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Derivation of Linear-Tangent Steering Laws

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Prepared for COMMANDER SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract No. AF 04(695)-1001.

This report, which documents research carried out during the month of February 1966, was submitted on 29 November 1966 to Capt Michael A. Ikezawa, SSTDG, for review and approval.

Approved

for E. Levin, Director
Guidance and Control Subdivision
Electronics Division

Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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ABSTRACT

The derivation of the several forms of the linear-tangent steering program is presented. The mathematical form of these equations is shown to be totally independent of the variation of thrust acceleration with time. Insofar as position and velocity changes caused by thrust only are concerned, the linear-tangent law is the precise mathematical optimum. Its utility and advantages are explained, as well as its limitations. The derivation of the thrust-direction steering law is given, but guidance equations are not.

CONTENTS

FIGURE

SECTION I

INTRODUCTION AND SUMMARY

The derivation of the several forms of the linear-tangent steering program is presented. The mathematical form of these equations is shown to be totally independent of the variation of thrust acceleration with time. Insofar as position and velocity changes caused by thrust only are concerned, the linear-tangent law is the precise mathematical optimum. It yields minimum thrusting time and, hence, maximum payload-carrying capability for any desired position and velocity change. Even when forces other than thrust are included, the linear-tangent program is often optimum or near optimum.

Linear-tangent steering may be used for such maneuvers as orbital plane changes in space as well as for the exoatmospheric portion of ascent of large multistage rocket vehicles. Guidance equations specifically for this latter application are presented in Reference 1. When used for guidance equations, the linear-tangent steering has the advantage of not requiring intermediate position and velocity aim points and hence not requiring a trial-and-error trajectory design study to minimize propellant consumption for multistage vehicles. Another advantage is that major maneuvers in yaw position and velocity are accomplished integrally with pitch steering in a manner which minimizes propellant consumption.

Although the linear-tangent pitch program was derived and first presented to the literature ten years ago (Reference 2), there are still many people engaged in trajectory, performance, and guidance work who do not understand its utility or validity. The purpose here is to explain the limitations of this steering law and to present a derivation similar to the procedure explained in Reference 2. It is felt that the derivation by means of this perturbation technique is simpler, far more concise, and yields a better physical understanding of the phenomenon than the classical LaGrange technique presented in later papers (References 3 and 4). This report is confined to the derivation of the thrust-direction steering law and does not contain any guidance equations.

SECTION II

APPLICATIONS AND APPROACH

A. APPLICATION TO MULTISTAGE BOOSTER

The linear-tangent steering program is applicable to that portion of the ascent of a multistage booster that takes place above the drag sensible atmosphere. When one considers that such a vehicle may burn out between 500 and 1500 n mi downrange from the launch site and yet clears the atmosphere generally less than 40 n mi downrange (altitude approximately 20 n mi), it is evident that the thrust direction versus time program during this exoatmospheric portion of flight is very important to over-all propellant economy. The linear-tangent program is not used within the atmosphere because to minimize aerodynamic load and heating problems, it is necessary to utilize a zero lift or near-zero lift trajectory for that first portion of ascent. Sometime after the principal burnout mentioned above, additional impulses may take place to change orbits, but these are ideally of short duration and could be considered as part of space flight rather than part of the launch trajectory

B. NATURE OF THE STEERING PROGRAM

As the vehicle emerges from the atmosphere, it has a (present) position and velocity, and it is desired to achieve a different position and velocity at thrust termination. The only two forces that can cause the vehicle to accomplish the desired position and velocity changes are gravity and thrust. Suppose the over-all integrated effect of gravity for the steering program to be used were known in advance. Then these total gravity position and velocity vectors could be subtracted from the vector difference between the burnout and present conditions to yield the total position vector and total velocity vector contributed by thrust alone. A velocity hodograph of this type is illustrated in Reference 2.

