have to make quite a break with the classical theories reviewed here. These theories are meant to achieve no more than an idealized representation of the instantaneous structure of a vortex ring, and should not be used mechanistically to infer the rates of decay. E.g., a ring which at a given instant can be fairly well represented by Fig. 4 may, after a certain time, be better represented by Fig. 3 (with a larger radius a and a smaller strength S); but this is not the same as to say that Fig. 4 actually evolves into Fig. 3. The dissipation depends upon the diffusion of vorticity, viz., upon the gradients of the surface $\sigma(x,y)$; and the actual surface is not the truncated cylinder assumed in these theories, but more nearly a "hump" rounded off everywhere. Attempts to apply theories, such as Navier-Stokes, that involve second derivatives of functions, to discontinuous functions (here, σ) are fraught with spurious mathematical difficulties. It is with the object of illustrating this type of difficulties (at first entirely unsuspected by us) that we find it instructive to review, in the Appendix, the only thoroughly solved analogous problem, the decay of a straight vortex. Experimental approach to this problem has been badly neglected.

Some inkling of the sort of results that a thorough theory will furnish may be gathered from common observations and conjecture. Rings seem to have a finite range, and in their travel describe a sort of trumpet-like surface, first moving fast with little decay, presently expanding and practically stopping their forward motion, and finally starting to distort; sometimes, depending upon the manner of the formation of the ring, this history is preceded by a brief regime of convergence of a ring. One may suppose that in a fast ring the predominant process is the growth of the core of a Helmholtz's ring; while in an old and weak ring the predominant process is the induction of the ambient fluid into the ring. There are a number of ways in which the mechanism of decay may be interpreted [17], but perhaps the best way, suggested in particular by Turner [15], is the latter process, the induction of ambient fluid. The key process here is that as soon as the "ambient" fluid acquires some vorticity, it is likely to become a part of the vortex ring (it may also pass into a wake). In particular, the "full" Hill-O'Brien vortices seem rather improbable, since the outer pathlines in the transported fluid in such rings will probably consist of nearly-irrotational recently-induced fluid. In the experiment cited in [18] the induction of the ambient fluid and the formation of the "evanescent cylinder" is prevented by the surface tension. The process seems particularly interesting from the viewpoint of transportation of the fluid or aerosol in the ring: the ring so to say builds its own envelope, thus counteracting the diffusion and the centrifugal precipitation of the aerosol, and so reducing the loss of the original contents of the ring in the traversed space.

In principle, today's computing machines allow us a fairly rigorous "brutal numerical" approach to the decay of the vortex ring [24, 25]. For example, by postulating a plausible instantaneous structure of the ring, it is possible to compute its subsequent history, including its decay; and in particular, if a reasonable variety of such plausible structures lead to substantially the same subsequent history, this subsequent history may indeed be identified with the actual ring. In practice, though, such an undertaking remains a large and expensive job, is often fraught with unsuspected difficulties, and may distract the attention from the long-needed - and in the long range, unavoidable - experimental approach.

The most immediate present need is for development of experimental techniques for evaluating a passing ring. Techniques for measuring a and U are obvious enough [13, 15], but new techniques are needed for measuring S and δ, the latter concept possibly needing a refinement which would amount to some specification of the surface $\sigma(x,y)$. The basic technique might be simply photographing a finite-duration flash, delivered as a thin sheet in the meridional plane of the ring, with the ring formed in a dust-laden atmosphere, and/or passing through the ambient atmosphere laden with dust of possibly different kind. Length of the streaks would indicate the velocity, and the circulation $2\pi S$ would be given by integrating this around any closed path. There are many interesting possibilities as to the development of the detailed techniques for analyzing the photographs. Optical sweep of the image with the expected velocity U would allow the direct photographing of pathlines instead of the pattern of the type of Fig. 1. Some automation is needed for taking the integral of the velocities. One should expect that eventually most of the labor of the analysis would consist of twirling of a few knobs so as to achieve the matching of the photograph to a certain series of patterns built into a special-purpose computer. With the development of such techniques it will become possible to start testing formulas such as (3), replacing them by empirical formulas, formulating the decay qualitatively, optimizing the designs of vortex-ring generators, and generally, improving the theory.

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APPENDIX

STRAIGHT VORTICES

Necessary Mathematical Preliminary to Vortex Rings

Foreword. In compiling this review we found that much critical reinspection of the fundamentals is necessary. This is so not merely because simplifying analogies often help; rather, this is so because certain analogies which persistently suggest themselves tend to distract our attention from the more essential differences, and actually cause difficulties to the student.

a. Patterns of flow in a straight vortex pair are unquestionably similar to those of Figs. 1-6; in fact, we found the theory of vortex pairs an indispensable exercise. But the similarity conceals the more essential difference in the mechanisms of a pair and a ring. In the straight pair the translational velocity of an element of a core is in no way caused by the core itself: it is fully the result of the existence of the opposite core. In a ring, however, the diametrally-opposite elements of the core contribute very little to the velocity of an element of the core; most of the velocity is due to "nearly-neighboring" elements of the core, or to the curvature of the core. Thus there is a sharp and rather artificial division in the theory between the straight and curved vortices. In reality, of course, the straightness, just as the perfect circularity, is but an abstraction; a better theory would "blend" these cases, allowing a slight curvature of a nearly-straight vortex. It does not seem spelled out in any text, however, that the theory of straight vortices achieves its simplifications not by eliminating some small refinement, but by the very radical step of bypassing the basic mathematical difficulty, the indeterminacy.

b. Attempt to compute the velocity of an idealized Helmholtz's ring leads to "improper" integrals; attempt to resolve the indeterminacy by purely mathematical means - without the aid of physical commonsense assumptions - leads to an even worse absurdity, infinite velocity. Practically all of the labor and all of the controversies of the theory thus amount to the resolution of rather artificial mathematical indeterminacies by rather arbitrary physical assumptions; this is all traceable to tradition and to the historical development of the ideas and the technological means at our disposal. Were our object no more that the production of the specific results, or engineering, we could arrange the whole approach so that these indeterminacies just do not arise; we tried to do so in the section entitled " Velocity of Helmholtz's Vortex Ring". But when our object is to lay grounds for tackling some larger problems, or education, we feel that a mere veiling of the difficulties will not do; something useful can come from learning that those indeterminacy pitfalls exist, and from learning that there are analogous problems - here, the straight vortices - which are free from those pitfalls.

c. Complex potential method applied to straight vortices appears to the student at first as a powerful general method; but the texts seem to gloss over, and the student is somewhat disappointed to learn, the fact that this method is not applicable to rings. Clebsch's transformation is usually mentioned incidentally, as a not-especially-powerful method; its analogy to the method of complex potential seems to mathematicians too obvious to deserve mention, and may remain entirely unsuspected by the student. Such bits of information, even if not directly applicable at once, seem of value.

d. Practically nothing is known on the decay of rings; and it is legitimate to see what is known on the decay of straight vortices. Even there the problem is solved only for that peculiar abstraction, the straight "single" vortex; on inspection this turns out to be not "single", and full of mathematical indeterminacies of its own. Now the artificial difficulties arise merely from postulating discontinuities in a function governable by a second-order partial differential equation. Some future theory of ring decay will bypass such difficulties; but today the student is better off if warned.

In view of such considerations we feel that a review of the state-of-the-art on vortex rings must be in the nature of a "student's summary".

Electromagnetic Analogy. The relation between the field of fluid velocity and vortex lines is practically the same as between the magnetic field and the lines of steady electric current. This fact is sometimes viewed as a curious coincidence [1], sometimes dismissed as irrelevant, and sometimes used to assert a sort of priority of mathematics over physics. The legitimacy of this analogy is particularly obvious if this analogy is viewed in the historic perspective, roughly sketchable as follows:

1820: basic discoveries of electromagnetics (Faraday, Oersted, Biot, Savart, Ampere, Weber), done against a background in which the rudiments of vector analysis exist (Newton, Gauss), but concepts of rotation and deformation are still limited mainly to the rigid body (Euler).

1837: a "proprietary" form of vector analysis (Hamilton).

1858: adaptation to fluid mechanics (Helmholtz, Stokes).

1865: effective vector analysis for electromagnetic theory (Maxwell), and the multiple-algebra approach (Grassman).

1879: hydrodynamics comes of age: first edition of Lamb's.

1900: vector analysis as we now know it (Gibbs).

