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DISTRIBUTION FUNCTION AND TEMPERATURES IN A MONATOMIC GAS UNDER STEADY EXPANSION INTO A VACUUM*

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Abstract

The distribution function for the spherical source is determined by integrating the B-G-K model equation, where the local temperature is determined by the moment equations under the hypersonic approximation. In the far field, the axial distribution is nearly Maxwellian. A free molecule limit exists, but does not define the lateral temperature and higher moments properly. The lateral temperature is determined by far field collisions, and is largely contained in the tail of the lateral distribution. A Mach number or a Reynolds number similarity in the distribution function is shown.

1. Introduction

Since Ashkenas and Sherman[1] pointed out that the flow along the centerline of a jet issuing from a sonic orifice into a vacuum could be approximated by a supersonic spherical source flow, and since, in the far field, the viscous effects can be shown to be significant, there has been interest in the analysis of source flow, with the transition to free molecule flow of concern. Narasimha[2] presented a "collisionless" solution to the problem, and noted the far field properties of a limiting Mach number and a nearly uniaxial, radially directed distribution function. To

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study transition from continuum to a nearly collisionless region, Brook and Oman[3] adopted the B-G-K model of the Boltzmann equation for a monatomic gas which was further simplified by the hypersonic approximation and integrated numerically. There was, however a question regarding the omission of several terms from the basic equation. The problem was subsequently analysed by Hamel and Willis[4] for a Maxwell molecule, and by Edwards and Cheng[5] using the B-G-K equation. If the collision frequency An in the B-G-K model is taken to be p/μ, the two methods yield equal moments of the collisional terms to second order, and, with the approximation introduced, the results of References 4 and 5 are equivalent. In both studies, the flow was assumed to have reached hypersonic speeds before the dissipative effects became significant, and, under the hypersonic approximation, it was shown that the moment equations could be truncated at second order. Again, a limiting temperature and uniaxial distribution function were obtained. Cylindrical source flows were also studied by the same techniques in both papers.

Recently, Edwards and Rogers[6] have applied the same approach to the study of the streamline structure in an axisymmetric free jet. It was concluded that, in the far field, the flow along the axis of symmetry is reducible to the spherically symmetrical source flow.

Borisov[7] has solved a two dimensional, time dependent, rotationally symmetric problem for Maxwell molecules using the method of Grad. If the problem is assumed to be independent of z, there exists a solution to the moment equations in which the flow is directed away from the z-axis with a speed proportional to the distance from the axis and inversely proportional to the time. With these assumptions, the moment equations admit a solution in which the pressure tensor is independent of position and has rotational symmetry about an axis parallel to the z-axis, and in which the third order moments are zero. This unsteady problem is mathematically equivalent to the hypersonic source flow problem considered here if one lets $t = r/\xi$ and if terms of $O(1/M)$ are omitted. Borisov obtains an equation for temperature which is identical in form to Eq. 3 of this paper, with ω set equal to one.

In this paper, the character of the distribution function for the spherical source flow is studied using the B-G-K model equations. The observations and conclusions on the analytical behavior of the distribution function in the far field are presumably not limited by the model equation, and the knowledge should be helpful in interpreting and determining distribution functions in free jet experiments. The solution presented here may also serve as a basis for detailed assessment of the B-G-K model equation.

2. Review of Moment Solution

In the notation of Ref. 5 the following equations may be obtained directly by taking moments up to second order of the Boltzmann B-G-K equation in spherical coordinates with spherical symmetry.

$$\left.\begin{aligned} \frac{d}{dr}(r^2 P_{rr}) &= \frac{An}{\bar{\xi}} r^2 (\rho RT - P_{rr}) + \Delta_1 \\ r\frac{d}{dr}(RT) &= \frac{2}{3}\left(\frac{P_{rr}}{\rho} - 3RT\right) + \Delta_2 \end{aligned}\right\} \qquad (1)$$

where

$$\left.\begin{aligned} P_{rr} &\equiv m\iiint(\xi-\bar{\xi})^2 f \, d\xi \, d\eta \, d\zeta \\ 3nT &\equiv \iiint[(\xi-\bar{\xi})^2+\eta^2+\zeta^2] f \, d\xi \, d\eta \, d\zeta \\ \bar{\xi} &\equiv \frac{1}{n}\iiint \xi f \, d\xi \, d\eta \, d\zeta \end{aligned}\right\} \qquad (2)$$

Note that ξ, η, ζ are the orthogonal components of particle velocity in the direction of increasing r, θ, φ respectively. Δ_1 and Δ_2 are errors due to truncation of the moment equations and contain moment terms higher than the second order. From the B-G-K solution to f, Δ_1 and Δ_2 can be shown to belong to an order at least $1/M$ higher than that of the terms which are kept. $\bar{\xi}$ is constant to $O(1/M^2)$. An represents the collision frequency term and is taken to be proportional to $\rho RT/\mu$.

Assume $\mu \propto T^{\omega}$, and $A = (amRT_1^{\omega}/\mu_1)T^{1-\omega}$ where a is a constant of order unity. Introduce $\Theta \equiv T/T_{\infty}$, and $s \equiv (\bar{\xi}/Anr) = \mu_1 r\bar{\xi}/amRn_1 r_1^2 T_1^{\omega} T_{\infty}^{1-\omega}$. The subscript 1 designates a reference upstream point where the flow is hypersonic, but still in equilibrium. The subscript ∞ designates conditions far downstream. Then Θ satisfies the equation[4]

$$s^2 \frac{d^2}{ds^2}\Theta + [3s + \Theta^{1-\omega}]\frac{d}{ds}\Theta + \frac{4}{3s}\Theta^{2-\omega} = 0 \qquad (3)$$

Let

$$T_\perp \equiv \frac{1}{2Rn}\iiint[\eta^2+\zeta^2]\, f\, d\xi\, d\eta\, d\zeta = \frac{3}{2}T - \frac{p_{rr}}{2\rho R} \tag{4}$$

Then[4]

$$\Theta_\perp \equiv \frac{T_\perp}{T_\infty} = -\frac{3}{4}\, s\frac{d}{ds}\Theta \tag{5}$$

If $\omega = 1$ (Maxwell molecules), Eq. 3 may be solved in terms of confluent hypergeometric functions, and a unique solution can be displayed which satisfies the conditions: $\Theta(\infty) \to 1$, and, as $s \to 0$, it approaches the isentropic solution $\Theta \propto s^{-4/3}$. In Fig. 1, solutions for Θ, $\Theta_\perp$ and $\Theta_{isen.}$ are shown as functions of the dimensionless radius s; for $\omega = 0.5$ and $\omega = 1$. These solutions were obtained by numerical integration of Eq. 3. It will be noticed that in terms of $\Theta = T/T_\infty$, for a given ω, transitions from an isentropic to a frozen temperature are represented by a single curve, irrespective of the reservoir pressure; and that $\Theta_\perp$ follows $\Theta_{isen.}$ closely for a considerable distance beyond the point at which Θ diverges from the isentropic value. At very large distances, however, $\Theta_\perp$ will diverge from $\Theta_{isen.}$, since the isentropic value decays as $s^{-4/3}$, while, the solution to Eq. 3 has the expansion for large s:

$$\Theta \to 1 + \frac{4}{3}\frac{1}{s} + \cdots \quad ; \quad \Theta_\perp \to \frac{1}{s} + \cdots \tag{6}$$

The limiting Mach number can be correlated explicitly with the reservoir conditions, or, equivalently, the ratio of the throat temperature to limiting temperature may be expressed as a function of the throat Reynolds number:

$$T_\infty / T^* = f(\omega)\left[Re^*\right]^{-4/[3+4(1-\omega)]} \tag{7}$$

where $f(\omega)$ is approximately 2.5 and 10 for $\omega = 1/2$ and 1, respectively.

