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**CHARACTERISTIC EQUATIONS FOR A  
SUPERSONIC FLOW PROBLEM WITH  
MAGNETOHYDRODYNAMIC EFFECTS**

**F. C. Loper and M. B. Lightsey**  
**ARO, Inc.**

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## FOREWORD

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Marion L. Laster  
Aerospace Engineer  
Research Division  
Directorate of Plans and Technology

Edward R. Feicht  
Colonel, USAF  
Director of Plans and Technology

**ABSTRACT**

A derivation of the characteristic equations for supersonic inviscid flow with magnetohydrodynamic (MHD) forces present is given for the two-dimensional and axisymmetric cases. Workable forms of the equations relating the MHD effects are indicated for the axisymmetric problem.

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## NOMENCLATURE

a	Sonic speed
B	Angular component of $\vec{B}$
$\vec{B}$	Magnetic induction
$\vec{E}$	Electric intensity
$E_x$	x component of $\vec{E}$
$E_y$	y component of $\vec{E}$
H	Total enthalpy
h	Enthalpy
$\vec{J}$	Current density
$J_x$	x component of $\vec{J}$
$J_y$	y component of $\vec{J}$

$p$	Pressure
$s$	Entropy
$u$	x component of $\vec{v}$
$v$	y component of $\vec{v}$
$\vec{v}$	Velocity
$x$	Axial coordinate
$y$	Radial coordinate
$\mu_0$	Permeability of free space
$\rho$	Density
$\sigma$	Conductivity
$\phi$	Set of variables ( $x, y, u, v, p,$ and $h$ )

**SECTION I**  
**INTRODUCTION**

The conservation equations for the axisymmetric supersonic inviscid flow problem with magnetohydrodynamic (MHD) forces present are taken to be

$$\rho v + y \frac{\partial \rho v}{\partial y} + y \frac{\partial \rho u}{\partial x} = 0 \quad (1)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} = J_y B \quad (2)$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = -J_x B \quad (3)$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \vec{J} \cdot \vec{E} \quad (4)$$

The primary purpose of the work that follows is deriving the characteristic equations corresponding to the above system and noting an appropriate set of relations for the MHD effects that would be solved simultaneously with the characteristic equations.

The manner in which the results of the following derivation can be made applicable to the two-dimensional problem is indicated in Section V.

Coupled with the conservation equations are five auxiliary relations which must also hold.

$$H = h + \frac{1}{2} (u^2 + v^2) \quad (5)$$

$$\rho = f_0(p, h) \quad (6)$$

$$J_y B = \underline{f_1}(\phi) \quad (7)$$

$$J_x B = f_2(\phi) \quad (8)$$

$$\vec{J} \cdot \vec{E} = f_3(\phi) \quad (9)$$

Equation (5) is the definition of total enthalpy. Equation (6) is the equation of state. It is assumed that  $f_0$  has continuous first derivatives with respect to  $p$  and  $h$ . Equations (7), (8), and (9) are intended to assert that the MHD forces are known continuous functions of  $\phi$ .



The reader is reminded that, for the axisymmetric problem,  $f_1$ ,  $f_2$ , and  $f_3$  must obey the appropriate laws of symmetry.<sup>1</sup>

An explicit set of conditions for the MHD relations is noted in Section VI. For the derivation of the characteristic equations, however, it is convenient not to introduce these cumbersome expressions.

As this is a problem of deriving the characteristic equations corresponding to Eqs. (1), (2), (3), and (4) and not one of actually solving the equations, boundary and initial conditions need not be specified.

## SECTION II CHOICE OF VARIABLES

The independent variables in Eqs. (1), (2), (3), and (4) are  $x$  and  $y$ . The dependent variables are  $u$ ,  $v$ ,  $p$ , and  $h$ , partly as a matter of choice. It is desirable, therefore, to rewrite the system of partial differential equations so that derivatives with respect to the independent variables of only the dependent variables appear. For example, since  $\rho$  is not a dependent variable and in lieu of Eq. (6)

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial \rho}{\partial h} \frac{\partial h}{\partial x}$$

$\frac{\partial \rho}{\partial p}$  and  $\frac{\partial \rho}{\partial h}$  can be obtained from Eq. (6), and  $\frac{\partial p}{\partial x}$  and  $\frac{\partial h}{\partial x}$  are derivatives of dependent variables with respect to an independent variable (dependent derivatives).

