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ASYMPTOTICALLY OPTIMAL STATISTICS IN SOME MODELS WITH INCREASING FAILURE RATE AVERAGES

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ABSTRACT

Let F and G be defined by $F(t) = H(\gamma t)$ and $G(t) = H(\theta t)$ where H is unknown and $H(0) = 0$. For testing the equality of the means of F and G in the two-sample problem; it is shown that the Savage (The Annals of Mathematical Statistics (1956) pp 590-615) statistic maximizes the minimum power over IFRA (or IFR) distributions asymptotically. Asymptotic uniqueness holds only in a class of rank tests. The results are extended to censored samples, the problem of estimating the ratio of the means, and the k-sample problem.

1. INTRODUCTION AND SUMMARY. Birnbaum, Esary and Marshall (1966) have shown that the class $\mathcal{F}$ of distributions with increasing failure rate averages (IFRA) characterizes the concept of wear-out in the sense that $\mathcal{F}$ is the smallest class that contains the exponential distributions and is closed under the formation of coherent systems.

In this note, statistical inference for models in which the distributions are unknown and IFRA will be considered. Let F and G be defined by

$$F(t) = H(t/\theta) \quad \text{and} \quad G(t) = H(t/\gamma) \tag{1.1}$$

where H is an unknown IFRA distribution with $H(0) = 0$. Then, for the two-sample problem where one tests the equality of the means of F and G, it is shown that the Savage (1956) statistic maximizes the minimum power over IFRA distributions asymptotically. This asymptotic minimax solution is extended to censored samples and it turns out that the Gastwirth (1965) modified version of the Savage statistic is asymptotically minimax for this case. Asymptotic uniqueness of these minimax solutions holds only in a class of rank tests. The results are extended to obtain an estimate of the ratio of the means that minimizes the maximum asymptotic variance over IFRA distributions.

Finally, the results are shown to hold also for distributions with increasing failure rates (IFR), and extensions to the k - sample problem are given.

2. THE TWO-SAMPLE LIFE-TESTING PROBLEM. $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are independent random samples from populations with life distributions F and G. $N = m + n$, $F(t) = H(t/\theta_N)$, $G(t) = H(t/\gamma_N)$, H has the density h and is IFRA, i.e., $H(0) = 0$ and for each $t > 0$,

$$\frac{d}{dt}\left\{-\log\left[1 - H(t)\right]/t\right\} = \frac{\ln\,[1-H(t)]}{t^2} + \frac{h(t)}{t[1-H(t)]} \geq 0 \tag{2.1}$$

$r_1, \ldots, r_m$ denote the ranks of the x's in the combined sample. The level α Savage (1956) test ψ_N of H_o: $\Delta_N = (\theta_N/\gamma_N) = 1$ against $\Delta_N > 1$ rejects for large values of the statistic

(2.2) $$S_N = \frac{1}{m} \sum_{i=1}^{m} - \ln\left(1 - \frac{r_i}{N+1}\right)$$

It is assumed throughout that

(2.3) $$0 < \lim_{N\to\infty} (m/N) = \lambda < 1$$

Let $0 \leq c \leq \infty$ and consider sequences of alternatives $\{\Delta_N\}$ satisfying

(2.4) $$\lim_{N\to\infty} N^{\frac{1}{2}} (\Delta_N - 1) = c$$

Then <u>the asymptotic power function</u> $\beta(c;\varphi,H)$ of a test φ_N is defined as the limit of the power for such alternatives, i.e.

(2.5) $$\beta(c;\varphi,H) = \liminf_{N\to\infty} \beta_N(\varphi_N|H)$$

where $\beta_N(\varphi_N|H) = E(\varphi_N|F_N,G_N)$ denotes the power of φ_N when $F_N(t) = H(t/\theta_N)$, $G_N(t) = H(t/\gamma_N)$ and $\Delta_N = \theta_N/\gamma_N$ satisfies (2.4).

Let Φ be the standard normal distribution function. Then the results of Chernoff and Savage (1958), Fatou's Lemma, and a few computations yield

<u>Lemma 2.1.</u> <u>Suppose</u> H <u>has a density</u> h <u>and that</u> $H(o) = 0$, <u>then the asymptotic power function of the level</u> α <u>Savage test</u> ψ_N <u>is given by</u>

(2.6) $$\beta(c;\psi,H) = \Phi\left(\Phi^{-1}(\alpha) + c\,[\lambda(1-\lambda)]^{\frac{1}{2}} \int_0^\infty \frac{th(t)}{1-H(t)}\, d\,H(t)\right)$$

The next result shows that ψ and the exponential distribution $K_\sigma(x) = 1 - \exp(-x/\sigma)$ is a saddle point for the asymptotic power function $\beta(c;\varphi,H)$. In other words, ψ is worst for the exponential distrібtuion, but is better than all other tests for this distribution.