1903: aerodynamics gets its real impetus from aeronautics.

today: computer revolution in progress.

While this analogy is no less than the identity of the basic mathematics, it by no means constitutes an identity of the two branches of physics. Rather, one is more impressed by the imperfections of this analogy:

One artificial difference is the traditional, but arbitrary, factor of 2, resulting essentially from a definition of vorticity as distinguished from angular velocity. Sommerfeld's avoidance of this factor, and Lamb's use of the symbol ω for vorticity, in practice only serve to aggravate the possible confusion for the student, unless the analogy be constantly kept in mind. Other differences are more substantive:

The surface of a wire is sharply defined; a vortex is basically diffuse.

A wire can be extremely thin; vortices of interest are rather "fat".

A wire can be fixed mechanically; vortices basically change and move. The obvious and time-dependent process of the motion of some "liquid" wire is not identical with the motion of vortices.

The magnetic field in the precise center of the wire is of little specific interest; but the velocity of a vortex is an essential concept.

Magnetic moment is a rather subtle concept; angular momentum is one of the most essential physical quantities, is conserved, and is never infinite.

The easily-produced steady current is in fact a cause of the magnetic field; but after all is said and done, a vortex is but an expression of the structure of the field of fluid's velocity.

Vorticity. In principle, the vorticity of a fluid is nothing but the angular velocity of a small element of the fluid, with two qualifications. The first is the traditional, rather arbitrary, and somewhat bothersome factor of 2. In a rigid body the angular velocity is $\underline{\omega} = \nabla \times \underline{V} /2$ vorticity is traditionally defined as $\underline{\sigma} = \nabla \times \underline{V} = 2\underline{\omega}$ The second qualification is simply that vorticity specifies the rotation of an element that may at the same time be changing its shape. For example, in two-dimensional flow vorticity is sometimes defined as the sum (rather than mean) of the angular velocities of two instantaneously perpendicular lines in the fluid. Vorticity is a vector, a property of each element of the fluid (independent of the choice of origin, or of translational velocity of the coordinate system), and its field is fully defined by the field of velocity.

Circulation is the line integral of velocity around a closed circuit; or, by Stoke's theorem, this is the integral of vorticity over the area inclosed by this circuit, viz., the "flux" of vorticity. Thus vorticity is the aerial density of circulation. As a line integral, circulation is fully analogous to magnetomotive force; as an area integral, to magnetic flux (the electromagnetic analogy to fluid mechanics makes no distinction between H and B). Circulation is a scalar property of the circuit, or of the area inclosed by this circuit; but insofar as the circuit, or its area, has some direction associated with it, circulation does have the corresponding aspects of a vector. In application to the straight vortex, in particular, a circuit is automatically associated with each point in space; hence the circulation there may also be viewed as a vector property of the element of fluid. The dimensions of circulation are those of moment of velocity, or angular momentum of the unit mass of fluid. Strength of a vortex is defined as (circulation)/2π, merely for convenience of saving the factor of 2π. The Biot-Savart formula for the straight vortex then takes the simple form $V = S/r$ (corresponding to $H = 2i/r$ for the straight wire - here is our factor 2 again), so that strength S corresponds to 2i, twice the current; circulation, of course, corresponds to $4\pi i$, and hence vorticity corresponds to (current density)/4π.

Shear. When we "strip off" from the motion of the fluid its rotation ($\sigma/2$) and dilatation (which is neglected in hydrodynamics, as opposed to gas dynamics), what remains might be spoken of as "shear". Strictly, this is the rate of shear strain deformation, a concept well known from the theory of strength of materials and easily visualizable, but unfortunately far from simple mathematically. In principle, the shear strain causes, through viscosity, shear stresses in the fluid; and these stresses combine with the "inertial" forces (due to variation of pressure) to cause changes in velocity in such ways as to reduce the shear strain, and to cause a redistribution of both vorticity and shear. It is a peculiar feature of fluid mechanics, however, that the state of shear in the fluid is practically never exhibited. In the elementary theory (the hydrodynamics of an inviscid fluid, which is our principal concern) the exhibit of the state of shear is not necessary simply because this shear is supposed to be without effect: shear stresses are neglected. In the full theory (Navier-Stokes treatment), which in effect does consider the shear strain and stress, it is more convenient to arrange the mathematics so that one deals with vectors; and the need for exhibiting the state of shear strain - which is a concept more complicated than a vector - just does not arise. One simply determines the changes in the field vectors $\underline{V}$ and $\underline{\sigma}$. In the primitive configuration of a straight vortex, however, the state of shear in the fluid can be described very simply, by a single scalar function. This function is easy to visualize, and is quite helpful in achieving an understanding of just what has been omitted in the simplified theory, and what will be the problems of the full theory.

A little digression might be useful here, to show what sort of animal is the shear strain, if it is not a vector. The spatial change in the instantaneous velocity V (= u,v,w) at a point x,y,z is given by

$$du = (\partial u/\partial x)dx + (\partial u/\partial y)\,dy + \cdots$$

$$dv = (\partial v/\partial x)dx + \cdots\cdot$$

$$dw = \cdots\cdot$$

so that the specification of the flow requires no less than nine numbers, the coefficients of dx, dy, dz. The 3 × 3 matrix of these numbers, denotable by the dyadic $\nabla\underline{V}$ (meaning that the above three equations are writable as the components of the vector equation $d\underline{V} = \nabla\,\underline{V}.d\underline{r}$) is then broken into the antisymmetric and symmetric parts,

$$\begin{bmatrix} 0 & (\partial u/\partial y - \partial v/\partial x)/2 & \cdots \\ (\partial v/\partial x - \partial u/\partial y)/2 & 0 & \cdots \\ \cdots & \cdots & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \partial u/\partial x & (\partial u/\partial y + \partial v/\partial x)/2 & \cdots \\ \cdots\cdot & \partial v/\partial y & \cdots \\ \cdots & \cdots & \partial w/\partial z \end{bmatrix}$$

denotable by $(\nabla \underline{V} - \underline{V}\nabla)/2$ and $(\nabla \underline{V} + \underline{V}\nabla)/2$. The first is a set of three numbers recognizable at once as the vector of angular velocity, $\underline{\sigma}/2$. The second is a set of six numbers representing the deformation. It can be simplified, but not much. With one scalar, the dilatation, stripped off, the shear strain is represented by the "deviator", a set of five numbers. The deviator can be "diagonalized", by a proper choice of the principal axes x,y,z - such that the off-diagonal elements vanish, and in the motion of our element of fluid the tree axes remain instantaneously perpendicular to each other. The rotation of the coordinate system uses up three scalars, and the diagonalized deviator - a set of three numbers adding to zero - is really a set of two numbers. In each configuration considered, the state of shear strain can be easily worked out much as in the theory of strength of materials; but the adaptation of this picture to vectorial treatment remains clumsy. So the advanced treatment simply does not use this tensor picture, and uses the shortcuts provided by the vectorial Navier-Stokes equation in the components of vectors $\underline{V}$ and σ.

Straight single vortex. It is only with these preliminaries that we are now ready to consider that idealized constituent of a vortex ring, the single straight vortex. Etymologically, this is a rotating cylinder; in practice, however, by "vortex flow" one usually means the irrotational flow around such a vortex. It is easy to see that the fluid might flow in concentric circles around a straight line, and still have zero vorticity if the velocity varies with the radius as $V = S/r$, where S is a constant; Fig. 12 shows this case. Fig. 13 shows that the shearing (Couette) flow in parallel lines is not irrotational, while the flow in circles subject to that condition, is. In reality the circulation on an inner circle might be slightly less than on the outer; we then say that the annulus contains vorticity, and the idealized irrotational flow can be imagined as a result of our "sweeping the vorticity" inward. The process cannot be carried to the limit $r = 0$, so one generally postulates a core rotating as a rigid body, viz., without shear. Fig. 14 further illustrates these two "extreme" regimes of flow. Curiously, the irrotational flow cannot be extended to $r \to \infty$, either: for this represents the absurdity of infinite angular momentum (and infinite, though only logarithmically infinite, energy). It is thus necessary to terminate the flow at some radius; but the transition region constitutes a vortex sheet (or a "tubular vortex annulus" - not to be confused with "vortex tube", such as the core), and hence a return vortex, entirely analogous to a coaxial cable. The assembly of the core and its return vortex is certainly not exactly a "single" vortex. The mathematical difficulty with the single core at $r \to \infty$ is peculiar to that idealized configuration; in all other configurations, such as the vortex pair or vortex ring, the velocity at great distances vanishes more rapidly

