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REMARKS ON OPTIMAL CONTROL I:
THE STANDARD SUFFICIENCY THEORY
FOR THE LEAST TIME PROBLEM

L. C. Young

MRC Technical Summary Report #654
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ABSTRACT

The sufficiency theory treated in this report concerns a least time problem of optimal control in which the suspected solutions, which obey a strengthened form of the Pontryagin maximum principle, cover a certain set E in n-space in a particular manner that we describe. The set E need not be a domain, and the covering need not be simple. In spite of this, we are able to develop a theory similar to that of Caratheodory in the classical Calculus of Variations. This theory is, however, now valid in the large, and in circumstances which differ rather radically from those which occur in the classical case. The main tools are a greatly strengthened theorem of Malus on the one hand, and the use, on the other hand, of a new and more powerful Hilbert independence integral, whose integrand is now, in effect, many-valued. The very general situations, to which the theory is applicable, require moreover, many new concepts and definitions, which are, in part, of a topological nature, and which make it possible to avoid restrictions to one-to-one maps with non-singular Jacobians, restrictions which are normally made in the Calculus of Variations, but which would here be quite out of place.

REMARKS ON OPTIMAL CONTROL I: THE STANDARD SUFFICIENCY THEORY FOR THE LEAST TIME PROBLEM

L. C. Young

1. Introduction. The object of this paper is to indicate how the classical sufficiency theory of the calculus of variations should be adapted to the problems of optimal control. The ideas are far from new: they are to be found in any good book on the calculus of variations. However, there are quite a number of ways in which optimal control differs radically from the classical calculus of variations, and the main difficulty to be overcome is that of adapting the old ideas to the new context. For simplicity, we limit ourselves here to a least time problem, in which we seek a trajectory, subject to certain controls, which leads in least time, from a given point, to a target, which may be either a point, or a sufficiently smooth set of points, not necessarily connected.

In this problem, it may well happen that the positions of the initial point, from which it is possible to reach the target at all, occupy a set R which need not by any means be the whole space outside the target, nor even a domain outside the target. In quite simple cases R is a set of lower dimension. This fact, of itself, renders a number of changes necessary in the classical method. Another important difference arises from the fact that the solutions need not, even in simple cases, provide a one-to-one covering of any substantial part of R, whereas the notion of a one-to-one covering played a vital role in the classical

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context. Finally, the classical theory was always rather lavish in smoothness assumptions, for supposedly practical reasons, and this is just what optimal control, for genuinely practical reasons, cannot afford. In this paper, smoothness will be reduced to mean simply: continuous differentiability (of the first order). Moreover, some of our basic assumptions concern, not smoothness, but piecewise smoothness, and some quite basic quantities $T(x)$, $y(x)$ are not even assumed to be continuous.

In spite of these far reaching changes, we refer to the sufficiency theory, here presented, as standard. This is because it corresponds still to the most elementary form of the sufficiency theory of the classical calculus of variations. There are no conjugate points, no Morse numbers, no generalized curves, such as would be needed in a deeper study. However, for many practical purposes, the theory here presented is adequate.

2. Background. For slightly greater generality, we leave the control space quite unspecified. Control values will simply be certain labels denoted by u, while x denotes a point of a finite dimensional Euclidean space, and t is the time. We denote by $g(x,u)$ a smooth function of x depending on the control value u, and we term admissible control a function $u(t)$ defined on a relevant time-interval, and which again belongs to a quite arbitrary class of functions, whose values lie in the control space. With this generality, the reader can be sure that our discussion will apply to measurable controls, with values in a cube, or some simple figure, and equally to generalized controls, whose values are probability measures on such a figure.

We consider an autonomous time-optimal problem, for which the admissible trajectories $x(t)$ and controls $u(t)$ satisfy the differential equation

$$\dot{x} = g(x, u), \tag{2.1}$$

where the time t does not appear explicitly. Here an admissible trajectory $x(t)$ is an absolutely continuous function connected with a definite admissible control $u(t)$ by the differential equation (2.1), which is understood to hold almost everywhere on the time-interval of definition. Further this time-interval will be supposed to end at $t = 0$, and the corresponding terminal value $x(0)$ of $x(t)$ will be supposed to lie on a given set, which we term the target, and which we take to be a sufficiently elementary configuration, without interior points. Since $x(t)$ is absolutely continuous, its derivative $\dot{x}$ is measurable in t, etc. This introduces restrictions in practice on the utilizable controls $u(t)$, but we need make no such restrictions explicitly.

Our problem is to determine, if possible, the least time from a given position to the target, along admissible trajectories. Our object is not to solve this problem - this could only be done subject to additional hypotheses - but rather to show how to complete the solution, after certain preliminary steps have been taken. These preliminary steps are here the raw material of our sufficiency theory. We do not have to concern ourselves with the question whether these steps need have been taken: in practice they are taken. Our question is what we can do next.

The first step, which we suppose already taken, is to select, from the inconveniently large class, consisting of all admissible trajectories, a much smaller class, where we suspect to find those which lead to the target, from their initial points, in least time. This is done, in practice, by making use of so-called necessary conditions. However, we can afford to be here more particular in our choice, and to impose slightly stronger conditions, which we do not claim to be necessary, and which we list below. The admissible trajectories, thus selected, will be termed lines of flight. We shall moreover confine ourselves to trajectories situated in a certain set R.

We shall suppose that every point $x \in R$ is the initial point of at least one line of flight, and that the selection of lines of flight is consistent in the following sense: an admissible trajectory C is a line of flight if and only if, for every point x interior to C, the portion of C from x to the target is a line of flight. Further, just as all lines of flight terminate on the target at the time $t = 0$, we shall suppose that all lines of flight, which start at a same point $x \in R$, do so at the same time t. (If they did not, this would provide a reason for excluding some of them, since our problem concerns the least time.) We shall term this condition that of synchronization. Of course, it does not ensure that, automatically, the quantity $|t|$ is then the desired least time from the point x. For there might very well be no solution to the problem, or we might have selected the wrong trajectories as our lines of flight.

