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Michel Sakarovitch

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# THE $k$ SHORTEST ROUTES AND THE $k$ SHORTEST CHAINS IN A GRAPH

by

Michel Sakarovitch

## 1. Introduction

In a given graph  $G = (N, A)$  where  $N$  represents the set of nodes and  $A$  the set of directed arcs, a length  $a(x, y)$  is associated to every arc  $(x, y) \in A$ . A route from  $x_0$  to  $x_n$  is a sequence of nodes and arcs

$$P = [x_0, (x_0, x_1), x_1, (x_1, x_2), \dots, x_{n-1}, (x_{n-1}, x_n), x_n]$$

such that

$$\begin{aligned} x_i &\in N & i &= 0, \dots, n \\ (x_{i-1}, x_i) &\in A & i &= 1, \dots, n \end{aligned}$$

A route  $P$  will also be considered as a partial subgraph (see [2]) of  $G$  containing nodes  $x_0, \dots, x_n$  and arcs  $(x_0, x_1), \dots, (x_{n-1}, x_n)$ . The length of a route  $P$  is:

$$l(P) = \sum_{i=1}^n a(x_{i-1}, x_i)$$

A chain is a route in which no node is repeated. A cycle is a route in which no node is repeated except that  $x_0 = x_n$ .  $O$  and  $D$  will be two particular nodes.

$P_q(P'_q)$  will denote the  $q^{\text{th}}$  shortest route (chain) from  $O$  to  $D$  and:

$$G_q = (N_q, A_q) = \bigcup_{j=1}^q P_j$$

will be the partial subgraph of  $G$  defined by:

$$\begin{aligned} x \in N_q & \text{ iff } x \in P_j \text{ for some } j \leq q \\ (x, y) \in A_q & \text{ iff } (x, y) \in P_j \text{ for some } j \leq q \\ \bar{A}_q & = A - A_q \end{aligned}$$

In a similar way  $G'_q$ ,  $A'_q$ ,  $\bar{A}'_q$  and  $N'_q$  are defined.  $k_q$  is the number of routes in  $G_q$  which we need to generate as  $P_{q+1}^q, \dots, P_{q+k_q}^q$  i.e.  $k_q$  is the min of  $(k - q)$  and (max number of routes in  $G_q - q$ ).

Our central problem will be the construction of the  $k$  shortest routes ( $k$  shortest chains) from 0 to  $D$ . In this form, the problem is slightly more general than the one dealt with in [1], [6] and [3] and reviewed in [7] in two respects:

- Negative lengths are allowed provided that no: "negative cycle" (a cycle along which  $\sum a(x,y) < 0$ ) exist.
- Attention is not restricted to chains.

Moreover, this paper is based on the following simple idea:

At any stage of the algorithm, suppose we know the  $q$  ( $q < k$ ) shortest routes from 0 to  $D$  and  $P_{q+1}^q, \dots, P_{q+k_q}^q$  which are the  $(q+1)^{st}, \dots, (q+k_q)^{th}$  shortest routes in  $G_q$  ordered according to increasing lengths.  $\bar{P}_{q+1}^q$  is a shortest route from 0 to  $D$  which has at least one arc in  $\bar{A}_q$ . The  $(q+1)^{st}$  shortest route is obtained by:

$$P_{q+1}^q = \begin{cases} P_{q+1}^q & \text{if } \ell(P_{q+1}^q) \leq \ell(\bar{P}_{q+1}^q) \\ \bar{P}_{q+1}^q & \text{if } \ell(\bar{P}_{q+1}^q) < \ell(P_{q+1}^q) \end{cases}$$

$G_q$  and the list  $P_{q+1}^q$  is updated and the process begins again.

The algorithm is initiated by finding the shortest routes from 0 to every node using an algorithm such as the one in [4] if all lengths are nonnegative and the one in [5] if there are negative lengths.

In the next section an algorithm to construct the  $k$  shortest routes from 0 to  $D$  is described and it is justified in Section 3. Since the problem of finding the  $k$  shortest chains is of interest in many applications, a variant of the algorithm presented in Section 2 which exhibits the  $k$  shortest chains is described

in Section 4. Attention is drawn to the fact that minor modifications of these two algorithms allow to find the routes (chains) shorter than a given length. If capacities were to be considered, an immediate application of this last problem could be: what quantity can be shipped from 0 to D if we demand that the maximum time of travel be less than or equal to a certain given number.