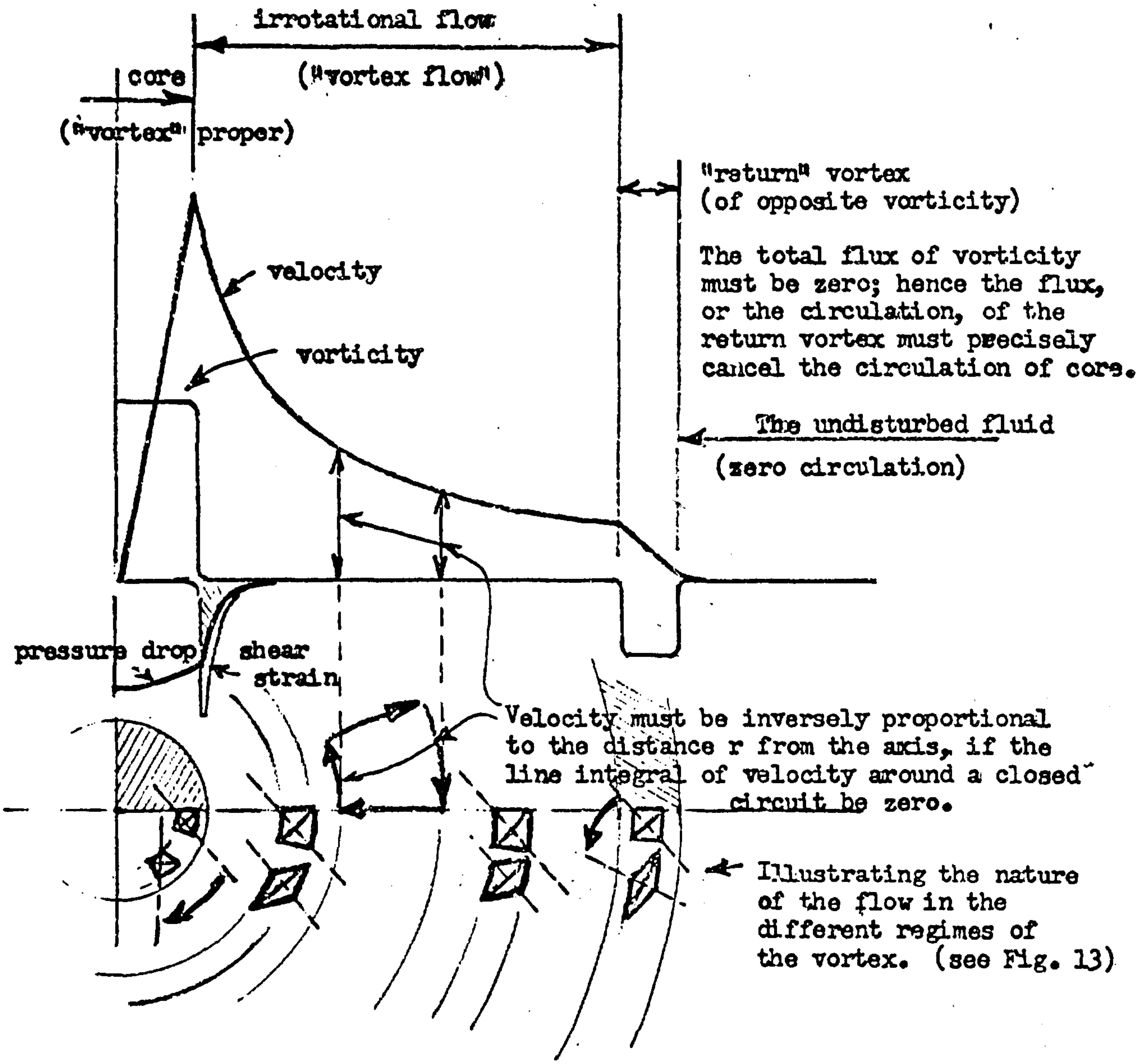

Figure 12. The conventional idealized structure of a straight vortex (The region of irrotational flow is imagined enlarged, perhaps by the process of "sweeping the vorticity" out of that region; the core is imagined small, and of uniform vorticity. The return vortex, necessary if an infinite angular momentum is to be avoided, is usually imagined at a very large radius, and often is "omitted" in the discussion).

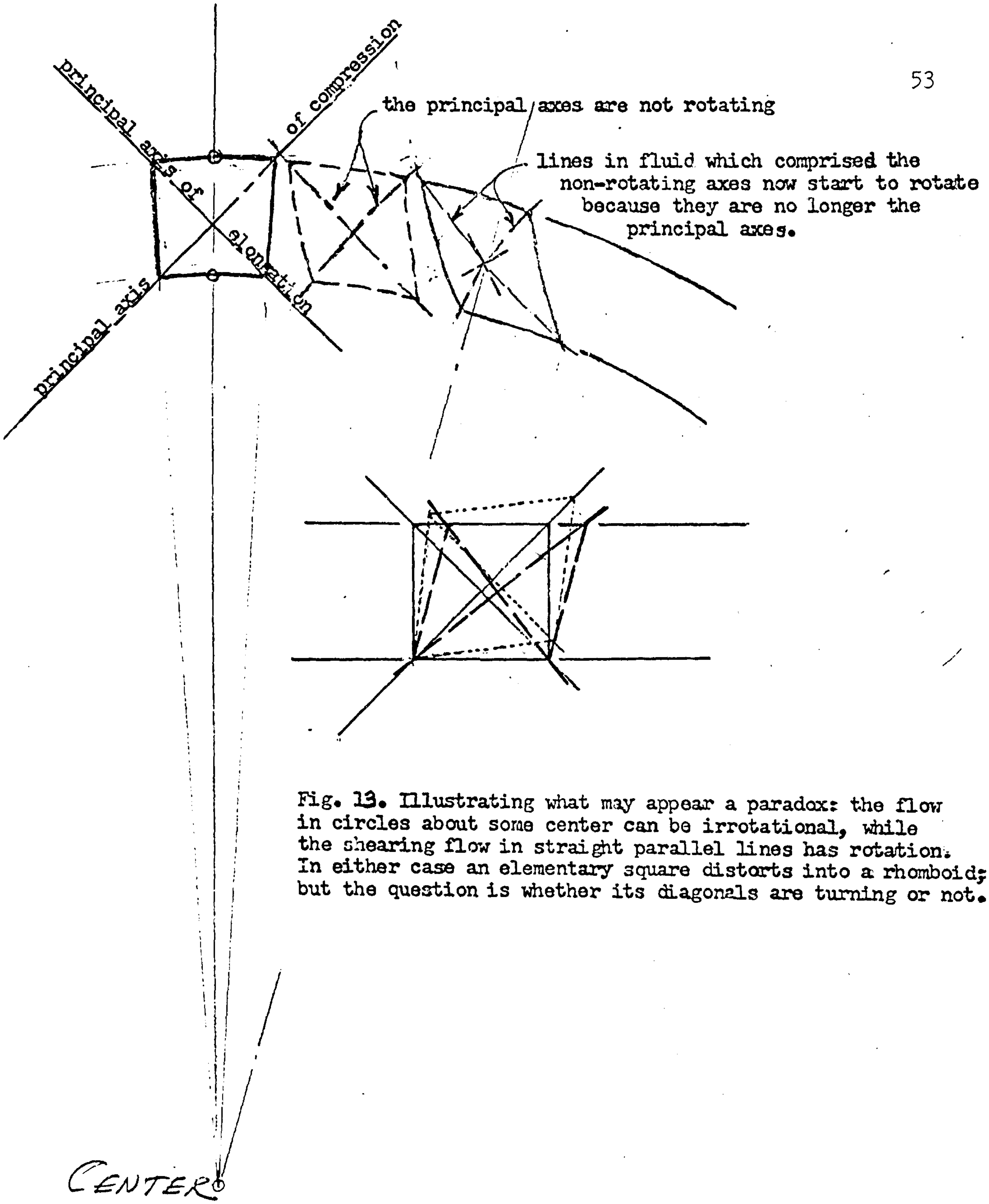

Fig. 13. Illustrating what may appear a paradox: the flow in circles about some center can be irrotational, while the shearing flow in straight parallel lines has rotation. In either case an elementary square distorts into a rhomboid; but the question is whether its diagonals are turning or not.

