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TAYLOR INSTABILITY OF A FLAT PLATE

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ABSTRACT

A flat plate experiences a transverse acceleration under the action of a pressure that also produces plastic yielding throughout the plate. Taylor instability is considered on the assumptions of a single degree of freedom, incompressible flow, and the Prandtl-Reuss constitutive equations in the limiting cases of (A) infinitesimal perturbations and (B) rigid-plastic flow (yield stress/shear modulus → 0).

1. Introduction

A flat plate is subjected to a uniform pressure, say p_o, over its lower surface, say $y = 0$, and is otherwise unconstrained. The resulting acceleration,

$$g = p_o/\rho h \ , \tag{1.1}$$

is directed along the positive y-axis in a reference frame that moves with the plate; ρ is the mean equilibrium density, and h is the equilibrium thickness. We consider the dynamic stability of this equilibrium configuration with respect to a small, periodic perturbation, such that the displacement of the lower face is

$$\eta_o(x,t) = q(t) \cos kx \quad (|q| \ll h) \tag{1.2}$$

in a reference frame that moves with the plate.

We assume that

$$p_o \gg s_1 \ , \tag{1.3}$$

where s_1 is the yield stress in shear, and that the material obeys the Prandtl-Reuss constitutive equations, with the yield stress and shear modulus, G, as constitutive parameters. We also assume that the material is homogeneous (thereby neglecting the y-dependence of the equilibrium density ρ) and incompressible. Admitting only a single degree of freedom and invoking energy considerations, we establish the differential equations for $q(t)$ in two limiting cases, namely:

$$\text{(A)} \quad (G/s_1)k|q|_{max} \to 0$$

and

$$\text{(B)} \quad s_1/G \to 0 \ .$$

2. Kinematics

Taylor's results for an accelerated slab of inviscid, incompressible fluid[1] yield the normal modes

$$\Phi_{\pm}(x,y,t) = f_{\pm}(t)e^{\pm ky}\cos kx \tag{2.1}$$

for the velocity potential corresponding to a disturbance of wavenumber k. The Φ_- mode is unstable, with an exponential growth rate of $(gk)^{\frac{1}{2}}$; the Φ_+ mode is stable, with an angular frequency $(gk)^{\frac{1}{2}}$. Guided by these results, w we base our analysis on the single-degree-of-freedom approximation

$$\Phi(x,y,t) = -k^{-1}\dot{q}(t)e^{-ky}\cos kx\ , \tag{2.2}$$

which satisfies the boundary condition implied by (1.2), namely

$$\partial\Phi/\partial y = \partial\eta_o/\partial t \quad (y = \eta_o \doteq 0)\ . \tag{2.3}$$

The corresponding motion of the upper surface, $y \doteq h$, is given by

$$\eta_h(x,t) = e^{-kh}\eta_o(x,t)\ . \tag{2.4}$$

The potential energy associated with the displacements of (1.2) and (2.4) in the accelerated reference frame is given by

$$\partial V/\partial x = \tfrac{1}{2}\rho g(\eta_h^2 - \eta_o^2) \tag{2.5a}$$

$$= -\tfrac{1}{2}\rho g(1 - e^{-2kh})q^2(t)\cos^2 kx. \tag{2.5b}$$

Averaging (2.5b) over one or more wavelengths, we obtain

$$\langle V\rangle = -\tfrac{1}{4}\rho g(1 - e^{-2kh})q^2(t)\ . \tag{2.6}$$

The corresponding average of the kinetic energy is

$$\langle T\rangle = \tfrac{1}{2}\rho\int_o^h \langle \Phi_x^2 + \Phi_y^2 \rangle\, dy \tag{2.7a}$$

$$= \tfrac{1}{4}\rho k^{-1}(1 - e^{-2kh})\dot{q}^2(t)\ . \tag{2.7b}$$

[1] G.I. Taylor, Proc. Roy. Soc. (London) A201, 192(1950); Scientific Papers (Cambridge University Press, 1960) vol. 3, p. 532. Note that the acceleration in Taylor's paper is directed along the negative y-axis, so that the roles of $\Phi_{\pm}$ in respect to stability are reversed.

