

ORC 66-21  
November 1966

AD 642822  
MULTI OBJECTIVE LINEAR PROGRAMMING

by  
David Savir

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This research was supported by the Office of Naval Research under Contract Nonr-222(83), the National Science Foundation under Grant GP-4593 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

## ABSTRACT

Let us consider a linear program with several objective functions. The traditional approach has been either to "trade off" by weighting each function, or if a "trade off" vector cannot be provided to ignore all but the most significant. We are interested in classes of programs whose members possess some common characteristics. Examples are sequences of production, refining, inventory problems over time at one installation. If sufficient conditions exist an estimate of a "trade-off" vector can be made. This estimate improves over the sequence.

A set  $X^*$  exists which contains the solutions obtained by optimizing with respect to all nonnegative combinations of objective functions. A decision maker is not indifferent to these solutions but can characterize preferred solutions. A method is presented whereby he can direct a finite sequence of solutions,  $(x_i)$ , over  $X^*$  towards a preferred solution. As the estimate of the "trade-off" vector improves, the expected length of the sequence  $(x_i)$  diminishes, and the efficiency of solution increases.

## 1. Introduction

In this paper we will investigate linear programs with more than one objective function:

$$(1) \quad \begin{aligned} &\text{Objective function } z = Qx \\ &\text{subject to } Ax = b \\ &\quad \quad \quad x \geq 0 \end{aligned}$$

where  $z$  is a  $p$ -vector

$x$  an  $n$ -vector

$b$  an  $m$ -vector

$Q$  a  $p \times n$  matrix =  $(c^1, \dots, c^k, \dots, c^p)'$

$A$  an  $n \times m$  matrix.

We will define a preferred set  $S$  such that only if  $x$  is in  $S$  will the objective be satisfied. Under certain conditions a semipositive linear transformation  $(\lambda)$  on  $z$  can be made such that  $\lambda Qx$  will be minimized and the minimizer  $x^0$  will be in  $S$ . We will define a set  $Z$  of weakly undominated  $z$  and show that under arbitrary  $\lambda$

$$\min_x \lambda Qx \quad \text{and } z = Qx, \quad z \in Z$$

will be attained by the same  $x$ . Also for any  $x^1$  such that  $Qx^1$  is not in  $Z$ ,  $Qx^1$  can be strictly decreased to a point  $Qx^2$  which is in  $Z$ . Hence the simplex method will always yield points in  $Z$  as optimal basic feasible solutions.

Having established that arbitrary  $\lambda$  will produce an extreme point  $x$  from a set of extreme points which also contains a preferred extreme point (assuming hopefully that  $S$  contains an extreme point!) we will find a  $\lambda^0$  which will yield the desired  $x$  in  $S$ . For this we make certain assumptions, in particular that  $\lambda$  is statistically estimable. If an immediate solution is not found an

algorithm is used to search for the preferred extreme point while minimizing the squared deviation of  $\lambda$  from its estimate.

The assumption that  $\lambda$  is estimable is not unrealistic if the approach taken in this paper is applied to a class of problems with certain similarities, for example: production planning over sequential periods where decisions are made at each period.

## 2. The Presentation of the Problem.

The objectives of many linear programs can be defined adequately only as vector functions. For example, in addition to minimizing cost, a decision maker might also wish to minimize execution time of a project, or to maximize employment, etc., or any combination of these.

Suppose that there exist  $p$  objective functions  $z_k = c^k x$ ,  $k=1, \dots, p$ , to a linear program subject to the constraint  $x \in T$  where  $T = \{x | Ax = b, x \geq 0, x \in R^n\}$ . We will assume throughout this paper that  $T$  is nonempty and bounded. In general no  $x^0 \in T$  exists such that  $z_k^0 = c^k x^0$  is an optimum<sup>†</sup> for all  $k$ . We construct the multiple objective vector function  $z = (z_1, \dots, z_k, \dots, z_p) \in R^p$ . After we exclude the case just mentioned, it is obvious that  $z$  cannot be minimized. Although other approaches can no doubt be taken, it seems both legitimate and consistent, since the subject is linear programming, to transform  $z$  to a scalar function which lends itself to minimization.

We define a pair of both physical and mathematical significance: the decision maker (DM) and the linear program problem solver (LP). First, mathematically, DM is a function from  $R^p$  to  $R$  with value  $\lambda z$  transforming the components of  $z$  by semipositive linear combinations to a scalar objective function. LP is a correspondence from  $R$  to  $T$ : given a real valued linear objective function

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<sup>†</sup>Optimum is used to mean maximum or minimum. In subsequent sections the minimum problem only is treated.

$DM(z)$  ,  $x^* \in LP(DM) \subset T$  such that  $DM$  is minimized. We call such  $x^*$  an optimal solution. If the decision maker is satisfied with some optimal solution and terminates the problem, the solution is defined to be preferred. If he is unsatisfied but no feasible optimal solution satisfies him more, then the solution is almost preferred.<sup>†</sup> Clearly, the latter definition implies that:

Given the existence of one of the solutions,  
either a preferred solution or an almost  
preferred solution exists, but not both.

We shall prove in Section 4 that under certain conditions one of these solutions does indeed exist. A solution which is either preferred or almost preferred will be abbreviated to a POAP solution. No implication has been made as to the uniqueness of any solution, only to the uniqueness of a set of solutions.

Secondly, physically, the problem thus far stated (1) can be decomposed between  $DM$  and  $LP$ . We can consider that  $DM$  generates by some means a vector of weights by which he constructs a linear combination of objective functions. He has then produced a linear program for  $LP$  to solve.  $LP$ , characteristically a computer, can produce an optimal feasible solution by conventional linear programming techniques.

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<sup>†</sup>The concept "satisfied" is unmathematical intuitively and undefined explicitly. In the light of the assumption  $DM2$  appearing at the end of the section we can define the following:

- (i)  $DM$  is satisfied if he does not specify any vector  $d$ , given  $x$ .
- (ii)  $DM$  is unsatisfied if he specifies some vector  $d$ , given  $x$ .
- (iii)  $DM$  is unsatisfied but no  $x'$  satisfies him more if he specifies some vector  $d$ , given  $x$ ; but no  $x'$  exists such that  $d(x',S) < d(x,S)$ .

Both the decision maker and the decision maker function will be called  $DM$  for notational simplicity. Similarly, both the linear program problem solver and its correspondence will be called  $LP$ . No ambiguity will occur.

(Recall that  $T$  is nonempty and bounded.) DM now evaluates the solution. If he is satisfied he terminates the problem; otherwise he tries a new vector and sends the data back to LP.

Suppose a common unit could be established in which the value of  $z_k$  would be expressed. Then, in the above example, DM could assign  $\alpha_1$  units of value to one unit of cost,  $\alpha_2$  units of value to one unit of execution time, etc., and all the components would be transformed to a unique additive measure  $\mu$  in units of value,  $\mu = \alpha'z$ , where  $\alpha$  is well defined and constant. We shall require that  $\lambda$  be normalized so  $DM(z) = \alpha'z / \|\alpha\|$ , which would then be completely determined. The problem would then reduce to a conventional linear program. We will assume that the decision maker cannot establish such a unit of value.

We can now define (1) more explicitly:

(2) Find  $(x, DM)$ ,  $x \in T$ ,  $DM \in \langle \{z_k\} \rangle^{\dagger\dagger}$  such that  $x \in LP(DM)$  and  $x$  POAP.

Equivalently:

$$(2') \quad \begin{array}{l} \min_{x} DM = \lambda Qx \\ \text{subject to } Ax = b \\ \quad \quad \quad x \geq 0 \\ \quad \quad \quad x \text{ POAP} \end{array}$$

for each  $\lambda \geq 0$ ,  $\sum_{k=1}^P \lambda_k = 1$

$$\text{where } Q = (c^1, \dots, c^k, \dots, c^P)' \quad . \quad \dagger$$

<sup>†</sup>"For each  $\lambda$ " seems to imply that the set of  $\lambda$  is at most countable, which is not true. However in Section 6 we will show that there exist a finite number of sets  $\Lambda_i$ , and any  $\lambda \in \Lambda_i$  will generate the same solution set  $\{x\}$  as any  $\lambda' \in \Lambda_i$ . Thus "for each  $\lambda$ " is to be interpreted "for any  $\lambda$  from each  $\Lambda_i$ ."

<sup>††</sup>" $\langle \{z_k\} \rangle$ " reads "the convex hull of the set of all  $z_k$ ."

We will approach (2') eventually as a two stage problem: firstly, for some  $\lambda$ , minimizing  $DM$ ; secondly, determining the given solution's POAP properties, and revising  $\lambda$  if necessary. Call then the first stage, i.e. (2'), neglecting the  $x$  POAP condition: (2'').