Figure 13. Illustrating what may appear a paradox: the flow in circles about some center can be irrotational, while the shearing flow in straight parallel lines has rotation. In either case an elementary square distorts into a rhomboid; but the question is whether its diagonals are turning or not.

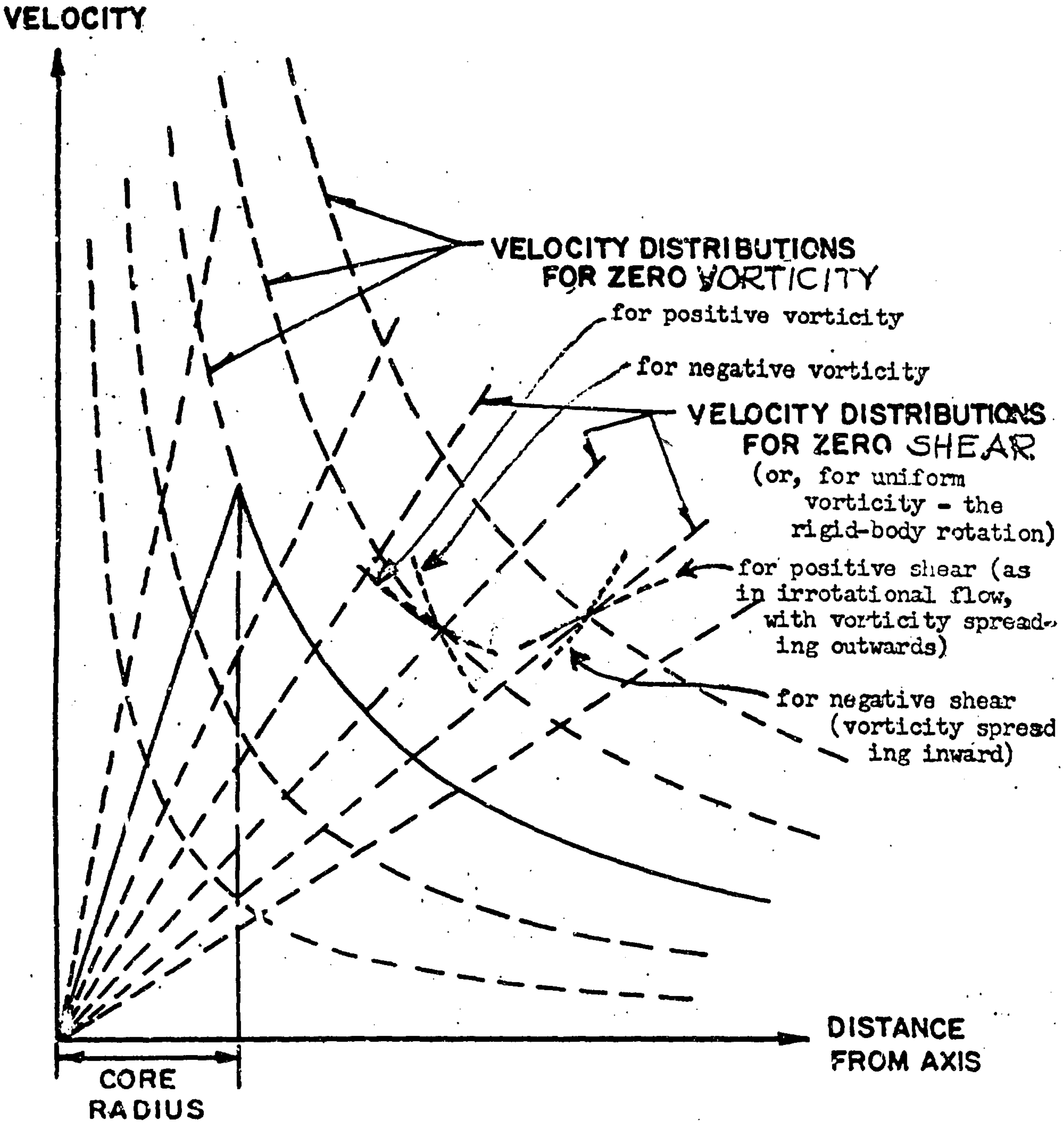

Figure 14. Illustrating the two extreme regimes of flow (zero vorticity and zero shear), and the possible departure of the velocity distribution from these extremes.

than inversely with r, and both energy and angular momentum are finite.

A number of questions which arise in the theory of vortex rings may be answered in a preliminary and primitive manner by inspecting the abstraction of the straight vortex.

For example, the pressure in a vortex in an incompressible fluid can easily be shown to vary as r^{-2}. Thus the pressure drop is concentrated in a very narrow region near the core. Cavitation may occur, but it would not affect the outer flow, for the actual presence of a fluid in the region containing vorticity is not needed. In principle, even "coreless" flow may exist, for it does not matter whether the vorticity is spread uniformly through the cavity, or concentrated in a vortex sheet at the surface of the cavity. Thus the incompressibility of the fluid is not a very bad assumption.

As to the shear in this flow, if the fluid rotated as a rigid body, with $\omega = V/r = S/r^2$, the increment of velocity would have been $dV = + (S/r^2)dr$; but this increment is $[d(S/r)/dr]dr = - (S/r^2)dr$. Thus the shear strain rate, or the angular velocity with which a square on Fig. is distorting itself into a parallelogram, is $2S/r^2$ in magnitude. Again, practically all the shear is concentrated in a narrow region outside the core. Were viscosity μ suddenly introduced into this flow, there would arise a shear stress $2\mu S/r^2$, and the dissipation of energy per unit cube of fluid would be $4\mu S^2/r^4$, indeed concentrated in a very narrow region outside the core. Outside some such narrow region the inviscidity of the fluid does not appear a bad assumption, either.

It is of interest to inspect briefly the decay of the straight vortex; and in particular, to inspect what can be inferred about this decay from the classical theory of the vortex in inviscid fluid. In brief, it turns out that the standard idealized structure of the vortex (a rigid core, an annular region of irrotational flow, and a return vortex sheet) is for this purpose not a good approximation, and not even a good initial condition.

To begin with, the decay of the straight vortex cannot be represented by a mere increase of the radius δ of its core, with the preservation of the structure of the vortex and of its strength S within some return vortex sheet of radius R. The angular momentum per unit mass in the irrotational flow is $(S/r)r = S$; while the average angular momentum per unit mass of the core is

$$(1/\pi\delta^2) \int_0^\delta r(\frac{S}{\delta} \cdot \frac{r}{\delta})\, 2\pi r\, dr = S/2$$

Thus, were we to assume S unchanged, with an increase of δ we would have an absurdity, a loss of the total angular momentum; were we to compensate for this loss by increasing S, we would have to explain how such an increase can come about; while the consideration of the shear in the irrotational flow shows us that the energy, and hence S, is decreasing throughout this flow. Our consideration of shear is useful only insofar as it shows that the decay starts simply by rounding off the corners in the curves of $V(r)$ and $\sigma(r)$.

The exact treatment [1] requires the solution of the equation $\partial\sigma/\partial t = \nu\nabla^2\sigma$, where ν is the kinematic viscosity, $\nabla^2\sigma$ is simply $\partial^2\sigma/\partial r^2$, $\sigma = \partial V/\partial r - V/r$, and shear is $\partial V/\partial r + Vr$. One can indeed expect that some sensible function $V(r)$ may emerge from the solution of this equation even though the function $V(r)$ postulated as the initial condition be rather artificial. It turns out, however, that postulating the standard idealized structure of the vortex as the initial condition brings out a quite strange feature of this equation. Ideally, at $t = 0$, σ can change neither in the core (where σ is constant) nor in the outer flow (where σ is zero); but at the core boundary we have the preposterous necessity of taking a second derivative of a discontinuous function. Even if we do round off the corners of the functions $\sigma(r)$ and $V(r)$, and so obtain a numerical solution of this "heat flow" equation, this "nonrelativistic" equation gives a very peculiar formal result (the infinitesimal changes in σ travel with infinite velocity) that is not consistent with the intended physical significance of this equation. The viscosity at once starts diminishing the fluid's velocity, and hence the circulation, too, everywhere in the irrotational region; hence vorticity must be passing through the perimeter of each concentric circle - and yet it must come from the core. Thus if the vorticity is to be visualized as something that cannot be generated within the fluid, but must travel from some boundary (according to Helmholtz's dictum), we must recognize that it "travels" at an infinite velocity. It is simpler to visualize the vorticity "disappearing" in one place and being simultaneously "generated" some other place. A similar physical qualification on the numerical solution of this equation is provided by the requirement that a return vortex exists: an infinite angular momentum may be permitted mathematically, but is repugnant physically.