3. Integration of the Distribution Function

The moment solutions as reviewed in Section 2 will be used to compute the distribution function.

Consider the Boltzmann B-G-K equation in spherical coordinates with spherical symmetry, i.e.

$$\left[\xi\frac{\partial}{\partial r} + \frac{\eta^2+\zeta^2}{r}\frac{\partial}{\partial \xi} + \frac{\zeta^2\cot\theta - \xi\eta}{r}\frac{\partial}{\partial \eta} - \zeta\,\frac{\xi+\eta\cot\theta}{r}\frac{\partial}{\partial \zeta}\right] f = An(F-f) \tag{8}$$

with

$$F \equiv \frac{n}{(2\pi RT)^{3/2}}\exp\left\{-\frac{(\xi-\bar{\xi})^2+\eta^2+\zeta^2}{2RT}\right\}$$

With the limiting speed taken for ξ, and the condition $n\bar{\xi}r^2$ = const., F is completely defined by the solution to Eq. 3 for temperature. Since the problem is spherically symmetrical, $\theta = \pi/2$ may be chosen. With this choice, η and ζ enter symmetrically, and the variable $\rho \equiv \sqrt{\eta^2 + \zeta^2}$ is sufficient to define the lateral velocity components.

Three independent variables α, β and r are introduced into Eq. 8. They are:

$$\left.\begin{aligned} \alpha &= \sqrt{\xi^2 + \rho^2} \\ \beta &= r\rho \end{aligned}\right\} \qquad (9)$$

and Eq. 8 takes the form:

$$\sqrt{\alpha^2 - \beta^2/r^2}\,\frac{\partial f}{\partial r} = An(F - f) \qquad (10)$$

α and β are, in fact, two of the characteristic variables for the partial differential Eq. 8. Constant values of α and β correspond to constant particle kinetic energy and angular momentum about the origin respectively. With no collisions, the α and β associated with any particle would be fixed, and, as can be seen from Eq. 10, the distribution function would be independent of r, and a function of α and β alone. The invariance of $\beta = r\rho$ in a collision-free solution signifies that the scale of the abscissa of the lateral distribution function will reduce with increasing r like 1/r.

A formal integration of Eq. 10, starting with a radius r_o, yields:

$$f = f_0' + f_1 \qquad (11)$$

where

$$f_0' \equiv f_0(\alpha,\beta)\,\exp.\left\{-\int_{r_0}^{r}\frac{An\,dr_1}{\sqrt{\alpha^2 - \beta^2/r_1^2}}\right\}$$

$$f_1 \equiv \int_{r_0}^{r}\exp.\left\{-\int_{r_1}^{r}\frac{An\,dr_2}{\sqrt{\alpha^2 - \beta^2/r_2^2}}\right\}\frac{An^2}{(2\pi RT)^{3/2}}\,\frac{\exp.\left\{-\left[\alpha^2 + \bar{\xi}^2 - 2\bar{\xi}\sqrt{\alpha^2 - \beta^2/r_1^2}\right]/2RT\right\}}{\sqrt{\alpha^2 - \beta^2/r_1^2}}\,dr_1$$

with $f_o(\alpha,\beta)$ representing the distribution function at $r = r_o$.

In this study, since the flow is assumed to be in equilibrium up to a hypersonic speed, that portion of the distribution function which has a negative value of ξ will be ignored, and all integration with respect to r will be performed in a forward direction. The analysis which follows will be presented in two parts: first, an analytical study of the lateral temperature in the far field, and second a study of numerical results and other details of the distribution function.

4. The Lateral Temperature and Collisions in the Far Field

Eq. 11 is valid for any initial value r_0, provided the flow is hypersonic at that point. The solution will first be studied when r_0 is within the continuum regime.

The exponential in f_0' may be expressed in the form subject to a relative error of $O(1/M)$,

$$\int_{r_0}^{r} \frac{A n \, dr_1}{\sqrt{\alpha^2 - \beta^2/r_1^2}} = \int_{r_0}^{r} \frac{20}{3} Re^* \left(\frac{r^*}{r_1}\right)^2 \left(\frac{T_1}{T^*}\right)^{1-\omega} \frac{dr_1}{r^*} \qquad (12)$$

with r^* defined by $r^{*2} = n\bar{\xi} r^2/n^*\bar{\xi}^*$. Where r and n are evaluated at any point where the flow is hypersonic; n^* and ξ^* represent the isentropic Mach one values of density and mass averaged velocity; $(\alpha^2-\beta^2/r^2)$ was replaced by $\bar{\xi}$ since f_0 has a significant value only near $\xi = \bar{\xi}$ for $M \gg 1$.

For $r_0/r^* \ll (Re^*)^{1/1+4(1-\omega)/3}$, the integral in Eq. 12 is large, and the f_0 term is rapidly damped out. Consequently, when $r/r_0 \gg 1$, the distribution function is given by f_1.

The existence of a limit of f_1 as $r \to \infty$ is easily shown for $\alpha^2-\beta^2/r^2$ greater than zero, since the An^2 times a constant will dominate the integrand. This limiting f_1 can be called the collision-free limit since it is a function of α and β; but it does not give the correct far field lateral temperature. One observes that, while f_1 converges to the collision-free limit uniformly in β as $r \to \infty$, $\beta^3 f$, as well as the ratio of f_1 to its collision-free limit, do not. Consider now a representation in which r_0/r^* is large compared to Re^*. The integral in Eq. 12 is then very small compared to 1, and, in Eq. 11, $f_0' \approx f_0(\alpha,\beta)$. f_0' then behaves as a nearly collision-free distribution. The number density and lateral temperature associated with f_0' would then both be shown to be porportional to $1/r^2$. For example

$$n_0' \equiv \iiint_{-\infty}^{\infty} f_0' \, d\xi \, d\eta \, d\zeta = 2\pi \int_{-\infty}^{\infty}\int_{0}^{\infty} \rho f' \, d\xi \, d\rho = 2\pi \int_{0}^{\infty}\int_{\frac{\beta}{r}}^{\infty} \frac{\alpha\beta f' \, d\alpha \, d\beta}{r^2\sqrt{\alpha^2-\beta^2/r^2}} \qquad (13)$$

and, with the hypersonic assumption, in the region of interest, $\beta^2/r^2 \ll \alpha^2$, hence,

$$n_0' \approx \frac{2\pi}{r^2}\int_0^\infty\int_0^\infty \beta f \, d\alpha d\beta = \frac{r_0^2}{r^2} n_0 \qquad (14)$$

Similarly,

$$T_{\perp_0}' = \frac{1}{2n_0'R}\iiint(\eta^2+\varsigma^2) f \, d\xi d\eta d\varsigma = \frac{2\pi r^4}{2n_0'R}\int_0^\infty\int_{-\infty}^\infty \beta^3 f \, d\alpha d\beta \approx \frac{r_0^2}{r^2} T_{\perp_0} \qquad (15)$$