The desired form of Eqs. (1), (2), (3), and (4) is<sup>2</sup>

$$\rho y \frac{\partial u}{\partial x} + \rho y \frac{\partial v}{\partial y} + uy \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial x} + yv \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial y} + yu \frac{\partial \rho}{\partial h} \frac{\partial h}{\partial x} + yv \frac{\partial \rho}{\partial h} \frac{\partial h}{\partial y} = -\rho v \quad (10)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = J_y B \quad (11)$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = -J_x B \quad (12)$$

$$\rho u^2 \frac{\partial u}{\partial x} + \rho uv \frac{\partial u}{\partial y} + \rho uv \frac{\partial v}{\partial x} + \rho v^2 \frac{\partial v}{\partial y} + \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \vec{J} \cdot \vec{E} \quad (13)$$

---

<sup>1</sup>i. e.,  $f_1$  and  $f_3$  are even functions of  $y$ , and  $f_2$  is odd.

<sup>2</sup>It is assumed until otherwise indicated that  $y \neq 0$ . A limiting process will be employed in Section IV, Part III to determine certain conditions on the  $x$ -axis.

Notice that the above equations can be interpreted as a system of four linear algebraic equations in eight unknowns, the unknowns being the dependent derivatives. Also, by definition,

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du \tag{14}$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = dv \tag{15}$$

$$\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = dp \tag{16}$$

$$\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy = dh \tag{17}$$

whenever these differentials exist. Equations (10) through (17) are, therefore, a system of eight linear algebraic equations in eight unknowns.

This concept, that of coupling the given differential equations with the equations defining the dependent differentials, and interpreting the results as a system of linear algebraic equations, is fundamental to this method of deriving the characteristic equations.

As much of what follows is directly related to the system of eight equations, they are presented according to a convenient format in the following illustration.<sup>3</sup>

$\frac{\partial u}{\partial x}$	$\frac{\partial u}{\partial y}$	$\frac{\partial v}{\partial x}$	$\frac{\partial v}{\partial y}$	$\frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial y}$	$\frac{\partial h}{\partial x}$	$\frac{\partial h}{\partial y}$	=
$\rho y$	0	0	$\rho y$	$y u \frac{\partial \rho}{\partial p}$	$y v \frac{\partial \rho}{\partial p}$	$y u \frac{\partial \rho}{\partial h}$	$y v \frac{\partial \rho}{\partial h}$	$-\rho v$
$\rho u$	$\rho v$	0	0	1	0	0	0	$J_y B$
0	0	$\rho u$	$\rho v$	0	1	0	0	$-J_x B$
$\rho u^2$	$\rho u v$	$\rho u v$	$\rho v^2$	0	0	$\rho u$	$\rho v$	$J \cdot E$
dx	dy	0	0	0	0	0	0	du
0	0	dx	dy	0	0	0	0	dv
0	0	0	0	dx	dy	0	0	dp
0	0	0	0	0	0	dx	dy	dh

<sup>3</sup>Shapiro made use of such a format in Ref. 1, p. 518.

### SECTION III DEFINITION OF THE CHARACTERISTICS

With the introduction of a limited amount of matrix notation, the definition of the characteristic equations can be conveniently and simply stated.

Let  $\vec{X}$  be the 8 by 1 column vector, the components of which are the dependent derivatives, and let  $\vec{Y}$  be the 8 by 1 column vector whose elements are the right-hand side of the system of equations.

$$\vec{X} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \vdots \\ \frac{\partial h}{\partial y} \end{pmatrix} \quad \vec{Y} = \begin{pmatrix} -\rho v \\ J_y B \\ \vdots \\ dh \end{pmatrix}$$

Let  $\tilde{A}$  be the 8 by 8 coefficient matrix. Then Eqs. (10) through (17) can simply be written

$$\tilde{A} \vec{X} = \vec{Y} \quad (18)$$

The  $\tilde{A}$  augmented matrix will be denoted by  $\tilde{C}$ . That is,  $\tilde{C}$  is the 8 by 9 matrix obtained by including the elements of  $\vec{Y}$  as an additional column to the matrix  $\tilde{A}$ .