Nevertheless, an exact and consistent solution of this equation is known; it is given in [4, 9], and is illustrated in Fig. 15. This solution starts by postulating a more sensible structure of the vortex, whereby the core and the return vortex are nicely blended: the core is rigid only at the very axis, and the irrotational flow exists only in a thin annulus. The region of $\sigma = 0$, which may be viewed as the boundary of a "core" spreads with the velocity $2\nu/r$ (note that the

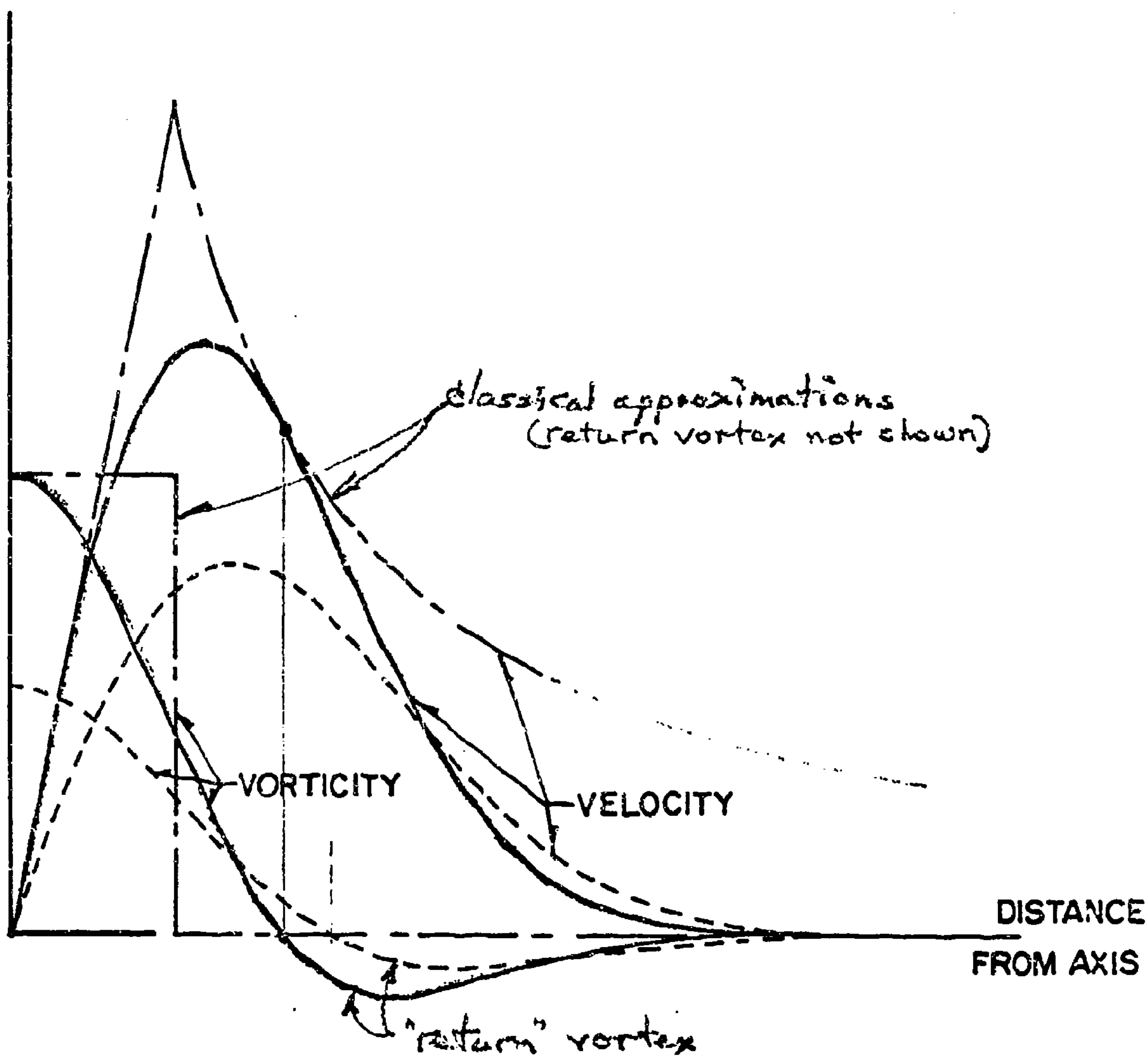

Figure 15. Theoretical structure of a coaxial single vortex. As the vortex spreads, the profiles of velocity and vorticity retain certain proportions; the equations (see reference 4) are

$$v = (Ar/2\nu t^2)\ \exp(-\ r^2/4\nu t)$$

$$\xi = (A/\nu t^2)(1 - r^2/4\nu t)\ \exp(-\ r^2/4\nu t)$$

where A is an arbitrary constant

At the point where vorticity changes sign, the quantity $s + 2v/r$ vanishes, but its Laplacian is not zero, and vorticity there does grow. See reference 4 for the survey of literature on the (coaxial) single vortex.

proportionate rate of growth of r is $2\nu/r^2$, i.e., that the thicker cores are the more "permanent", while the thin cores dissipate particularly rapidly). There is no longer any "straining" of the Helmholtz's dictum, since the disappearance and the appearance of vorticity occurs in contiguous infinitesimal regions: vorticity is indeed "traveling". One should expect that any real vortex, even if its instantaneous structure differs from this solution, will be decaying in such a way that its structure will be approaching this solution. As it happens, this is the only known rigorous solution of vortex decay. As regards its possible application to the decay of vortex pairs and vortex rings, this solution seems handicapped by the presence of the return vortex, made necessary only by the indeterminacies of the classical vortex as $r \to \infty$ while both the vortex pair and the vortex ring in effect provide their own return vortices, and do not, strictly, call for a "coaxial" sheath of negative vorticity. Still, such a sheath - not necessarily strong enough to cancel out the circulation due to the core - may be quite real. For instance, it may provide a natural concept missing in the classical theory: a weak and slow true ring, with a velocity distribution somewhat as sketched in Fig. 16. This could be viewed as a superposition of two rings of opposite sense, and may be useful in broaching the theory of the travel of a single slightly-curved vortex, and the changes in the manner of the dissipation of the ring, as evidenced by the behaviour of water drop (Fig. 19).

Vortex pair. Well enough illustrated in an airplane vapor trail, the vortex pair provides an excellent introduction to the vortex ring. Idealized in the same way as the Helmholtz's flow field, viz., to the field of two line vortices, it has a remarkably neat theory. If the strength of each vortex is S and their separation 2a, and the dimensional factor S/a omitted through the choice of units, the velocity of the vortex is 1/2, simply because each vortex is in the field of the other one, located at the distance 2. The velocities in the field of flow in the stationary coordinates (Fig. 17, corresponding to Fig. 1) can easily be shown to be

$$u = 2(x^2 - y^2 + 1)/[(x^2 - y^2 + 1)^2 + (2xy)^2]$$

$$v = 4xy/(\text{same denominator})$$

The equation of the steamlines is $dx/dy = (x^2 - y^2 + 1)/2xy$, with the solution $x^2 + (y - B)^2 = B^2 - 1$. The streamlines are circles with respect to which the two vortices are mutually inverse points; from this a number of simple geometrical constructions of these circles are possible.