3. Constitutive Equations

Let s_{ij} and $\dot{e}_{ij}$ be the stress and strain-rate deviators,

$$J_2 = s_{ij}s_{ij} \quad \text{and} \quad I_2 = \dot{e}_{ij}\dot{e}_{ij} \tag{3.1a,b}$$

their second principal invariants, and

$$\dot{W} = s_{ij}\dot{e}_{ij} \tag{3.2}$$

the rate at which work is done in deforming a unit volume of material.† We recall that, by definition, the first principal invariants vanish:

$$J_1 = s_{ii} = 0 , \qquad I_1 = \dot{e}_{ii} = 0 . \tag{3.3a,b}$$

The Prandtl-Reuss hypotheses imply $J_2 \le 2s_1^2$, with $J_2 = 2s_1^2$ as the yield condition. The constitutive equations are[2]

$$\dot{s}_{ij} + \lambda s_{ij} = 2G\dot{e}_{ij} , \tag{3.4}$$

where

$$\lambda = (G/s_1^2)\dot{W} \quad (J_2 = 2s_1^2 \;\&\; \dot{W} > 0) \tag{3.5a}$$

$$= 0 \quad (J_2 < 2s_1^2 \;\text{ or }\; J_2 = 2s_1^2 \;\&\; \dot{W} < 0) . \tag{3.5b}$$

The conditions $J_2 = 2s_1^2$ and $\dot{W} > 0$ imply plastic flow; $J_2 < 2s_1^2$ implies elastic deformation; $J_2 = 2s_1^2$ and $\dot{W} < 0$ imply unloading at the yield limit.

The Prandtl-Reuss equations are indeterminate for a state of static equilibrium on the yield surface. In the present instance, the plastic compression of a flat plate, we invoke the yield condition and symmetry considerations to obtain

$$s_{xx} = s_{zz} = s_1/\sqrt{3} , \qquad s_{yy} = -2s_1/\sqrt{3} , \qquad s_{xy} = s_{yz} = s_{xz} = 0 \tag{3.6a}$$

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†We note that the development in §3, and also in §4 through (4.9b), does not require the assumption of incompressibility. The invocation of this assumption renders the total strain-rate tensor and the strain-rate deviator identical.

[2]W. Prager & P. G. Hodge, Theory of Perfectly Plastic Solids (John Wiley & Sons, New York, 1951), pp. 27-30.

or, more briefly,

$$s_{ij} = (s_1/\sqrt{3})\delta_{ij} \lfloor 1, -2, 1 \rfloor , \tag{3.6b}$$

where $\lfloor 1, -2, 1 \rfloor$ comprises the non-zero terms in a diagonal tensor. We note that these results overlook a small neighborhood of the upper surface, of the order of $(s_1/p_o)h$ in thickness, in which the material has not yielded.

4. Small-perturbation Approximation

We now consider small perturbations about the equilibrium state of (3.6), say

$$s_{ij} = s_{ij}^{(o)} + s_{ij}^{(1)} \quad \left(|s_{ij}^{(1)}| \ll s_1\right) , \tag{4.1}$$

where $s_{ij}^{(1)}$ is the perturbation stress associated with $\dot{e}_{ij}$. The first approximation to $\dot{W}$ on this assumption is

$$\dot{W}^{(1)} = s_{ij}^{(o)}\dot{e}_{ij} \tag{4.2a}$$

$$= (s_1/\sqrt{3})(\dot{e}_{xx} - 2\dot{e}_{yy} + \dot{e}_{zz}) \tag{4.2b}$$

$$= - \sqrt{3}s_1\dot{e}_{yy} , \tag{4.2c}$$

where (4.2b) and (4.2c) follow from (4.2a) by virtue of (3.6b) and (3.3b). The corresponding first (linearized) approximation to (3.4) is

$$\dot{s}_{ij}^{(1)} + (G/s_1^2)[\dot{W}^{(1)}]s_{ij}^{(o)} = 2G\dot{e}_{ij} , \tag{4.3}$$

where the square bracket imply the positive part of the bracketed term--i.e.