Definition<sup>4</sup>: The set of characteristic equations consists of all equations that follow on requiring that  $R(\tilde{A}) = R(\tilde{C}) \leq 7$ .

This definition is motivated by the question, "Do there exist curves within the domain of interest on which Eq. (18) has multiple solutions?" It is known from the theory of linear algebra that if such curves exist, then it is necessary that  $R(\tilde{A}) \leq 7$  (Ref. 2, p. 61). It is known also, that no solution can exist unless  $R(\tilde{A}) = R(\tilde{C})$  (Ref. 3, p. 15).

### SECTION IV DERIVATION OF THE CHARACTERISTIC EQUATIONS

The purpose of this section is to determine the equations that follow on requiring that  $R(\tilde{A}) = R(\tilde{C}) \leq 7$ . These results will be derived in

---

<sup>4</sup>For any matrix  $\tilde{K}$ ,  $R(\tilde{K})$  denotes the rank of  $\tilde{K}$ .

Part I and Part II. Certain special cases will be treated in Part III.

#### 4.1 PART I

Since  $\tilde{C}$  is the  $\tilde{A}$  augmented matrix, it follows that  $R(\tilde{A}) \leq R(\tilde{C})$  necessarily (Ref. 3, p. 15).

Let  $\tilde{B}$  be the 8 by 8 matrix obtained by replacing any column of  $\tilde{A}$  (say the first column for definiteness) with the elements of  $\vec{Y}$ . Then  $R(\tilde{C}) \leq 7$  only if  $\det^5(\tilde{A}) = 0$  and  $\det(\tilde{B}) = 0$  simultaneously.

The result of setting  $\det(\tilde{A}) = 0$  and simplifying is

$$(udy - vdx)^2 [(u^2 - a^2) dy^2 - 2uvdydx - (v^2 - a^2) dx^2] = 0 \quad (19)$$

where, because of the choice of dependent variables<sup>6</sup>,

$$a^2 = \frac{\rho}{\frac{\partial \rho}{\partial h} + \rho \frac{\partial \rho}{\partial p}} \quad (20)$$

From Eq. (19) either

$$udy - vdx = 0 \quad (21)$$

or

$$(u^2 - a^2) dy^2 - 2uvdydx + (v^2 - a^2) dx^2 = 0 \quad (22)$$

Equation (21) is the equation of a streamline. Equation (22), a quadratic in  $dy/dx$ , is commonly referred to as the equation of a left or right running characteristic. Setting  $\det(\tilde{B}) = 0$  gives

$$\begin{aligned} & (udy - vdx) \left\{ \frac{\rho}{a^2} [(J_y B dy - \rho v du)(udy - vdx) + v dy dp] \right. \\ & + \frac{\partial \rho}{\partial h} dy (J_x B v dy + \vec{J} \cdot \vec{E} dy - J_y B u dy) + \frac{\rho^2 v}{y} dy^2 \left. \right\} \\ & + \rho dy (dx dp + \rho u dy dv + J_x B dy dx) \\ & - \rho dx^2 (J_y B dy - \rho v du) = 0 \end{aligned} \quad (23)$$

<sup>5</sup>The determinant of  $\tilde{A}$

<sup>6</sup>Appendix I gives a proof that (a) as defined by Eq. (20) is the speed of sound provided additional assumptions on the nature of the fluid are made.