Finally we shall suppose that every line of flight satisfies the maximum principle of Pontryagin, in a strengthened form described below. This form is understood to include the transversality conditions for $t = 0$, at the points of the target. It states that, along a line of flight $x(t)$ with control $u(t)$, there exists a conjugate vector function $y(t)$, absolutely continuous in t, such that

$$
(2.2)\quad \begin{cases} \text{(a)} & \dot{y}(t) = -y(t)g_x[x(t), u(t)], \\ \text{(b)} & y(t)g[x(t), u] \leq 1 \text{ for all } u, \\ & \text{with equality when } u = u(t), \\ \text{(c)} & \text{for } t = 0, \text{ the vector } y \text{ is normal} \\ & \text{to the target at the point } x(0). \end{cases}
$$

This differs from the original principle of Pontryagin in that, in the inequality (2.2)(b), the right-hand side is here unity, instead of a quantity $H \geq 0$, which is constant in t. By replacing H by unity, i.e. y/H by y, we do not affect our trajectories, except for excluding the case where $H = 0$. This last exclusion we insist on here: there are sound reasons for it in a sufficiency theory as opposed to a necessity theory. (A policeman, instructed to round up every conceivable suspect, is liable to include certain persons whose behavior attracted his attention simply because they were unusual persons, e.g. poets, mathematicians, chess players, and the like; these rather innocuous individuals correspond to what we exclude here.)

We shall term canonical line of flight, a trio of functions

$$x(t), \quad y(t), \quad u(t)$$

such that $x(t)$, $u(t)$ defines a line of flight, and $y(t)$ is a corresponding conjugate vector-valued function subject to (2.2). An open arc of a line of flight, or of a canonical line of flight, will be termed an arc of flight, or a canonical arc of flight, respectively. In general, the conjugate function $y(t)$ is not uniquely determined by the line of flight $x(t)$, $u(t)$ but may depend on auxiliary parameters ρ, which can represent, for instance, initial or terminal conditions. Thus, a line, or an arc, of flight, is in general the projection of a whole family of canonical lines, or arcs, of flight which depend on the additional parameter ρ. This is one of the basic facts that our discussion has to allow for, and we note that when lines of flight merge or separate, so do the corresponding sets of values of ρ.

3. The notion of a satisfactory map. We noted at the end of the preceding section the intrusion of certain parameters ρ, whose values are irrelevant in a family of arcs of flight. On this account, we particularly wish to avoid a condition, much used in classical analysis, by which certain smooth maps are required to have non-singular Jacobian matrices. The intrusion of irrelevant parameters, which play an important part in our discussion, would clearly disturb such a condition.

Generally, a map f^* of a set $Q^* = Q \times I$ may be said to possess the irrelevant parameter $c \in I$, if for $q^* = (q, c)$ where $q \in Q$, $c \in I$, the value $f^*(q^*)$ has the form $f(q)$ independent of c.

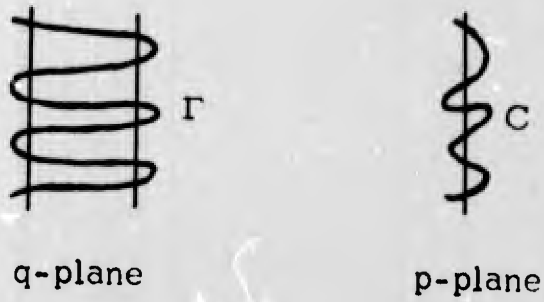
A map f from a set Q to a set $P = f(Q)$, where Q, P lie in two metric spaces, will be termed satisfactory, if, for each $q \in Q$,

(3.1) $\left\{\begin{array}{l} \underline{\text{given any rectifiable curve}}\ C \subset P\ \underline{\text{issuing from}} \\ f(q),\ \underline{\text{there exists a rectifiable curve}}\ \Gamma \subset Q \\ \underline{\text{issuing from}}\ q,\ \underline{\text{such that every small arc of}}\ A \\ \underline{\text{issuing from}}\ f(q)\ \underline{\text{is the image under the map}} \\ f\ \underline{\text{of a small arc of}}\ \Gamma\ \underline{\text{issuing from}}\ q. \end{array}\right.$

We note that, if the map f is satisfactory, so is the map f^* defined above, provided that I lies in a metric space. An example of a map which is not satisfactory, is obtained by choosing, in the complex plane, the function

$$f(q) \begin{cases} = q+1 & \text{if} \quad \Re q < -1 \, , \\ = q-1 & \text{if} \quad \Re q > 1 \, , \\ = q - \Re q & \text{if} \quad -1 \leq \Re q \leq 1 \, . \end{cases}$$

A curve C which crosses the imaginary axis of p corresponds to a curve Γ which crosses the strip $-1 \leq \Re q \leq 1$.



4. <u>The notion of a spray of flights</u>. Suppose defined, on an open Euclidean set, whose points we denote by σ, a pair of extended real-valued functions $t^-(\sigma)$, $t^+(\sigma)$ where $-\infty \leq t^-(\sigma) < t^+(\sigma) \leq 0$. The points σ at which $t^-(\sigma) \neq -\infty$ are to constitute an open set, and the two functions are to be continuous, except in the case of $t^-(\sigma)$, at any points where the value $-\infty$ is taken. Further, we shall suppose that the local restriction of the function $t^+(\sigma)$ to small segments parallel to the σ-axes is of bounded variation on each such segment.

We suppose further, that the open set of σ is the projection of a certain set of (σ, ρ), situated in a higher dimensional Euclidean space. Of this set of (σ, ρ) we need not assume that it is open: instead we shall suppose that if (σ_0, ρ_0) is any one of its points, and γ is a small enough curve of the σ set, issuing from σ_0, then there exists along it a continuous function $\rho(\sigma)$ which reduces to ρ_0 at σ_0, such that all the points $(\sigma, \rho(\sigma))$ for $\sigma \epsilon \gamma$ lie in our (σ, ρ) set.

We shall denote by S^-, S, S^+ the sets of (t, σ) for which σ is as before, and t is subject to the corresponding condition $-\infty < t^-(\sigma) = t$, or $t^-(\sigma) < t < t^+(\sigma)$, or $t = t^+(\sigma)$. We denote by $[S]$ the union of these three sets. Similarly, we denote by S^{*-}, S^*, S^{*+} the sets of (t, σ, ρ) for which t is subject to these respective conditions, and (σ, ρ) is as before. We write $[S^*]$ for the union of the three sets.

