THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
DEPARTMENT OF NUCLEAR ENGINEERING

Technical Report

Collisions in Ionized Gases

JORDAN D. LEWIS

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ABSTRACT

This dissertation is a study of the description and effects of particle interactions in ionized gases. The principal results are: (i) An expansion theorem for the linearized Fokker-Planck collision operator for each component of a two-component fully ionized gas, and (ii) A description of photon scattering from a partially ionized gas.

It is shown that the Fokker-Planck collision operator generates a complete, continuous set of velocity-space eigenfunctions, for which high-speed asymptotic forms are found. Since the set is continuous, the expansion formula has the form of a generalized Fourier integral.

The effect of neutral atoms on the spectrum of photons scattered by electrons in a partially ionized gas is shown to be primarily a reduction in height and increase in width of the two electron-plasma "wings." The scattered photon spectrum is described for several characteristic cases.

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I. INTRODUCTION

The purpose of this dissertation is to investigate certain aspects of the kinetic behavior of ionized gases. The emphasis here is primarily on the description and effects of particle interactions. Two principal results are obtained. The first, developed in Chapter III, is an expansion theorem for the linearized Fokker Planck collision operator for a two component fully ionized gas. The second, developed in Chapter IV, is a description of photon scattering from a partially ionized gas.

The function of Chapter I is twofold: it presents a brief discussion of the present state of the art in the treatment of classical many-particle systems, thereby setting the stage for the rest of the work; in addition it contains an outline of this work designed to guide the reader. Chapter II contains a survey of some of the more pertinent results of plasma kinetic theory.

The reader interested primarily in the results of this dissertation may choose to skip the first two chapters and proceed directly to Chapters III and IV, which are essentially self-contained.

1. The Many-Particle Problem

The theory of classical many-particle systems may be studied from three points of view. One may begin with the macroscopic, or fluid, equations with parameters such as density, mass velocity, and temperature as independent variables, and involving various transport

coefficients, e.g., viscosity, heat conductivity, etc. Examples are the Euler and Navier-Stokes equations.[1] An altogether different approach involves the use of more fundamental and general microscopic formalisms. On the one hand, one may work with equations describing the evolution of one particle distribution functions, the well-known Boltzmann equation[2] being a prime example. On the other hand, one may employ equations relating one-, two-, etc. particle distribution functions such as the hierarchy of equations derived from the Liouville equation,[3] which latter treats the evolution of the distribution function for all N particles in the system.

The fluid equations are generally considered adequate for treating wide classes of problems in gas dynamics. In this range the microscopic theory would not yield significantly different results. In fact, subject to certain conditions which define their range of applicability, the fluid equations, together with explicit formulas for the various transport coefficients entering into them, are derivable from the microscopic theory.[2,4]

There are, however, many important situations in which the macroscopic theory does not give a correct description. In general, this occurs when the length or time scales characterizing phenomena of interest are not long as compared with the scales on which the microscopic quantities fluctuate. Examples are the propagation of high-frequency or short wavelength waves, and behavior near boundaries. In these cases one must properly begin with the microscopic equations.

The solution of the Boltzmann equation (or, more generally, any equation involving only the single particle distribution function) is, in general, a matter of considerable difficulty even in cases

corresponding to the physically simplest situations. Significant progress has been confined almost entirely to the study of two limiting cases in which two different approximation procedures can be applied. A criterion for the range of validity of the approximate methods is provided by the comparison of a characteristic time τ or length L for the relevant process with the average time τ_c or mean free path L_c between particle encounters.

For high densities ($\tau \gg \tau_c$ or $L \gg L_c$) the Chapman-Enskog theory[4,5] may be used. The first approximation of the theory consists in assuming collisions to be sufficiently frequent to maintain a local thermodynamic equilibrium. The next approximation corrects the distribution function by terms proportional to gradients in temperature T, flow velocity $\underset{\sim}{q}$, and density n; this corresponds to the fluid equations with transport coefficients for heat conduction, viscosity, and diffusion. This high density region is in fact the range in which the fluid equations provide an adequate description. Higher approximations of the Chapman-Enskog theory lead to correction terms proportional to higher derivatives of $T, \underset{\sim}{q}, n$. The successive approximations of the Chapman-Enskog theory correspond[4] to an expansion of the distribution function in powers of the mean free path L_c. For example if we consider sound waves with wavelength $L \gg L_c$, the first and second approximations are already sufficient to give all significant features of the process. When L becomes comparable to L_c however, it is necessary to go to the third and even higher approximations to obtain adequate results; the third approximation already involves formidable labor and has been used to solve only the simplest problems.[3] Consideration of higher approximations is, in any case, of doubtful value since the

entire procedure breaks down in just the range where the contributions from these higher-order terms becomes important. A different approach, using expansions in terms of Hermite polynomials in velocity space, has been given by Grad.[6] He uses some moments of low order in addition to the usual ones representing n, $\underline{q}$ and T. The procedure involves a gain in simplicity over the Chapman-Enskog theory but is still quite complicated. In any event, it is basically a high-density theory.

The opposite limiting case of low densities ($\tau \ll \tau_c$ or $L \ll L_c$) has been studied using iterative schemes beginning with the solution of the "collisionless" equation.[7,8] In the case of ionized gases immersed in strong electromagnetic fields, such theories have been used extensively.[7] As with the approximation schemes employed in the high-density case, these iterative procedures become unwieldy if more than one iteration is necessary.

It would naturally be very desirable to have a method capable of treating the microscopic equations over the whole range from low to high densities. Unfortunately the describing equations are generally non-linear, and even when linearized are extremely intractable, a prime source of difficulty being the term representing interparticle collisions. Relatively little work has been done in this intermediate density region, often referred to as the kinetic regime. Attention has generally focused on mathematical properties of the collision operator,[10] on test particle treatments,[5] or on a numerical solution of the kinetic equation in a few simplified situations.[11] A notable exception is in the work of Chang and Uhlenbeck.[12,13] These authors treated the propagation of small amplitude sound waves in a monatomic gas composed of atoms interacting via an inverse-fifth power force law, i.e., Maxwell

molecules. They were able to show that the linearized Boltzmann collision operator for this case generated a complete set of velocity-space eigenfunctions. Upon expanding the perturbed distribution function in terms of these eigenfunctions, they obtained from the linearized Boltzmann transport equation an infinite set of coupled algebraic equations which they solved by successive approximation. Their results were in quite good agreement with experimental observations on collections of neutral atoms.

In view of the difficulties involved in solving the microscopic equations in the kinetic region, considerable interest has recently been focused on the mathematical properties of the terms representing collision effects. Motivation in this direction has been based in part on the feeling that a knowledge of the spectral properties of the (linearized) collision operator would lend insight into the kinetic behavior of the system.[10,14] Grad[10] has considered the linearized Boltzmann collision operator for particles interacting via the general inverse power force law,

$$F = K/r^{s}$$

where K is a constant and r is the interparticle separation. Grad found that in order to obtain mathematical results it was necessary to assume the interparticle force extended over a finite range; i.e., angular integrations in the collision integral were truncated at small deflections. On the basis of this assumption he was able to show that the spectrum consists of two parts, a discrete spectrum and a continuous spectrum. The latter is bounded away from zero for "hard" potentials ($s > 5$), and approaches zero for "soft" potentials ($s < 5$). For the special case of the Maxwell molecule ($s = 5$) there is only a

discrete spectrum. For $s < 3$ Grad was unable to find the spectrum. Recently Ferziger[15] has used Grad's results for inverse power-law molecules to show that the linear Boltzmann collision operator generates a complete set of eigenfunctions. The form of the eigenfunctions was not, however, obtained. In any event, the spectral and completeness properties were not obtained for the Coulomb potential ($s = 2$).

A major problem in plasma physics is that of determining the properties of an isolated hot plasma; any material probe introduces impurities, while for a fully ionized plasma the emitted radiation is only moderately informative, any line structure arising from undesirable impurities.

The interaction of an incident beam of radiation with a plasma has proved to be a useful method for determining the electron density; since the plasma acts roughly as a dielectric with coefficient

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}, \qquad \omega_{pe}^2 = 4\pi n e^2/m,$$

transmission is cut off below the plasma frequency. Radiation above the plasma frequency may also be used as a plasma probe, and several experiments have used the modificiation in phase velocity produced by the dielectric coefficient ϵ as a measure of electron density.[16]

It has been known for some time that the scattering of photons or material particles from a system of interacting particles yields detailed information on the structure of the scattering system.[17] With the advent of intense light sources such as pulsed ruby lasers, considerable attention has been given to the scattering of photons by free electrons in ionized gases. Several authors have presented analyses of this phenomenon, usually basing their descriptions on

semi-intuitive derivations, and employing collisionless kinetic theories. Notable among these are Salpeter,[18] and Rosenbluth and Rostoker.[19] Shortly thereafter considerations of relativistic effects[20] and nonlinear scattering[21] appeared.

The first experimental observation of photon scattering from an ionized gas was reported by Bowles,[22] who observed the scattering of a radar beam from the ionosphere. More recently many workers have reported the measurement of optical photon spectra produced by scattering from ionized gases in the laboratory. The observations are generally in remarkable qualitative and quantitative agreement with theoretical predictions, which is a rarity in plasma physics. An illustration of this close agreement is given in the recently reported work of Anderson.[23]

Provided the photon wavelength λ is of the order of the Debye length λ_D or larger, the scattered spectrum is characterized by a narrow central peak located at the incident frequency, and by two symmetrically placed satellite peaks separated from the central peak by $\Delta\omega \simeq \omega_{pe}$, the electron plasma frequency. The central peak reflects the strong coupling of the electrons to the ions characteristic of long wavelength plasma phenomena, while the satellites are attributed to the resonant scattering of photons from longitudinal electron plasma oscillations.[18,19] As the photon wavelength becomes large in comparison with the Debye length, the satellites become narrower and rapidly increase in height. This has been attributed[19] to a decrease in the effect of Landau damping on long wavelength plasma oscillations.

Recently Ron, Dawson, and Oberman[24] and Fante[25] have computed

the scattered photon spectrum for a fully ionized gas including the effect of collisions. They find the difference between their results and the collisionless treatments to be very small, of order Λ^{-1}, where $\Lambda = n\lambda_D^3$ is generally very large; in fact Λ is generally assumed large for the various theoretical models to be valid.

In recent years considerable effort has been expended on the production and diagnosis of gases that are only partially ionized. Examples range from low temperature gas discharges to relatively high temperature ($\sim 30ev$) and high density plasmas generated by the laser bombardment of solids. Since photon scattering has been proven to be a most useful tool in the diagnostics of fully ionized gases, it is natural to expect this usefulness may be extended to include systems containing significant numbers of neutrals. In addition, since the Thomson scattering cross-section for electrons is several orders of magnitude larger than the ion cross-section or the Rayleigh scattering cross-section for neutrals,[26] we would expect that photon scattering from electrons should be observable even when neutral densities exceed electron densities. The primary difference between fully- and partially-ionized gases in this respect would then be in the effects of neutral atoms on the scattering process, since charge-neutral collision frequencies may often be considerably larger than their Coulomb counterparts.

2. Outline of this Work

The purpose of this dissertation is twofold. In the first part of this work we obtain the spectrum and prove an expansion theorem for the linearized Fokker-Planck collision operator[27] for particles interacting via an inverse-square force law. In the second

part of this work we employ a simple collision model to study the effects of collisions on the spectrum of photons scattered from a partially ionized gas. The plan of this paper is as follows.

In Chapter II we present a brief survey of plasma kinetic theory, with particular emphasis on the description and effects of charged particle interactions. Due to the long range nature of the Coulomb interaction, when an ionized gas is in the kinetic regime, a great number of particles are "colliding" simultaneously. In this case the simple binary collision models employed to treat collections of neutral particles are often deemed to be inadequate. As a consequence many attempts have been made to develop suitable kinetic descriptions for plasmas, and a comparison of the various treatments has often led to confusion. The purpose of this chapter is to compare a few of the better known kinetic models in an attempt to cast some light on their similarities and differences.

In Part 1 of Chapter II we give a very brief phenomenological derivation of the Boltzmann collision integral, and of the Fokker-Planck collision operator for inverse-square law forces. Emphasis is on the difference between short- and long-range interactions.

In Part 2 of Chapter II we briefly discuss the hierarchy of equations generated by the Liouville equation. The effect of correlations between particles is studied by, on the one hand, neglecting correlations altogether and, on the other, retaining two particle correlations. In the first case we obtain the Boltzmann equation for short-range interactions, and the Vlasov[28] equation for long-range interactions. In the second case we obtain a kinetic equation developed by Lenard,[29] Guernsey,[30] and Balescu.[31] With proper assumptions this

equation reduces to the Fokker-Planck equation.

In view of the mathematical difficulties involved in the solution of the various kinetic descriptions, it is often advantageous to replace the more accurate and less manageable collision descriptions by a model that simplifies the solution of the kinetic equations. In Part 3 of Chapter II we consider such a simplified collision model, generally referred to as the Krook model.[32]

The ultimate test of any theory lies in a comparison of the predictions thereof with experimental observation. Due to the scarcity of relevant experimental and theoretical information in plasma physics, it is often instructive to compare the results of the various theories. One hopes, in so doing, to acquire physical insight into both the structure of the theories and the as yet unobserved properties of nature. In the last three parts of Chapter II we make such comparisons for a few illustrative cases.

In Part 4 we review some recent numerical treatments of relaxation to equilibrium in velocity space. In some cases it is possible to compare the Fokker-Planck, Lenard-Guernsey-Balescu, and Krook descriptions. From the information currently available we observe a negligible difference between the predictions of the first two treatments. The limitations of a single parameter Krook-type model are discussed and compared with other results.

In Part 5 of Chapter II we discuss the phenomenon of electron runaway and the contribution of collisions to plasma transport parameters. The failure of the Krook model in the description of runaway is noted, and the results of a computation using the Fokker-Planck description are discussed. The majority of this section is devoted to

a comparison of the different collision descriptions in the computation of the plasma electrical conductivity. For field frequencies small compared to the collision frequency, the Krook, Boltzmann, and Fokker-Planck results are identical. For frequencies above the collision frequency but below the electron plasma frequency ω_{pe}, the Boltzmann and Fokker-Planck results differ only slightly. For frequencies above ω_{pe} these collision descriptions break down; the reasons for this failure are discussed. For high frequencies conductivity computations based on the Vlasov equation and the first two members of the BBGKY hierarchy give similar results, and match the results of the collision description just below ω_{pe}.

In the final section of Chapter II we present a brief review of some recent work on collisional effects in plasma collective behavior. Since relatively little work has been done in this area, only a few comparisions of the different collision descriptions are possible. In general it is found that for wavelengths long compared to the Debye length damping is primarily collisional; for wavelengths of the order of the Debye length or less Landau damping[33] predominates. Moreover, if a plasma is inherently stable, collisions increase the damping of small amplitude oscillations. In contrast if a plasma is unstable, collisions may increase the growth rate of the instability.

In Chapter III we obtain the spectral properties, and develop an expansion theorem for the linear Fokker-Planck collision operator for a two-component fully ionized gas. In Part 1 of this chapter we take advantage of the small electron-ion mass ratio to decouple the equations for each species. For convenience we then concentrate our attention on the ion collision operator and later discuss the extension

of our results to the electron case.

To develop the expansion formula we follow the standard method of assuming solutions to the kinetic equation of the form

$$f(\underset{\sim}{v}, t) = g_\lambda(\underset{\sim}{v}) \exp(-\lambda t) .$$

This reduces the equation to the form

$$L(\underset{\sim}{v})\, g_\lambda(\underset{\sim}{v}) = -\lambda\, g_\lambda(\underset{\sim}{v})$$

where L is a three-dimensional integrodifferential operator. In Part 1 we show that $Re\lambda \geq 0$ and $Im\lambda = 0$ as we would expect physically. In Part 2 we introduce a spherical harmonic expansion which replaces the three dimensional equation by an infinite set of uncoupled equations,

$$L_{nlm}\, g_{nlm} = -\lambda_{nlm}\, g_{nlm}$$

where $L_{nlm}(v)$ is a singular integrodifferential operator. These are then cast into a self-adjoint form in Part 3 by introducing a suitable algebraic transformation on the functions $g_{n'l'm'}(v, \cdot_{nlm})$. With boundary conditions obtained by combining the original kinetic equation with the conservation laws, we proceed to find the spectrum which is continuous and for $l = 0, 1$ consists of all $\lambda_{nlm} \geq 0$, $-l \leq m \leq l$, and for $l \geq 2$ consists of all $\lambda_{nlm} > 0$, $-l \leq m \leq l$.

Although the spectral resolution theorem[34] implies the existence of an expansion theorem for self-adjoint operators, there always remains the task of constructing the expansion explicitly. We turn to this task in Part 4. Since L_{nlm} is singular at $v = 0$ and $v \to \infty$, we temporarily replace the interval $0 \leq v \leq \infty$ by the interval $0 < v_1 \leq v \leq v_2 < \infty$ and use a result of Tamarkin[35] to show that L_{nlm} generates a complete orthonormal set on this interval. To return to the original interval and

thus obtain the desired expansion theorem, we use the above completeness property together with an extension of the theory of singular differential equations.[36,37] This finally yields a continuous, normalizable set of functions $\Psi_{n\ell m}(\underline{v}, \lambda_{n\ell m})$ that is complete with respect to functions square integrable in velocity space. Since the set is continuous, the expansion has the form of a generalized Fourier integral. For $v >> (2\Theta/m)^{1/2}$ we have found asymptotic forms of the expansion functions. Finally, in Parts 5 and 6 the extension to the electron kinetic equation is discussed, as well as certain implications of our results.

In Chapter IV of this dissertation we develop a theory of photon scattering from partially ionized gases. The starting point for the present discussion is the description developed recently by Osborn,[38] which treats photon scattering from a fully ionized gas. The primary concern in this work[38] was with the establishment of a relationship between the observed distribution of scattered photons and the dynamical and statistical characteristics of the scattering plasma. It was assumed that (i) the dynamical variables of the plasma obeyed the classical equations of motion, and that (ii) the plasmas in question were sufficiently highly ionized that the presence of neutrals could be neglected.

In Chapter IV we examine the second assumption described above. Our motivation in this direction is twofold. First, it is not clear a priori when the presence of neutrals will be truly negligible. Further, when their effect is significant, it must then be incorporated in a description of the scattering process. Vineyard,[39] Salpeter,[18] and Feyer[40] have discussed the contribution of neutrals to the plasma

scattering function, but each based his discussion on semi-intuitive arguments and in no case was a quantitative description developed.

It was shown in reference 38 that, neglecting relativistic effects, the cross-section presented by an electron to a photon having frequency ω and direction $\underline{\Omega}$ for the scattering of that photon into $(\omega', \omega' + d\omega')$ and $(\underline{\Omega}', \underline{\Omega}' + d\underline{\Omega}')$ is given by

$$\sigma(\omega, \underline{\Omega}; \omega', \underline{\Omega}') = \frac{\omega'}{\omega} \sigma_T(\theta) S(\underline{\kappa}, \Delta\omega)$$

where σ_T is the Thomson cross-section. The so-called scattering function $S(\underline{\kappa}, \Delta\omega)$ is given by

$$S = \frac{1}{2\pi N^e} \int_{-\infty}^{\infty} dt\, e^{-it\Delta\omega} G^{ee}(\underline{\kappa}, t)$$

where

$$G^{ee} = \langle \bar{\rho}^{e\dagger}(\underline{\kappa}, 0)\, \bar{\rho}^{e}(\underline{\kappa}, t) \rangle_T .$$

The normalization is such that N^e is the total number of electrons in the scattering volume, and the function $\bar{\rho}^e$ is the Fourier-transformed configuration space electron density operator, with transform variable $\underline{\kappa}$.

In Part 1 of Chapter II we briefly review some of the more pertinent aspects of reference 38. Following Osborn,[38] we employ a classical representation of the density operators and develop and discuss various aspects of the classical scattering function. Certain properties of a thermal equilibrium plasma relevant to the computation of the scattering function are also discussed.

In general neutrals may influence the scattered photon spectrum in two ways. The first, and perhaps more obvious, contribution is significant when the number of photons scattered by neutrals into the

frequency range of interest is not small compared with the number scattered by electrons. Secondly, neutrals may bias the scattered spectrum through their influence on the electron density operator. We investigate this latter effect in Part 2 of Chapter IV. We approach the problem along two somewhat different paths. The first is a simple extension of the treatment in ref. 38, to include a neutral species in the description. We show that this approach does not lead to a noticable contribution from neutrals unless exceptionally high neutral densities are present. At this point we take a somewhat different tack, employing the simple Krook model to represent interactions with neutrals, and leading eventually to a modified scattering function displaying significant contributions from neutrals at neutral densities at least five or six orders of magnitude lower than in our earlier treatment.

Part 3 of Chapter IV is devoted to a presentation and discussion of the classical scattering function, with and without the effects of neutrals, for several experimental configurations. Our results are in agreement with earlier predictions[39,18,40] in that the effect of neutrals is to heighten and narrow the central spectral peak, while lowering and broadening the so-called electron plasma wings.

In Part 4 we discuss the implications and limitations of the approximations and assumptions employed in the development of our results: a few recent experiments are cited as illustrations. Extensions of the present theory are suggested where such modifications may lead to significant differences.

II. A BRIEF SURVEY OF PLASMA KINETIC THEORY

1. Phenomenological Kinetic Equations

One of the earliest and still most successful descriptions of a collection of free particles is the famous Boltzmann equation[1,2]

$$\frac{\partial F^A}{\partial t} + \underline{v}_1 \cdot \frac{\partial F^A}{\partial \underline{r}_1} + \frac{1}{m_A} \underline{\mathcal{F}}^A \cdot \frac{\partial F^A}{\partial \underline{v}_1} = \sum_B (\delta F^A/\delta t)_B$$

which expresses how F^A changes in time due to streaming and to encounters with other particles. Here $F^A d^3r_1 d^3v_1$ is the expected number of type A particles in $d^3r_1 d^3v_1$ about $\underline{r}_1, \underline{v}_1$ at time t, and $\underline{\mathcal{F}}^A$ is the force on a particle of type A and $(\delta F^A/\delta t)_B$ represents the time rate of change of F^A produced by inter-particle encounters. For our purposes we assume that only elastic encounters are important.

In the classical theory of non-uniform neutral gases[1,2] $(\delta F^A/\delta t)_B$ is taken to be the net number of particles of type A entering the phase space volume element $d^3r_1\, d^3v_1$ per unit time due to instantaneous binary encounters. Thus if $\underline{g} = \underline{v}_2 - \underline{v}_1$ is the relative velocity of a colliding pair, then the flux of particles of type B having velocity $\underline{v}_2$ incident on any particle of type A having velocity $\underline{v}_1$ is $d^3v_2\, F^B(\underline{r}_1, \underline{v}_2, t)\, |\underline{v}_1 - \underline{v}_2|$.

If $\sigma_{AB}(g, \theta)$ is the differential scattering cross-section, then the number of particles of type A scattered out of the phase space volume element $d^3r_1\, d^3v_1$ into $d\underline{\Omega}$ during dt by collisions with type B particles is

$$d^3r_1 d^3v_1 F^A(\underline{r}_1, \underline{v}_1, t) d^3v_2 F^B(\underline{r}_1, \underline{v}_2, t) g \sigma_{AB}(g,\theta) d\underline{\Omega}\, dt .$$

The number scattered into this same volume in dt is

$$d^3r_1 d^3v_1' F^A(\underline{r}_1, \underline{v}_1', t) d^3v_2' F^B(\underline{r}_1, \underline{v}_2', t) g' \sigma_{AB}(g',\theta) d\underline{\Omega}\, dt .$$

As a consequence of momentum and energy conservation in the collision we have[2]

$$d^3v_1 d^3v_2 |\underline{v}_2 - \underline{v}_1| \sigma_{AB}(g,\theta) = d^3v_1' d^3v_2' |\underline{v}_2' - \underline{v}_1'| \sigma(g',\theta)$$

and hence

$$\left(\frac{\delta F^A}{\delta t}\right)_B d^3r_1 dt =$$

$$\int d^3v_2 \int d\Omega\, \sigma_{AB}(g,\theta)\, g \left[F^A(\underline{v}_1') F^B(\underline{v}_2') - F^A(\underline{v}_1) F^B(\underline{v}_2)\right] d^3r_1 dt,$$

the well-knwon Boltzmann collision integral.

A binary encounter is often described in terms of the impact parameter b, the distance of closest approach if no interaction is present. This latter quantity is related to the differential scattering cross-section through the relation[3]

$$b\, db\, d\phi = \sigma(g,\theta) d\Omega$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$.

The derivation of the Boltzmann collision integral rests on three basic assumptions, and as yet there is considerable debate as to their significance:

(i) The collection of particles is sufficiently rarefied so that only binary collisions need be considered;

(ii) The probability of finding two particles in d^3r about $\underline{r}$ simultaneously is proportional to the product of

their individual distribution functions;

(iii) the force $\underset{\sim}{\mathcal{H}}^A$ is distinct from the interparticle forces, and it does not affect the collision process.

Assumption (i) implies that the interparticle potential is so localized that, for a sufficiently dilute gas, the chances of finding more than two particles within "range" of each other simultaneously is negligible.

While this assumption seems reasonable for dilute gases composed of neutral atoms or molecules, it causes considerable difficulty in the treatment of charged particles. The difficulty arises because electrostatic forces, being proportional only to the inverse square of the distance, permit many particles to be within range of each other at a given time. To see this, note that the effective interparticle potential for a fully ionized gas not far from thermal equilibrium is[4]

$$V^{AB}(r) \simeq \frac{e_A e_B}{r} e^{-r/\lambda_D}$$

where the Debye screening length λ_D is given by[4]

$$\lambda_D^{-2} = \frac{4\pi}{\Theta} \sum_B n_B e_B^2$$

where Θ is the temperature in energy units. It follows that the number of particles interacting simultaneously is roughly

$$n \lambda_D^3 \equiv \Lambda$$

or (for a singly ionized gas)

$$\Lambda = \left[(\Theta / 8\pi e^2) n^{-1/3} \right]^{3/2}$$

which is usually quite large.[4] In contrast, charged particles will suffer large angle deflections only when[3] the impact parameter is of the order of e^2/Θ, the distance of closest approach in a head-on collision.

Since interparticle spacings are $\sim n^{-1/3}$, the fraction of particles making such encounters at any instant is

$$\sim \left(\frac{e^2/9}{n^{-1/3}} \right)^3 \simeq \Lambda^{-2} .$$

Thus the binary collision assumption seems inadequate since it does not appear to account for the many overlapping long-range encounters. This was recognized by Chapman[5,6] in the calculation of transport coefficients for an ionized gas. Chapman found that integrals over the impact parameter diverged for $b \to \infty$ due to the long range nature of the Coulomb force. He overcame this difficulty by cutting off the integration at $b_{max} \sim n^{-1/3}$, and assuming that the resultant force on a particle due to more distant encounters could be represented by an internal electrostatic force in the streaming term of the Boltzmann equation.

Cohen, Spitzer, and Routly[7] sought to overcome Chapman's difficulties by adopting a treatment developed by Chandrasekhar[8,9] in a study of stellar dynamics. Chandrasekhar's work was based on Jeans'[10] demonstration that when particles interact through inverse-square forces, the cumulative effect of the weak deflections resulting from the relatively distant encounters is more important than the effect of occasional large deflections. Chandrasekhar noted the strong similarity between the Brownian motion of a colloidal particle and the motions of particles interacting via inverse-square forces; his treatment is based on a description of Brownian motion due to A. D. Fokker[11] and M. Planck.[12] While Cohen, et al[7] restricted themselves to slightly anisotropic velocity distributions, the general case of arbitrary distribution functions was considered by Rosenbluth, MacDonald, and Judd.[13] We present a brief

derivation, following Chandrasekhar[9] and Rosenbluth, et al.[13]

Assume that there exist time intervals Δt long enough for a particle to suffer a large number of weak deflections but short enough for the net mean square change in velocity, $\langle |\Delta \underline{v}|^2 \rangle$, to be small compared with the mean square velocity. Let $P^A(\underline{v}, \Delta \underline{v})$ denote the probability that in the time interval Δt a particle of type A having velocity $\underline{v}$ undergoes a displacement $\Delta \underline{v}$. Assuming that P^A does not depend explicitly on time, the distribution function for the A^{th} species is then given by[9]

$$F^A(\underline{x}, \underline{v}, t) = \int d^3(\Delta \underline{v})\, F^A(\underline{x}, \underline{v} - \Delta \underline{v}, t - \Delta t)\, P^A(\underline{v} - \Delta \underline{v}, \Delta \underline{v}). \quad (2.1)$$

Since Δt and $\Delta \underline{v}$ are both assumed small, the integrand is expanded in a Taylor's series:

$$F^A(\underline{x}, \underline{v}, t) =$$

$$\int d^3(\Delta \underline{v}) \left[F^A(\underline{x}, \underline{v}, t)\, P^A(\underline{v}; \Delta \underline{v}) - \Delta t \frac{\partial F^A}{\partial t} P^A(\underline{v}; \Delta \underline{v}) - \right.$$

$$- \Delta \underline{v} \cdot \left\{ \frac{\partial F^A}{\partial \underline{v}} P^A(\underline{v}; \Delta \underline{v}) + F^A \frac{\partial}{\partial \underline{v}} P^A(\underline{v}; \Delta \underline{v}) \right\} +$$

$$\left. + \tfrac{1}{2} \Delta \underline{v} \Delta \underline{v} : \left\{ \frac{\partial^2 F^A}{\partial \underline{v} \partial \underline{v}} P^A + 2 \frac{\partial F^A}{\partial \underline{v}} \frac{\partial P^A}{\partial \underline{v}} + F^A \frac{\partial^2 P^A}{\partial \underline{v} \partial \underline{v}} \right\} + \cdots \right]. \quad (2.2)$$

Using the fact that

$$\int d^3(\Delta \underline{v})\, P^A(\underline{v}; \Delta \underline{v}) = 1,$$

equation (2.2) gives for the time rate of change of F^A resulting from the cumulative effect of small deflections

$$\Delta t \left(\frac{\partial F^A}{\partial t} \right)_{coll} = - \frac{\partial}{\partial \underline{v}} \cdot \left[F^A \left\{ \Delta \underline{v} \right\}_A - \frac{1}{2} \frac{\partial}{\partial \underline{v}} \cdot F^A \left\{ \Delta \underline{v}\, \Delta \underline{v} \right\}_A + \cdots \right] \quad (2.3)$$

where

$$\left\{\Delta \underset{\sim}{v}\right\}_A = \int d^3(\Delta \underset{\sim}{v})\, P^A(\underset{\sim}{v};\Delta \underset{\sim}{v})\,\Delta \underset{\sim}{v},$$

$$\left\{\Delta \underset{\sim}{v}\,\Delta \underset{\sim}{v}\right\}_A = \int d^3(\Delta \underset{\sim}{v})\, P^A(\underset{\sim}{v};\Delta \underset{\sim}{v})\,\Delta \underset{\sim}{v}\,\Delta \underset{\sim}{v} \qquad (2.4)$$

and so forth.

Since $\left\{\Delta \underset{\sim}{v}\right\}_A$ represents the mean change in $\Delta \underset{\sim}{v}$ resulting from encounters during Δt, we write

$$\left\{\Delta \underset{\sim}{v}\right\}_A = \sum_B \int d^3 v'\, F^B(\underset{\sim}{v}') \int d\Omega\, \sigma_{AB}(g,\Theta)\, g\, \Delta \underset{\sim}{v}_A\, \Delta t$$

$$\equiv \left\langle \Delta \underset{\sim}{v} \right\rangle_A \Delta t \qquad (2.5)$$

and

$$\left\{\Delta \underset{\sim}{v}\,\Delta \underset{\sim}{v}\right\}_A = \sum_B \int d^3 v'\, F^B(\underset{\sim}{v}') \int d\Omega\, \sigma_{AB}(g,\Omega)\, g\, \Delta \underset{\sim}{v}_A\, \Delta \underset{\sim}{v}_A\, \omega t$$

$$\equiv \left\langle \Delta \underset{\sim}{v}\,\Delta \underset{\sim}{v} \right\rangle_A \Delta t\,, \qquad (2.6)$$

etc. for higher order terms, where $g = |\underset{\sim}{v} - \underset{\sim}{v}'|$ is the magnitude of the relative velocity. Rosenbluth, et al[13] evaluated the Fokker-Planck coefficients (2.5) and (2.6) assuming a two-particle Coulomb cross-section. Since the integrals over Ω diverge logarithmically at large impact parameters, they followed Cohen et al[7] and cut off the integration at $b_{max} \sim \lambda_D$, the Debye screening length. After some algebraic manipulation Rosenbluth et al obtain

$$\left(\frac{\partial F^A}{\partial t}\right)_{coll} = \sum_B \frac{\partial}{\partial \underset{\sim}{v}} \left[\frac{1}{2} \frac{\partial}{\partial \underset{\sim}{v}} \left\{ F^A(\underset{\sim}{v}) \int d^3 v'\, \Gamma_{AB}\, F^B(\underset{\sim}{v}') : \frac{\partial^2 g}{\partial \underset{\sim}{v}\,\partial \underset{\sim}{v}} \right\} - \right.$$

$$\left. - F^A(\underset{\sim}{v}) \frac{m_A + m_B}{m_B} \int d^3 v'\, \Gamma_{AB}\, F^B(\underset{\sim}{v}') \frac{\partial (1/g)}{\partial \underset{\sim}{v}} \right] \qquad (2.7)$$

where

$$\Gamma_{AB} = \frac{4\pi e_A^2 e_B^2}{m_A^2} \ln \frac{\frac{1}{2} \mu_{AB} g^2}{c_A e_B / \lambda_D}$$

and μ_{AB} is the reduced mass. Assuming the logarithm to be a sufficiently slowly varying function of the relative velocity, the approximation is made

$$\Gamma_{AB} \simeq \frac{4\pi e_A^2 e_B^2}{m_A^2} \ln \frac{3\theta}{e_A e_B / \lambda_D}$$

$$\equiv \frac{4\pi e_A^2 e_B^2}{m_A^2} \ln \Lambda_{AB} . \qquad (2.8)$$

(It can be shown[14] that this approximation introduces an error of less than one percent in the determination of transport coefficients based on (2.7).)

In the derivation of (2.7) only those terms porportional to $\ln \Lambda_{AB}$ have been retained from the expansion (2.3) and in the evaluation of the Fokker-Planck coefficients (2.5) and (2.6). All other terms can be shown[13] to be down by a factor $(\ln \Lambda_{AB})^{-1}$. Thus the Fokker-Planck equation may be viewed as an expansion in powers of the ratio of mean kinetic energy to potential energy at a separation of λ_D, and is sound for $\ln \Lambda_{AB}$ sufficiently large, or roughly

$$\theta^3(ev) \gtrsim 10^{-14}\, n\,(cm^{-3})$$

Equation (2.7) may be transformed into a more symmetric form. Since

$$\frac{\partial}{\partial \underline{v}} \int d^3v'\, F^B(\underline{v}') g^{-1} = -\frac{1}{2} \int d^3v'\, F^B(\underline{v}') \frac{\partial}{\partial \underline{v}'} \nabla_{\underline{v}'}^2 g$$

$$= \frac{1}{2} \int d^3v'\, \frac{\partial F^B(\underline{v}')}{\partial \underline{v}'} \cdot \frac{\partial^2 g}{\partial \underline{v} \partial \underline{v}} ,$$

(2.7) can be rewritten in the form

$$\left(\frac{\partial F^A}{\partial t}\right)_{coll} =$$

$$\sum_B \frac{1}{2} \Gamma_{AB} \frac{\partial}{\partial \underline{v}} \int d^3v' \left[F^B(\underline{v}') \frac{\partial F^A}{\partial \underline{v}} - \frac{m^A}{m_B} F^A(\underline{v}) \frac{\partial F^B(\underline{v}')}{\partial \underline{v}'} \right] : \frac{\partial^2 g}{\partial \underline{v} \partial \underline{v}} \qquad (2.9)$$

This form of the Fokker-Planck equation was published by Landau[15] in 1936.

While the above analysis treats the effects of a large number of overlapping small-angle deflections occurring in the time interval Δt, the form of the probability employed in (2.5) and (2.6) still assumes that these small angle deflections are themselves due to binary encounters. Moreover, we have retained the assumption (ii) of the joint probability being proportional to the product of the two singlet probabilities and have thereby ignored possible correlation effects. In addition, the present description explicitly ignores the effects of the relatively infrequent large angle deflections. Both Cohen[7] and Rosenbluth[13] and their co-workers suggested the inclusion of a Boltzmann collision operator for impact parameters below an unspecified critical value. At the same time, they suggested the effect would usually be negligible. Finally, we note in passing that the Fokker-Planck equation as displayed above may be obtained[4] by Taylor expanding the integrand of the Boltzmann collision integral with Coulomb cross-section, cutting off the integrals at $b_{max} = \lambda_D$, and retaining only dominant terms.

Several authors have presented descriptions which take into account the electrostatic properties of the plasma.[16,17,18] The method is to consider a test particle as being subject to local fluctuating electric fields, and then calculating the Fokker-Planck coefficients on this basis. In all cases the results are quite similar, and are tantamount to including a dielectric constant in the functions Γ_{AB} in (2.7). An advantage of this work is that the Debye length enters the description as a natural cutoff distance, without having to be introduced in an <u>ad hoc</u> fashion as before. In contrast, the treatment yields a

divergence at small impact parameters due to an improper handling of close encounters. To overcome this a cutoff is postulated at an impact parameter of the order of the distance of closest approach $\sim e^2/\theta$. Hubbard[19] surmounted this latter difficulty by retaining the entire infinite series in the expansion (2.2).