Since  $\det(\tilde{A}) = 0$  and  $\det(\tilde{B}) = 0$  must hold simultaneously, the form of Eq. (23) can be simplified somewhat by imposing Eq. (22) as a side condition.

$$\begin{aligned} & (udy - vdx)(udv - vdu) + (udy - vdx)^2 \frac{v}{y} \\ & + \frac{dp}{\rho} (vdy + udx) + \frac{1}{\rho^2} \frac{\partial \rho}{\partial h'} (v \overset{\pi_v}{J_x} \tilde{B}) + \vec{J} \cdot \vec{E} - \mu J_y B (udy - vdx)^2 \\ & + \frac{B}{\rho} (J_x dx + J_y dy) (udy - vdx) = 0 \end{aligned} \quad (24)$$

Equation (24) is known as the compatibility equation along a left or right running characteristic.

When Eqs. (22) and (24) are satisfied simultaneously,  $R(\tilde{B}) = R(\tilde{A}) = 7$  by inspection and  $R(\tilde{C}) = 7$  (Ref. 4, p. 54).

#### 4.2 PART II

Along a streamline (when  $udy - vdx = 0$ )  $R(\tilde{A}) = 6$  by inspection, so that  $R(\tilde{C})$  is at most seven. In this case it can be shown that  $R(\tilde{C}) = 6$  if and only if

$$\frac{dp}{\rho} - udu + vdv + (J_x B dy - J_y B dx) / \rho = 0 \quad (25)$$

and

$$dh + udu + vdv = \frac{\vec{J} \cdot \vec{E}}{\rho} \frac{dx}{v} \quad (26)$$

Satisfying Eqs. (25) and (26) does not affect  $R(\tilde{A})$ ; hence Eqs. (24), (25), and (26) are the desired relations on the streamline.

#### 4.3 PART III

The characteristic equations for the two-dimensional case are exactly the same as those previously derived except that the term containing the factor  $v/y$  in Eq. (24) is replaced with zero.

For the axisymmetric case, Eq. (24) is singular on the  $x$  axis because of the  $v/y$  term. Applying l'Hospital's rule and making use of Eq. (1), it can be shown that

$$\lim_{y \rightarrow 0} \frac{v}{y} = - \frac{1}{2\rho} \frac{\partial \rho u}{\partial x} \Big|_{y=0} \quad (27)$$

Since  $\rho$  is not a dependent variable, a more workable form is

$$\lim_{y \rightarrow 0} \frac{v}{y} = \frac{1}{2} \left\{ \left( \frac{u^2}{a^2} - 1 \right) \frac{\partial u}{\partial x} - \frac{1}{\rho} \left( u J_y B \frac{\partial \rho}{\partial p} + \frac{\vec{j} \cdot \vec{E}}{\rho} \frac{\partial \rho}{\partial h} \right) \right\} \Big|_{y=0} \quad (28)$$

Equation (28) was obtained by differentiating the product in Eq. (27) and utilizing Eqs. (6), (20), (25), and (26) (note that  $y = 0$  is a streamline).

Define

$$K = \begin{cases} \frac{v}{y} & \text{if } y \neq 0 \\ \lim_{y \rightarrow 0} \frac{v}{y} & \text{if } y = 0 \\ 0 & \end{cases} \quad \left. \begin{array}{l} \text{axisymmetric case} \\ \text{two-dimensional case} \end{array} \right\} \quad (29)$$

The characteristic equations for either the axisymmetric or two-dimensional case are now summarized for convenience. Along a left or right running characteristic

$$(u^2 - a^2) dy^2 - 2uvdydx - (v^2 - a^2) dx^2 = 0 \quad (30)$$

$$\left. \begin{aligned} & (udy - vdx)(udv - vdu) + (udy - vdx)^2 K \\ & + \frac{dp}{\rho} (vdy + udx) + \frac{1}{\rho^2} \frac{\partial \rho}{\partial h} (vJ_x B + \vec{j} \cdot \vec{E} - uJ_y B)(udy - vdx) \\ & + \frac{B}{\rho} (J_x dx + J_y dy)(udy - vdx) = 0 \end{aligned} \right\} \text{Compatibility on characteristics} \quad (31)$$

and along a streamline

$$udy - vdx = 0 \quad \text{Equation of Streamline} \quad (32)$$

$$\text{Momentum Equat. } \frac{dp}{\rho} - udu + vdv + (J_x B dy - J_y B dx) / \rho = 0 \quad (33)$$

$$\text{Energy Equation: } dh + udu + vdv = \frac{\vec{j} \cdot \vec{E}}{\rho} \frac{dx}{u} \quad (34)$$

An independent check of the above compatibility equations is given in Appendix II.