$$[\dot{W}] = \dot{W} \quad (\dot{W} > 0) \tag{4.4a}$$

$$= 0 \quad (\dot{W} < 0) \tag{4.4b}$$

Substituting (3.6b) and (4.2c) into (4.3), we obtain

$$\dot{s}_{ij}^{(1)} = 2G\dot{e}_{ij} - G[-\dot{e}_{yy}]\delta_{ij}\{1, -2, 1\} . \tag{4.5}$$

Integrating (4.5, we obtain

$$s_{ij}^{(1)} = 2Ge_{ij} - G\delta_{ij}\{1, -2, 1\}\int_{t_o}^{t} [-\dot{e}_{yy}]dt , \tag{4.6}$$

where we define e_{ij} and t_o such that

$$s_{ij}^{(1)} = e_{ij} = 0 \quad \text{at} \quad t = t_o . \tag{4.7}$$

We require, for the construction of the linearized equations of motion, the second (quadratic) approximation to $\dot{W}$, say

$$\dot{W} = \dot{W}^{(1)} + \dot{W}^{(2)} . \qquad (4.8)$$

Multiplying (4.6) through by $\dot{e}_{ij}$ and invoking (3.3b), we obtain

$$\dot{W}^{(2)} = 2Ge_{ij}\dot{e}_{ij} - G(\dot{e}_{xx} - 2\dot{e}_{yy} + \dot{e}_{zz}) \int_{t_o}^{t} [-\dot{e}_{yy}]dt \qquad (4.9a)$$

$$= 2Ge_{ij}\dot{e}_{ij} + 3G\dot{e}_{yy} \int_{t_o}^{t} [-\dot{e}_{yy}]dt . \qquad (4.9b)$$

The strain-rate tensor implied by (2.2) is

$$\dot{e}_{ij} = \Phi_{,ij} \equiv k\dot{q}(t)e^{-ky}M_{ij}(x) , \qquad (4.10)$$

where

$$M_{ij} = \begin{bmatrix} \cos kx & -\sin kx & 0 \\ -\sin kx & -\cos kx & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad (4.11)$$

Substituting (4.10), together with its temporal integral from t_o, into (4.6) and (4.9b), we obtain

$$s_{ij}^{(1)} = Gke^{-ky} \left(2q(t)M_{ij} - \delta_{ij} \{1, -2, 1\} \int_{t_o}^{t} [\dot{q}(t)\cos kx]dt \right) \qquad (4.12)$$

and

$$\dot{W}^{(2)} = Gk^2e^{-2ky}\dot{q}(t) \left(4q(t) - 3\cos kx \int_{t_o}^{t} [\dot{q}(t) \cos kx]dt \right) , \qquad (4.13)$$

where, by definition,

$$q(t_o) = 0 . \qquad (4.14)$$

Comparing (4.12) and (4.1), we infer that (4.12) and (4.13) are valid approximations if

$$k|q|_{max} \ll s_1/G \ , \tag{4.15}$$

as anticipated in 1.

Remarking that the mean value of $\dot{W}^{(1)}$ vanishes by virtue of the periodicity of $\dot{e}_{yy}$, we obtain

$$\langle\dot{W}\rangle = \langle\dot{W}^{(2)}\rangle = \frac{5}{2} Gk^2 e^{-2ky} q(t)\dot{q}(t) \tag{4.16}$$

Substituting (4.16), together with (2.6) and (2.7b), into the energy-balance equation

$$\frac{\partial}{\partial t} (\langle T\rangle + \langle V\rangle) + \int_0^h \langle\dot{W}\rangle dy = 0 \tag{4.17}$$

and dividing the result through by $\frac{1}{2}(\rho/k)(1 - e^{-2kh})\dot{q}$, we obtain the linearized equation of motion

$$\ddot{q}(t) + \left(\frac{13}{4}\frac{G}{\rho} k^2 - gk\right) q(t) = 0 \ . \tag{4.18}$$

Substituting g from (1.1) into (4.18), we infer the stability of small perturbations for which

$$kh > 4p_o/13G \ . \tag{4.19}$$

Combining (4.15) and (4.19), we obtain

$$|q|_{max}/h \ll 13s_1/4p_o \tag{4.20}$$

as a necessary condition for the validity of the stability criterion (4.19). This is a rather severe restriction in consequence of the a priori condition (1.3).