2. Kintetic Equations Derived From the Liouville Equation

The construction of kinetic theories on the basis of phenomenological considerations naturally raises questions concerning the validity and range of applicability of the various descriptions. Perhaps the most satisfactory scheme for surmounting these difficulties is to begin with the most complete (and intractable) description available, the Liouville equation.

For simplicity we consider a collection of N indistinguishable particles, occupying a volume V , with no external forces. The generalization to a multicomponent system is straightforward but tedious, and will be indicated by reference where appropriate.

We define $F_N(\underset{\sim}{r}_i, \underset{\sim}{v}_i, t)$, $i = 1, 2, \cdots, N$ as the N-particle distribution function such that

$$F_N \prod_{i=1}^{N} d^3r_i d^3v_i$$

is the probability of finding the i^{th} particle in $d^3r_i d^3v_i$ for each of the N particles, all at time t . Clearly

$$\int \prod_{i=1}^{N} d^3r_i d^3v_i \, F_N = 1.$$

According to Liouville's theorem,[20] F_N is governed by the equation

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{N} \dot{\underset{\sim}{r}}_i \cdot \frac{\partial F_N}{\partial \underset{\sim}{r}_i} + \sum_{i=1}^{N} \dot{\underset{\sim}{v}}_i \cdot \frac{\partial F_N}{\partial \underset{\sim}{v}_i} = 0 \qquad (2.10)$$

where

$$\dot{\underline{r}}_i = \underline{v}_i , \qquad \dot{\underline{v}}_i = -\frac{1}{m}\frac{\partial}{\partial \underline{r}_i} V(\underline{r}_i),$$

$$V(\underline{r}_i) = \sum_{j=1, j\neq i}^{N} V^{ij}(|\underline{r}_i - \underline{r}_j|)$$

and V^{ij} is the two-body potential. Thus we have

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{N} \underline{v}_i \cdot \frac{\partial F_N}{\partial \underline{r}_i} - \frac{1}{m}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{\partial V^{ij}}{\partial \underline{r}_i} \cdot \frac{\partial F_N}{\partial \underline{v}_i} = 0. \qquad (2.11)$$

The information contained in the Liouville equation (2.10) or (2.11) is all inclusive but in general is inaccessible because of its complexity. A method for extracting useful information has been developed independently by Bogolyubov, Born, Green, Kirkwood, and Yvon, and is summarized in Montgomery and Tidman[4] or in any good text on statistical mechanics such as de Boer and Uhlenbeck.[21]

To develop the so-called BBGKY hierarchy, we define reduced probability distributions

$$F_s \equiv V^s \int \prod_{i=s+1}^{N} d^3r_i\, d^3v_i\, F_N \quad , \; s < N. \qquad (2.12)$$

Multiplying (2.11) by V^s and integrating as in (2.12) we obtain an expression for F_s which involves F_{s+1}. In particular, for $s = 1$ and $s = 2$ we have

$$\frac{\partial F_1}{\partial t} + \underline{v}_1 \cdot \frac{\partial \underline{r}_1}{\partial \underline{r}_1} = \frac{N-1}{mV} \int d^3r_2\, d^3v_2 \frac{\partial V^{12}}{\partial \underline{r}_1} \cdot \frac{\partial F_2}{\partial \underline{v}_1} \qquad (2.13)$$

and

$$\frac{\partial F_2}{\partial t} + \left(\underline{v}_1 \cdot \frac{\partial}{\partial \underline{r}_1} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{r}_2}\right) F_2 - \frac{1}{m}\left(\frac{\partial V^{12}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{v}_1} + \frac{\partial V^{12}}{\partial \underline{r}_2} \cdot \frac{\partial}{\partial \underline{v}_2}\right) F_2 =$$

$$= \frac{N-2}{mV} \int d^3r_3\, d^3v_3 \left(\frac{\partial V^{13}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{v}_1} + \frac{\partial V^{23}}{\partial \underline{r}_2} \cdot \frac{\partial}{\partial \underline{v}_2}\right) F_3 . \qquad (2.14)$$

While the Liouville equation uniquely determines the evolution of F_N from the initial condition $F_N(t=0)$, the derived relations for F_S do not uniquely determine the evolution of F_S from $F_S(t=0)$. Instead, the interactions of our S particles with the remaining $N-S$ particles are summarized in terms of a typical $S+1$ particle and we require a knowledge of F_{S+1} to solve the problem.

We shall now see how, for a given set of circumstances, the relations for F_1 and F_2 can be simplified. We will make assumptions consistent with the physical conditions which we desire to treat and on the basis of these assumptions introduce approximations which are designed to retain the pertinent elements of information.

Our first approximation is of quite general validity. We assume N to be a large number and expand in terms of N^{-1}. At the same time, we allow the volume of the system to become arbitrarily large, but such that $N/V = n$ remains finite. This removes interactions with the boundary, and reduces (2.13) and (2.14) to

$$\frac{\partial F_1}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial F_1}{\partial \underset{\sim}{r}_1} = \frac{n}{m} \int d^3r_2\, d^3v_2 \frac{\partial V^{12}}{\partial \underset{\sim}{r}_1} \cdot \frac{\partial F_2}{\partial \underset{\sim}{v}_1} \qquad (2.15)$$

and

$$\frac{\partial F_2}{\partial t} + \left(\underset{\sim}{v}_1 \cdot \frac{\partial}{\partial \underset{\sim}{r}_1} + \underset{\sim}{v}_2 \cdot \frac{\partial}{\partial \underset{\sim}{r}_2}\right) F_2 - \frac{1}{m}\left(\frac{\partial V^{12}}{\partial \underset{\sim}{r}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} + \frac{\partial V^{12}}{\partial \underset{\sim}{r}_2} \cdot \frac{\partial}{\partial \underset{\sim}{v}_2}\right) F_2 =$$

$$= \frac{n}{m} \int d^3r_3\, d^3v_3 \left(\frac{\partial V^{13}}{\partial \underset{\sim}{r}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} + \frac{\partial V^{23}}{\partial \underset{\sim}{r}_2} \cdot \frac{\partial}{\partial \underset{\sim}{v}_2}\right) F_3 . \qquad (2.16)$$

A. The Absence of Correlations. The Boltzmann and Vlasov Equations.

Writing $F_2(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t)$ in the form

$$F_2 = F_1(\underset{\sim}{r}_1, \underset{\sim}{v}_1, t) F_1(\underset{\sim}{r}_2, \underset{\sim}{v}_2, t) + g(\underset{\sim}{r}_1, \underset{\sim}{v}_1, \underset{\sim}{r}_2, \underset{\sim}{v}_2, t)$$

where g is a correlation function, and taking $g = 0$, the system (2.15), (2.16) reduces to an equation for F_1,

$$\frac{\partial F_1}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial F_1}{\partial \underset{\sim}{r}_1} = \frac{n}{m} \int d^3r_2 d^3v_2 \frac{\partial V^{12}}{\partial \underset{\sim}{r}_1} \cdot \frac{\partial F_1(1)}{\partial \underset{\sim}{r}_1} F_1(2). \quad (2.17)$$

It is interesting to consider (2.17) in two contrasting cases. In the first, we consider a dilute gas with short-range interparticle portentials V^{12}, such that the effective range of the potential is much smaller than the mean interparticle separation. Following Schonberg[22] and Osborn[23], we rewrite (2.17) in the form

$$\frac{\partial F_1}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial F_1}{\partial \underset{\sim}{r}_1} = n \int d^3r_2 d^3v_2 \left[\frac{1}{m} \frac{\partial V^{12}}{\partial \underset{\sim}{r}_1} \cdot \frac{\partial}{\partial \underset{\sim}{v}_1} + \frac{1}{m} \frac{\partial V^{12}}{\partial \underset{\sim}{r}_2} \cdot \frac{\partial}{\partial \underset{\sim}{v}_2} \right] F_1(1) F_1(2), \quad (2.18)$$

which is equivalent to (2.17) since the additional terms are zero as may be verified by partial integration. Following Osborn[23] we now introduce a set of assumptions designed to lend plausibility to the succeeding argument.

(i) The potential V^{12} is appreciable only over a sufficiently well defined region of radius $\sim a$ such that the $\underset{\sim}{r}_2$ integration in (2.18) is effectively confined to the volume $|\underset{\sim}{r}_1 - \underset{\sim}{r}_2| < a$.

(ii) The length, a, is small compared to the mean interparticle spacing, i.e., $a n^{1/3} \ll 1$. This implies that a can be chosen such that the probability of finding more than two particles within a distance a of each other at any time is negligible, i.e., we choose a so that the binary collision assumption is in some sense justified.

(iii) The above choice of a leads to a sufficiently restricted region for the spatial integration in (2.18) so

that $F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_2, \underline{v}_2, t) \approx F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t)$ for all $\underline{r}_2$ defined by $|\underline{r}_1 - \underline{r}_2| < a$.

(iv) The product $F_1(1) F_1(2)$ does not vary appreciably over time intervals of order

$$\tau = a / |\underline{v}_1 - \underline{v}_2| .$$

(v) Two particles within a distance a of each other will be presumed to be interacting so strongly that they may be regarded as effectively decoupled from their environment.

Assumption (v) permits us to interpret

$$\frac{1}{m} \frac{\partial V^{12}}{\partial \underline{r}_1} = - \underline{a}_{12} \simeq - \underline{a}_1$$
$$\frac{1}{m} \frac{\partial V^{12}}{\partial \underline{r}_2} = - \underline{a}_{21} \simeq - \underline{a}_2 \qquad (2.19)$$

where $\underline{a}_1$ and $\underline{a}_2$ are the accelerations experienced by particles 1 and 2 respectively throughout the duration of their close encounter. The approximation (2.19) plus assumptions (iii) and (iv) then enables us to approximate the integrand in (2.18) as follows:

$$\frac{1}{m} \frac{\partial V^{12}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{v}_1} F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t) +$$
$$+ \frac{1}{m} \frac{\partial V^{12}}{\partial \underline{r}_2} \cdot \frac{\partial}{\partial \underline{v}_2} F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t) \simeq$$
$$\simeq - \underline{a}_1 \cdot \frac{\partial}{\partial \underline{v}_1} F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t) - \underline{a}_2 \cdot \frac{\partial}{\partial \underline{v}_2} F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t)$$
$$\simeq \frac{1}{\tau} \left[F_1(\underline{r}_1, \underline{v}_1 - \underline{a}_1 \tau, t) F_1(\underline{r}_1, \underline{v}_2 - \underline{a}_2 \tau, t) - F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t) \right].$$

Evidently we may interpret the velocities $(\underline{v}_1 - \underline{a}_1 \tau, \underline{v}_2 - \underline{a}_2 \tau)$ as the precollision velocities of a pair of particles entering into a strong binary interaction, whereas $(\underline{v}_1, \underline{v}_2)$ are the post-collision velocities of the same pair. Introducing the notation

$$(\underline{v}_1 - \underline{a}_1 \tau, \underline{v}_2 - \underline{a}_2 \tau) = (\underline{v}_1', \underline{v}_2')$$

we see that we may now approximate (2.18) as

$$n \int_{|\underline{r}_1 - \underline{r}_2| < a} d^3 r_2 \int d^3 v_2 \; \frac{|\underline{v}_1 - \underline{v}_2|}{a} \left[F_1(\underline{r}_1, \underline{v}_1', t) F_1(\underline{r}_1, \underline{v}_2', t) - F_1(\underline{r}_1, \underline{v}_1, t) F_1(\underline{r}_1, \underline{v}_2, t) \right] \quad (2.20)$$

Note now that nothing in the integrand of (2.20) depends on $\underline{r}_2$. Thus the space integration is readily performed if we introduce the variable change $\underline{R} = \underline{r}_1 - \underline{r}_2$ whence

$$\int_{|\underline{r}_1 - \underline{r}_2| < a} d^3 r_2 = \int_{|\underline{R}| < a} d^3 R \quad . \quad (2.21)$$

If we take $\underline{R}$ to be the distance between the two particles during their close encounter and introduce a cylindrical coordinate system (see figure (2.1)) with z -axis parallel to the pre-collision relative velocity, then (2.21) may be written as

$$\int_{\phi=0}^{2\pi} \int_{b=0}^{\sim a/2} b \, db \, d\phi \int_{-a/2}^{a/2} dz \; . \quad (2.22)$$

The quantity $b\,db\,d\phi$ is clearly the center of mass differential scattering cross-section introduced earlier, which we may write as $\sigma \, d\Omega$ Thus (2.21) may be written as

$$\int_{|\underline{r}_1 - \underline{r}_2| < a} d^3 r_2 = a \int_{\Omega} \sigma \, d\Omega \; . \quad (2.23)$$

It is understood that the restriction on the range of impact parameters as displayed in (2.22) implies a corresponding limitation on the range of the angular integration in (2.23). In these terms then, equation (2.18) becomes, for the dilute short-range approximation,

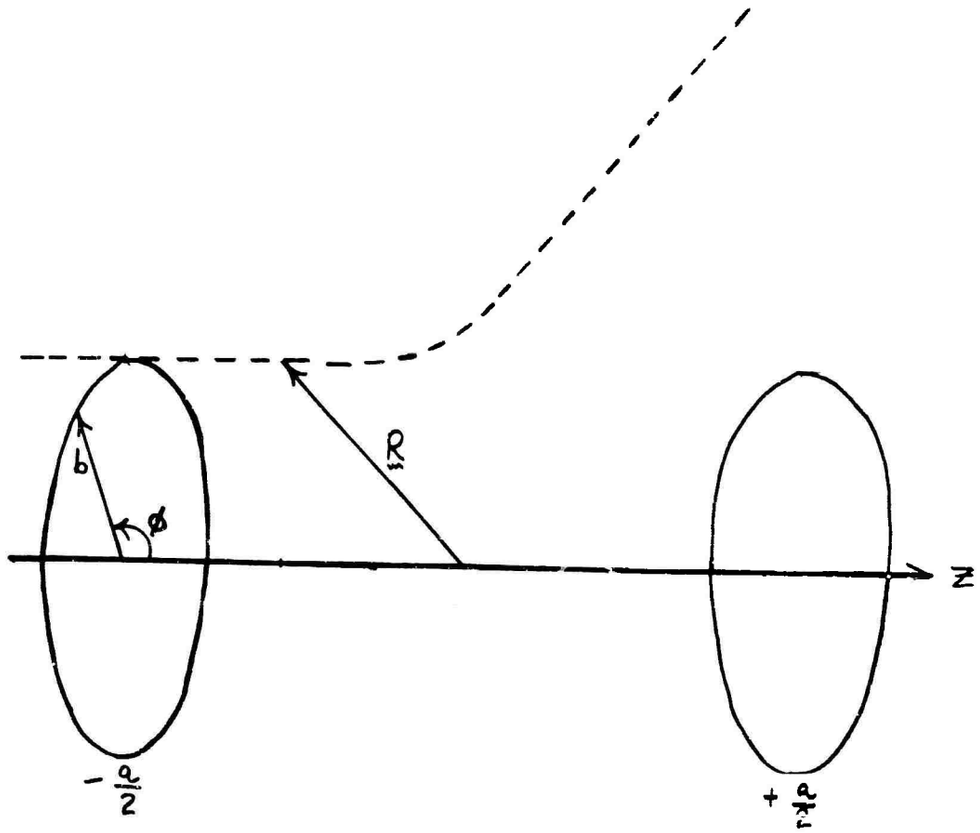


Figure 2.1

Details of Binary Interaction

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial f_1}{\partial \underset{\sim}{r}_1} = n \int d^3 v_2 \int d\Omega \, |\underset{\sim}{v}_1 - \underset{\sim}{v}_2| \, \sigma(\times)$$

$$(\times)\left[F_1(\underset{\sim}{r}_1, \underset{\sim}{v}_1', t) F_1(\underset{\sim}{r}_1, \underset{\sim}{v}_2', t) - F_1(\underset{\sim}{r}_1, \underset{\sim}{v}_1, t) F_1(\underset{\sim}{r}_1, \underset{\sim}{v}_2, t)\right] \quad (2.24)$$

Equation (2.24) is, apart from normalization differences, the Boltzmann equation in the absence of external forces.

Finally, let us consider again equation (2.17) when V^{12} is the Coulomb potential. In this case, as we have seen earlier, the notion of a binary interaction length a is indistinct when the number of particles in a Debye sphere is large. We could pick a length a say of order a few times the distance of closest approach, but the choice is vague. In this case the potential term would be divided into a "binary" term and a "collective" term, this latter corresponding to the force term in the Boltzmann equation. For example if we consider <u>only</u> collective effects, then we may regard the potential term in (2.17) as producing an internal electric field which is a function of $\underset{\sim}{r}_1$ alone. We define this field by

$$\underset{\sim}{E}(\underset{\sim}{r}_1, t) = \frac{n_i}{e} \int d^3 r_2 \, d^3 v_2 \, \frac{\partial V^{12}}{\partial \underset{\sim}{r}_1} F_1(\underset{\sim}{r}_2, \underset{\sim}{v}_2, t)$$

and write (2.17) in the form

$$\frac{\partial F_1}{\partial t} + \underset{\sim}{v}_1 \cdot \frac{\partial F_1}{\partial \underset{\sim}{r}_1} - \frac{e \underset{\sim}{E}}{m} \frac{\partial F_1}{\partial \underset{\sim}{v}_1} = 0 \quad (2.25)$$

which is known as the Vlasov equation.

B. The Inclusion of Correlations. The Lenard-Guernsey-Balescu Equation.

It is clear from the foregoing discussions that the neglect of correlations in the description of an ionized gas is a simplification

which is difficult to justify, and which leads to problems such as the divergence at large impact parameters. Since the BBGKY hierarchy (2.15), (2.16)..., includes correlations, we might hope that the kinetic equations we derive from it will be intrinsically free from this divergence. In the following we present the salient points of an important kinetic equation derived from the hierarchy which includes two-particle correlations. We outline the work of Lenard[24] and Guernsey,[25] which is based on earlier results of Bogolyubov.[26]

We begin with Mayer cluster expansions[27] of the distribution functions, similar to that used in the previous section. We have, with $(i) = (\underline{r}_i, \underline{v}_i)$,

$$F_2(1,2,t) = F_1(1,t)\,F_1(2,t) + g(1,2,t) \tag{2.26}$$

and

$$\begin{aligned} F_3(1,2,3,t) = {} & F_1(1,t)\,F_1(2,t)\,F_1(3,t) + \\ & + F_1(1,t)\,g(2,3,t) + F_1(2,t)\,g(1,3,t) + \\ & + F_1(3,t)\,g(1,2,t) + h(1,2,3,t) \end{aligned} \tag{2.27}$$

where h is a three-particle correlation function. We work in the so-called plasma limit, $n\lambda_D^3 >> 1$, and we further assume the three-particle correlations to be negligible ($h \simeq 0$). We consider only homogeneous plasmas, still with no external forces, so that the various $F_1(i)$ are independent of position and the correlation functions depend on $\underline{r}_i$ and $\underline{r}_j$ only in the combination $r = |\underline{r}_i - \underline{r}_j|$.

Bogolyubov[26] assumed that as a consequence of the assumption $n\lambda_0^3 >> 1$, the correlation functions would vary much more rapidly in time than any of the distribution functions $F_1(i,t)$. He thus

suggested that

(i) The distribution functions $F_1(i,t)$ may be considered time independent in solving for $g(i,j,t)$.

(ii) The asymptotic value of the correlation function, $g(i,j,\infty)$, represented as a functional of the $F_1(i,t)$, may be used in solving for these latter quantities.

(iii) As a consequence of (ii) the information contained in the initial condition would not appear in the $F_1(i,t)$ and hence we may take $g(i,j,0) = 0$.

Employing these assumptions, Bogolyubov[26] was able to reduce (2.15) and (2.16) to

$$\frac{\partial F}{\partial t} = \frac{1}{m} \int d^3r \, \frac{\partial V^{12}(r)}{\partial \underset{\sim}{r}} \cdot \int d^3v_2 \, \frac{\partial G(\underset{\sim}{r}, \underset{\sim}{v}_1, \underset{\sim}{v}_2)}{\partial \underset{\sim}{v}_1} \tag{2.28}$$

where G is a time independent function of $F(\equiv F_1)$, and is the solution of the integral equation

$$G(\underset{\sim}{r}, \underset{\sim}{v}_1, \underset{\sim}{v}_2) = \underset{\sim}{K}(\underset{\sim}{r}, \underset{\sim}{v}_1 - \underset{\sim}{v}_2) \cdot \left[\frac{\partial F(\underset{\sim}{v}_1)}{\partial \underset{\sim}{v}_1} F(\underset{\sim}{v}_2) - \frac{\partial F(\underset{\sim}{v}_2)}{\partial \underset{\sim}{v}_2} F(\underset{\sim}{v}_1) \right] +$$

$$+ \int d^3\xi \int d^3\eta \, \underset{\sim}{K}(\underset{\sim}{r} - \underset{\sim}{\xi}, \underset{\sim}{v}_1 - \underset{\sim}{v}_2) \cdot$$

$$\cdot \left[\frac{\partial F(\underset{\sim}{v}_1)}{\partial \underset{\sim}{v}_1} G(-\underset{\sim}{\xi}, \underset{\sim}{v}_2, \underset{\sim}{\eta}) - \frac{\partial F(\underset{\sim}{v}_2)}{\partial \underset{\sim}{v}_2} G(\underset{\sim}{\xi}, \underset{\sim}{v}_1, \underset{\sim}{\eta}) \right] \tag{2.29}$$

whose kernel is

$$\underset{\sim}{K}(\underset{\sim}{r}, \underset{\sim}{v}) = \frac{1}{m} \int_0^\infty dt \, \frac{\partial V^{12}(|\underset{\sim}{r} - \underset{\sim}{v}t|)}{\partial \underset{\sim}{r}} .$$

The normalization is $\int d^3v \, F = n$ and we take $V^{12} = e^2/r$.

Lenard[24] and Guernsey[25] reduced the system of equations (2.28), (2.29) to a single equation for F by following Bogolyubov's suggestion[26]

of introducing spatial Fourier transforms. Their result was also obtained by Balescu[28] using a diagram technique;

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{v}_1} \cdot \int d^3 v_2 \, \underset{\approx}{Q}(\underset{\sim}{v}_1, \underset{\sim}{v}_2) \cdot \left[\frac{\partial F(\underset{\sim}{v}_1)}{\partial \underset{\sim}{v}_1} F(\underset{\sim}{v}_2) - \frac{\partial F(\underset{\sim}{v}_2)}{\partial \underset{\sim}{v}_2} F(\underset{\sim}{v}_1) \right] \qquad (2.30)$$

where $\underset{\approx}{Q}$ is a symmetric second rank tensor whose components are given by

$$Q_{ij}(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = -\frac{1}{8\pi^2} \int d^3 k \, \delta(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \underset{\sim}{k} \cdot \underset{\sim}{v}_2) \frac{k_i k_j \tilde{\phi}^2(k)}{\left| 1 + \tilde{\phi}(k) \Psi(\underset{\sim}{k}, \underset{\sim}{k} \cdot \underset{\sim}{v}_1) \right|} \cdot \qquad (2.31)$$

Here $\underset{\sim}{\kappa} = \underset{\sim}{k}/|\underset{\sim}{k}|$, and

$$\tilde{\phi}(k) = \frac{1}{m} \int d^3 r \, e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}} V^{12}(r) = \frac{4\pi e^2}{m k^2} ,$$

$$\Psi = \int \frac{d^3 v_2 \, \underset{\sim}{\kappa} \cdot \partial F / \partial \underset{\sim}{v}_2}{\underset{\sim}{\kappa} \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2) - i\epsilon} .$$

The integrand in (2.31), aside from the δ-function, becomes independent of k for $k \to 0$ (large impact parameters), so that the integral converges in this limit. Thus the incorporation of correlations in the kinetic description removes, in a natural way, the troublesome divergence at large impact parameters. For k large, the integrand behaves as k^{-2} and hence the integral diverges for $k \to \infty$. This is the same divergence encountered in the electrostatic treatments[17,18] of the Fokker-Planck equation discussed above, and is due to the neglect of the three-particle correlation h when two of the three are of the order of e^2/θ apart. In this case h is the same order of magnitude as the binary correlation g, and cannot properly be ignored. To achieve convergence in (2.31) it is therefore necessary to make a short range cutoff, i.e., integrate only within the sphere

$$k \leq k_0 \simeq \theta/e^2 .$$

Lenard[24] simplified (2.31) somewhat by showing that, if one neglects contributions to Q_{ij} coming from speeds greater than a few times $(2\Theta/m)^{1/2}$, $\underset{\approx}{Q}$ reduces to

$$\underset{\approx}{Q}\ (\underset{\sim}{v}_1, \underset{\sim}{v}_2) = -\frac{\pi e^4}{m^2} \ln (n\lambda_D^3)\ \frac{\partial^2 g}{\partial \underset{\sim}{v}_1 \partial \underset{\sim}{v}_1} \cdot \qquad (2.32)$$

The approximation introduced in going from (2.31) to (2.32) requires that $F(\underset{\sim}{v})$ must be small, for velocities greater than a few times the thermal speed, as compared with its values at lower speeds. Equation (2.30) with (2.32) is, aside from a missing factor of 2, just the Landau form (2.9) of the Fokker-Planck equation.

We might consider the foregoing as a rather convincing argument supporting the use of the Fokker-Planck equation for ionized gases with distribution functions satisfying the above condition, and which satisfy the Bogolyubov assumptions (i) through (iii). These assumptions are violated, for example, by systems in which F and g vary on the same time scale. Examples are the interaction of high-frequency waves with a plasma, or rapidly growing instabilities.

The generalization of equation (2.30) for a multicomponent system may be found in Montgomery and Tidman.[4] The generalization to a constant, uniform magnetic field was worked out by Rostoker,[30] and is considerably more complex than (2.30), (2.31).

Working with a multiple time scale theory developed by Bogolyubov for certain problems in nonlinear mechanics, Frieman and Book[31] have developed a kinetic equation for homogeneous field-free systems that is free of divergences for all impact parameters. For small impact parameters their result resembles the Boltzmann collision integral; elsewhere

it is similar to equation (2.30).

The development of plasma kinetic equations is an active field, and it is beyond our purpose here to survey the topic in its entirety. In concluding this section we mention only briefly some of the problems of current interest in this area, and related work.

In our derivation of equation (2.30) we assumed a nonogeneous system with no external forces. We noted earlier that the phenomenological kinetic equations generally assume that inhomogeneities and local force fields do not significantly interfere with interparticle encounters. While these may often be valid assumptions, we would expect them to fail when length and time scales characteristic of inhomogeneities or of local forces are small compared with λ_D or λ_D / v_T (the time required for the establishment of a Debye screening cloud about each particle) where v_T is the thermal speed of the particles. At this writing a kinetic equation that includes these generally neglected effects is not available. Many attempts have been made to overcome these difficulties. Perhaps the most noteworthy are in the work of Bohm and Pines,[32] Rostoker and Rosenbluth,[33] Dupree,[34] and Frieman.[35] While none of these authors have presented a treatment satisfactory to all, their work lends considerable insight into the problem at hand, and often presents novel and significant mathematical tools. This work is certainly a good starting point for the interested reader.

3. A Simple Collision Model

We have so far been concerned with the development of a kinetic equation for the description of ionized gases. While a satisfactory theory is yet unavailable, it is apparent from our earlier discussion

that the more rigorous the treatment, the less tractable it is in terms of analytic solution. In point of fact, this problem is not unique to the study of ionized gases. The Boltzmann transport equation has so far been solved only for a few special cases, namely the inverse-fifth-power force law suggested by Maxwell, and for certain scattering kernels relevant to neutron transport theory.[36] In view of these difficulties, it is often advantageous to replace the more accurate and less manageable collision descriptions by a model that simplifies the solution of the kinetic equations. With the current paucity of experimental information on fully ionized gases, such a model may often be a good starting point for the interpretation of the little information that is available. We thus consider briefly a collision model designed to satisfy the conservation laws and an H-theorem, and which considerably simplifies the mathematical analysis.

One of the earliest of these models, generally referred to as the Krook model, was developed by Bhatnagar, Gross, and Krook.[37] A similar but somewhat simpler model was suggested independently by Welander.[38] The first Krook model, for a single component system, is[37]

$$\left(\frac{\delta F}{\delta t}\right)_{coll} = \sigma N(\underset{\sim}{r},t)\left[-F(\underset{\sim}{r},\underset{\sim}{v},t) + N(\underset{\sim}{r},t)\,\Phi(\underset{\sim}{r},\underset{\sim}{v},t)\right] \qquad (2.3?)$$

with

$$\Phi = \left[\frac{m}{2\pi\Theta(\underset{\sim}{r},t)}\right]^{3/2} \exp\left[-\frac{m}{2\Theta(\underset{\sim}{r},t)}\left\{\underset{\sim}{v} - \underset{\sim}{q}(\underset{\sim}{r},t)\right\}\right],$$

$$N(\underset{\sim}{r},t) = \int d^3v\, F(\underset{\sim}{r},\underset{\sim}{v},t), \qquad \int d^3v\, \Phi = 1,$$

$$\underset{\sim}{q}(\underset{\sim}{r},t) = \frac{1}{N(\underset{\sim}{r},t)} \int d^3v\, \underset{\sim}{v}\, F(\underset{\sim}{r},\underset{\sim}{v},t)$$

and

$$\frac{3\Theta(\underline{r},t)}{m} = \frac{1}{N(\underline{r},t)} \int d^3v\, (\underline{v}-\underline{q})^2 F$$

where $\underline{q}$ and Θ define the flow velocity and kinetic temperature (in energy units) at $(\underline{r},t)$, and σ is the parameter of the model.

The dimensions of $N\sigma$ are inverse time, and σ is generally chosen to yield an appropriate collision frequency on the basis of phenomenological considerations.[39] For an ionized gas, σ is generally chosen[39] such that

$$\sigma \simeq \frac{8\pi m e^4 \ln \Lambda}{(3m\Theta)^{3/2}} \quad , \quad \Lambda = n\lambda_D^3 .$$

While the Krook model is highly nonlinear, it is considerably simpler than, e.g., the Boltzmann collision operator, since the distribution function enters in (2.33) in a simple way: as

$$F, \quad \int d^3v\, F, \quad \int d^3v\, \underline{v}\, F, \quad \int d^3v\, v^2 F .$$

For the particular case of small amplitude perturbations near equilibrium, the linearized form of the Krook model permits solution of the kinetic equation in closed form for several interesting cases.[37,40] To linearize (2.33) we write

$$N(\underline{r},t) = N_0 + \epsilon n(\underline{r},t)$$
$$F(\underline{r},\underline{v},t) = F_0(\underline{v}) + \epsilon f(\underline{r},\underline{v},t)$$
$$\Theta(\underline{r},t) = \Theta_0 + \epsilon\Theta(\underline{r},t),$$

where ϵ is a small parameter, and where

$$F_0(\underset{\sim}{v}) = (\pi v_0^2)^{-3/2} \exp(-v^2/v_0^2),$$

$$n(\underset{\sim}{r},t) = \int d^3v\, f(\underset{\sim}{r},\underset{\sim}{v},t),$$

$$v_0^2 = 2\Theta_0/m,$$

$$N_0 \frac{\Theta(\underset{\sim}{r},t)}{\Theta_0} = \frac{1}{3v_0^2}\int d^3v\, v^2 f(\underset{\sim}{r},\underset{\sim}{v},t) - n(\underset{\sim}{r},t). \qquad (2.34)$$

We thus obtain, neglecting terms in ϵ^2,

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = \sigma N_0 \left[-f(\underset{\sim}{r},\underset{\sim}{v},t) + F_0(\underset{\sim}{v})\, n(\underset{\sim}{r},t) + \right.$$

$$\left. + N_0 F_0(\underset{\sim}{v}) \left\{ \frac{2}{v_0^2}\underset{\sim}{v}\cdot\underset{\sim}{q}(\underset{\sim}{r},t) + \left(\frac{v^2}{v_0^2} - \frac{3}{2}\right)\frac{\Theta(\underset{\sim}{r},t)}{\Theta_0} \right\} \right] \qquad (2.35)$$

with

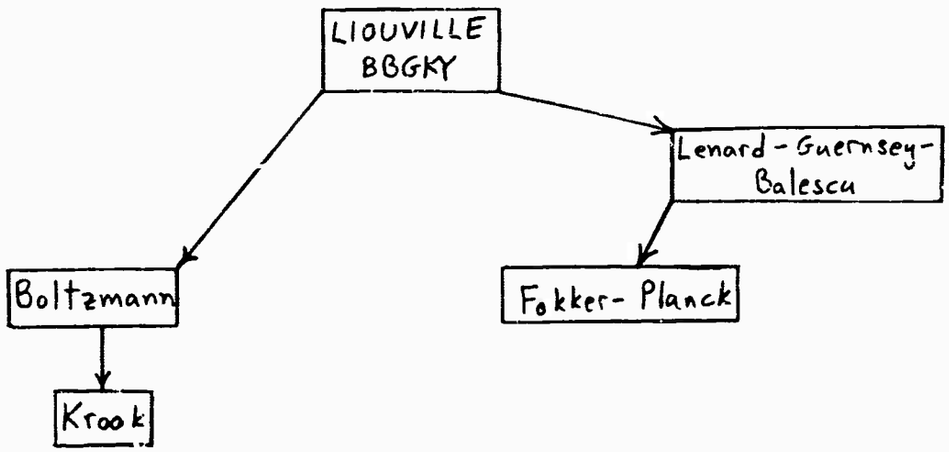
$$\underset{\sim}{q}(\underset{\sim}{r},t) = \frac{1}{N_0}\int d^3v\, \underset{\sim}{v}\, f(\underset{\sim}{r},\underset{\sim}{v},t).$$

The extension of the Krook model to a two-component system was first given by Gross and Krook,[40] together with an application of the linearized version to oscillations in a fully ionized gas. More recently Sirovich,[41] Liboff,[42] and Oppenheim[43] have presented similar collision models for a general multicomponent system. While all of these models are similar in form, they are not identical.

The relationship between the Krook model and the more sophisticated collision models rests primarily on intuitive grounds. In one special case, however, a more direct relationship has been demonstrated. For the case of a Maxwell molecule, Grosc and Jackson[44] showed that the linearized Boltzmann collision operator yields the linearized version of (2.33) if all the non-zero eigenvalues of the collision operator are approximated by a single constant which appears as σ in (2.33).

In Figure (2.2) we have indicated for convenience the relations between the kinetic theories discussed above.

Fig. 2.2 Relations Between Kinetic Theories Discussed in Text



4. Relaxation in Velocity Space

The ultimate test of a theory lies in a comparison of the predictions thereof with experimental observation. At the present writing, the scarcity of relevant experimental and theoretical information is a major source of difficulty in kinetic physics.[2] This is particularly so in the physics of fully ionized gases[4] where on the one hand, the

maintenance of a plasma and reliable observation techniques still present many unsolved problems and on the other hand, tractable theories allowing for inhomogeneities, rapid temporal variation, and boundary effects, are generally unavailable.

In the presence of these difficulties it may be instructive to compare the results of the various theories, with the hope that this would lend physical insight into both the structure of the theories and the as yet unobserved properties of nature. In this and the following two sections we intend to make such comparisons for a few illustrative cases. Our division of these topics into three separate categories is not intended to imply that they are mutually exclusive; it has been effected for convenience alone.

The problem of determining how a homogeneous expanse of gas behaves as it approaches equilibrium, i.e. "relaxes," is perhaps the simplest problem in the kinetic theory of gases. The problem is of interest here because it focuses attention on the collision operator. Perhaps the simplest description of the relaxation process is that obtained from the homogeneous isotropic Krook model (2.35). Thus if f_o is the value of the distribution function at $t = 0$, then

$$f(t) = N_o F_o + \left[f_o - N_o F_o \right] e^{-N_o \sigma t} \qquad (2.36)$$

where F_o is given in (2.34). The characteristic time $(N_o\sigma)^{-1}$ appearing in (2.36) is generally referred to as a relaxation time. Due to the complexity of the more sophisticated models, a study of relaxation via analytic solution has generally not been achieved. While the simple Krook relaxation time may often be a sufficiently accurate estimate of

relevant time scales, it is easily recognized that the description of a relaxation process by a single parameter could often be misleading. For example, the rate at which a given distribution becomes isotropic in velocity space if initially anisotropic, could be significantly different than the rate at which it relaxes to a Maxwellian.

One method of estimating such rates without actually solving the kinetic equation is to single out and consider a single "test" particle in the gas. This procedure has been employed by Spitzer,[39] who analyzed various aspects of the relaxation of the electron and ion components of an ionized gas, such as (i) removal of angular anisotropy, (ii) energy exchange, (iii) loss of energy of a particle by "dynamical friction." Bohm and Aller[45] have similarly presented a detailed analysis on the relative importance of electron-electron collisions in establishing the velocity distribution of electrons in gaseous nublae and stellar atmospheres. Montgomery and Tidman[4] perform a test particle analysis by assuming all particles except the test particle have a known (equilibrium) distribution. The kinetic equation (in this case Fokker-Planck) is then "linearized" about the test particle "distribution," and velocity moments of the linear equation are obtained. Relaxation times τ_M are then obtained by defining these as the ratio of the velocity moment M in question to its time derivative, i.e.,

$$|\tau_M| \equiv \frac{M}{\partial M/\partial t} \ .$$

The relaxation times obtained from the test particle approach generally depend on the initial speed of the test particle, the relative temperatures of the species present, and the relative masses. For a

two component ionized gas with equal electron and ion temperatures, it is found that electrons become isotropic primarily through collisions with ions; collisions with electrons play a small role in ion relaxation, and the relaxation to equilibrium of an isotropic electron distribution is primarily due to encounters with other electrons.

