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A. Deprit

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STABILITY OF THE TRIANGULAR LAGRANGIAN POINTS

by

A. Deprit

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SUMMARY

Leontovich has proved that the triangular equilibrium positions in the planar Restricted Problem of Three Bodies are stable for almost all admissible mass ratios. It is shown here that the set of exceptional mass ratios for which stability remains to be proved or invalidated contains only one point besides the critical mass ratios of order two and three.

1. INTRODUCTION

We consider a conservative dynamical system with two degrees of freedom defined by the equations of motion

$$\dot{z}_i = \frac{\partial \mathcal{H}}{\partial z_{i+2}} \qquad \dot{z}_{i+2} = -\frac{\partial \mathcal{H}}{\partial z_i} \qquad (i=1,2)$$

with the Hamiltonian function $\mathcal{H}$ analytic at the origin in the four dimensional phase space.

Let us suppose that the origin is a position of equilibrium for the system. Thus, in the neighborhood of the origin, the Hamiltonian can be expanded as a series

$$\mathcal{H} = \sum_{n\geq 2} \mathcal{H}_n$$

of powers of z_1, z_2, z_3, z_4, the term $\mathcal{H}_n$ being the homogeneous component of the series with degree n.

We assume that there exists a completely canonical linear mapping $(z_1,z_2,z_3,z_4) \rightarrow (Z_1,Z_2,Z_3,Z_4)$ reducing $\mathcal{H}_2$ to the form

$$\mathcal{H}_2 = \frac{1}{2}\,\omega_1^2(Z_1^2+Z_3^2) + \frac{1}{2}\,\omega_2^2(Z_2^2+Z_4^2)$$

where ω_1 and ω_2 are real numbers. Thus the equilibrium is said to be of the elliptic kind, and the numbers ω_1 and ω_2 represent a set of basic frequencies for the linear dynamical system described by the Hamiltonian function $\mathcal{H}_2$.

Given a fixed integer n such that $n \geq 4$, let us consider the following hypothesis:

(A_n) ("Restricted Condition of Irrationality")--

$$k_1\omega_1 + k_2\omega_2 \neq 0$$

for all pairs (k_1,k_2) of rational integers such that

$$|k_1| + |k_2| \leq n.$$

Under the condition (A_n), there exists at least one canonical transformation $(Z_1,Z_2,Z_3,Z_4) \to (\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ which decomposes the original Hamiltonian into the sum

$$\mathcal{H} \equiv \mathcal{H}(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = \bar{\mathcal{H}} + \mathcal{P}^{(n+1)} \tag{1}$$

where $\bar{\mathcal{H}}$ is a polynomial in the variables

$$I_1 = \zeta_1^2 + \zeta_3^2, \qquad I_2 = \zeta_2^2 + \zeta_4^2$$

of the form

$$\bar{\mathcal{H}} = \omega_1 I_1 + \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \cdots$$

and where $\mathcal{P}^{(n+1)}$ is a power series in ζ_1, ζ_2, ζ_3, ζ_4 beginning with terms of degree $n + 1$ and convergent in the neighborhood of the origin.

The coefficients in the polynomial $\bar{\mathcal{H}}$ do not depend on the integer n nor on the manner by which the normalization transformation is obtained. Thus, in particular, the determinant

$$D = \det(a_{ij})_{1\leq i,j\leq 3}$$

whose elements are defined as follows

$$a_{ij} = \left(\frac{\partial^2 \bar{\mathcal{H}}}{\partial I_i \partial I_j}\right)_{I_i=I_j=0} \qquad (i,j=1,2),$$

$$a_{i3} = a_{3i} = \left(\frac{\partial \bar{\mathcal{H}}}{\partial I_i}\right)_{I_i=I_j=0} \qquad (i=1,2),$$

$$a_{33} = 0,$$

is an invariant of the Hamiltonian $\mathcal{H}$ with respect to the canonical transformations leading to the normal decomposition (1).

At last, let us still consider two more hypotheses:

(A_∞) ("General Condition of Irrationality")--For any pair (k_1,k_2) of rational integers, $k_1\omega_1 + k_2\omega_2 \neq 0$.

(B) The determinant D is not zero.

Arnol'd (1961) has established the following theorem. If the conditions (A_∞) and (B) are verified, then on each energy manifold $\mathcal{H} = h$ in the neighborhood of the equilibrium, there exist invariant tori of quasi-periodic motions which divide the manifold. In consequence, the equilibrium is stable.

As an application of this theorem, Leontovich (1962) deduced that the triangular equilibrium positions in the planar Restricted Problem of

Three Bodies are stable for all permissible mass ratios but a set of measure zero.

However, Moser has shown that Arnol'd's theorem is still valid under the weaker conditions (A_4) and (E).

We propose here to apply this weaker form of the theorem to the triangular positions of the planar Restricted Problem, in order to determine the mass ratios for which there is stability.