In Section 3 we will prove that solutions to (2'') exist under certain conditions, and that the simplex method will find a subset of these solutions.

The method proposed for finding a solution has little practical significance for a single problem, since the value of  $\lambda$  needed to obtain a POAP solution is known only after the POAP solution has been found. (This is clear from the definitions of preferred and almost preferred.) However, if we have a class  $\mathcal{C}$  of problems possessing certain properties, and the first  $T-1$  sequential problem can be solved more efficiently when LP has available the values  $\lambda_t$ ,  $t=1, \dots, T-1$ , used to obtain the respective POAP solutions, than otherwise. The required properties are:

Let  $C_t \in \mathcal{C}$  be the  $t^{\text{th}}$  sequential problem of form (2') possessing a POAP solution.

- a) Let  $T_t \triangleq T$  for  $C_t$ . Then  $\bigcap_t T_t \neq \emptyset$  where  $t$  is the index set  $\{1, 2, \dots\}$ .
- b) Let  $x_t \triangleq x \in T_t$  for  $C_t$ . Then  $x_t \in R^n \forall t$  and  $\sum_t x_t \in R^n$ .
- c) If  $x^*$  is preferred for some  $C_t$ , and if  $x^*$  is optimal for some  $C_{t'}$ ,  $t' \neq t$ , then  $x^*$  is preferred for  $C_{t'}$ . Equivalently.
- c') There exists a domain  $S \subset R^n$  such that if optimal  $x_t^* \in S$  it is preferred.
- d)  $z$  is common to  $C_t$ ,  $\forall t$ .
- e) The distribution of  $T_t$  is such that a maximum likelihood estimate of  $\lambda$  exists. This estimate is the mean of  $\lambda_t$  over  $t$ .



Now we require some assumptions on LP and on DM. Our recognition of LP as a linear programming problem solver should be sufficient: we know that it will read and interpret submitted data without error, that it will produce an optimal feasible solution with certainty, and that it will yield DM the results with typographic perfection. As for DM we assume the following:

DM1 (Recognition). DM can always recognize an optimal solution  $x \in S$  to be preferred. He can always recognize an optimal solution  $x \notin S$  to be not preferred.

DM2 (Direction). Given an optimal solution  $x \notin S$ , let the distance between  $x$  and the closest point in  $S$  be  $d(x, S)$ . Then DM can specify a vector  $d$  such that  $d(x + \epsilon d, S) < d(x, S)$  for arbitrarily small  $\epsilon > 0$ . No knowledge of any point  $s \in S$  is assumed.

Using the assumptions DM will be able to direct LP towards  $S$  and terminate if LP produces  $x \in S$ .

### 3. The Existence of a Solution.

We define a vector  $z^0$  to be weakly undominated if  $z \not\prec z^0$ † in the domain of  $z$ . This definition will enable us to replace the scalar "minimize" by the vector "find weakly undominated" in the vector optimization problem:

$$(3) \text{ Find weakly undominated } z = Qx$$

$$\text{subject to } \left. \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} \text{ or } x \in T$$

where symbols are defined in (1).

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† " $z \not\prec z^0$ " reads "it is not true that  $z_k < z_k^0$ ,  $k=1, \dots, p$ ."

We can now find a set of  $x$  which will solve (3).

Let  $z^{\min k}$  be any vector  $z$  such that  $z_k^{\min k} \leq z_k$ , for any  $z$  in the range of  $x$ . Then  $z^{\min k}$  is weakly undominated. Thus we can produce  $p$  such weakly undominated vectors, each of which is minimal in one component. How can we produce the inverse images of such  $z_k^{\min k}$ ? Consider one such problem:

$$(4) \quad \min_{\text{w.r.t. } k} z = Qx, \quad x \in T.$$

Clearly  $z_{k'}$ ,  $k' \neq k$ , can adopt any value and is independent of the minimizing process. Hence (4) is equivalent to

$$(5) \quad \min z_k = c^k x, \quad x \in T$$

which is a linear program. Hence by solving  $p$  linear programs of form (5) we can find  $p$  basic feasible solutions to (3), each of which has the minimal property in a different component. Call a feasible solution to (5)  $x^{k^*}$ .

("Solution" is here used according to the linear programming convention to mean a value of  $x$  at which the objective is minimized. No ambiguity results. All solutions are assumed feasible.) Let  $X^{k^*} = \{x^{k^*}\}$ . From well-known results  $X^{k^*}$  is convex.

Now to dispose of the simplest case:

Theorem 1 (3) has a minimum solution  $x^{**}$  if and only if  $\bigcap_k X^{k^*} \neq \emptyset$ , and any point in the nonempty intersection is such a solution.

Proof. Suppose  $x^{**}$  is a minimum solution to (3). Then  $z^{**} = Qx^{**}$  is not only weakly undominated but also a minimum. Hence  $z^{**} \leq z$ ,  $x \in T$ . It follows that  $z_k^{**} \leq z_k$ ,  $k=1, \dots, p$ . so  $x^{**}$  solves (5) for each of the components, and  $x^{**} = x^{k^*}$ ,  $k=1, \dots, p$ . Consequently  $x^{**} \in X^{k^*}$  for all  $k$ , and  $x^{**} \in \bigcap_k X^{k^*}$ . Suppose  $\bigcap_k X^{k^*} \neq \emptyset$ . Let

$x^0 \in \cap X^{k^*}$ . Then  $x^0$  solves (5) for each of the components,  
 $Qx^0 = z^0 \leq z$ ,  $x \in T$ ,  $z^0$  is a minimum vector and  $x^0$  solves (3)  
 to a minimum.  $x^0$  was chosen arbitrarily from  $\cap X^{k^*}$ , so any  
 $x^{0'} \in \cap X^{k^*}$  solves (3). ///

Now a minimum solution to (3) seems the best we can hope for, and indeed will solve (2') provided we can show that it is POAP. Theorem 3 will give us that answer. Meanwhile let us consider the case where  $QX^{k^*} = \emptyset$ . We have shown that there is no minimum solution to (3).

Now we have shown that a set  $\cup_k X^{k^*}$  of solutions to (3) exists, and we would like to discover, if possible, a maximal such set from which to select some preferred or almost preferred  $x$ . Also, since we have the simplex method in mind, we would like to know if we can move from  $x$  to a better  $x'$ .<sup>†</sup>

We will show that this maximal set  $X^*$  exists and  $x^* \in X^*$  is a solution to  $\min_x \lambda Qx$ ,  $x \in T$ , for some  $\lambda \geq 0$ . We would like also to be able to specify  $\lambda$  and thus generate a specific solution. If the rows of  $Q$  are not positively linearly independent we can find some  $\lambda^0$  such that  $\lambda^0 Qx = 0$  for all  $x \in T$ ! Legislating this trivial solution out of existence we require:  
Positive linear independence of  $c^k$ , i.e.  $\sum_k \lambda_k c^k = 0$ ,  $\lambda_k \geq 0 \Rightarrow \lambda = 0$ .

We can now construct a closed convex cone  $Y = \{ \langle kQ^i \rangle, k \geq 0, \text{ a scalar} \}$ , the minimal cone containing  $c^k$ . Let  $X^* = \{ x^* \mid x^* \text{ minimizes } c^\lambda x = z_\lambda, x \in T, \text{ for some } c^\lambda \in Y \}$ . Clearly  $x^{k^*} \in X^*$  for all  $k$ . Also for any  $c^\lambda \in Y$  there exists the corresponding  $x^*$  by the compactness of  $T$ .

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<sup>†</sup>Analogously to linear programming we can say that  $x'$  is better than  $x$  if  $z' = Qx' \leq Qx = z$ .

Theorem 2

If there exists  $x^{00} \notin X^*$ ,  $x \in T$ , and  $z^{00} = Qx^{00}$ , where  $Q$  is a matrix of  $p$  positively linearly independent rows, the  $k^{\text{th}}$  row being  $c^k$ , there exists  $z^0 < z^{00}$  (i.e.  $z_k^0 < z_k^{00}$ ,  $\forall k$ ) such that  $x^0 \in X^*$ .

Proof

$x^{00} \notin X^*$ .

$z_k^{00} = c^k x^{00}$ ,  $k = 1, \dots, p$ .

(6) Referring to (5):  $z_k^{00} > z_k^*$  else  $x^{00} \in X^{k*} \subset X^*$ .

$p$  hyperplanes  $c^k x = z_k^{00}$  pass through  $x^{00}$ . Hence there exist  $p$  closed half-spaces  $c^k x \leq z_k^{00}$  with nonempty intersection  $\mathcal{L}$ , and, by the positive linear independence of  $c^k$  and the compactness of  $T$ ,  $\mathcal{L}$  has an interior.  $(\text{int}\mathcal{L}) \cap T \neq \emptyset$  by (6).