Although the qualitative conclusions reached in a test particle treatment should generally be correct, they do not display all of the information available in the kinetic equation. Hence, in lieu of an analytic solution of the kinetic equations, several authors have presented numerical treatments for various situations. MacDonald, Rosenbluth, and Chuck[46] have presented a numerical solution of the Fokker-Planck equation (2.7) for an isotropic electron gas imbedded in a positive neutralizing background. They assumed an initial Gaussian-shaped distribution peaked in the vicinity of the speed $(2e/m)^{1/2}$. They found the time required for the distribution to come within a few percent of the final Maxwellian, throughout the range from zero energy to several times the average energy, is about ten times the self-collision time defined by Spitzer[39] (the mean time required for a thermal particle to eventually suffer a 90° change in direction due to the cumulative effects of many small angle encounters with like particles). MacDonald, et al also found, as could be expected, that it takes considerably longer to fill out the high velocity "tail" of the Maxwell distribution.

Recently Wu, Levans, and Primack[47] have studied numerically the relaxation of a two-component plasma with initially anisotropic electron and ion temperatures, and with initially isotropic (but unequal) component temperatures, using the Lenard-Guernsey-Balescu equation (2.30), (2.31). They assumed that the distribution functions maintain a

Maxwellian character throughout the relaxation process, having the form

$$F^A(\underline{v},t) = \left(\frac{m_A}{2\pi}\right)^{3/2} \frac{1}{\theta_{A\parallel}^{1/2}(t)\,\theta_{A\perp}(t)} \exp\left[-\frac{m_A v_\parallel}{2\theta_{A\parallel}(t)} - \frac{m_A v_\perp^2}{2\theta_{A\perp}(t)}\right] \quad (2.37)$$

where

$$\theta_{A\parallel}(t) = m_A \int d^3v\, v_\parallel^2 F^A, \quad \theta_{A\perp} = \tfrac{1}{2} m_A \int d^3v\, v_\perp^2 F^A,$$

and that

$$\left|\frac{\theta_{A\parallel} - \theta_{A\perp}}{\theta_{A\parallel}}\right| << 1, \qquad \left|\frac{\theta_{A\parallel} - \theta_{A\perp}}{\theta_{A\perp}}\right| << 1.$$

The results are in qualitative agreement with the predictions of the test particle theory, except when $\theta_e^c/\theta_i > 10^2$. In this case, the anisotropic electron temperature relaxation is governed by collective phenomena. Since Wu, et al[47] constrained their distribution functions to the form (2.37), a comparison of their isotropic relaxation results with those of MacDonald, et al[46] would not be fruitful. We can only note that the collective effects manifest in (2.30), (2.31) are important in the relaxation process, under certain conditions of anisotropy

A direct comparison of the Fokker-Planck and Lenard-Guernsey-Balescu equations has been achieved in another numerical relaxation study, performed by Dolinsky.[48] Dolinsky solved both equations for several different initial conditions, for an isotropic electron gas in a neutralizing background. A comparison of the solutions showed a difference of less than two percent, for all speeds and for all time.

5. Transport Phenomena; Electrical Conductivity

Many of the interesting phenomena in kinetic theory involve systems that are inhomogeneous or are subject to external fields. As we discussed in Chapter I, the role of particle interactions in such cases may or may not be important relative to other phenomena, depending on the nature of the system under consideration. A simple and yet interesting illustration is the phenomenon of electron runaway, which occurs when an ionized gas is subject to a sufficiently strong electric field.

Kruskal and Bernstein[49] have studied electron runaway using a transport equation with Fokker-Planck collision operator. For simplicity they neglected electron-electron collisions, and assumed the electron-ion mass ratio to be zero. Their analysis leads to a decomposition of velocity space into three regions, for electric fields greater than a critical value. In the first of these, the low velocity domain, the form of the electron distribution function is dominated by collisions and hence almost isotropic. The second regior, one of intermediate velocity, is characterized by "quasi-steady" flow in velocity space, for which the low velocity region provides the source. Lastly there is a high velocity region, fed by the intermediate region, in which the electrons accelerate or "run away" almost freely under the action of the electric field, with only a very weak diffusion due to collisions.

The phenomenon of runaway, like the relaxation of high speed electrons discussed earlier, reflects the rapid decrease of the Coulomb cross-section with increasing relative velocity. It is apparent that a simplified collision model that does not take into proper account the

nature of the interacting particles would here lead to erroneous results.

As we noted in Chapter I, there are many interesting circumstances wherein the macroscopic properties of a system exhibit only small variations in times of the order of the inverse collision frequency ν_c or in space over distances of the order of the mean free path $L_c \sim v/\nu_c$. It is then possible to approximate the kinetic description by a fluid description treating macroscopic quantities such as density, mean velocity, pressure, etc., where flows are linearly related to the generalized forces driving them.[2,4,50] For example the electric current is given by the product of the electric field and the conductivity. For an isotropic system the conductivity is a scalar; more generally it is a tensor.

Since we are concerned in this work with the description and effects of particle interactions, it is instructive to consider the calculation of transport parameters briefly. While all transport coefficients are sensitive to particle interactions, the phenomena of interest here can be illustrated by a consideration of the electrical properties of a plasma. In the following discussion we will assume for simplicity that the system being considered is free from magnetic fields, temperature gradients, and inhomogeneities. We will further assume that the applied field is spatially uniform; i.e., that $\lambda >> \lambda_D$ where λ is a length characterizing the field.

Considerable attention has been turned in recent years to determining the conductivity of a fully ionized gas. The subject is not only of interest as a problem in kinetic theory, but is also of practical importance in that from a knowledge of the a.c. conductivity one can

compute immediately the absorption coefficient for radiation in a plasma and hence, by Kirchoff's law, the emission properties.[4,50,51]

The earliest calculations of electrical conductivity were based on phenomenological considerations.[2,39] Thus one simply assumes that the current carrying electrons suffer, on the average, equal accelerations by the electric field and decelerations due to collisions. Using such considerations, Spitzer[39] calculates a conductivity assuming all current to be carried by the electrons, and neglecting interactions between electrons. He finds the conductivity σ to be

$$\sigma = \frac{N_0 e^2}{m \nu_c} \tag{2.38}$$

where N_0 and m are the electron number density and mass, respectively, and ν_c is the electron-ion collision frequency. We can easily obtain similar results with a simplified Krook model. The d.c. conductivity of a fully ionized gas has also been computed using a Fokker-Planck collision operator. Spitzer, et al[7,52] obtained a numerical result very close to that given by (2.38) with ν_c the "self-collision" frequency defined earlier.

As we noted above, a significant contribution to the collisional processes in a plasma arises from long-range Coulomb encounters, and the duration of these encounters is quite sensitive to the relative speed of the particles. While a Krook-type collision model seems satisfactory for a fully ionized gas subject to low frequency or d.c. fields, we would not expect such a simple representation of particle interactions to suffice for frequencies ω of order ν_c or higher.

The earliest treatments of the a.c. response of a plasma attempted to overcome these difficulties by employing velocity dependent

collision frequencies[54] or collision operators that described a diffusion in velocity space.[2] These approaches were not, however, founded on a consideration of the nature of the interacting particles.

Several authors have computed the impedance $\mathcal{Z} = \sigma^{-1}$ of a plasma employing the various kinetic theories considered earlier in this chapter. Their results provide an effective means for comparing the various theories. In Fig. 2.3 we have displayed the quantity R_{Ac}/R_{Dc} where R is the real part of the plasma impedance $\mathcal{Z}$, as a function of ω/ω_p. The results are given for a fully ionized hydrogen plasma with $\Lambda = 2^{-1/2} \times 10^7$, where Λ is given by $N_0 \lambda_D^3$, and are based on a similar display due to DeWolf.[55]

For frequencies ω well below the collision frequency the purely resistive impedance is constant and the results of the simple Krook theory[51] agree with the Fokker-Planck calculations.[52] The low frequency resistivity has been computed[51,52] both including and excluding (Lorentz gas) encounters between electrons. The effect of including these is to increase the low frequency resistivity by a factor ~ 1.7 as is evident in the figure.

Bernstein and Trehan,[56] Robinson and Bernstein,[57] Kauffmann,[58] and Shkarofsky[59] have obtained the a.c. plasma impedance using a Fokker-Planck collision operator. Their results are summarized in Shkarofsky, Bernstein, and Robinson.[60] Marshall[61] performed a similar analysis using the linearized Boltzmann collision operator. None of these authors included the effects of internal "self-consistent" fields; i.e., they did not include the Maxwell equations in their analysis. The results for the Boltzmann and Fokker-Planck collision operators agree within a few percent, and this difference is likely due to different

computational procedures. For frequencies below ν_c these results match the d.c. results, as is evident in the resistivity diagram. (For frequencies not small compared to the collision frequency the impedance has a reactive part[2,54,60,61] reflecting inertial effects of the conducting charges.) As the frequency increases past ν_c the resistivity increases and eventually approaches a constant, independent of ω. For frequencies well above ν_c electron-electron collisions are seen to be insignificant as compared with electron-ion collisions.

When the field frequency exceeds the plasma frequency Scheuer[62] has argued that the resistivity should decrease, in contrast with the Fokker-Planck or Boltzmann results.[60,61] When $\omega > \omega_p$ Scheuer suggested that the maximum effective impact parameter should decrease from the Debye length v_T/ω_p to the length v_T/ω. At distances larger than v_T/ω, encounters do not contribute to the resistivity since they are much longer in duration than the oscillations themselves. Dawson and Oberman[63,64] computed the high frequency impedance of a plasma using the simple Vlasov equation including the internal electrostatic field. Their results (see Fig. 2.3)) agree with Scheuer's reasoning[62] for $\omega > \omega_p$ and join the Fokker-Planck and Boltzmann results[60,61] for $\omega < \omega_p$. They observed a slight bump in the resistivity near $\omega = \omega_p$ which they attribute to the generation of longitudinal plasma oscillations. For very low frequencies their resistivity does not decrease, in contrast with the collisional treatment.[60,61]

This latter difference in the predictions of the two treatments might be interpreted by reasoning as follows. For frequencies below the plasma frequency the dielectric response of the plasma is fast compared with the period of the imposed oscillation. For frequencies in the

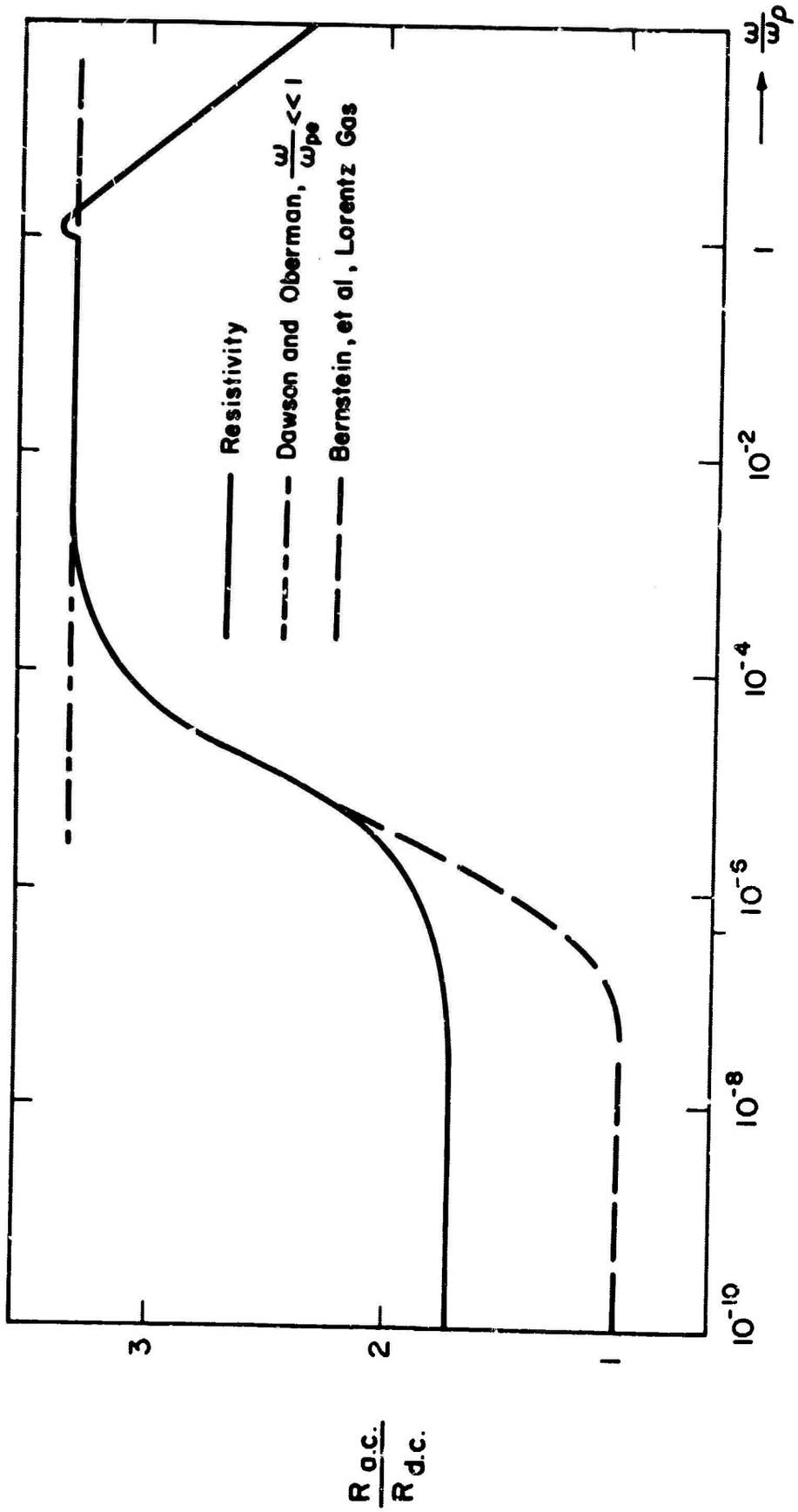


Figure 2.3 Resistivity vs. frequency for a fully ionized hydrogen gas. Taken from reference 54, $\Lambda = 2\text{-}1/2 \times 10^7$.

range $\nu_c << \omega < \omega_p$, collisions are too slow to affect the plasma response and a description incorporating only dielectric effects, i.e., the Vlasov equation, yields results that are insensitive to frequency. For lower frequencies the collisions become important and the response is frequency sensitive. Evidently there is a range of frequencies below ω_p where the collisional and collective, or dielectric, descriptions produce similar results.

Oberman, Ron, and Dawson[65] have computed the high frequency conductivity of a fully ionized plasma by solving the first two members (2.13), (2.14) of the BBGKY hierarchy using a method due to Guernsey.[66] We note that the Bogoliubov hypothesis was not employed; i.e., the two particle correlation function was allowed to vary on the same time scale as the one-particle distributions. The results of Oberman, et al[65] are in complete agreement with the predictions of the much simpler Vlasov treatment.[63,64]

In concluding this section we note some other computations of plasma transport parameters for the interested reader. In references 57 through 60 the low frequency ($\omega < \omega_p$) thermal diffusion and conductivity, and the viscosity have been computed as well as the electrical conductivity, for a plasma having small temperature and density gradients and immersed in a constant uniform magnetic field. Kivelson and Dubois[67] have found the electrical conductivity for finite wavelengths using the kinetic equation (2.30) of Lenard, et al. Berk[68] has obtained the conductivity for finite wavelengths. His approach was similar to that of Dawson and Oberman.[63,64] Oberman and Shure[69] used the first two BBGKY equations as in ref. 65 to compute the high frequency conductivity with a magnetic field. The high frequency

electrical conductivity has been computed quantum mechanically by DuBois, Gilinsky, and Kivelson[70] and by Ron and Tzoar.[71] Oberman and Ron[72] extended this work to include a magnetic field. The results are in agreement with the classical descriptions.

We summarize briefly the effects of finite wavelengths, and magnetic fields. For finite wavelengths it is convenient to refer to the phase speed of the wave, $v_p = \omega/k$ where $k = 2\pi/\lambda$. For $v_p \simeq v_T$ or less, and $\omega \gtrsim \omega_p$, the principal contribution to the conductivity is electrostatic.[68,70] For greater speeds collisional effects predominate. At high frequencies, $\omega > \omega_p$, collisions are unimportant. In the presence of a magnetic field[69] the conductivity in the direction of the field is unaffected. In contrast the transverse components decrease with increasing field strength until in the limit of infinite field strength no current flows across the field.

Finally, Klevens, Primack, and Wu[73] have computed the a.c. conductivity for $\omega > \omega_p$ using the Lenard-Guernsey-Balescu equation (2.30). Two specific cases are considered in detail: in the first, the unperturbed plasma has different electron and ion temperatures; in the second, the unperturbed plasma is characterized by a relative drift between electrons and ions. For the first case they find that for θ_e/θ_i or $\theta_i/\theta_e \gtrsim 10^4$, the real part of the conductivity becomes negative. For the second case they find that if the electron drift speed exceeds 1.37 times the electron thermal speed, and if $\theta_i > 1.07\theta_e$, the conductivity is again negative.

6. Collisional Effects on Small Amplitude Plasma Oscillations

Until recently, most studies of plasma oscillations have been concerned with relatively high temperatures and low densities, as in thermonuclear devices, or with very weakly ionized systems such as the ionosphere. For systems in the first category the collision frequencies are generally very small compared with the oscillatory frequencies of interest, and this is used as a basis for disregarding collisional effects. For systems in the second category collisions with neutrals often predominate, and a simple Krook-type model is employed to account for these.

In recent years considerable experimental attention has been given to fully ionized, relatively low temperature plasmas for which the foregoing collisionless or simple collision model assumptions are thought to be unsound. In addition, in the study of high temperature unstable plasmas it has been recognized that an inclusion of even very weak collisions can have a significant effect on the growth rate of the instability. For these reasons there has appeared an incentive to treat collective phenomena including the effects of particle interactions.

In the following discussion we present a brief review of some recent work on collisional effects in plasma collective behavior. Since relatively few theoretical results are available, the work summarized here should be considered as a first step in the direction of understanding these phenomena. In keeping with the objective of the present treatise, our emphasis is on the nature and description of collisional effects. The multitude of possible collective modes that a plasma may

support often makes a generalization of specific results very difficult, and few attempts to do so are made here. Since a considerable effort has gone into the analysis and classification of collisionless oscillations, the reader having more than a passing interest in collective phenomena would probably benefit by consulting the collisionless literature first.[56,74-83]

In the analysis of small amplitude plasma collective phenomena it is frequently convenient to Fourier-Laplace transform the governing equations, together with the Maxwell equations for the electromagnetic field, and then to solve the transformed equations for the internal electric field, from which all other field quantities may be determined. The result is then displayed in the form[83]

$$\underset{\sim}{E}(\underset{\sim}{k},s) = \frac{\underset{\sim}{\epsilon}^{A} \cdot \underset{\sim}{a}(\underset{\sim}{k})}{|\underset{\sim}{\epsilon}(\underset{\sim}{k},s)|} \tag{2.39}$$

where $\underset{\sim}{\epsilon}$ is the plasma dielectric tensor (or constant for isotropic systems), the elements of $\underset{\sim}{\epsilon}^{A}$ are the cofactors of their counterparts in $\underset{\sim}{\epsilon}$, and $\underset{\sim}{a}(\underset{\sim}{k})$ is a vector incorporating the initial conditions. The dependence of the electric field is given by the inverse Laplace transform of (2.39), and since $\underset{\sim}{\epsilon}^{A}$ and $\underset{\sim}{a}$ are entire functions of s and $\underset{\sim}{k}$ for many interesting cases,[83] one is usually interested in the zeroes of the determinant $|\underset{\sim}{\epsilon}(\underset{\sim}{k},s)|$. Thus setting this quantity equal to zero yields a relation between the wave vector $\underset{\sim}{k}$ and the Laplace variable $s = i\omega + \gamma$, and hence one estimates the growth or decay rates, etc. of various collective modes (it should be noted, however, that the dispersion relation $|\underset{\sim}{\epsilon}| = 0$ does not necessarily imply a one-to-one correspondence between frequency and wavelength[84]).

In the following review the reader will note in some cases a remarkable similarity between the collisional effects on plasma collective behavior, and the collisional effects on transport phenomena discussed above. This is, of course, more than fortuitous. It can be shown[70,85] that the longitudinal and transverse dielectric and conductivity tensors are related; for $S = i\omega$ we have[70]

$$\underset{\approx}{\epsilon}_L(\underset{\sim}{k},\omega) = \underset{\approx}{I} + 4\pi i\,\underset{\approx}{\sigma}_L(\underset{\sim}{k},\omega)/\omega,$$

$$\underset{\approx}{\epsilon}_T(\underset{\sim}{k},\omega) = \underset{\approx}{I} + 4\pi i\,\underset{\approx}{\sigma}_T(\underset{\sim}{k},\omega)/(\omega^2 - c^2k^2)$$

with $\underset{\approx}{I}$ the unit dyadic.

In an early attempt to treat longitudinal plasma oscillations including collisions, Bhatnagar, Gross, and Krook[37] employed the simple collision model discussed earlier. They treated a one component plasma consisting of electrons with fixed ions with no external fields, and neglected collisions between electrons. (The assumption of fixed ions implies the wave frequency is large compared to the ion plasma frequency.) Their results may be summarized as follows: (i) For wavelengths long compared to the Debye length or the mean free path a small change in the oscillation frequency was observed as the collision frequency varied from zero to infinity; the damping was slow (i.e., $\nu/\omega \ll 1$) and reached its maximum when the collision frequency equalled the plasma frequency. (ii) For wavelengths shorter than both the Debye length and the mean free path the damping was heavy and was primarily electrostatic, or Landau damping.

Lenard and Bernstein[36] treated the problem studied by Bhatnagar, et al[37,38] using a pseudo Fokker-Planck collision operator designed to

represent a diffusion in velocity space, and which conserved electron number density and yielded the Maxwell distribution for the equilibrium state. Their velocity dependent "diffusion coefficients" however, increased with velocity in contrast with the true Fokker-Planck coefficients. Their results are in general agreement with Bhatnagar, et al.[37]

Comisar,[87] Gorman and Montgomery,[88] Burgers,[89] and Wu and Klevans[90] have treated collisional damping of longitudinal electron oscillations in a one component plasma, including both electron-electron and electron-ion collisions. Comisar[87] used the linearized Fokker-Planck collision operator, Gorman and Montgomery[88] used Guernseys reduction[66] of the first BBGKY equations, Burgers[89] solved a Boltzmann-like equation with the Debye potential replacing the Coulomb potential, and Wu and Klevans[90] approximated the first two BBGKY equations and then employed a Guernsey-like reduction. All of these authors obtained similar results, which were restricted to weak collisions and long wavelengths. The results may be summarized as follows: (i) A wavelength-independent damping constant was found for electron-ion collisions, (ii) a damping constant porportional to k^2 was found for both electron-electron and electron-ion collisions, (iii) collision damping dominated Landau damping, (iv) electron-ion collisions dominate the damping, and (v) a small, wavelength independent correction to the oscillation frequency was found. In each of the first two cases the damping constant γ was found proportional to the respective collision frequencies given by Spitzer.[39] The work of Comisar[87] has been extended by Buti and Jain[91] to treat high frequency transverse plasma oscillations. Their results are essentially the same as Comisar's.

The collisional damping of electron plasma oscillations is

easily described in terms of momentum transfer out of the collective modes, due primarily to electron-ion collisions, and the damping increases with increasing electron-ion collision frequency. In contrast, we might expect the effects of collisions on low frequency ion waves to be somewhat different, since momentum transfer to electrons is small. Bhadra and Varma[92] have investigated collisional damping of longitudinal ion waves using a simple Krook model and neglecting ion-electron collisions. For equal electron and ion temperatures, the damping decreased monotonically with increasing collision frequency. Their interpretation of this result is that since collisions do not transfer momentum out of the wave, their only affect is to enhance the propagation.

Kulsrud and Shen[93] have investigated the propagation of ion waves using a Fokker-Planck collision operator in the limit of weak ion-ion collisions. They found the spatial damping to decrease with increasing collision frequency as with the time damping treated by Bhadra and Varma,[92] and calculated the relation between wave speed and collision frequency for comparison with experiments on ion waves performed by Motley and Wong.[94] Their results are in fair quantitative and qualitative agreement with the experimental results, but they suggest this may be only fortuitous since they attempted to extrapolate a time-damping theory to explain spatial damping lengths.

We have so far been concerned with waves in isotropic plasmas. Liboff[42] and Oppenheim[95] have treated longitudinal electron plasma oscillations in the presence of a constant uniform magnetic field. Liboff used a Krook model to represent collisions, while Oppenheim employed a pseudo Fokker-Planck collision operator similar to that used by Lenard and Bernstein.[86] The two treatments give similar results

for long wavelengths and low temperatures, and for magnetohydrodynamic modes, in the absence of the magnetic field. The results differ, however, in the case of "microscopic Larmor resonance" modes, in the parameter range where wavelength is much longer than both the Larmor resonance and the collision length. Liboff's Krook model[42] gives an infinite number of damped Larmor modes only at propagation precisely perpendicular to the applied magnetic field. Oppenheim's model, in contrast, gives an infinity of Larmor modes at arbitrary directions, except parallel to the field. Oppenheim suggests this difference reflects the velocity-space diffusion property of his collision operator. The damping constants found by Oppenheim and Liboff were quite similar, being proportional to the collision frequency in each case.

It is well known[83] that small amplitude disturbances of a homogeneous plasma near thermal equilibrium are stable; i.e., any such disturbances tend to decay in time. In addition, this inherent stability is not affected by the inclusion or exclusion of collisional effects in the describing equations, or by the imposition of a uniform magnetic field. In contrast the presence of currents or spatial gradients is known[96] to be sufficient to induce unstable plasma behavior.

The study of plasma instabilities is a relatively new field but nevertheless has received prominant attention in regard to both laboratory and extra-terrestrial phenomena, prime examples being the containment of hot plasmas[96] and the growth mechanism of stellar flares.[97,98] Due to the considerable complexity of the equations employed, the analysis of plasma instabilities has generally been restricted to collisionless treatments. Only within the past two years have attempts been made to include collisional effects. While these efforts have

been few in number, the results indicate that these effects may have a profound influence on plasma behavior. Certainly more work is needed in this area.

In an early treatment including collision effects, Kuckes[99] analyzed the propagation of low frequency ion waves in a current carrying plasma without a magnetic field. Using a simple one-parameter collision model, he showed that "collisional effects of the electrons can lead to growth mechanisms for these oscillations," while "the thermal motions of the ions leads to a damping." More recently Bhadra,[100] and Kulsrud and Shen[93] have reported studies of ion acoustic waves, employing Fokker-Planck collision operators in an iterative weak collision analysis. Bhadra[100] treated waves propagating parallel to a strong magnetic field with a perpendicular density gradient, and Kulsrud and Shen[93] assumed a homogeneous plasma with a small external electric field. Bhadra found electron-electron collisions to have a destabilizing effect, while electron-ion collisions tended to stabilize. Kulsrud and Shen, in contrast, observed electron-ion collisions to decrease the critical current; electron-electron collisions had negligible effect. Bhadra[100] also used a simple Krook model for purposes of comparison; he found only a slight difference in growth rates under some conditions.

III. AN EXPANSION THEOREM FOR THE LINEARIZED FOKKER-PLANCK EQUATION

1. Properties of the Equation

In the first three sections of this chapter certain spectral properties of the collision operator are established. While these properties (apart from the reality of the spectrum) are not necessary for the later development of the expansion theorem, they are both useful by themselves, and enable certain conclusions to be drawn regarding the final form of the expansion.

For our purposes it will prove convenient to write the Fokker-Planck equation in the Landau form (2.9);

$$\frac{\partial F^A}{\partial t} = \sum_{B=e,i} \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' \left[F^{B'} \frac{\partial F^A}{\partial \underline{v}} - \frac{m_A}{m_B} F^A \frac{\partial F^{B'}}{\partial \underline{v}'} \right] \cdot \underline{\underline{Q}}_{AB} \qquad (3.1)$$

where

$$\underline{\underline{Q}}_{AB}(\underline{v},\underline{v}') = \Gamma_{AB}\, g^{-3}(\underline{\underline{I}} - \underline{g}\,\underline{g}),$$

Γ_{AB} is a positive constant, $\underline{g} = \underline{v} - \underline{v}'$, and $\underline{\underline{I}}$ is the unit dyadic. It is not difficult to show[1] that the Maxwell distributions F_M^A, F_M^B satisfy (3.1) for $\partial F^j/\partial t = 0$, $j = A, B$. In the vicinity of equilibrium we may write $F^j = F_M^j \left[1 + f_j(\underline{v},t)\right]$. Neglecting terms quadratic in f_j we then obtain from (3.1) the linear equation

$$F_M^A \frac{\partial f_A}{\partial t} = \sum_B \frac{\partial}{\partial \underline{v}} \cdot \int d^3v\, F_M^{B'} F_M^A \left[\frac{\partial f_A}{\partial t} - \frac{m_A}{m_B} \frac{\partial f_B'}{\partial t} \right] \cdot \underline{\underline{Q}}_{AB}. \qquad (3.2)$$

We will refer to the quantity $F_m^j f_j$ as the perturbation from equilibrium.

Equation (3.2) as it stands is in fact a pair of coupled equations for f_A and f_B. Due to the quite small value of the electron-ion mass ratio the equations are however only very weakly coupled. Thus for example the effect of the ion perturbation on the electron perturbation is small when compared with the effect of the ions and electrons in the unperturbed equilibrium distributions. In addition, it can be shown[2] that in the approximation $m_i >> m_e$, the ions act like a single component gas. In the following we will consider the equation for the ions. The treatment of the electron equation is quite similar, and the modifications necessary for this case will be indicated later. We have then,

$$F_m^i \frac{\partial f_i}{\partial t} = \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' F_m^{i\prime} F_m^i \left[\frac{\partial f_i}{\partial \underline{v}} - \frac{\partial f_i'}{\partial \underline{v}'} \right] \cdot \underline{\underline{Q}}_{ii} \qquad (3.3)$$

We will for convenience drop the subscript "i" from F_m^i and f_i. If f satisfies the conditions

$$\lim_{v \to 0} v f = 0, \qquad \lim_{v \to \infty} F_m \frac{\partial f}{\partial v} = 0, \qquad (3.4)$$

it is possible to show that (3.4) conserves number, momentum, and kinetic energy densities.

Introducing $f(\underline{v}, t) = g_\lambda(\underline{v}) \exp(-\lambda t)$ in (3.3), we find

$$-\lambda F_m g_\lambda = \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' F_m F_m' \left[\frac{\partial g_\lambda}{\partial \underline{v}} - \frac{\partial g_\lambda'}{\partial \underline{v}'} \right] \cdot \underline{\underline{Q}} \, . \qquad (3.5)$$

Multiplying (3.5) by g_λ^* and integrating over $\underline{v}$ we have, after a parts integration,

$$\lambda \int d^3v\, F_M |g_\lambda|^2 = \int d^3v \frac{\partial g_\lambda^*}{\partial \underline{v}} \cdot \int d^3v' F_M F_M' \left[\frac{\partial g_\lambda}{\partial \underline{v}} - \frac{\partial g_\lambda'}{\partial \underline{v}'}\right] \cdot \underline{\underline{Q}} -$$

$$- \int d^3v \frac{\partial}{\partial \underline{v}} \cdot g_\lambda^* \int d^3v' F_M F_M' \left[\frac{\partial g_\lambda}{\partial \underline{v}} - \frac{\partial g_\lambda'}{\partial \underline{v}'}\right] \cdot \underline{\underline{Q}}. \quad (3.6)$$

The second term on the right in (3.6) vanishes provided g_λ satisfies the second of conditions (3.4) and

$$\lim_{v \to 0} v^{1/2} g_\lambda(\underline{v}) = 0. \quad (3.7)$$

Assuming these conditions hold we exchange $\underline{v}$ and $\underline{v}'$ in (3.6), noting that $\underline{\underline{Q}}(\underline{v}, \underline{v}') = \underline{\underline{Q}}(\underline{v}', \underline{v})$. We add the result to (3.6), obtaining

$$\lambda \int d^3v\, F_M |g_\lambda|^2 =$$

$$\iint d^3v\, d^3v' F_M F_M' \left[\frac{\partial g_\lambda^*}{\partial \underline{v}} - \frac{\partial g_\lambda^{*\prime}}{\partial \underline{v}'}\right] \cdot \underline{\underline{Q}} \cdot \left[\frac{\partial g_\lambda}{\partial \underline{v}} - \frac{\partial g_\lambda'}{\partial \underline{v}'}\right]. \quad (3.8)$$

Since $\underline{\underline{Q}}$ is a real positive quadratic form it follows that the right side of (3.8) is real and positive or zero. Hence $\mathrm{Im}\,\lambda = 0$ and $\lambda \geq 0$.

Employing standard methods[3] we can find from (3.8) the most general form of g_λ when $\lambda = 0$;

$$g_0(\underline{v}) = a_1 v^2 + \underline{a}_2 \cdot \underline{v} + a_3, \quad (3.9)$$

with a_1, $\underline{a}_2$ and a_3 arbitrary, but necessarily independent, constants

2. Expansion in Spherical Harmonics

From (3.5) we have

$$-\lambda F_M g_\lambda = \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[F_M \frac{\partial g_\lambda}{\partial \underset{\sim}{v}} \cdot \int d^3v' F_M' \underset{\sim}{\underset{\sim}{Q}} \right] -$$

$$- \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[F_M \int d^3v' F_M' \frac{\partial g_\lambda'}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} \right] . \qquad (3.5)$$

To perform a parts integration on the second term, consider the quantity

$$\frac{\partial}{\partial \underset{\sim}{v}'} \cdot \left[F_M' g_\lambda' \underset{\sim}{\underset{\sim}{Q}} \right] = F_M' g_\lambda' \frac{\partial}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} + F_M' \frac{\partial g_\lambda'}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} + g_\lambda' \frac{\partial F_M'}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} . \qquad (3.10)$$

With $F_M = N_0 (\alpha_i/\pi)^{3/2} \exp(-\alpha_i v^2)$, $\alpha_i = m_i/2\theta$, we have $\partial F_M'/\partial \underset{\sim}{v}' = -2\alpha_i \underset{\sim}{v}' F_M'$. Also $\underset{\sim}{v}' \cdot \underset{\sim}{\underset{\sim}{Q}} = \underset{\sim}{v} \cdot \underset{\sim}{\underset{\sim}{Q}}$, so (3.10) gives

$$\int d^3v' F_M' \frac{\partial g_\lambda'}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} = \int d^3v' \frac{\partial}{\partial \underset{\sim}{v}'} \cdot \left[F_M' g_\lambda' \underset{\sim}{\underset{\sim}{Q}} \right] -$$

$$- \int d^3v' F_M' g_\lambda' \frac{\partial}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} +$$

$$+ 2 \frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d^3v' F_M' g_\lambda' \underset{\sim}{\underset{\sim}{Q}} . \qquad (3.11)$$

The first term on the right in (3.11) vanishes if g_λ satisfies (3.4). Using the relations

$$\underset{\sim}{\underset{\sim}{Q}} = \Gamma_{ii} \frac{\partial^2 g}{\partial \underset{\sim}{v} \partial \underset{\sim}{v}} , \quad \frac{\partial}{\partial \underset{\sim}{v}'} \cdot \underset{\sim}{\underset{\sim}{Q}} = -2\Gamma_{ii} \frac{\partial}{\partial \underset{\sim}{v}} \left(\frac{1}{g} \right) ,$$

and introducing a dimensionless time τ and dimensionless velocity $\underset{\sim}{c}$ in (3.3),

$$\tau (= \tau_i) = 4\pi N_0 \Gamma_{ii} (\alpha_i/\pi)^{3/2} t , \quad \underset{\sim}{c} = \alpha_i^{1/2} \underset{\sim}{v} ,$$

the kinetic equation takes the form

$$-4\pi\lambda e^{-c^2} g_\lambda(\underline{c}) =$$

$$= \frac{\partial}{\partial \underline{c}} \cdot \left[e^{-c^2} \frac{\partial g_\lambda}{\partial \underline{c}} \cdot \frac{\partial^2}{\partial \underline{c} \partial \underline{c}} \int d^3c' e^{-c'^2} |\underline{c} - \underline{c}'| \right] -$$

$$- 2 \frac{\partial}{\partial \underline{c}} \cdot \left[e^{-c^2} \frac{\partial}{\partial \underline{c}} \int d^3c' e^{-c'^2} g_\lambda' |\underline{c} - \underline{c}'|^{-1} \right] -$$

$$- 2 \frac{\partial}{\partial \underline{c}} \cdot \left[\underline{c}\, e^{-c^2} \frac{\partial^2}{\partial \underline{c} \partial \underline{c}} \int d^3c' e^{-c'^2} g_\lambda' |\underline{c} - \underline{c}'| \right]. \qquad (3.12)$$

The time τ is measured in units of the "Spitzer self-collision time,"[4] and $c = |\underline{c}|$ is in units of the rms thermal speed.