With the decomposition, f_1 and its moments will be approximated in the far field where $T \sim T_\infty$, $An^2 = (C/r^4)$, and $r_0 \gg Re^* r^*$. C can be represented as $(3/5) Re_0 \lambda_\infty \bar{\xi} r_0^3$ but it is independent of r_0. Then

$$f_1 \approx \int_{r_0}^{r} \frac{C}{r_1^4} \frac{1}{(2\pi R T_\infty)^{3/2}} \frac{1}{(\alpha^2 - \beta^2/r_1^2)^{1/2}} \exp\left\{-\frac{\alpha^2 + \bar{\xi}^2 - 2\bar{\xi}(\alpha^2 - \beta^2/r_1^2)^{1/2}}{2RT_\infty}\right\} dr_1 \qquad (16)$$

and

$$\iiint \rho^j f_1 \, d\xi d\eta d\varsigma \approx \frac{2\pi}{r^{j+2}}\int_0^\infty\int_0^\infty \beta^{j+1} f_1 \, d\alpha d\beta \qquad (17)$$

The integral in Eq. 17 can be evaluated by using Eq. 16 and interchanging the order of integration to obtain:

$$\frac{2\pi}{r^{j+2}}\iint \beta^{j+1} f_1 \, d\alpha d\beta = \frac{2\pi}{r^{j+2}}\int_{-\infty}^{\infty}\int_0^\infty\int_{r_0}^{r} \frac{C}{r_1^4} \frac{r_1^{j+2}\rho^{j+1}}{(2\pi R T_\infty)^{3/2}} \frac{\exp\left\{-\frac{(\xi_1-\bar{\xi})^2+\rho^2}{2RT_\infty}\right\}}{(\xi_1^2+\rho^2)^{1/2}} dr_1 d\rho_1 d\xi_1$$

where the subscript 1 has been used to designate ξ and ρ at r_1. Again, with the hypersonic approximation, Eq. 17 yields:

for m = 0, $$n_1 = \iiint f_1 \, d\xi d\eta d\varsigma = \frac{C}{r^2\bar{\xi}}\left(\frac{1}{r_0} - \frac{1}{r}\right) \qquad (18)$$

and for m = 2, $$2n_1 R T_{1\perp} = \iiint \rho^2 f_1 \, d\xi d\eta d\varsigma = \frac{C(2RT_\infty)}{r^4\bar{\xi}}(r - r_0) \qquad (19)$$

where T_1 has been introduced to designate the lateral temperature of the scattered molecules. If $r \gg r_0$

$$n_1 = \frac{C}{r_0\bar{\xi} r^2} \qquad \text{and} \qquad T_{1\perp} = \frac{r_0}{r} T_\infty \qquad (20)$$

These results are strongly dependent on r_o, but the lateral pressure component is not, since

$$n_1 T_{1\perp} \sim C T_\infty / \bar{\xi} r^3 \qquad (21)$$

The lateral temperature of the combined distribution will be given by:

$$T_\perp = \frac{n_0' T_{0\perp}' + n_1 T_{1\perp}}{n_0' + n_1} \sim \frac{n_0 T_{0\perp} r_0^4/r^4 + C T_\infty / \bar{\xi} r^3}{n} \qquad (22)$$

For r sufficiently large,

$$T_\perp \approx C T_\infty / c r^3 \bar{\xi} = A n r T_\infty / \bar{\xi} = T_\infty / S. \qquad (23)$$

This is precisely that given in Eq. 6 of Part 2. It has, thus, been demonstrated that the lateral temperature at large distances from the source is governed by collisions in the far field. The number density of the particles which furnish the lateral temperatures is, on the other hand, small compared to the total number density. Specifically,

$$\frac{n_1}{n_0'} = \frac{C/(r^2 \bar{\xi} r_0)}{n_0 r_0^2/r^2} = \frac{3}{20} \frac{r^*}{r_0} Re^* \left(\frac{T_0}{T^*}\right)^{1-\omega} \qquad (24)$$

which, by assumption is small compared to 1. The lateral velocity distribution in the far field would then be representable as the sum of two distributions. The first would contain most of the particles, and would peak near $\eta = \xi = 0$, with a peak width proportional to $1/r$ and a lateral temperature proportional to $1/r^2$. The second part would contain considerably fewer particles, but would have a total lateral pressure which is higher than that of the peaked distribution. The resultant lateral temperature is then manifested primarily in a thickening of the tail of the distribution function rather than in the pulse width. The foregoing study indicates clearly that the distribution functions in lateral velocity is not adequately represented by the collision-free solution in the far field. (In fact, the collision-free limit resulting from the full solution to Eq. 10 would yield infinite lateral temperature.)

While the foregoing is based on the B-G-K model, similar results can be obtained for the hard sphere with relatively simple arguments that follow. Suppose that the flow is nearly collisionless, radially directed, with a limiting temperature T_∞, and a limiting Mach number $M_\infty \gg 1$. The distribution function is assumed to be nearly uniaxial and aligned with the flow. The number of collisions per unit time and volume is proportional to $\pi\sigma^2 n^2 \bar{C}$. Here $\bar{C}$ is a mean random speed, and $\pi\sigma^2$ is the collision cross section. After collision, a mean lateral speed of approximately $2/3\bar{C}$ will be induced. If a lateral speed is induced at $r = r_1$, it will be decreased by a factor r_1/r_2 at $r = r_2$ since, with no further collisions, β = const.

Let us consider then the "lateral energy" at $r = r_2$ due to collisions which have taken place in $r_0 < r < r_2$. Let $r_0 \gg r^* Re^*$. The major portion of the flow will then be nearly collisionless. The flux in lateral energy through the surface of the sphere at $r = r_2$ due to collisions in the prescribed range is

$$W = K_1 \int_{r_0}^{r} \pi\sigma^2 n^2 \bar{c}\, 4\pi r^2 \frac{4}{9} \bar{c}^2 \left(\frac{r}{r_2}\right)^2 dr \qquad (25)$$

where K_1 is a constant of proportionality. This integral expresses the fact that the lateral energy induced at r is transported outwards and reduced in magnitude by the ratio r^2/r_2^2 in the process. If $n = n_0 r_0^2/r^2$, then

$$W = K\pi\sigma^2 (n_0^2 r_0^4) \frac{4}{9} \bar{c}^3 \frac{4\pi}{r_2^2} (r_2 - r_0) \qquad (26)$$

and

$$n_1 T_{1\perp}\Big|_{r=r_2} = \frac{K_1}{2} \frac{W}{4\pi r_2^2 \bar{\xi} R} \approx \frac{2K}{9} \frac{\pi\sigma^2 n^2}{\bar{\xi}} \bar{c}^3 r \qquad (27)$$

Since n is proportional to $1/r^2$

$$T_{1\perp} = \frac{n_1 T_{1\perp}}{n} \propto \frac{1}{r}. \qquad (28)$$

5. More Detailed Properties of the Distribution Function and Numerical Results

The dimensionless radial distance s is introduced as in Eq. 3. Let

$$\alpha' \equiv \frac{\alpha}{\bar{\xi}}, \quad \rho' \equiv \frac{\rho}{\bar{\xi}}, \quad \beta' \equiv \rho s, \quad \xi' \equiv \frac{\xi}{\bar{\xi}} \qquad (29)$$

Then Eq. 7 becomes:

$$\sqrt{\alpha'^2 - \beta'^2/s^2}\; \frac{\partial f}{\partial s} = \frac{\Theta^{1-\omega}}{s^2} (F - f) \qquad (30)$$

where

$$F(\alpha, \beta, s) = \frac{K}{s^2 \Theta^{3/2}} \exp\left\{ - \frac{M_\infty^2}{2} \frac{\alpha'^2 + 1 - 2\sqrt{\alpha'^2 - \beta'^2/s^2}}{\Theta} \right\}$$

$$K \equiv \frac{n^{*} r^{*2}}{(2\pi R T_{\infty})^{3/2}} \left(\frac{S}{r}\right)^{2}, \quad \Theta \equiv \frac{T}{T_{\infty}}, \quad M_{\infty} \equiv \frac{\xi}{\sqrt{RT_{\infty}}}$$