Further suppose that a steering program (thrust attitude versus time) were derived to achieve these known desired thrust velocity and position vectors by means of minimum burning time, and that gravity did not enter into the derivation at all. Such a steering program would be precisely optimum from the standpoint of position and velocity changes caused by thrust alone. It would, however, not necessarily be precisely the over-all optimum for the following reasons. Any departure from this program would result in losses in position and velocity contributed by thrust. However, a departure judiciously selected could also result in a vehicle position-time history which would reduce the gravity losses. Whether or not an over-all gain could be realized would depend upon whether the saving in propellant caused by reducing the gravity losses would exceed the increased propellant required to overcome thrust losses caused by departure from this suboptimal program. The linear-tangent pitch programs derived herein are of the type just described. It is felt that any net propellant saving caused by departing from these programs is probably not worth the effort to find it. However, an approximate method of realizing most of this gain has been devised, but is not presented herein. This refinement has never been tested.

C. TWO-POINT PERTURBATION TECHNIQUE FOR CURVED-EARTH CASE

In Reference 2, the two-point perturbation technique was explained, but the actual derivation for the curved-earth case was not presented. The final equation for the curved-earth case, however, was presented. It amounts to diminishing the tangent of the pitch angle as measured above burnout horizontal linearly with time. Unfortunately, in the paper, the pitch angle was measured above launch horizontal so the fact that the tangent of the angle above burnout horizontal was being diminished at a constant rate was not readily visible from the equation. The reason the pitch angle was referenced to launch horizontal was that guidance equations were not being considered and, for performance calculations, the known horizontal at the launch site was a more convenient reference.

D. THE PROBLEM

After subtracting the expected gravity position and velocity changes by the hodograph technique mentioned earlier and presented in Reference 2, the problem is reduced to one of achieving certain position and velocity changes attributable to thrust alone by means of a steering program that results in minimum burning time. These desired position and velocity changes will in general have pitch, yaw, and downrange components which may be vectorially combined to yield one desired position vector and one desired velocity vector. All thrust direction turning will take place in the plane defined by these two vectors. This steering plane will in general be canted to both the pitch and yaw planes. A linear-tangent steering program in this canted plane geometrically results in another linear-tangent program in the pitch plane and another in the yaw plane. Although the linear-tangent steering program applies to both pitch and yaw, for simplicity of explanation its derivation will be carried out for the pitch plane only.

SECTION III

PROGRAM DERIVATIONS

A. PITCH STEERING

Because of its practical importance, the case selected first is that in which it is desired to achieve a certain altitude, velocity, and flight path angle at thrust termination without specifically controlling downrange distance.

1. THE GEOMETRY

The geometry of the problem is illustrated in Figure 1. (It should be noted that this figure refers to a thrust-caused phenomenon only.)

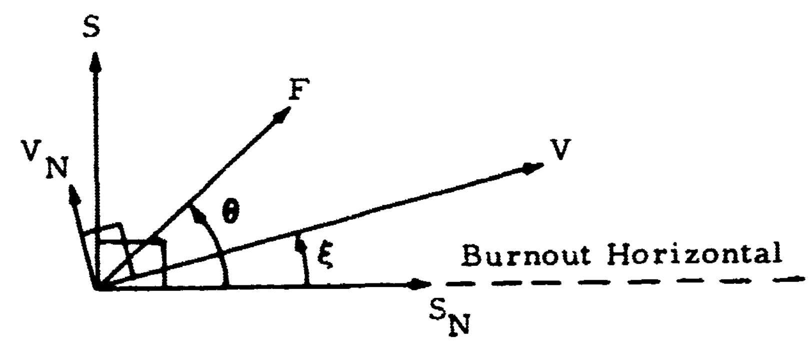

Figure 1. Thrust-caused Position and Velocity Components

F	thrust (any function of time)
S	position change in direction of burnout vertical
S_N	position change normal to S
V	velocity change in desired direction
V_N	velocity change normal to V
θ	pitch attitude measured above burnout horizontal
ξ	angle between desired velocity direction and burnout horizontal

In terms of Figure 1, the object is to find θ as a function of time, which will minimize burning time while achieving V and S subject to the constraint that V_N is zero. Minimum burning time means minimum propellant required, which, in turn, means maximum payload-carrying capability.