It is necessary and sufficient to show that there exists  $x^0 \in (\text{int}\mathcal{L}) \cap T$  satisfying  $x^0 \in X^*$  since  $x^0 \in (\text{int}\mathcal{L}) \cap T \Rightarrow z^0 < z^{00}$ .

Select a point  $x^{01}$  in  $(\text{int}\mathcal{L}) \cap T$ . Define  $\mathcal{L}^i = \{x | c^k x \leq z_k^{0i}, \forall k\}$ . Then  $\mathcal{L}^1 \cap T \subseteq (\text{int}\mathcal{L}) \cap T$ . Construct a sequence  $x^{00}, x^{01}, \dots, x^{0i}$  such that  $(\text{int}\mathcal{L}) \cap T \supseteq \mathcal{L}^1 \cap T \supseteq \dots \supseteq \mathcal{L}^i \cap T \supseteq \dots \mathcal{L}^i, T$  are closed, so  $\mathcal{L}^i \cap T$  is closed.  $\mathcal{L}^i \cap T$  is a decreasing monotone sequence of sets so  $\lim_{i \rightarrow \infty} \mathcal{L}^i \cap T = \bigcap_{i=1}^{\infty} (\mathcal{L}^i \cap T) = x_1^0$ .  $x_1^0$  lies on a supporting hyperplane, since  $x_1^{0i}$  lies on the boundary of  $T$ , for if not a smaller set could be found containing points of  $\mathcal{L}$  and  $T$ . The boundary of  $T$  is a set of hyperplanes. So  $x_1^0$  must lie on at least one.

Construct  $p-1$  other different sequences yielding  $x_k^0$  ( $k=1, \dots, p$ ) arbitrarily close and lying on the same hyperplane. Either all  $x_k^0$  are different or  $\lim_{i \rightarrow \infty} \mathcal{L}^i \cap T$  is unique over all sequences. If  $x_1^0 = x_2^0 = \dots = x_k^0 = \dots = x_p^0 = x^0$ , for any  $x^0 + \Delta x \in T$ ,  $z^0 + \Delta z > z^0$ . Hence at  $x^0$ ,  $z$  attains a local minimum. By the linearity of the

space and every objective function, a local minimum is global. Hence  $x^0 \in X^{k*}$  (for all  $k$ )  $\subset X^*$ .

If all  $x_k^0$  are different we will find  $\alpha$  such that

$$\alpha' z_k^0 = \text{a constant}$$

$$\alpha' Z^0 = \bar{z}, \text{ a constant vector}$$

$$\text{where } Z^0 = (z_1^0, \dots, z_p^0)$$

$$\text{Let } z_k^0 = (z_k^0, z_k^0, \dots, z_k^0)$$

$$\text{Then } \alpha' (Z^0 - z_k^0) = 0.$$

Transposing, and by Farkas' Lemma

$$(Z^0 - z_k^0)' y \geq 0, 0 \cdot y < 0 \text{ has no solution}$$

$$\text{therefore } (Z^0 - z_k^0)' \alpha = 0, \alpha \geq 0 \text{ has a solution}$$

$$0 = \alpha' (Z^0 - z_k^0) = \alpha' (QX^0 - QX_k^0)$$

$$[\text{where } X^0 = (x_1^0, x_2^0, \dots, x_p^0)$$

$$X_k^0 = (x_k^0, x_k^0, \dots, x_k^0)]$$

$$= \alpha' Q(X^0 - X_k^0)$$

But  $(X^0 - X_k^0)$  is a  $(p-1)$ -tuple of vectors in one supporting hyperplane, so  $v'(X^0 - X_k^0) = 0$  is the equation of the hyperplane,

$v \neq 0$ . Therefore  $Q' \alpha = v \neq 0$ , and  $\alpha \neq 0$ .  $\alpha \geq 0, \alpha \neq 0 \Rightarrow \alpha \geq 0$ .

Now  $\alpha' Z^0 = \bar{z}$  implies that at each  $x_k^0$  is attained the value

$\bar{z} = \alpha' QX$ , a constant on the boundary hyperplane; therefore at

$x_k^0$  is attained a local minimum. By the previous argument it is

also global.  $\alpha' Q$  is a semipositive linear combination of  $c^k$ , hence

$\alpha' Q \in Y$ . Consequently  $x_k^0 \in X^*$ . ///

from the conditions of Theorem 2 falls out that

Corollary 1  $X^*$  is the set of all solutions to (2'')

Proof  $X^* = \{x^* | x^* \text{ minimizes } c^\lambda x = z_\lambda, \text{ for any } c^\lambda \in Y, x \in T\}$ .

By the convexity of  $Y$ ,  $c^\lambda = \sum_k \lambda_k c^k$  where  $c^k$  is an extreme ray of  $Y$  and  $\lambda_k \geq 0$ ,  $\sum \lambda_k > 0$ . But  $c^k$  are the independent rows of  $Q$ , so  $c^\lambda = \sum_{k=1}^p \lambda_k c^k$  where  $\lambda_k = 0$  if  $c^k$  is dependent. So  $x^*$  minimizes  $\lambda Qx$ ,  $\lambda \geq 0$ .  $x^*$  also minimizes  $a\lambda Qx$  where  $a$  is a scalar  $> 0$ . Let  $a = \frac{1}{\sum_{k=1}^p \lambda_k}$ , and  $x^*$  is a solution to (2'').

Hence  $X^*$  is the set of all solutions to (2''). ///

We have now found  $X^*$ , the set of all solutions to (2''). We show that (2'') then is equivalent to (3) and a search for a weakly undominated  $z$  replaced by a simple linear program. Let  $Z$  be the set of all weakly undominated  $z$ .

Corollary 2  $x \in X^*$  if and only if  $z \in Z$ .

Proof Suppose  $x \in X^*$ . Then either

(i) at least one component of  $z$  is minimal. Then  $z$  is weakly undominated; or

(ii)  $\alpha'z$  is minimal for some  $\alpha \geq 0$ . Suppose there exists some  $z^0 < z$ . Then  $\alpha'z^0 < \alpha'z$  which contradicts the hypothesis.

Suppose  $z \in Z$ . Then  $z < z^{00} \notin Z$  and  $x \in X^*$  by Theorem 2. ///

Thus far we have produced a set of solutions  $X^*$ , each element of which is equally satisfactory. The choice will be narrowed by restricting (or attempting to restrict)  $x$  to a preferred set  $S$ . Now that we know that the search for  $x \in S$  can be done by linear programming techniques we must assume:

LPI (Finiteness). Let  $x^1 \in X^*$  be an optimal solution. Then  $x^2 \in X^*$ , another optimal solution, can be produced in a finite number of steps.

#### 4. $X^*$ and the Preferred Set

In Section 2 we have assumed the existence of a preferred set  $S$ . We also

constructed an interaction between DM and LP, whereby LP proposes optimal solutions to a linear program, and DM accepts them or tries to steer them towards  $S$  (assumptions DM1, DM2). In Section 3 we showed that the set of points that LP can propose is indeed that set of points from which DM wishes to select a preferred point. Now what about  $S$ ?

In this paper we choose not to make any assumptions on  $S$ . Such assumptions would involve economic concepts such as insatiability of preferences, convexity of preferences, etc., which are not appropriate here. However, it is important to grasp that although LP has a linear objective, to find some  $x \in X^*$ , DM has no such thing. Now suppose that we have a set  $X^*$  obtained from solving (2') or from solving (3), we will show that a complete solution to (2') exists. Before proving the next theorem we require a lemma:

Lemma  $X^* = \langle X^* \rangle \cap \text{bdry } T$

Comment We proved in Theorem 2 that  $X^* \subset \text{bdry } T$ . We now require that  $X^*$  be simply connected over the boundary i.e. that there be no holes. In Theorem 3 we will have sequences of solutions starting in  $X^*$  and ending in  $X^*$ . These sequences must stay in  $X^*$ .

Proof Let  $x^0 \in \text{bdry } T$ . Then

(A)  $x^0 = \sum \mu_i x_i$ , where  $\{i\}$  is the index set of all extreme points of  $T$  contained in the supporting hyperplane containing  $x$ , and  $\mu_i \geq 0$ ,  $\sum \mu_i = 1$ .