2. FIRST ORDER

In the canonical units and with respect to the synodical barycentric system of Cartesian coordinates, the planar Restricted Problem of Three Bodies is described by the Lagrangian function

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + (x\dot{y}-\dot{x}y) + (1-\mu)\left(\frac{1}{\rho_1} + \frac{1}{2}\rho_1^2\right) + \mu\left(\frac{1}{\rho_2} + \frac{1}{2}\rho_2^2\right).$$

For the sake of convenience, we put

$$\gamma = 1 - 2\mu.$$

The substitution $x \to x + \gamma/2$, $y \to y + \sqrt{3}/2$ translates the origin of the coordinate system to the triangular equilibrium position L_4. There we expand the Lagrangian function in power series of x and y, and we find (Deprit 1966) :

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4,$$

$$\mathcal{L}_2 = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + (x\dot{y}-\dot{x}y) + \frac{3}{8}x^2 + \frac{3}{4}\gamma\sqrt{3}\,xy + \frac{9}{8}y^2,$$

$$\mathcal{L}_3 = \frac{7}{16}\gamma x^3 - \frac{3}{16}\sqrt{3}\,x^2y - \frac{33}{16}\gamma xy^2 - \frac{3}{16}\sqrt{3}\,y^3,$$

$$\mathcal{L}_4 = -\frac{37}{128}x^4 - \frac{25}{32}\gamma\sqrt{3}\,x^3y + \frac{123}{32}x^2y^2 + \frac{45}{32}\gamma\sqrt{3}\,xy^3 + \frac{3}{128}y^4.$$

Thus to the first order, the equations of motion are to be derived from $\mathcal{L}_2$. They are

$$\ddot{x} - 2\dot{y} - \frac{3}{4}x - \frac{3}{4}\gamma\sqrt{3}y = 0,$$

$$\ddot{y} + 2\dot{x} - \frac{3}{4}\gamma\sqrt{3}\,x - \frac{9}{4}y = 0.$$

To them belongs the characteristic polynomial

$$s^4 + s^2 + \frac{27}{16}(1-\gamma^2).$$

Its roots are distinct and all purely imaginary if and only if

$$\mu(1-\mu) < 1/27.$$

In order that this be true, the mass ratio μ must be strictly smaller than Routh's critical mass ratio

$$\mu_1 < \frac{1}{2}(1 - \frac{1}{9}\sqrt{69}) = 0.038520896\ldots$$

in which case, the four characteristic roots at the equilibrium L_4 are as follows:

$$s_1 = i\omega_1, \qquad s_2 = i\omega_2, \qquad s_3 = -i\omega_1, \qquad s_4 = -i\omega_2$$

where the real numbers ω_1 and ω_2 are characterized unambiguously by the set of relations

$$0 < \omega_2 < 1/\sqrt{2} < \omega_1 < 1, \tag{2}$$

$$\omega_1^2 + \omega_2^2 = 1,$$

$$16\omega_1^2\omega_2^2 = 27(1-\gamma^2).$$

These classical results define the stability and the periods of the linear oscillations about the equilibrium L_4 omitting the nonlinear terms in the Lagrangian function. What interests us more now is, however, to investigate the perturbations in the coordinates x,y and in the basic frequencies arising from the presence of quadratic and cubic terms on the right-hand members of equations. In order to do so, let us hereafter put, for the sake of simplicity,

$$k = (2\omega_1^2-1/2)^{1/2} = (1-2\omega_2^2)^{1/2},$$

$$\ell_i = (9+4\omega_i^2)^{1/2} \qquad (i=1,2),$$

$$m_i = 1 + 4\omega_i^2 \qquad (i=1,2),$$

$$n_i = 9 - 4\omega_i^2 \qquad (i=1,2).$$

Then, following Breakwell and Pringle (1966), we introduce the symplectic matrix

$$\mathcal{A} = (a_{ij})_{1\leq i,j\leq 4}$$

whose elements are defined as follows:

$$a_{11} = 0, \qquad a_{21} = -4\omega_1/k\ell_1,$$
$$a_{12} = 0, \qquad a_{22} = -4\omega_2/k\ell_2,$$
$$a_{13} = \ell_1/2k\omega_1, \qquad a_{23} = -3\gamma\sqrt{3}/2k\ell_1\omega_1,$$
$$a_{14} = -\ell_2/2k\omega_2, \qquad a_{24} = 3\gamma\sqrt{3}/2k\ell_2\omega_2,$$

$$a_{31} = -m_1\omega_1/2k\ell_1, \qquad a_{41} = 3\gamma\sqrt{3}\omega_1/2k\ell_1,$$
$$a_{32} = -m_2\omega_2/2k\ell_2, \qquad a_{42} = 3\gamma\sqrt{3}\omega_2/2k\ell_2,$$
$$a_{33} = 3\gamma\sqrt{3}/2k\ell_1\omega_1, \qquad a_{43} = n_1/2k\ell_1\omega_1,$$
$$a_{34} = -3\gamma\sqrt{3}/2k\ell_2\omega_2, \qquad a_{44} = -n_2/2k\ell_2\omega_2.$$