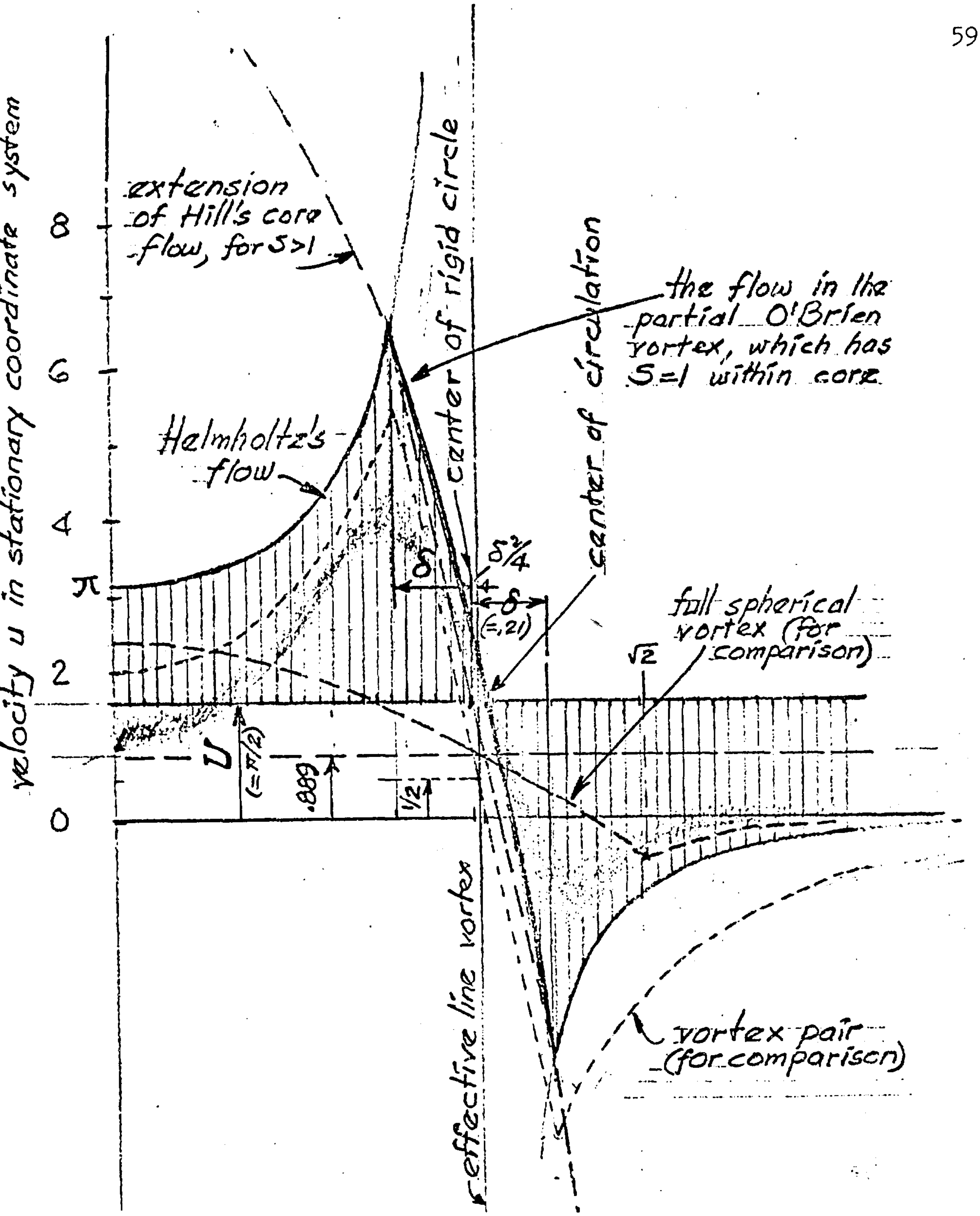

Figure 16. Plots of longitudinal velocity u in the central plane (x = 0). Several unrelated velocity distributions are shown. Solid lines illustrate the distribution in the pattern of Fig. 3 (for U = 1.57, δ = .215). Dotted lines refer to patterns of Fig. 8 (full spherical vortex) and Fig. 18 (vortex pair). The thick blurred line suggests a possible velocity distribution in an actual "weak true ring", in which the main core is sheathed in a partial return vortex analogous to that of Fig. 15.

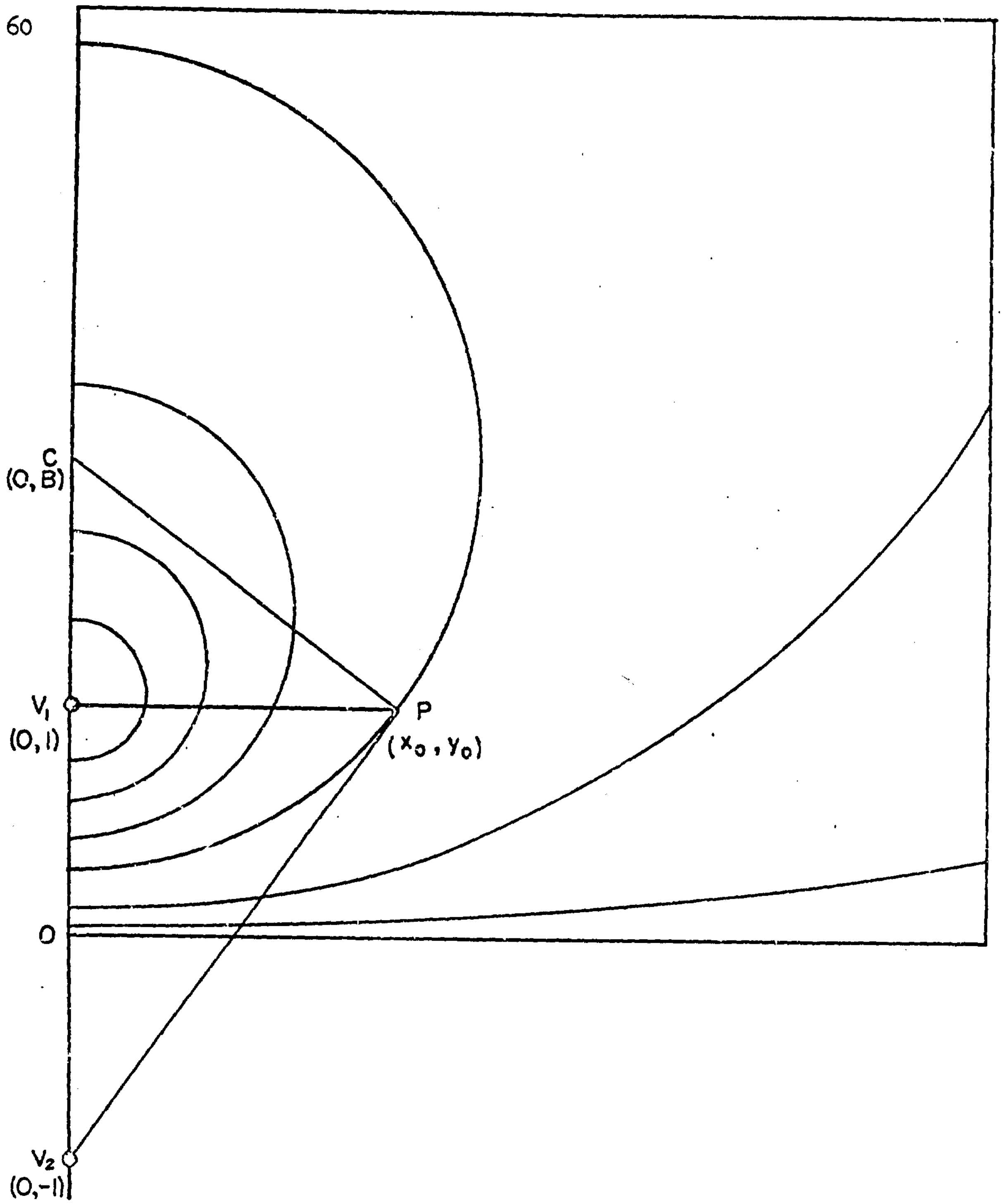

Figure 17. Streamlines of a vortex pair in a stationary coordinate system at the instant when the vortex pair passes through the plane x = 0.

<u>Note</u>. The streamlines are circles with respect to which the two vortices are mutually inverse points (Lamb). I.e., if the ordinate of the center of the circle is B, and its radius is R, then $(B - 1)(B + 1) = R^2$. From this relation a number of simple geometrical constructions of these circles is possible. For instance:

(1) To draw a circle passing through a point P on the line y = 1: draw V_2P, and the normal to it, PC; C is the center, and PC the radius.

(2) To draw a circle passing through any point P: find the center C at the distance $V_2C = (V_2P)^2/2y_o$.

Transferring to the coordinate system moving with velocity $U = 1/2$ we get the pathline pattern of Fig. 18. For our present purposes the interesting features of this pattern are first, that there is only one such pattern (while with vortex rings there are many such patterns); and second, that even quite thick cores are very nearly circles centered on the vortex. These pathlines have been investigated particularly by Bradley [17]. As regards "very thick" cores there apply a number of considerations quite analogous, and perhaps properly preliminary, to our matching of Helmholtz's and Hill's theories for vortex rings. On each pathline the vorticity is constant; so $\sigma = \sigma(\Psi)$ where Ψ is the two-dimensional stream function, the imaginary part of the complex potential. It is very natural to assume, with Hill, the vorticity is constant throughout the core. The core then must be exactly a circle; but the outer irrotational flow is no longer exactly the flow caused by the two line vortices. Thus for thick cores we must either change this irrotational flow, or not accept the assumption of the uniform vorticity (this is another case where the analogy between the vortex ring and straight vortices fails to work).