## 2. Algorithm For Finding The k Shortest Routes

Step 0. - Find all the shortest routes from 0 to D :  $P_1, \dots, P_q$  .

If  $q \geq k$  , terminate.

Determine for all  $x \in N$  :

$\pi(x)$  = shortest distance from 0 to  $x$  .

$\eta(x)$  = shortest distance from  $x$  to D .

Note that  $P_{q+1}^q$  does not exist (the length of a route which does not exist is taken as infinite).

$k_q = 0$  .

Step 1. - Let:

$$(1) \quad \zeta_q(x) = \begin{cases} \text{Min}_{(y,x) \in \bar{A}_q} \{ \pi(y) + a(y,x) \} & \text{for } x \in N_q \\ \pi(x) & \text{for } x \notin N_q \end{cases}$$

$$(2) \quad E = \{ x \in N \mid \zeta_q(x) + \eta(x) = \text{Min}_{y \in N} [\zeta_q(y) + \eta(y)] = \bar{\pi}_q \}$$

If  $\bar{\pi}_q = \infty$  and  $k_q + q < k$  : terminate, the problem is infeasible.

Determine  $m$  by:

$$l(P_{q+m}^q) \leq \bar{\pi}_q < l(P_{q+m+1}^q)$$

Let:

$$P_{q+i} = P_{q+i}^q \quad i = 1, \dots, m$$

$$G_{q+m} = G_q$$

$$P_{q+m+i}^{q+m} = P_{q+m+i}^q \quad i = 1, \dots, (k_q - m)$$

$$q = q + m$$

$$j = 0$$

For each  $x \in E$  generate all the routes  $R_{(0x)}$  from 0 to  $x$ , the last arc of which belongs to  $\bar{A}_q$ , and, the length of which is  $\zeta_q(x)$ ; this is easily done by backtracking in the definition of  $\zeta_q(x)$  and  $\eta(x)$ . Similarly generate all the chains  $C_{(xD)}$  from  $x$  to  $D$  of length  $\eta(x)$ .

By combining a route  $R_{(0x)}$  with a chain  $C_{(xD)}$  and repeating the operation for all  $x \in E$ , all the routes from 0 to  $D$  of length  $\bar{\pi}_q$  can be generated. Lets call them  $\bar{P}_{q+1}^q, \dots, \bar{P}_{q+n}^q$ . We need only to generate at most  $k - q$  of them.

If  $n \geq k - q$  let  $P_{q+j} = \bar{P}_{q+j}^q$   $j = 1, \dots, k - q$ ; terminate, solution has been obtained.

Go to Step 2.

Step 2. - Let  $j = j + 1$

If  $j > n$  go to Step 1.

$$P_{q+1} = \bar{P}_{q+j}^q$$

$$G_{q+1} = G_q \cup \bar{P}_{q+j}^q$$

If the set of arcs in  $G_{q+1} - G_q$  is empty let:

$$p_{q+1+i}^{q+1} = p_{q+i}^q \quad 1 \leq i \leq k_q$$

$$q = q + 1$$

Go to Step 2.

If the set of arcs in  $G_{q+1} - G_q$  is not empty, they form a simple route  $C_j$  from  $O_j$  to  $D_j$  ( $O_j, D_j \in G_q$ ) (of Lemma 5).

Let  $R = (R_1, \dots, R_r)$ ,  $S = (S_1, \dots, S_s)$ ,  $T = (T_1, \dots, T_t)$  be respectively a set of routes from 0 to  $O_j$ , from  $D_j$  to  $O_j$  and from  $D_j$  to  $D$ , obtained by considering parts of:

- Those routes from 0 to  $D$  which contain  $O_j$  ;
- Those routes from 0 to  $D$  which contain  $D_j$  and then  $O_j$  ;
- Those routes from 0 to  $D$  which contain  $D_j$  .

Generate new routes in  $G_q$ , according to increasing lengths, by taking the union of a part of a route in  $R$ , from 0 to  $O_j$ , route  $C_j$  and a part of a route in  $T$  from  $D_j$  to  $D$ . Insert those newly created routes in the sequence  $p_{q+i}^q$  so that the new sequence is still ordered according to increasing lengths. Generate new routes until exhaustion of possibilities or until the

$$p_q = k - q - n + j - 1$$

first elements of this new sequence have been created.