This being so, we consider a family Σ of arcs of flight with corresponding controls, given by functions

$$x(t, \sigma), \quad u(t, \sigma) \quad (t, \sigma) \epsilon S .$$

Here σ is the label, which distinguishes a member of the family, i.e. σ remains constant on an arc of flight of Σ, and this arc then corresponds to the open time-interval $t^-(\sigma) < t < t^+(\sigma)$. We shall denote further by Σ^* a family of canonical arcs of flight, which correspond to the arcs of Σ, and which are obtained by giving, with the above functions, a further conjugate vector-function

$$y(t, \sigma, \rho) \quad (t, \sigma, \rho) \epsilon S^* .$$

The definition of the functions $x(t, \sigma$, $y(t, \sigma, \rho)$ will be supposed extended to the sets $[S]$, $[S^*]$. This means defining them for $t = t^+(\sigma)$ and $t = t^-(\sigma) > -\infty$, where the values of x, y correspond to the end-points of our arcs. The sets of values of $x(t,\sigma)$ in the (t,σ) sets $S^-, S, S^+, [S]$ will be written E^-, E, E^+, $[E]$, and those of the pair $x(t,\sigma)$, $y(t,\sigma,\rho)$ in the (t,σ,ρ) sets S^{*-}, S^*, S^{*+}, $[S^*]$ will be $E^{*-}, E^*, E^{*+}, [E^*]$.

Finally we write, when $(t,\sigma) \in S$ and when x is a point of E sufficiently near to $x(t,\sigma)$,

$$h(t,\sigma), \quad g(x,t,\sigma), \quad g_x(x,t,\sigma)$$

for the expressions

$$g[x(t,\sigma), u(t,\sigma)], \; g(x, y(t,\sigma)], \; g_x[x, u(t,\sigma)].$$

We now suppose the following conditions satisfied:

(4.1)

(i) <u>the function</u> $h(t,\sigma)$ <u>and for each fixed</u> $x \in E$ <u>the function</u> $g(x,t,\sigma)$ [<u>when</u> (t,σ) <u>is near to the values at which</u> $x(t,\sigma)$ <u>takes the value</u> x], <u>are smooth in</u> S <u>and satisfy at</u> $x = x(t,\sigma)$ <u>the relation</u>

$$\frac{\partial h}{\partial \sigma} = g_x(x,t,\sigma)x_\sigma + \frac{\partial g(x,t,\sigma)}{\partial \sigma};$$

(ii) <u>the function</u> $y(t,\sigma,\rho)$ <u>is continuous in</u> $[S^*]$;

(iii) <u>the function</u> $x(t,\sigma)$ <u>is smooth in</u> $[S]$;

(iv) <u>the maps</u> $S^- \to E^-$, $S \to E$ <u>defined by the function</u> $x(t,\sigma)$ <u>are satisfactory</u>.

These conditions, together with those in the definitions of the functions $t^-(\sigma)$, $t^+(\sigma)$, and the stipulations about the corresponding sets of (t,σ,ρ)

play a basic part in our discussion. When they are satisfied, we term

Σ a spray of flights from E^- to E^+, and Σ^* a canonical spray of flights from E^{*-} to E^{*+}. We term E^- or E^{*-} the source, and E^+ or E^{*+} the destination, of Σ or Σ^*, while E or E^* will be termed the corresponding flight-corridor. This terminology must not make us forget that some arcs of Σ do not originate in E^-; they are the ones that start at $t = -\infty$. All arcs of flight of Σ terminate in E^+, but only those which start at a finite time originate in E^-. It should also be emphasized that we attach to each spray Σ a definite corresponding canonical spray Σ^*, i.e. we distinguish formally between two sprays defined by a same family of arcs of flight, associated with different families of canonical arcs. We also wish to emphasize once more the possibility of various forms of degeneracy in a spray: for instance, Σ may very well consist of subarcs of a given arc of flight.

We shall use the notion of spray of flights in much the same way as that of field of extremals in the calculus of variations.

5. The Hilbert independence integral. We denote by $T(x)$, for $x \in R$, and term flight-time from x, the length of the time-interval for a line of flight issuing from the point x. By our synchronisation condition, this flight-time

depends only on x . A set of points in which $T(x)$ is bounded will be termed a set of bounded flight-time.

Given a subset A of R, we shall term canonical set corresponding to A, the set $A^{\#}$ of the points (x, y) of $2n$-space, each of which lies on a canonical line of flight, such that their projections x lie in A. By a canonical set $A^{\#}$ of bounded flight-time, we mean the canonical set corresponding to a subset A of R of bounded flight-time. We shall denote further by

$$\mathfrak{y}(x) \; ,$$

for $x \epsilon R$, the set of the values of the conjugate vector y for which (x, y) lies in the canonical set $R^{\#}$ corresponding to R. By a function

$$y(x) \; \epsilon \; \mathfrak{y}(x) \qquad x \; \epsilon \; R \; ,$$

we shall mean a function defined in R, whose value at each point x lies in the corresponding set $\mathfrak{y}(x)$ at this point. We term such a function $y(x)$ a momentum in R, and we term $\mathfrak{y}(x)$ the momentum range at the point x.

A similar notation will also be used in relation to a given spray of flights Σ. We shall write

$$\mathfrak{y}_{\Sigma}(x) \; ,$$

to mean, for $x \epsilon [E]$, the set of the values of $y(t, \sigma, \rho)$ at those points $(t, \sigma, \rho) \; \epsilon \; [S^{*}]$, for which $x(t, \sigma)$ takes the given value x. We shall write

$$\mathfrak{y}_{\Sigma}(x) \qquad x \; \epsilon \; [E] \; ,$$

to mean a function defined in $[E]$, whose value at each point x lies in the corresponding set $\mathfrak{Y}_{\Sigma}(x)$ at this point. We shall refer to $y_{\Sigma}(x)$ as a momentum for Σ, and to $\mathfrak{Y}_{\Sigma}(x)$ as the momentum range for Σ at the point x.