Equation (3.12), in three dimensions, may be replaced by a set of uncoupled equations in one dimension by introducing the spherical harmonic expansion

$$g_\lambda(c) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{nlm}(c, \lambda_{nlm})\, Y_l^m(\theta, \phi).$$

We find (see appendix A)

$$\int d^3c' e^{-c'^2} g_\lambda' |\underline{c} - \underline{c}'|^{-1} = \sum_{l,m} \frac{4\pi}{2l+1} Y_l^m R_{nl}, \qquad (3.13)$$

$$\int d^3c' e^{-c'^2} g_\lambda' |\underline{c} - \underline{c}'| = \sum_{l,m} \frac{4\pi}{4l^2-1} Y_l^m S_{nl}, \qquad (3.14)$$

$$\int d^3c' e^{-c'^2} |\underline{c} - \underline{c}'| = 4\pi T \qquad (3.15)$$

where

$$R_{nl} = \int_0^c dc' c'^3 \left(\frac{c'}{c}\right)^{l+1} e^{-c'^2} g_{nl}' + \int_c^\infty dc' c' \left(\frac{c}{c'}\right)^{l} e^{-c'^2} g_{nl}',$$

$$S_{nl} = \int_0^c dc' c'^3 \left(\frac{c'}{c}\right)^{l-1} \left[\frac{2l-1}{2l+3}\left(\frac{c'}{c}\right)^2 - 1\right] e^{-c'^2} g'_{nl} +$$

$$+ \int_c^\infty dc' c'^3 \left(\frac{c}{c'}\right)^{l} \left[\frac{2l-1}{2l+3}\left(\frac{c}{c'}\right)^2 - 1\right] e^{-c'^2} g'_{nl},$$

$$T = \frac{1}{4} e^{-c^2} + \left(c + \frac{1}{2c}\right) \frac{\pi^{1/2}}{4} \operatorname{erf}(c),$$

with $\operatorname{erf}(c) \equiv 2\pi^{-1/2} \int_0^c dx \exp(-x^2)$. The expressions for R_{nl} and S_{nl} were found by Rosenbluth et al.,[5] using a different method, for the axially symmetric case $m = 0$.

Combining (3.12) through (3.15) we find

$$-\lambda e^{-c^2} \sum_{l,m} Y_l^m g_{nlm} = \sum_{l,m} \frac{\partial}{\partial \underline{c}} \cdot \left[e^{-c^2} \frac{\partial}{\partial \underline{c}} \left(Y_l^m g_{nlm}\right) \cdot \frac{\partial^2 T}{\partial \underline{c}\, \partial \underline{c}}\right] -$$

$$- \sum_{l,m} \frac{2}{2l+1} \frac{\partial}{\partial \underline{c}} \cdot \left[e^{-c^2} \frac{\partial}{\partial \underline{c}} \left(Y_l^m R_{nl}\right)\right] -$$

$$- \sum_{l,m} \frac{2}{4l^2-1} \frac{\partial}{\partial \underline{c}} \cdot \left[\underline{c}\, e^{-c^2} \cdot \frac{\partial^2}{\partial \underline{c}\, \partial \underline{c}} \left(Y_l^m S_{nl}\right)\right]. \quad (3.16)$$

Performing the indicated angular differentiations in (3.16) and then employing the orthogonality property of the spherical harmonics, we obtain the uncoupled equation

$$-\lambda_{nlm} g_{nlm} = T'' \frac{d^2 g_{nlm}}{dc^2} + \left[T''' + \left(\frac{2}{c} - 2c\right) T''\right] \frac{dg_{nlm}}{dc} -$$

$$- \frac{l(l+1)}{c^3} T' g_{nlm} - \frac{2}{2l+1} \left[R''_{nl} + \left(\frac{2}{c} - 2c\right) R'_{nl} - \frac{l(l+1)}{c^2} R_{nl}\right] - \quad (3.17)$$

$$- \frac{2}{4l^2-1} \left[c\, S'''_{nl} + (3 - 2c^2)\, S''_{nl} - \frac{l(l+1)}{c^2} \left(c\, S'_{nl} - S_{nl}\right)\right].$$

The index m is clearly superfluous and will be deleted in the following. Performing the primed differentiations in (3.17) we find

$$
\begin{aligned}
-\lambda_{nl} g_{nl} = &\left[\frac{\pi^{1/2}}{4c^3}\operatorname{erf}(c) - \frac{1}{2c^2}e^{-c^2}\right]\frac{d^2 g_{nl}}{dc^2} + \\
&+\left[\left(\frac{1}{2c^3}+\frac{2}{c}\right)e^{-c^2} - \left(\frac{1}{c^4}+\frac{2}{c^2}\right)\frac{\pi^{1/2}}{4}\operatorname{erf}(c)\right]\frac{dg_{nl}}{dc} + \\
&+\left[2e^{-c^2} - \frac{l(l+1)}{c^3}\left\{\frac{1}{4c}e^{-c^2} + \left(1-\frac{1}{2c^2}\right)\frac{\pi^{1/2}}{4}\operatorname{erf}(c)\right\}\right] g_{nl} + \\
&+\frac{4c^3(l+1)(l+2)}{(2l+1)(2l+3)}\int_0^c dc' e^{-c'^2}\left(\frac{c'}{c}\right)^{l+4} g_{nl}(c') - \\
&-\frac{4c}{2l+1}\left[1+\frac{c^2 l(l-1)}{2l-1}\right]\int_0^c dc' e^{-c'^2}\left(\frac{c'}{c}\right)^{l+2} g_{nl}(c') - \\
&-\frac{4c^3 l(l-1)}{4l^2-1}\int_c^\infty dc' e^{-c'^2}\left(\frac{c}{c'}\right)^{l-3} g_{nl}(c') - \\
&-\frac{4c}{2l+1}\left[1-\frac{c^2(l+1)(l+2)}{2l+3}\right]\int_c^\infty dc' e^{-c'^2}\left(\frac{c}{c'}\right)^{l-1} g_{nl}(c').
\end{aligned}
\tag{3.18}
$$

For boundary conditions we will use conditions (3.4), which were obtained from the conservation laws. Although (3.7) is stronger than the first of conditions (3.4) we will see below that the solutions of (3.18) which satisfy (3.4) also satisfy (3.7).

3. Spectrum of the Radial Equation

If we introduce the transformation

$$
g_{nl}(c,\lambda_{nl}) = c^{-1} e^{c^2/2}\,\psi_{nl}(c,\lambda_{nl}) \tag{3.19}
$$

we obtain from (3.18) the formally self-adjoint equation

$$\frac{d}{dc}\left[P\frac{d\psi_{nl}}{dc}\right] + \left[Q_l + \lambda_{nl}\right]\psi_{nl} + \int_0^\infty dc' K_l(c,c')\psi_{nl}(c') = 0 \tag{3.20}$$

with

$$P(c) = \frac{\pi^{1/2}}{4c^3}\,\mathrm{erf}(c) - \frac{1}{2c^2}e^{-c^2};$$

$$Q_l(c) = \left(\frac{3}{c^3} - \frac{1}{c}\right)\frac{\pi^{1/2}}{4}\mathrm{erf}(c) + \left(\frac{7}{2} - \frac{3}{2c^4} - \frac{1}{c^2}\right)e^{-c^2} -$$

$$- \frac{l(l+1)}{c^3}\left[\frac{1}{4c}e^{-c^2} + \left(1 - \frac{1}{2c^2}\right)\frac{\pi^{1/2}}{4}\mathrm{erf}(c)\right];$$

$$K_l(c,c') = \frac{4}{2l+1}e^{-\frac{1}{2}(c^2+c'^2)}\left[\frac{(l+1)(l+2)}{2l+3}\binom{c^3}{c'^3} -\right.$$

$$\left. - \binom{c}{c'} - \frac{l(l+1)}{2l-1}\binom{cc'^2}{c'c^2}\right]\begin{bmatrix}(c/c')^l & (c\le c')\\ (c'/c)^l & (c'\le c)\end{bmatrix}.$$

Transforming (3.4) via (3.19) we have

$$\lim_{c\to\infty} e^{-c^2/2}\frac{d\psi_{nl}}{dc} = 0 \tag{3.21}$$

$$\lim_{c\to 0}\psi_{nl} = 0. \tag{3.22}$$

We will later show (Appendix B) that the problem (3.20)-(3.22) is self-adjoint.

The spectrum of (3.20) is that set of numbers $\{\lambda_{nl}\}$ such that (3.20) has non-trivial solutions which satisfy (3.21) and (3.22). We have already seen that the λ_{nl} must be real, and must be positive for $l \ge 2$ and positive or zero for $l < 2$.

We can find the spectrum of (3.20) by first considering the related problem

$$\frac{d}{dc}\left[P\frac{dy_{nl}}{dc}\right] + \left[Q_l + \lambda_{nl}\right] y_{nl} = 0 \qquad (3.23)$$

with conditions on the functions $y_{nl}(c)$ identical to (3.21) and (3.22). Clearly P, dP/dc, and Q_l are bounded and continuous for all finite c except possibly near $c = 0$. For $c \ll 1$ we have

$$P(c) = \frac{1}{3} - \frac{1}{5}c^2 + O(c^4),$$

$$Q_l(c) = \frac{17}{5} + O(c^2) - l(l+1)\left[\frac{1}{3c^2} - \frac{1}{15} + O(c^2)\right].$$

In general we can write

$$P(c) = \frac{1}{2c^3}\int_0^c dx\,(e^{-x^2} - e^{-c^2})$$

and thus $P(c) > 0$ for all $c < \infty$. It follows that for $l \neq 0$, (3.23) has a regular singular point at $c = 0$.

For c small, (3.23) has the asymptotic solutions

$$y_{nl}(c) \simeq c^{l+1},\ c^{-l} \qquad (c \ll 1). \qquad (3.24)$$

The first of these satisfies (3.22) for all l. It also satisfies the stronger condition obtained from (3.7), $\lim_{(c\to 0)} c^{-1/2} y_{nl}(c) = 0$. The second solution satisfies neither condition. This is clear for $l \neq 0$. For $l = 0$, the second solution is a constant which cannot be zero since the solutions (3.24) are linearly independent.

For c sufficiently large and for $\lambda_{nl} \neq 0$, (3.23) takes the asymptotic form

$$\frac{d^2 y_{nl}}{dc} - \frac{3}{c}\frac{dy_{nl}}{dc} + \frac{4\lambda_{nl} c^3}{\pi^{1/2}} y_{nl} = 0 \qquad (c \text{ large}, \lambda_{nl} \neq 0).$$

We find ($\lambda_{nl} > 0$)

$$y_{nl}(c) \sim A_{nl}\, c^{3/4} \cos\left(\nu_{nl} c^{5/2} - \gamma_{nl}\right) \tag{3.25}$$

with $\nu_{nl} \equiv 4\lambda_{nl}^{1/2}/5\pi^{1/4}$.

Given l and λ_{nl}, equation (3.23) has only one solution which satisfies the condition at $c=0$. This solution thus contains only one arbitrary constant and it follows that A_{nl} and γ_{nl} in (3.25) are not independent. Whatever the relation between A_{nl} and γ_{nl} is, (3.25) satisfies (3.21) for all positive λ_{nl}. Hence the spectrum of (3.20) contains all positive λ_{nl}, for each l.

To determine the spectrum of the integrodifferential equation (3.20) with (3.21) and (3.22) we note that $K_l(c,c')$ is a Hilbert-Schmidt kernel and the symmetric integral operator in (3.20) is consequently completely continuous.[6] According to Weyl's perturbation theorem,[6] the addition of a completely continuous symmetric operator cannot alter the continuous spectrum of any symmetric operator to which it is added. Since (3.23) is self-adjoint (see Appendix B) it is symmetric, and it follows that the spectrum of each l-component of the linearized Fokker-Planck equation contains all positive real λ_{nl}. For $l=0$ and $l=1$ we found $\lambda_{nl}=0$ belongs to the spectrum; this corresponds to a shift to an equilibrium different from that originally postulated. For $l \geq 1$ the spectrum is $(2l+1)$-fold degenerate, corresponding to the $(2l+1)$ different spherical harmonics of order l.

4. Outline of the Expansion Theorem

We now proceed with the development of an expansion theorem based on (3.20). Our method is essentially an extension of the theory due to Weyl and Levinson,[7] to include singular integrodifferential equations with Hilbert-Schmidt kernels. We give a brief outline in this section. The details are left to the appendix.

Since the eigenfunctions of the kinetic equation are bounded and continuous on every finite interval, it is natural to pursue an expansion formula for functions $u(c)$ square integrable on the interval $\Delta: 0 \le c < \infty$. As with the earlier theory we first establish an expansion formula on a finite subinterval δ of Δ, $\delta: a \le c \le b$, $0 < a$, $b < \infty$, so that the singularities of the linear operator are external to δ. The expansion formula we seek is then obtained by taking $\delta \to \Delta$ in a suitable manner. In the following we will mean by L_Δ the integrodifferential operator in (3.20) and by L_δ the operator obtained when the lower and upper limits of the integral in (3.20) are replaced by a and b, respectively. In the following the index ℓ will be retained only where it is necessary to avoid confusion.

We have already seen that P, P' and Q are continuous on δ and that $K_\ell(c,c')$ is bounded and integrable on the square $a \le c \le b$, $a \le c' \le b$. Tamarkin[8] has shown that subject to these conditions, the solutions of $L_\delta \phi = -\lambda \phi$ which satisfy homogeneous bondary conditions at $c = a$ and $c = b$ form a complete orthogonal and normalizable set of eigenfunctions $\{h_{\delta n}\}$ on δ, with an associated denumerable sequence of real eigenvalues $\{\lambda_{\delta n}\}$. Assuming the $h_{\delta n}$ to be normalized, the expansion formula on δ is thus

$$u(c) = \sum_{n=0}^{\infty} h_{\delta n} \int_a^b dc\, u(c)\, h_{\delta n}^*(c) \tag{3.26}$$

where $u(c)$ is any function square integrable on δ.

We now use the Weyl-Levinson theory to take $\delta \to \Delta$. Since the subsequent development of the expansion theorem is in every respect a duplication of the earlier theory, we will display only the salient features.

Given λ, the most general solution of $L_\delta \phi = -\lambda \phi$ is a linear combination of the two linearly independent solutions, say ϕ_1, ϕ_2. Thus we can write

$$h_{\delta n}(c, \lambda_{\delta n}) = r_{\delta n 1}\, \phi_1(c, \lambda_{\delta n}) + r_{\delta n 2}\, \phi_2(c, \lambda_{\delta n}) \tag{3.27}$$

where $r_{\delta n 1}$ and $r_{\delta n 2}$ are complex constants. With (3.27), (3.26) becomes

$$u(c) = \sum_{n=0}^{\infty} \sum_{j,k=1}^{2} r_{\delta n j}\, r_{\delta n k}^*\, \phi_j \int_a^b dc\, u(c)\, \phi_k^* . \tag{3.28}$$

Following Levinson[7] we define an Hermitian, positive semidefinite matrix ρ_δ, called the spectral matrix, with elements $\rho_{\delta j k}$ which consist of step functions with jumps at the eigenvalues $\lambda_{\delta n}$ given by

$$\rho_{\delta j k}(\lambda_{\delta n} + 0) - \rho_{\delta j k}(\lambda_{\delta n} - 0) = r_{\delta n j}\, r_{\delta n k}^* .$$

Let $\rho_\delta(\lambda + 0) = \rho_\delta(\lambda)$, and let $\rho_\delta(0)$ be the zero matrix. We use the spectral matrix to replace the infinite series in (3.28) by a Lebesgue-Stieltjes integral

$$u(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \phi_j(c, \lambda)\, \tilde{u}_k(\lambda)\, d\rho_{\delta j k}(\lambda) \tag{3.29}$$

where

$$\tilde{u}_k = \int_a^b dc\, u(c)\, \phi_k^*(c,\lambda).$$

As $\delta \to \Delta$ (that is, $a \to 0$, $b \to \infty$), ρ_δ approaches a limit matrix ρ_Δ. To find ρ_Δ let $\lambda = \mu + i\omega$, $\omega > 0$, and let $\chi_a = \phi_1 + m_a(\lambda)\phi_2$ be a solution of $L_\delta \phi = -\lambda\phi$ satisfying the homogeneous boundary condition

$$\cos\alpha\, \chi(a) + \sin\alpha\, P(a)\, \chi'(a) = 0,$$

and similarly let $\chi_b = \phi_1 + m_b(\lambda)\phi_2$ be a solution of the same equation satisfying

$$\cos\beta\, \chi(b) + \sin\beta\, P(b)\, \chi'(b) = 0.$$

Clearly $m_a = r_{\delta n 2}(a)/r_{\delta n 1}(a)$ and similarly for m_b.

As $a \to 0$ and $b \to \infty$, m_a and m_b approach limiting values in the complex m plane denoted respectively by $m_0(\lambda)$ and $m_\infty(\lambda)$. These limiting values are clearly determined by the behavior of ϕ_1 and ϕ_2 for small and large c, for λ complex.

For ϕ_1 and ϕ_2 to be linearly independent it is necessary and sufficient that their Wronskian equal a nonzero constant, say one:

$$P\left[\phi_1 \phi_2' - \phi_2 \phi_1'\right] = 1. \tag{3.30}$$

This last will be satisfied if ϕ_1 and ϕ_2 satisfy the conditions

$$\phi_1(s,\lambda) = \sin\sigma \qquad \phi_2(s,\lambda) = \cos\sigma$$
$$P(s)\,\phi_1'(s,\lambda) = -\cos\sigma \qquad P(s)\,\phi_2'(s,\lambda) = \sin\sigma,$$

where s is an interior point of δ and $0 \le \sigma < \pi$. These conditions are also sufficient to ensure that ϕ_1, ϕ_2 are entire functions of λ

for each fixed c on δ (this follows from Tamarkin[8]).

With these properties secured we can find[7] the limit values m_0 and m_∞ and hence the limit matrix ρ_Δ, whose elements are given by

$$\rho_{\Delta jk}(\lambda) - \rho_{\Delta jk}(\eta) = \lim_{\omega \to 0+} \frac{1}{\pi} \int_\eta^\lambda \operatorname{Im} M_{jk}(\mu + i\omega)\, d\mu \tag{3.31}$$

where

$$M_{11}(\lambda) = \frac{1}{m_0 - m_\infty}$$

$$M_{12}(\lambda) = M_{21}(\lambda) = \frac{1}{2} \frac{m_0 + m_\infty}{m_0 - m_\infty}$$

$$M_{22}(\lambda) = \frac{m_0 m_\infty}{m_0 - m_\infty} \quad .$$

To find the M_{jk} we need asymptotic forms of ϕ_1 and ϕ_2 for large and small c. These are given by (3.24) and (3.25), as may be verified by direct substitution. Taking ϕ_1, ϕ_2 to be asymptotic respectively to c^{-l}, c^{l+1} for c small, we apply the homogeneous boundary condition to χ_a and then take $a \to 0$ to find

$$m_0 = \infty \quad (l \neq 0)$$

$$m_0 = -\cot\alpha \quad (l = 0). \tag{3.32}$$

Thus for $l \neq 0$ only M_{22} can have a nonzero imaginary part and consequently only ϕ_2 will contribute to the expansion formula (3.29). When $l = 0$ both solutions are regular at $c = 0$ and the limit matrix is not determined until we specify α. The boundary condition (3.22) dictates the choice $\alpha = 0$.

For c large we take

$$\phi_2 \simeq A_{n\ell}\, c^{3/4} \cos\left(\nu_{n\ell} c^{5/2} - \gamma_{n\ell}\right) \qquad \nu_{n\ell} = 4\lambda_{n\ell}^{1/2}/5\pi^{1/4},\ \lambda_{n\ell} > 0$$

and find ϕ_1 by integrating (3.30):

$$\phi_1 \simeq -\frac{8\, c^{3/4} \sin\left[\nu_{n\ell}\left(c^{5/2} - c_0^{5/2}\right)\right]}{5\pi^{1/2} A_{n\ell}\, \nu_{n\ell} \cos\left(\nu_{n\ell} c_0^{5/2} - \gamma_{n\ell}\right)}$$

where c_0 is a constant of integration. Applying the homogeneous boundary condition to $\chi_b \equiv \phi_1 + m_b(\lambda)\phi_2$ and then taking $b \to \infty$ with $\operatorname{Im}\lambda > 0$ we find

$$m_\infty = \frac{8\ell \exp\left[i\left(\nu_{n\ell} c_0^{5/2} - \gamma_{n\ell}\right)\right]}{5\pi^{1/2} A_{n\ell}^2\, \nu_{n\ell} \cos\left(\nu_{n\ell} c_0^{5/2} - \gamma_{n\ell}\right)} \,. \tag{3.33}$$

Combining(3.31) through (3.33) we have finally

$$d\rho(\lambda_{n\ell}) \equiv d\rho_{\Delta 22}(\lambda_{n\ell}) = \frac{2\, d\lambda}{\pi^{5/2} A_{n\ell}^2(\lambda_{n\ell})\, \lambda_{n\ell}^{1/2}},\ \lambda_{n\ell} > 0. \tag{3.34}$$

Since the spectrum is empty for $\lambda_{n\ell} < 0$, $\rho(\lambda_{n\ell})$ is constant on this range. The expansion formula (3.29) becomes

$$u(c) = \int_0^\infty \phi_2(c, \lambda_{n\ell})\, \tilde{u}(\lambda_{n\ell})\, d\rho(\lambda_{n\ell}) \tag{3.35}$$

with

$$\tilde{u} = \int_0^\infty u(c)\, \phi_2^*(c, \lambda_{n\ell})\, dc \,.$$

The expansion converges in the mean for all functions $u(c)$ square integrable on $(0, \infty)$. If the spectral function ρ is not continuous at $\lambda_{n\ell} = 0$, this point will contribute to the integral in (3.35).

We return to the description of perturbations from equilibrium. If $\exp(-c^2) f(\xi, 0)$ is square integrable in velocity space, then from

(3.19) and (3.35) we have

$$e^{-c^2} f(\xi,\tau) = \sum_{l,m} \int_0^\infty d\rho(\lambda_{nl})\, \Psi_{nlm}\, \tilde{F}_{nlm}\, e^{-\lambda_{nl}\tau} \qquad (3.36)$$

with

$$\tilde{F}_{nlm} = \frac{2l+1}{2} \int d^3c\, \Psi^*_{nlm}\, f(\xi,0)$$

and

$$\Psi_{nlm}(\xi,\lambda_{nlm}) = c^{-1} e^{-c^2/2}\, \psi_{nl}(c,\lambda_{nl})\, Y_l^m(\theta,\phi).$$

The functions ψ_{nl} correspond to ϕ_2 in (3.35) and are the solutions of (3.20) satisfying (3.22).

We have defined the density, mean velocity, and kinetic temperature of the ion gas as being proportional respectively to the first three moments of the equilibrium distribution F_M. If ρ is continuous at $\lambda_{nl} = 0$ this point will not contribute to the expansion formula (3.36) and the eigenfunctions (3.9) for $\lambda_{nl} = 0$ will not be contained in the expansion. By virtue of the conservation laws the functions (3.9) will then be orthogonal to (3.36). Thus (3.36) is complete only if ρ has a jump at $\lambda_{nl} = 0$ for $l = 0, 1$. It follows that the exclusion of (3.9) from (3.36) yields an expansion which is complete with respect to all square integrable perturbations conserving n, $\langle m\underline{v}\rangle$, and θ.

5. The Electron Kinetic Equation

We have developed an expansion theory based on the uncoupled kinetic equation (3.3) for the ions. The extension to the electron kinetic equation is straightforward and requires only a little algebra.

As we indicated earlier we decouple the electron kinetic equation from the ion equation by dropping the term $(m_e/m_i)\,\partial f_i'/\partial \underline{v}'$ in (3.2). This amounts to neglecting the effect of the ion perturbation on the electron perturbation, but retains the effects of encounters with ions in the thermal distribution.

The uncoupled equation conserves electron number density provided (3.4) holds, but does not conserve momentum or kinetic energy in the electron gas. This is as it should be, since a substantial portion of the electron momentum, and a small amount of the energy, is lost to the ions.

Applying the methods of section 1 we find as before $\mathrm{Im}\,\lambda = 0$, $\lambda \geq 0$, and for $\lambda = 0$ we find $g_0 =$ const., corresponding to (3.9). The remainder of the development proceeds as before. A spherical harmonic expansion yields a set of singular integrodifferential equations, and the transformation (3.19) brings these into self-adjoint form. As before, the expansion formula has the form of a generalized Fourier integral.

If, for example, the ions are protons, then we can take $\Gamma_{ee} = \Gamma_{ei}$. If we use α_e in place of α_i in the definition of τ and c, then the electron equations may be obtained from the ion equations by replacing $T(c)$ in (3.15) by $T(c) + \alpha^{1/2} T(\alpha^{-1/2} c)$ where $\alpha = \alpha_e/\alpha_i = m_e/m_i$.

6. Discussion

We have used boundary conditions obtained by requiring the solutions of the kinetic equation (3.3) to be consistent with the conservation laws. The Hilbert space then emerged as a natural function space for the framework of the mathematical development. The question

persists (see e.g., the discussion of Uhlenbeck and Ford[9]) as to whether square integrability should be a requirement on the distribution functions from the beginning. In the light of the present work this condition does not appear to be necessary, and for our purposes it would not have been sufficient. To see this we note the condition $\int d^3v\, |F_M f|^2 < \infty$ leads to

$$\lim_{v \to 0} v^{3/2} f = 0 , \tag{3.37}$$

which is weaker than the corresponding condition (3.4). Since both solutions of (3.20) satisfy (3.37) for $\ell = 0$, it would be possible to have an expansion theorem for solutions of the kinetic equation which are square integrable but do not satisfy the conservations laws.

IV. THE SCATTERING OF PHOTONS FROM A PARTIALLY IONIZED GAS

1. Some General Properties of the Scattering Function

In the first part of this chapter we present a brief review of a classical derivation of the scattering function, and discuss certain properties of an equilibrium gas relevant to the computation of the scattering function. The rest of the chapter treats photon scattering from a partially ionized gas.

The photon scattering can be characterized[1] by a cross-section describing the effective area that a particle in the sample presents to an incident photon, having direction $\underset{\sim}{\Omega}$ and energy $\hbar\omega$, for the scattering of that photon into a small solid angle about the direction $\underset{\sim}{\Omega}'$ and into a small energy increment about $\hbar\omega'$. It can be shown that, neglecting relativistic and dispersion effects, the electron cross-section is given by[1]

$$\sigma(\omega, \underset{\sim}{\Omega}; \omega', \underset{\sim}{\Omega}') = \frac{\omega'}{\omega}\sigma_T(\theta)\, S(\underset{\sim}{K}, \Delta\omega) \tag{4.1}$$

where $\sigma_T(\theta)$ is the Thomson cross-section and $\underset{\sim}{K}$, θ and $\Delta\omega$ are given by

$$\underset{\sim}{K} = \underset{\sim}{k} - \underset{\sim}{k}', \qquad \cos\theta = \underset{\sim}{k}\cdot\underset{\sim}{k}'/|\underset{\sim}{k}\cdot\underset{\sim}{k}'|, \qquad \Delta\omega = \omega - \omega'$$

with $\omega = ck$, $\omega' = ck'$, and

$$\underset{\sim}{k}/|\underset{\sim}{k}| = \underset{\sim}{\Omega}, \qquad \underset{\sim}{k}'/|\underset{\sim}{k}'| = \underset{\sim}{\Omega}'.$$

The so-called scattering function S is given by

$$S(\underline{k}, \Delta\omega) = \frac{1}{2\pi N^e} \int_{-\infty}^{\infty} dt\, e^{-it\Delta\omega}\, G^{ee}(\underline{k}, t) \tag{4.2}$$

where

$$G^{ee}(\underline{k}, t) = \langle \bar{\rho}^{e\dagger}(\underline{k}, 0)\, \bar{\rho}^{e}(\underline{k}, t) \rangle_T . \tag{4.3}$$

The normalization is such that[1] N^e is the total number of electrons in the scattering volume. It can be shown[1] that the scattering cross-section for ions with mass m_i is of order $(m_e/m_i)^2$ smaller than the electron cross-section (4.1), where m_e is the electron mass. We will thus neglect the photon scattering from ions, assuming local charge neutrality in the scattering system. Scattering from neutrals will be considered later.

The function $\bar{\rho}^e$ in (4.3) is the Fourier-transformed electron density operator. Since the derivation of (4.1) was necessarily quantum mechanical, it follows that the density operators should be described quantum-mechanically. It was argued in reference 1 that the difference between the quantum and classical descriptions of the density operators will often have negligible quantitative significance. On this basis the somewhat simpler classical description was employed. We will continue to assume the validity of this approximation here. The reader interested in a quantum mechanical description of the scattering function would do well to consult the work of Rosenbaum, Zweifel, et al.[2,3]

We are clearly concerned with the electron density operators $\rho^e(\underline{x}, t)$, where

$$\rho^e(\underline{x}, t) = \int d^3v\, g^e(\underline{x}, \underline{v}, t) \tag{4.4}$$

and

$$g^e(\underline{x},\underline{v},t) = \sum_{\alpha}^{N^e} \delta(\underline{x}-\underline{x}^\alpha(t))\,\delta(\underline{v}-\underline{v}^\alpha(t)). \tag{4.5}$$

We seek ultimately the thermal average of the product of the Fourier-transformed electron density operators as displayed in (4.3); i.e., with (4.4) we wish to obtain

$$G^{ee}(\underline{k},t) = \int d^3v\, d^3v'\, d^3x\, d^3x'\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} \langle g^{e*}(\underline{x}',\underline{v}',0)\, g^e(\underline{x},\underline{v},t)\rangle_T .$$

To compute the thermal average above we follow Osborn[1] and generate a set of equations for the phase-space density operators, which we then solve subject to certain well-defined approximations. Since the procedure for generating these equations has been delineated elsewhere,[1] we present only a brief summary here.

Assuming the dynamical variables of the system obey the classical equations of motion, we have

$$\dot{x}_j^\alpha = \frac{\partial H}{\partial p_j^\alpha} = \{x_j^\alpha, H\},$$

$$p_j^\alpha = -\frac{\partial H}{\partial x_j^\alpha} = \{p_j^\alpha, H\}$$

where

$$H = T + V$$

is the plasma Hamiltonian. The symbol $\{\ \}$ means Poisson bracket, and for any function A of the system dynamical variables,

$$\{A, H\} = \sum_\alpha \left[\frac{\partial A}{\partial x_j^\alpha}\frac{\partial H}{\partial p_j^\alpha} - \frac{\partial A}{\partial p_j^\alpha}\frac{\partial H}{\partial x_j^\alpha}\right].$$

It thus follows that

$$\dot{A} = -\{H, A\}$$
$$\equiv -LA$$

and hence

$$A(t) = e^{-tL} A(0).$$

In particular, the phase-space density operator g^A for particles of kind A is given by

$$g^A(\underset{\sim}{x}, \underset{\sim}{v}, t) = e^{-tL} \sum_{\alpha}^{N^A} \delta(\underset{\sim}{x} - \underset{\sim}{x}^\alpha(0)) \delta(\underset{\sim}{v} - \underset{\sim}{v}^\alpha(0)) \tag{4.6}$$

If the plasma Hamiltonian is taken to be

$$H = \frac{1}{2} \sum_{A\,(\text{species})} \sum_{\alpha}^{N^A} \frac{(\underset{\sim}{p}^\alpha)^2}{2m_\alpha} + \frac{1}{2} \sum_{A} \sum_{\alpha,\beta}^{N^A} V^{AA}(|\underset{\sim}{x}^\alpha - \underset{\sim}{x}^\beta|) +$$
$$+ \sum_{\substack{A,B \\ A \neq B}} \sum_{\alpha,\beta}^{N^A, N^B} V^{AB}(|\underset{\sim}{x}^\alpha - \underset{\sim}{x}^\beta|), \tag{4.7}$$

then it is a straightforward matter to show that $g^A(\underset{\sim}{x}, \underset{\sim}{v}, t)$ satisfies the equation

$$\frac{\partial g^A}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial g^A}{\partial \underset{\sim}{x}} -$$
$$- \frac{1}{m_A} \frac{\partial g^A}{\partial \underset{\sim}{v}} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \sum_{B} \int d^3x' d^3v' V^{AB}(|\underset{\sim}{x} - \underset{\sim}{x}'|) g^B(\underset{\sim}{x}', \underset{\sim}{v}', t) = 0. \tag{4.8}$$

Equation (4.8) is similar to eqn. (II.19) in reference 1, but is now generalized to include any number of species in the scattering system. Now as in reference 1 we let the average of g^A be F^A, i.e.,

$$F^A(\underset{\sim}{x},\underset{\sim}{v},t) = \langle g^A(\underset{\sim}{x},\underset{\sim}{v},t)\rangle_T , \tag{4.9}$$

and further define the fluctuation operator δg^A,

$$\delta g^A(\underset{\sim}{x},\underset{\sim}{v},t) \equiv g^A(\underset{\sim}{x},\underset{\sim}{v},t) - F^A(\underset{\sim}{x},\underset{\sim}{v},t). \tag{4.10}$$

Still proceeding as in reference 1 we combine (4.8), (4.9) and (4.10) to obtain an equation for the fluctuation operators δg^A, and then approximate this equation by neglecting terms quadratic in the fluctuation operators. We obtain

$$\frac{\partial \delta g^A}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial \delta g^A}{\partial \underset{\sim}{x}} - \frac{1}{m_A}\frac{\partial}{\partial \underset{\sim}{v}}\cdot\frac{\partial}{\partial \underset{\sim}{x}}\sum_B \int d^3x' d^3v' V^{AB}(|\underset{\sim}{x}-\underset{\sim}{x}'|)\,(\times)$$

$$(\times)\left[F^A(\underset{\sim}{x},\underset{\sim}{v},t)\,\delta g^B(\underset{\sim}{x}',\underset{\sim}{v}',t) + \delta g^A(\underset{\sim}{x},\underset{\sim}{v},t)\,F^B(\underset{\sim}{x}',\underset{\sim}{v}',t)\right] = 0. \tag{4.11}$$

Equation (4.11) is now further simplified by assuming that the target plasma is in the thermodynamic state, and furth that the singlet densities F^A, F^B are independent of space and time, and are Maxwellian functions of the velocity. Equation (4.11) now reduces to

$$\frac{\partial \delta g^A}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial \delta g^A}{\partial \underset{\sim}{x}} - \frac{1}{m_A}\frac{\partial F_M^A}{\partial \underset{\sim}{v}}\cdot\frac{\partial}{\partial \underset{\sim}{x}}\,(\times)$$

$$(\times)\sum_B \int d^3x' d^3v' V^{AB}(|\underset{\sim}{x}-\underset{\sim}{x}'|)\,\delta g^B(\underset{\sim}{x}',\underset{\sim}{v}',t) = 0. \tag{4.12}$$

It follows from (4.2) and (4.3) that we must solve the system of equations (4.12) for the Fourier-transformed electron fluctuation operators as functions of time for all t, $-\infty < t < \infty$. To this end we introduce respective Laplace transformations[4] for $t > 0$ and $t < 0$;

$$\delta g_+^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},p) = \int_0^{\infty} dt\, e^{-pt}\, \delta g^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},t) \qquad p = \sigma + i\nu,\ \sigma > \sigma_+$$

$$\delta g_+^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, e^{pt}\, \delta g_+^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},p) \qquad (t>0)$$

$$= 0 \qquad (t<0) \tag{4.13}$$

$$\delta g_-^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},p) = \int_{-\infty}^{0} dt\, e^{-pt}\, \delta g^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},t) \qquad p = \sigma + i\nu,\ \sigma < \sigma_-$$

$$\delta g_-^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, e^{pt}\, \delta g_-^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},p) \qquad (t<0)$$

$$= 0 \qquad (t>0) \tag{4.14}$$

where

$$\delta g^{A}(\underset{\sim}{x},\underset{\sim}{v},t) = \delta g_+^{A} + \delta g_-^{\wedge} \tag{4.15}$$

and further introduce the Fourier transformation

$$\delta g_{\pm}^{A}(\underset{\sim}{k},\underset{\sim}{v},p) = \int d^3x\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, \delta g_{\pm}^{\wedge}(\underset{\sim}{x},\underset{\sim}{v},p). \tag{4.16}$$

We thereby obtain from (4.12)

$$(p - i\underset{\sim}{k}\cdot\underset{\sim}{v})\, \delta g_{\pm}^{\wedge}(\underset{\sim}{k},\underset{\sim}{v},p) + \frac{1}{m_A}\, i\underset{\sim}{k}\cdot\frac{\partial F_{..}^{\wedge}}{\partial \underset{\sim}{v}}(x)$$

$$(x)\sum_B V^{AB}(\underset{\sim}{k}) \int d^3v'\, \delta g^{B}(\underset{\sim}{k},\underset{\sim}{v}',p) = \pm\, \delta g^{\wedge}(\underset{\sim}{k},\underset{\sim}{v},0). \tag{4.17}$$

We now divide (4.17) by $(p - i\underset{\sim}{k}\cdot\underset{\sim}{v})$, integrate over $\underset{\sim}{v}$, and define

$$h_{\pm}^{A}(\underline{k},p) \equiv \int d^3v\, \delta g_{\pm}^{A}(\underline{k},\underline{v},p) \tag{4.18}$$

and

$$\Lambda_{AB} \equiv \frac{V^{AB}(\underline{k})}{m_A}\, i\underline{k}\cdot \int \frac{d^3v\, \partial F_M^A/\partial \underline{v}}{p - i\underline{k}\cdot\underline{v}}\,, \tag{4.19}$$

obtaining

$$(1+\Lambda_{AA})\, h_{\pm}^{AA} + \sum_{B\neq A} \Lambda_{AB}\, h_{\pm}^{B} = \pm \int \frac{d^3v\, \delta g'(\underline{k},\underline{v},0)}{p - i\underline{k}\cdot\underline{v}}\,. \tag{4.20}$$

Now let Δ represent the determinant implicit in the system of equations (4.20) and let Δ_B be the cofactor of $h_{\pm}^{B}$. Then solving for $h_{\pm}^{e}$ we find

$$h_{\pm}^{e}(\underline{k},p) = \pm \sum_{B} \frac{\Delta_B(\underline{k},p)}{\Delta(\underline{k},p)} \int d^3v\, \frac{\delta g^B(\underline{k},\underline{v},0)}{p - i\underline{k}\cdot\underline{v}}\,. \tag{4.21}$$

Finally, we multiply this last expression by $\delta g^{e*}(\underline{k},\underline{v}',0)$, thermal average the product, and integrate over $\underline{v}'$. These operations yield

$$\int d^3v\, d^3v' \langle \delta g^{e*}(\underline{k},\underline{v}',0)\, \delta g_{\pm}^{e}(\underline{k},\underline{v},p)\rangle_T = \pm \sum_{B} \frac{\Delta_B}{\Delta} \int \frac{d^3v\, d^3v'}{p - i\underline{k}\cdot\underline{v}}\, G^{Be}(0) \tag{4.22}$$

where

$$G^{Be}(0) \equiv \langle \delta g^{e*}(\underline{k},\underline{v}',0)\, \delta g^{B}(\underline{k},\underline{v},0)\rangle_T\,. \tag{4.23}$$

We now perform the inverse Laplace transformations on (4.22), as per (4.13) and (4.14), and employ the Laplace convolution relations,[4] obtaining

$$\pm \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dp\, e^{pt}}{p - i\underline{k}\cdot\underline{v}} \frac{\Delta_B(\underline{k},p)}{\Delta(\underline{k},p)} = \begin{cases} \int_{-\infty}^{\infty} d\tau\, D_+^B(\underline{k},\tau)\, e^{i\underline{k}\cdot\underline{v}(t-\tau)} & (\sigma > \sigma_+,\ t>0) \\ \int_{-\infty}^{\infty} d\tau\, D_-^B(\underline{k},\tau)\, e^{i\underline{k}\cdot\underline{v}(t-\tau)} & (\sigma < \sigma_-,\ t<0) \end{cases} \tag{4.24}$$

where we have defined

$$D_\pm^B(\underline{k},\tau) \equiv \pm \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp\, e^{p\tau} \frac{\Delta_B(\underline{k},p)}{\Delta(\underline{k},p)} \qquad \begin{matrix} + \text{ if } \sigma > \sigma_+ \\ - \text{ if } \sigma < \sigma_- \end{matrix} . \tag{4.25}$$

Writing

$$D^B = D_+^B + D_-^B$$

we can easily show (provided the interparticle potentials depend only on the magnitude of the separation), since Δ_B and Δ are functions of the Δ_{AB} only, that D^B obeys the symmetry relations

$$D^B(\underline{k},\tau) = D^B(\underline{k},-\tau) = D^B(-\underline{k},\tau) = D^B(-\underline{k},-\tau) \tag{4.26}$$

and further

$$D^B(\underline{k},\tau) = D^{B*}(\underline{k},\tau) . \tag{4.27}$$

Combining (4.3), (4.4), (4.10), and (4.15) we have, with

$$\int d^3v\, d^3v'\, d^3x\, d^3x'\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} F_M^e(\underline{v}) F_M^e(\underline{v}') = (2\pi)^3 n^e N^e \delta(\underline{k}),$$

where N^e = n^e x scattering volume,

$$G^{ee}(\underline{\kappa},t) = (2\pi)^3 n^e N^e \delta(\underline{\kappa}) +$$

$$+\sum_B \int d^3v\, d^3v'\, e^{i\underline{\kappa}\cdot\underline{v}t} \int_{-\infty}^{\infty} d\tau\, e^{-i\underline{\kappa}\cdot\underline{v}\tau}\, D^B(\underline{\kappa},\tau) G^{Be}(\underline{\kappa},\underline{v},\underline{v}';0). \quad (4.28)$$

Fourier transforming with respect to the time variable as per (4.2) we find

$$S(\underline{\kappa},\Delta\omega) = (2\pi)^3 n^e \delta(\underline{\kappa})\delta(\Delta\omega) + \frac{1}{N^e}\sum_B \int d^3v\, d^3v'\, \delta(\underline{\kappa}\cdot\underline{v}-\Delta\omega)(x)$$

$$(x)\, G^{Be}(\underline{\kappa},\underline{v},\underline{v}';0) \int_{-\infty}^{\infty} d\tau\, e^{-i\underline{\kappa}\cdot\underline{v}\tau}\, D^B(\underline{\kappa},\tau). \quad (4.29)$$

To complete the description, i.e., to portray a given experiment, we must specify the interparticle potentials and the quantity $G^{Be}(0)$ defined in (4.23). The complete specification of this latter quantity requires a fairly detailed knowledge of the scattering system, and is generally a formidable computational task. Nevertheless certain general properties of $G^{Be}(0)$ are readily established and are germane to a description of the structure of $S(\underline{\kappa},\Delta\omega)$.