If  $S = \emptyset$  go to Step 4.

Let  $u = 1$ . We will call route of order  $u$  a route which contains  $(u+1)$  times route  $C_j$  (and  $u$  times an element of  $S$ ). For instance a route of order 1 is the union of a route in  $R$ , route

$C_j$ , a route in  $S$ , route  $C_j$ , a route in  $T$ .

Go to Step 3.

Step 3. - Generate routes of order  $u$  according to increasing lengths and their elements in the sequence just generated in such a position that the sequence is still ordered according to increasing lengths either until exhaustion of possibilities or until the  $p_q$  first elements of the sequence have been generated.

If the first route of order  $u$  is among the  $p_q$  first elements let  $u = u + 1$  and go to Step 3.

If not, go to Step 4.

Step 4. - Let  $q = q + 1$

Denote by  $P_{q+i}^q$  the elements of the sequence just generated and by  $k_q$  their number.

Go to Step 2.

### 3. Justification Of The Algorithm

Let

$$\tilde{A} = \{(x,y) \in A \mid \pi(y) - \pi(x) = a(x,y)\}$$

$$\tilde{G} = (N, \tilde{A})$$

Lemma 1. - A necessary and sufficient condition for a cycle to be of 0 length (0-cycle) is that it be contained in  $\tilde{G}$ .

Proof:

(i) If a cycle  $[x_0, x_1, \dots, x_{n-1}, x_n = x_0]$  belongs to  $\tilde{G}$ , it is a 0 cycle.

$$\begin{array}{r} \pi(x_1) - \pi(x_0) = a(x_0, x_1) \\ \vdots \\ \pi(x_0) - \pi(x_{n-1}) = a(x_{n-1}, x_0) \\ \hline 0 = \sum_{i=1}^n a(x_{i-1}, x_i) \end{array}$$



(ii) Let  $C$  be a 0 cycle. For all the arcs  $(x,y) \in C$  we have by definition of  $\pi$  :

$$\pi(y) - \pi(x) \leq a(x,y)$$

If for one arc we had strict inequality by adding these relations for all arcs of  $C$ , we would obtain  $0 < \ell(C)$  which contradicts the assumption.

q.e.d.

Lemma 2. - All routes in  $\tilde{G}$  from 0 to a given node  $z$  have the same length:  $\pi(z)$ . Any route not completely included in  $\tilde{G}$  has a larger length.

Proof:

Obvious, since along any route of  $\tilde{G}$

$$\pi(y) - \pi(x) = a(x,y)$$

and for a route not completely included in  $\tilde{G}$ , we would have on at least one arc:  $\pi(y) - \pi(x) < a(x,y)$ .

q.e.d.

Lemma 3. - A necessary and sufficient condition for  $n$  to be finite at the end of Step 1 is that  $P_{q+j}^q$  do not contain a 0-cycle.

Proof:

Suppose  $\bar{P}_{q+j}^q$  contains a 0-cycle, then the routes obtained by the union of  $\bar{P}_{q+j}^q$  and  $i$  times this 0-cycle are of same length  $\bar{\pi}_q$  for any positive integer  $i$ .

If there is an infinite number of routes of length  $\bar{\pi}_q$ , there is an infinite subset of these routes the elements of which differ from one another by the addition of a cycle a certain number of times since the number of chains from 0

to  $D$  is finite. The length of this cycle is thus 0 .

q.e.d.

Lemma 4. -  $\bar{\pi}_q$  , as defined in (2) , is the shortest length of all routes not completely included in  $G_q$  .

Proof:

It is sufficient to show that for each route  $P^*$  which contains at least an arc  $(x,y)$  in  $\bar{A}_q$  ,  $l(P^*) \geq \bar{\pi}_q$  .

From the definition of  $\pi$  and  $\eta$  we have:

$$(3) \quad l(P^*) \geq \pi(x) + a(x,y) + \eta(y)$$

But from (1) :

$$\pi(x) + a(x,y) \geq \zeta_q(y)$$

and the property of shortest distances that:

$$\pi(y) \leq \pi(x) + a(x,y)$$

Then, from (2) :

$$\zeta_q(y) + \eta(y) \geq \bar{\pi}_q$$

which together with (3) implies:

$$l(P^*) \geq \bar{\pi}_q .$$

q.e.d.