On any rectifiable curve C of bounded flight-time in R, we define the curvilinear integral

$$\int_C y(x)\,dx = \int y(x)\,\frac{dx}{dz}\,ds$$

for any momentum in R such that $y(x)\,dx/ds$ is a bounded measurable function of the arc-length s along C. The functional, defined by this integral for the class of curves C and momenta $y(x)$ $x \epsilon R$ specified above, will be termed the Hilbert independence integral.

We shall study the case in which this integral exists for every momentum $y(x) \epsilon \mathfrak{Y}(x)$, $x \epsilon R$, and is independent, not only of the choice of this momentum but also, in large measure, independent of the choice of the curve C, provided that we fix the ends of C.

To this effect, we introduce still further definitions.

At a point $x \epsilon R$, we term direction of univalence, a direction θ such that all the vectors $y \epsilon \mathfrak{Y}(x)$ have the same projection $y\theta$ on this direction. Further, we term curve of univalence, a rectifiable curve $C \subset R$, such that at almost all points of C, the direction of the tangent to C is a direction of univalence. Finally, we term set of univalence, a subset A of R, such that all rectifiable curves $C \subset A$, of bounded flight-time, are curves of univalence.

For any rectifiable curve C, of bounded flight-time, situated in a set A of univalence, we express the Hilbert independence integral in terms of the arc-length s along C, by writing

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$$\int_C y(x)\,dx = \int y[x(s)]\,\theta(s)\,ds \ ,$$

where $x(s)$ is the representation of C in terms of arc-length, and where $\theta = \theta(x) = dx/ds$. This integral does not depend on the choice of the function $y(x) \in \mathcal{Y}(x) \quad x \in R$.

A subset A of R will, further, be termed set of exact integrability, or simply an exact set, if it is a set of univalence, and if, in addition, for every rectifiable curve $C \subset A$, of bounded flight-time, we have, for each $y(x) \in \mathcal{Y}(x)$, $x \in R$,

$$\int_C y(x)\,dx = T(x_1) - T(x_2) \ ,$$

where x_1, x_2 are the initial and final points of C.

Our discussion of univalence and exactness will use corresponding notions relative to a spray Σ, whose source, destination, flight-corridor, and so forth, we denote as previously.

At a point $x \in [E]$, we term direction of relative univalence, a direction θ such that all the vectors $y \in \mathcal{Y}_\Sigma(x)$ have the same projection $y\theta$ on this direction. We term curve of relative univalence, a rectifiable curve $C \subset [E]$, such that, at almost all points of C, the direction of the tangent to C is a direction of relative univalence. We term set of relative univalence, a subset A of $[E]$, such that all rectifiable curves $C \subset A$, of bounded flight-time, are curves of relative univalence. A subset A of $[E]$ will, further, be termed a set of relative exact integrability, or simply a relative exact set, if it is a set of relative univalence, and if, in addition, for every rectifiable curve $C \subset A$, of bounded flight-time, we have, for each $y_\Sigma(x) \in \psi_\Sigma(x)$ $x \in [E]$,

$$\int_C y_\Sigma(x)\,dx = T(x_1) - T(x_2) ,$$

where x_1, x_2 are the initial and final points of C.

A more restricted form of realtive exactness arises when we consider only those curves C which are images of (t, σ) curves under the map $x(t, \sigma)$, and only those momenta which have the form $y(t, \sigma, \rho)$ along them. In that case, by (2.2)(b) and (2.1), the Hilbert integral takes the form

$$\int_\Gamma yx_t\,dt + yx_\sigma\,d\sigma = \int_\Gamma dt + \int_\Gamma yx_\sigma\,d\sigma$$

where Γ is a (t, σ) curve whose image is C, so that the relative exactness condition reduces to the vanishing of the integral of $yx_\sigma\,dx_\sigma$, or equivalently to the vanishing of yx_σ. This condition will therefore play a part in the sequel.

6. _Preliminary lemmas._ In this section Σ is fixed. Our first lemma partly makes up for the fact that we make no assumptions about $T(x)$, $y_\Sigma(x)$. The other two relate the vanishing of yx_σ to our other conditions.

(6.1) _Lemma._ _Let_ C _be a rectifiable curve in_ E^-, _or in_ E, _together with its end-points._ _Then_ C _is of bounded flight-time, and there exists on_ C _a bounded momentum_ $y_\Sigma(x)$ _relative to_ Σ, _which is Borel measurable._

(6.2) _Lemma._ _If_ E^+ _is a relative exact set, then_ yx_σ _vanishes in_ S^{*+}.

(6.3) _Lemma._ _If_ yx_σ _vanishes in_ S^{*-}, _then_ E^- _is a relative exact set._ _If it vanishes in_ S^* _then_ E _is so._

Proof of (6.1). By (4.1)(iv), we can attach to each point of C, including the end-points, a neighborhood on C, which is the image of a curve Γ. We can suppose t bounded on each such Γ, and it then follows from Borel's covering theorem that $T(x)$ is bounded on C. In proving the second assertion, we may suppose C small enough, so that there is a corresponding (t, σ) curve, which we again denote by Γ, and we can choose on Γ a continuous, and therefore bounded, function of the form $y(t, \sigma, \rho(\sigma))$. To each point $x \in C$ we now attach the first point (t, σ) of Γ at which $x(t, \sigma) = x$. By substitution in $y(t, \sigma, \rho(\sigma))$ we obtain a Borel measurable $y_\Sigma(x)$ on C, which is also bounded, as asserted.

Proof of (6.2). Let Γ be the small rectifiable curve in S^+, which corresponds, by setting $t = t^+(\sigma)$, to a small segment parallel to one of the σ-axes. Let C be the image of Γ in E^+, and let $y_\Sigma(x)$ be a momentum relative to Σ. Since E^+ is firstly a set of relative univalence, and secondly, a relative exact one, we may write firstly

$$\int_C y_\Sigma(x)\, dx = \{\int_\Gamma yx_t\, dt + yx_\sigma\, d\sigma\},$$

and secondly equate the common value of the two sides of this formula to the difference Δt of $t^+(\sigma)$ between the ends of Γ. Here y now stands for $y(t, \sigma, \rho)$, and ρ for a continuous function of σ, suitably chosen; moreover, by (2.2)(b) and (2.1), we have $yx_t = 1$. Hence we find that

$$\int_\Gamma yx_\sigma\, d\sigma = 0 .$$

This must now hold for Γ, however we choose the small segment parallel to a σ-axis through an initial σ_0, and however we choose the initial value $\rho_0 = \rho(\sigma_0)$ of a corresponding continuous function $\rho(\sigma)$ along Γ. Since yx_σ is continuous in $[S^*]$, we find that each of its components yx_β must vanish at the point $[t^+(\sigma_0), \sigma_0, \rho_0]$, i.e. at an arbitrary point of S^{*+}, as asserted.