If the integration for f will start at $s = \beta'/\alpha'$, Eq. 10 for $\xi > 0$ becomes:

$$f(\alpha',\beta',s) = \int_{\beta'/\alpha'}^{S} \frac{\Theta^{1-\omega} f}{x^{2}\sqrt{\alpha'^{2}-\beta'^{2}/x^{2}}} \exp\left\{-\int_{x}^{S} \frac{\Theta^{1-\omega}\, dy}{y^{2}\sqrt{\alpha'^{2}-\beta'^{2}/y^{2}}}\right\} dx$$

$$+ \exp\left\{-\int_{\beta'/\alpha'}^{S} \frac{\Theta^{1-\omega}\, dx}{x^{2}\sqrt{\alpha'^{2}-\beta'^{2}/x^{2}}}\right\} \cdot F\left(\alpha',\beta',\frac{\beta'}{\alpha'}\right) \qquad (31)$$

Two types of results obtained for the distribution function will be presented. First, the centerline distribution $f(\alpha',o,s)$ will be determined. Second, the integral $\int f d\xi$ will be evaluated as a function of β' and s, and the moments which are associated with this distribution will be studied. For example, $\int f d\xi d\eta$ or $\int \rho^{3} f d\xi d\rho$ can be displayed as a function of s. Note that, if the flow is hypersonic, $\int_{-\infty}^{\infty} f d\xi \approx \int_{0}^{\infty} f d\xi$.

By the argument given in Section 3, one can ignore the second term of Eq. 31 if the integration is started in the equilibrium region. Then

$$\frac{f(\xi',0,s)}{K} = \int_{0}^{S} \frac{1}{\xi' x^{4}\, \Theta^{\frac{1}{2}+\omega}} \exp\left\{-\frac{M_{\infty}^{2}}{2}\frac{(\xi'-1)^{2}}{\Theta} - \int_{x}^{S} \frac{\Theta^{1-\omega}\, dy}{\xi' y^{2}}\right\} \qquad (32)$$

This equation has been integrated numerically for $\omega = 1$, and $M_{\infty} = 15$. The results are shown in Fig. 2. The rather high degree of symmetry with respect to $\xi' = 1$ is expected from Eq. 32, since except in $M^{2}(\xi'-1)^{2}$, the ξ' under the integral sign can be replaced by $\xi' = 1$, subject to an error of order 1/M. This indicates that the similarity parameter is $M_{\infty}(\xi'-1)$ which is, therefore, employed in the presentation. Note that, in view of Eq. 7,

$$M_{\infty}(\xi'-1) \propto (\xi'-1)[Re^{*}]^{2/[3+4(1-\omega)]} .$$

At large values of s, the centerline distribution is seen to be nearly independent of s, and is well represented in the far field by one component of a Maxwell distribution with a temperature which is three times the limiting temperature. The factor of three is necessary to account for the fact that the distribution function is nearly uniaxial.

The lateral distribution will be characterized by:

$$\frac{1}{K}\int_0^{\infty} f\,d\xi = \int_0^{\infty} d\xi \int_{\beta'/\alpha'}^{S} g(s;\alpha',\beta',x)\,dx \qquad (33)$$

where

$$g(s;\alpha',\beta',x) \equiv \frac{\exp\left\{-\frac{M_\infty^2}{2\Theta}\left(\alpha'^2+1-2\sqrt{\alpha'^2-\beta'^2/x^2}\right)-\int_x^{S}\frac{\Theta^{1-\omega}\,dy}{y^2\sqrt{\alpha'^2-\beta'^2/y^2}}\right\}}{x^4\,\Theta^{\frac{1}{2}+\omega}\sqrt{\alpha'^2-\beta'^2/x^2}}$$

But

$$\int_0^{\infty}\frac{f}{K}d\xi = \bar{\xi}\int_{\beta'/S}^{\infty}\int_{\beta'/\alpha'}^{S}\frac{g(s,\alpha',\beta',x)\,\alpha'}{\sqrt{\alpha'^2-\beta'^2/s^2}}\,dx\,d\alpha' \qquad (34)$$

$$= \bar{\xi}\int_0^{S}dx\int_{\beta'/x}^{\infty}\frac{g(s,\alpha',\beta',x)\,\alpha'\,d\alpha'}{\sqrt{\alpha'^2-\beta'^2/s^2}} \qquad (35)$$

Consistent with the hypersonic approximation, the interior integral in Eq. 35 is evaluated by Laplace's method to yield, (subject to an error of order $1/M^2$).

$$\int_0^{\infty}\frac{f\,d\xi}{K} = \bar{\xi}\int_0^{S}\frac{\exp\left\{-\frac{M_\infty^2\beta'^2}{2\Theta\,x^2}-\int_x^{S}\frac{\Theta^{1-\omega}\,dy}{y^2\sqrt{1+\beta'^2/x^2-\beta'^2/y^2}}\right\}}{x^4\,\Theta^{\omega}\sqrt{1+\beta'^2/x^2-\beta'^2/s^2}}\,dx \qquad (36)$$

Eq. 36 can be further simplified. With the integrand again approximated to within order $1/M^2$, one obtains (for all β')

$$\int_0^{\infty}\frac{f\,d\xi}{K} \approx \bar{\xi}\int_0^{S}\frac{\exp\left\{-\frac{M_\infty^2\beta'^2}{2\Theta\,x^2}-\int_x^{S}\frac{\Theta^{1-\omega}}{y^2}dy\right\}}{x^4\,\Theta^{\omega}}\,dx \qquad (37)$$

The approximation follows from the fact that if $M_\infty\beta'$ is of order one or less, the radicals are $1+O(1/M^2)$. If $M_\infty\beta' \gg 1$, the only contribution comes from the neighborhood of $x = s$ where the two terms in the radical cancel, again with error of order $1/M^2$. Note that, in Eq. 37 the lateral velocity enters only in the combination $M_\infty\beta'$ or $M_\infty\rho'$. Also note that both ranges of $M_\infty\beta'$ considered will contribute to the integral.

Eq. 37 has been integrated numerically for $\omega = 1$, and values of Θ obtained from the integration of Eq. 3. The results of these calculations appear in Figs. 3, 4, and 5. In Fig. 3, $\int f d\xi$ is plotted against the similarity parameter $M_\infty \beta' \propto \beta (Re^*)^{2/[\beta+4(1-\omega)]}$. It will be noted that the curves corresponding to s = 7.46, 20 and ∞ are quite close over much of the range of $M_\infty \beta'$ plotted. This is to be expected, since, from Eq. 37, it can be shown that the solution will depart from the limiting distribution when $M_\infty \beta'$ is of order s. The curves presented for large s cannot be meaningfully fitted to a Maxwellian-type distribution $Ae^{-\rho^2/B^2}$ (the dash curve); the scale B bears no relationship to the lateral temperature.

Fig. 4 displays $\rho s \int f d\xi$ as a function of $M_\infty \beta'$ or $M_\infty s \rho'$. The area under these curves is independent of s, since this area represents $s^2 n$. Again the existence of the limit is indicated. Note the range of $M_\infty \beta'$ contributing to the area, hence, the number density, is $M_\infty \beta' = 0(1) \neq 0$.