2. SYMBOLS

The discussion will be aided by use of the following symbols.

a_T	thrust acceleration = F/M
M	vehicle mass
t	time from start of pitch program
t_f	burnout time from start of pitch program
t'	fractional time = t/t_f
Δt	an infinitesimal time increment
Δ	an increment in <u>burnout</u> conditions caused by thrust acting during Δt
$\delta\theta$	a perturbation in θ acting only over Δt
δ	an increment in <u>burnout</u> conditions caused by $\delta\theta$
$\triangleq$	equals by definition

<u>Subscripts</u>

a, b, c	general points along the trajectory
o	initial
f	final

3. THE DERIVATION TECHNIQUE

The derivation technique will be first to assume that the pitch program is optimum, then to investigate certain properties derived from the assumption and, finally, to determine the optimum pitch program from these properties. By definition the optimum program is one that will achieve the desired position and velocity changes with the minimum burning time. Then, if the burning time were held fixed at this minimum, it would be impossible to modify the steering program in any way that would result in increasing any of the desired thrust position or velocity components except at the expense of others. In other words, it would be impossible to alter the pitch angle as a function of time in any way that would result in increasing one of the position or velocity components while maintaining the others unchanged.

The minute increments in the components of burnout position and velocity caused by thrust acceleration acting over an infinitesimal time increment Δt at any point along the trajectory are as follows (see Figure 1):

$$\Delta V = a_T \Delta t \cos(\theta - \xi) \tag{1}$$

$$\Delta V_N = a_T \Delta t \sin(\theta - \xi) \tag{2}$$

$$\Delta S = a_T \Delta t \sin \theta (t_f - t) \tag{3}$$

$$\Delta S_N = a_T \Delta t \cos \theta (t_f - t) \tag{4}$$

It will be observed that the thrust velocity ($a_T \Delta t$) generated during the time increment Δt appears as a multiplier in all four equations. Therefore, it may be expected that the mathematical form of the final optimum pitch program will be completely independent of thrust acceleration. This can be seen from the fact that, regardless of the value of a_T at any point, the value of the product $a_T \Delta t$ can be adjusted to any value by changing the magnitude of the infinitesimal time increment Δt. Equations (1) through (4) show that the division of impulse between the velocity components depends upon the angle only, while that going into position is a function of both angle and time. Hence, the form of the final equation will present θ as a function of time independent of the variation of thrust acceleration with time.

Partial differentiation of Eqs. (1) through (4) yields the changes in burnout conditions caused by an infinitesimal perturbation in θ acting only during the time interval Δt. Thus

$$\delta \Delta V = - a_T \Delta t \sin (\theta - \xi) \delta\theta \tag{5}$$

$$\delta \Delta V_N = a_T \Delta t \cos (\theta - \xi) \delta\theta \tag{6}$$

$$\delta \Delta S = a_T \Delta t \cos \theta (t_f - t) \delta\theta \tag{7}$$

$$\delta \Delta S_N = - a_T \Delta t \sin \theta (t_f - t) \delta\theta \tag{8}$$

Equations (4) and (8) are not pertinent to this derivation but enter into the subsequent discussion.

<u>If</u> the values of perturbations in θ at three perfectly general points a, b, c along the trajectory could be adjusted to cause no change in the controlled burnout conditions, the following three equations would apply.

$$\sum \delta \Delta V = \alpha_a \delta\theta_a + \alpha_b \delta\theta_b + \alpha_c \delta\theta_c = 0 \tag{9}$$

$$\sum \delta \Delta V_N = \beta_a \delta\theta_a + \beta_b \delta\theta_b + \beta_c \delta\theta_c = 0 \tag{10}$$

$$\sum \delta \Delta S = \gamma_a \delta\theta_a + \gamma_b \delta\theta_b + \gamma_c \delta\theta_c = 0 \tag{11}$$