Let  $x \in \langle X^* \rangle$ . Then

(B)  $x^0 = \sum \lambda_j x_j^*$ , where  $\{j\}$  is the index set of all extreme points of  $T$  in  $X^*$ , and  $\lambda_j \geq 0$ ,  $\sum \lambda_j = 1$

But the extreme point representation of any  $x \in T$  is unique, and if  $x^0 \in \langle X^* \rangle \cap \text{bdry } T$  then if  $\mu_k > 0$ ,  $\lambda_k = \mu_k$ . Similarly

if  $\lambda_k > 0$ ,  $\lambda_k = \mu_k$ . Thus the extreme points of (A) are the same as those of (B) and  $x^0 = \sum \mu_i x_i^*$ . But all  $x_i^*$ ,  $i \in \{i\}$  minimize  $\alpha Qx$  for some  $\alpha$ . Hence  $x^0$  also minimizes  $\alpha Qx$ , and  $x^0 \in X^*$ . Suppose  $x^0 \in X^*$ . Then  $x^0 \in \langle X^* \rangle$  and  $x^0 \in \text{bdry } T$ . So  $x^0 \in \langle X^* \rangle \cap \text{bdry } T$  and  $X^* = \langle X^* \rangle \cap \text{bdry } T$ . ///

Theorem 3 will show under what conditions a preferred solution can be found. The effects of pathological circumstances, those in which the linear program and the decision maker are opposed, will become clear.

The lemma has shown that  $X^*$  is a simply connected part of the boundary of  $T$ , intuitively a partly open convex set without an interior. If then we required conditions for a preferred solution to be produced, it would seem immediate that would suffice firstly: convexity of  $S$ , and secondly:  $X^* \cap S \neq \emptyset$ . However Figure 1 shows a case where these conditions are insufficient. Using the decision maker's axioms, and starting from the left of the Figure, almost preferred solutions are encountered twice, at  $x_1$  and at  $x_2$  before a preferred solution is reached at  $x_3$ . Clearly the choice between  $x_1$ ,  $x_2$ , or  $x_3$  is determined by the original trial solution and the direction of search. The situation is pathological because we do not expect a decision maker to be satisfied with a solution that is nonoptimal.

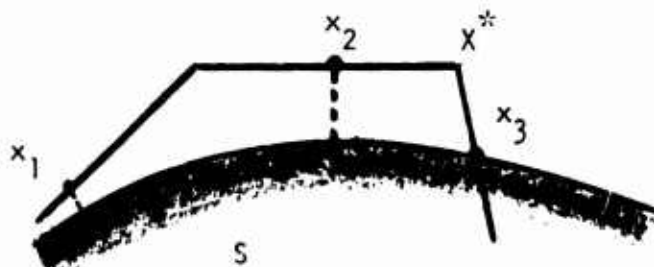


Fig. 1.



Even if the diagram is less pathological (Fig.2) there is still and almost preferred solution at  $x_4$ . The third condition on  $S$  which will guarantee a preferred solution is that the asymptotic cone of  $S$ ,  $A(S)$ , contains  $Y$

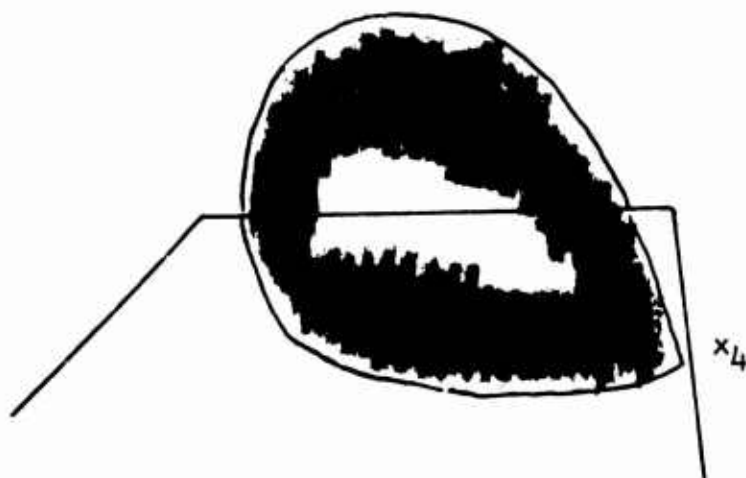


Fig. 2.

This implies insatiability of preferences in the direction of every possible optimizing vector (Fig.3). That it is insufficient merely to have nonempty intersection of the cones is shown in Fig. 4, where there is an almost preferred solution at  $x_5$ . It is evident that these three conditions are not necessary for a preferred solution - merely sufficient.

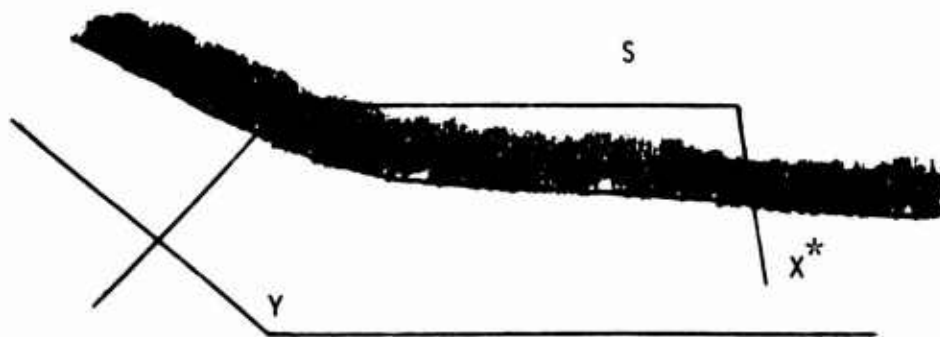


Fig. 3.

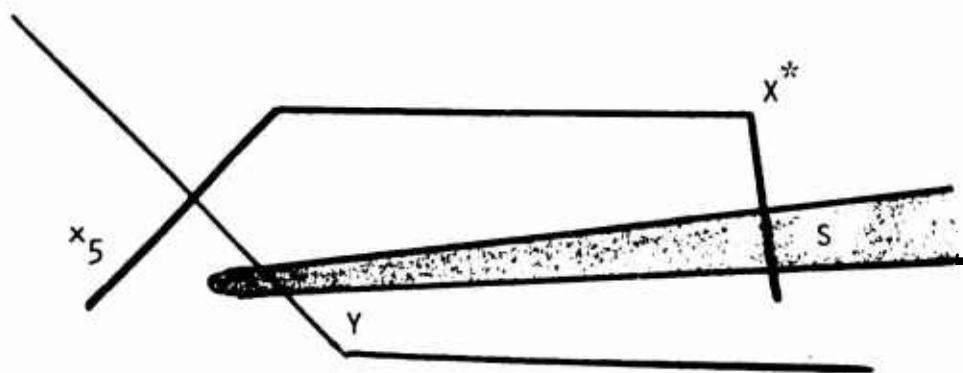


Fig. 4.

Theorem 3

Let  $X^*$  be the set of solutions to (3). Let  $x^0 \in X^*$  be any such solution. Then

- (i) If  $S$  is convex,  $X^* \cap S \neq \emptyset$  and  $Y \subset A(S)$  a preferred  $x$  can be produced.
- (ii) If  $X^* \cap S = \emptyset$  an almost preferred  $x$  can be produced.
- (iii) Otherwise, either a preferred or an almost preferred  $x$  can be produced.

Proof

- (i) Suppose  $x^0 \notin S$ . The vector  $d^i$  can be specified by DM2, and  $k^i > 0$  is a scalar. Let  $x^0 + k^0 d^0 \in \bar{S}$  be the closest point of  $\bar{S}$  to  $x^0$ . If  $x^0 + k^0 d^0 \in X^*$  let  $x^1$  be the closest point of  $X^*$  to  $x^0 + k^0 d^0$ . If  $x^0 + k^0 d^0 \notin X^*$  let  $x^1 = x^0 + k^0 d^0$ . Suppose  $x^1 \notin S$ . By the convexity of  $S$  there exists a unique point  $x^1 + k^1 d^1 \in \bar{S}$  which is the closest point of  $\bar{S}$  to  $x^1$ . If  $x^1 + k^1 d^1 = x^0 + k^0 d^0$  a hyperplane separating  $X^*$  from  $S$  could be defined, for by the lemma  $X^*$  is a simply connected subset of a convex body either wholly contained in  $S$  or closed with respect to  $S$  since  $Y \subset A(S)$ . Hence the points  $x^i + k^i d^i$  are

not in  $T$ . Also  $d(x^1 + k^1 d^1, x^1) < d(x^1, x^0 + k^0 d^0) < d(x^0 + k^0 d^0, x^0)$ . Consequently a sequence of pairs  $(x^i, x^{i-1} + k^{i-1} d^{i-1})$  for  $i \geq 1$  can be defined. Suppose  $x^i \notin S$ . Then the sequence is countable and  $d(x^i, x^{i-1} + k^{i-1} d^{i-1})$  is strictly decreasing. Hence  $\lim_{i \rightarrow \infty} x^i \in S$  and some  $x^{\infty}$  arbitrarily close to the limit is in  $X^* \cap S$ . Let  $x^* = x^i$  if  $x^i \in S$ , or  $x^{\infty}$  otherwise. Then by LPI  $x^*$  can be produced in a finite number of iterations, and by DMI it is recognized as preferred.