Lemma 5. - In Step 2 the set of arcs in  $G_{q+1} - G_q$  is either empty or forms a simple route  $C_j$  from  $O_j$  to  $D_j$  ( $O_j, D_j \in G_q$ ) . This route is either a chain or a simple cycle in which case  $D_j = O_j$  .

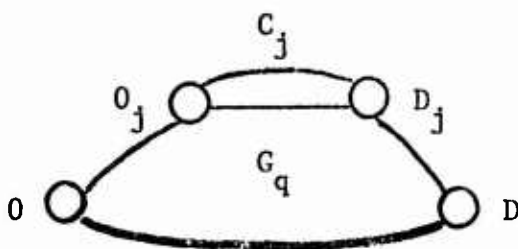


Figure 1

Proof:

If the set of arcs in  $G_{q+1} - G_q$  is not empty,  $\bar{P}_{q+j}^q$  includes successive sequences of arcs of  $\bar{A}_q$ ; let  $C_j$  be the last of these sequences when  $\bar{P}_{q+j}^q$  is described from 0 to D.  $C_j$  forms a route from  $0_j (\in G_q)$  to  $D_j (\in G_q)$  since  $C_j$  is a subset of  $\bar{P}_{q+j}^q$ .

The part of  $\bar{P}_{q+j}^q$  from 0 to  $0_j$  is included in  $\bar{G}$  for if it were not, it would be possible, from Lemma 2, to construct another route shorter than  $\bar{\pi}_q$  and which would have the arcs of  $C_j$  outside of  $G_q$  thus contradicting Lemma 4 and similarly for  $D_j$  to D, proving  $\exists$  a single path  $C_j$ .

Suppose that a proper subset of  $C_j$  forms a cycle. If this cycle was a 0-cycle from Lemma 3,  $n$  would be infinite and the algorithm would have been terminated at the end of Step 1.

If this cycle had a positive length, it would be possible by deleting it from  $\bar{P}_j^q$  to construct a route not completely included in  $\bar{G}_q$  and of length  $< \bar{\pi}_q$ , thus contradicting Lemma 4 again. Thus if there is a cycle, it can only be simple.

q.e.d.

The following lemma is not crucial for the proof of the theorem, however, it helps understanding of the algorithm.

Lemma 6. -  $\bar{\pi}_q > \bar{\pi}_{q-1}$

Proof:

It is impossible that  $\bar{\pi}_q < \bar{\pi}_{q-1}$  since  $\bar{P}_{q+j}^q$  is not among the  $q$  shortest

routes.

Suppose  $\bar{\pi}_q = \bar{\pi}_{q-1}$ . Then  $\exists (x,y) \in \bar{A}_q \cap \bar{P}_{q+1}^q$  for which  $\zeta_q(y) + \eta(y) = \bar{\pi}_q = \bar{\pi}_{q-1}$  but then  $\bar{P}_{q+1}^q$  would have been discovered at a previous passage and  $(x,y)$  would have been added to  $\bar{A}_q$ , thus a contradiction.

q.e.d.

### Theorem

The algorithm leads to a solution in a finite number of steps, or shows that no solution exists.

### Proof:

In Steps 2 and 3 the sequence  $P_{q+1}^q, \dots, P_{q+k_q}^q$  is generated so that this sequence is at hand at beginning of Step 1. Then  $P_{q+1}^q$  is either the shortest route completely included in  $G_q$ ,  $P_{q+1}^q$ , or the shortest route containing some arc in  $\bar{A}_q$ ,  $\bar{P}_{q+1}^q$  (Lemma 4).

If the number of routes in  $G_q$ ,  $(k_q + q)$ , is less than  $k$  and if there does not exist routes outside those in  $G_q$  ( $\bar{\pi}_q = \infty$ ) the problem is infeasible.

Each time Step 1 is described, some arcs are added to  $G_q$  so that we can only go a finite number of times through Step 1.

We cannot cycle indefinitely in Step 3 since each time we go through Step 3, at least, a new route is added to the updated sequence and we do not want more than  $p_q$  routes.

Now we can go only a finite number of times through Step 2 since at each passage the index  $j$  is increased by one unit and when  $j > n$  we go to Step 1.

q.e.d.