where

$$\alpha \triangleq a_T \Delta t \sin(\theta - \xi) \tag{12}$$

$$\beta \triangleq a_T \Delta t \cos(\theta - \xi) \tag{13}$$

$$\gamma \triangleq a_T \Delta t \cos \theta (t_f - t) \tag{14}$$

The absolute magnitude of the $\delta\theta$'s is not important but only their relative ratios of one to another. This can be seen by the fact that, if all the $\delta\theta$'s were multiplied by the same constant, Eqs. (9), (10), and (11) would still balance. Therefore, it will be assumed that $\delta\theta_c$ is fixed at an arbitrary small value. The three equations may now be viewed as three straight-line equations in two unknowns, $\delta\theta_a$ and $\delta\theta_b$. A glance at the coefficients, Eqs. (12), (13), and (14), shows that these lines are not parallel. However, for a common solution to exist, all three lines would have to intersect at the same point ($\delta\theta_a$, $\delta\theta_b$). The next step is to fix the values of $\delta\theta_a$ and $\delta\theta_b$ at those that satisfy Eqs. (9) and (10). It now remains to determine whether or not these same values do not in fact also satisfy Eq. (11).

With the three angular perturbations determined as just explained, Eq. (11) is either greater than zero, less than zero, or equal to zero as shown. No other possibilities exist. In physical considerations, the possibility of the net change in S [Eq. (11)] being greater than zero may be ruled out. If it were greater then zero, it would mean that putting all three perturbations into effect would result in an increase in burnout altitude without any loss in either of the burnout velocity components. This would violate the assumption that these perturbations were being imposed upon a pitch program that was already optimum (minimum thrusting time). The possibility of Eq. (11) being less than zero may also be ruled out, because, in that event, simply changing the sign of each $\delta\theta$ would convert the loss in burnout altitude to a gain of like absolute magnitude. The only remaining possibility is that Eq. (11) equals zero as shown for the same perturbations in θ that satisfy Eqs. (9) and (10).

Now that it has been established that, in physical fact, Eqs. (9), (10), and (11) each equals zero, the next step is to again fix $\delta\theta_c$ at an arbitrarily small value and view $\delta\theta_a$ and $\delta\theta_b$ as variables. This time the products of the $\delta\theta_c$'s and their respective coefficients will be transposed as constants to the right-hand side of the equations to form the following set of functions f_1, f_2, f_3.

$$f_1 = \alpha_a \delta\theta_a + \alpha_b \delta\theta_b = -\alpha_c \delta\theta_c \tag{15}$$

$$f_2 = \beta_a \delta\theta_a + \beta_b \delta\theta_b = -\beta_c \delta\theta_c \tag{16}$$

$$f_3 = \gamma_a \delta\theta_a + \gamma_b \delta\theta_b = -\gamma_c \delta\theta_c \tag{17}$$

The coefficients of $\delta\theta_a$ and $\delta\theta_b$ in the set of Eqs. (15), (16), and (17) form a 3×2 matrix. Since the rank of this matrix is less than 3, the three functions form a linearly dependent set, and so

$$f_1 + C_1 f_2 + C_2 f_3 = 0 \tag{18}$$

where C_1 and C_2 are as yet undetermined constants. The expressions with the subscript c in Eqs. (15) through (17) may now be used to replace the f values of Eq. (18). The resulting equation may then be divided by $a_{T_c} \Delta t_c \delta\theta_c$ to remove these quantities.

Since point c could be anyplace along the trajectory, the subscript may be dropped to yield

$$\sin(\theta - \xi) + C_1 \cos(\theta - \xi) + C_2 \cos\theta(t_f - t) = 0 \tag{19}$$

In Eq. (19), C_1 is found by setting t equal to t_f with θ equal to θ_f, and then C_2 is found by setting t to 0 and θ to θ_o. After trigonometric simplification, the equation reduces to

$$\tan\theta = \tan\theta_o - (\tan\theta_o - \tan\theta_f)t' \tag{20}$$

where

$$t' \triangleq \frac{t}{t_f}$$

Equation (20) is the linear-tangent pitch program used in the guidance equations of Reference 1.

Use of this steering law, Eq. (20), reduces the problem of trajectory optimization for the vacuum portion of ascent to merely finding the unique set of constants θ_o, θ_f, and t_f that yield the desired altitude, velocity, and flight-path angle at thrust termination. The independence of Eq. (20) from thrust acceleration enables it to be applied as a single steering program through multiple propulsion stages regardless of the variation of thrust and propellant flow rate. The single program applies even through periods of zero thrust. Of course, the values of the constants θ_o, θ_f, and t_f will depend upon the thrust acceleration versus time profile and upon the burnout conditions dictated by the mission.