Fig. 5 shows $\rho'^3 s^3 \int f d\xi$ as a function of $M_\infty \rho'$ corresponding to two values of s. For large s, although the curves are not similar, the areas under the curves appear to be approximately equal. With this area fixed, the lateral temperature varies as 1/s. The main contribution to the area, hence, the lateral temperature, is seen to be in the range of $M_\infty \rho' = 0(1) \neq 0$. The limit as $s \to \infty$ can be obtained analytically, and is given by the dashed curve.

By an extended application of the method of Laplace it is possible to find a rather general moment of the distribution function, again with accuracy $1/M^2$. For simplicity, consider only $\omega = 1$,

$$H^{ij} = \iiint (\xi - \bar{\xi})^i \rho^j f \, d\xi d\eta d\zeta$$
$$= 2\pi K(\bar{\xi})^{i+j+3} \int_0^\infty b^{j+1} e^{-b^2} db \cdot e^{\frac{1}{s}} \left(\frac{2}{M_\infty^2 s^2}\right)^{1+j/2} \times$$
$$\times \int_0^S \Theta^{1+j/2} x^{j+2} e^{-\frac{1}{x}} P_i \, dx \qquad (38)$$

where $$P_0 = \frac{\sqrt{2\pi}}{M_\infty} \frac{1}{x^4 \Theta}, \qquad P_2 = \frac{\sqrt{2\pi}}{M_\infty^3 x^4}$$

Specializing to $s \gg 1$, sample calculations show that H^{00}, H^{02} and H^{20} reduce, indeed, to n, $2nRT_\infty/s$ and $3nRT_\infty$ respectively. Note that for $j \geq 1$, the last integral of Eq. 38 increases with s. This confirms that moments with $j \geq 1$ are defined mainly by collisions in the far field.

6. Conclusion

In the far field of a spherical source, the B-G-K model equation admits a solution which possesses a limiting temperature and a limiting speed. The distribution function is nearly uniaxially and radially oriented, but the lateral temperature in the far field is governed by far field collisions. Because of these collisions, the lateral temperature decays as $1/r$ rather than $1/r^2$. However, this lateral temperature and higher moments will be manifested in a thickening of the tail of the distribution function rather than in a broadening of the spike. The primary contribution to the lateral temperature comes from the region where $M_\infty \rho'$ is O(1). The spike width decreases as $1/r$, and a lateral temperature measurement based on spike width would indicate a temperature decay which is proportional to $1/r^2$.

For hypersonic flow, a Mach or Reynolds number independence is obtained for both axial and lateral distributions, with M_∞ or $(Re^*)^{2/3+4(1-\omega)}$ appearing only as a scaling factor for the peculiar velocities. The axial distribution in the far field looks much like one component of a Maxwell distribution which possesses a temperature three times the limiting temperature.

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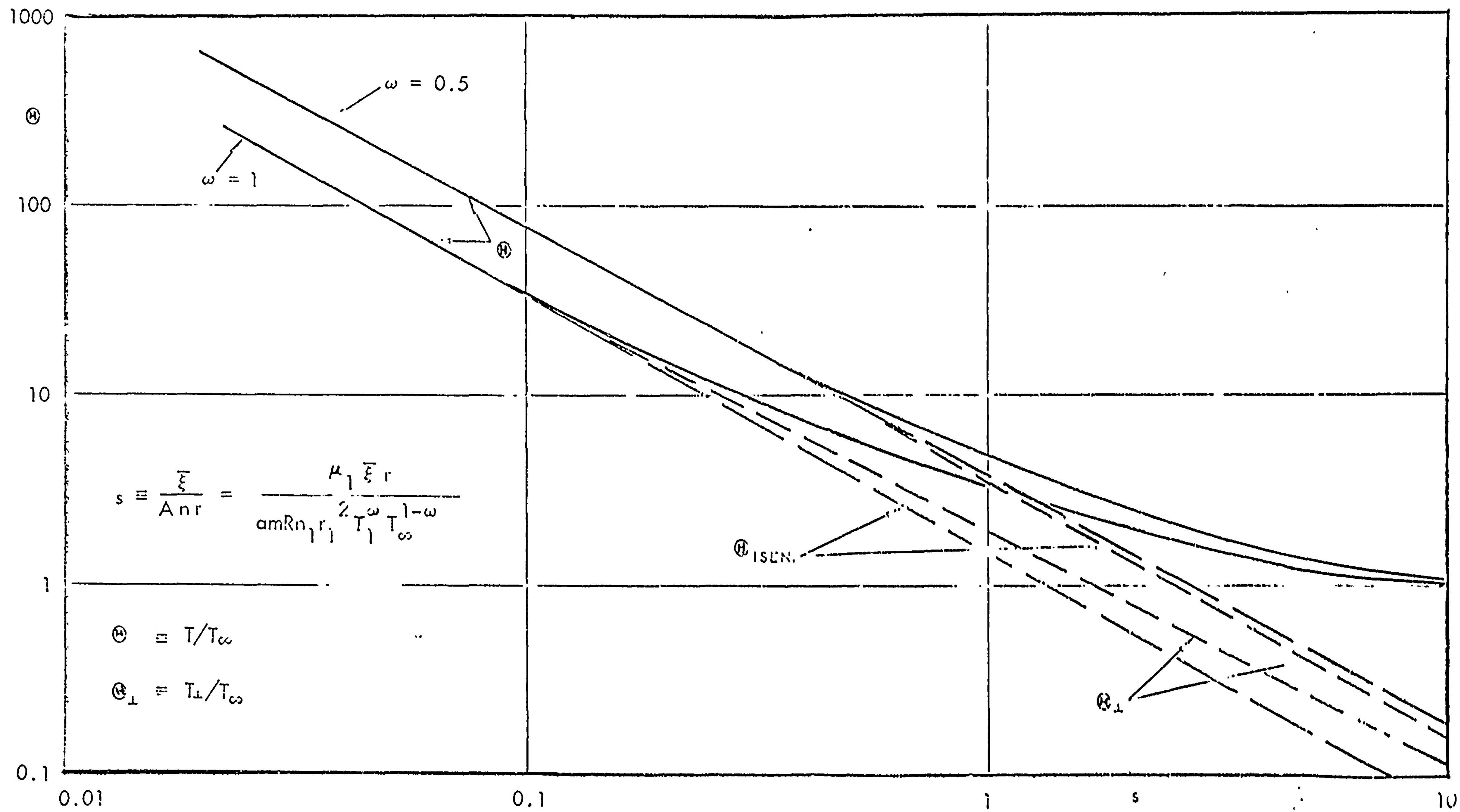


Fig. 1 Temperatures T, $T_\perp$ and T_{isen} vs. local Knudsen variable s for ω = 1/2 and 1.