Accordingly the transformation from the phase space (x,y,p_x,p_y) into the phase space product of the angle-coordinates (ϕ_1,ϕ_2) and of the two action momenta (I_1,I_2), as it is defined by the equations

$$x = a_{13}P_1 + a_{14}P_2,$$
$$y = a_{21}Q_1 + a_{22}Q_2 + a_{23}P_1 + a_{24}P_2,$$
$$p_x = a_{31}Q_1 + a_{32}Q_2 + a_{33}P_1 + a_{34}P_2,$$
$$p_y = a_{41}Q_1 + a_{42}Q_2 + a_{43}P_1 + a_{44}P_2,$$

wherein

$$Q_i = (2I_i/\omega_i)^{1/2}\sin\phi_i, \qquad (i=1,2)$$

$$P_i = (2I_i\omega_i)^{1/2}\cos\phi_i, \qquad (i=1,2)$$

is completely canonical. Moreover, in the new phase variables, the second order part of the Hamiltonian assumes the normal form

$$\mathcal{H}_2 = \omega_1 I_1 - \omega_2 I_2,$$

and its general solution is

$$\left.\begin{aligned}\phi_i &= \omega_i t + \text{const.},\\ I_i &= \text{const.}\end{aligned}\right\} \qquad (i=1,2)$$

If the oscillations about L_4 were exactly linear, the integrals of motion would, in fact, be represented by the above relations and the corresponding orbits would be given by the formulae

$$x = \frac{\ell_1}{k\sqrt{2\omega_1}}\, I_1^{1/2}\cos\phi_1 - \frac{\ell_2}{k\sqrt{2\omega_2}}\, I_2^{1/2}\cos\phi_2$$

$$y = -\frac{3\gamma\sqrt{3}}{k\ell_1\sqrt{2\omega_1}}\, I_1^{1/2}\cos\phi_1 + \frac{3\gamma\sqrt{3}}{k\ell_2\sqrt{2\omega_2}}\, I_2^{1/2}\cos\phi_2$$

$$- \frac{4\sqrt{2\omega_1}}{k\ell_1}\, I_1^{1/2}\sin\phi_1 - \frac{4\sqrt{2\omega_2}}{k\ell_2}\, I_2^{1/2}\sin\phi_2.$$

3. SECOND ORDER NORMALIZATION

J. Henrard (1966) has shown how to carry on in a straightforward manner Birkhoff's normalization without introducing generating functions and without inverting power series.

The coordinates (x,y) are to be expanded in double d'Alembert series:

$$x = \sum_{n\geq 1} B_n^{1,0}, \qquad y = \sum_{n\geq 1} B_n^{0,1} \tag{3}$$

where the homogeneous components x_n and y_n of degree n are of the form

$$\sum_{0\leq m\leq n} I_1^{(n-m)/2} I_2^{m/2} \sum_{(p,q)} [C_{n-m,m,p,q}\cos(p\phi_1+q\phi_2)+S_{n-m,m,p,q}\sin(p\phi_1+q\phi_2)]. \tag{4}$$

In (4) the double summation over the indices p and q is subject to the following conventions:

a) p runs over those integers in the interval $0 \leq p \leq n-m$ that have the same parity as $n-m$;

b) q runs over those integers in the interval $-m \leq q \leq m$ that have the same parity as m.

In the developments (3), the quantities I_1 and I_2 are to be taken as constants of integration, while ϕ_1 and ϕ_2 are to be determined as linear functions of the time in such a way that

$$\dot{\phi}_1 = \omega_1 + \sum_{n\geq 1} f_{2n}(I_1,I_2),$$

$$\dot{\phi}_2 = -\omega_2 + \sum_{n\geq 1} g_{2n}(I_1,I_2)$$

where, for any $n \geq 1$, f_{2n} and g_{2n} are homogeneous polynomials of degree n in the actions I_1 and I_2. The canonical character of the transformation will be ensured formally by requesting that the double d'Alembert series (3) satisfy the identities

$$\begin{aligned} &(x;y) = 0, \\ &(x;\dot{x}) = 1, \quad (y;\dot{x}) = 0, \\ &(x;\dot{y}) = 0, \quad (y;\dot{y}) = 1, \quad (\dot{x},\dot{y}) = 0 \end{aligned}$$

where the left-hand members stand for the Poisson bracket with respect to the phase variables (ϕ_1,ϕ_2,I_1,I_2), thus for instance

$$(x;y) = \frac{\partial x}{\partial \phi_1}\frac{\partial y}{\partial I_1} - \frac{\partial x}{\partial I_1}\frac{\partial y}{\partial \phi_1} + \frac{\partial x}{\partial \phi_2}\frac{\partial y}{\partial I_2} - \frac{\partial x}{\partial I_2}\frac{\partial y}{\partial \phi_2} ,$$

and where $\dot{x},\dot{y}$ should be taken as the composition derivatives

$$\dot{x} = \dot{\phi}_1 \frac{\partial x}{\partial \phi_1} + \dot{\phi}_2 \frac{\partial x}{\partial \phi_2} ,$$

$$\dot{y} = \dot{\phi}_1 \frac{\partial y}{\partial \phi_1} + \dot{\phi}_2 \frac{\partial y}{\partial \phi_2} .$$

In this way, a Birkhoff normalizing transformation can be constructed entirely by the method of undetermined coefficients.