5. Rigid-plastic Approximation

We can avoid the restriction (4.20) by letting $s_1/G \to 0$ at the outset to obtain the rigid-plastic (von Mises) constitutive equations[3]

$$s_{ij} = s_1(2/I_2)^{\frac{1}{2}}\, \dot{e}_{ij} \tag{5.1}$$

and

$$\dot{W} = s_1(2I_2)^{\frac{1}{2}} \tag{5.2}$$

in place of (3.4) and (3.5). We emphasize that (5.1) does not allow for elastic unloading from the yield surface and therefore may imply stress discontinuities in both space and time. We also emphasize that $\dot{W}$, as given by (5.2), is non-negative.

Evaluating I_2 from (3.1b) and (4.10), we obtain

$$\dot{W} = 2s_1ke^{-ky}|\dot{q}(t)| \ . \tag{5.3}$$

Substituting (5.3), together with (2.6) and (2.7b), into (4.17), we obtain

$$\ddot{q}(t) - gkq(t) + 4(1 + e^{-kh})^{-1}(ks_1/\rho)\,\mathrm{sgn}\dot{q}(t) = 0 \ . \tag{5.4}$$

The differential equation (5.4) is similar to that for a Coulomb-damped oscillator (but with the important difference that the square of the angular frequency, $-gk$, is now negative). Introducing the dimensionless variables f and τ according to

$$q(t) = \delta h f(\tau) \tag{5.5}$$

and

$$\tau = (gk)^{\frac{1}{2}}t \tag{5.6}$$

where $[s_1/\rho gh \equiv s_1/p_o$ by virtue of (1.1)]

$$\delta = 4(1 + e^{-kh})^{-1}(s_1/p_o) \ , \tag{5.7}$$

[3] Prager, op. cit. ante, p. 31.

we rewrite (5.4) in the normalized form (dots now imply differentiation with respect to τ)

$$\ddot{f}(\tau) - f(\tau) + \mathrm{sgn}\dot{f} = 0 . \tag{5.8}$$

We emphasize that the wave number no longer appears explicitly in (5.8) and that it enters primarily through the time scale $(gk)^{-\frac{1}{2}}$; the variation of the factor $(1 + e^{-kh})^{-1}$, from $\frac{1}{2}$ to 1 as kh varies from 0 to ∞, is small compared with the uncertainties implied by the antecedent approximations.

Transforming (5.8) to

$$\frac{d\dot{f}}{df} = \frac{f - \mathrm{sgn}\dot{f}}{\dot{f}} , \tag{5.9}$$

we obtain the integral curves

$$(f - \mathrm{sgn}\dot{f})^2 - \dot{f}^2 = (f_o - \mathrm{sgn}\dot{f}_o)^2 - \dot{f}_o^2 , \tag{5.10}$$

where f_o and $\dot{f}_o$ are initial values. These are sections of rectangular hyperbolae as sketched in Fig. 2; the arrows designate the direction in which a given integral curve must be transversed (with increasing τ). There are six distinct possibilities, namely:

(A) $\dot{f}_o > 0$

(A1) $f_o < 1 - \dot{f}_o$

(A2) $1 - \dot{f}_o < f_o < 1 + \dot{f}_o$

(A3) $f_o > 1 + \dot{f}_o$;

(B) $\dot{f}_o < 0$ and

(B1) $f_o > -1 - \dot{f}_o$

(B2) $-1 + \dot{f}_o < f_o < -1 - \dot{f}_o$

(B3) $f_o < -1 + \dot{f}_o$.