**SECTION V**  
**MHD RELATIONS FOR THE AXISYMMETRIC PROBLEM**

Maxwell's equation along with Ohm's law are the only relations needed to obtain the MHD terms appearing in Eqs. (31), (33), and (34) assuming adequate boundary conditions are given and assuming that  $\sigma$  is a known function of  $\phi$ .

$$\nabla \cdot \vec{E} = 0 \quad (35)$$

$$\nabla \times \vec{E} = 0 \quad (36)$$

$$\nabla \cdot \vec{B} = 0 \quad (37)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad (38)$$

$$\vec{J} = \sigma (\vec{E} + \vec{V} \times \vec{B}) \quad (39)$$

Equations (35) and (36), an elliptic system of equations, can be uncoupled from the remaining equations. If "reasonable" boundary conditions are imposed on these two equations, a closed form solution can be obtained giving  $E_x$  and  $E_y$  as functions of position only. Hence, assuming  $\vec{E}$  to be a known function of position, an attempt can be made to solve the scalar equations corresponding to Eqs. (37), (38), and (39) simultaneously with the system of characteristic equations (Eqs. (30) through (34)).

Equation (37) holds identically, and Eqs. (38) and (39) can be written

$$\frac{\partial yB}{\partial x} = -\mu_0 y J_y \quad (40)$$

$$\frac{\partial yB}{\partial y} = \mu_0 y J_x \quad (41)$$

$$J_x = \sigma (E_x + vB) \quad (42)$$

$$J_y = \sigma (E_y - uB) \quad (43)$$

The total system of MHD relations can be summarized as

$$\sigma = g_1(\phi) \quad (44)$$

$$E_x = g_2(x, y) \quad (45)$$

$$E_y = g_3(x, y) \quad (46)$$

$$J_x = \sigma (E_x + vB) \quad (47)$$

$$J_y = \sigma (E_y - uB) \quad (48)$$

$$d(yB) = \mu_0 \sigma \left\{ (E_x + vB) y dy - (E_y - uB) y dx \right\} \quad (49)$$

Equation (49) was obtained by using Eqs. (40) and (41) to obtain the total differential of  $y_B$  and by using Eqs. (42) and (43) to eliminate  $J_x$  and  $J_y$ .

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**APPENDIXES**

**I. SONIC SPEED RELATIONS**

**II. CHECK ON COMPATIBILITY EQUATIONS**

**APPENDIX I**  
**SONIC SPEED RELATIONS**

It was mentioned earlier that the only assumption necessary on the type of gas being considered is that  $\rho$  be a given function of  $p$  and  $h$  with continuous first derivatives.

$$\rho = f_0(p, h) \quad (I-1)$$

In order to prove that (a) as given by the relation

$$a^2 = \frac{\rho}{\left(\frac{\partial \rho}{\partial h}\right)_p + \rho \left(\frac{\partial \rho}{\partial p}\right)_h} \quad (I-2)$$

is the sonic speed, certain additional assumptions are made.

It is assumed that a differentiable relation defining entropy exists

$$s = g_0(p, h) \quad (I-3)$$

and that unique inverse relations giving  $p$  and  $h$  as functions of  $\rho$  and  $s$  also exist. The definitions of  $\rho$  and  $a$  are taken to be

$$\rho = \left(\frac{\partial p}{\partial h}\right)_s \quad (I-4)$$

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \quad (I-5)$$

Finally, it is assumed that

$$\rho \neq 0 \quad \left(\frac{\partial s}{\partial h}\right)_p \neq 0$$

It will now be shown that the terms on the right-hand side of Eqs. (I-2) and (I-5) are equivalent.