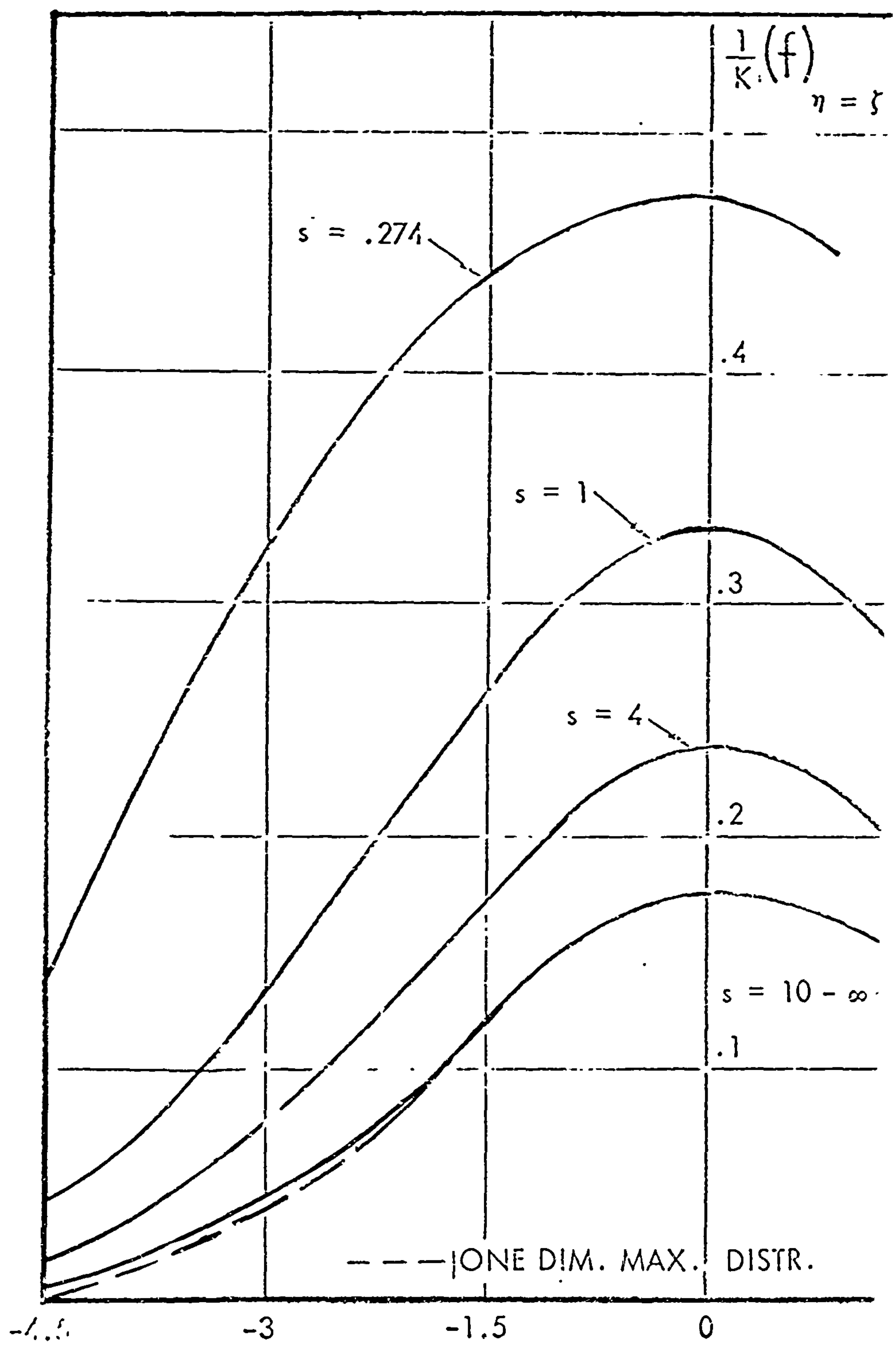
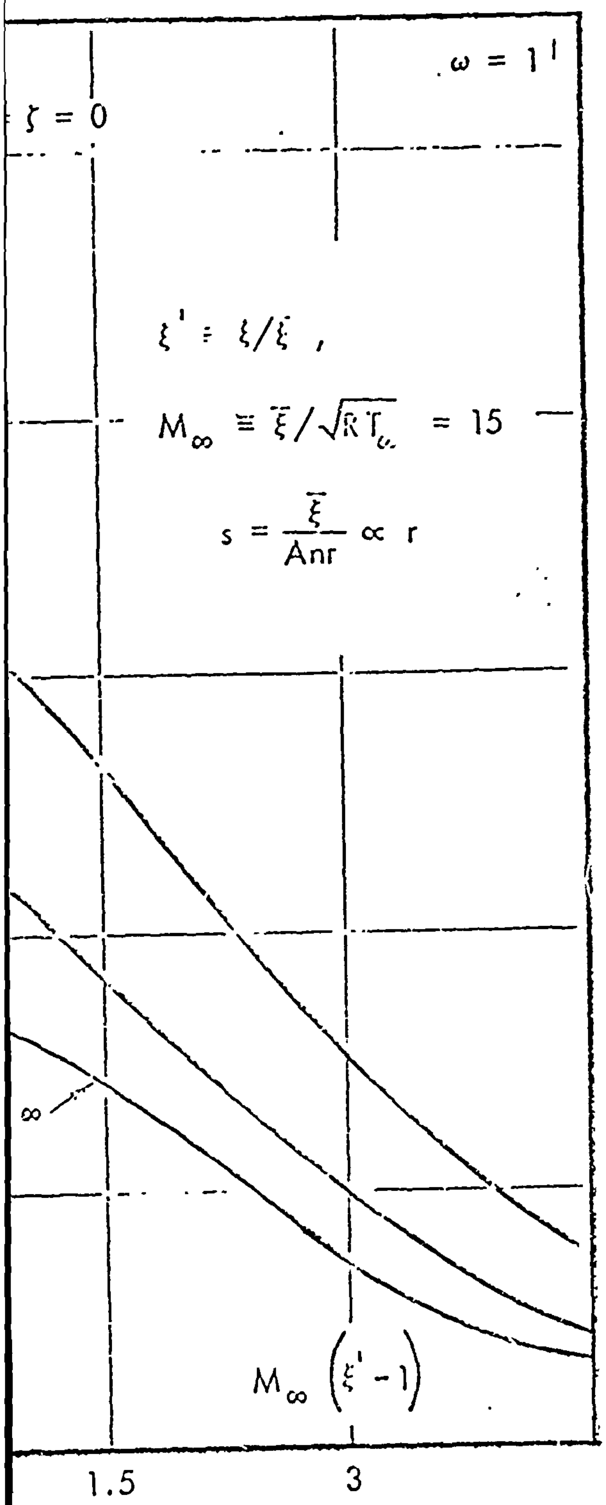


Fig. 2 The centerline distribution fun
as function of $M(\xi'-1)$ and s for $\omega = 1$.