(ii) Let  $x^i + k^i d^i$  be defined as in case (i). Construct a similar sequence  $(x^i, x^{i-1} + k^{i-1} d^{i-1})$ . Since there is no convexity assumption on  $S$  and since  $X^* \cap S = \emptyset$ , there may be some  $x^k + k^k d^k = x^{k+1} + k^{k+1} d^{k+1}$ . This implies that  $x^k = x^{k+1} = \dots = x^{k+i} = \dots$ ,  $i \geq 1$ . Let  $x^{k_0} = x^k = x^{k+1} = \dots = x^{k+i} = \dots$ . If  $x^{k_0}$  does not exist then  $d(x^i, x^{i+1} + k^{i+1} d^{i+1})$  is a sequence strictly decreasing to some limit  $\mathcal{L} \geq 0$ . As  $d$  approaches  $\mathcal{L}$ ,  $x^i$  approaches some limit  $x^{\infty}$  since  $X^*$  is bounded. Let  $x^* = x^{k_0}$  or  $x^{\infty}$  whichever is defined. Then by LPI  $x^*$  can be produced in a finite number of steps, and by DMI it is recognized to be not preferred.

(iii) This case follows case (ii) except that either

a)  $x^i \in S$  or

b)  $\mathcal{L} = 0$  may occur since  $X^* \cap S \neq \emptyset$ . Referring to

case (i), if either a) or b) occurs a preferred solution is the outcome; otherwise, as in case (ii) the solution is almost preferred. ///

The converse to Theorem 3 is trivial:

Corollary 3 If a solution to (2') exists then it is some  $x \in X^*$ .

Before attacking the method of solution proposed in Section 5, the reader is cautioned to recall that any  $x \in X^*$  is a solution to (2''), given the appropriate  $\lambda$ . However, the simplex method produces only extreme point solutions! If  $S$  contains an extreme point LP will find it, but if  $S$  contains no extreme point and  $S \cap X^* \neq \emptyset$  the simplex method will have LP oscillating across  $S$  indefinitely. Clearly DM must take some linear combination of extreme points himself in this case.

### 5. The Method of Solution

If we accept the assumptions and conditions on  $T$ ,  $S$ ,  $Q$ , DM, and LP so far discussed we have shown that a solution  $x$  POAP can indeed be found. Without loss of generality we can restrict ourselves to the case in which  $S$  contains an extreme point of  $T$ . If  $S$  does not contain an extreme point DM can terminate when by DM2 he must reverse direction; and if an almost preferred solution is produced DM can terminate when LP can propose no solution in the direction specified.

By solving (2') we are trying to define an optimum in terms of the decision maker's preference, and then to find a function which will yield this optimum, rather than to find an optimizing function and insist that the decision-maker like it. We are granting him intuition and knowledge that he has withheld from the computer, unwittingly or otherwise--after all, it was this intuition and knowledge that made him a decision maker in the first place.

Why then bother with  $\lambda$  at all? A very simple approach could be select

any extreme point  $x^1$ , rejecting it and selecting  $x^2$  and some direction of improvement  $d^1$  such that  $d^1(x^2 - x^1) > 0$ , and continuing until no  $x^{n+1}$  such that  $d^n(x^{n+1} - x^n) > 0$  can be found. Then  $x^n$  is clearly optimal by the convexity of  $T$ .

If we follow this approach, though, we find that the solution to the  $t^{\text{th}}$  sequential problem is a lengthy in computation and as tedious and demanding on the decision maker as the first problem, even though the solutions may be very close. For example; in January a manufacturer of automobiles decides, after investigating many extreme point solution, that it is preferred to produce 700 Rolls Royces and 400 Bentleys. Now in February he notices that management policy remains the same and that all his unit costs and unit profits are in the same proportion. Consequently, he decides that rather than reinvestigating all the extreme points anew he will again select  $x^1 = (700, 400)$ . However, he discovers very quickly that either  $x^1$  is infeasible for February, or that it is not optimal: he could find some better  $x^2$  which is preferred. He realizes that conditions (a) through (e) of Section 2 hold. He sees that the January problem could have been solved by (2') for some  $\lambda^0$ , and if he knew  $\lambda^0$ , (2') using  $\lambda^0$  and  $T$  for February would offer him a solution close to  $x^1$  that would be both feasible and optimal. Further reflection, and an understanding of linear programming, tells him that in general  $\lambda^0$  is a point in a closed convex set  $\Lambda$  and any  $\lambda \in \Lambda$  would have solved the January problem. Also, calling  $\Lambda$  in January  $\Lambda_1$ , there must be some  $\Lambda_2$  for February, and it is immediate that  $\Lambda_1$  and  $\Lambda_2$  are not necessarily equal. It would be folly to select arbitrarily  $\lambda^1$  from  $\Lambda_1$  and to hope that  $\lambda^1$  would also be in  $\Lambda_2$ . But now he recalls condition (e) which tells him that there is some  $\bar{\lambda}$  which is most likely to be in the intersection of many of the  $\Lambda_t$ , and that  $\bar{\lambda}$

is the mean of all successful  $\lambda^t$  to date.

We can find a way to estimate  $\bar{\lambda}$ . Suppose we are solving the first sequential problem. We start with  $\lambda_1^1$ , an arbitrary estimate of  $\lambda$ , and produce an  $x^1$  which minimizes  $\lambda_1^1 Qx$ . Suppose  $x^1$  is not POAP. We modify  $\lambda_1^1$  to  $\lambda_1^2$  and produce an  $x^2$  which minimizes  $\lambda_1^2 Qx$ . . . until  $x^{n_1}$  which minimizes  $\lambda_1^{n_1} Qx$  is POAP. When we solve the second problem clearly the best value of  $\lambda$  with which to begin is  $\lambda_1^{n_1}$ , since all the sequential problems are of the same class. If then the solution to the second problem is attained using  $\lambda_2^{n_2}$ , we will begin the third problem using  $\lambda_3^1 = \frac{1}{2}(\lambda_1^{n_1} + \lambda_2^{n_2})$ .

Similarly the  $t+1^{\text{th}}$  problem would be entered using  $\lambda_{t+1}^1 = \frac{1}{t} \sum_{i=1}^t \lambda^{n_i} = \bar{\lambda}_t$ .

(If we were to waive the assumption of time invariance of  $S$  the mean  $\bar{\lambda}$  might be a weighted mean.) As  $t$  becomes large  $\bar{\lambda}_t$  approaches  $\bar{\lambda}$  in Cesaro limit, and  $\bar{\lambda}$  is the maximum likelihood estimate of  $\lambda$  which will minimize  $\lambda Qx$  and produce a POAP  $x$ .

Consider the  $t+1^{\text{th}}$  sequential problem. The first solution offered  $x^1$ , minimizes  $\bar{\lambda}_t Qx$  over  $T_{t+1}$ . Henceforth the subscript on  $\bar{\lambda}$  will be dropped where no ambiguity can result. If  $x^1$  is preferred the problem is solved and  $\bar{\lambda}$  remains unchanged for the  $t+2^{\text{th}}$  problem. If  $x^1$  is not preferred,  $x^2$  such that  $d^1(x^2 - x^1) > 0$  is to be found.

## 6. The Generation of $x^{i+1}$

The problem to which  $x^i$  is the solution, using  $\lambda^i$ , is

$$\begin{aligned} & \min z \\ \text{Subject to } & Ax = b \\ & x \geq 0, \\ & \lambda^i Qx = z \end{aligned}$$

Rearranging columns so that the first  $m$  columns are the basis  $B_i$  we have

$$\begin{bmatrix} B_i & C_i \\ \lambda^i Q_{B_i} & \lambda^i Q_{C_i} \end{bmatrix} \begin{bmatrix} x_{B_i} \\ x_{C_i} \end{bmatrix} = \begin{bmatrix} b \\ z \end{bmatrix}$$

where subscript  $B$  corresponds to basic variables and subscript  $C$  corresponds to nonbasic variables. The solution  $x^i$  is found by premultiplying both sides by

$$\begin{bmatrix} B_i^{-1} & 0 \\ -\lambda^i Q_{B_i} B_i^{-1} & 1 \end{bmatrix} \quad \text{yielding}$$

$$(7) \quad \begin{bmatrix} 1 & B_i^{-1} C_i \\ 0 & \lambda^i Q_{C_i} - \lambda^i Q_{B_i} B_i^{-1} C_i \end{bmatrix} \begin{bmatrix} x_{B_i} \\ x_{C_i} \end{bmatrix} = \begin{bmatrix} B_i^{-1} b \\ z - \lambda^i Q_{B_i} B_i^{-1} b \end{bmatrix}$$

$$x^i = (B_i^{-1} b, 0).$$

Now suppose that DM specifies direction vector  $d = (d_{B_i}, d_{C_i})$ . That is to say that  $d(x^{i+1} - x^i) > 0$  can be considered an additional constraint to the problem. Of course, if we were to try to resolve the problem using this constraint we would succeed merely in shifting  $x^i$  an infinitesimal distance in the required direction; but we will use the constraint to determine which variables can enter the basis  $(B_{i+1})$  for  $x^{i+1}$ .