Theorem 2.1. For all $o \leq c \leq \infty$ and all $\sigma > o$,

(2.7) $$\sup_{\varphi} \beta(c;\varphi,K_{\sigma}) = \beta(c;\psi,K_{\sigma}) = \inf_{H} \beta(c;\psi,H)$$

where H ranges over the class of IFRA distribtuions with a density, and φ_N ranges over the class of all level α tests.

Proof. The left hand equality was proved by Capon (1961) by essentially comparing ψ_N with the Neyman - Pearson test for K_{σ}. To prove the right equality, note that (2.1) yields

(2.8) $$\frac{th(t)}{1-H(t)} \geq - \ln [1-H(t)] \quad ,$$

thus

$$\int_{0}^{\infty} \frac{th(t)}{1-H(t)} \, d\,H(t) \geq \int_{0}^{\infty} - \ln [1-H(t)] \, d\,H(t) = 1$$

The equality signs hold if and only if H has a constant failure rate average, i.e., if and only if H is exponential, thus

Corollary 2.1. If H is IFRA, has a density, and is not exponent'al, then

(2.9) $$\beta(c;\psi,K_{\sigma}) < \beta(c;\psi,H)$$

The minimax property of the Savage statistic now follows at once from Theorem 2.1.

Theorem 2.2. The level α Savage test ψ_N is asymptotically minimax over the class Ω of all IFRA distributions with a density, i.e. if H ranges over Ω, then

(2.10) $$\inf_{H} \beta(c;\psi,H) \geq \inf_{H} \beta(c;\varphi,H)$$

for all level α tests φ_N.

Remarks

(i) H is said to have increasing failure rate (IFR) [1] if $H(0)=0$ and $h(t)/[1-H(t)]$ is nondecreasing in $t > 0$. The class of IFR distributions contains the class of exponential distributions and is contained in the class of IFRA distributions. It follows that Theorem 2.1, Corollary 2.1 and Theorem 2.2 holds also for this class.

(ii) The results of this section are stronger than the minimax results of [6] in the sense that no conditions such as bounds on the Kolmogorov distances or variances of the distributions are needed.

(iii) The lim inf in (2.5) can be replaced by a limit if one assumes conditions as in Lemma 3 of Hodges and Lehmann (1963). The results hold if lim inf is replaced by lim sup or partially replaced by lim sup as in [6].

(iv) An asymptotically equivalent form of the Savage statistic is (see [9,p. 1127]),

$$\sum_{i=1}^{m} J_o(r_i) \text{ , where}$$

(2.11)

$$J_o(k) = \sum_{j=N-k+1}^{N} \frac{1}{j}$$

(v) The results in this section hold if one, instead of considering level α tests, considers φ_N with asymptotic level α, i.e. tests for which $E(\varphi_N | \theta = \gamma) \to \alpha$ as $N \to \infty$.

(vi) The one-sided alternative $\Delta > 1$ can be replaced by the two-sided alternative $\Delta \neq 1$.

(vii) For the k-sample problem with model $F_i(x) = H(x/[1+\theta c_i])$; $i=1,\dots,k$; the Puri (1964) extention of the Savage statistic is asymptotically minimax for testing $H_o^k : \theta = 0$ against $\theta > 0$ (or $\theta \neq 0$).

3. EFFICIENCY OF THE BEST TEST FOR EXPONENTIAL MODELS. When H equals an exponential distribution $K_\sigma(t) = 1 - \exp(-t/\sigma)$, then the uniformly most powerful level α test [7] φ_N^* of $\theta = \gamma$ against $\theta > \gamma$ rejects when

(3.1) $$T = \frac{1}{m}\sum_{i=1}^{m} X_i \Big/ \frac{1}{n}\sum_{i=1}^{n} Y_i > F_{2m,2n}(\alpha)$$

where $F_{2m,2n}(\alpha)$ is obtained from the tables of the F distribution. In this section the performance of T is investigated when the assumption of exponentiality is violated and H is an IFRA distribution.