function $(f)\ \rho = 0$

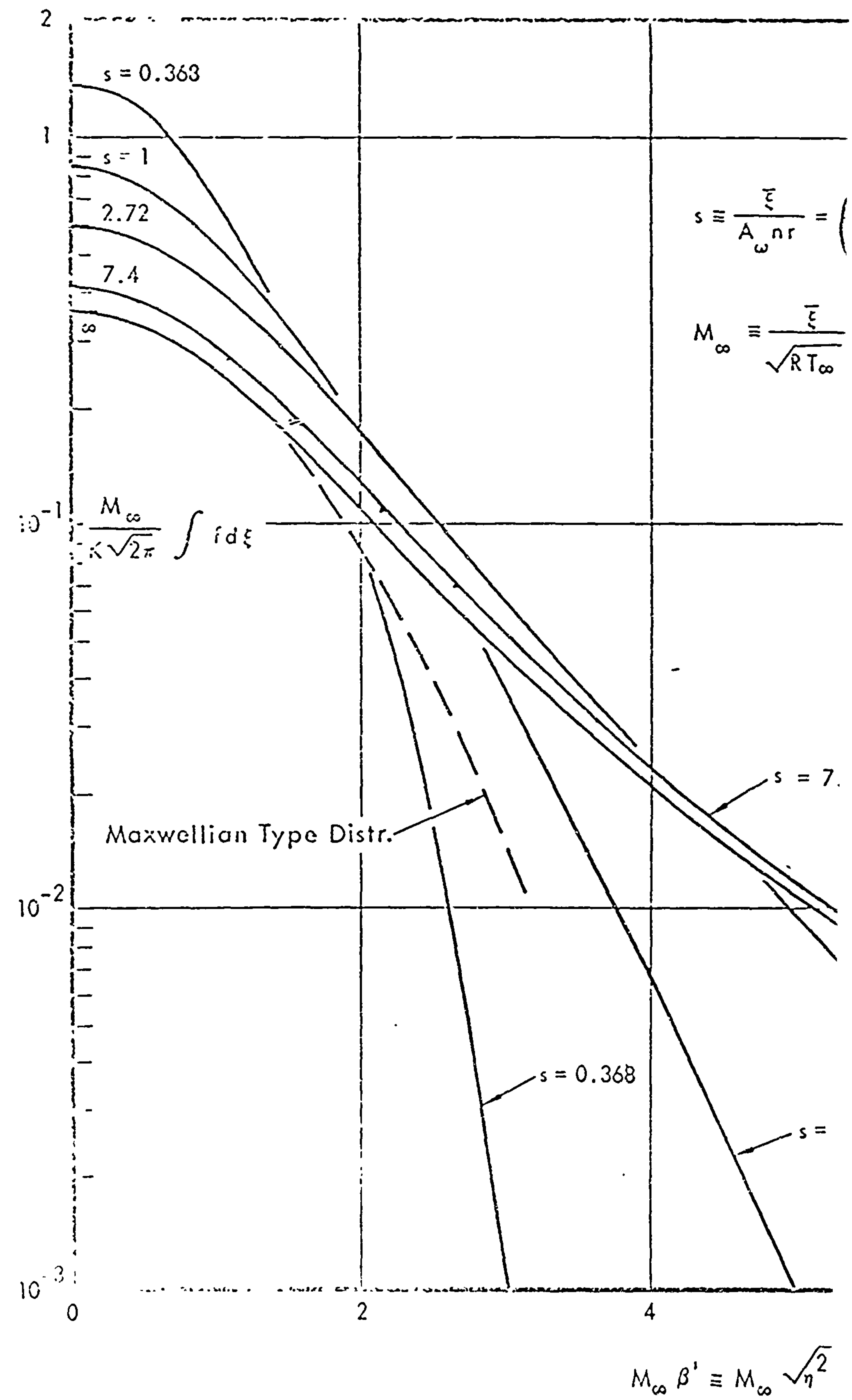
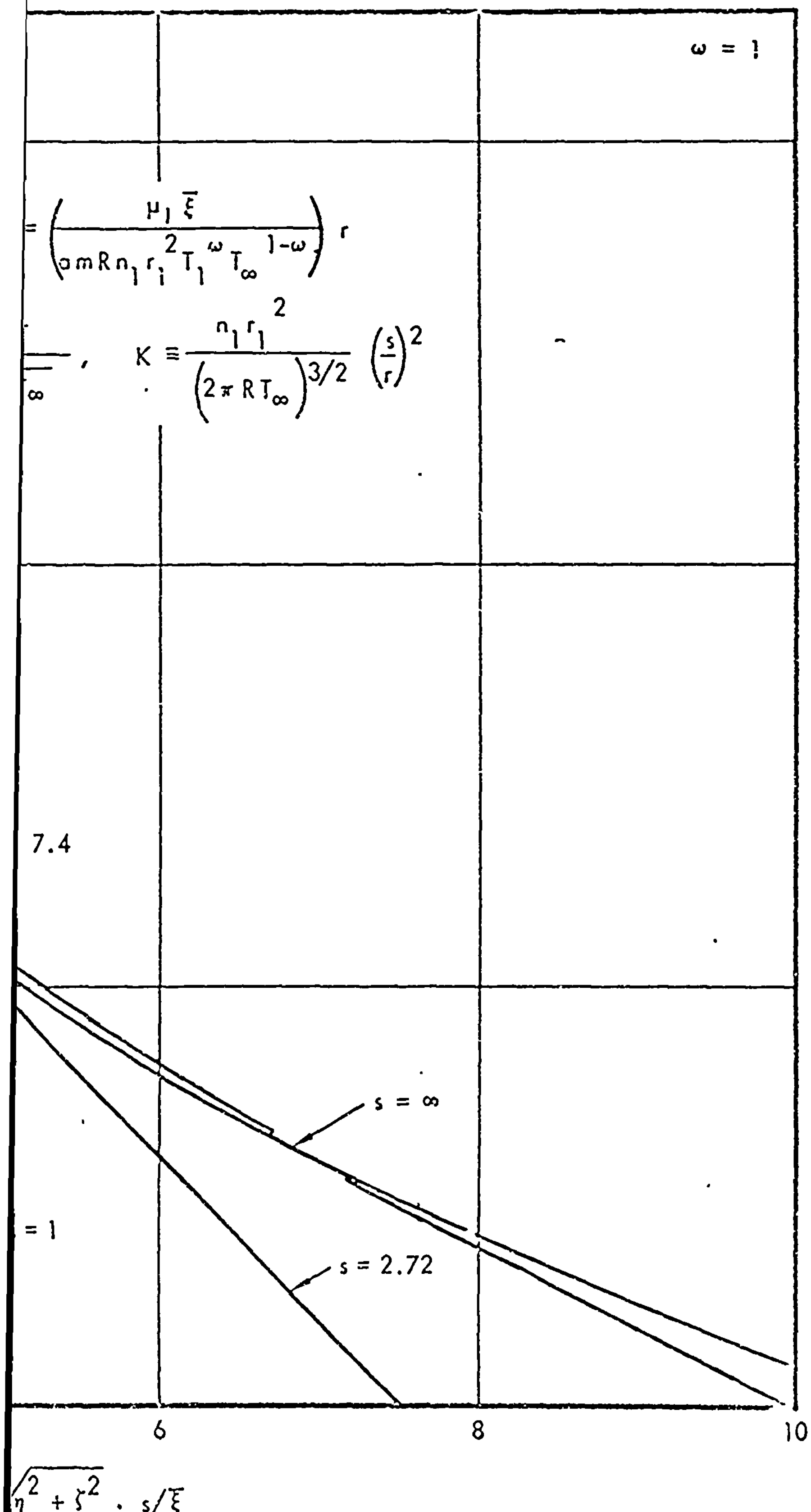


Fig. 3 The integral $\int f d\xi$ as function of $M_\infty \beta'$ and s for $\omega = 1$.