There is no further need to maintain the iteration affix  $(i)$  everywhere, since henceforth the analysis will be based on solution  $x^i$ . We rewrite (7):

$$(7') \quad \begin{bmatrix} 1 & B^{-1} C \\ 0 & \lambda(Q_C - Q_B B^{-1} C) \end{bmatrix} \begin{bmatrix} x_B \\ x_C \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ z - \lambda Q_B B^{-1} b \end{bmatrix}$$

Theorem 4 If a solution  $x^{i+1}$  exists  $x^e$  will enter the basis  $(B_{i+1})$  for  $x^{i+1}$  only if  $d_C^e - d_B B^{-1} C^e > 0$ , where  $e$  denotes a column of  $C$ .

Proof Rewrite  $d(x^{i+1} - x^i) > 0$

as  $dx^{i+1} > dx^i$ .

we have  $x^i = (B^{-1}b, 0)$ ; yielding

$dx - s = d_B B^{-1}b + \epsilon$  ( $\epsilon > 0$ ), where  $s$  is a nonnegative slack variable.

Add the constraint into (7'), yielding

$$(8) \quad \begin{bmatrix} I & B^{-1}C & 0 \\ d_B & d_C & -1 \\ 0 & \lambda(Q_C - Q_B B^{-1}C) & 0 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ s \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ d_B B^{-1}b + \epsilon \\ z - \lambda Q_B B^{-1}b \end{bmatrix}$$

Reducing (8) to canonical form by premultiplying both sides by

$$\begin{bmatrix} I & 0 & 0 \\ -d_B & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{we have}$$

$$(9) \quad \begin{bmatrix} I & B^{-1}C & 0 \\ 0 & d_C - d_B B^{-1}C & -1 \\ 0 & \lambda(Q_C - Q_B B^{-1}C) & 0 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ s \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ \epsilon \\ z - \lambda Q_B B^{-1}b \end{bmatrix}$$

The  $m+1$  row of the system (second row of the matrix (9)) is infeasible since the value of the nonnegative slack  $s$  is  $-\epsilon$ , ( $\epsilon > 0$ ). To restore feasibility  $x^e$  must enter the basis with positive value; hence  $d_C^e - d_B B^{-1}C^e > 0$ . ///



Corollary 4 If  $d_C^e - d_B B^{-1} C^e \leq 0$  for all  $e=m+1, \dots, n$  then no feasible  $x^{i+1}$  can be found that will minimize (2') subject to  $d(x^{i+1} - x^i) > 0$ .

So corollary 4 can be the criterion by which the decision maker determines  $x$  to be almost preferred and terminates.

Now suppose that some  $x^e$  can be found to enter  $B_{i+1}$ . As  $\epsilon$  increases so does  $x^e$  until some  $x^{\ell}, \ell=1, \dots, m$ , is reduced to zero and is driven out of the basis.

$x^i$  is the solution to (2'') using a given  $\lambda^i$ . Since we are going to modify  $\lambda^i$  to produce our next solution  $x^{i+1}$ , we must have some properties of  $\lambda$ .

Let  $\Lambda = \{\lambda \mid \lambda_k \geq 0, \sum_{k=1}^p \lambda_k = 1\}$

Let  $\lambda_\nu$  be some value of  $\lambda$  such that  $x_\mu$  minimizes  $\lambda_\nu Qx$ ,  $x_\mu \in X^*$ .

Let  $\Lambda_\nu = \{\lambda \mid x_\mu \text{ minimizes } \lambda Qx, x_\mu \in X^*\}$ ;  $\lambda_\nu \in \Lambda_\nu$ .

Theorem 5 The number of distinct  $\Lambda_\nu$  is finite.  $\Lambda_\nu$  is closed and convex.

$\Lambda = \bigcup_{\nu} \Lambda_\nu$ . No  $\Lambda_\nu$  is disjoint.

Proof Suppose  $x_\mu$  is an extreme point of  $T$ . The number of extreme points is finite. Suppose  $x_\mu$  is not an extreme point, i.e.,  $x_\mu$  is a linear combination of extreme points. From the proof of Theorem 2, given a linear combination of extreme points  $x_\mu$  supporting  $T$ ,  $(\sum_{\mu} \beta_{\mu} x_{\mu}, \beta_{\mu} \geq 0, \sum \beta_{\mu} = 1)$ , the set of  $\alpha$  as defined in Theorem 2 is independent of the positive weights  $(\beta_{\mu})$  in the combination. The number of combinations of  $x_{\mu}$  with positive  $\beta_{\mu}$  is finite.  $\lambda = \frac{\alpha}{\|\alpha\|}$  yields the same minimizer as  $\alpha$ . Suppose

$\lambda_1 Qx$  is minimized by  $x_\mu$  and  $\lambda_2 Qx$  is also minimized by  $x_\mu$ .  
 Let  $\lambda_1$  and  $\lambda_2 \in \Lambda_\nu$ ,  $\lambda_1 Q = c_1$ ,  $\lambda_2 Q = c_2$ . It is a well-known  
 result that  $[\alpha c_1 + (1-\alpha) c_2]x$  is also minimized by  $x_\mu$ , where  
 $0 \leq \alpha \leq 1$ . Hence  $[\alpha \lambda_1 + (1-\alpha) \lambda_2] Qx$  is minimized by  $x_\mu$  and  
 $\Lambda_\nu$  is convex. Suppose the sequence  $\lambda_1 Qx, \lambda_2 Qx, \dots, \lambda_t Qx, \dots$   
 is minimized by  $x_\mu$ , where  $\lambda_1, \lambda_2, \dots, \lambda_t, \dots \rightarrow \lambda$ ;  $\lambda_1, \lambda_2, \dots,$   
 $\lambda_t, \dots \in \Lambda_\nu$ . Then  $c_1 x, c_2 x, \dots, c_t x, \dots$  is minimized by  $x_\mu$ .  
 If  $c_1, c_2, \dots, c_t, \dots \rightarrow c$  then  $cx$  is minimized by  $x_\mu$  since  
 both  $T$  and  $z = cx$  are closed. Hence  $\Lambda_\nu$  is closed.

From the definitions  $\Lambda_\nu \subset \Lambda$ . Suppose there exists some  
 $\lambda^0 \in \Lambda \cap (\cup_\nu \Lambda_\nu)^c$ . Then no  $x_\mu$  is minimized by  $\lambda^0 Qx$ , which  
 contradicts the compactness of  $T$ . Suppose some  $\Lambda_\nu$  is disjoint.  
 Then  $\Lambda \sim \Lambda_\nu$  is not closed; but the union of closed sets is closed.///

Suppose that  $\lambda^i$  is an element of  $\Lambda_\nu$ . We must find some  $\lambda \in \Lambda_\nu$ , which  
 will generate  $x^{i+1}$  subject to the conditions of Theorem 4. We know that  
 $\bar{\lambda}$  is the most likely value to use to achieve our aim, but we also know that  
 $\bar{\lambda}$  will not work because we have already tried it. Consequently, we try some  
 $\lambda \in \Lambda_\nu$ , such that  $||\lambda - \bar{\lambda}||$  is minimized, considering only those  $\lambda$  which  
 generate  $x^{i+1}$  subject to the constraints of Theorem 4.

Let us formalize the conditions on the new  $\lambda$ . Let  $E$  be the set of  
 columns of  $C$  satisfying the conditions of Theorem 4. For notational simplicity  
 we drop the superscript of  $\lambda$ ; where no affix exists we refer only to the new  
 $\lambda$  that we are seeking. Suppose that we can generate  $x^i$  using  $\lambda \in \Lambda_\nu$ . Then  
 the canonical set of equations is still (7'), although the value of  $\lambda$  is not  
 identical. Let the value of  $\lambda^i$  used to generate  $x$  originally be in  $\Lambda_\nu$ .  
 What can we say from this? Firstly, the solution is  $x^i$  since  $x = (B^{-1}b, 0)$ .

Secondly, since  $x^i$  is optimal  $\lambda(Q_C - Q_B B^{-1}C) \geq 0$ . Thirdly,  $\lambda \in \Lambda_V$ . But from Theorem 5 we know that there exists some  $\lambda \in \Lambda_V \cap \Lambda_{V_1}$ . Consequently, this choice of  $\lambda$  from  $\Lambda_V \cap \Lambda_{V_1}$  is possible and justified. We can say then that

$$(10) \quad \begin{aligned} \lambda(Q_C - Q_B B^{-1}C) &\geq 0 \\ \lambda &\geq 0 \\ \sum_{k=1}^P \lambda_k &= 1 \end{aligned}$$

and also (a) that there is at least one equality in the first row of (10), and further (b) that equality corresponds to a column in  $E$ . In addition we have an objective -- to minimize  $\sum(\lambda_k - \bar{\lambda}_k)^2$ . So we have something that looks like a quadratic program. From the optimality properties of  $x$  it is clear that there is a one to one correspondence between the columns of  $C$  and the rows of (10). It follows that if column  $j$  is in  $E$  then row  $j$  of the first line of (10) is eligible for equality. For simplicity of notation we will write that row  $j$  is in  $E$ .