Trajectories that originate in either (A_1) or (B_1), i.e. in the corridor

$$(f + \dot{f})\operatorname{sgn}\dot{f} < 1 , \tag{5.11}$$

must terminate on the interval $|f| < 1$ and $\dot{f} = 0$ and correspond to stable motions, whereas trajectories originating outside of this corridor diverge according to

$$f(\tau) = \operatorname{sgn}\dot{f}_o(1 - \cosh\tau) + f_o\cosh\tau + \dot{f}_o\sinh\tau \tag{5.12a}$$

$$\sim \tfrac{1}{2}(f_o + \dot{f}_o - \operatorname{sgn}\dot{f}_o)e^{\tau} . \tag{5.12b}$$

Considering further only those disturbances that originate from a state of rest ($\dot{f} = 0\pm$), we conclude that disturbances for which

$$|q_o| > \delta h \equiv (q_o)_{cr} \tag{5.13}$$

will diverge according to

$$q(t) \sim \tfrac{1}{2}[|q_o| - (q_o)_{cr}]\exp[(gk)^{\frac{1}{2}}t]. \tag{5.14}$$

We observe that disturbances of short wavelength (large k) have a higher threshold of instability (although this is a relatively small effect) but grow more rapidly if this threshold is exceeded. We emphasize, however, that the basic hypotheses on which these conclusions are based cannot remain valid for disturbances of very short wavelength; in particular, viscous effects must be stabilizing for sufficiently large k.

6. Two-degree-of-freedom Problem

We consider briefly the extension of the preceding analysis to accomodate two degrees of freedom, with (1.2) replaced by

$$\eta_o(x,t) = q_1(t)\cos kx + q_2(t)\sin kx \ . \qquad (6.1)$$

We obtain the corresponding generalizations of $\langle V\rangle$, $\langle D\rangle$, and $\langle\dot{W}^{(2)}\rangle$ simply by replacing q^2 by $q_1^2 + q_2^2$ in (2.6), $\dot{q}^2$ by $\dot{q}_1^2 + \dot{q}_2^2$ in (2.7b), and $q\dot{q}$ by $q_1\dot{q}_1 + q_2\dot{q}_2$ in (4.16), in consequence of which each of q_1 and q_2 satisfies (4.18) independently--i.e., q_1 and q_2 are normal coordinates in the small-perturbation approximation.

The position is otherwise for the rigid-plastic approximation, in that q_1 and q_2 are coupled through the plastic forces. The above generalizations of $\langle V\rangle$ and $\langle D\rangle$ remain valid, but $|\dot{q}|$ must be replaced by the non-negative radical $(\dot{q}_1^2 + \dot{q}_2^2)^{\frac{1}{2}}$ in (5.3) to generalize $\dot{W}$. The analogy with the Coulomb-damped oscillator continues to hold, however, by virtue of the fact that the rigid-plastic constitutive equation (5.1) postulates a stress tensor s_{ij} that is parallel to the strain-rate tensor $\dot{e}_{ij}$ but independent of its magnitude $I_2^{\frac{1}{2}}$; this is a simple generalization of the constitutive equation for Coulomb friction, which postulates a force vector that is parallel to the velocity vector but independent of its magnitude. Invoking this analogy, we find that the two-degree-of-freedom generalization of (5.4) is

$$\ddot{q}_i - gkq_i + 4(1 + e^{-kh})^{-1}(ks_1/\rho)(\dot{q}_1^2 + \dot{q}_2^2)^{-\frac{1}{2}}\,\dot{q}_i = 0 \qquad (i = 1,2) \ . \qquad (6.2)$$

Introducing the dimensionless variables $f(\tau)$ and $\theta(\tau)$ according to

$$q_1 = \delta hf\cos\theta \ , \qquad q_2 = \delta hf\sin\theta \ , \qquad (6.3)$$

where τ and δ are defined by (5.6) and (5.7), we obtain

$$\ddot{f} - f - f\dot{\theta}^2 + (\dot{f}^2 + f^2\dot{\theta}^2)^{-\frac{1}{2}}\dot{f} = 0 \qquad (6.4a)$$

and

$$f\ddot{\theta} + 2\dot{f}\dot{\theta} + (\dot{f}^2 + f^2\dot{\theta}^2)^{-\frac{1}{2}}f\dot{\theta} = 0 \ . \qquad (6.4b)$$