Upon writing

(3.2) $$\sqrt{N}\,(T - \Delta) = \sqrt{N}\,(\overline{X} - \Delta\overline{Y}) / \overline{Y} ,$$

it is clear that $\sqrt{N}\,(T - \Delta)$ has an asymptotic normal distribution with mean zero and variance

(3.3) $$\sigma^2(T) = \Delta^2\,\sigma^2(H) / \lambda(1-\lambda)\,\mu^2(H) \quad \text{where}$$
$$\mu(H) = \int_0^\infty t\,dH(t) \quad \text{and} \quad \sigma^2(H) = \int_0^\infty t^2 dH(t) - \mu^2(H) .$$

When H is exponential, then $\sigma^2(H) = \mu^2(H)$. It follows that when H is such that $\sigma^2(H) \neq \mu^2(H)$, then φ_N^* does not have level α asymptotically, in fact

(3.4) $$E(\varphi_N^* \mid \theta = \gamma) \to \Phi\,(\Phi^{-1}(\alpha)\,\mu(H)/\sigma(H)) \quad \text{as} \quad N \to \infty$$

Thus when $\alpha < \frac{1}{2}$ and $\mu(H) > \sigma(H)$, then the asymptotic level of φ_N^* is less than α . Barlow, Marshall and Proschan (1963) have essentially shown that for IFRA distributions, $\mu(H) \geq \sigma(H)$. The asymptotic power function of φ_N^* is

$$(3.5) \qquad \beta(c;\varphi^* , H) = \Phi(\{\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)]^{\frac{1}{2}}\} \mu(H)/\sigma(H))$$

φ_N^* can easily be modified to have asymptotic level α by dividing

$\sqrt{N}$ (T-1) by a consistent estimate of

$r(H) = \sigma(H)/\mu(H)$; e.g.

$\hat{r}(H) = \hat{\sigma}(H)/\hat{\mu}(H)$ with

$\hat{\mu}(H) = \frac{1}{N} (\Sigma x_i + \Sigma y_i)$ and

$\hat{\sigma}(H) = \frac{1}{N} (\Sigma x_i^2 + \Sigma y_i^2) - \hat{\mu}^2(H)$.

For this test, $\hat{\varphi}_N$, one has

$$(3.6) \qquad \beta(c;\hat{\varphi},H) = \Phi(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda]^{\frac{1}{2}}\mu(H)/\sigma(H))$$

Since $\mu(H) \geq \sigma(H)$ [1] when H is IFRA, $\mu(k_\sigma) = \sigma(k_\sigma)$, and since $\beta(c;\psi,k_\sigma) = \beta(c,\hat{\varphi},k_\sigma)$, then (2.7) and remark (i) of Section 2 yields

<u>Theorem 3.1.</u> <u>For all</u> $o \leq c \leq \infty$ <u>and all</u> $\sigma > o$,

$$(3.7) \qquad \sup_\varphi \beta(c;\varphi,k_\sigma) = \beta(c;\hat{\varphi},k_\sigma) = \inf_H \beta(c,\hat{\varphi},H)$$

<u>where</u> H <u>ranges over the class of</u> IFRA <u>distributions and</u> φ_N <u>ranges over the class of all tests with asymptotic level</u> α .

Thus $\hat{\varphi}_N$ is asymptotically minimax in the sense of Theorem 2.2 for the class of IFRA distributions and the class of tests with asypmtotic level α . To see that this is not true for φ_N^* , let H be an IFRA distribution with $\mu(H) > \sigma(H)$, then for each $\alpha < \frac{1}{2}$,

$$(3.8) \qquad \beta(c;\varphi^*,H) < \beta(c;\hat{\varphi},k_\sigma) \quad \text{for} \quad 0 \leq c < \sigma(H)/\mu(H) .$$

Let Pitman asymptotic efficiency be as defined in [10]. It follows from (2.6) and (3.6) that the Pitman efficiency of the Savage test ψ_N to the modified classical test $\hat{\varphi}_N$ is

$$(3.9) \qquad e(\psi,\hat{\varphi}) = \frac{\sigma^2(H)\left[\int_0^\infty xq(x)dH(x)\right]^2}{\mu^2(H)}$$

where $q(x) = h(x)/[1-H(x)]$ is the failure rate of H.