As it is shown elsewhere (Deprit *et al* 1966b), the homogeneous components of order 2 in the coordinates x and y are solutions of the partial differential equations

$$\left[\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right)^2 - \frac{3}{4}\right] B_2^{1,0} - \left[2\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right) + \frac{3}{4}\gamma\sqrt{3}\right] B_2^{0,1} = X_2,$$

$$\left[\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right)^2 - \frac{9}{4}\right] B_2^{0,1} + \left[2\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right) - \frac{3}{4}\gamma\sqrt{3}\right] B_2^{1,0} = Y_2. \qquad (5)$$

The right-hand members are the homogeneous components of order 2 obtained on substituting in the derivatives

$$\frac{\partial\mathcal{L}_2}{\partial x} = \frac{21}{16}\gamma x^2 - \frac{3}{8}\sqrt{3}\,xy - \frac{33}{16}\gamma y^2,$$

$$\frac{\partial\mathcal{L}_2}{\partial y} = -\frac{3}{16}\sqrt{3}\,x^2 - \frac{33}{8}\gamma xy - \frac{9}{16}\sqrt{3}\,y^2$$

the first order expressions already obtained for x and y. From eliminating in turn $B_2^{0,1}$ and $B_2^{1,0}$ from the system (5), we arrive at the partial differential system

$$\Delta_1\Delta_2 B_2^{1,0} = \Phi_2,$$

$$\Delta_1\Delta_2 B_2^{0,1} = \Psi_2$$

where the differential operators on the left-hand members are

$$\Delta_1 = \left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right)^2 + \omega_1^2,$$

$$\Delta_2 = \left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right)^2 + \omega_2^2$$

and the functions on the right-hand members are to be constructed as follows:

$$\Phi_2 = \left[\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right)^2 - \frac{9}{4}\right] X_2 + \left[2\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right) + \frac{3}{4}\gamma\sqrt{3}\right] Y_2,$$

$$\Psi_2 = \left[\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right)^2 - \frac{3}{4}\right] Y_2 - \left[2\left(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}\right) - \frac{3}{4}\gamma\sqrt{3}\right] X_2.$$

It is quite essential to remark that no term in $\cos\phi_1$, $\sin\phi_1$ or $\cos\phi_2$, $\sin\phi_2$ appears in X_2 and Y_2, so that Φ_2 and Ψ_2 are also free of such terms. Hence the homogeneous components $B_2^{1,0}$, $B_2^{0,1}$ can be obtained in a straightforward manner, on applying the differentiation rules:

$$\Delta_1\Delta_2 \cos(p\phi_1 + q\phi_2) = \Delta_{p,q} \cos(p\phi_1 + q\phi_2),$$

$$\Delta_1\Delta_2 \sin(p\phi_1 + q\phi_2) = \Delta_{p,q} \sin(p\phi_1 + q\phi_2),$$

where

$$\Delta_{p,q} = [\omega_1^2 - (p\omega_1 - q\omega_2)^2][\omega_2^2 - (p\omega_1 - q\omega_2)^2].$$

Since we assumed that the irrationality condition (A_3) is verified, none of the divisors $\Delta_{2,0}$, $\Delta_{1,1}$, $\Delta_{1,-1}$, $\Delta_{0,2}$ is zero. Notice that, in view of the inequalities (2), the condition (A_3) is fulfilled if and only if, in the interval $0 < \mu < \mu_1$, the mass ratio does not take the critical value

$$\mu_2 = \frac{1}{2}\left[1 - \frac{1}{45}\sqrt{1833}\right] = 0.024293897\ldots .$$

A simple sequence of algebraic manipulations leads eventually to the components $B_2^{1,0}$ and $B_2^{0,1}$. It has been checked that they actually transform the Hamiltonian function $\mathcal{H}_3 = -\mathcal{L}_3$ into the zero function. Both components are listed in Table I.