ω = 1
7.4
s = ∞
= 1
s = 2.72
6
8
10

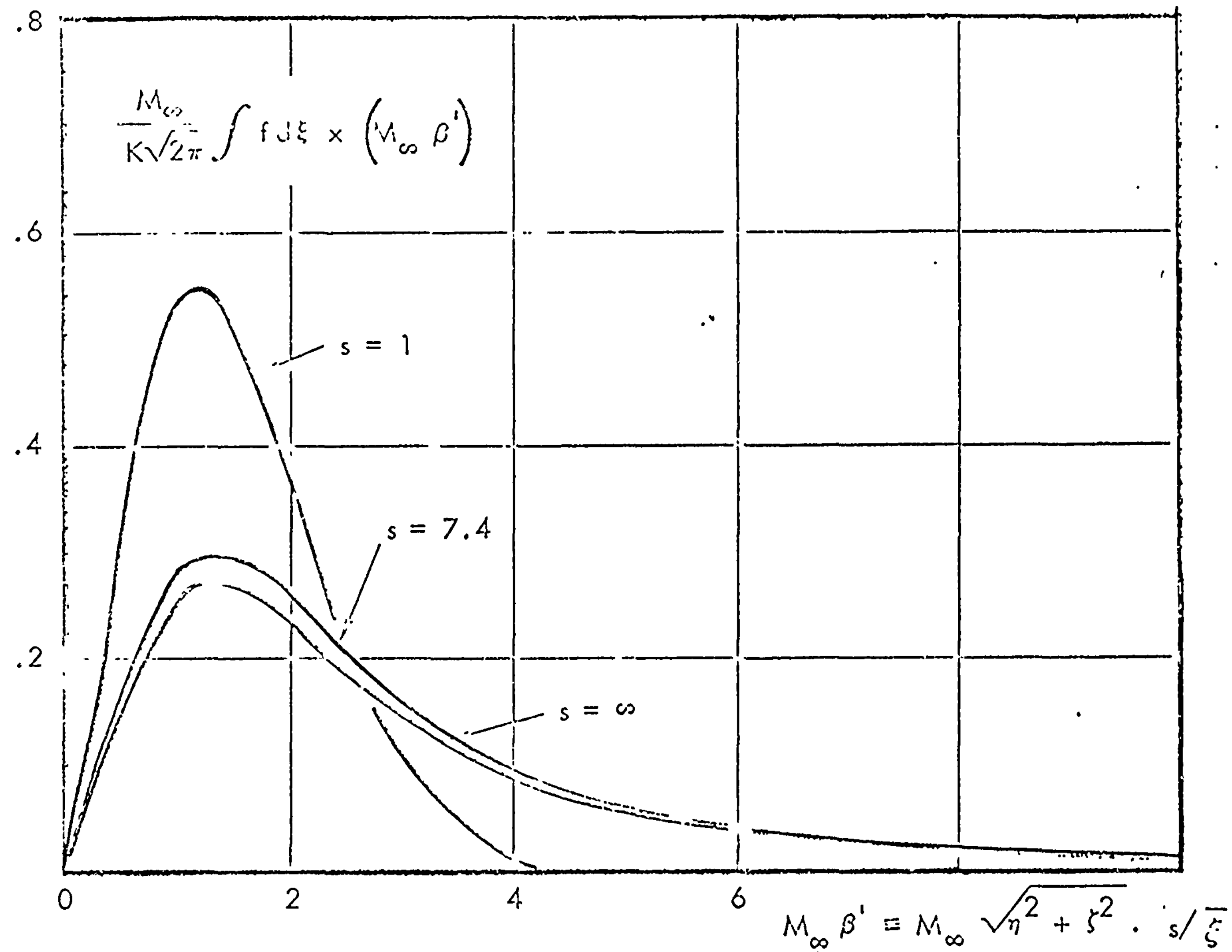


Fig. 4 The product $\rho' \int f d\xi$ as function of $M_\infty \beta'$ and s for $\omega = 1$.

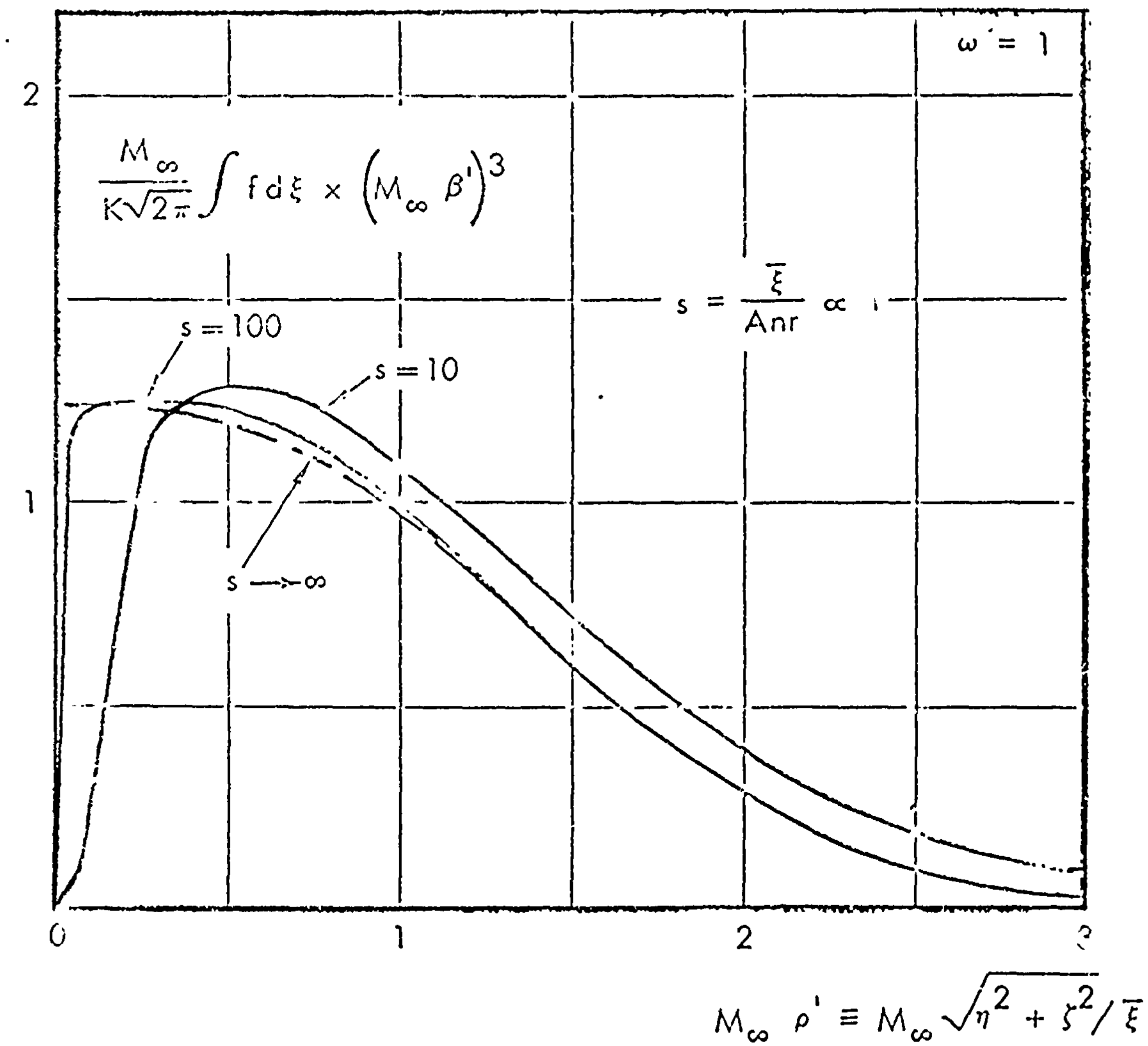


Fig. 5 The product $(\rho')^3 \int f d\xi$ as function of $M_\infty \beta'$ and s for $\omega = 1$.

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13 ABSTRACT

The distribution function for the spherical source is determined by integrating the B-G-K model equation, where the local temperature is determined by the moment equations under the hypersonic approximation. In the far field, the axial distribution is nearly Maxwellian. A free molecule limit exists, but does not define the lateral temperature and higher moments properly. The lateral temperature is determined by far field collisions, and is largely contained in the tail of the lateral distribution. A Mach number or a Reynolds number similarity in the distribution function is shown.

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Freezing of Translational Temperature						
Kinetic Theory of Source Flow						
Expansion into Vacuum						
Parallel and Lateral Temperature						
B-G-K Equation						
Nonisotropy in Distribution Function						
Hypersonic Approximation						