The Weibull distribution is defined by

$$(3.10) \qquad \hat{H}(x) = 1-e^{-ax^b} \; ; \; a,b > 0; \; x \geq 0$$

If μ_k denotes the kth moment about zero, then

$$(3.11) \qquad \mu_k = a^{-\frac{k}{b}} \Gamma\left(\frac{k}{b}+1\right), \; q(x) = ab\, x^{b-1} ,$$

$$\text{and} \quad \int_0^\infty xq(x)d\hat{H}(x) = ab\, \mu_b = b$$

Thus for the Weibull distribution

$$(3.12) \qquad e_b(\psi,\hat{\varphi}) = \frac{b^2\left[\Gamma\left(\frac{2}{b}+1\right) - \Gamma^2\left(\frac{1}{b}+1\right)\right]}{\Gamma^2\left(\frac{1}{b}+1\right)}$$

For $b = 1$, the Weibull distribution coinsides with exponential distribution and $e_1(\psi,\hat{\varphi}) = 1$. For $b = 2$, one has the linear failure rate $q(x) = 2ax$ and (3.12) becomes

$$(3.13) \qquad e_2(\psi,\hat{\varphi}) = \frac{16}{\pi} - 4 \doteq 1.093$$

Using L' Hospitals rule, one finds that

(3.14) $$\lim_{b\to\infty} e_b(\psi,\hat{\varphi}) = \infty$$

When $b < 1$, the failure rate is decreasing. Stirling's approximation shows that if $k = (1/b)$ is large, then (3.12) is approximately $2^{2k+\frac{1}{2}} k^{k-2} e^{-k} - k^{-2}$, and that

(3.15) $$\lim_{b\to 0} e_b(\psi,\hat{\varphi}) = \infty$$

If k is an integer, then

(3.16) $$e_b(\psi,\hat{\varphi}) = \frac{(2k)!}{k^2(k!)^2} - \frac{1}{k^2}$$

For $k = 2$ and 3 (3.16) becomes 1.25 and (19/9) = 2.11 respectively.

It is easy to show ([15] and [9]) that ψ_N is asymptotically most powerful and locally most powerful for the Weibull distribution. Thus

(3.17) $$e_b(\psi,\varphi) \geq 1 \quad \text{for all} \quad b > 0$$

and for all test φ_N for which this efficiency is computable. In particular (3.17) holds for $\hat{\varphi}_N$. Thus the Savage test ψ_N is uniformly more efficient than the adjusted classical test $\hat{\varphi}_N$ for the Weibull distribution. Moreover, the Savage test is much better when the failure rate parameter b is large or close to zero. It is conjectured that the Savage statistic is uniformly more efficient than $\hat{\varphi}_N$ for all distributions with monotone failure rates.

4. CENSORED SAMPLES. Fix $M \leq N$ and wait until a total of M X's and Y's have failed. Let $m' \leq m$ be the number of X's observed, then the ranks $r_1, \ldots, r_{m'}$ can be computed from the data. The Gastwirth (1965) modified Savage statistic is

$$(4.1) \qquad S_M = -\frac{1}{M}\left[\sum_{i=1}^{m'} \ln\left(1-\frac{r_i}{N+1}\right) + m' + (m-m')\ln\left(1-\frac{M}{N+1}\right)\right]$$

It is assumed that

$$(4.2) \qquad o < \lim_{N\to\infty} (M/N) = p < 1$$

The asymptotic power function of the level α test ψ_M that rejects for large values of S_M can be computed using [9] and [8]. One gets

$$(4.3) \qquad \beta(c;\psi_p,H) = \Phi\left(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{\frac{1}{2}} \int_o^{H^{-1}(p)} \frac{th(t)}{1-H(t)}\, dH(t)\right)$$

From (2.8), it follows that when H ranges over the class of IFRA distributions, then

$$(4.4) \qquad \begin{aligned} \inf_H \beta(c;\psi_p,H) &= \beta(c;\psi_p,K_\sigma) \\ &= \Phi\left(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{\frac{1}{2}} [p+\ln(1-p)(1-p)]\right) \end{aligned}$$

Since Hájek (1962) and Gastwirth (1965) has shown that

$$(4.5) \qquad \beta(c;\psi_p,K_\sigma) \geq \beta(c;\varphi,K_\sigma)$$

for all level α tests φ_N, then the results of Section 2 holds for ψ_M.

5. ASYMPTOTIC UNIQUENESS. Stein (1956) and Hájek (1962) has shown that one can obtain asymptotically optimal statistics by estimating the underlying distribution. Although these statistics are impractical, they show that one can not hope for asymptotic uniqueness in the class of all tests with asymptotic level α.