Let us simplify (10) into a quadratic program by removing the additional constraints, (a) and (b):

$$(11) \quad \begin{aligned} \min \quad & \sum(\lambda_k - \bar{\lambda}_k)^2 \\ \text{subject to} \quad & R\lambda \leq 0 \\ & \lambda \geq 0 \\ & \sum \lambda_k = 1 \\ \text{where} \quad & R = -(Q_C - Q_B B^{-1}C) \end{aligned}$$

By inspection the optimal solution is  $\lambda = \bar{\lambda}$ , the point from which we started. The constraint set has an interesting geometrical interpretation. In  $R^P$  all

solutions  $\lambda$  lie on the intersection of the hyperplane  $\sum \lambda_k = 1$  and the nonnegative orthant. A solution to (10) is found when a point  $\lambda$  lies on one of the admissible feasible  $R\lambda = 0$  (row  $\in E$ ) while remaining as close to  $\bar{\lambda}$  as possible. If no such point can be found then case (ii) of Theorem 3 holds. If  $\lambda$  lies on the axis of the orthant a solution (in general) does not exist there since  $\Lambda_v \cap \Lambda_v$  is not necessary at the boundary of  $\Lambda$ , hence a change of basis in (7) does not occur.

We set up the system of equations to solve (11) by Wolfe's method of quadratic programming, adding slack variables  $\sigma \geq 0$ , multipliers  $\pi$  and complementary slack variables  $v$ , complementary to  $(\lambda, \sigma)$ :

$$(12) \quad \begin{bmatrix} R & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ -2I & 0 & -R' & -I' & I & 0 \\ 0 & 0 & -I' & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \pi \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2\bar{\lambda} \\ 0 \end{bmatrix}$$

The matrix contains  $2(n-m)+p+1$  rows and  $3(n-m)+2p+1$  columns. We have already established that there exists a solution in nonnegative  $\lambda, \sigma$ , unrestricted  $\pi$  and zero  $v$  which is optimal both a fortiori and by the Wolfe optimality criterion of  $(\lambda, \sigma)'v = 0$ . Hence by Gauss-Jordan elimination or simplex phase I we make the first  $n-m+k$  columns basic, and immediately obtain the tableau corresponding to the solution  $\lambda = \bar{\lambda}$ :

$$(13) \quad \begin{array}{ccccccc|c} 1 & & Y_{1,s+1} & \dots & \dots & \dots & Y_{1,s+n-m+p} & B_1^1 \\ & \cdot & \vdots & & & & \vdots & \vdots \\ & \cdot & \vdots & & & & \vdots & B_j^2 \\ & \cdot & \vdots & & & & \vdots & \vdots \\ & & \vdots & & & & \vdots & B_s^3 \\ & & 1 & Y_{s,s+1} & \dots & \dots & Y_{s,s+n-m+p} & B_s^3 \end{array}$$

where  $s=2(n-m)+p+1$

$B^1$  are nonnegative values of  $\lambda$

$B^2$  are nonnegative values of  $\sigma$

$B^3$  are unrestricted values of  $\pi$

### 7. The Computation of $\lambda$

We recall in (11) that it is sufficient for one inequality to be an equation, provided its row  $e \in E$ , to yield a solution for  $\lambda$ . Suppose row  $j \in E$ , and the  $j^{\text{th}}$  inequality of (10) is an equation. Then  $\sigma_j = 0$  and  $v_{p+j}$  enters the basis. From the last row of (12):

$$\pi_j = v_{p+j}$$

$\sigma_j$  and  $v_{p+j}$  cannot simultaneously be in the basis hence the pivot operation entering  $v$  and expelling  $\sigma$  must be on  $\gamma_{p+j, s+p+j}$ . From Kuhn-Tucker theory we obtain that the multipliers  $\pi_j$  are nonpositive, hence  $v_{p+j} \leq 0$  (since the solution will be nonoptimal with respect to the Wolfe algorithm). Since  $\sigma_j \geq 0$  before the pivot and  $v_{p+j} \leq 0$  after the pivot

$$\gamma_{p+j, s+p+j} < 0 .$$

From the arithmetic of pivot operations, referring to (13)

$$B_k^1 \rightarrow B_k^1 - \frac{\gamma_{k, s+p+j}}{\gamma_{p+j, s+p+j}} B_{p+j}^2$$

$$B_{p+j'}^2 \rightarrow B_{p+j'}^2 - \frac{\gamma_{p+j', s+p+j}}{\gamma_{p+j, s+p+j}} B_{p+j}^2 \quad \text{where } j' \neq j$$

$$B_{p+j}^2 \rightarrow \frac{B_{p+j}^2}{\gamma_{p+j, s+p+j}}$$

$$\begin{aligned} \text{To maintain feasibility} \quad & 0 \leq B_k^1 \leq 1 \\ \text{and} \quad & 0 \leq B_k^2 \end{aligned}$$

We have thus found the conditions by which  $\lambda_j$  may leave the basis.

The steps are:

- 1) Identify  $j \in E$
- 2) Eliminate  $j \rightarrow Y_{p+j, s+p+j} \geq 0$
- 3) Eliminate  $j \rightarrow B_k^1 < 0$  or  $B_k^1 > 1$
- 4) Eliminate  $j \rightarrow B_{j'}^2 < 0, j' \neq j$ .

Those that survive the test are feasible and we wish to minimize the deviation from  $\bar{\lambda}$ . It is sufficient to minimize  $\sum |\lambda_k - \bar{\lambda}_k|$  since the expression is monotone with respect to  $\sum (\lambda_k - \bar{\lambda}_k)^2$ .

We have found values of  $\lambda$ , if they exist, which will make a new basis in (11) as desired as the current basis. In order to precipitate the change we replace  $\lambda$  by  $\lambda + \epsilon(\lambda - \lambda^i)$  where  $\epsilon$  is small and positive. Clearly  $\epsilon(\lambda - \lambda^i)$  sums to zero. Now suppose another iteration is required.  $\lambda$  computed on the first iteration becomes  $\lambda^{i+1}$  and a new value of  $\lambda$  is desired.  $R$  will of course be different since it is derived from the  $x^i$ . We follow the exact procedure as in the first iteration, forcing a basic solution in the first  $n-m+p$  columns. This will not be optimal since  $\bar{\lambda}$  is excluded from the constraint set. Consequently there will be at least one value of  $\sigma < 0$ . Again using Theorem 4 we find that value of  $v$ , if any, which satisfies the conditions for pivoting.

## 8. Termination

There are three ways of terminating the procedure.

Case I. The decision maker accepts  $x$  produced by minimizing  $\lambda^n Qx$ . Then

$$\bar{\lambda}_{t+1} = (t\bar{\lambda}_t + \lambda_{t+1}^n)/(t+1)$$

Case II. The program can find no  $v$  to bring into the basis which satisfies the pivoting rules. Then case ii of Theorem 3 applies and the last  $x$  is the best that can be provided.

$$\bar{\lambda}_{t+1} = (t\bar{\lambda}_t + \lambda_{t+1}^n)/(t+1)$$

as in case I.

Case III. The decision maker causes cycling between a number  $\geq 2$  of solutions. This implies that  $S$  does not contain an extreme point and the decision maker must interpolate points of the cycle  $\xi$ .

$$\bar{\lambda}_{t+1} = (t\bar{\lambda}_t + \sum_{\xi} \lambda_{t+1}^n / ||\xi||)/(t+1) .$$

9. Numerical Example see figure 5.

A very simple example is illustrated:

A constraint set  $T_t \subset C_t \subset \mathcal{C}$

$$-x_1 + 2x_2 \leq 4$$

$$x_1 + 1/2x_2 \leq 6$$

$$2x_1 - x_2 \leq 4$$

yielding a minimum at  $(4,4)$  under

$$c^1 x = -x_1$$

$$c^2 x = -x_2$$

$$\lambda^1 = \bar{\lambda} = (1/2, 1/2)$$

The  $t+1$  sequential problem contains  $T_{t+1} \subset C_{t+1}$  :

$$-x_1 + 2x_2 \leq 4$$

$$x_1 + 2x_2 \leq 6$$

$$2x_1 - x_2 \leq 4$$

The optimal solution is:

$$\begin{bmatrix} 1 & 0 & 0 & 1/5 & 2/5 \\ 0 & 1 & 0 & 2/5 & -1/5 \\ 0 & 0 & 1 & -3/5 & 4/5 \\ 0 & 0 & 0 & 3/10 & 1/10 \end{bmatrix} x = \begin{bmatrix} 14/5 \\ 8/5 \\ 18/5 \\ 11/5 \end{bmatrix}$$

producing  $x = (14/5, 8/5)$  which is not in  $S$ . We wish to increase at least  $x_2$  and write  $\Delta x = x_2 > 8/5$ .