B. YAW STEERING

The linear-tangent pitch program is precisely optimum from the standpoint of what is contributed to the trajectory by thrust only. Therefore, another linear-tangent pitch program will be optimum in the same sense for yaw steering as well. In the case of yaw steering, the tangent angle would be measured out of the orbital plane at burnout. As already explained, a linear-tangent program in both pitch and yaw is equivalent to one linear-tangent steering program in a plane canted to both the pitch and yaw planes. This canted plane is defined by one desired thrust velocity vector and one desired thrust position vector, each of which is the vector sum of the desired pitch plane and yaw components.

C. DOWNRANGE DISTANCE CONTROL

Equation (20) does not control downrange distance. The most practical way of controlling this, if needed, is by means of a variable-duration zero-thrust period. This amounts to changing the thrust acceleration versus time profile, of which the coast period is a part. If, however, it is desired to control downrange distance by means of the steering program, a bi-linear-tangent program could be used.

1. FOR A FLAT EARTH

The bi-linear-tangent steering law was presented in Reference 4 for a flat earth in the form

$$\tan\theta = \frac{A + Bt}{C + Dt} \tag{21}$$

where A, B, C, and D are unspecified constants.

2. FOR A CURVED EARTH

The derivation for a curved earth in terms of the notation used herein may be accomplished similarly to the derivation just completed, as follows.

To the set of Eqs. (9), (10), (11), add

$$\sum \delta S_N = \eta_a \delta\theta_a + \eta_b \delta\theta_b + \eta_c \delta\theta_c = 0 \tag{22}$$

Eta is obtained from Eq. (8). Then

$$\eta \triangleq a_T \Delta t \sin\theta (t_f - t) \tag{23}$$

The equation corresponding to Eq. (18) becomes

$$f_1 + C_1 f_2 + C_2 f_3 + C_3 f_4 = 0 \tag{24}$$

which leads to the solution

$$\tan\theta = \frac{\tan\theta_o - [\tan\theta_o - (1 + K)\tan\theta_f]t'}{1 + Kt'} \tag{25}$$

The additional constant K in Eq. (25) is required to control the downrange distance to burnout. When the desired downrange distance happens to be that which would result without control, K goes to zero and Eq. (25) reduces to Eq. (20). In fact, with chemical propulsion, if the downrange distance is not close to this value, the cost in propellant may be excessive.

Equation (25) is the bi-linear-tangent program of Eq. (21) with a superfluous constant removed. It is generally recognized that this is the optimum steering program resulting from the calculus of variations. It is not, however, generally known that all bi-linear-tangent programs may be reduced to mono-linear tangent ones by properly selecting the axis from which the angle θ is measured. For example, Eqs. (21) or (25) may be reduced to

$$\tan \theta' = A' + B' t \tag{26}$$

where

$$\theta' \triangleq \theta + \arc \tan \frac{D}{B} \tag{27}$$

in which D and B are in terms of Eq. (21). The constant arc tangent D/B represents the angle between the original and revised reference axes. This angle may be used to control downrange distance to burnout, if desired.

D. PROGRAM FOR SPECIAL SITUATIONS

A third linear-tangent steering program, which may have some application in special situations, is to maximize the relationship between one velocity and one position vector without any constraints on the components normal to each. The final equation is the same as Eq. (20), except that the terminal thrust direction is made to lie along the desired thrust velocity direction. In other words, θ_f becomes the known constant

$$\theta_f = \xi \tag{28}$$

SECTION IV

CONCLUSIONS

In conclusion, it may be said that the linear-tangent steering programs are precise optimums insofar as the contributions of thrust only are concerned.

The linear-tangent programs are the practical optimum solutions to any space maneuvers in which a departure from these programs cannot be found that will favorably modify the over-all effect of nonthrust forces such as gravity to an extent that will appreciably exceed the losses in thrust-caused values that result from departing from linear tangent steering.

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Steering Law Derivation
Thrust Direction
Downrange Distance Control

Abstract (Continued)