Proof of (6.3). The two assertions are proved in the same manner, and we shall limit ourselves to the one concerning E^- and S^{*-}. We assume then that $yx_\sigma = 0$ in S^{*-}. We denote by C any small rectifiable curve in E^-, of bounded flight-time. (If no such curve exists, we have nothing to prove.) We represent C in terms of its arc-length s, by a function $X(s)$, and we denote by $\theta(s)$ the direction of the tangent at the corresponding point, for almost every s. As origin for s, we choose a value at which $\theta(s)$ is approximately continuous, and we denote by $\hat{x}, \hat{\theta}$ the corresponding values of $X(s)$, $\theta(s)$. Further we denote by $\hat{y}$ any vector in $\psi_\Sigma(\hat{x})$ and by $(\hat{t}, \hat{\sigma}, \hat{\rho})$ a point S^{*-} for which $x(\hat{t}, \hat{\sigma}) = \hat{x}$, $y(\hat{t}, \hat{\sigma}, \hat{\rho}) = \hat{y}$.

Approximate continuity of $\theta(s)$ implies that, given $\epsilon > 0$, there exists a closed set B of values of s, such that, for every sufficiently small interval I of the form $0 \leq s \leq \delta$, we have

(i) $|\theta(s) - \hat{\theta}| < \epsilon$ whenever $s \in B \cap I$,

(ii) $\text{meas}\,(I - B) < \epsilon \cdot \text{meas}\,(I)$.

We now denote by Γ a rectifiable curve in S^-, such that small arcs of C issuing from $\hat{x}$ are, in accordance with (4.1)(iv), the images under the map $x(t, \sigma)$ of small arcs γ of Γ, issuing from $(\hat{t}, \hat{\sigma})$. We represent Γ in terms of its arc-length λ, by functions $t(\lambda)$, $\sigma(\lambda)$, so that the point $(\hat{t}, \hat{\sigma})$ corresponds to $\lambda = 0$. We can then define a continuous increasing function $s(\lambda)$, which vanishes at $\lambda = 0$, and which gives rise to the corresponding arc-length along C, i.e. which satisfies the relation

$$X[s(\lambda)] = x[t(\lambda), \sigma(\lambda)] .$$

We shall denote by Λ the set of λ for which $s(\lambda) \epsilon B$.

This being so, let $\Delta s, \Delta T$ denote the difference in s and in $T(x)$ at the ends of a small arc of C. We wish to show that

$$\begin{cases} \text{(a) the ratio } \dfrac{\Delta T}{\Delta s} \text{ is bounded,} \\ \text{(b) for an arc of } C \text{, issuing from } \hat{x}, \\ \quad \text{which shrinks to this point,} \\ \quad \operatorname{Lim} \dfrac{\Delta T}{\Delta s} = - \hat{y}\hat{\theta} . \end{cases}$$

We remark that (a) and (b) together imply the assertion of (6.3). In fact (b) implies, on the one hand, that $\hat{\theta}$ is a direction of relative univalence at $\hat{x}$, and on the other hand, since $\hat{x}$ was any point of C at which $\theta(s)$ is approximately continuous, and so almost any point of C, that for every $y_\Sigma(x) \epsilon \mathcal{Y}_\Sigma(x)$, $x \epsilon [E]$, we have, almost everywhere along C,

$$\frac{dT[X(s)]}{ds} = - y_\Sigma[X(s)] \frac{dX}{ds} .$$

Moreover (a) implies that we can integrate here in s, to obtain the relation which defines relative exactness. The proof of (6.3) is thus reduced to that of (a) and (b), and we now establish these two statements.

To this effect, let J be the interval of λ corresponding to an arc γ of Γ, and suppose γ mapped by $x(t,\sigma)$ onto our small arc of C. Clearly $\Delta T = -\Delta t$, where Δt is the difference of t at the ends of γ. On the other hand, we have along γ, by hypothesis, $yx_\sigma = 0$, and, by (2.2)(b) and (2.1), $yx_t = 1$. Hence

$$\Delta t = \int_J yx_t\, dt(\lambda) + yx_\lambda\, d\sigma(\lambda) = \int_J y\theta ds(\lambda) .$$

Evidently this implies the boundedness of the ratio $\Delta t/\Delta s$, and so (a). Further, is we take γ to issue from $(\hat{t}, \hat{\sigma})$ and the corresponding arc of C to be small, γ and so J will be small. Moreover, in the expression found for Δt, the vector y is a continuous function of λ, obtained by taking $y = y(t, \sigma . \rho)$ along γ, while θ is the direction $\theta(s)$, where $s = s(\lambda)$. In terms of these functions, if we set $\varphi = \varphi(\lambda) = y\theta - \hat{y}\hat{\theta}$, we thus have

$$\frac{\Delta t}{\Delta s} - \hat{y}\hat{\theta} = \frac{1}{\Delta s}\int_J \varphi ds(\lambda) = \frac{1}{\Delta s}\int_{J\cap\Lambda} + \frac{1}{\Delta s}\int_{J-\Lambda} .$$

For small J these last two terms cannot exceed certain fixed multiples of an arbitrarily small positive ϵ: for on the one hand, φ is bounded in $J-\Lambda$ and this set has $s(\lambda)$-measure less than $\epsilon\Delta s$, by (ii) above; and on the other hand, $J\cap\Lambda$ has $s(\lambda)$-measure at most Δs, while in it, by (i) above and by the continuity (and boundedness) of y, the absolute value $|\varphi|$ of the integrand is at most a fixed multiple of ϵ. This completes the proof.