Consider the thermal average of the product of the time-independent density operators for the species (A , B). With (4.5) or (4.10) this quantity may be written either as

$$\langle g^A(Q)\, g^B(Q')\rangle_T = \Big\langle \sum_\alpha^{N^A} \sum_\beta^{N^B} \delta(Q-Q^\alpha)\,\delta(Q'-Q^\beta)\Big\rangle_T \quad (4.30)$$

or as

$$\langle g^A(Q)\, g^B(Q')\rangle_T = F^A(Q)\,F^B(Q') +$$

$$+ \langle \delta g^A(Q)\, \delta g^B(Q')\rangle_T \quad (4.31)$$

where $Q \equiv \underline{x}, \underline{v}$. As before we assume the functions F^A, F^B are space-and-time independent Maxwellian functions of velocity F_m^A, F_m^B. For later convenience we denote

$$G^{AB}(Q,Q') \equiv \langle \delta g^A(Q)\, \delta g^B(Q') \rangle_T . \tag{4.32}$$

Since

$$\int d^3Q\, d^3Q' \langle g^A(Q)\, g^B(Q') \rangle_T = N^A N^B \tag{4.33}$$

by definition of the density operators, it follows that

$$\int d^3Q\, d^3Q'\, G^{AB}(Q,Q') = 0. \tag{4.34}$$

Our interest here is in the functions G^{AB}, the Fourier transform $G^{AB}(\underline{k})$ being needed for (4.29).

We begin our analysis by separating (4.30) into two terms;

$$\langle g^A(Q) g^B(Q') \rangle_T = \delta_{AB}\, \delta(Q-Q') \Big\langle \sum_{\alpha}^{N^A} \delta(Q-Q^\alpha) \Big\rangle_T +$$

$$+ \Big\langle \sum_{\alpha}^{N^A} \sum_{\substack{\beta \neq \alpha \\ \text{if } A=B}}^{N^B} \delta(Q-Q^\alpha)\, \delta(Q'-Q^\beta) \Big\rangle_T . \tag{4.35}$$

We take the system Hamiltonian $H = T+V$ to be as given in (4.7), and define

$$Z_T \equiv \int d^3v^N e^{-T/\theta} , \quad Z_V \equiv \int d^3x^N e^{-V/\theta}$$

where the respective integrations run over the coordinates of all N particles in the system. We now write the second term in (4.35) as

$$< \sum_{\alpha,\beta,\,\beta\neq\alpha \text{ if } A=B}^{N^A,N^B} \delta(Q-Q^\alpha)\,\delta(Q'-Q^\beta) \Big>_T =$$

$$= \int d^6Q^N Z_T^{-1} Z_V^{-1} e^{-(T+V)/\theta} \sum_{\alpha,\beta,\,\beta\neq\alpha \text{ if } A=B}^{N^A,N^B} \delta(Q-Q^\alpha)\,\delta(Q'-Q^\beta)$$

$$= M^A(\underline{r})\,M^B(\underline{r}') \sum_{\alpha,\beta,\,\beta\neq\alpha \text{ if } A=B}^{N^A,N^B} \int d^3x^N Z_V^{-1} e^{-V/\theta}\,\delta(\underline{x}-\underline{x}^\alpha)\,\delta(\underline{x}'-\underline{x}^\beta)$$

$$\equiv M^A(\underline{r})\,M^B(\underline{r}')\,n_2^{AB}(\underline{x},\underline{x}'), \tag{4.36}$$

where

$$n^A M^A(\underline{r}) = F_M^A(\underline{r})$$

and n^A is the number density of the A^{th} species.

To determine the functions n_2^{AB} defined in (4.36) we take the gradient of (4.36) with respect to $\underline{x}$.

$$\frac{\partial n_2^{AB}(\underline{x},\underline{x}')}{\partial \underline{x}} = \sum_{\alpha,\beta,\,\beta\neq\alpha \text{ if } A=B}^{N^A,N^B} \int d^3x^N Z_V^{-1} e^{-V/\theta}\,\delta(\underline{x}-\underline{x}^\alpha)\,\delta(\underline{x}'-\underline{x}^\beta)$$

$$= -\frac{1}{\theta} \sum_{\alpha,\beta,\,\beta\neq\alpha \text{ if } A=B}^{N^A,N^B} \int d^3x^N Z_V^{-1}\,\delta(\underline{x}-\underline{x}^\alpha)\,\delta(\underline{x}'-\underline{x}^\beta)\, e^{-V/\theta} \frac{\partial V}{\partial \underline{x}^\alpha} \cdot \tag{4.37}$$

From (4.7) we have

$$V = \frac{1}{2} \sum_{C} \sum_{\eta,\sigma}^{N^C,N^C} V^{CC}(|\underline{x}^\eta - \underline{x}^\sigma|) + \sum_{\substack{C,D \\ C\neq D}} \sum_{\eta,\sigma}^{N^C,N^D} V^{CD}(|\underline{x}^\eta - \underline{x}^\sigma|)$$

and hence

$$\sum_{\alpha}^{N^A} \frac{\partial V}{\partial \underline{X}^\alpha} = \sum_{\alpha}^{N^A} \sum_{D} \sum_{\substack{\sigma \\ \sigma \neq \alpha \text{ if } D=A}}^{N^D} \frac{\partial V^{AD}(|\underline{X}^\alpha - \underline{X}^\sigma|)}{\partial \underline{X}^\alpha} + \frac{1}{2} \sum_{\alpha,\sigma}^{N^A,N^A} \frac{\partial V^{AA}(|\underline{X}^\sigma - \underline{X}^\alpha|)}{\partial \underline{X}^\alpha} \cdot \tag{4.38}$$

In the following we will neglect the second term on the right in (4.38). This term is proportional to the force exerted on a particle by that same particle, and we expect that the effects of neglecting this term will not be manifest in any observable results.

The relation (4.37) now becomes

$$\frac{\partial n_2^{AB}(\underline{X},\underline{X}')}{\partial X} =$$

$$= -\frac{1}{\Theta} \sum_{D} \sum_{\substack{\alpha,\beta,\sigma \\ \beta \neq \alpha \text{ if } B=A \\ \sigma \neq \alpha \text{ if } D=A}}^{N^A,N^B,N^D} \int d^3X^N Z_V^{-1} \delta(\underline{X}-\underline{X}^\alpha)\delta(\underline{X}'-\underline{X}^\beta) e^{-V/\Theta} \frac{\partial V^{AD}(|\underline{X}^\alpha - \underline{X}^\sigma|)}{\partial \underline{X}^\alpha}$$

$$= -\frac{1}{\Theta} \sum_{D} \sum_{\substack{\alpha,\beta,\sigma \\ \beta \neq \alpha \text{ if } B=A \\ \sigma \neq \alpha \text{ if } D=A}}^{N^A,N^B,N^D} \int d^3X^N Z_V^{-1} \delta(\underline{X}-\underline{X}^\alpha)\delta(\underline{X}'-\underline{X}^\beta) \int d^3X'' \delta(\underline{X}''-\underline{X}^\sigma)(*) e^{-V/\Theta} \frac{\partial V^{AD}(|\underline{X}^\alpha - \underline{X}''|)}{\partial \underline{X}^\alpha} \cdot$$

We now separate this last into two terms; one for $\sigma = \beta$ (and hence $B = D$) and one for $\sigma \neq \beta$:

$$\frac{\partial n_2^{AB}(\underline{X},\underline{X}')}{\partial \underline{X}} =$$

$$-\frac{1}{\Theta} \sum_{\substack{\alpha,\beta \\ \beta \neq \alpha \text{ if } B=A}}^{N^A,N^B} \int d^3X^N d^3X'' Z_V^{-1} \delta(\underline{X}-\underline{X}^\alpha)\delta(\underline{X}'-\underline{X}^\beta)\delta(\underline{X}''-\underline{X}^\beta) e^{-V/\Theta} \frac{\partial V^{AB}(|\underline{X}-\underline{X}''|)}{\partial \underline{X}} -$$

$$-\frac{1}{\theta}\sum_{D}\sum_{\substack{\alpha,\beta,\sigma \\ \beta\neq\alpha \text{ if } B=A,\ \sigma\neq\alpha \text{ if } D=A,\ \sigma\neq\beta}}^{N^A,N^B,N^D}\int d^3x''\,\frac{\partial V^{AD}(|\underline{x}-\underline{x}''|)}{\partial \underline{x}}\int d^3x^N\,\delta(\underline{x}-\underline{x}^\alpha)\,\delta(\underline{x}'-\underline{x}^\beta)\,\delta(\underline{x}''-\underline{x}^\sigma)\,(x)\,Z_V^{-1}e^{-V/\theta}\,. \tag{4.39}$$

Recalling tne definition (4.36) of $n_2^{AB}(\underline{x},\underline{x}')$ and introducing

$$n_3^{ABD}(\underline{x},\underline{x}',\underline{x}'') \equiv \sum_{\substack{\alpha,\beta,\sigma \\ \beta\neq\alpha \text{ if } B=A;\ \sigma\neq\alpha \text{ if } D=A;\ \sigma\neq\beta}}^{N^A,N^B,N^D}\int d^3x^N\,\delta(\underline{x}-\underline{x}^\alpha)\,\delta(\underline{x}'-\underline{x}^\beta)\,\delta(\underline{x}''-\underline{x}^\sigma)\,Z_V^{-1}\,e^{-V/\theta}\,,$$

the relation (4.39) becomes

$$\frac{\partial n_2^{AB}(\underline{x},\underline{x}')}{\partial \underline{x}} + \frac{1}{\theta}\,n_2^{AB}\,\frac{\partial V^{AB}(|\underline{x}-\underline{x}'|)}{\partial \underline{x}} +$$

$$+\frac{1}{\theta}\sum_{D}\int d^3x''\,\frac{\partial V^{AD}(|\underline{x}-\underline{x}''|)}{\partial \underline{x}}\,n_3^{ABD}(\underline{x},\underline{x}',\underline{x}'') = 0\,. \tag{4.40}$$

To compute the two-particle correlation functions n_2^{AB}, we write n_3^{ABD} as

$$n_3^{ABD}(\underline{x},\underline{x}',\underline{x}'') = n^A n^B n^D + n^A\,n_2^{BD}(\underline{x}',\underline{x}'') + n^B\,n_2^{AD}(\underline{x},\underline{x}'') +$$

$$+ n^D\,n_2^{AB}(\underline{x},\underline{x}') + h_3^{ABD}(\underline{x},\underline{x}',\underline{x}'') \tag{4.41}$$

where h_3^{ABD} is a three particle correlation. Inserting (4.41) in (4.40) and neglecting h_3 then yields the system or equations

$$\frac{\partial n_2^{AB}(\underline{x},\underline{x}')}{\partial \underline{x}} + \frac{1}{\theta}\,n_2^{AB}\,\frac{\partial V^{AB}(|\underline{x}-\underline{x}'|)}{\partial \underline{x}} +$$

$$+\frac{n^A}{\theta}\sum_{D}\int d^3x''\,\frac{\partial V^{AD}(|\underline{x}-\underline{x}''|)}{\partial \underline{x}}\,n_2^{BD}(\underline{x}',\underline{x}'') = 0\,. \tag{4.42}$$

Thus given the system potentials, the correlations may in principle be computed from (4.42).

Two pertinent properties of the n_2^{AB} can be demonstrated without specifying the potentials. To obtain the first of these we introduce a change of variables in (4.42) according to

$$\underset{\sim}{r} \equiv \underset{\sim}{x} - \underset{\sim}{x}', \qquad \underset{\sim}{r}' \equiv \underset{\sim}{x}'' - \underset{\sim}{x}' .$$

This gives

$$\frac{\partial n_2^{AB}}{\partial \underset{\sim}{r}} + \frac{1}{\theta} n_2^{AB} \frac{\partial V^{AB}(|\underset{\sim}{r}|)}{\partial \underset{\sim}{r}} + \frac{n^A}{\theta} \int d^3 r' \frac{\partial V^{AD}(|\underset{\sim}{r} - \underset{\sim}{r}'|)}{\partial \underset{\sim}{r}} n_2^{BD}(\underset{\sim}{r}') = 0. \tag{4.43}$$

Equation (4.43) is clearly invariant under the transformation $\underset{\sim}{r} \to -\underset{\sim}{r}$. If we now add a constant vector $\underset{\sim}{a}$ to $\underset{\sim}{r}$ such that

$$|\underset{\sim}{r} + \underset{\sim}{a}| = |\underset{\sim}{r}|,$$

we find that (4.43) is unchanged. It follows that n_2^{AB} is a function of $|\underset{\sim}{r}|$ alone; i.e.,

$$n_2^{AB} = n_2^{AB}(|\underset{\sim}{r}|) = n_2^{AB}(|\underset{\sim}{x} - \underset{\sim}{x}'|). \tag{4.44}$$

In the sequel it will be necessary to have on hand information regarding the normalization of n_2^{AB}. Since the normalization is already specified by (4.33), we merely combine (4.30), (4.31), (4.35), and (4.36) to find $G^{AB}(Q,Q')$ as a function of n_2^{AB}:

$$G^{AB}(\underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{v}, \underset{\sim}{v}') = \delta_{AB} \delta(\underset{\sim}{x} - \underset{\sim}{x}') \delta(\underset{\sim}{v} - \underset{\sim}{v}') n^A M^A(\underset{\sim}{v}) +$$

$$+ M^A(\underset{\sim}{v}) M^B(\underset{\sim}{v}') \left[n_2^{AB}(\underset{\sim}{x}, \underset{\sim}{x}') - n^A n^B \right]. \tag{4.45}$$

Integrating now over ($\underset{\sim}{x}$, $\underset{\sim}{x}'$, $\underset{\sim}{v}$, $\underset{\sim}{v}'$) we find, with (4.34),

$$\int d^3x\, d^3x'\, n_2^{AB}(\underline{x},\underline{x}') = N^A(N^B - \delta_{AB}) \tag{4.46}$$

or

$$\int d^3r\, n_2^{AB}(r) = n^A(N^B - \delta_{AB}).$$

We return now to our discussion of the scattering function.

Fourier transforming $G^{Be}(\underline{x},\underline{x}',\underline{v},\underline{v}')$, we have

$$G^{Be}(\underline{k},\underline{v},\underline{v}',o) = \int d^3x\, d^3x'\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\, G^{Be}(\underline{x},\underline{x}',\underline{v},\underline{v}')$$

where $G^{Be}(o)$ is given in (4.23). With (4.44) it follows that

$$G^{Be}(\underline{k},\underline{v},\underline{v}',o) = G^{Be}(-\underline{k},\underline{v},\underline{v}',o), \tag{4.47}$$

and since n_2^{AB} is real we also have

$$G^{Be}(\underline{k},\underline{v},\underline{v}',o) = G^{Be*}(\underline{k},\underline{v},\underline{v}',o). \tag{4.48}$$

Combining (4.29 and (4.47) gives

$$S(\underline{k},\Delta\omega) = S(-\underline{k},-\Delta\omega). \tag{4.49}$$

It follows easily with (4.27), (4.29) and (4.48) that $S(\underline{k},\Delta\omega)$ is real, as we would expect.[5] With (4.28) we also have

$$G^{ee}(\underline{k},t) = G^{ee}(\underline{k},-t) = G^{ee*}(\underline{k},t). \tag{4.50}$$

Combining (4.2) and (4.50) we find

$$S(\underline{k},\Delta\omega) = \frac{1}{\pi N^e}\, Re \int_0^\infty dt\, e^{-it\Delta\omega}\, G^{ee}(\underline{k},t). \tag{4.51}$$

Hence when G^{ee} satisfies (4.50) we can compute the scattering function

having only $\mathcal{S}^{ee}$ for $t > 0$.

At this point, instead of inverting the Laplace transform as in (4.24) and (4.25), an operation straight-forward in principle but generally Herculean in practice, we adopt a procedure due to Rostoker.[6] Thus we consider first the identity

$$\begin{aligned} S(\underline{k},\nu) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu' \int_{-\infty}^{\infty} dt\, e^{i(\nu'-\nu)t} S(\underline{k},\nu') \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu' \left[\int_{0}^{\infty} dt\, e^{i(\nu'-\nu)t} + \int_{-\infty}^{0} dt\, e^{i(\nu'-\nu)t}\right] S(\underline{k},\nu') \\ &\equiv S_+(\underline{k}, i\nu) + S_-(\underline{k}, i\nu) \end{aligned} \tag{4.52}$$

where

$$S_\pm = \frac{S(\underline{k},\nu)}{2} \pm \frac{i}{2\pi} P\int_{-\infty}^{\infty} \frac{d\nu'}{\nu'-\nu} S(\underline{k},\nu') \tag{4.53}$$

since[4]

$$\begin{aligned} \int_0^\infty dt\, e^{i(\nu'-\nu)t} &= \pi\delta(\nu'-\nu) + iP\frac{1}{\nu'-\nu} \\ \int_{-\infty}^0 dt\, e^{i(\nu'-\nu)t} &= \pi\delta(\nu'-\nu) - iP\frac{1}{\nu'-\nu} \end{aligned} \tag{4.54}$$

with P indicating principal value.

Now let $S_+(\underline{k}, p)$, $p = \sigma + i\nu$ be the Laplace transform of some function, and consider further the following inverse transformation

$$\lim_{\sigma\to 0+} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, e^{pt} S_+(\underline{k}, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{i\nu t} S_+(\underline{k}, i\nu),$$

$$\lim_{\sigma\to 0+} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, e^{pt}\, S_+(\underline{\kappa},p) =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{i\nu t}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu' \int_0^{\infty} dt'\, e^{i(\nu'-\nu)t'}\, S(\underline{\kappa},\nu')$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu'\, S(\underline{\kappa},\nu')\int_0^{\infty} dt'\, e^{i\nu' t'}\,\delta(t-t')$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{i\nu t}\, S(\underline{\kappa},\nu),\ t>0\ ;\ =0,\ t<0. \qquad (4.55)$$

In arriving at (4.55) we have used the definition (4.53) of the function $S_+(\underline{\kappa},i\nu)$. Substituting (4.2) into (4.55) we have

$$\lim_{\sigma\to 0+} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, e^{pt}\, S_+(\underline{\kappa},p) = \frac{1}{(2\pi)^2 N^e}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\nu\, e^{i\nu(t-t')}\, G^{ee}(\underline{\kappa},t')$$

$$= \frac{1}{2\pi N^e}\, G^{ee}(\underline{\kappa},t),\ t>0;\ =0,\ t<0. \qquad (4.56)$$

It follows that S_+ and the Laplace-transformed correlation function for $t>0$ are related via

$$\lim_{\sigma\to 0+} S_+(\underline{\kappa},\sigma+i\nu) = \lim_{\sigma\to 0+} \frac{1}{2\pi N^e}\, G_+^{ee}(\underline{\kappa},\sigma+i\nu). \qquad (4.57)$$

From (4.53) and the knowledge that the scattering function is real we have, with (4.51),

$$S(\underline{\kappa},\Delta\omega) = \frac{1}{\pi N^e}\, \mathrm{Re}\, \lim_{\sigma\to 0+} G_+^{ee}(\underline{\kappa},\sigma+i\Delta\omega).$$

We now combine this last with (4.3), (4.4), (4.10), and (4.22). After Laplace transforming the forward scattering term $(2\pi)^3 n^e N^e \delta(\underline{\kappa})$ and noting that

$$\frac{1}{\pi N^e}\,\mathrm{Re}\lim_{\sigma\to 0+}\frac{(2\pi)^3 n^e N^e \delta(\underline{\kappa})}{\sigma + i\Delta\omega} = (2\pi)^3 n^e \delta(\underline{\kappa})\,\mathrm{Re}\left[\delta(\Delta\omega) + \frac{i}{\pi}P\frac{1}{\Delta\omega}\right]$$

$$= (2\pi)^3 n^e \delta(\underline{\kappa})\delta(\Delta\omega)$$

we obtain

$$S(\underline{\kappa},\Delta\omega) = (2\pi)^3 n^e \delta(\underline{\kappa})\delta(\Delta\omega) +$$

$$+ \frac{1}{\pi N^e}\,\mathrm{Re}\lim_{\sigma\to 0+}\sum_B \frac{\Delta_B}{\Delta}\int \frac{d^3v\, d^3v'\, G^{Be}(\underline{\kappa},\underline{v},\underline{v}',0)}{\sigma + i\Delta\omega - i\underline{\kappa}\cdot\underline{v}} \quad . \quad (4.58)$$

2. The Electron Scattering Function For A Partially Ionized Gas

In this section we derive the electron scattering function for a system composed of electrons and one species, respectively, of positive ions and neutral atoms. The extension to a more general multicomponent system is straightforward but adds considerably to the algebraic complexity. We will continue to assume that the equilibrium plasma is characterized by a single temperature common to each species, and is free from spatial gradients or external fields. We present two different treatments, with somewhat different results.

A. Reversible Theory

From (4.58) we have, neglecting forward scattering,

$$S(\underline{\kappa},\Delta\omega) = \frac{1}{\pi N^e}\,\mathrm{Re}\lim_{\sigma\to 0+}\frac{1}{\Delta}(x)$$

$$(x)\left[\left\{1 + \Lambda_{ii}\left(1 + \Lambda_{nn} + \frac{\Lambda_{ni}}{\Lambda_{ii}} - \Lambda_{ni}\frac{\Lambda_{in}}{\Lambda_{ii}}\right\}G_o^{ee} -\right.\right.$$

$$- \Lambda_{ei}\left(1 + \Lambda_{nn} - \Lambda_{ni}\frac{\Lambda_{en}}{\Lambda_{ei}}\right)G_o^{ie}\left\{1 -\right.$$

$$\left.\left. - \frac{\Lambda_{in} - (\Lambda_{en}/\Lambda_{ei})(1+\Lambda_{ii})}{1 + \Lambda_{nn} - \Lambda_{ni}(\Lambda_{en}/\Lambda_{ei})}\right\}\frac{G_o^{ne}}{G_o^{ie}}\right] \quad (4.59)$$

where

$$\Delta = 1 + \Lambda_{ee}\left(1+\Lambda_{nn}-\Lambda_{ni}\Lambda_{in}\right) +$$

$$+ \Lambda_{ii}\left(1+\Lambda_{nn} - \Lambda_{ne}\Lambda_{en} + \frac{\Lambda_{nn}}{\Lambda_{ii}} - \Lambda_{en}\frac{\Lambda_{ne}}{\Lambda_{ii}} - \Lambda_{ni}\frac{\Lambda_{in}}{\Lambda_{ii}}\right)$$

and

$$G_o^{Be}(\underline{k},p) \equiv \int \frac{d^3v\, d^3v'}{p - i\underline{k}\cdot\underline{v}} \left\langle \delta g^B(\underline{k},\underline{v},0)\, \delta g^e(\underline{k},\underline{v}',0)\right\rangle_T .$$

In writing (4.59) we have made use of the identities

$$\Lambda_{ei}\Lambda_{ie} = \Lambda_{ee}\Lambda_{ii}$$

$$\Lambda_{\alpha\beta}\Lambda_{\beta\gamma}\Lambda_{\gamma\alpha} = -\Lambda_{\beta\alpha}\Lambda_{\gamma\beta}\Lambda_{\alpha\gamma}.$$

We have displayed the scattering function in (4.59) in a form that will facilitate an estimate of the significance of the terms involving the neutrals. To this end it will prove useful to write Λ_{AB} in the form

$$\lim_{\sigma\to 0^+} \Lambda_{AB} = \frac{V^{AB}(k)}{\theta} n^A \left[1 - \frac{\Delta\omega}{k v_{oA}} \pi^{-1/2} \int_{\Lambda -\infty}^{\infty} \frac{du\, e^{-u^2}}{u + \frac{\Delta\omega}{k v_{oA}}}\right] \qquad (4.60)$$

where

$$v_{oA} = (2\theta/m_A)^{1/2}$$

and the sumbol "Λ" next to the integral means the path of integration is deformed above the singularity.

To estimate $V^{AB}(k)$ when either or both of the pair (A , B) is a neutral atom, we assume the potential may be approximated by a Yukawa potential, $V^{AB}(r) \simeq c_{AB}\, r^{-1} \exp(-r/a)$

where a is the effective range of the potential,

$$a \sim 10^{-8}\,cm$$

and C_{AB} is a constant to be determined. We have easily

$$V^{AB}(\underline{k}) = 4\pi C_{AB}\,\frac{a^2}{1+k^2a^2}\ .$$

With $k^{-1} \simeq \lambda >> 10^{-8}\,cm$, $k^2a^2 << 1$ and thus

$$V^{AB}(\underline{k}) \simeq 4\pi a^2 C_{AB}\ . \tag{4.61}$$

To estimate the constant C_{AB} we recall that the center-of-mass differential elastic scattering cross-section for the pair (A , B) having relative momentum $\hbar\underline{k}$ is given by[7]

$$\sigma^{AB}(\theta) = \left|\frac{\mu_{AB}}{2\pi\hbar^2}\,V^{AB}(\underline{k})\right|^2$$

where μ_{AB} is the relative mass. From (4.61) it follows that, in the energy range of interest, $\sigma^{AB}(\theta)$ is approximately isotropic. Writing

$$\sigma^{AB} \equiv \int d\Omega\,\sigma^{AB}(\theta) = 10^{-16}\,\sigma_0\ (cm^2)$$

where[8] σ_0 is of order $1-10\,cm^2$, we have

$$C_{AB}\ (erg\cdot cm) \sim \frac{\frac{1}{2}\times 10^{-62}\,\sigma_0^{1/2}\ (cm)}{a^2\ (cm^2)\,\mu_{AB}\ (gms)}\ .$$

To estimate the various terms in (4.59) we assume $m_n \simeq m_p$ and take m_n to be the mass of the C^{12} atom, $m_n \simeq 2.5\times 10^{-23}\,gm$.

Assuming for simplicity equal electron and ion densities, we have, for $\lambda = 6943\ \text{Å}$.

$$\frac{\Lambda_{nn}}{\Lambda_{ii}} \simeq \frac{n^n}{n^e}\frac{\Lambda_{in}}{\Lambda_{ii}} \propto \frac{c_{nn}a^2}{(e^2/\kappa^2)}\frac{n^n}{n^e}$$

$$\propto 2\times10^{-13}\frac{n^n}{n^e}\sigma_0^{1/2}(cm)\sin^2\frac{\theta}{2},$$

$$\frac{\Lambda_{ne}}{\Lambda_{ii}} \propto \frac{n^n}{n^e}\frac{\Lambda_{en}}{\Lambda_{ei}} \propto 0.9\times10^{-8}\frac{n^n}{n^e}\sigma_0^{1/2}(cm)\sin^2\frac{\theta}{2}. \qquad (4.62)$$

Here θ is the laboratory scattering angle,

$$|\underline{K}| = |\underline{k} - \underline{k}'| \propto 4\pi\lambda^{-1}\sin\frac{\theta}{2}$$

and appears only in the argument of the sine, in contrast with the kinetic temperature, also denoted by θ .

Noting that[9] the quantity in square brackets in (4.60) is of order one or less for all values of $\Delta\omega/\kappa v_0$, we find (for $\lambda = 6943\ \text{Å}$)

$$\lim_{\sigma\to 0+}\begin{bmatrix}\Lambda_{nn}\\ \Lambda_{ni}\\ \Lambda_{in}\end{bmatrix} \simeq 1.6\times10^{-27}(cm^2 ev)\frac{\sigma_0^{1/2}(cm)}{\theta(ev)}\begin{bmatrix}n^n\\ n^n\\ n^e\end{bmatrix},$$

$$\lim_{\sigma\to 0+}\begin{bmatrix}\Lambda_{en}\\ \Lambda_{ne}\end{bmatrix} \propto 4.6\times10^{-23}(cm^2 ev)\frac{\sigma_0^{1/2}(cm)}{\theta(ev)}\begin{bmatrix}n^e\\ n^n\end{bmatrix}, \qquad (4.63)$$

and

$$\lim_{\sigma\to 0+}\Lambda_{ii}\frac{\Lambda_{en}}{\Lambda_{ei}} \simeq 4.6\times10^{-23}(cm^2 ev)\frac{\sigma_0^{1/2}(cm)}{\theta^2(ev^2)}n^e, \quad n^i \text{ in } cm^{-3}.$$

It is clear that the quantities in (4.62) and (4.63) are smaller for larger m_n (or m_i) and are largest (about a factor of twelve larger than the abofe values) for a hydrogeneous scattering system.

To determine the significance of electron-neutral correlations in the scattering function, we must estimate the magnitude of the quantity (see (4.45) and (4.59))

$$\frac{G_o^{ne}}{G_o^{ie}} = \frac{\int d^3r\, e^{i\underline{k}\cdot\underline{r}} \left[n_2^{ne}(r) - n^n n^e \right]}{\int d^3r\, e^{i\underline{k}\cdot\underline{r}} \left[n_2^{ie}(r) - n^i n^e \right]} \quad , \tag{4.64}$$

where we have assumed $m_n = m_i$. To facilitate computation of the pair correlation functions $n_2^{\beta e}$ appearing in (4.64) we introduce the assumption that the average distance between any pair of particles is large compared with the effective range of the charge-neutral or neutral-neutral potentials. This range being typically of the order of 10^{-8} cm, the assumption implies particle densities small compared with $10^{24}\ cm^{-3}$. Under this assumption it follows that the contribution of neutrals to the correlation between charged particles may be neglected. We thereby obtain from (4.43) a pair of equations for $n_2^{ee}(r)$, $n_2^{ie}(r)$. Taking advantage of (4.44) we can write these as

$$\frac{dn_2^{ee}}{dr} - \frac{e^2}{\Theta}\frac{1}{r^2} n_2^{ee} + \frac{n^e e^2}{\Theta}\frac{d}{dr}\int d^3r' \frac{1}{|\underline{r}-\underline{r}'|} n_2^{ee}(r') - \\ - \frac{n^e q_i e^2}{\Theta}\frac{d}{dr}\int d^3r' \frac{1}{|\underline{r}-\underline{r}'|} n_2^{ie}(r') = 0$$

$$\frac{dn_2^{ie}}{dr} + \frac{q_i e^2}{\Theta}\frac{1}{r^2} n_2^{ie} - \frac{n^e e^2}{\Theta}\frac{d}{dr}\int d^3r' \frac{1}{|\underline{r}-\underline{r}'|} n_2^{ee}(r') + \\ + \frac{n^e q_i e^2}{\Theta}\frac{d}{dr}\int d^3r' \frac{1}{|\underline{r}-\underline{r}'|} n_2^{ie}(r') = 0$$

where $q_i e$ is the ionic charge and $n^e = q_i n^e$. The equations above are similar to a pair of equations treated by Lamb[10] for a singly ionized

gas; with $q_i = 1$ Lamb's equations coincide with ours. Following Lamb, we first perform the angular integrations and then differentiate with respect to r. The result is a pair of differential equations

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dn_2^{ee}}{dr}\right) - \frac{e^2/\theta}{r^2}\frac{dn_2^{ee}}{dr} + \frac{4\pi n^e e^2}{\theta}\left(q_i n_2^{ie} - n_2^{ee}\right) = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dn_2^{ie}}{dr}\right) + \frac{q_i e^2/\theta}{r^2}\frac{dn_2^{ie}}{dr} + \frac{4\pi n^e e^2}{\theta}\left(n_2^{ee} - q_i n_2^{ie}\right) = 0. \quad (4.65)$$

The system (4.65) permits a non-trivial constant solution. Taking

$$n_2^{ee} = A_1, \quad n_2^{ie} = A_2,$$

we find

$$A_1 = q_i A_2 .$$

Lamb[10] has shown that for distances r large compared with e^2/θ, the first derivatives in (4.65) contribute negligibly to the solutions. Neglecting these terms, it is then a simple matter to show by direct substitution that the functions

$$f^{ee} = \frac{B_1}{r}e^{-r/\lambda_D}, \quad f^{ie} = \frac{B_2}{r}e^{-r/\lambda_D}$$

satisfy (4.65) provided $B_1 = -q_i B_2$, and where

$$\lambda_D^{-2} = \frac{4\pi n^e e^2(1+q_i)}{\theta}.$$

We thus take the solutions of (4.65) to be

$$n_2^{ee}(r) = -\frac{q_i B_2}{r}e^{-r/\lambda_D} + q_i A_2$$

$$n_2^{ie}(r) = \frac{B_2}{r}e^{-r/\lambda_D} + A_2 . \qquad r >> \frac{q_i e^2}{\theta}$$

The constants A_2, B_2 are determined easily with the normalization conditions (4.46), so that finally we have

$$n_2^{ee}(r) = n^e n^e \left[-\frac{e^2}{\theta} \frac{e^{-r/\lambda_D}}{r} + 1 - \frac{1}{N} \right]$$

$$n_2^{ie}(r) = n^i n^e \left[\frac{q_i e^2}{\theta} \frac{e^{-r/\lambda_D}}{r} + 1 - \frac{1}{N} \right] \qquad (4.66)$$

where $N = N^e + N^i$.