Consider the class of one-sided level α rank tests $\mathcal{J}$ [5] based on statistics of the form

$$(5.1) \qquad T_M = T_M(J_N) = \frac{1}{m} \sum_{i=1}^{m} J_N\left(\frac{r_i}{N+1}\right)$$

where there exist a function J which is continuous except for possibly a finite number of jump discontinuities and which satisfies

$$(5.2) \qquad \int_0^1 J^2(u)du < \infty \text{ and } \lim_{N\to\infty} \int_0^1 [J_N(u)-J(u)]^2 du = 0$$

and the conditions of Comment 3.8 of Hájek (1962). Let $\mathcal{F}'$ be the class of IFRA distributions H with a density h which has the Radon - Nykodim derivative h' with respect to Lebesque measure and satisfies

$$(5.3) \qquad \int_0^\infty [x^2 h'(x)/h(x)]^2 \, dH(x) < \infty \quad .$$

For these classes one has

<u>Theorem 5.2.</u> <u>The Savage - Gastwirth test</u> ψ_p <u>is asymptotically uniquely minimax for</u> $\mathcal{J}$ <u>and</u> $\mathcal{F}'$, i.e., if $\varphi_o = \varphi_o(J_N) \epsilon \mathcal{J}$, <u>if</u> H <u>ranges over</u> $\mathcal{F}'$, <u>and if</u>

$$(5.4) \qquad \inf_H \beta(c;\varphi_o,H) \geq \inf_H \beta(c;\varphi,H)$$

<u>for all</u> $\varphi \in \mathcal{J}$, <u>then there exists constants</u> a_N and b_N <u>such that</u>

$$(5.5) \qquad \sqrt{N}\,[S_M - (a_N T_M(J_N) + b_N)] \to 0$$

<u>in probability as</u> $N \to \infty$ <u>provided</u> (2.3),(4.2) <u>and</u> (2.4) <u>hold with</u> $c < \infty$.

<u>Proof.</u> (2.7) and (5.4) show that $\beta(c;\varphi_o,K_\sigma) = \beta(c;\psi_p,K_\sigma)$. Thus φ_o is asymptotically optimal for K_σ . From Hájek (1962), it follows that the correlation coefficient satisfies

$$(5.6) \qquad \rho_N(S_M,T_M \mid K_\sigma;\Delta = 1) \to 1 \quad \text{as} \quad N \to \infty$$

This implies that for regression coefficients a_N and b_N ,

(5.7) $$E(N[S_M-(a_N T_M+b_M)]^2 \mid K_\sigma;\Delta=1) \to 0 \quad .$$

Since S_M and T_M are distribution free, (5.7) holds not only for k_σ, but for general H . The result now follows from the contiquity arguments of LeCam and Hájek (e.g., [9]).

6. ESTIMATION. Barlow and Prochan (1966) have shown that the estimates of the mean that are optimal for exponential models are not robust for IFR distributions. Here an asymptotically robust estimate of the ratio μ_1/μ_2 of the means of X and Y is constructed using the methods of Hodges and Lehmann (1963). Write $x = (x_1,\ldots,x_m), y = (y_1,\ldots,y_n)$, $ax = (ax_1,\ldots,ax_n)$ etc., and let

(6.1) $$s(x,y) = S_M$$

be the Savage - Gastwirth statistic (4.1). $\mu_1/\mu_2 = \theta\mu(H)/\gamma\mu(H) = \theta/\gamma = \Delta$, so one estimates Δ .

Note that $\sqrt{N}\, s\,(X\,,\,\Delta Y)$ asymptotically tends to be normally distributed about the point 0 [9] . Let

(6.2) $$\Delta^* = \sup\{\Delta:\ s(x,\Delta y) \geq 0\} \text{ and } \Delta^{**} = \inf\{\Delta:\ s(x,\Delta y) \leq 0\}$$

and define the estimate $\hat{\Delta}$ of Δ by

(6.3) $$\hat{\Delta} = \hat{\Delta}(x,y) = \tfrac{1}{2}(\Delta^* + \Delta^{**}) \quad .$$

Since $s(ax,ay) = s(x,y)$ by the invariance properties of ranks, then

(6.4) $\hat{\Delta}\,(ax,ay) = \hat{\Delta}\,(x,y)$ for all $a > 0$, i.e., $\hat{\Delta}$ is scale invariant.