7. <u>The theorem of Malus.</u> This theorem of geometrical optics was reformulated for the classical calculus of variations. We need to reformulate it again here, and to establish it, under greatly reduced smoothness assumptions. We shall need the following:

(7.1) <u>Lemma.</u> <u>In a canonical spray, we have</u>

$$\frac{\partial}{\partial t}\left(y \frac{\partial x}{\partial \sigma}\right) = 0 .$$

In this statement, x and y stand for the functions $x(t, \sigma)$, $y(t, \sigma, \rho)$, and the relation asserted is for $(t, \sigma, \rho) \in S^*$. The lemma asserts incidentally the existence of the left-hand side, although we do not assume second derivatives to exist.

Proof of (7.1.) We denote by $(\hat{t}, \hat{\sigma}, \hat{\rho})$ and $\hat{x}, \hat{y}, \hat{u}$, a point of S^* and the corresponding values of x, y, u; further, by β any coordinate of σ and by c the value at $(\hat{t}, \hat{\sigma}, \hat{\rho})$ of

$$y \frac{\partial}{\partial \beta}\{h(t, \sigma) - g[x(t, \sigma), \hat{u}]\} .$$

By performing in different orders the operations of integration in t and differentiation in β, on the relation (2.1), and then differentiating in t, we obtain successively

$$x_\beta(t, \sigma) - x_\beta(\hat{t}, \sigma) = \int_{\hat{t}}^{t} \frac{\partial}{\partial \beta} h(\tau, \sigma)\, d\tau ,$$

$$\frac{\partial}{\partial t} x_\beta(t, \sigma) = \frac{\partial}{\partial \beta} h(t, \sigma) .$$

This last we multiply scalarwise by y, with $(\hat{t}, \hat{\sigma}, \hat{\rho})$ for (t, σ, ρ). We then add at this same point, for $x = \hat{x}$, the relation

$$x_\beta(t, \sigma)\frac{\partial}{\partial t}y(t, \sigma, \rho) = -\hat{y}\, g_x(x, u)x_\beta(t, \sigma)$$

which there follows from (2.2)(a). We thus find that

$$\frac{\partial}{\partial t}\{y\,\frac{\partial x}{\partial \beta}\} = c$$

at $(\hat{t}, \hat{\sigma}, \hat{\rho})$. It only remains to prove that $c = 0$.

However c, by its definition and by (4.1)(i), is the value for $(t, \sigma) = (\hat{t}, \hat{\sigma})$ of

$$\frac{\partial}{\partial \beta}\,\hat{y}\, g(\hat{x}, t, \sigma),$$

and this vanishes by (2.2)(b), since the set of σ is open.

(7.2) Corollary. On each arc of Σ^*, the quantity $y x_\sigma$ is constant.

In fact, the proof of (7.1) shows that x_β is, for constant σ, absolutely continuous in t, since its difference is an integral. The function y is also absolutely continuous in t by section 2. It follows that $y x_\sigma$ is absolutely continuous in t, and so constant by (7.1), for constant σ, ρ.

(7.3) Theorem of Malus. Let Σ be a spray of flights with a relative exact destination E^+. Then Σ possesses a relative exact source E^-, and a relative exact flight-corridor E.

Proof. By (7.2), $y x_\sigma$ is constant on each arc of Σ^*, and by (4.1) it is continuous in $[S^*]$. The constant value on each arc is 0 by (6.2) and the assertion of (7.3) then follows from (6.3).

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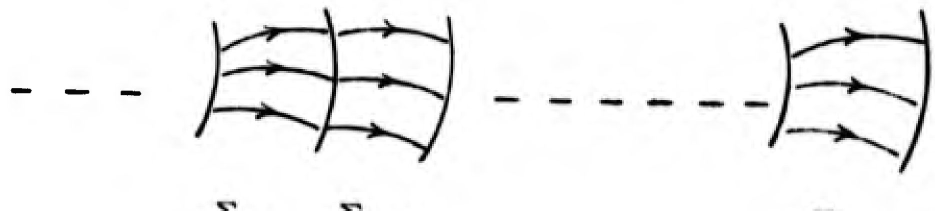
8. <u>Chains of flights</u>. The relationship between the classical calculus of variations and geometrical optics, which shows itself in the classical form of the theorem of Malus, becomes particularly close in the least time problems of optimal control. In geometrical optics, the source E^-, and the destination E^+, of a spray of flights Σ, would correspond to a pair of consecutive lenses or mirrors, and Σ to the family of light-rays passing from one to the other. Such a system must then be studied as part of a whole complex of such families of rays, fitted together between parts of optical instruments. Just as we fit together such families of rays in geometrical optics, we shall here fit together different sprays of flights.

A finite or countable sequence of sprays of flights in R,

$$\Sigma_1, \Sigma_2, \ldots, \Sigma_N, \ldots$$

will be termed a chain of flights, and the corresponding sequence of canonical sprays a canonical chain, if, for $r = 1, 2, \ldots, N-1, \ldots$, they "fit together" in inverse order, so that the source of each Σ_r^* contains the destination of Σ_{r+1}^*. It is thus the canonical sprays that must fit, not only their projections the Σ_r.



Σ_N Σ_{N-1} Σ_1

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The destination of Σ_1 is termed the destination of the chain. A chain of flights whose destination is a subset of the target, will be termed a chain of flights to the target. A finite chain, consisting of N sprays, has also a source: the latter is defined as the source of Σ_N. Generally, the sources and flight-corridors of the individual sprays of a chain of flights will be termed its constituent sets. We do not mention destinations: they are subsets of sources of succeeding sprays, except for the destination of Σ_1 which is that of the whole chain. If the source or flight-corridor of an individual spray Σ_r is a relative exact set for this spray, we term it a relative exact constituent set for the given chain. In the case of the source, this clearly implies that the destination of Σ_{r+1} is a relative exact set for Σ_r, and a fortiori for Σ_{r+1}, since the set $\mathfrak{U}_\Sigma(x)$ for an $x \in E^+_{r+1}$, contracts when Σ passes from Σ_r to Σ_{r+1}.