We compute  $d_C^e - d_B B^{-1} C^e$  for  $x_4$  and  $x_5$ .

$$d^4 - d_B B^{-1} C^4 = 0 - [0 \ 1 \ 0] \begin{bmatrix} 1/5 \\ 2/5 \\ -3/5 \end{bmatrix} = -2/5$$

$$d^5 - d_B B^{-1} C^5 = 0 - [0 \ 1 \ 0] \begin{bmatrix} 2/5 \\ -1/5 \\ 4/5 \end{bmatrix} = 1/5$$

Only  $x_5$  is permitted to enter the basis.

We calculate  $Q_B B^{-1} C - Q_C$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \\ -3/5 & 4/5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$



We add slacks  $\sigma \geq 0$  yielding (11)

$$\min (\lambda_1 - \frac{1}{2})^2 + (\lambda_2 - \frac{1}{2})^2$$

$$\text{subject to } \begin{bmatrix} -1/5 & -2/5 & 1 & 0 \\ -2/5 & 1/5 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We have seen that only  $x_5$  can enter the next basis.  $x_5$  is the second column of  $C$ , so  $E = \{2\}$ . We reach the tableau (13); showing  $\gamma_{ij}$  and the RHS:

$$\begin{bmatrix} 1 & -1/4 & 1/4 & 1/20 & -3/20 \\ . & 1/4 & -1/4 & -1/20 & 3/20 \\ . & 1/20 & -1/20 & -1/100 & 3/100 \\ . & -3/20 & 3/20 & 3/100 & -9/100 \\ . & 0 & 0 & -1 & 0 \\ . & 0 & 0 & 0 & -1 \\ . & -1 & -1/2 & -3/10 & -1/10 \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \\ \pi \\ v \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 3/10 \\ 1/10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$p=2$ ,  $j=2$  so we inspect  $\gamma_{4,11} = -9/100$

So far a pivot is feasible.

$$B^1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} -3/20 \\ 3/20 \end{bmatrix} \left( \frac{1/10}{-9/100} \right) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \frac{10}{9} \begin{bmatrix} -3/20 \\ 3/20 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$B^2 = 3/10 - (3/100)(-10/9) = 1/3$$

$$\text{So } \lambda^2 = (1/3 - \epsilon/6, 2/3 + \epsilon/6)$$

We re-enter the primal problem with objective  $\min ((-1/3 + \epsilon/6)x_1 + (-2/3 - \epsilon/6)x_2)$  and obtain  $x = (1, 5/2)\epsilon S$ .

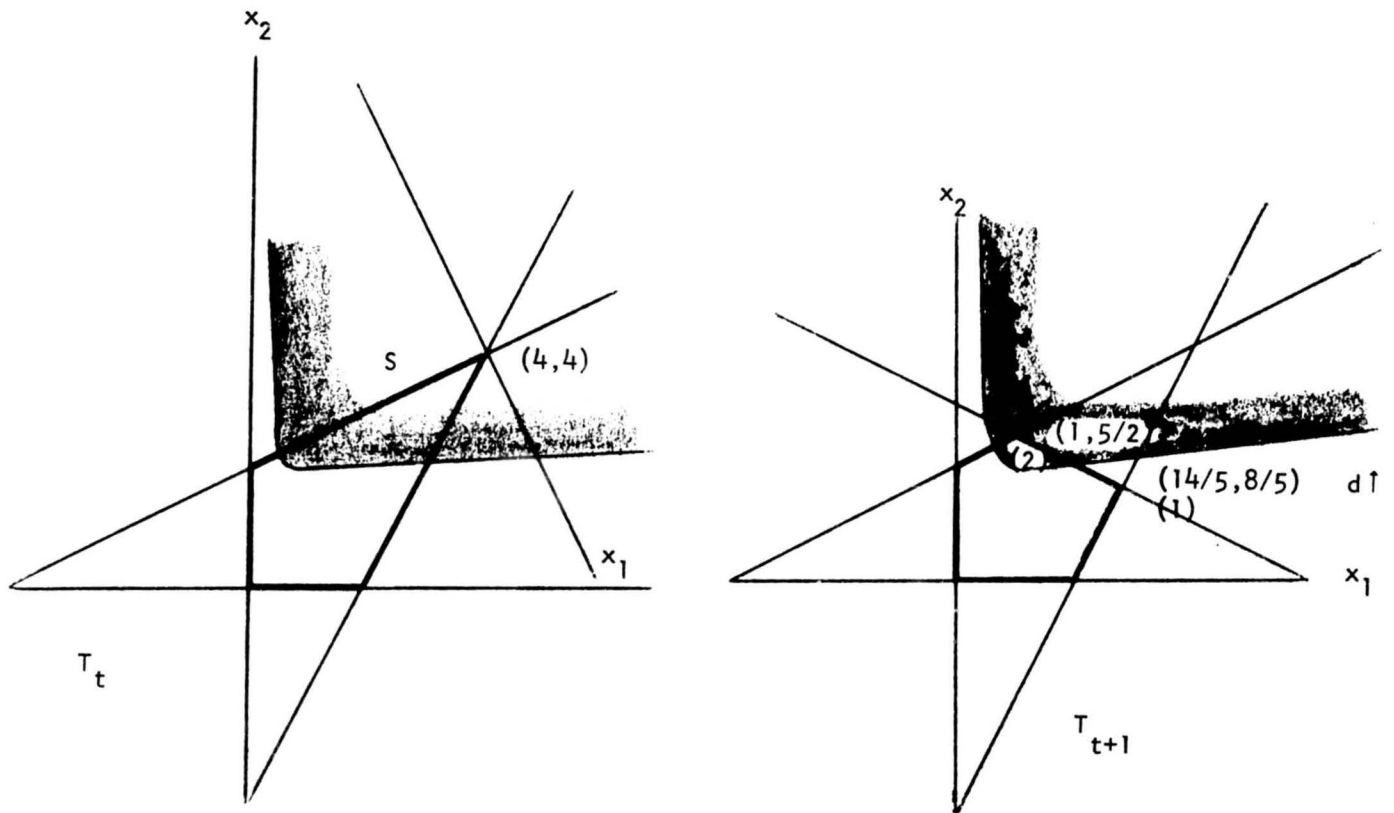


Fig. 5.

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1 ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley		2a REPORT SECURITY CLASSIFICATION Unclassified	
		2b GROUP	
3 REPORT TITLE MULTI OBJECTIVE LINEAR PROGRAMMING			
4 DESCRIPTIVE NOTES (Type of report and inclusive dates) Research Report			
5 AUTHOR(S) (Last name, first name, initial) Savir, David			
6 REPORT DATE November 1966		7a TOTAL NO OF PAGES 36	7b. NO OF REFS 2
8a CONTRACT OR GRANT NO Nonr-222(83)		9a ORIGINATOR'S REPORT NUMBER(S) ORC 66-21	
b PROJECT NO NR 047 033			
c Research Project No. RR 003-07-01		9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d			
10 AVAILABILITY/LIMITATION NOTICES Available without limitation on dissemination			
11 SUPPLEMENTARY NOTES Also supported by the National Science Foundation under Grant GP-4593		12 SPONSORING MILITARY ACTIVITY Mathematical Science Division	
13 ABSTRACT <p>Let us consider a linear program with several objective functions. The traditional approach has been either to "trade off" by weighting each function, or if a "trade-off" vector cannot be provided to ignore all but the most significant. We are interested in classes of programs whose members possess some common characteristics. Examples are sequences of production, refining, inventory problems over time at one installation. If sufficient conditions exist an estimate of a "trade-off" vector can be made. This estimate improves over the sequence.</p> <p>A set <math>X^*</math> exists which contains the solutions obtained by optimizing with respect to all nonnegative combinations of objective functions. A decision maker is not indifferent to these solutions but can characterize preferred solutions. A method is presented whereby he can direct a finite sequence of solutions, <math>(x_i)</math>, over <math>X^*</math> towards a preferred solution. As the estimate of the "trade-off" vector improves, the expected length of the sequence <math>(x_i)</math> diminishes, and the efficiency of solution increases.</p>			

14	KEY WORDS	LINK A		LINK B		LINK C	
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	<p>Multi Objective Functions Linear Programming Semi-Positive Linear Transformation</p>						

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