To estimate the significance of electron-neutral correlations as per (4.64), we write $n_2^{ne}(r)$ in the form

$$n_2^{ne}(r) = n^n n^e \left[\phi^{ne}(r) + 1 - \frac{1}{N} \right]$$

where ϕ^{ne} is a function that we expect will differ appreciably from zero only for $r \lesssim a \sim 10^{-8}$ cm. ...ge of the electron-neutral potential. The normalization condition (4.46) gives

$$\int d^3r \, \phi^{ne}(r) = \frac{V}{N} = (n^e + n^n)^{-1} .$$

Substituting the above form of n_2^{ne} together with n_2^{ie} from (4.66) into (4.64) now gives, neglecting the contribution from terms in $1/N$ (which corresponds to the neglect of forward scattering)

$$\frac{\sigma_o^{ne}}{\sigma_o^{ie}} = n^n (1+q_i)(1+K^2\lambda_D^2) \int d^3r \, e^{iK\cdot r} \phi^{ne}(r) .$$

With $K \sim \lambda^{-1} \sim (7 \times 10^{-5} \text{ cm})^{-1}$ where λ is the photon wavelength, we approximate the integral above;

$$\int d^3r\, e^{i\underline{k}\cdot\underline{\Sigma}}\,\phi^{ne}(r) = \int d^3r\, e^{ia\underline{k}\cdot\underline{\Sigma}/a}\,\phi^{ne}(r)$$

$$a \sim 10^{-8}\ \text{cm}$$

$$\simeq \int_{|\underline{\Sigma}|<a} d^3r\,\phi^{ne}(r)$$

$$\simeq (n^n + n^e)^{-1}\ .$$

Thus finally we have, with $\lambda = 7\times10^{-5}$ cm ,

$$\frac{\sigma_o^{ne}}{\sigma_o^{ie}} \simeq \left(1+\frac{n^e}{n^n}\right)^{-1}(1+q_i)(1+k^2\lambda_D^2)$$

$$\simeq \left(1+\frac{n^e}{n^n}\right)^{-1}\left[1+0.9\times10^{16}\,\frac{\Theta(ev)}{n^e(cm^{-3})}\sin^2\frac{\theta}{2}\right] \qquad (q_i=1).$$

We are now in a position to estimate the magnitudes of the terms involving neutrals in (4.59). First, we note from (4.62) that Δ_{nn}/Δ_{ii} is negligible unless $n^n/n^e \simeq 10^{12}$ or larger. In such cases the total light intensity scattered by the electrons is greatly exceeded by that scattered by the neutrals. Since our interest here is in the influence of neutral atoms on the <u>electron</u> scattering function, and since we do not anticipate an experiment in which scattering from electrons could be observed at such extreme density ratios, we will not consider these extreme cases here.

It follows from (4.63) that (4.59) may be further simplified provided

$$\frac{\sigma_o^{1/2}(cm)\,n^n(cm^{-3})}{\Theta(ev)}\ ,\quad \frac{\sigma_o^{1/2}(cm)\,n^e(cm^{-3})}{\Theta(ev)} \ll 10^{27}\ . \qquad (4.67)$$

Since particle densities do not generally exceed $10^{23}\,cm^{-3}$ (e.g., $n^n \simeq 10^{23}\,cm^{-3}$ for graphite; $n^n \simeq 10^{18}\times\Theta(ev)\,cm^{-3}$ at atmospheric pressure),

we would expect (4.66) to hold for almost all plasmas of interest here. Thus when (4.62) and (4.63) are satisfied (4.59) reduces to

$$S(\underline{K}, \Delta\omega) \simeq$$

$$\frac{1}{\pi N^e} \mathrm{Re} \lim_{\sigma \to 0+} \frac{1}{\Delta} \left[(1+\Lambda_{ii}) G_o^{ee} - \Lambda_{ei} G_o^{ie} \left\{ 1 + \frac{\Lambda_{en}}{\Lambda_{ei}} \Lambda_{ii} \frac{G_o^{ne}}{G_o^{ie}} \right\} \right] \quad (4.68)$$

where now

$$\Delta \simeq 1 + \Lambda_{ee} + \Lambda_{ii}(1 - \Lambda_{ne}\Lambda_{en}) .$$

The dependence of the scattering function displayed in (4.68) upon electron-neutral correlations (the term containing G_o^{ne}) disappears when

$$\frac{\sigma_o^{1/2}(cm)\, n^e(cm^{-3})}{\theta(ev)} << 10^{23} . \quad (4.69)$$

If in addition the condition

$$\frac{\sigma_o(cm^2)\, n^e(cm^{-3})\, n^n(cm^{-3})}{\theta^2(ev)} << 10^{45} \quad (4.70)$$

holds, then $\Lambda_{en}\Lambda_{ne} << 1$ and (4.68) reduces to the result[1] for a fully ionized gas. For $n^n \simeq 10^{23}\, cm^{-3}$, (4.70) becomes

$$\frac{\sigma_o(cm)\, n^e(cm^{-3})}{\theta^2(ev)} << 10^{22} . \quad (4.71)$$

It is thus apparent that, unless very high neutral <u>and</u> electron densities are present, together with relatively low temperatures, the present theory does not predict an observable effect of neutral atoms on the electron scattering function.

Before abandoning our quest for an observable effect of neutrals in the spectrum of electron-scattered photons, we turn to a somewhat different and more realistic formulation of the scattering function.

B. Irreversible Theory

It was noted in Chapter II above that, since there are generally many charged particles within range of each other simultaneously for most plasmas, the effect of close binary encounters may often be neglected. In this case one may represent the effects of particle interactions by an appropriate electric field term in the kinetic equation. In contrast, we would not expect such a "field representation" to be suitable for the representation of encounters between particles having ranges of interaction that are small compared with the mean interparticle distance. In the following discussion we adopt a scheme outlined in Chapter II designed to give a more realistic treatment of the interactions between charges and neutrals than that employed above.

We begin with equation (4.11) for the fluctuation operators of the A^{th} species;

$$\frac{\partial \delta g^A}{\partial t} + \underline{v} \cdot \frac{\partial \delta g^A}{\partial \underline{x}} - \frac{1}{m_A} \sum_B \int d^3x'' d^3v'' \frac{\partial V^{AB}(|\underline{x} - \underline{x}''|)}{\partial \underline{x}} \cdot$$

$$\cdot \frac{\partial}{\partial \underline{v}} \left[F^A(\underline{x}, \underline{v}, t)\, \delta g^B(\underline{x}'', \underline{v}'', t) + \delta g^A(\underline{x}, \underline{v}, t)\, F^B(\underline{x}'', \underline{v}'', t) \right] = 0. \tag{4.72}$$

At this point in ref. 1 and in Part 1 above, the assumption of thermal equilibrium was introduced for the target plasma. Before doing so here it is convenient to exploit the difference between the relatively

long-range Coulomb forces between charged particles and the relatively short-range charged-neutral and neutral-neutral forces. To this end, and with an eye on our ultimate goal, we multiply (4.72) by $\delta g^e(\underline{x}', \underline{v}', 0)$, average the result, and introduce

$$\Gamma^{Ae}(\underline{x}, \underline{x}', \underline{v}, t) \equiv \int d^3v' \langle \delta g^e(\underline{x}', \underline{v}', 0)\, \delta g^A(\underline{x}, \underline{v}, t) \rangle_T . \tag{4.73}$$

We thus obtain from (4.72)

$$\frac{\partial \Gamma^{Ae}}{\partial t} + \underline{v} \cdot \frac{\partial \Gamma^{Ae}}{\partial \underline{x}} - \frac{1}{m_A} \sum_B \int d^3x''\, d^3v'' \, \frac{\partial V^{AB}(|\underline{x} - \underline{x}''|)}{\partial \underline{x}} \cdot$$

$$\cdot \frac{\partial}{\partial \underline{v}} \left[F^A(\underline{x}, \underline{v}, t) \Gamma^{Be}(\underline{x}', \underline{x}'', \underline{v}'', t) + F^B(\underline{x}'', \underline{v}'', t) \Gamma^{Ae}(\underline{x}, \underline{x}', \underline{v}, t) \right] = 0. \tag{4.74}$$

We now introduce a change of variables according to

$$\underline{r} = \underline{x} - \underline{x}', \quad \underline{r}' = \underline{x}'' - \underline{x}',$$

so (4.74) takes the form

$$\frac{\partial \Gamma^{Ae}}{\partial t} + \underline{v} \cdot \frac{\partial \Gamma^{Ae}}{\partial \underline{r}} - \frac{1}{m_A} \sum_B \int d^3r'\, d^3v'' \, \frac{\partial V^{AB}(|\underline{r} - \underline{r}'|)}{\partial \underline{r}} \cdot$$

$$\cdot \frac{\partial}{\partial \underline{v}} \left[F^A(\underline{r}, \underline{v}, t) \Gamma^{Be}(\underline{r}', \underline{v}'', t) + F^B(\underline{r}', \underline{v}'', t) \Gamma^{Ae}(\underline{r}, \underline{v}, t) \right] = 0. \tag{4.75}$$

In writing (4.75) we have suppressed the dependence on the variable $\underline{x}'$. Our identification of the integrand in

$$\Gamma^{Ae}(0) = \int d^3v' \langle \delta g^A(\underline{x}, \underline{v}, 0)\, \delta g^e(\underline{x}', \underline{v}', 0) \rangle_T \tag{4.76}$$

as the equilibrium phase-space correlation function for the pair (A, e) implies that $\Gamma^{Ae}(0)$ is a function of $\underline{r}$ and $\underline{v}$ only. It

follows from (4.75) that $\Gamma^{Ae}(t)$ is a function of ξ, v, and t alone. The identification of $G^{Ae}(\xi, t)$,

$$G^{Ae} = \int d^3v\, \Gamma^{Ae}(\xi, v, t) \tag{4.77}$$

as a time dependent correlation function, and the relationship between the scattering function and the space-time Fourier transform of G^{ee} has been exploited above. The description of Γ^{Ae} by a linear transport equation, as in (4.75), has been suggested on the basis of a semi-intuitive argument by Nelkin and Ghatak,[11] and has been employed by them and by Yip and Nelkin[12] in a study of slow neutron scattering from liquids and dense gases. Recently Van Leeuwen and Yip[13] have derived a similar kinetic equation for Γ^{AB}, for short range potentials, from the cluster expansion of a one-particle distribution function.

At this point we introduce an approximation into the treatment of the charged-neutral and neutral-neutral interactions. This leads us ultimately to the binary collision description attained by Van Leeuwen and Yip,[13] and employed by Nelkin, et al.[11,12] We thus adopt the treatment outlined in Chapter II in going from equation (2.17) to the Boltzmann collision integral (2.24). Thus the terms in (4.75) involving the relatively short-range neutral interaction potentials $V^{An}(|\xi - \xi'|)$ are approximated as in Chapter II by a linear Boltzmann collision integral, and (4.75) becomes

$$\frac{\partial \Gamma^{ee}}{\partial t} + v \cdot \frac{\partial \Gamma^{ee}}{\partial \xi} - \frac{1}{m_e} \sum_{A=e,i} \int d^3r'\, d^3v'' \, \frac{\partial V^{eA}(|\xi - \xi'|)}{\partial \xi} \cdot$$

$$\cdot \frac{\partial}{\partial v} \left[F^e(\xi, v, t)\, \Gamma^{Ae}(\xi', v'', t) + F^A(\xi', v'', t)\, \Gamma^{ee}(\xi, v, t) \right] =$$

$$= \int d^3v'' \int d\Omega\, |\underline{v} - \underline{v}''|\, \sigma_{en} \Big[F^e(\underline{r}, \underline{v}_o, t)\, \Gamma^{ne}(\underline{r}, \underline{v}_o'', t) +$$

$$+ F^N(\underline{r}, \underline{v}_o'', t)\, \Gamma^{ee}(\underline{r}, \underline{v}_o, t) - F^e(\underline{r}, \underline{v}, t)\, \Gamma^{ne}(\underline{r}, \underline{v}'', t) -$$

$$- F^N(\underline{r}, \underline{v}'', t)\, \Gamma^{ee}(\underline{r}, \underline{v}, t) \Big], \qquad (4.78)$$

and similarly for Γ^{ie}. At this point we introduce the assumption that the target plasma is in the thermodynamic state, and hence that the singlet densities F^A, F^N are independent of space and time and are Maxwellian functions of velocity.

We have argued in the preceding section that electron-neutral correlations should, under most conditions, contribute negligibly to the electron scattering function. It is worth noting that Salpeter[14] has argued semi-intuitively that pair separations which are small compared with both the photon wavelength and the Debye length contribute negligibly to the spectrum of photons scattered by electron density fluctuations. This is in good agreement with the experimental observations of Ramsden and Davies.[15] This suggests that, since the electron-neutral correlation is significant only for separations of order 10^{-8} cm or less, it would be reasonable to ignore electron-neutral correlations in our computation of the scattering function. A somewhat different argument in support of this assumption is suggested by the observation that fluctuations induced in the neutral distribution by recoiling electrons should be insignificant. Referring to (4.73) we see that neglecting neutral fluctuations implies neglecting electron-neutral correlations. With these assumptions (4.78) becomes

$$\frac{\partial \Gamma^{ee}}{\partial t} + \underline{v} \cdot \frac{\partial \Gamma^{ee}}{\partial \underline{r}} - \frac{n^e}{m_e} \frac{\partial M^e(\underline{v})}{\partial \underline{v}} \cdot \frac{\partial}{\partial \underline{r}} \sum_{A=e,i} \int d^3r' d^3v'' V^{eA}(|\underline{r}-\underline{r}'|)\, \Gamma^{Ae}(\underline{r}', \underline{v}'', t) =$$

$$= n^n \int d^3v'' \int d\Omega\, |\underline{v}-\underline{v}''| \sigma_{en} \left[M^n(\underline{v}_0) \Gamma^{ee}(\underline{r}, \underline{v}_0, t) - M^n(\underline{v}'') \Gamma^{ee}(\underline{r}, \underline{v}, t) \right] \quad (4.79)$$

and similarly for Γ^{ii} . It is apparent that the neutral species collision operator in (4.79) conserves both electron and ion number densities, but does not conserve total momentum and kinetic energy in the system, in contrast with the collision operator in (4.78) which satisfies all three conservation laws. This, of course, results from our neglect of fluctuations in the neutral distribution.

An important consequence of the collision approximation manifest in (4.78) or (4.79) is that while (4.75) is invariant under the time reversal transformation $t \to -t$, $\underline{v} \to -\underline{v}$, these last two equations are not invariant. We will return to consider the consequences of this irreversibility shortly.

We now introduce a further approximation into the collision description in order to avoid the complexities of the collision operators in (4.79) in their present form. We thus replace these operators by the linearized version of the simple single parameter collision model first proposed by Bhatnagar, Gross, and Krook,[16] and discussed in Chapter II above. The model is constructed to satisfy, in this case, the requirement of number conservation for each species, preserves the irreversible nature of the above description, and provides a considerable simplification for the subsequent analysis. Finally then, the kinetic equations that we use are

$$\frac{\partial J^{ee}}{\partial t} + \vec{\nabla}\cdot\frac{\partial J^{ee}}{\partial \vec{r}} - \frac{e^2}{m_e}\int d^3r'\,d^3v''\,\frac{\partial V^{ee}(|\vec{r}-\vec{r}'|)}{\partial \vec{r}}\cdot\frac{\partial f^e}{\partial \vec{v}}\,w_e(\vec{r})\,J^{ee}_e(\vec{r}',\vec{v}'',t) -$$

$$-\frac{e^2}{m_i}\int d^3r'\,d^3v''\,\frac{\partial V^{ei}(|\vec{r}-\vec{r}'|)}{\partial \vec{r}}\cdot\frac{\partial f^i}{\partial \vec{v}}\,w_e(\vec{r})\,J^{ie}_i(\vec{r}',\vec{v}'',t) =$$

$$= \nu^{ne}\left[w_n(\vec{r})\int d^3v'\,J^{ee}_e(\vec{r}',\vec{v}',t) - J^{ee}_e(\vec{r},\vec{v},t)\right] \qquad (4.80)$$

$$\frac{\partial J^{ie}}{\partial t} + \vec{v}\cdot\frac{\partial J^{ie}}{\partial \vec{r}} - \frac{i^2}{m_i}\int d^3r'\,d^3v''\,\frac{\partial V^{ii}(|\vec{r}-\vec{r}'|)}{\partial \vec{r}}\cdot\frac{\partial f^i}{\partial \vec{v}}\,w_i(\vec{r})\,J^{ie}_i(\vec{r}',\vec{v}'',t) -$$

$$-\frac{i^2}{m_i}\int d^3r'\,d^3v''\,\frac{\partial V^{ie}(|\vec{r}-\vec{r}'|)}{\partial \vec{r}}\cdot\frac{\partial f^e}{\partial \vec{v}}\,w_i(\vec{r})\,J^{ee}_e(\vec{r}',\vec{v}'',t) =$$

$$= \nu^{ni}\left[w_n(\vec{r})\int d^3v'\,J^{ie}_i(\vec{r}',\vec{v}',t) - J^{ie}_i(\vec{r},\vec{v},t)\right]. \qquad (4.81)$$

The parameters ν^{ne}, ν^{ni} are clearly lectron-neutral and ion-neutral collision frequencies. Fourier-Laplace transforming (4.80) and (4.81) as before, we eventually find, with (4.77)

$$G^{ee}_+(\vec{k},p) = \Delta^{-1}\left(1+\Lambda_{ii}-\nu^{ni}\Omega^n\right)\int \frac{d^3v'\,J^{ee}_e(\vec{k},\vec{v}',0)}{p-i\vec{k}\cdot\vec{v}'+\nu^{ne}} -$$

$$-\Delta^{-1}\Lambda_{ei}\int \frac{d^3v'\,J^{ie}_i(\vec{k},\vec{v}',0)}{p-i\vec{k}\cdot\vec{v}'+\nu^{ni}} \qquad (4.82)$$

where

$$\Omega^A = \int \frac{d^3v'\,w_A(\vec{v}')}{p-i\vec{k}\cdot\vec{v}'+\nu^{nA}}, \qquad (4.83)$$

$$\Delta = \left(1+\Lambda_{ee}-\nu^{ne}\Omega^n\right)\left(1+\Lambda_{ii}-\nu^{ni}\Omega^n\right)-\Lambda_{ie}\Lambda_{ei}, \qquad (4.84)$$

and

$$\Lambda_{AB} = n^A\,\frac{V^{AB}(\vec{k})}{\theta}\left[1-(p+\nu^{nA})\Omega^A\right]. \qquad (4.85)$$

The functions $\Gamma^{Ae}(\underline{k}, \underline{v}, o)$ are obtained from (4.76) and the Fourier-transformed equilibrium phase-space correlations. Thus with (4.45) and (4.66) we have, neglecting forward scattering,

$$\Gamma^{ee}(\underline{k}, \underline{v}, o) = N^e M^e(\underline{v}) \left[1 - (1+q_i)^{-1} \left\{ 1 + (k\lambda_D)^2 \right\}^{-1} \right] \tag{4.86}$$

and

$$\Gamma^{ie}(\underline{k}, \underline{v}, o) = N^e M^i(\underline{v}) (1+q_i)^{-1} \left[1 + (k\lambda_D)^2 \right]^{-1} \tag{4.87}$$

where $q_i e$ is the average ionic charge.

As a result of our having employed an irreversible theory in the description of the correlation function, it is readily shown that this quantity is not symmetric under the interchange $t \to -t$, in contrast with (4.28). In addition $G^{ee}(\underline{k}, t)$ diverges exponentially for $t \to -\infty$ so that the integral in (4.2) does not exist. To overcome this difficulty we follow Nelkin, et al.,[11,12] and prescribe a behavior for negative times different from that we would obtain by solving the system (4.78), (Γ^{ee}, Γ^{ie}) for $t < o$. The prescription ensures the convergence of (4.2), yields a real scattering function, as it must, and the result is symmetric in $\underline{k}$ and $\Delta\omega$ as required for classical systems.[5] The prescription is

$$G^{ee}(\underline{k}, -t) = G^{ee}(k, t). \tag{4.88}$$

Fourier transforming (4.78) for both species we can show

$$G^{ee}(\underline{k}, t) = G^{ee}(-\underline{k}, t). \tag{4.89}$$

With the symmetry property (4.83) we can now compute the

scattering function as before using only the Fourier-Laplace transformed electron correlation function for $t > 0$. Thus with (4.59) and (4.82) we eventually obtain

$$S(\underline{\kappa},\Delta\omega) = \pi^{-1}\,\mathrm{Re}\lim_{\sigma\to 0+}\Delta^{-1}(x)$$

$$(x)\left[(1-X)\Omega^{e}\left[1+\frac{q_i}{(1+q_i)(\kappa\lambda_D)^2}\left\{1-(p+\nu^{ni})\Omega^{i}\right\}-\nu^{ni}\Omega^{n}\right]+\right.$$

$$\left.+X\Omega^{i}\left[\frac{q_i}{(1+q_i)(\kappa\lambda_D)^2}\left\{1-(p+\nu^{ne})\Omega^{e}\right\}\right]\right] \qquad (4.90)$$

where

$$\Delta = \left[1+\frac{1}{(1+q_i)(\kappa\lambda_D)^2}\left\{1-(p+\nu^{ne})\Omega^{e}\right\}-\nu^{ne}\Omega^{n}\right](x)$$

$$(x)\left[1+\frac{q_i}{(1+q_i)(\kappa\lambda_D)^2}\left\{1-(p+\nu^{ni})\Omega^{i}\right\}-\nu^{ni}\Omega^{n}\right]-$$

$$-q_i\left[\frac{1}{(1+q_i)(\kappa\lambda_D)^2}\right]^2\left[1-(p+\nu^{ne})\Omega^{e}\right]\left[1-(p+\nu^{ni})\Omega^{i}\right]$$

and

$$X \equiv \frac{1}{(1+q_i)\left[1+(\kappa\lambda_D)^2\right]} .$$

It will prove convenient for computational purposes to write (4.90) in terms of dimensionless variables. We thus introduce the new variables

$$\lambda \equiv (2\kappa\lambda_D)^{-1}, \quad \xi \equiv \Delta\omega/\omega_{pe} , \quad \eta^{A} \equiv \nu^{nA}/\omega_{pe} \qquad (4.91)$$

where

$$\omega_{pe}^{2} = \frac{4\pi ne^2}{m_e} , \quad \lambda_D^{-2} = \frac{4\pi ne^2}{\theta}(1+q_i) .$$

We obtain from (4.83)

$$\lim_{\sigma \to 0^+} \Omega^A = \frac{i}{\omega_{pe}} \left(\frac{m_A}{m_e}\right)^{1/2} \lambda' \left[\pi^{-1/2} \int_{-\infty}^{\infty} \frac{du e^{-u^2}}{u - \left(\frac{m_A}{m_e}\right)^{1/2} \lambda' (\xi + i\eta^A)}\right]^*$$

$$= \frac{i}{\omega_{pe}} \left(\frac{m_A}{m_e}\right)^{1/2} \lambda' Z^* \left[\left(\frac{m_A}{m_e}\right)^{1/2} \lambda' (\xi + i\eta^A)\right]$$

$$= \frac{i}{\omega_{pe}} \left(\frac{m_A}{m_e}\right)^{1/2} \lambda' Z_A^* \qquad (4.92)$$

where the complex function

$$Z(x+iy)$$

is the plasma dispersion function tabulated by Fried and Conte[9] and

$$\lambda' \equiv \left(\frac{2}{1+q_i}\right)^{1/2} \lambda . \qquad (4.93)$$

The scattering function now takes the form

$$S(\underline{\kappa}, \Delta\omega) = (\pi\kappa)^{-1} \left(\frac{m_e}{2\Theta}\right)^{1/2} (4\lambda^2+1)^{-1} \operatorname{Re} i\, \Delta^{-1}(x)$$

$$(x) \left[(2q_i\lambda'^2+1) Z_e^* \left\{1 - q_i\lambda'^2 Z_i^* - i\eta^i \left(\frac{m_e}{m_i}\right)^{1/2} \lambda' Z_n^*\right\} - \right.$$

$$\left. - 2q_i \left(\frac{m_i}{m_e}\right)^{1/2} \lambda'^4 Z_i^* \dot{Z}_e^*\right] \qquad (4.94)$$

with

$$\Delta = \left[1 - \lambda'^2 \dot{Z}_e^* - i\eta^e \lambda' Z_n^*\right]\left[1 - q_i\lambda'^2 \dot{Z}_i^* - i\eta^i \lambda' \left(\frac{m_i}{m_e}\right)^{1/2} Z_n^*\right]$$

$$- q_i\lambda'^4 \dot{Z}_e^* \dot{Z}_i^*$$

and[9]

$$\dot{Z}(t) = -2\left[1 + t\,Z(t)\right], \quad \text{all } t .$$

The scattering function displayed in (4.94) will differ negligibly from the result for a fully ionized gas[1] when the imaginary parts of the arguments of the functions $Z(t), \dot{Z}(t)$ are small compared to unity;

$$\left(\frac{m_A}{m_e}\right)^{1/2} \eta^{nA} \lambda \ll 1 . \tag{4.95}$$

To secure an estimate of when this condition holds, we write the collision frequencies ν^{nA} as

$$\nu^{nA} \simeq \sigma_0^{nA}(cm^2)\, n^n \left(\frac{2\theta}{m_A}\right)^{1/2} \times 10^{-16} \tag{4.96}$$

where σ_0^{nA} is of order one to ten cm^2 as previously. With (4.91), (4.95) becomes

$$\frac{\sigma_0^{nA}\, n^n \times 10^{-16}}{k} \ll 1$$

or

$$\frac{\lambda(cm)\,\sigma_0^{nA}(cm^2)\, n^n(cm^{-3})}{\sin\frac{\theta}{2}} \ll 3.5 \times 10^{17} . \tag{4.97}$$

For photons emitted by a ruby laser ($\lambda = 6943\,\text{Å}$), this can be written as

$$\frac{\sigma_0^{nA}(cm^2)\, n^n(cm^{-3})}{\sin\frac{\theta}{2}} \ll 4.6 \times 10^{21}\, cm^{-1} . \tag{4.98}$$

Comparing (4.98) with (4.66) and (4.70) we find that the scattering function (4.94) obtained with the irreversible theory will

display neutral atom effects at neutral densities at least five or six orders of magnitude lower than those necessary to see these effects in the reversible result (4.59). In the following section we describe and discuss the photon scattering function as displayed in (4.94).

3. Photon Scattering From a Partially Ionized Gas

In this section we discuss the results of the previous section for various plasma configurations. For convenience we take $m_i = m_n$.

To obtain quantitative information regarding the scattering function, it is necessary to claculate the real part in (4.94). To this end we write

$$Z_A^* = R_A - i I_A , \qquad \dot{Z}_A^* = \dot{R}_A - i \dot{I}_A \tag{4.99}$$

and define $m \equiv (m_i/m_e)^{1/2}$, and

$$a_1 \equiv (1 - q_i \lambda'^2 R_i - \eta^i m \lambda' I_i) \qquad a_3 \equiv (R_i \dot{R}_e - I_i \dot{I}_e)$$

$$a_2 \equiv (q_i \lambda'^2 \dot{I}_i - \eta^i m \lambda' R_i) \qquad a_4 \equiv (R_i \dot{I}_e + I_i \dot{R}_e)$$

$$b_1 \equiv (1 - \lambda'^2 \dot{R}_e - \eta^e \lambda' I_i) \qquad b_4 \equiv (q_i \lambda'^2 - \eta^i \lambda' m R_i)$$

$$b_2 \equiv (1 - q_i \lambda'^2 \dot{R}_i - \eta^i \lambda' m I_i) \qquad b_5 \equiv (\dot{R}_e \dot{R}_i - \dot{I}_e \dot{I}_i)$$

$$b_3 \equiv (\lambda'^2 \dot{I}_e - \eta^e \lambda' R_i) \qquad b_6 \equiv (\dot{I}_e \dot{R}_i + \dot{R}_e \dot{I}_i)$$

(4.100)

with $\lambda' \equiv (2/1+q_i)^{1/2} \lambda$ as per (4.93). After some algebraic

manipulation (4.94) becomes

$$S(k,\Delta\omega) = (\pi\kappa)^{-1}\left(\frac{m_e}{2\theta}\right)^{1/2}\frac{1}{u^2+1}\;\frac{N_I\Delta_R - N_R\Delta_I}{\Delta_R^2+\Delta_I^2} \qquad (4.101)$$

where

$$N_R \equiv (2q_i\lambda'^2+1)(R_e a_1 + I_e a_2) - 2q_i m\lambda'^4 a_3$$

$$N_I \equiv (2q_i\lambda'^2+1)(R_e a_2 - I_e a_1) + 2q_i m\lambda'^4 a_4$$

$$\Delta_R \equiv b_1 b_2 - b_3 b_4 - q_i\lambda'^4 b_5$$

$$\Delta_I \equiv b_1 b_4 + b_2 b_3 + q_i\lambda'^4 b_6 \,. \qquad (4.102)$$

Using tabulated values[9] of R_A, $\dot{R}_A$, I_A and $\dot{I}_A$ we have computed S as a function of $\xi = \Delta\omega/\omega_{pe}$ for different values of λ and η^e, η^i. Some typical results are plotted in Figures 4.1 through 4.3. To enhance our understanding of these results, we now develop approximations to (4.103) in various limits. To this end we employ both the power series and asymptotic expansions of the plasma dispersion function. These are, respectively,[9] with $t = x + iy$,

$$Z(t) = i\pi^{1/2}e^{-t^2} - 2t\left[1 - 2t^2/3 + 4t^4/15 - \cdots\right] \qquad (4.103)$$

and

$$Z(t) = i\,r\,\pi^{1/2}e^{-t^2} - t^{-1}\left[1 + (1/2t^2) + (3/4t^4) + \cdots\right] \qquad (4.104)$$

where

$$r = \begin{cases} 0 & , \; y > 0 \\ 1 & , \; y = 0 \end{cases} \,. \qquad (4.105)$$

Consider first S as a function of λ. For $\lambda \ll 1$ collective effects play a negligible role in the scattering process. With (4.103), (4.101) reduces under this condition to

$$S(\underline{k}, \Delta\omega) \simeq (\pi k)^{-1} \left(\frac{m_e}{2\theta}\right)^{1/2} Re\, i \frac{Z_e^*}{1 - i\eta^e \lambda' Z_n^*}$$

$$= (\pi k)^{-1} \left(\frac{m_e}{2\theta}\right)^{1/2} \frac{I_e(1-\eta^e\lambda' I_n) - \eta^e \lambda' R_e R_n}{(1-\eta^e\lambda' I_n)^2 + (\eta^e \lambda' R_n)^2} \,. \quad (4.106)$$

This is identical in form with the result of Ghatak and Nelkin.[11] For $\eta^e = 0$, (4.106) reduces to the well-known ideal gas form

$$S(\underline{k}, \Delta\omega) \simeq k^{-1} \left(\frac{m_e}{2\pi\theta}\right)^{1/2} \exp\left(-\lambda\xi\right)^2$$

$$= k^{-1} \left(\frac{m_e}{2\pi\theta}\right)^{1/2} \exp\left[-m_e(\Delta\omega/k)^2/2\theta\right] \,. \quad (4.107)$$

For $\eta^e > 0$, the scattering function given in (4.106) becomes narrower and increases in height as η^e increases in magnitude.[11]

For $\lambda \gg 1$, $m\xi \ll 1$, $m = (m_i/m_e)^{1/2}$, and $\eta^e\lambda', \eta^i m\lambda' \sim 1$ or less, (4.101) becomes

$$S(\underline{k}, \Delta\omega) \simeq 2q_i \lambda'^2 (\pi k)^{-1} \left(\frac{m_i}{2\theta}\right)^{1/2} Re\, i \frac{Z_i^*}{1 - i\eta^i m\lambda' Z_i^*} \,. \quad (4.108)$$

This last is similar to (4.106) but now the scattering electrons are strongly coupled to the ions, and the ion mass replaces the electron mass in the scattering function, resulting in a narrower scattered photon spectrum.

Next consider S with the condition

$$\lambda' m |\xi - i\eta^i| \ll 1, \quad \eta^e \simeq m\eta^i \,. \quad (4.109)$$

When (4.109) holds we find

$$S(\underline{k},\Delta\omega) \simeq k^{-1}(4\lambda'^2+1)^{-1}\left(\frac{m_e}{2\pi\epsilon}\right)^{1/2}(x)$$

$$(x)\left[(2q_i\lambda'^2+1)^2 e^{\lambda'^2(\eta^{e2}-\xi^2)} + 4q_i\lambda'^4 m\, e^{m^2\lambda'^2(\eta^{i2}-\xi^2)}\right]. \quad (4.110)$$

In this case the center of the scattered photon spectrum is dominated by a relatively narrow "ion peak" provided

$$4q_i\lambda'^4 m > (2q_i\lambda'^2+1)^2$$

or

$$\lambda'^2 > \tfrac{1}{2}\left[(mq_i)^{1/2}-1\right]^{-1}. \quad (4.111)$$

Under the appropriate conditions the scattering function will have a resonance at the electron and/or ion plasma frequencies. Thus for $\lambda' \gtrsim 3$, $\eta^e \ll 1$ and $\xi \sim 1$ or larger, S takes the approximate forms*

$$S(\underline{k},\Delta\omega) \simeq \left(\frac{m_e}{2\pi\theta}\right)^{1/2}\frac{(2q_i\lambda'^2+1)}{4\lambda'^2+1}(x)$$
$$(\eta^e = 0)$$

$$(x)\ \frac{\exp\left[-(\lambda'\frac{\Delta\omega}{\omega_{pe}})^2\right]}{\left[1-(\frac{\omega_{pe}}{\Delta\omega})^2\right]^2 + \pi\left(\frac{\Delta\omega}{\omega_{pe}}\right)^2(2q_i\lambda'^2)^3\exp\left[-2(\lambda'\frac{\Delta\omega}{\omega_{pe}})^2\right]} \quad (4.112)$$

$$S(\underline{k},\Delta\omega) \simeq \frac{\eta^e}{2\pi k}\left(\frac{m_e}{2\theta}\right)^{1/2}\left[\lambda'\left(\frac{\Delta\omega}{\omega_{pe}}\right)^4\right]^{-1}(x)$$
$$(\eta^e \neq 0)$$

$$(x)\left[\left\{1-\left(\frac{\omega_{pe}}{\Delta\omega}\right)^2\right\}^2 + \left(\frac{\eta^e\omega_{pe}}{\Delta\omega}\right)^2\left\{1+2\left(\frac{\omega_{pe}}{\Delta\omega}\right)^2\right\}\right]^{-1}. \quad (4.113)$$

---

* The reader will note that, because of the two different asymptotic forms (4.105), the form (4.112) cannot be obtained from (4.113) with $\eta^e \to 0$.

The resonance at $\Delta\omega = \omega_{pe}$ is apparent in (4.112) and (4.113). It is clear from (4.113) that the effect of collisions is to inhibit the resonance effect, as we might expect. The height of the resonance is inversely proportional to η^e while the width increases as η^e increases.

For $\lambda >> 1$, $m\xi \sim 1$, and $\eta^i = 0$, we find

$$S(\underline{k}, \Delta\omega) \approx 2q_i \lambda'^4 k^{-1} \left(\frac{m_i}{2\pi\Theta}\right)^{1/2} (\star)$$

$$(\star)\ \frac{\exp\left[-q_i(\lambda'\frac{\Delta\omega}{\omega_{pi}})^2\right]}{\left[1-(\frac{\omega_{pi}}{\Delta\omega})^2\right]^2 + \pi(2q_i^2\lambda'^2)^3(\frac{\Delta\omega}{\omega_{pi}})^2 \exp\left[-2q_i(\lambda'\frac{\Delta\omega}{\omega_{pi}})^2\right]} \qquad (4.114)$$

where

$$\omega_{pi}^2 = \frac{4\pi n e^2 q_i}{m_i}$$

and $n = n^e$ is the electron number density as before. For $\eta^i \neq 0$ we find a form similar to (4.113).

Finally, consider the scattering function in the limit of very strong collisions. With $\lambda\eta^e$, $\lambda m\eta^i >> 1$, and $m\xi \gtrsim 1$, S takes the approximate form

$$S(\underline{k}, \Delta\omega) \approx$$

$$(\pi k)^{-1}\left(\frac{m_e}{2\Theta}\right)^{1/2} \frac{2q_i\lambda'^2+1}{4\lambda^2+1}\, Re\, i\, \frac{Z_e^*}{1-i\eta^e\lambda' Z_n^*}\ . \qquad (4.115)$$

Hence in the strong collision limit collective phenomena are unimportant (except possibly for $m\xi \approx 1$; see (4.110)), as we would expect. The effects of electron-neutral collisions dominate, and the approximate form (4.115) of the scattering function is similar

to (4.106).