In reality, a "full" vortex pair is improbable, for the ambient fluid is being induced into the pair, mostly in the rear, in much the same way as with a vortex ring in [15, 12]. A reasonably thorough theory of vortex pair in a real fluid will have to start with a continuous and rounded-off surface of $\sigma(x,y)$, much as with the theory of single vortex; it will be free of the necessity to provide a return vortex enveloping each core separately; and it will most probably constitute an essential preliminary to a reasonably thorough theory of the vortex ring.

Ampere's law in fluid mechanics. The classical theory of vortices may be based practically wholly on Ampere's law [17], and many qualitative considerations based on this law survive in the more advanced fluid mechanics. Our resort to this law has been criticized as contradicting the modern trend of basing fluid mechanics on Navier-Stokes equations, and indeed must be "defended". In electromagnetics this law defines a force acting at a distance; in fluid mechanics, by a seeming legerdemain, it defines the velocity of a particle, while we would normally wish to have the velocity determined from an acceleration. The student may well ask: is this law being proposed in any sense as a competing alternative to Newton's law? or, how does a fluid particle "know" that there is a vortex somewhere at a distance, and in particular, how does it "know" enough to adjust its velocity as that vortex moves? The formal answer, of course, is that Ampere's law is merely an elegant solution of the properly-idealized Navier-Stokes equations, standing in much the same relation to these equations

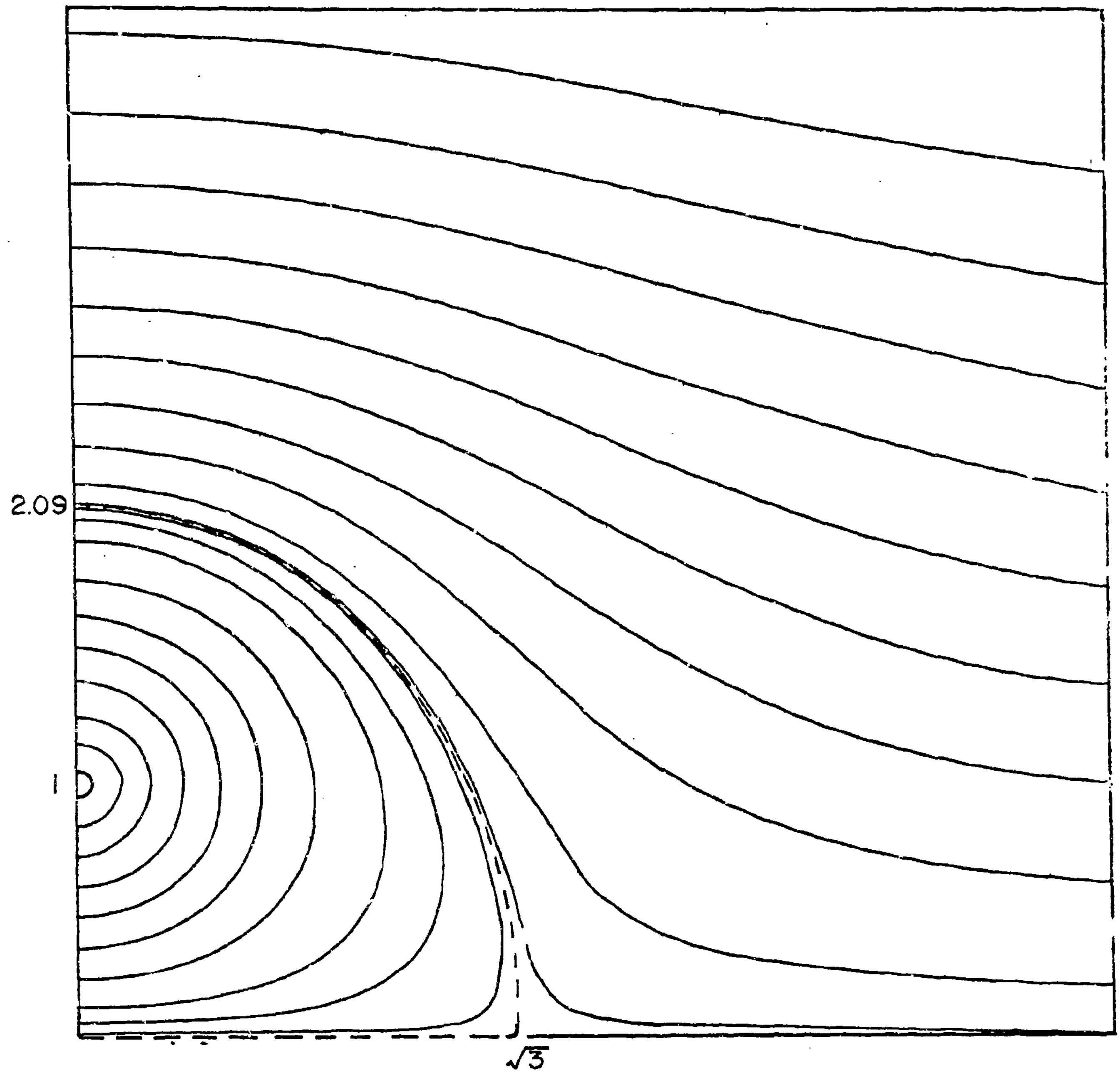

Figure 18. Pathlines in a vortex pair; i.e., streamlines in the coordinate system moving with the pair. The separatrix between the two families of curves defines the fluid moving with the pair. This separatrix has the shape of an oval with semi-axes of 2.09 and $\sqrt{3} = 1.73$ (Lamb); [1, 17]. Note that the trajectories near the vortex are very nearly concentric circles; this seems to be a good justification of the rigid-core hypothesis.

as the "law of conservation of momentum" or "the law of conservation of the center of gravity" stand to Newton's law. Unfortunately, the rigorous proof of this fact (by Sommerfeld) seems awkward both heuristically and historically: it uses very advanced concepts (the vortex as a succession of dipoles) to prove something which is both simpler and older. Subjectively, we find the following "explanation" rather satisfying and helpful.

When a vortex ring is produced by a short plunge of a piston, the outer irrotational flow, extending to great distances, is established by purely "inertial" forces, not involving vorticity or viscosity; this is roughly a half of the flow from an isotropic source, and can be viewed, approximately, as a superposition of the flow from an isotropic source with that of a dipole, or of a vortex ring; the former dies out rapidly with the square of the distance, the latter survives essentially by itself. Meanwhile the plunge necessarily produces also some vorticity: all of it concentrated in a thin vortex-sheet ribbon at the circumference of the piston. This ribbon is at once rolled up into a substantially-round "core", and this core is carried by the irrotational flow; thus the core may be viewed as a consequence, or an expression, of the irrotational flow - rather than its "cause". The individual particle in the outer flow "does not know" about the existence of the core, and changes its velocity according to the instantaneous pressure gradient at its location. Ampere's law merely asserts that the velocity so changed remains in a certain relation to the motion of the core carried by the rest of the flow. Certainly, to determine the motion of the particle by the Navier-Stokes equations would be to shut our eyes to the existence of a powerful and elegant answer.

Some texts seem to "prefer" the Biot-Savart law, apparently because of its apparent simplicity; this is merely the application of Ampere's law to a straight wire or vortex. Curiously, the classical experiment (repulsion and attraction of coaxial coils) already implied the grasp of the more sophisticated, differential, relation.

It can hardly be doubted that Ampere's law was viewed by the pioneers as simply a sort of "polarized" inverse-square law. Just as the latter is based on the purely geometrical fact that the surface of the sphere is proportional to the square of the radius, one should expect that there exists a purely geometric basis for Ampere's law. It may perhaps be sketched as follows.