Differentiating Eqs. (I-1) and (I-3) with respect to  $\rho$  at constant  $s$  yields

$$\left(\frac{\partial g_0}{\partial p}\right)_h \left(\frac{\partial p}{\partial \rho}\right)_s + \left(\frac{\partial g_0}{\partial h}\right)_p \left(\frac{\partial h}{\partial \rho}\right)_s = 0$$

$$\left(\frac{\partial f_0}{\partial p}\right)_h \left(\frac{\partial p}{\partial \rho}\right)_s + \left(\frac{\partial f_0}{\partial h}\right)_p \left(\frac{\partial h}{\partial \rho}\right)_s = 1$$

Since the Jacobian related to the above equations cannot vanish, it follows that

$$\begin{aligned}
 \left. \frac{\partial p}{\partial \rho} \right)_s &= \frac{\left. \frac{\partial \xi_0}{\partial h} \right)_p}{\left. \frac{\partial \xi_0}{\partial h} \right)_p \left. \frac{\partial f_0}{\partial p} \right)_h - \left. \frac{\partial \xi_0}{\partial p} \right)_h \left. \frac{\partial f_0}{\partial h} \right)_p} \\
 &= \frac{\rho}{\rho \left. \frac{\partial \rho}{\partial p} \right)_h - \rho \left. \frac{\partial \rho}{\partial h} \right)_p \left[ \left. \frac{\partial s}{\partial p} \right)_h / \left. \frac{\partial s}{\partial h} \right)_p \right]} \\
 &= \frac{\rho}{\rho \left. \frac{\partial \rho}{\partial p} \right)_h - \rho \left. \frac{\partial \rho}{\partial h} \right)_p / \left. \frac{\partial p}{\partial h} \right)_s} \\
 &= \frac{\rho}{\left. \frac{\partial \rho}{\partial h} \right)_p + \rho \left. \frac{\partial \rho}{\partial p} \right)_h}
 \end{aligned}$$

This completes the proof.

**APPENDIX II**  
**CHECK ON THE COMPATIBILITY EQUATIONS**

Because of the involved process of obtaining the characteristic equations, an independent and relatively simple check of the compatibility equations is offered.

Consider Eq. (34) which, according to a previous assertion, is a compatibility equation on a streamline. This equation can be checked as follows:

1. Multiply Eq. (4) through by  $dx$ .
2. Impose the condition  $udy = vdx$  (Eq. (32)) on the result.
3. Note the definition of total enthalpy (Eq. (5)).
4. Equation (34) follows.

Equation (33), the other compatibility equation on a streamline, can be checked in much the same way:

1. Multiply Eqs. (2) and (3) by  $dx$  and  $dy$ , respectively.
2. Impose the condition  $udy = vdx$  on each equation.
3. Add the resulting equations.
4. Equation (33) follows.

The check on the compatibility equation along a left or right running characteristic (Eq. (31)) is somewhat more involved:

1. Multiply Eq. (1) by  $(udy - vdx)^2 \left( \frac{1}{\rho y} \right)$ .
2. Multiply Eq. (2) by  $\left[ \frac{u}{\rho} \frac{\partial \rho}{\partial h} (udy - vdx)^2 - dy (udy - vdx) \right] \frac{1}{\rho}$ .
3. Multiply Eq. (3) by  $\left[ \frac{v}{\rho} \frac{\partial \rho}{\partial h} (udy - vdx)^2 + dx (udy - vdx) \right] \frac{1}{\rho}$ .
4. Multiply Eq. (4) by  $\left[ - \frac{1}{\rho} \frac{\partial \rho}{\partial h} (udy - vdx)^2 \right] \frac{1}{\rho}$ .
5. Add the results of items 1 through 4 above and simplify.
6. Impose Eq. (30) as a side condition on the result of the above.
7. Equation (31) follows.

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13 ABSTRACT  A derivation of the characteristic equations for supersonic inviscid flow with magnetohydrodynamic (MHD) forces present is given for the two-dimensional and axisymmetric case. Workable forms of the equation relating the MHD effects are indicated for the axisymmetric problem.			

14 KEY WORDS  magnetohydrodynamic forces supersonic flow problems two dimensional axisymmetric characteristic equations	LINK A		LINK B		LINK C	
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