In Figure 4.1 we have plotted the normalized scattering function, for a single ionized ($q_i = 1$) gas of carbon atoms, vs. ξ as a function of the parameter λ for $\eta^e = \eta^i = 0$ (zero effective neutral density). The general qualitative dependence of S on the value of λ is clearly apparent. For $\lambda << 1$, S has the Gaussian form (4.107). For $\lambda \gtrsim 1$ the resonance at $\xi \simeq 1$ is present, becoming narrower and higher as λ increases, as per (4.112). When ξ is small the effect of ions dominates in S as suggested by (4.110) while an incipient ion-plasma resonance is evident in the vicinity of $m\xi = 1.48 \times 10^2 \xi \simeq 1$. For $\lambda >> 1$, S approaches the "strong coupling" form (4.108).

In Figure 4.2 we display the normalized scattering function, as in Figure 4.1, for $\lambda = 1.18$ as a function of the dimensionless collision parameters η^e , η^i . For convenience we have assumed $\eta^e = m\eta^i$; from (4.96) this implies equal electron-neutral and ion-neutral collision cross-sections, i.e. $\sigma_o^{ne} = \sigma_o^{ni}$. As η^e increases relative to λ^{-1} collective effects are seen to disappear; the resonance near $\Delta\omega = \omega_{pe}$ becomes lower and broader as per (4.113). When $\lambda\eta^e >> 1$, S approaches the form (4.115). Figure 4.3 is similar to Figure 4.2, only for $\lambda = 2.35$.

To understand the behavior of the scattering function when $\eta^e \neq m\eta^i$, we have computed S for a few values of ξ with $\lambda = 1.18$, when (i) $\lambda\eta^e = 0.3$, $\lambda m\eta^i = 0.1$, and when (ii) $\lambda\eta^e = 0.1$, $\lambda m\eta^i = 0.3$. The results are displayed below in tabular form. It is evident that changes in the ion collision parameter have only a very small effect on S in the vicinity of $\xi = 1$, while changes in the electron parameter similarly do not significantly disturb S for ξ small.

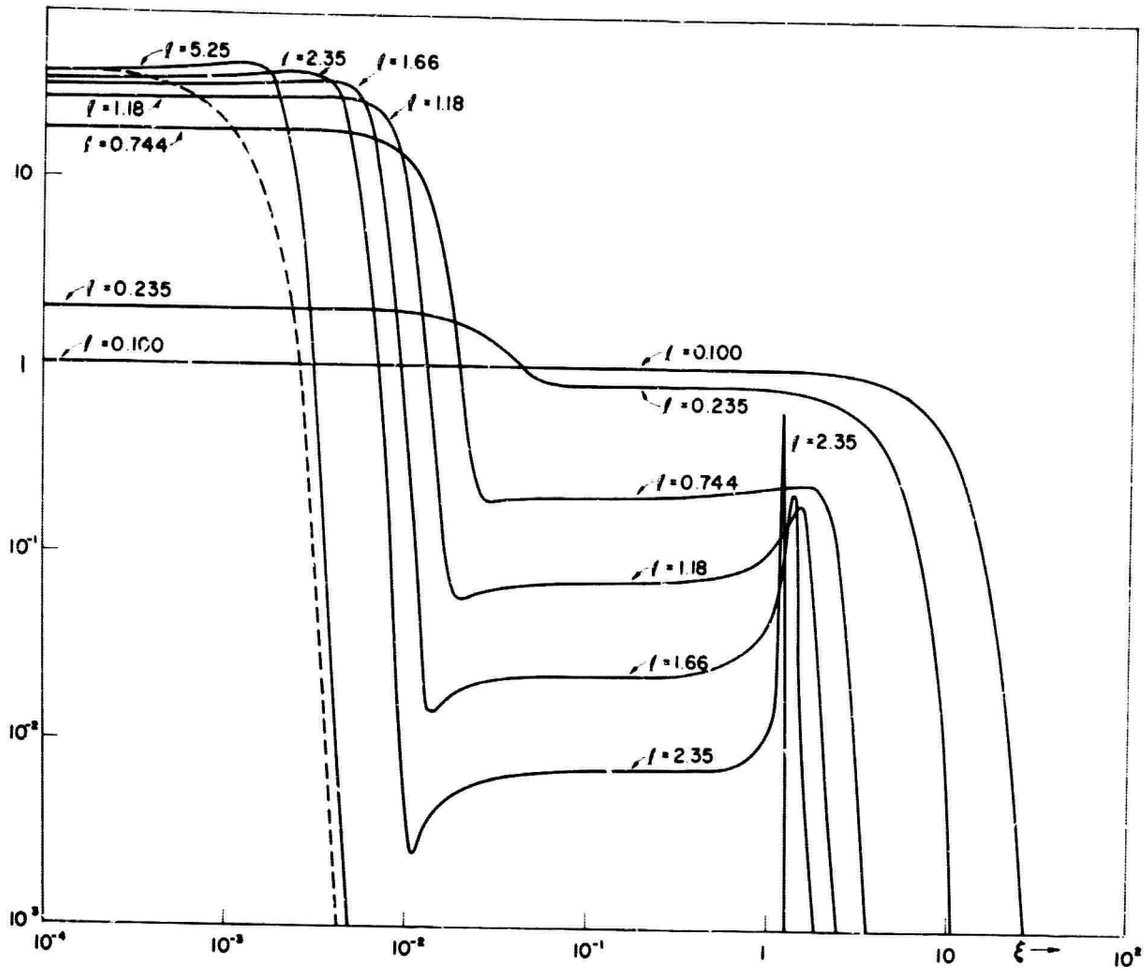


Figure 4.1 Electron scattering functions for a singly ionized carbon plasma, showing dependence on the parameter $l = (2\kappa\lambda_D)^{-1}$. The results have been normalized to the ideal electron gas scattering function at $\xi = \Delta\omega/\omega_{pe} = 0$. Broken line represents ideal ion gas scattering function for $l = 5.25$. Effective neutral density is zero for all cases.

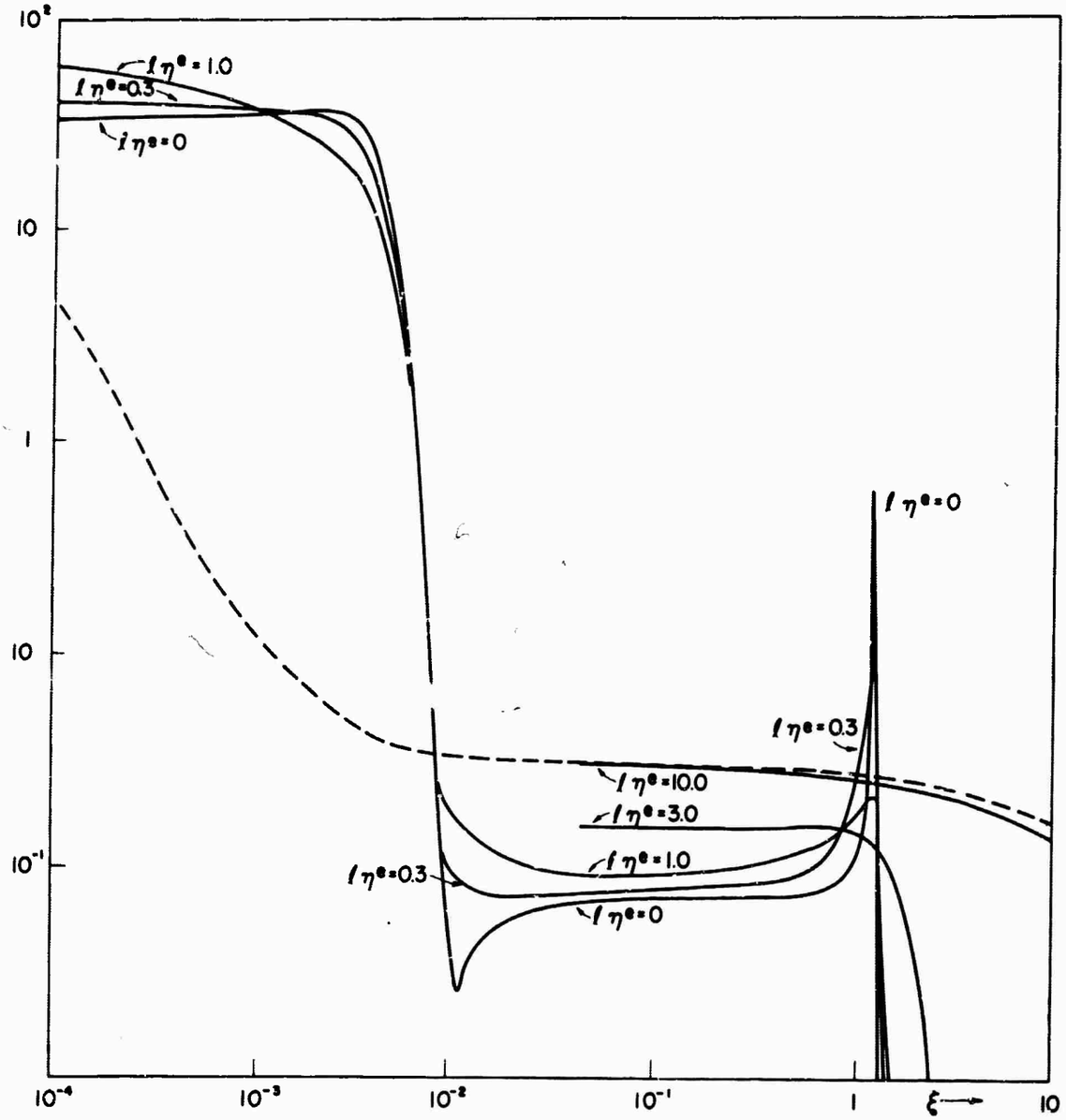


Figure 4.2 Electron scattering functions for a singly ionized carbon plasma, showing the effect of neutrals for $l = (2\kappa\lambda_D)^{-1} = 2.35$. The results have been normalized to the ideal electron gas scattering function at $\xi = \Delta\omega/\omega_{pe} = 0$. The parameter η^e is defined by $\eta^e = \nu^{ne}/\omega_{pe}$ and it is assumed that $\nu^{ne} = (m_i/m_e)^{1/2}\nu^{ni}$. The broken line represents the approximate form (eqn. (4.115)) for $l\eta^e = 10$.

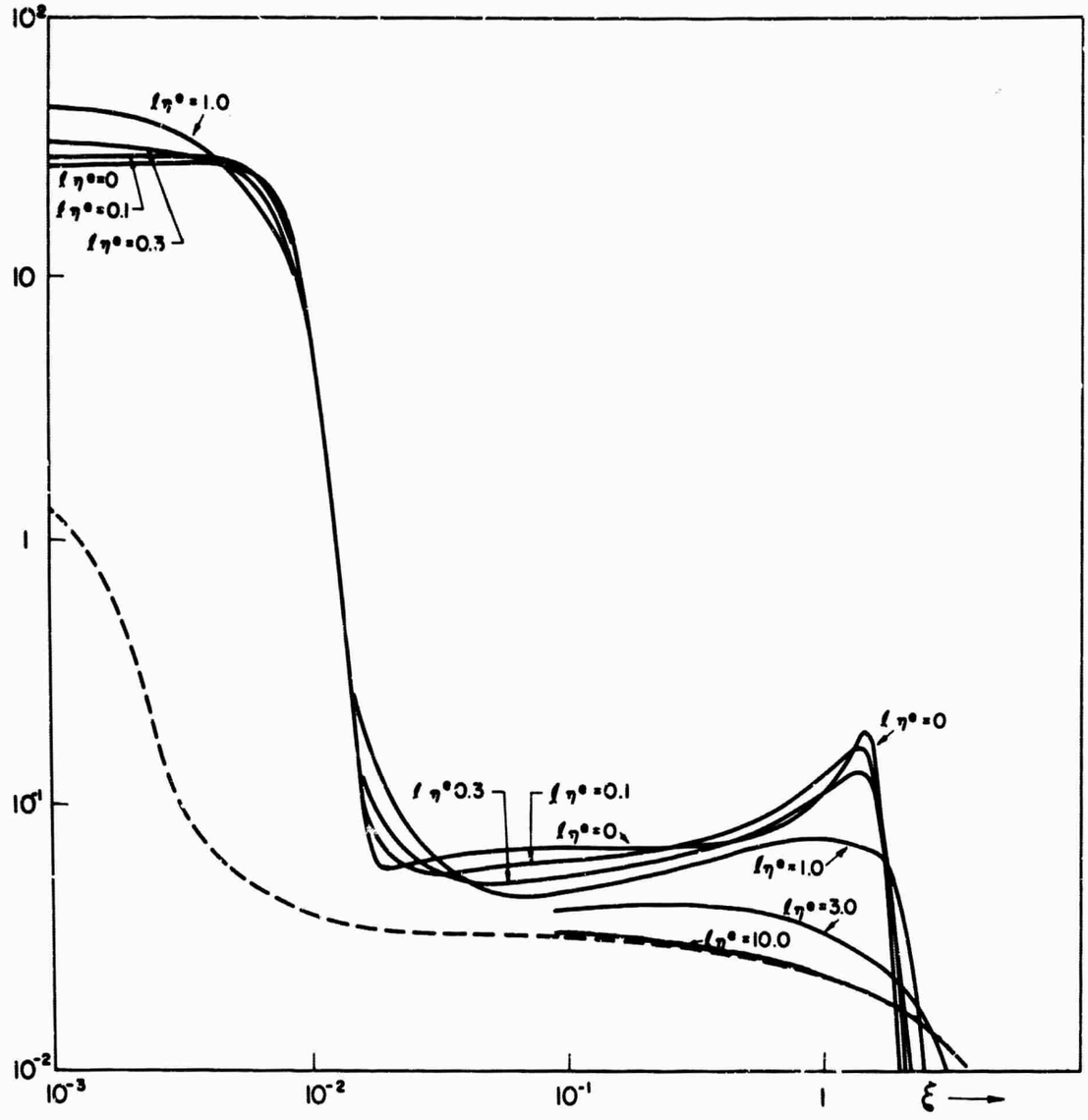


Figure 4.3 Electron scattering functions for a singly ionized carbon plasma, showing the effect of neutrals for $\ell = (2\kappa\lambda_D)^{-1} = 1.18$. The results have been normalized to the ideal electron gas scattering function at $\xi = \Delta\omega/\omega_{pe} = 0$. The parameter η^e is defined by $\eta^e = \nu^{ne}/\omega_{pe}$. The broken line represents the approximate form (eqn. (4.115)) for $\ell\eta^e = 10$.

Table 4.1

Comparison of Scattering Functions for
Different Collision Parameters

$S = S(k_i e, \lambda_m \gamma_i); \lambda = 1.18$ \ ξ =	10^{-3}	5×10^{-2}	1
$\frac{S(0.1, 0.3)}{S(0.1, 0.1)}$	1.21	1.14	0.986
$\frac{S(0.1, 0.3)}{S(0.3, 0.3)}$	0.992	0.85	1.23
$\frac{S(0.3, 0.1)}{S(0.1, 0.1)}$	0.998	1.09	0.81
$\frac{S(0.3, 0.1)}{S(0.3, 0.3)}$	0.83	1.12	0.989

This is as we would expect; when

$$m\lambda'\xi = \frac{\Delta\omega/k}{v_{oi}} >> 1$$

where $v_{oi} = (2\theta/m_i)^{1/2}$ the function Z_l is well approximated by the leading terms in its asymptotic form (4.105) and electron dynamics dominate ion dynamics in the scattering function. Similarly when

$$\lambda'\xi = \frac{\Delta\omega/k}{v_{oe}} << 1$$

ion dynamics play the dominant role. These observations are reflected in the approximate forms (4.110), 4.112), and (4.113) of the scattering function.

In most experimental situations the scattered spectrum is observed as a function of the shift $\Delta\lambda$ in wavelength from the wavelength of the incident photons. Consider for example the differential

photon scattering cross-section as a function of $\Delta\lambda$ for ruby laser photons ($\lambda = 6943\ \text{Å}$), for a singly ionized carbon plasma with $n_e = 10^{18}\ cm^{-3}$ and

$$\ell = (2K\lambda_D)^{-1} = \frac{\lambda}{\lambda_D}\left(1+\frac{\Delta\lambda}{\lambda}\right)^{1/2}\left(8\pi \sin\frac{\theta}{2}\right)^{-1} + O\left(\frac{\Delta\lambda}{\lambda}\right)^2 \quad (4.116)$$

From (4.1) we have, converting to wavelengths,

$$\sigma(\lambda, \Delta\lambda, \Omega') = 2\pi c \left(1+\frac{\Delta\lambda}{\lambda}\right)^{-3} \omega^{-1} S(K, \Delta\lambda)\sigma_T(\theta) \quad (4.117)$$

where $\Delta\lambda = \lambda' - \lambda$. It is clear from (4.116) and (4.117) that retaining the terms in $\Delta\lambda/\lambda$ yields a cross-section that is asymmetric about $\Delta\lambda = 0$, for constant K .

Comparing (4.116) with the scattering function as displayed in Figure 4.1, it is evident that, for constant K , retaining $\Delta\lambda/\lambda$ in (4.116) has the effect of <u>increasing</u> the height of the peak at $\xi = +1$ and <u>lowering</u> the peak at $\xi = -1$. In addition, the upper peak is shifted nearer $\Delta\lambda = 0$ while the lower peak is shifted further away. Since it is not the scattering function but the cross-section that is measured, however, we must also account for the factor $(1+\Delta\lambda/\lambda)^{-3}$ in (4.117). Clearly this factor will contribute to a <u>lowering</u> of the upper peak and an <u>increase</u> in height of the lower peak. The two effects are thus competitive.

For the example at hand we have found the net effect to be a 5.9 per cent decrease in amplitude of the upper resonance, and an equal increase in amplitude of the lower resonance. In addition, the location of the upper resonance is shifted about 1.3 percent closer to $\Delta\lambda = 0$ while the lower resonance is shifted the same amount further

away.

We have so far neglected contributions to the scattered photon intensity due to elastic (Rayleigh) scattering of photons from the neutral atoms. For simplicity we characterize the spectrum of neutral-scattered photons by the ideal gas, or Doppler scattering function. The ratio of light intensity scattered by neutrals to that scattered by electrons is then

$$\frac{\sigma_R\, n^n K^{-1} \left(\frac{m_n}{2\pi\theta}\right)^{1/2} \exp\left[-m_n(\Delta\omega/k)^2/2\theta\right]}{\sigma_T\, n^e\, S(\underline{k}, \Delta\omega)} \simeq$$

$$\simeq \frac{\sigma_R}{\sigma_T}\frac{n^n}{n^e}\begin{cases} (m_n/m_e)^{1/2}, & \lambda \ll 1 \\ 1 & \lambda \sim 1 \\ \lambda^{-2} & \lambda >> 1 \end{cases} \qquad (4.118)$$

where $S(\underline{k}, \Delta\omega)$ is the electron scattering function discussed above, and

$$\sigma_R \simeq 10^{-28}\ \text{cm}^2\ \text{sterad}^{-1}$$

is the Rayleigh cross-section.[17] It is evident that scattering from neutrals is unimportant when

$$\frac{n^n}{n^e}\begin{Bmatrix} (m_n/m_e)^{1/2} \\ 1/\lambda^2 \end{Bmatrix} << 10^2 \begin{cases} \lambda \ll 1 \\ \lambda \sim 1 \\ \lambda >> 1 \end{cases} . \qquad (4.119)$$

Moreover when scattering from neutrals is significant, it is clear that the effect will be manifest in the observed spectrum only in the vicinity of the central ion peak; i.e., when $\Delta\omega << \omega_{pe}$.

From the foregoing discussion it is evident that the scattered photon spectrum will contain electron plasma wings whenever

$$\lambda \equiv (2k\lambda_D)^{-1} \gtrsim 1,$$

$$\lambda \simeq 5.3\times10^{-5} \frac{\lambda(cm)}{\sin\frac{\theta}{2}} \left[\frac{n^e(cm^{-3})(1+q_i)}{\theta(ev)}\right]^{1/2}.$$

In addition, collisions with neutrals eliminate the wings when

$$l_{ne} = l \frac{\nu^{ne}}{\omega_{pe}} \gtrsim 1.$$

It thus follows, with (4.96), that the wings should be observable whenever

$$\sigma_o^{ne}(cm^2)\,\theta^{1/2}(ev)\,n^n(cm^{-3}) \gtrsim 10^{13}\left[n^e(cm^{-3})\right]^{1/2}.$$

4. Discussion

In the preceding analysis we have employed a treatment based on a temporally irreversible kinetic theory to describe the effect of neutral atoms on the spectrum of photons scattered from electron density fluctuations in a partially ionized gas. Our analysis was quite similar to that suggested by Yip et al.[12,13] for a description of neutral particle (i.e. photons or neutrons) scattering from moderately dense neutral gases. We remarked in Section 2 that an analysis based on a reversible kinetic theory predicted neutral atom effects only at unusually high neutral densities.

A recent experiment reported by Greytak and Benedek[18] provides striking quantitative support for the irreversible treatment. These authors observed the spectrum of 6328 Å photons scattered from thermal fluctuations in neutral gases near standard temperature and pressure.

Their observation of a symmetric pair of spectral lines located at $\Delta\lambda \simeq \pm 10^{-3}$ Å is in excellent agreement with the theoretical prediction of Yip and Nelkin[12,19] for scattering from neutral gases. These results clearly contradict the predictions of the reversible theory, i.e., a Gaussian-shaped spectrum, at the relatively low densities ($n \simeq 2\times10^{19}$ cm^{-3}) involved.

We have assumed in our computation of the classical scattering function that each of the particle species could be characterized by a Maxwell velocity distribution with a common temperature for all. For many experiments this assumption is invalid, and could lead to erroneous conclusions. The extension of the present work to allow for different component temperatures is straightforward but adds considerably to the algebraic complexity. While such considerations are beyond the scope of our purpose here, we note that several authors[14,20,21] have investigated the effects of unequal temperatures for a fully ionized two-component plasma. They showed that the scattering function for such a system can be qualitatively different from that computed with a single temperature model. In a recent experiment Kronast, et al[22] have employed Salpeter's results[14] in a measurement of electron and ion temperatures in a theta pinch. For their particular experiment they found $\Theta_e/\Theta_i \simeq 0.4$.

In our development of the electron correlation function in Part 2, we tacitly assumed that the inclusion of close encounters (collisions) between charged particles could be neglected. To lend support to this assumption we employ a simple Krook model to estimate the significance of Coulomb collisions. From the analysis of Part 2 it is evident that these effects should be negligible provided

$$\left(\frac{m_A}{m_e}\right)^{1/2} \eta^A l << 1 \tag{4.95}$$

where now

$$\eta^A = \frac{\nu^c}{\omega_{pe}} \tag{4.120}$$

with ν^c the Spitzer collision frequency[23] for charged particles,

$$\nu^c \simeq \frac{8\pi n e^4 \mu \ln \Lambda_1}{(3\mu\Theta)^{3/2}} \tag{4.121}$$

Here μ is the reduced mass for the pair in question and we assumed for simplicity a singly ionized gas. Combining (4.95), (4.120), and (4.121), the condition for neglecting Coulomb collisions becomes

$$\frac{l \ln \Lambda}{\Lambda} << 1 \tag{4.122}$$

where $\Lambda = n\lambda_D^3$. Since Λ is generally a very large number,[23] we would generally not expect Coulomb encounters to be significant here. (Ron, Dawson and Oberman[24] and Fante[25] have recently estimated the effects of Coulomb encounters on the electron scattering function using somewhat different analyses than the simple model employed here. They found the inclusion of these effects produced a change in S of the order of Λ^{-1}).

The principal result of this chapter is the electron scattering function for a partially ionized gas, as discussed and displayed in Part 3. It is apparent that for given values of η^e , η^i , $\bar{r}_i$, l and n , the scattering function as a function of $\Delta\lambda$ is uniquely determined. Even so, we would not expect that a single experiment

could serve to measure all of these quantities for a given plasma.

Our results apply for instruments with infinitely sharp spectral resolving power and uniform average plasma density over the scattering volume. Average density nonuniformities and the finite resolving power of instruments together with the natural width of the incident photon beam will add to the width of the observed spectral structure and it may be necessary to take these into account in a given experiment.

In addition to the extension of this work to allow for different componenent temperatures, it would be most interesting to consider the effects of magnetic fields and small spatial gradients, for fully or partially ionized gases.

APPENDIX A. INTEGRALS

Let Θ be the angle between $\underline{\xi}$ and $\underline{\xi}'$. From the generating function relation for Legendre polynomials we have

$$|\underline{\xi} - \underline{\xi}'|^{-1} = \left[c^2 + c'^2 - 2cc' \cos\Theta\right]^{-1/2}$$

$$= c^{-1} \sum_{l=0}^{\infty} \left(\frac{c'}{c}\right)^l P_l(\cos\Theta) \qquad (c' \le c) \tag{A1}$$

and similarly for $c \le c'$. Writing $g_l(\xi) = \sum g_{lm} Y_l^m$ and employing the addition theorem for Legendre polynomials, we obtain (3.13).

To find (3.14) we use the relation

$$\left[1 + x^2 - 2xy\right]^{1/2} = \int dx \, x \left[1 + x^2 - 2xy\right]^{-1/2} -$$

$$- y \int dx \left[1 + x^2 - 2xy\right]^{-1/2} . \tag{A2}$$

Combining (A1) and (A2) we have

$$|\underline{\xi} - \underline{\xi}'| = \left[c^2 + c'^2 - 2cc' \cos\Theta\right]^{1/2} \qquad (c' \le c)$$

$$= c^{-1} \sum_{l=0}^{\infty} \left[\frac{(c'/c)^{l+2}}{l+2} - \frac{(c'/c)^{l+1}}{l+1} \cos\Theta\right] P_l(\cos\Theta) \tag{A3}$$

and similarly for $c \le c'$. From the pure recurrence relation for Legendre polynomials we have

$$\cos\Theta \, P_l(\cos\Theta) = \frac{l}{2l+1} P_{l-1}(\cos\Theta) + \frac{l+1}{2l+1} P_{l+1}(\cos\Theta). \tag{A4}$$

Combining (A3) and (A4) and then using the addition theorem as before we find (3.14).

Finally (3.15) is obtained fror (3.14) with $g_{nlm} = \delta_{l0}\,\delta_{m0}$ with δ_{jk} the Kronecker delta.

APPENDIX B. THE SELF-ADJOINT PROPERTY

Let L_δ be either (i) the differential operator in (3.20) or (ii) the integrodifferential operator in (3.20) defined as in Chapter III, Part 4 on the closed interval $\delta: a \le c \le b$, $0 < a$, $b < \infty$. Let L_Δ be similarly defined on $(0, \infty)$. Then in either case (i) or (ii) there exists a complete orthonormal set of functions $\{h_{\delta n}\}$ on δ , generated by $L_\delta \phi = -\lambda \phi$ with homogeneous boundary conditions at a and b .[1,2]

Let f and g be any nonzero functions square integrable on $(0, \infty)$, and consider the inhomogeneous problems

$$L_\delta u + \Lambda u = f, \quad L_\delta v + \Lambda^* v = g, \tag{B1}$$

with the same homogeneous boundary conditions at a and b as in the above homogeneous problem. Let $\operatorname{Im} \Lambda \neq 0$ so that Λ will not belong to the spectrum of the set $\{h_{\delta n}\}$. Then the problems (B1) have nontrivial solutions

$$u(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \frac{\phi_j(c,\lambda)\, \tilde{u}_k(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda},$$

$$v(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \frac{\phi_j(c,\lambda)\, \tilde{v}_k(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda^* - \lambda},$$

with

$$\tilde{u}_k = \int_a^b f(c)\, \phi_k^*(c,\lambda)\, dc, \quad \tilde{v}_k = \int_a^b g(c)\, \phi_k^*(c,\lambda)\, dc. \tag{B2}$$

Now let

$$u_\mu = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\phi_j \tilde{u}_k \, d\rho_{\delta jk}}{\Lambda - \lambda} ,$$

$$v_\mu = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\phi_j \tilde{v}_k \, d\rho_{\delta jk}}{\Lambda^* - \lambda}$$

with $\tilde{u}_k$ and $\tilde{v}_k$ as in (B2). Multiplying u_μ by g^* and v_μ^* by f and then integrating over δ we have

$$\int_a^b g^*(c) u_\mu(c) dc = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\tilde{u}_k(\lambda) \, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda} \int_a^b g^*(c) \phi_j(c,\lambda) dc$$

$$= \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\tilde{u}_k(\lambda) \tilde{v}_j^*(\lambda) \, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda} \qquad \text{(B3)}$$

and similarly

$$\int_a^b f(c) v_\mu^*(c) dc = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\tilde{v}_k^*(\lambda) \tilde{u}_j(\lambda) \, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda} . \qquad \text{(B4)}$$

Since (3.24) and (3.25) are asymptotic solutions for both the differential and integrodifferential equations, the limit matrix ρ_Δ is the same in either case. Taking $\delta \to \Delta$ in (B3) and (B4) we have $\rho_{\delta jk} = \rho_{\Delta 22}$ and thus

$$\int_0^\infty g^*(c) u_\mu(c) dc = \int_{-\mu}^{\mu} \frac{\tilde{u}_2 \tilde{v}_2^* \, d\rho_{\Delta 22}}{\Lambda - \lambda} = \int_0^\infty f(c) v_\mu^*(c) dc . \qquad \text{(B5)}$$

After taking $\mu \to \infty$ and employing (B1), (B5) becomes

$$\int_0^\infty (L_\Delta v)^* u \, dc = \int_0^\infty (L_\Delta u) v^* \, dc$$

which is the desired result.

APPENDIX C. PROOF OF THE EXPANSION THEOREM

This proof is based largely on the so-called Weyl-Stone-Titchmarsh-Kodaira-Levinson theorem, as outlined by Yosida,[1] and by Coddington and Levinson.[2] Our goal is an expansion theorem for real valued continuous functions $u(c)$ in $(0,\infty)$ with $\int_0^\infty dc\,|u(c)|^2<\infty$. The expansion functions are to be the solutions of the linear integro-differential equation

$$\frac{d}{dc}\left(P\frac{d\phi}{dc}\right)+(Q+\Lambda)\phi+\int_0^\infty dc'\,K(c,c')\,\phi(c')=0.$$

Here $P=P(c)$, P', and $K=K_\ell$ are real and continuous and $P>0$, $(0,\infty)$, and $\int_0^\infty\int_0^\infty dc\,dc'\,|K(c,c')|^2<\infty$. The function $Q=Q_\ell(c)$ is real and continuous, is regular at $c=0$ for $\ell=0$, and has a regular singularity at $c=0$ for $\ell\neq 0$.

An expansion theorem based on the solutions of

$$\frac{d}{dc}\left(P\frac{d\phi}{dc}\right)+(Q+\Lambda)\phi+\int_a^b dc'\,K(c,c')\,\phi(c')=0 \qquad (C1)$$

which satisfy homogeneous boundary conditions at $0<a$, $b<\infty$ has been established by J. D. Tamarkin.[3] Our task here is to extend the interval (a,b) to the interval $(0,\infty)$.

Preliminaries, Tamarkin's Results

Let L_ℓ represent the linear integrodifferential operator in (C1). The following properties have been demonstrated by Tamarkin.[3]

(i) For a fixed complex Λ let ϕ_1, ϕ_2 represent a pair of

linearly independent solutions of $L_\delta \phi = -\Lambda\phi$, real for real Λ , satisfying the conditions ( )

$$\begin{aligned} \phi_1(s,\Lambda) &= 1 & \phi_2(s,\Lambda) &= 0 \\ P(s)\,\phi_1'(s,\Lambda) &= 0 & P(s)\,\phi_2'(s,\Lambda) &= 1 \,. \end{aligned} \tag{C2}$$

Then ϕ_1 , ϕ_2 , ϕ_1' , ϕ_2' are entire in Λ for every fixed c on δ .

(ii) For the self-adjoint boundary value problem

$$\begin{aligned} & L_\delta \phi = -\Lambda\phi \\ & \cos\alpha\,\phi(a) + \sin\alpha\, P(a)\,\phi'(a) = 0 \\ & \cos\beta\,\phi(b) + \sin\beta\, P(b)\,\phi'(b) = 0 \end{aligned} \tag{C3}$$

there exists a sequence of real eigenvalues $\{\lambda_{\delta n}\}$ and a complete orthonormal set of eigenfunctions $\{h_{\delta n}\}$. In terms of these functions the expansion formula for any $u \in \mathcal{L}^2(\delta)$ is

$$u(c) = \sum_n h_{\delta n}(c) \int_a^b dc'\, u(c')\, h_{\delta n}^*(c') . \tag{C4}$$

To extend the interval $\delta:(a,b)$ to $\Delta:(0,\infty)$ we proceed as follows. Since ϕ_1 , ϕ_2 form a basis for the solutions of $L_\delta\phi = -\Lambda\phi$ we can write

$$h_{\delta n}(c) = r_{\delta n1}\,\phi_1(c,\lambda_{\delta n}) + r_{\delta n2}\,\phi_2(c,\lambda_{\delta n}) \tag{C5}$$

where $r_{\delta n1}$, $r_{\delta n2}$ are complex constants. With (C5), (C4) becomes

$$u(c) = \sum_n \sum_{j,k=1}^{2} r_{\delta nj}\, r_{\delta nk}^*\, \phi_j \int_a^b dc\, u(c)\, \phi_k^*(c) . \tag{C6}$$

Now define an Hermitian, positive semidefinite matrix ρ_δ , called the

spectral matrix, with elements $\rho_{\delta jk}$ consisting of step functions with jumps at the eigenvalues $\lambda_{\delta n}$ given by

$$\rho_{\delta jk}(\lambda_{\delta n}+0) - \rho_{\delta jk}(\lambda_{\delta n}-0) = r_{\delta nj} r^*_{\delta nk} .$$

We let $\rho_\delta(0)$ be the zero matrix and define ρ_δ away from the eigenvalues by $\rho_\delta(\lambda+0) = \rho_\delta(\lambda)$. We employ the spectral matrix to replace the infinite series in (C6) by a Lebesgue-Stieltjes integral;

$$u(\epsilon) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \phi_j(\epsilon,\lambda)\, \tilde{u}_{\delta k}(\lambda)\, d\rho_{\delta jk}(\lambda) \qquad \text{(C7)}$$

where

$$\tilde{u}_{\delta k} = \int_a^b du\, u(\epsilon)\, \phi_k(\epsilon,\lambda) . \qquad \text{(C8)}$$

As $\delta \to \Delta$ (that is, $a \to 0$, $b \to \infty$), ρ_δ approaches a limit matrix ρ_Δ. Our task to to find the matrix ρ_Δ and to prove the convergence of the expansion (C7), (C8) in the limit.

Weyl's Limit Point and Limit Circle Theory

For any number m_b, the expression $\chi_b = \phi_1 + m_b \phi_2$ satisfies equation (C1). We now choose m_b so that χ_b satisfies the boundary condition

$$\cos\beta\, \chi_b + \sin\beta\, P(b)\, \chi_b' = 0 \qquad \text{(C9)}$$

at the point b. Then m_b must satisfy

$$m_b(\lambda) = - \frac{\cos\beta\, \phi_1(b,\lambda) + P(b)\phi_1'(b,\lambda)}{\cos\beta\, \phi_2(b,\lambda) + P(b)\phi_2'(b,\lambda)} .$$

Note that the one-point boundary condition (C9) has not restricted Λ to real values. Since ϕ_1 , ϕ_2 , ϕ_1' , and ϕ_2' are all entire functions of Λ , $m_b(\Lambda)$ is a meromorphic function of Λ . It is readily shown that every zero of the entire function χ_b is real, and hence all the poles of $m_b(\Lambda)$ lie on the real axis of the Λ-plane. Consider m_b as a function of Λ , b , and β . If we let $z = \omega + i\beta$ and maintain Λ and b at fixed values, we can write m_b as

$$m_b = - \frac{Az + B}{Cz + D} \quad . \tag{C10}$$

Since

$$|AD - BC| = |\phi_1(b,\Lambda)\,\phi_2'(b,\Lambda) - \phi_2(b,\Lambda)\,\phi_1'(b,\Lambda)|\,P(b)$$

$$\equiv W_b(\phi_1, \phi_2) \neq 0$$

where $W_b(\phi_1, \phi_2)$ is the Wronskian[2] of ϕ_1 , ϕ_2 evaluated at b , the transformation (C10) is a one-to-one conformal mapping which transforms the real axis of the z -plane into a circle C_b in the complex $m(\Lambda)$ plane. Therefore if $\mathrm{Im}\,\Lambda = \omega \neq 0$, then $m_b(\Lambda, z)$ varies on the circle $C_b(\Lambda)$ with a finite radius, as z varies over the real axis of the z plane.