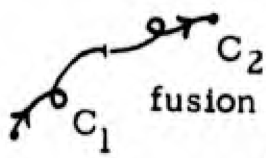
By an obvious induction, we thus conclude from Malus's theorem (7.3), that if the destination of Σ_1 is for Σ_1 a relative exact set, then all the constituent sets of the chain of flights are relative exact for the chain. This is in particular the case, by the transversality condition (2.2)(c), if the destination of Σ_1 lies in the target. Thus

(8.1) Theorem. For a chain of flights to the target, all the constituent sets are relative exact.

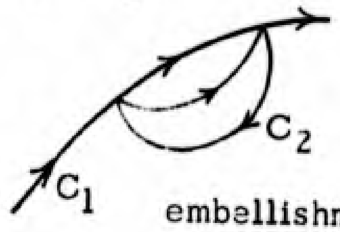
9. Piecing together fragments of curves. Let $\mathfrak{K}$ be the class of rectifiable curves in R, of bounded flight-time, and let R_ν, where ν describes a set of positive integers, be a finite or countable system of disjoint subsets of R,

whose union is R. A curve $C \in \mathcal{K}$ will be termed fragmentary, or a fragment, if its interior portion lies in some R_ν. The class of such fragments will be denoted by $\mathcal{K}_0$. More generally, subject to conditions described below, $\mathcal{K}$ could denote a given class of rectifiable curves in a metric space, and $\mathcal{K}_0$ a subclass. We wish to describe a situation in which $\mathcal{K}$ can be derived from $\mathcal{K}_0$ by simple operations of addition and subtraction of curves.

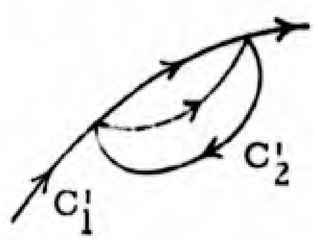
By a restricted algebra of such operations, we mean one in which we allow them only in so far as they lead from curves of $\mathcal{K}$ to curves of $\mathcal{K}$. We shall define two such forms of addition:



fusion



embellishment



alternative embellishment leading to the same curve

(i) if the terminal point of C_1 is the initial point of C_2, we term fusion of C_1 and C_2 a curve C made up of two adjacent arcs, consisting of C_1 and C_2;

(ii) if C_2 is a closed curve which intersects C_1, we term embellishment of C_1 by C_2 a curve C which describes, first an arc of C_1 up to an intersection, then C_2, and then the remaining arc of C_1.

In each case C_1, C_2, C are supposed to lie in the class $\mathcal{K}$.

We define correspondingly two subtraction operations, which we term, respectively, those of cutting and trimming C_2 from C. We term C_1 the result of this cutting or trimming, if C is expressible as the fusion of C_1 and C_2, or as the embellishment of C_1 by C_2. Of course, as the figures show, a given C may be expressible in more than one way as the embellishment of one curve by another.

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It should be made clear that, from the point of view of classical analysis, the operation of embellishment is not uniquely defined: if C_2 is a closed curve which meets C_1 in more than one point, there are at least two classical parametric curves, each of which is, by our definition, the embellishment of C_1 by C_2, provided that it is a member of $\mathcal{K}$. This means that the classical notion of curve is not suited to the algebra of curves, and our definition must not be based on it. Accordingly, we here identify any two classical rectifiable curves, if they give rise to the same operation of curvilinear integration, for continuous integrands $f(x, \theta)$ which are functions of the point x and the direction θ.

In the sequel, the classes $\mathcal{K}$ and $\mathcal{K}_0$ will be such that if a curve is a member, so is each arc, and also the inverse arc. Such classes of curves we term hereditary and reversible. In that case, the operation of cutting can be omitted: it can be carried out in two stages by fusion with an inverse arc, and trimming by a closed curve consisting of an arc and its inverse. Moreover, the operation of fusion is then associative and we can define the fusion of a finite number of members of $\mathcal{K}$ whose end-points agree in pairs where the definition requires this. On the other hand, we shall allow the two operations of embellishment and trimming to be carried out countably often.

We denote by $\mathcal{K}_1$ the subclass of $\mathcal{K}$ whose members are obtained from those of $\mathcal{K}_0$ by finite fusion and at most countable embellishment. We shall term the members of $\mathcal{K}_1$ reconstituted curves. From $\mathcal{K}_1$ we now define a class $\mathcal{K}_2$ whose members are obtained by at most countable trimming: we term them trimmed reconstituted curves.

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If $\mathcal{K}_2$ coincides with $\mathcal{K}$, we term $\mathcal{K}_0$ a reparable class of fragments, and the decomposition of R into the disjoint sets R_ν will be termed a reparable decomposition. In that case we say that R is the unimpaired union of the sets R_ν. More generally, a class of subsets P of R, not necessarily disjoint or countable, will be said to have R as its unimpaired union, if it has the union R, and if further, there exists a reparable decomposition of R into disjoint R_ν, whose number is at most countable, such that each P is the union of those R_ν which are its subsets.

We shall illustrate the notion of a reparable decomposition in the case in which R is the plane and $\mathcal{K}$ is the class of rectifiable curves in R. $\mathcal{K}_0, \mathcal{K}_1$, $\mathcal{K}_2$ are defined as before in terms of an at most countable decomposition of R.

Clearly the decomposition into two sets, consisting of the rational points, and the irrational points, is not reparable.

On the other hand, the decomposition into three sets, which are a line and the corresponding open half-planes, is reparable. This we see as follows.

We need only verify that each $C \in \mathcal{K}$ belongs to $\mathcal{K}_2$, and in so doing, we may suppose the end-points of C on the given line. Then C meets each of the two half-planes in at most countably many open arcs C_ν, and the line in a closed set. We denote by γ the directed segment with the same ends as C, by γ_ν the directed segment with the same ends as C_ν, by γ_ν^* the opposite segment to γ_ν, by Γ_ν the closed curve consisting of the fusion of γ_ν, γ_ν^*, and by Γ the embellishment of C by the finite or countable system of the Γ_ν. We then find that Γ is also the embellishment of γ by the finite or countable system of closed curves which are fusions of C_ν, γ_ν^*. Thus $\Gamma \in \mathcal{K}_1$, and by trimming off the Γ_ν we find that $C \in \mathcal{K}_2$.