Introducing

$$u(f) = \dot{f}, \qquad v(f) = f^2\dot{\theta}; \tag{6.5}$$

we obtain the first-order differential equations

$$du/df = fu^{-1} + f^{-3}u^{-1}v^2 - f(f^2u^2 + v^2)^{-\frac{1}{2}} \tag{6.6a}$$

and

$$dv/df = -fvu^{-1}(f^2u^2 + v^2)^{-\frac{1}{2}}. \tag{6.6b}$$

We may assume $u \geq 0$ without loss of generality (cf. Fig. 2, which illustrates the symmetry between motions with positive and negative values of $\dot{f}_o$). We therefore may infer from (6.6a) that du/df is positive-definite, and hence that integral curves in a u,f-plane diverge, for $f > 1$; accordingly,

$$(q_{10}^2 + q_{20}^2)^{\frac{1}{2}} > oh, \tag{6.7}$$

a simple generalization of (5.13), is a sufficient condition for instability. We have been unable to obtain a generalization of (5.11), but the form of the differential equations strongly suggests that motions for which $|v| > 0$ (two degrees of freedom) are less stable than comparable motions for which $v = 0$ (one degree of freedom). On the other hand, we infer from (6.4b) that v must be a monotonically decreasing function of τ (unless $v \equiv 0$), in consequence of which the spatial phase difference between the two degrees of freedom can have only a transient effect.

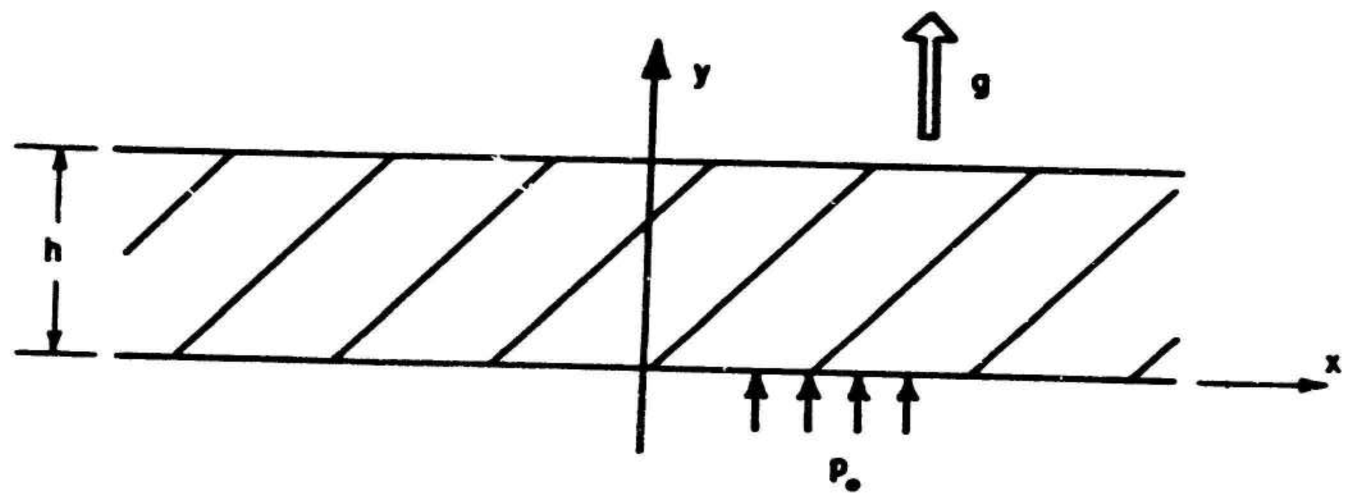

Fig. 1--Basic configuration in accelerated reference frame.