The equation of the image of the real axis, $\mathrm{Im}\,z = 0$, is found from (C10);

$$(A^* + C^* m_b^*)(B + D m_b) - (A + C m_b)(B^* + D^* m_b^*) = 0,$$

or

$$W_b(\chi_b, \chi_b^*) = 0, \tag{C11}$$

which is the equation for $C_b(\Lambda)$. It follows easily that the center of C_b is $\hat{m}_b$,

$$\hat{m}_b \equiv - \frac{W_b(\phi_1, \phi_2^*)}{W_b(\phi_2, \phi_2^*)}$$

and the radius is

$$r_b(\Lambda) = \left| \frac{W_b(\phi_1, \phi_2)}{W_b(\phi_2, \phi_2^*)} \right| . \tag{C12}$$

For the moment let ϕ_1, ϕ_2 satisfy $L_S \phi_1 = -\Lambda_1 \phi_1$, $L_S \phi_2 = -\Lambda_2 \phi_2$ with $\Lambda_1 \neq \Lambda_2$. Then the symmetry of L_S permits demonstration of the Greens formula,

$$(\Lambda_2 - \Lambda_1) \int_S^b \phi_1 \phi_2 \, dc = W_S(\phi_1, \phi_2) - W_b(\phi_1, \phi_2) .$$

Now with $\Lambda_1 = \Lambda_2 = \Lambda$ and by virtue of (C2),

$$W_S(\phi_1, \phi_2) = W_b(\phi_1, \phi_2) = 1. \tag{C13}$$

Further, with $\phi_1(c, \Lambda^*) = \phi_1^*(c, \Lambda)$, $\phi_2(c, \Lambda^*) = \phi_2^*(c, \Lambda)$, and making use of Greens formula, we have

$$\begin{aligned} 2\omega \int_S^b |\phi_2(c, \Lambda)|^2 dc &= 2\omega \int_S^b \phi_2(c, \Lambda) \phi_2(c, \Lambda^*) dc \\ &= i W_S \left[\phi_2(c, \Lambda), \phi_2(c, \Lambda^*) \right] - \\ &\qquad - i W_b \left[\phi_2(c, \Lambda), \phi_2(c, \Lambda^*) \right] \\ &= -i W_b \left[\phi_2(c, \Lambda), \phi_2(c, \Lambda^*) \right] \qquad (\omega = \mathrm{Im}\, \Lambda). \end{aligned} \tag{C14}$$

Combining (C12), (C13), and (C14) we obtain

$$r_b(\Lambda) = \left[2|\omega| \int_s^b |\phi_2(c,\Lambda)|^2 dc\right]^{-1}, \quad \mathrm{Im}\,\Lambda = \omega \neq 0; \tag{C15}$$

Lemma 1. If $\omega = \mathrm{Im}\,\Lambda \neq 0$, then the interior of the circle $C_b(\Lambda)$ is mapped onto the lower half plane of the z-plane by the transformation (C10).

Proof. Since the real axis of the z-plane is the image of the circle $C_b(\Lambda)$ by the transformation (C10), the interior of $C_b(\Lambda)$ is mapped onto either the upper half plane or the lower half plane of the z-plane and further, the point at infinity of the m-plane is mapped onto the point $-P(b)\phi_2'(b,\Lambda)/\phi_2(b,\Lambda)$ of the z-plane.

On the other hand, we can write

$$\mathrm{Im}\left[-P(b)\phi_2'(b,\Lambda)/\phi_2(b,\Lambda)\right] = \frac{i}{2}\left[P(b)\left\{\frac{\phi_2'(b,\Lambda)}{\phi_2(b,\Lambda)} - \frac{\phi_2'^*(b,\Lambda)}{\phi_2^*(b,\Lambda)}\right\}\right]$$

$$= -\frac{i}{2}\,\frac{W_b(\phi_2,\phi_2^*)}{|\phi_2(b,\Lambda)|^2}$$

$$= \frac{\omega \int_s^b |\phi_2(c,\Lambda)|^2 dc}{|\phi_2(b,\Lambda)|^2} > 0. \tag{C16}$$

This means that $-P(b)\phi_2'(b,\Lambda)/\phi_2(b,\Lambda)$ belongs to the upper half plane of the z-plane. Hence the point at infinity, which is not contained in the interior of $C_b(\Lambda)$, is mapped into the upper half plane. This proves the lemma.

Since $W_s(\phi_1,\phi_2) = 1$, the transformation (C10) has an unique inverse which is given by

$$z = -P(b)\,\frac{\phi_2'(b,\Lambda)\,m_b(\Lambda) + \phi_1'(b,\Lambda)}{\phi_2(b,\Lambda)\,m_b(\Lambda) + \phi_1(b,\Lambda)}\,. \tag{C17}$$

In view of Lemma 1, if $\mathrm{Im}\,\Lambda = \omega > 0$, m belongs to the interior of

the circle $C_b(\Lambda)$ if and only if $\operatorname{Im} z<0$, namely, $i'(z-z^*)>0$.

From (C17) it follows that

$$i(z-z^*) = i\left[-P(b)\frac{\phi_2'(b,\Lambda)m+\phi_1'(b,\Lambda)}{\phi_2(b,\Lambda)m+\phi_1(b,\Lambda)} + P(b)\frac{\phi_2'^*(b,\Lambda)m^*+\phi_1'^*(b,\Lambda)}{\phi_2^*(b,\Lambda)m^*+\phi_1^*(b,\Lambda)}\right]$$

$$= \frac{W_b(\phi_1+m\phi_2,\ \phi_1^*+m^*\phi_2^*)}{|\phi_1+m\phi_2|^2}\ .$$

Therefore, $\operatorname{Im} z<0$ if and only if

$$iW_b(\phi_1+m\phi_2,\ \phi_1^*+m\phi_2^*)>0. \tag{C18}$$

By Green's formula we have

$$2\omega\int_0^b|\phi_1+m\phi_2|^2d\sigma =$$

$$= i\left[W_s(\phi_1+m\phi_2,\ \phi_1^*+m^*\phi_2^*) - W_b(\phi_1+m\phi_2,\ \phi_1^*+m\phi_2^*)\right] \tag{C19}$$

and from (C2),

$$W_s(\phi_1+m\phi_2,\ \phi_1^*+m\phi_2^*) = W_s(\phi_1,\phi_1^*) + W_s(\phi_2,\phi_1^*)m + W_s(\phi_1,\phi_2^*)m^* + W_s(\phi_2,\phi_2^*)|m|^2$$

$$= -m+m^*$$

$$= -2i\operatorname{Im} m. \tag{C20}$$

Combining (C18), (C19), and (C20) we have

Lemma 2. If $\omega=\operatorname{Im}\Lambda>0$, then m is interior to the circle $C_b(\Lambda)$ if and only if

$$\int_S^b |\phi_1(c,\Lambda) + m\,\phi_2(c,\Lambda)|^2 dc < \frac{\operatorname{Im} m}{\omega} ,$$

and m lies on $C_b(\Lambda)$ if and only if

$$\int_S^b |\phi_1(c,\Lambda) + m\,\phi_2(c,\Lambda)|^2 dc = \frac{\operatorname{Im} m}{\omega} .$$

(Note: it is easily shown that Lemma 2 also holds when $\omega = \operatorname{Im}\Lambda < 0$.)

It follows that, if m is inside C_b, and $b' < b$, then

$$\int_S^{b'} |\phi_1 + m\,\phi_2|^2 dc < \int_S^b |\phi_1 + m\,\phi_2|^2 dc < \frac{\operatorname{Im} m}{\omega} .$$

Hence m is also in $C_{b'}$, even though the centers of C_b and $C_{b'}$ may not coincide. We thus have

Lemma 3. The circle $C_{b'}$ contains C_b for $b' < b$ and $\operatorname{Im}\Lambda \neq 0$.

It follows that, as $b \to \infty$, the circles C_b converge either to a limit-circle or to a limit-point. In the limit circle case we have from (C15) that ϕ_2 is $L^2(S,\infty)$; the same property is readily demonstrated for ϕ_1.

Next consider the boundary point a, $0 < a < b$. For an arbitrary real number α, the boundary condition

$$\cos\alpha\, \chi_a + \sin\alpha\, P(a)\, \chi_a' = 0$$

at the point a with $\chi_a \equiv \phi_1 + m_a \phi_2$, determines

$$m_a(\Lambda) = - \frac{\cot\alpha\, \phi_1(a,\Lambda) + P(a)\,\phi_1'(a,\Lambda)}{\cot\alpha\, \phi_2(a,\Lambda) + P(a)\,\phi_2'(a,\Lambda)}$$

and also the circle $C_a(\Lambda)$, described by the equation

$$W_a(\chi_a, \chi_a^*) = 0 . \qquad \text{(C21)}$$

Similarly as in Lemma 2 we can prove

Lemma 4. If $\omega = \operatorname{Im}\lambda \neq 0$, then m lies on the circle $C_a(\lambda)$ or in its interior depending on whether

$$\int_a^s |\phi_1(c,\lambda) + m\phi_2(c,\lambda)|^2 dc = -\frac{\operatorname{Im} m}{\omega}$$

or

$$\int_a^s |\phi_1(c,\lambda) + m\phi_2(c,\lambda)|^2 dc < -\frac{\operatorname{Im} m}{\omega}.$$

Similarly as in Lemma 3 we can prove

Lemma 5. The circle $C_{a'}$ contains C_a for $a<a'$ and $\operatorname{Im}\lambda \neq 0$.

As before, as $a \to 0$ the circles C_a converge either to a limit-circle or to a limit-point.

The Limit Matrix ρ_0

Recall that

$$L_s\phi \equiv \frac{d}{dc}\left(P\frac{d\phi}{dc}\right) + Q\phi + \int_a^b dc' K(c,c')\phi(c').$$

Let ϕ_1, ϕ_2 be solutions of $L_s\phi = -\lambda\phi$ satisfying the conditions

$$\begin{aligned} \phi_1(s,\lambda) &= 1 & \phi_2(s,\lambda) &= 0 \\ P(s)\phi_1'(s,\lambda) &= 0 & P(s)\phi_2'(s,\lambda) &= 1. \end{aligned} \tag{C2}$$

For the self-adjoint boundary value problem ($0 < a$, $b < \infty$)

$$\begin{aligned} &L_s\phi = -\lambda\phi \\ &\cos\alpha\,\phi(a) + \sin\alpha\, P(a)\phi'(a) = 0 \\ &\cos\beta\,\phi(b) + \sin\beta\, P(b)\phi'(b) = 0 \end{aligned} \tag{C3}$$

there exists a sequence of real eigenvalues $\{\lambda_{\delta n}\}$ and a complete orthonormal set of eigenfunctions $\{h_{\delta n}\}$. The expansion formula for any function $u \in \mathcal{L}^2(\delta)$ is

$$u(c) = \sum_n h_{\delta n}(c) \int_a^b dc'\, u(c')\, h^*_{\delta n}(c'). \tag{C4}$$

Multiplying (C4) by $u^*(c)$ and integrating over δ gives the Parseval relation

$$\int_a^b |u(c)|^2 dc = \sum_n \left| \int_a^b u(c')\, h^*_{\delta n}(c')\, dc' \right|^2. \tag{C22}$$

Similarly if $u_I(c), u_{II}(c) \in \mathcal{L}^2(\delta)$ then

$$\int_a^b dc\, u_I(c) u^*_{II}(c) = \sum_n \int_a^b dc\, u_I(c)\, h^*_{\delta n}(c) \left[\int_a^b dc\, u_{II}\, h^*_{\delta n} \right]. \tag{C23}$$

Employing the representation (C5) and the spectral matrix ρ_δ, (C22) may be rewritten as

$$\int_a^b |u(c)|^2 dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}^*_{\delta j}(\lambda)\, \tilde{u}_{\delta k}(\lambda)\, d\rho_{\delta jk}(\lambda) \tag{C24}$$

where

$$\tilde{u}_{\delta k} = \int_a^b dc\, u(c)\, \phi_k(c, \lambda).$$

Applying the Parseval relation (C22) to any continuous function u on $(0, \infty)$ which vanishes outside some interval δ_1, contained in δ, one obtains

$$\int_0^{\infty} |u(c)|^2 dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}^*_j(\lambda)\, \tilde{u}_k(\lambda)\, d\rho_{\delta jk}(\lambda) \tag{C25}$$

where

$$\tilde{u}_k(\lambda) = \int_0^\infty dc\, u(c)\, \phi_k(c,\lambda) .$$

Let $\chi_a \equiv \phi_1 + m_a \phi_2$ be a solution of $L_0\phi = -\lambda\phi$, $\mathrm{Im}\,\lambda \neq 0$ satisfying the boundary condition

$$\cos\alpha\, \phi(a) + \sin\alpha\, P(a)\, \phi'(a) = 0$$

and similarly let $\chi_b = \phi_1 + m_b \phi_2$ be a solution of the same equation satisfying

$$\cos\beta\, \phi(b) + \sin\beta\, P(b)\, \phi'(b) = 0 .$$

Then m_a and m_b lie on circles C_a and C_b in the complex m-plane whose equations are, respectively,

$$W_a(\chi_a, \chi_a^*) = 0, \quad W_b(\chi_b, \chi_b^*) = 0 . \tag{C26}$$

It is easily shown[1] that Green's function for the boundary value problem (C3) is ($\mathrm{Im}\,\lambda \neq 0$)

$$G_\delta(c,c',\lambda) = \begin{cases} \dfrac{\chi_a(c,\lambda)\,\chi_b(c',\lambda)}{m_a(\lambda) - m_b(\lambda)} & c \leq c' \\ \dfrac{\chi_a(c',\lambda)\,\chi_b(c,\lambda)}{m_a(\lambda) - m_b(\lambda)} & c > c' \end{cases}$$

where $m_a(\lambda) - m_b(\lambda) = W_c(\chi_a, \chi_b)$. The Parseval relation in the form (C23) is now applied to the functions

$$u_I(c) = \frac{\partial^j G_\delta}{\partial (c')^j}, \quad u_{II}(c) = \frac{\partial^k G_\delta}{\partial (c')^k} \qquad j,k = 0,1$$

yielding

$$\int_a^b \frac{\partial^j G_\delta}{\partial(c')^j}\,\frac{\partial^k G_\delta^*}{\partial(c')^k}\,dc = \sum_n \int_a^b \frac{\partial^j G_\delta}{\partial(c')^j}\,h_{\delta n}^*\,dc \left[\int_a^b \frac{\partial^k G_\delta}{\partial(c')^k}\,h_{\delta n}^*\,dc\right]^* . \tag{C27}$$

From the definition of G_δ it follows that

$$\begin{aligned} G_\delta(c,s,\Lambda) &= \frac{\chi_a(c,\Lambda)}{m_a(\Lambda)-m_b(\Lambda)} \qquad c \le s \\ &= \frac{\chi_b(c,\Lambda)}{m_a(\Lambda)-m_b(\Lambda)} \qquad c > s \end{aligned} \tag{C28}$$

and

$$\begin{aligned} \left.\frac{\partial G_\delta(c,c',\Lambda)}{\partial c'}\right|_{c'=s} &= \frac{m_b(\Lambda)\,\chi_a(c,\Lambda)}{P(s)\left[m_a(\Lambda)-m_b(\Lambda)\right]} \qquad c \le s \\ &= \frac{m_a(\Lambda)\,\chi_b(c,\Lambda)}{P(s)\left[m_a(\Lambda)-m_b(\Lambda)\right]} \qquad c > s . \end{aligned} \tag{C29}$$

Using (C28) and (C29) and Green's formula, the integrals in (C27) can be evaluated. For example

$$\begin{aligned} 2i\omega\int_a^b |G_\delta(c,s,\Lambda)|^2\,dc &= \frac{2i\omega}{|m_a(\Lambda)-m_b(\Lambda)|}\left\{\int_a^s |\chi_a(c,\Lambda)|^2\,dc + \right. \\ &\qquad \left. + \int_s^b |\chi_b(c,\Lambda)|^2\,dc\right\} \\ &= |m_a(\Lambda)-m_b(\Lambda)|^{-2}\left\{W_s(\chi_a,\chi_a^*) - \right. \\ &\qquad \left. - W_s(\chi_b,\chi_b^*)\right\} \\ &= \frac{2i\,\mathrm{Im}\left[(\cdot\,\cdot)-m_a(\Lambda)\right]}{|m_a(\Lambda)-m_b(\Lambda)|^2} \end{aligned}$$

(where we have made use of (C26) to arrive at the second step). Therefore

$$\int_a^b |G_\delta(c,s,\Lambda)|^2 dc = \frac{\mathrm{Im}\,[m_a(\Lambda) - m_b(\Lambda)]^{-1}}{\omega} . \qquad \text{(C30)}$$

Similarly

$$[m_a(\Lambda) - m_b(\Lambda)](\lambda - \lambda_{\delta n}) \int_a^b G_\delta(c,s,\Lambda) h^*_{\delta n}(c)\,dc =$$

$$= W_b(\chi_b, h^*_{\delta n}) - W_s(\chi_b, h^*_{\delta n}) + W_s(\chi_a, h^*_{\delta n}) -$$

$$- W_a(\chi_a, h^*_{\delta n})$$

$$= W_s[(\chi_a - \chi_b), h^*_{\delta n}] = [m_a(\Lambda) - m_b(\Lambda)] W_s(\phi_2, h^*_{\delta n})$$

$$= [m_b(\Lambda) - m_a(\Lambda)]\, r^*_{\delta n1}$$

and hence

$$\int_a^b G_\delta(c,s,\Lambda) h^*_{\delta n}(c)\,dc = \frac{r^*_{\delta n1}}{\lambda_{\delta n} - \Lambda} . \qquad \text{(C31)}$$

In arriving at (C31) we have used the fact that $W_a(\chi_a, h^*_{\delta n}) = 0$ which follows since both χ_a and $h^*_{\delta n}$ satisfy the same boundary condition at a ; similarly $W_b(\chi_b, h^*_{\delta n}) = 0$.

Recalling the definition of the spectral function $\rho_{\delta 11}$ we combine (C27), (C30), and (C31);

$$\int_{-\infty}^{\infty} \frac{d\rho_{\delta 11}(\lambda)}{|\Lambda - \lambda|^2} = \frac{\mathrm{Im}\, M_{\delta 11}(\Lambda)}{\omega}$$

where

$$M_{\delta 11}(\Lambda) = \frac{1}{m_a(\Lambda) - m_b(\Lambda)} . \qquad \text{(C32)}$$

We can similarly show that

$$\int_{-\infty}^{\infty} \frac{d\rho_{\delta jk}(\lambda)}{|\Lambda-\lambda|^2} = \frac{\operatorname{Im} M_{\delta jk}(\Lambda)}{\omega} \tag{C33}$$

where $M_{\delta 11}$ is given by (C32) and

$$M_{\delta 12}(\Lambda) = M_{\delta 21}(\Lambda) = \frac{1}{2}\,\frac{m_a(\Lambda)+m_b(\Lambda)}{m_a(\Lambda)-m_b(\Lambda)},$$

$$M_{\delta 22} = \frac{m_a(\Lambda)m_b(\Lambda)}{m_a(\Lambda)-m_b(\Lambda)}.$$

From Lemma 2 and Lemma 4 we have

$$\omega\int_a^s |\chi_a(\epsilon,\Lambda)|^2 d\epsilon = -\operatorname{Im} m_a(\Lambda),$$

$$\omega\int_s^b |\chi_b(\epsilon,\Lambda)|^2 d\epsilon = \operatorname{Im} m_b(\Lambda).$$

Thus for a fixed Λ, $\operatorname{Im}\Lambda \neq 0$; $m_a(\Lambda)$ and $m_b(\Lambda)$ are in oppostie half planes. Suppose $\Lambda = i$ in (C33). Then points $m_a(i)$ lie on a circle C_a which is in $C_{s/2}$ for $a < \frac{s}{2}$ whereas points $m_b(i)$ lie on C_b which is in $C_{3s/2}$ for $b > 3s/2$. Thus there is a constant $k_1 > 0$ such that $|m_a(i) - m_b(i)| > k_1$ for $a < \frac{s}{2}$, $b > \frac{3s}{2}$. Since $m_a(i)$ and $m_b(i)$ are uniformly bounded for $a < \frac{s}{2}$, $b > \frac{3s}{2}$, it follows from (C33), and the definition of the $M_{\delta jk}$ that

$$\int_{-\infty}^{\infty} \frac{|d\rho_{\delta jk}(\lambda)|}{1+\lambda^2} < k_2.$$

Thus for $\nu > 0$,

$$\int_{-\nu}^{\nu} |d\rho_{\delta jk}(\lambda)| < k(1+\nu^2).$$

This last together with $\rho_{\delta jk}(0) = 0$ gives

$$|\rho_{\delta jk}(\lambda)| < K(1+\lambda^2) \qquad -\infty < \lambda < \infty. \tag{C34}$$

We are now able to prove the existence of the limit matrix ρ_Δ. For this we need the

Helly Selection Theorem.[2] Let $\{h_n(\lambda)\}$, $n = 1, 2, \ldots$, be a sequence of real nondecreasing functions $-\infty < \lambda < \infty$, and let $H(\lambda)$ be a continuous nonnegative function on the same interval. If

$$|h_n(\lambda)| \le H(\lambda) \qquad n = 1, 2, \ldots; \; -\infty < \lambda < \infty$$

then there exists a subsequence $\{h_{n_k}\}$ and a non-decreasing function h such that

$$|h(\lambda)| \le H(\lambda) \qquad -\infty < \lambda < \infty$$

and

$$\lim_{k\to\infty} h_{n_k}(\lambda) = h(\lambda).$$

It follows from (C34) and the Helly Selection Theorem that there exists a sequence of intervals $\delta_n \; (a_n, b_n)$, $\delta_n \to (0, \infty)$ and corresponding boundary conditions prescribed by α_n, β_n, such that $\rho_{\delta_n jk}(\lambda)$ tends to a limit $\rho_{\Delta jk}(\lambda)$, $n \to \infty$. It is easily seen that the limit matrix ρ_Δ, like ρ_δ, is Hermitian, is positive semi-definite, and is of bounded total variation on every finite λ interval.

It remains to establish an explicit formulation for ρ_Δ. From (C33) we have, with $\Lambda = \mu + i\omega$ and $\delta \to \Delta$,

$$\frac{\operatorname{Im} M_{\Delta jk}(\Lambda)}{\omega} = \int_{-\infty}^{\infty} \frac{d\rho_{\Delta jk}(\lambda)}{(\mu-\lambda)^2+\omega^2}.$$

Let λ_1 , λ_2 be points of continuity of ρ_Δ . Then integrating the above with $\omega \neq 0$ held fixed and finally taking $\omega \to 0$, we have

$$\lim_{\omega\to 0} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} m_{\Delta jk}(\lambda)\, d\mu = \lim_{\omega\to 0} \int_{\lambda_1}^{\lambda_2} \int_{-\infty}^{\infty} \frac{d\rho_{\Delta jk}(\cdot)}{(\mu-\lambda)^2+\omega^2}$$

$$= \lim_{\omega\to 0} \int_{-\infty}^{\infty} \left[\tan^{-1}\left(\frac{\lambda_2-\lambda}{\omega}\right) - \tan^{-1}\left(\frac{\lambda_1-\lambda}{\omega}\right)\right] d\rho_{\Delta jk}$$

$$= \pi\left[\rho_{\Delta jk}(\lambda_2) - \rho_{\Delta jk}(\lambda_1)\right]$$

where $M_{\Delta jk}$ is given by

$$M_{\Delta 11} = \frac{1}{m_\infty(\lambda) - m_0(\lambda)}$$

$$M_{\Delta 12} = M_{\Delta 21} = \frac{1}{2}\,\frac{m_\infty(\lambda) + m_0(\lambda)}{m_\infty(\lambda) - m_0(\lambda)}$$

$$M_{\Delta 22} = \frac{m_\infty(\lambda)\, m_0(\lambda)}{m_\infty(\lambda) - m_0(\lambda)} \quad .$$

If both points $a \to 0$, $b \to \infty$ are in the limit point case, m_0 and m_∞ are unique and it follows easily that ρ_Δ is unique. If either point is on a limit circle, the spectral matrix is not unique without the specification of a boundary condition at the point in question. Whether a particular case is limit point or limit circle is readily determined from the asymptotic solution and the expression (C15) for the radius of C_b or its analog for C_a .

The Parseval Relation in the Limit $\delta \to \Delta$

Consider a function $u(\xi)$ having a continuous second derivative on $0 \le \xi < \infty$, and which vanishes outside some interval δ_1 , contained in δ . Then applying (C25) to $L_\delta u$ we have

$$\int_0^\infty |L_\delta u|^2 dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \int_0^\infty (L_\delta u)\phi_k \, dc \left[\int_0^\infty (L_\delta u)\phi_j \, dc\right]^* d\rho_{\delta jk}. \quad \text{(C35)}$$

Applying Green's formula,

$$\begin{aligned}\int_0^\infty (L_\delta u)\phi_j \, dc &= \int_0^\infty (L_\delta \phi_j) u \, dc \\ &= -\lambda \int_0^\infty \phi_j u \, dc \\ &= -\lambda \tilde{u}_{\delta j}(\lambda),\end{aligned}$$

and hence (C35) becomes

$$\int_0^\infty |L_\delta u|^2 dc = \int_{-\infty}^{\infty} \lambda^2 \sum_{j,k=1}^{2} \tilde{u}_{\delta k} \tilde{u}_{\delta j}^* \, d\rho_{\delta jk}(\lambda). \quad \text{(C36)}$$

Now for A large,

$$\begin{aligned}\left(\int_{-\infty}^{-A} + \int_A^\infty\right) \sum_{j,k=1}^{2} \tilde{u}_{\delta k} \tilde{u}_{\delta j}^* \, d\rho_{\delta jk} &\le A^{-2}\left(\int_{-\infty}^{-A} + \int_A^\infty\right) \lambda^2 \sum_{j,k=1}^{2} \tilde{u}_{\delta k} \tilde{u}_{\delta j}^* \, d\rho_{\delta jk} \\ &\le A^{-2} \int_{-\infty}^{\infty} \lambda^2 \sum_{j,k=1}^{2} \tilde{u}_{\delta k} \tilde{u}_{\delta j}^* \, d\rho_{\delta jk} \\ &= A^{-2} \int_0^\infty |L_\delta u|^2 dc, \quad \text{(C37)}\end{aligned}$$

this last following from (C36).

It is convenient to rewrite (C25) in the form

$$\int_0^\infty |u(c)|^2 dc = \left(\int_{-\infty}^{-A} + \int_A^\infty + \int_{-A}^{A}\right) \sum_{j,k=1}^{2} \tilde{u}_{\delta k} \tilde{u}_{\delta j}^* \, d\rho_{\delta jk}. \quad \text{(C25)}$$

Combining (C25) and (C37)

$$\left| \int_0^\infty |u(c)|^2 dc - \int_{-A}^{A} \sum_{j,k=1}^{2} \tilde{u}_{\delta k} \tilde{u}_{\delta j}^* \, d\rho_{\delta jk} \right| \le \frac{1}{A^2} \int_0^\infty |L_\delta u|^2 dc. \quad \text{(C38)}$$

To take the limit $\delta \to \Delta$ (that is, $a \to 0$, $b \to \infty$) in (C38) we need the following

Integration Theorem.[2] Suppose $\{h_n(\lambda)\}$ is a real, uniformly bounded, sequence of nondecreasing functions on a finite interval $d \le \lambda \le e$, and assume

$$\lim_{n\to\infty} h_n(\lambda) = h(\lambda) \qquad d \le \lambda \le e .$$

If f is any continuous function on $(d \le \lambda \le e)$, then

$$\lim_{n\to\infty} \int_d^e f(\lambda)\, dh_n(\lambda) = \int_d^e f(\lambda)\, dh(\lambda).$$

We established earlier the properties of $\rho_{\delta j k}$ required by the integration theorem. Thus letting $\delta \to \Delta$ through the sequence of intervals δ_n found above, it follows, using (C38) and the integration theorem, that

$$\left| \int_0^\infty |u(c)|^2 dc - \int_{-A}^{A} \sum_{j,k=1}^{2} \tilde{u}_k \tilde{u}_j^* \, d\rho_{\Delta jk} \right| \le \frac{1}{A^2} \int_0^\infty |L_\Delta u|^2 dc.$$

Now allowing $A \to \infty$, there results the Parseval equality

$$\int_0^\infty |u(c)|^2 dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}_k \tilde{u}_j^* \, d\rho_{\Delta jk} \qquad \text{(C39)}$$

for any $u(c)$ restricted as above. We now show that the Parseval equality holds for any $u(c)$ in $\mathcal{L}^2(0,\infty)$. First suppose $u(c) \in \mathcal{L}^2(0,\infty)$ and vanishes for c sufficiently large and sufficiently small. Then there exists a sequence of functions $u_n \in \mathcal{L}^2(0,\infty)$ possessing continuous second derivatives and vanishing near $c = 0$ and for all large c such that

$$\lim_{n\to\infty} \int_0^a |u_n - u|^2 dc = 0.$$

Applying (C39) to $u_n - u_m$,

$$\int_0^\infty |u_n - u_m|^2 dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} (\tilde{u}_{kn} - \tilde{u}_{km})(\tilde{u}_{jn}^* - \tilde{u}_{jm}^*)\, d\rho_{\Delta jk} . \qquad (C40)$$

Since the left side of (C40) tends to zero as $n, m \to \infty$, it follows that the sequence of vectors $\{\tilde{u}_n\}$, $\tilde{u}_n = \tilde{u}_{1n}, \tilde{u}_{2n}$ where

$$\tilde{u}_{jn} = \int_{-\infty}^{\infty} dc\, u_n(c)\, \phi_j(c,\lambda) \qquad (C41)$$

converges in the mean in $\mathcal{L}^2(\rho_\Delta)$, and since the latter space is complete[2] there exists a vector $\tilde{u}$ which is the limit in the mean of this sequence. It is clear from (C41) that the components of $\tilde{u}$ are the continuous functions

$$\tilde{u}_j = \int_{-\infty}^{\infty} dc\, u(c)\, \phi_j(c,\lambda) .$$

Returning to (C39)

$$\int_0^\infty |u(c)|^2 dc = \lim_{n\to\infty} \int_0^\infty |u_n(c)|^2 dc = \lim_{n\to\infty} \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}_{nj}^*\, \tilde{u}_{nk}\, d\rho_{\Delta jk}$$

$$= \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}_k\, \tilde{u}_j^*\, d\rho_{\Delta jk} ,$$

which proves the Parseval relation for any $u(c) \in \mathcal{L}^2(0,\infty)$ vanishing for all c sufficiently large and sufficiently small. Suppose now that $u(c)$ is <u>any</u> function of class $\mathcal{L}^2(0,\infty)$ and define

$$u_{xy}(c) = \begin{cases} u(c) & x \le c \le y \\ 0 & c < x,\ y < c \end{cases}$$

and

$$\tilde{\alpha}_{xyj}(\lambda) = \int_0^\infty u_{xy}(c)\,\phi_j(c,\lambda)\,dc$$

$$= \int_x^y u(c)\,\phi_j(c,\lambda)\,dc$$

and similarly for $u_{rs}(c)$. Since $(x<r,\ y<s)$

$$\int_{-\infty}^{\infty} \sum_{j,k=1}^{2} (\tilde{\alpha}^*_{xyj} - \tilde{\alpha}^x_{rsj})(\tilde{\alpha}_{xyk} - \tilde{\alpha}_{rsk})\,d\rho_{\Delta jk} = \left(\int_x^r + \int_y^s\right)|u(c)|^2 dc,$$

it follows that the set of vectors $\tilde{u}_{xy}$ converges (as $x \to 0$, $y \to \infty$) in the mean $\varepsilon\, \mathfrak{L}^2(\rho_\Delta)$ to a vector function $\tilde{u} \varepsilon\, \mathfrak{L}^2(\rho_\Delta)$. By letting $x \to 0$, $y \to \infty$ in

$$\int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}_{xyk}\,\tilde{u}^*_{xyj}\,d\rho_{\Delta jk}(\lambda) = \int_x^y |u(c)|^2 dc$$

there now follows the Parseval equality

$$\int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}_k\,\tilde{u}_j^*\,d\rho_{\Delta jk}(\lambda) = \int_0^\infty |u(c)|^2 dc, \qquad (C42)$$

for any $u(c) \varepsilon\, \mathfrak{L}^2(0,\infty)$.

The Expansion Theorem for the Singular Interval

With the Parseval relation established, the proof of the expansion theorem may now be given. Let $D \equiv (-\eta, \eta)$ and define

$$u_D(c) \equiv \int_{-\eta}^{\eta} \sum_{j,k=1}^{2} \phi_j(c,\lambda)\,\tilde{\alpha}_k(\lambda)\,d\rho_{\Delta jk}(\lambda). \qquad (C43)$$

If $u_I(c)$ and $u_{II}(c)$ are in $\mathfrak{L}^2(0,\infty)$, then the relation (C42) implies

$$\int_0^\infty u_I(c)\,u_{II}^*(c)\,dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \int_0^\infty u_I\,\phi_k^*\,dc \int_0^\infty u_{II}^*\,\phi_j\,dc, \tag{C44}$$

which follows since we can write

$$4u_I u_{II}^* = |u_I + u_{II}|^2 - |u_I - u_{II}|^2 + i|u_I + iu_{II}|^2 - i|u_I - iu_{II}|^2.$$

Now consider some function $T \in \mathcal{L}^2(0,\infty)$ which vanishes for $c < c_1$, $c_2 < c$, and represent the transform of T by the vector $\tilde{T}$. Multiplying (C43) by T^* and integrating we have

$$\begin{aligned}\int_{c_1}^{c_2} u_D(c)\,T^*(c)\,dc &= \int_{c_1}^{c_2}\left(\int_{-\eta}^{\eta} \sum_{j,k=1}^{2} \phi_j(c,\lambda)\,\tilde{u}_k(\lambda)\,d\rho_{\Delta jk}(\lambda)\right) T^*(c)\,dc \\ &= \int_{-\eta}^{\eta} \sum_{j,k=1}^{2} \tilde{u}_k(\lambda)\left(\int_{c_1}^{c_2} \phi_j\,T^*\,dc\right) d\rho_{\Delta jk}(\lambda) \\ &= \int_{-\eta}^{\eta} \sum_{j,k=1}^{2} \tilde{u}_k\,\tilde{T}_j^*\,d\rho_{\Delta jk}. \end{aligned} \tag{C45}$$

From (C44) for $u_I = u$ and $u_{II} = T$,

$$\int_{c_1}^{c_2} u\,T^*\,dc = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{u}_k\,\tilde{T}_j^*\,d\rho_{\Delta jk}. \tag{C46}$$

Subtracting (C45) from (C46), and using the Schwarz inequality,

$$\begin{aligned}\left|\int_{c_1}^{c_2}(u-u_D)T^*dc\right|^2 &\le \left(\int_{-\infty}^{\infty} - \int_{-\eta}^{\eta}\right) \sum_{j,k=1}^{2} \tilde{u}_k\,\tilde{u}_j^*\,d\rho_{\Delta jk} \left(\int_{-\infty}^{\infty} - \int_{-\eta}^{\eta}\right) \sum_{j,k=1}^{2} \tilde{T}_k\,\tilde{T}_j^*\,d\rho_{\Delta jk} \\ &\le \left(\int_{-\infty}^{\infty} - \int_{-\eta}^{\eta}\right) \sum_{j,k=1}^{2} \tilde{u}_k\,\tilde{u}_j^*\,d\rho_{\Delta jk} \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \tilde{T}_k\,\tilde{T}_j^*\,d\rho_{\Delta jk} \\ &= \left(\int_{-\infty}^{\infty} - \int_{-\eta}^{\eta}\right) \sum_{j,k=1}^{2} \tilde{u}_k\,\tilde{u}_j^*\,d\rho_{\Delta jk} \int_0^\infty |T(c)|^2\,dc. \end{aligned}$$

Applying this inequality to the function $T(c)$ given by

$$T(c) = \begin{cases} u(c) - u_0(c), & 0 < c_1 \le c \le c_2 < \infty \\ 0 & c < c_1, \; c_2 < c \end{cases}$$

we obtain

$$\int_{c_1}^{c_2} |u(c) - u_0(c)|^2 dc \le \left(\int_{-\infty}^{\infty} - \int_{-\eta}^{\eta} \right) \sum_{j,k=1}^{2} \tilde{u}_k \tilde{u}_j^* \, d\rho_{\Delta jk}$$

or finally

$$\int_{c_1}^{c_2} \left| u(c) - \int_{-\eta}^{\eta} \sum_{j,k=1}^{2} \phi_j(c,\lambda) \tilde{u}_k(\lambda) \, d\rho_{\Delta jk}(\lambda) \right|^2 dc \le$$

$$\le \left(\int_{-\infty}^{\infty} - \int_{-\eta}^{\eta} \right) \sum_{j,k=1}^{2} \tilde{u}_k \tilde{u}_j^* \, d\rho_{\Delta jk} .$$

Since the right side does not depend on c_1 , c_2 , the above holds with $c_1 \to 0$, $c_2 \to \infty$. Letting $\eta \to \infty$ yields the expansion

$$u(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \phi_j(c,\lambda) \tilde{u}_k(\lambda) \, d\rho_{\Delta jk}(\lambda)$$

which clearly converges in the mean in $\mathcal{L}^2(0,\infty)$.

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13. ABSTRACT

This dissertation is a study of the description and effects of particle interactions in ionized gases. The principal results are: (i) An expansion theorem for the linearized Fokker-Planck collision operator for each component of a two-component fully ionized gas, and (ii) A description of photon scattering from a partially ionized gas. It is shown that the Fokker-Planck collision operator generates a complete, continuous set of velocity-space eigenfunctions, for which high-speed asymptotic forms are found. Since the set is continuous, the expansion formula has the form of a generalized Fourier integral. The effect of neutral atoms on the spectrum of photons scattered by electrons in a partially ionized gas is shown to be primarily a reduction in height and increase in width of the two electron-plasma "wings." The scattered photon spectrum is described for several characteristic cases.

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