Table I

Homogeneous components of order 2 in the coordinates x and y

	$B_2^{1,0}$	$B_2^{0,1}$
I_1	$\frac{3}{4}\,\frac{3-\omega_1^2}{k^2\omega_1}\,\gamma$	$\frac{1}{12}\,\frac{27-31\omega_1^2+22\omega_1^4}{k^2\omega_1}\,\sqrt{3}$
$I_1\cos 2\phi_1$	$\frac{3}{4}\,\frac{27+91\omega_1^2-4\omega_1^4}{k^2\ell_1^2\omega_1}\,\gamma$	$\frac{1}{12}\,\frac{243+45\omega_1^2+74\omega_1^4+88\omega_1^6}{k^2\ell_1^2\omega_1}\,\sqrt{3}$
$I_1^{1/2}I_2^{1/2}\cos(\phi_1+\phi_2)$	$-\frac{3}{2}\,\frac{45-44\omega_1\omega_2+18\omega_1^2\omega_2^2}{k^2\ell_1\ell_2\sqrt{\omega_1\omega_2}}\,\gamma$	$-\frac{1}{6}\,\frac{297-108\omega_1\omega_2-224\omega_1^2\omega_2^2}{k^2\ell_1\ell_2\sqrt{\omega_1\omega_2}}\,\sqrt{3}$
$I_1^{1/2}I_2^{1/2}\cos(\phi_1-\phi_2)$	$-\frac{3}{2}\,\frac{45+44\omega_1\omega_2+18\omega_1^2\omega_2^2}{k^2\ell_1\ell_2\sqrt{\omega_1\omega_2}}\,\gamma$	$-\frac{1}{6}\,\frac{297+108\omega_1\omega_2-224\omega_1^2\omega_2^2}{k^2\ell_1\ell_2\sqrt{\omega_1\omega_2}}\,\sqrt{3}$
I_2	$\frac{3}{4}\,\frac{3-\omega_2^2}{k^2\omega_2}\,\gamma$	$\frac{1}{12}\,\frac{27-31\omega_2^2+22\omega_2^4}{k^2\omega_2}\,\sqrt{3}$
$I_2\cos 2\phi_2$	$\frac{3}{4}\,\frac{27+91\omega_2^2-4\omega_2^4}{k^2\ell_2^2\omega_2}\,\gamma$	$\frac{1}{12}\,\frac{243+45\omega_2^2+74\omega_2^4+88\omega_2^6}{k^2\ell_2^2\omega_2}\,\sqrt{3}$
$I_1\sin 2\phi_1$	$-\frac{1}{3}\,\frac{54-53\omega_1^2+44\omega_1^4}{k^2\ell_1^2}\,\sqrt{3}$	$3\,\frac{18+11\omega_1^2}{k^2\ell_1^2}\,\gamma$
$I_1^{1/2}I_2^{1/2}\sin(\phi_1+\phi_2)$	$\frac{1}{3}\,\frac{54+9\omega_1\omega_2-44\omega_1^2\omega_2^2}{k^2\ell_1\ell_2}\cdot\frac{\omega_1-\omega_2}{\sqrt{\omega_1\omega_2}}\,\sqrt{3}$	$-3\,\frac{18-11\omega_1\omega_2}{k^2\ell_1\ell_2}\cdot\frac{\omega_1-\omega_2}{\sqrt{\omega_1\omega_2}}\,\gamma$
$I_1^{1/2}I_2^{1/2}\sin(\phi_1-\phi_2)$	$\frac{1}{3}\,\frac{54-9\omega_1\omega_2-44\omega_1^2\omega_2^2}{k^2\ell_1\ell_2}\cdot\frac{\omega_1+\omega_2}{\sqrt{\omega_1\omega_2}}\,\sqrt{3}$	$-3\,\frac{18+11\omega_1\omega_2}{k^2\ell_1\ell_2}\cdot\frac{\omega_1+\omega_2}{\sqrt{\omega_1\omega_2}}\,\gamma$
$I_2\sin 2\phi_2$	$\frac{1}{3}\,\frac{54-53\omega_2^2+44\omega_2^4}{k^2\ell_2^2}\,\sqrt{3}$	$-3\,\frac{18+11\omega_2^2}{k^2\ell_2^2}\,\gamma$

4. SECOND ORDER COEFFICIENTS IN THE FREQUENCIES

The homogeneous components $B_3^{1,0}$ and $B_3^{0,1}$ of the third order in the coordinates x and y, as well as the homogeneous polynomials f_2 and g_2 of the second order in the frequencies $\dot{\phi}_1$ and $\dot{\phi}_2$, satisfy the partial differential equations

$$
\begin{aligned}
&\left[\left(\omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2}\right)^2 - \frac{3}{4}\right] B_3^{1,0} - \left[2\left(\omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2}\right) + \frac{3}{4}\gamma\sqrt{3}\right] B_3^{0,1} \\
&\qquad + 2f_2 \frac{\partial}{\partial \phi_1}\left[\omega_1 \frac{\partial}{\partial \phi_1} B_1^{1,0} - B_1^{0,1}\right] - 2g_2\left[\omega_2 \frac{\partial}{\partial \phi_2} B_1^{1,0} + B_1^{0,1}\right] = X_3, \\
&\left[\left(\omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2}\right)^2 - \frac{9}{4}\right] B_3^{0,1} + \left[2\left(\omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2}\right) - \frac{3}{4}\gamma\sqrt{3}\right] B_3^{1,0} \\
&\qquad + 2f_2 \frac{\partial}{\partial \phi_1}\left[\omega_1 \frac{\partial}{\partial \phi_1} B_1^{0,1} + B_1^{1,0}\right] - 2g_2\left[\omega_2 \frac{\partial}{\partial \phi_2} B_1^{0,1} - B_1^{1,0}\right] = Y_3.
\end{aligned}
\tag{6}
$$

The right-hand members are the homogeneous components of order 3 obtained on substituting in the derivatives

$$\frac{\partial}{\partial x}(\mathcal{L}_3 + \mathcal{L}_4), \qquad \frac{\partial}{\partial x}(\mathcal{L}_3 + \mathcal{L}_4)$$

the expansions obtained for the coordinates x and y up to the second order so far.