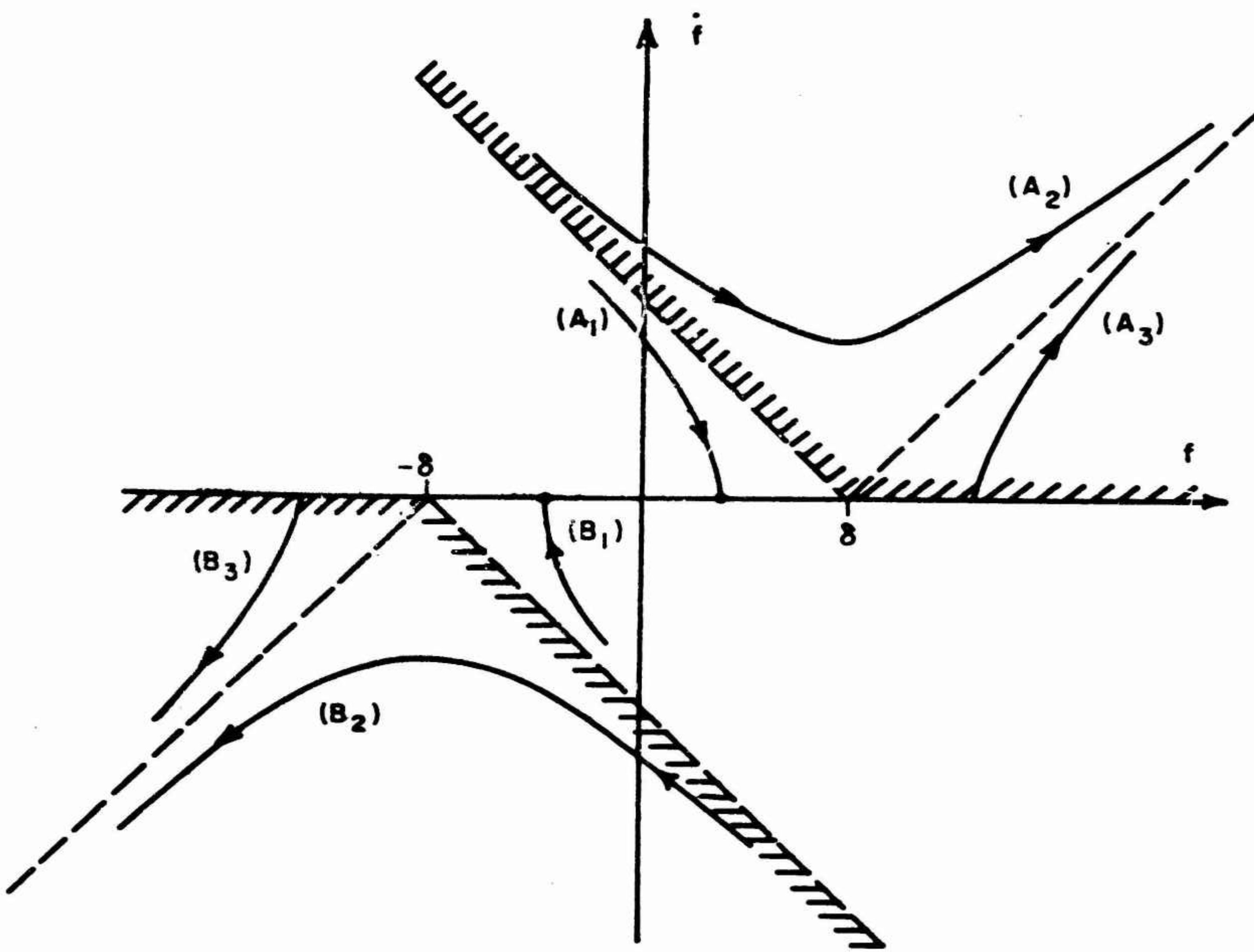

Fig. 2--Typical integral curves of (5.9). Trajectories that originate within the corridor bounded by the cross-hatched lines are stable; trajectories that originate outside of this corridor are unstable.