Moreover, using this, the definition (6.2), and noting that $s(x,\Delta y)$ is decreasing in Δ, one gets

$$(6.5) \quad \hat{\Delta}(ax,by) = (a/b)\,\hat{\Delta}(x,y) \quad ,$$

$$(6.6) \quad P_\Delta(\hat{\Delta}/\Delta \le t) = P_1(\hat{\Delta} \le t) \quad ,$$

$$(6.7) \quad \Delta^* \le \Delta^{**} \quad ,$$

$$(6.8) \quad P(\Delta^* < t) = P(s(x,ty) < 0) \quad ,$$

$$(6.9) \quad P(\Delta^{**} \le t) = P(s(x,ty) \le 0) \quad ,$$

$$(6.10) \quad P(s(x,ty) < 0) \le P(\hat{\Delta} \le t) \le P(s(x,ty) \le 0) \quad , \text{ and}$$

<u>Lemma 6.1.</u> <u>If</u> H <u>satisfies</u> (5.3) <u>and</u> H(0) = 0, <u>then</u>

$$\lim_{N\to\infty} P_\Delta(\sqrt{N}\,[(\hat{\Delta}/\Delta) - 1] \le t)$$
$$= \Phi\left(t[\lambda(1-\lambda)/p]^{\frac{1}{2}} \int_0^{H^{-1}(p)} \frac{xh(x)}{1-H(x)}\, dH(x)\right) .$$

<u>Proof.</u> (6.6) shows that one can let $\Delta = 1$. From (6.10) it follows that

$$\lim_{N\to\infty} P_1(N^{\frac{1}{2}}(\hat{\Delta} - 1) \le t) = \lim_{N\to\infty} P_1(\hat{\Delta} \le 1 + tN^{-\frac{1}{2}})$$
$$= \lim_{N\to\infty} P_1(s(\mathbb{X},(1 + tN^{-\frac{1}{2}})Y) \le 0)$$
$$= \lim_{N\to\infty} P_{\Delta_N}(s(\mathbb{X},Y) \le 0)$$

where $\Delta_N = 1/(1 + tN^{-\frac{1}{2}})$. Since $N^{\frac{1}{2}}(\Delta_N - 1) \to t$ as $N \to \infty$, the result follows from (4.3).

Lemma 6.1 shows that the asymptotic variance of $\sqrt{N}\,[(\hat{\Delta}/\Delta) - 1]$ is

$$(6.11) \qquad V(\hat{\Delta},H) = 1/[\lambda(1-\lambda)/p]^{\frac{1}{2}} \int_0^{H^{-1}(p)} \frac{th(t)}{1 - H(t)}\, dH(t) \; .$$

Moreover, (4.4) shows that the maximum asymptotic variance over IFRA distributions is

$$\text{(6.12)} \qquad \sup_H V(\hat{\Delta},H) = V(\hat{\Delta},K_\sigma) = 1/[\lambda(1-\lambda)/p]^{\frac{1}{2}}[p + \ln(1-p)(1-p)]$$

Let $\mathfrak{F}'$ be as in Section 5, then the results of the previous sections yield

<u>Theorem 6.1.</u> $\hat{\Delta}$ <u>is asymptotically minimax over</u> $\mathfrak{F}'$ <u>and the class</u> $\mathcal{E}$ <u>of scale invariant estimates that are asymptotically normal</u>; i.e., <u>if</u> $V(\tilde{\Delta},H)$ <u>denotes the asymptotic variance of the estimate</u> $\tilde{\Delta} \in \mathcal{E}$, <u>then</u>

$$\text{(6.13)} \qquad \sup_H \{V(\hat{\Delta},H): H \in \mathfrak{F}'\} \leq \sup_H \{V(\tilde{\Delta},H): \mathfrak{F}' \in H\}$$

$V(\hat{\Delta},H)$ also satisfies the saddle-point inequality

$$\text{(6.14)} \qquad \sup_H V(\hat{\Delta},H) = V(\hat{\Delta},K_\sigma) = \inf_{\tilde{\Delta}} V(\tilde{\Delta},K_\sigma)$$

where H ranges over $\mathfrak{F}'$ and $\tilde{\Delta}$ over $\mathcal{E}$.

A different approach to the problem of obtaining asymptotic minimax estimates is given by Huber (1963).

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Let F and G be defined by $F(t) = H(\gamma t)$ and $G(t) = H(\theta t)$ where H is unknown and $H(0) = 0$. For testing the equality of the means of F and G in the two-sample problem; it is shown that the Savage (The Annals of Mathematical Statistics (1956) pp 590-615) statistic maximizes the minimum power over IFRA(or IFR) distributions asymptotically. Asymptotic uniqueness holds only in a class of rank tests. The results are extended to censored samples, the problem of estimating the ratio of the means, and the k-sample problem.

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