As we did at the second order, we eliminate in turn $B_3^{0,1}$ and $B_3^{1,0}$ from the system (6) so that we obtain the equations

$$\Delta_1\Delta_2 B_3^{1,0} = \Phi_3 - 2f_2 P - 2g_2 Q,$$

$$\Delta_1\Delta_2 B_3^{0,1} = \Psi_3 - 2f_2 U - 2g_2 V,$$

where we put

$$\Phi_3 = \left[(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2})^2 - \frac{9}{4}\right] X_3 + \left[2(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}) + \frac{3}{4}\gamma\sqrt{3}\right] Y_3,$$

$$\Psi_3 = \left[(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2})^2 - \frac{3}{4}\right] Y_3 - \left[2(\omega_1 \frac{\partial}{\partial\phi_1} - \omega_2 \frac{\partial}{\partial\phi_2}) - \frac{3}{4}\gamma\sqrt{3}\right] X_3,$$

$$P = \frac{\partial}{\partial\phi_1}\left[(\omega_1^2 \frac{\partial^2}{\partial\phi_1^2} - \frac{9}{4})(\omega_1 \frac{\partial}{\partial\phi_1} B_1^{1,0} - B_1^{0,1}) + (2\omega_1 \frac{\partial}{\partial\phi_1} + \frac{3}{4}\gamma\sqrt{3})(\omega_1 \frac{\partial}{\partial\phi_1} B_1^{0,1} + B_1^{1,0})\right],$$

$$Q = \frac{\partial}{\partial\phi_2}\left[(\omega_2^2 \frac{\partial^2}{\partial\phi_2^2} - \frac{9}{4})(-\omega_2 \frac{\partial}{\partial\phi_2} B_1^{1,0} - B_1^{0,1}) + (-2\omega_2 \frac{\partial}{\partial\phi_2} + \frac{3}{4}\gamma\sqrt{3})(-\omega_2 \frac{\partial}{\partial\phi_2} B_1^{0,1} + B_1^{1,0})\right],$$

$$U = \frac{\partial}{\partial\phi_1}\left[(\omega_1^2 \frac{\partial^2}{\partial\phi_1^2} - \frac{3}{4})(\omega_1 \frac{\partial}{\partial\phi_1} B_1^{0,1} + B_1^{1,0}) - (2\omega_1 \frac{\partial}{\partial\phi_1} - \frac{3}{4}\gamma\sqrt{3})(\omega_1 \frac{\partial}{\partial\phi_1} B_1^{1,0} - B_1^{0,1})\right],$$

$$V = \frac{\partial}{\partial\phi_2}\left[(\omega_2^2 \frac{\partial^2}{\partial\phi_2^2} - \frac{3}{4})(-\omega_2 \frac{\partial}{\partial\phi_2} B_1^{0,1} + B_1^{1,0}) - (-2\omega_2 \frac{\partial}{\partial\phi_2} - \frac{3}{4}\gamma\sqrt{3})(-\omega_2 \frac{\partial}{\partial\phi_1} B_1^{1,0} - B_1^{0,1})\right].$$

We do not need to evaluate the components $B_3^{1,0}$ and $B_3^{0,1}$. For it will be sufficient for our purpose to compute the coefficients of $\cos\phi_1$, $\sin\phi_1$, $\cos\phi_2$ and $\sin\phi_2$ in the right-hand members of the partial equations. Those are the critical terms, because they belong to the kernels of the differential operator $\Delta_1\Delta_2$. Now they can be eliminated by a proper choice of the coefficients in the polynomials

$$f_2 = f_{2,0}I_1 + f_{0,2}I_2,$$

$$g_2 = g_{2,0}I_1 + g_{0,2}I_2.$$

As a matter of fact, it turns out that the system of 16 linear equations in the four unknowns $f_{2,0}$, $f_{0,2}$, $g_{2,0}$, $g_{0,2}$ to which we arrived is consistent and yields a unique solution. We have listed it in Table II.

Table II

Component of fourth order in the normalized Hamiltonian

$$f_{2,0} = A = \frac{1}{72}\,\frac{\omega_2^2(81-696\omega_1^2+124\omega_1^4)}{(1-2\omega_1^2)^2(1-5\omega_1^2)},$$

$$f_{0,2} = g_{2,0} = B = -\frac{1}{6}\,\frac{\omega_1\omega_2(43+64\omega_1^2\omega_2^2)}{(1-2\omega_1^2)(1-2\omega_2^2)(1-5\omega_1^2)(1-5\omega_2^2)},$$

$$g_{0,2} = C = \frac{1}{72}\,\frac{\omega_1^2(81-696\omega_2^2+124\omega_2^4)}{(1-2\omega_2^2)^2(1-5\omega_2^2)}.$$

These results have been checked. For the system Sun-Jupiter ($\mu = 0.000953875\ldots$), the Birkhoff's normalization has been carried numerically up to order thirteen (Deprit *et al* 1966b); there it was found that

$$A = 0.01135436,$$

$$B = -0.1551412,$$

$$C = 1.119733.$$

The same task has been performed (Deprit *et al* 1966a) for the system Earth-Moon ($\mu = 0.0121500\ldots$) where it was found that

$$A = 0.2313561,$$

$$B = -1.712630,$$

$$C = 0.6771718.$$

Both systems of values have been recovered from the definitions in Table II valid for any permissible mass ratios.