Consider two closed and linked curves in space, of lengths L and ℓ, and let $\underline{r}$ be the vector from the element $d\underline{L}$ to the element $d\underline{\ell}$; L might be a vortex of strength S, and ℓ a streamline. In the differential form of Ampere's law, $d\underline{V} = (S/2)d\ell \cdot \sin(d\underline{L}, d\underline{\ell})/r^2$ the factors $d\ell$ $\sin(\ldots)$ are obvious enough; the question is merely whether the factor $S/2r^2$ can be derived. If we now "relax" this law by postulating merely

$$\underline{V} = \int_L S\ f(r)\ \underline{r} \times d\underline{L}$$

whereby S is defined by

$$2\pi\ S = \int_\ell \underline{V} \cdot d\ell$$

we must have the identity

$$2\pi\ S = \int_\ell \left[\int_L S\ f(r)\ \underline{r} \times d\underline{L} \right] \cdot d\underline{\ell}$$

whence f(r) must be of the form $1/r^2$ from dimensional considerations; in this respect fluid mechanics has a simpler task than electromagnetics, for no new concepts (i, H or B) are involved. Now, it is a purely geometrical fact that the integral

$$\iint_{\ell\ L} \frac{\underline{r} \times d\underline{L}\ .\ d\ell}{r^3}$$

is 4π when the curves are linked right-handedly, and zero for non-linked curves. Hence $f(r) = 1/2r^3$, viz.,

$$\underline{V} = \int_L (S/2)\ \frac{\underline{r} \times d\underline{L}}{r^3}$$

and Ampere's law appears (at least in fluid dynamics) as a consequence of purely geometrical and dimensional considerations. Just as Gauss's theorem amounts to a definition of divergence, Stokes' theorem amounts to a definition of vorticity (either the source, or the curve of L, are given some finite thickness). The curious question, whether there is some contradiction here to the concept that physical laws are basically empirical, is outside our scope.

In the section entitled "Hill's Spherical Vortex" of this text we mentioned the "vortex analog of Faraday's cage"; the fact that vorticity on the outer pathlines of Hill's vortex does not "affect" the inner pathlines. A part of the proof is quite simple. The vorticity at a point on an outer pathline constitutes an elementary core which fails to link the circuit formed by the inner pathline;

so no distribution of vorticity outside that inner pathline can change the circulation on that circuit. But this is not the same as to assert that the inner pathline is "unaffected"; e.g., with an arbitrary change of vorticity outside, this circuit may cease to be a pathline. The point is that the effect of all elementary cores formed by the outer pathline cancel out at each point inside that outer pathline. This is more readily seen if we visualize the outer pathline as a toroidal vortex sheet constituting a boundary between an irrotational flow outside, and the stationary fluid inside; a removal of this sheet leaves the interior fluid stationary.

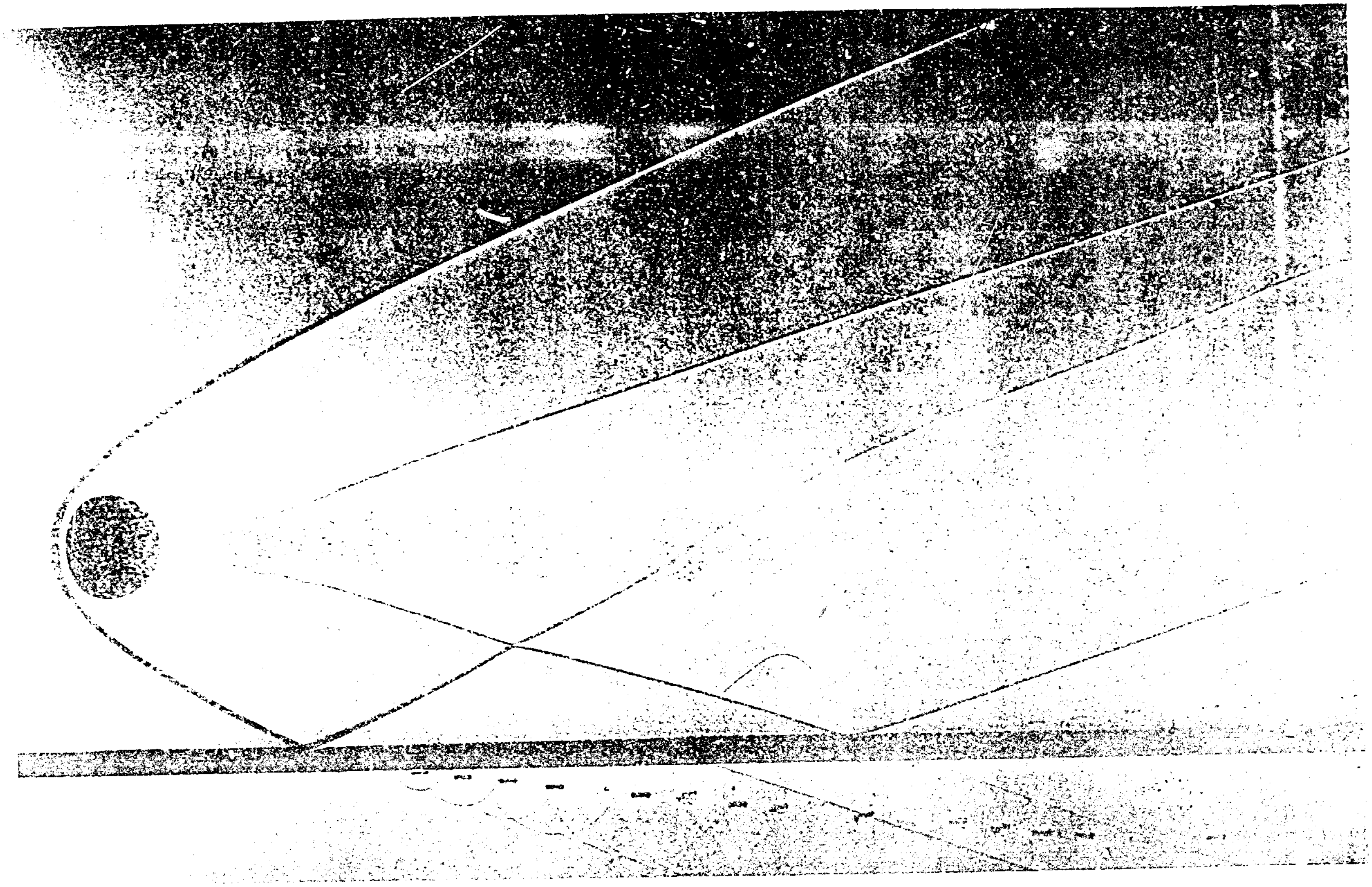

Figure 21. The commonly-used illustration of Huygens' principle: a straight Mach cone is produced by firing a ball over a row of holes in a plate. Note the vortex rings produced within each miniature shock wave; and note the flow in the opposite direction, caused by the wake wave. (Photo taken at BRL circa 1942.)

ERRATA

1. On Page 32 the following text should be inserted after "and rotate as a rigid ...":

"body in the x,y planes, there is still shear in the supposedly-irrotational region adjoining the core; and the outer filaments of the core have shear because of the continuous alteration of their length. Were we to assume that the situation in an actual vortex is more nearly similar to a case where the rate of the total energy dissipation associated with the presence of shear is minimized, we would require a diminution of this continuous alteration of the length of this outer filament of the core; and to this purpose, for a given range of the fore-and-aft travel of such a filament, we would require that the range of the radial travel be diminished. That is, the cross-section of the core would indeed be not quite circular, but elongated in the x-direction; from this point of view Hill's spherical vortex is not an entirely bad representation of reality. No detailed study of this question is known, but we have an impression that the inner pathlines of the spherical vortex are elongated more than is necessary for this purpose. Strictly, on the assumption of the inviscidity, such considerations do not apply at all, and it is desirable that the inner pathlines be circles.

The circularity of the inner pathlines can be achieved simply by resorting to a different proportion Q of the ellipse, with a relatively mild proviso that the size of the core be "small". Once this is done, there is no longer the need, suggested by Hill [28], to construct a special irrotational flow: the standard Helmholtz's flow will fit."

2. On Page 38 the section entitled "O'Brien's Oblate Vortices", should be read before the section entitled "Velocity of Helmholtz's Vortex Ring" on Page 34.

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13. ABSTRACT

The vortex ring is an essential and ubiquitous phenomenon that has been rather neglected in aerodynamics and technology, perhaps because it is usually construed, unfortunately, as a mere exercise in old-fashioned mathematics. The existing venerable theories of this phenomenon are at once little known, difficult, uncoordinated, insufficient and inconsistent. They are reviewed, modified and combined, and are thus made ready for the long-overdue experimental tests. An essential mathematical preliminary, the theory of straight vortices, is discussed in an appendix.

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It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.