### 4. Variant Of The Preceding Algorithm To Find The $k$ Shortest Chains

The basic idea of this variant is to perform the algorithm of Section 2 and to keep the routes thus generated only if they are chains. Two main differences

with the initial algorithm justify a rewriting of the algorithm:

- When a 0-cycle is met, no new chain will ever be constructed.
- Routes of order  $u \geq 1$  will not give rise to any chain and Step 3 can be skipped.

Step 0'. - Find all the shortest chains from 0 to D :  $P'_1, \dots, P'_q$ .

If  $q \geq k$ , terminate.

Determine for all  $x \in N$  :

$\pi(x)$  = shortest distance from 0 to  $x$ .

$\eta(x)$  = shortest distance from  $x$  to D.

$k'_q = 0$ .

Step 1'. - Define:

$$(1) \quad \zeta_q(x) = \begin{cases} \text{Min}_{(y,x) \in \bar{A}_q} \{ \pi(y) + a(y,x) \} & \text{for } x \in N_q \\ \pi(x) & \text{for } x \notin N_q \end{cases}$$

$$(2) \quad E = \{ x \in N \mid \zeta_q(x) + \eta(x) = \text{Min}_{y \in N} [\zeta_q(y) + \eta(y)] = \bar{\pi}_q \}$$

If  $\bar{\pi}_q = \infty$  and  $k'_q + q < k$  : terminate, the problem is infeasible.

Determine  $m$  by:

$$\ell(P'_{q+m}) \leq \bar{\pi}_q < \ell(P'_{q+m+1})$$

Let:

$$P'_{q+i} = P'_{q+i}{}^q \quad i = 1, \dots, m$$

$$G'_{q+m} = G'_q$$

$$P'_{q+m+i}{}^{q+m} = P'_{q+m+i}{}^q \quad i = 1, \dots, (k'_q - m)$$

$$q = q + m$$

$$j = 0$$

For each  $x \in E$  generate the routes  $R_{(0x)}$  from 0 to  $x$ , the last arc of which belongs to  $\bar{A}_q$  and the length of which is  $\zeta_q(x)$ ; this is easily done by backtracking in the definition of  $\zeta_q(x)$  and  $\pi(x)$ . Similarly generate all the chains  $C_{(xD)}$  from  $x$  to  $D$  of length  $\eta(x)$ .

By combining a route  $R_{(0x)}$  with a chain  $C_{(xD)}$  in such a way that no newly constructed route from 0 to  $D$  contains a cycle counted more than once, a finite number  $n$  of routes from 0 to  $D$  of length  $\bar{\pi}_q, \bar{p}_{q+1}^q, \dots, \bar{p}_{q+n}^q$  are generated. Among those are all the chains of length  $\bar{\pi}_q$  and having a nonempty intersection with  $\bar{A}_q$ .

Go to Step 2'.

Step 2'. -  $j = j + 1$

If  $j > n$  go to Step 1'.

If  $\bar{p}_{q+j}^q$  is a chain, let  $p'_{q+1} = \bar{p}_{q+j}^q$ .

Let

$$q' = \begin{cases} q + 1 & \text{if } \bar{p}_{q+j}^q \text{ is a chain} \\ q & \text{otherwise} \end{cases}$$

$$G'_{q'} = G'_q \cup \bar{p}_{q+j}^q$$

If the set of arcs in  $G'_{q'} - G'_q$  is empty, let

$$k'_{q'} = k'_q$$

$$p'_{q'+i} = p'_{q+i} \quad 1 \leq i \leq k'_q$$

$$q = q'$$

Go to Step 2'.

If the set of arcs in  $G'_q - G'_q$  is not empty, it forms a simple route  $C_j$  from  $O_j$  to  $D_j$ . Let  $C'_j$  be the subset of  $C_j$  which is a chain from  $O_j$  to  $D_j$ .

Let

$$R' = (R'_1 \dots R'_r), T' = (T'_1 \dots T'_s)$$

be respectively a set of chains from  $O$  to  $O_j$ , from  $D_j$  to  $D$  obtained by taking parts of:

- Those chains from  $O$  to  $D$  which contain  $O_j$ .
- Those chains from  $O$  to  $D$  which contain  $D_j$ .

Generate new routes in  $G'_q$ , according to increasing lengths, by routes taking union of a chain in  $R'$ , chain  $C'_j$  and a chain in  $T'$ . Keep only those routes which are chains and insert them in the sequence  $P'_{q+i}$  so that the new sequence be still ordered according to increasing lengths. Generate new chains until exhaustion of possibilities or until the  $k - q + j - 1$  first elements of the new sequence have been generated.