Another check is provided by the analytical expansions of the two natural families of periodic orbits issued from L_4, as they were performed by Pedersen (1935). There the author finds that the frequency along the family of short period orbits is a series

$$\dot{\phi}_1 = \omega_1 + E\varepsilon^2 + \cdots$$

in a certain orbital parameter ε. For the coefficient E, he gives the expression

$$E = -\frac{243-2007\omega_1^2-648\omega_1^4+2908\omega_1^6-496\omega_1^8}{2304\omega_1(1-2\omega_1^2)(1-5\omega_1^2)} .$$

If our computation is correct, our coefficient A should be traced in Pedersen's coefficient E. But this is the case, since

$$243 - 2007\omega_1^2 - 648\omega_1^4 + 2908\omega_1^6 - 496\omega_1^8 = \omega_2^2(3+4\omega_1^2)(81-696\omega_1^2+124\omega_1^4);$$

hence

$$E = -\frac{3+4\omega_1^2}{32\omega_1} A.$$

Birkhoff's normalization cannot be carried at order three unless the irrationality condition (A_4) is assumed. But (A_4) means that, from the interval $0 < \mu < \mu_1$ of permissible mass ratios, besides μ_2, the value

$$\mu_3 = \frac{1}{2}\,(1 - \frac{2}{45}\,\sqrt{117}) = 0.013516016...$$

should be excluded.

Thus A, B, C are functions of the mass ratio on the interval $0 < \mu < \mu_1$ out of which the critical values μ_2 and μ_3 have been taken. The main qualitative characteristics of these functions can be read from Figure 1 in which we have plotted respectively A, B, C *versus* the mass ratio.

5. STABILITY

Knowing the fourth order part

$$\mathcal{H}_4 = \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2)$$

of the normalized Hamiltonian, we compute the determinant D which decides about the stability at the equilibrium in the theorems of Arnol'd and Moser. Thus we find that

$$D = -(A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2),$$

a quantity which can easily be expressed as a rational function of the product $\omega_1^2\omega_2^2$:

$$D = -\frac{1}{8}\,\frac{36-541\omega_1^2\omega_2^2+644\omega_1^4\omega_2^4}{(1-4\omega_1^2\omega_2^2)(4-25\omega_1^2\omega_2^2)}\ .$$

It is easy to see that, in the interval $0 < \mu < \mu_1$, it possesses one, and only one, zero for the mass ratio

$$\mu_c = 0.01091367\ldots\ .$$

Thus, stability of the equilibrium at L_4 cannot be decided for this mass ratio from applying Moser's theorem.

The qualitative features of the determinant $-D$ as a function of the mass ratio μ are summarized in the Figure 2.