In the applications that we shall make of the notions of this section, the fact that we allow only finite fusion instead of countable fusion, amounts to a restriction on the notions of reparable decomposition and unimpaired union. This restriction could be removed if we assumed the function $T(x)$ continuous on each rectifiable curve of bounded flight-time, but we prefer not to do this.

10. <u>The fundamental theorem and its consequences.</u> We are not in a position to extend very considerably the results of section 8. A finite or countable system of chains of flight to the target, will be termed a concourse of flights, and the corresponding system of canonical chains, a canonical concourse. By the constituent sets of a concourse, or of a canonical concourse, we shall mean those of the individual chains, or canonical chains. Their union will be termed the zone of the concourse, or canonical concourse.

We shall make the following hypotheses:

(10.1) <u>A suitable concourse.</u> <u>We suppose that there exists a concourse of flights such that</u>

(i) R <u>is the unimpaired union of its constituent sets,</u>

(ii) $R^{\#}$ <u>is the zone of the corresponding canonical concourse.</u>

(10.2) <u>Bounded momentum.</u> <u>We suppose that there exists in</u> R <u>a momentum</u> $y(x)$ <u>which is bounded in each bounded subset of</u> R <u>of bounded flight-time.</u>

In practice, in many instances, the verification of the first hypothesis may well amount to no more than the often rather arduous task of giving a complete and adequate description of the lines of flight. For in order to describe them properly, there is really little else that we can do except to group them into families, which correspond to our chains, and to divide these up into families of smooth arcs, which correspond to our sprays. In any event, (10.1) directs us to carry out this preliminary work in such a manner.

The second hypothesis can usually be verified rather easily in practice, and in a much stronger form: for the lines of flight issuing from a subset A of R, which is bounded and of bounded flight time, normally remain, all the way to the target, in some bounded subset B of R, in which the function $g_x(x,u)$ is uniformly bounded for all relevant values of u. In that case, since the conjugate vectors along a line of flight obey (2.2)(a), we easily verify an inequality of the form

$$|y(x)| \leq e^{K|t|} = e^{KT(x)} ,$$

provided that $y(x) \in \psi(x)$ $x \in R$, and that $y(x)$ is bounded on the target, i.e. for $t = 0$. In that case, not only does there exist a momentum with the property required by (10.2), but every momentum has this property.

We now come to our main result:

(10.3) Fundamental theorem. If R is subject to (10.1) and (10.2), then R is exact.

Proof. By (10.1) there is a reparable decomposition of R into disjoint R_ν, each of which is a subset of every constituent set that it meets, of our concourse. We define the classes of curves $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2$ accordingly, taking for $\mathcal{K}$ that of the rectifiable curves in R, of bounded flight-time.

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Now, for any $C \in \mathcal{K}_0$, and any R_ν in which C lies, we have by theorem (8.1)

$$\int_C y_\Sigma(x)\,dx = T(x_1) - T(x_2),$$

where x_1, x_2 are the initial and final points of C, and where Σ is any spray of a chain of our concourse, such that R_ν meets the source, or the flight-corridor, of Σ. From this relation, we deduce that $T(x)$, regarded as function of the arc-length s along C, is absolutely continuous, and that its derivative in s is almost everywhere $y_\Sigma(x)\,dx/ds$. This is true simultaneously for all relevant Σ, since there are at most countably many sprays, provided that we exclude at most countably many sets of s of measure 0. This means that the derivative in question has almost everywhere the stated value for every Σ, and therefore that almost every point of C is a point of univalence on C, since, by (10.1)(ii), every $y \in \mathcal{Y}(x)$ has the form $y_\Sigma(x)$ for some Σ at the point x. Hence we may rewrite our relation

$$\int_C y(x)\,dx = T(x_1) - T(x_2),$$

where $y(x)$ is now any momentum in R. Moreover, the relation in this form is now proved for every $C \in \mathcal{K}_0$.

The relation thus proved extends at once by addition to the case in which C is a finite fusion of members of $\mathcal{K}_0$, and in particular the left-hand side vanishes in that case if C is closed. The relation is therefore unaffected by at most countable embellishment of C, since its left-hand side then continues to exist, by (10.2). The relation thus holds for all $C \in \mathcal{K}_1$. For the same reason, the relation is also unaffected by at most countable trimming, and must hold for all $C \in \mathcal{K}_2$, i.e. for all $C \in \mathcal{K}$.

This completes the proof.

(10.4) *Corollary.* *With the same hypotheses, let* C *be any rectifiable curve in* R *of bounded flight-time, whose initial and final points are* x_1, x_2, *and let* $y(x)$ *be any momentum in* R. *Then*

$$\int_C y(x)\,dx = T(x_1) - T(x_2) .$$

Further, if in particular, C *is a trajectory arc, which starts at the time* t_1 *and ends at the time* t_2, *then*

$$\int_C y(x)\,dx \leq t_2 - t_1 .$$

Proof. We need only justify the last statement, and this follows from the fact that $y(x)\,\dot{x} \leq 1$ along a trajectory, by (2.1) and (2.2)(b).

In particular, if we set $x_1 = x$, and suppose x_2 on the target, we obtain, as a further corollary, the following:

(10.5) *Theorem.* *With the same hypotheses, let* x *be any point of* R. *Then the flight-time* $T(x)$ *along a line of flight is the least time for transferring the point* x *to the target along a trajectory in* R.

Theorem (10.5) is the basic existence theorem provided by the method.

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MRC Technical Summary Report #654

REMARKS ON OPTIMAL CONTROL I: THE STANDARD SUFFICIENCY THEORY FOR THE LEAST TIME PROBLEM

BY L. C. Young

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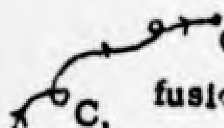

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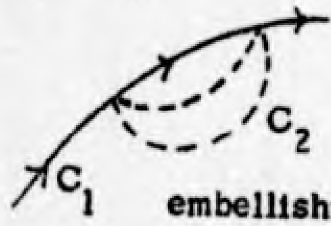

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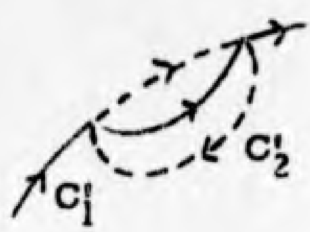

alternative embellishment leading to the same curve