Let

$$q = q'$$

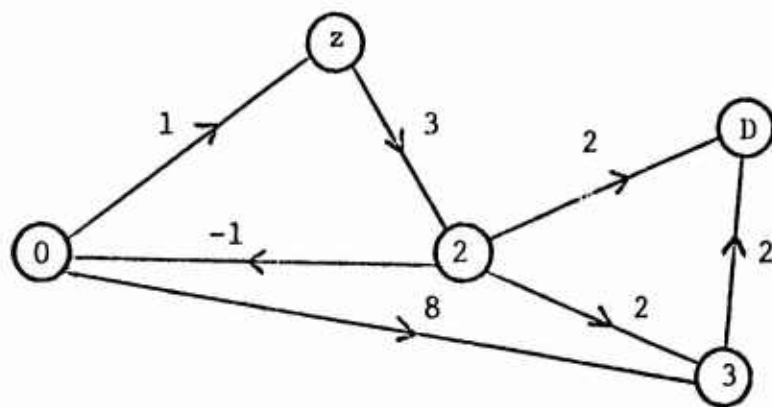
Denote by  $P'_{q+i}$  the elements of the sequence just generated and by  $k'_q$  their number.

Go to Step 2'.

This algorithm, as well as algorithm of Section 2, can be slightly modified in order to produce, respectively all the chains (routes) which are shorter than a given length  $L$  (or within  $x\%$  of the length of the shortest chain (route)).

### 5. Examples

Suppose we want to find the four shortest routes and the four shortest chains in the following graph:



#### a) Four Shortest Routes

1) Step 0. - The shortest route is unique  $P_1 = [0, 1, 2, D]$   $A_1 = \{(01), (12), (2D)\}$   $\ell(P_1) = 6$

2) Step 1. -

Nodes	$\pi(x)$	$\eta(x)$	$\zeta_1(x)$	$\eta(x) + \zeta_1(x)$
0	0	6	3	9
1	1	5	$\infty$	$\infty$
2	4	2	$\infty$	$\infty$
3	6	2	6	8
D	6	0	8	8

$$E = \{3, D\}$$

$$\bar{\pi}_1 = 8$$

$P_2'$  does not exist

$$\bar{P}_2' = [0, 1, 2, 3, D]$$

$$n = 1$$

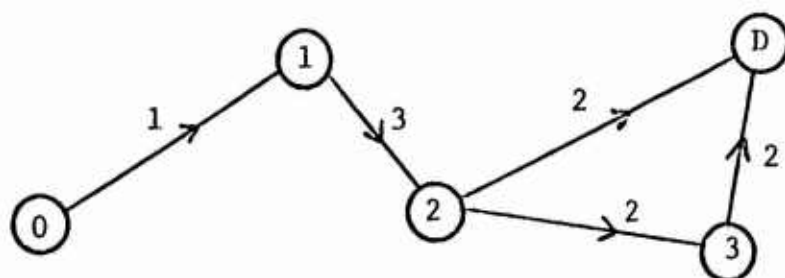


3) Step 2. -

$$j = 1$$

$$P_2 = \bar{P}_2^1$$

$$G_2 = G_1 \cup \bar{P}_2^1 \quad \text{i.e. :}$$



There is no other route in  $G_2$  than  $P_1$  and  $P_2$ , i.e.  $P_3^2$  does not exist.  $k_2 = 0$ .

Go to Step 2.

4) Step 2. -  $j = 2 > n$  : go to Step 1.

5) Step 1. -

Nodes	$\eta(x)$	$\zeta_2(x)$	$\eta(x) + \zeta_2(x)$
0	6	3	9
1	5	$\infty$	$\infty$
2	2	$\infty$	$\infty$
3	2	8	10
D	0	$\infty$	$\infty$

$$E = \{0\}$$

$$\bar{\pi}_2 = 9$$

Since  $P_3^2$  does not exist  $m = 0$

$$\bar{P}_3^2 = [0, 1, 2, 0, 1, 2, D]$$

$$n = 1$$

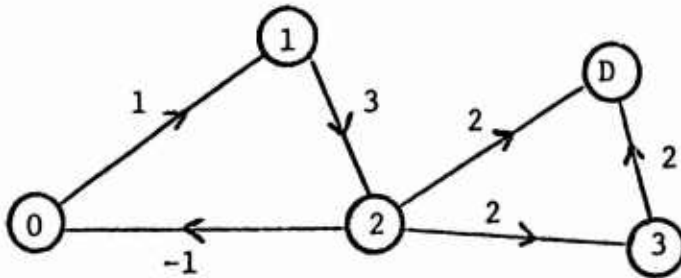
Go to Step 2.