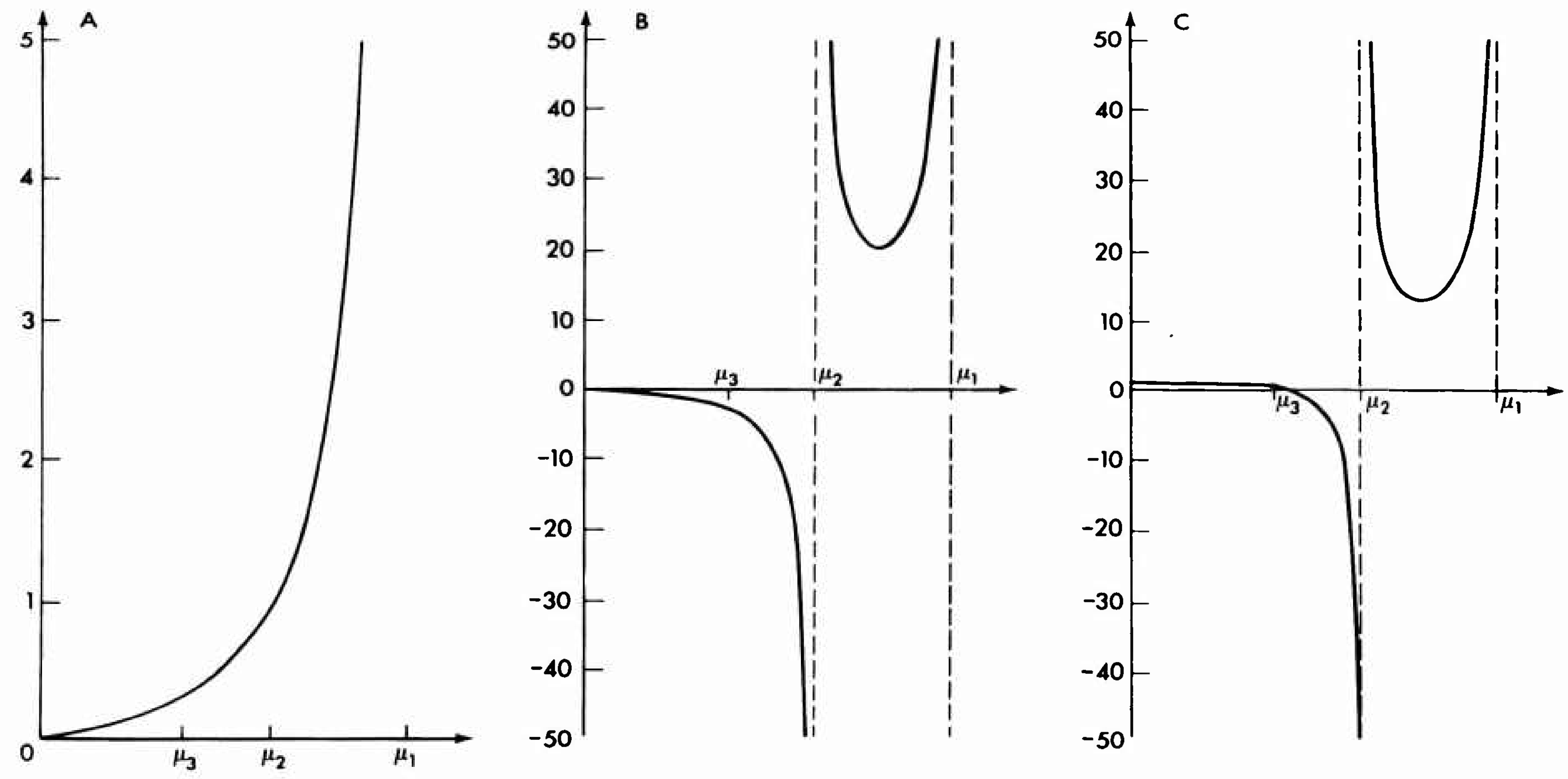

Fig. 1. The coefficients in the component of order 4 of the normalized Hamiltonian.

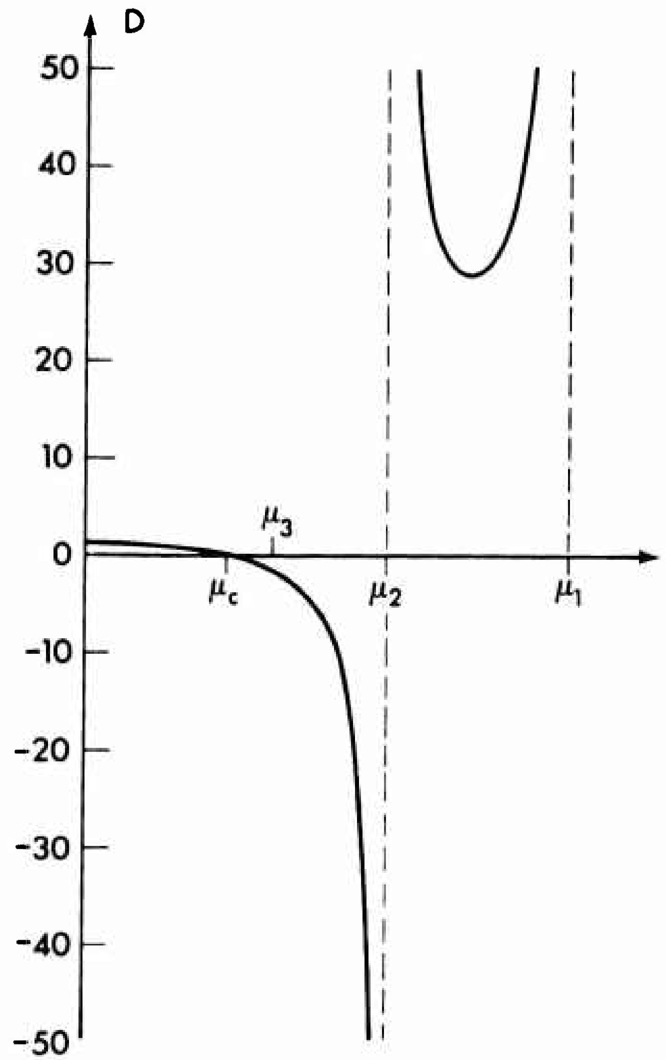

Fig. 2. Stability condition from the normalized Hamiltonian at order 4.

6. CONCLUSIONS

In the planar Restricted Problem of Three Bodies, the question of stability of the equilateral positions of equilibrium can be answered in the affirmative for all values of the mass ratio μ in the open interval $0 < \mu < \mu_1$ except at the critical mass ratios $\mu_2 = 0.024293...$ and $\mu_3 = 0.013516...$ and at a third point $\mu_c = 0.010913...$. At these three points, Moser's theorem does not apply.

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