6) Step 2. -

$$j = 1$$

$$P_3 = \bar{P}_3^2$$

$$G_3 = G_2 \cup \bar{P}_3^2 \quad \text{i.e.}$$



$$C_3 = (2, 0)$$

$$P_4^3 = [0-1-2-0-1-2-3-D]$$

$$l(P_4^3) = 11$$

If we go to Step 3 and generate the shortest route of order 1, we find

$$\tilde{P}_4^3 = [0-1-2-u-1-2-0-1-2-D]$$

$$l(\tilde{P}_4^3) = 12$$

and we insert it in the sequence after  $P_4^3$ .

Go to Step 2.

7) Step 2. -  $j = 2 > n - 1$ , go to Step 1.

8) Step 1. -

Nodes	$\eta(x)$	$\zeta_3(x)$	$\eta(x) + \zeta_3(x)$
0	6	$\infty$	$\infty$
1	5	$\infty$	$\infty$
2	2	$\infty$	$\infty$
3	2	8	10
D	0	$\infty$	$\infty$

$E = \{3\}$   
 $\bar{\pi}_3 = 11$

and

$$P_4 = \bar{P}_4^3 = [0-3-D]$$

terminate.

b) Four Shortest Chains

Up to stage (6) the two algorithms give the same result i.e.

$$P_1' = [0, 1, 2, D]$$

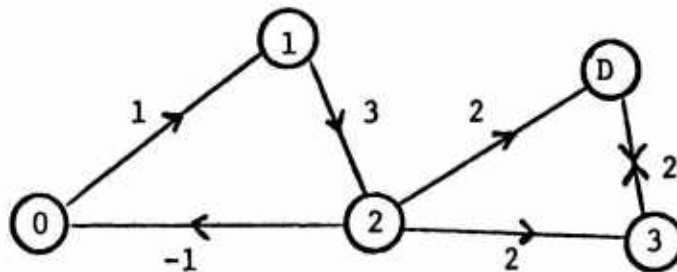
$$P_2' = [0, 1, 2, 3, D]$$

6') Step 2'. -

$$j = 1$$

$$q' = q = 2$$

$$G_2' = G_2' \cup \bar{P}_3^2$$



No new chain in  $G_2'$ ;  $\bar{P}_3^2$  does not exist.  $k_2' = 0$ .

Go to Step 2'.

7') Step 2'. -  $j = 2 > n = 1$  go to Step 1'.

8') Step 1'. -

Nodes	$\eta(x)$	$\zeta_2(x)$	$\eta(x) + \zeta_2(x)$
0	6	$\infty$	$\infty$
1	5	$\infty$	$\infty$
2	2	$\infty$	$\infty$
3	2	8	10
D	0	$\infty$	$\infty$

$$E = \{3\}$$

$$\bar{\pi}_3 = 10$$

$P_3'^2$  does not exist so  $m = 0$

$$\bar{P}_3^2 = [0, 3, D]$$

$$n = 1$$

Go to Step 2'.

9') Step 2'. -  $j = 1$

$\bar{P}_3^2$  is a chain:  $P_3' = \bar{P}_3^2 = [0, 3, D]$

$$q' = 3$$

$$G_3' = G_2' \cup \bar{P}_3^2 = G$$

There is no chain outside  $P_1' P_2' P_3'$  in  $G_3'$ ; i.e.,  $P_4'^3$  does not exist.  $k_3' = 0$ .

Go to Step 1'.

10') Step 1'. -

Nodes	$\eta(x)$	$\zeta_3(x)$	$\eta(x) + \zeta_3(x)$
0	6	$\infty$	$\infty$
1	5	$\infty$	$\infty$
2	2	$\infty$	$\infty$
3	2	$\infty$	$\infty$
D	0	$\infty$	$\infty$

$$\bar{\pi}_3 = \infty$$

$$k_3' + 3 = 0 + 3 < 4$$

Problem is infeasible.

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