DAVID TAYLOR MODEL BASIN
WASHINGTON, D. C.   20007

CALCULATIONS ON THE COLLAPSE OF A
SPHERICAL GAS-FILLED CAVITY
IN A COMPRESSIBLE LIQUID

by

Russel R. Lilliston

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NOTATION

B A constant which characterizes the adiabatic nature of the liquid medium (for water B = 3000 atmospheres)

C Isentropic sound speed in the medium as a function of ambient and transient pressures

c_∞ Sound speed in the undisturbed liquid medium

H Specific enthalpy of the liquid medium

n A constant which characterizes the adiabatic nature of the liquid medium (for water, n = 7)

P Pressure in the liquid at the bubble wall

p Pressure of the gas inside the sphere

P_0 Initial pressure of the gas inside the sphere

$\bar{p}$ Pressure in the liquid outside the bubble wall

p_∞ Pressure in the undisturbed liquid medium; ambient pressure

R Instantaneous radius of the imploding sphere

R_0 Initial radius of the imploding sphere

r Standoff (measured from the bubble center); component in the direction of the radial spherical coordinate

t Time

t_R Time measured at the bubble wall; time measured when the Eulerian position vector $r = R$

U Instantaneous velocity of the bubble wall

u Eulerian velocity in the fluid outside the bubble wall

v Instantaneous specific volume of the gas inside the bubble

v_0 Initial specific volume of the gas inside the bubble

γ Polytropic gas constant for an adiabatic process

ρ_∞ Density of the undisturbed liquid medium

ABSTRACT

This paper presents a method for calculating the instantaneous pressure, velocity, acceleration, and radius associated with the collapse of a spherical gas-filled cavity in an infinite compressible liquid. The method is an independent approach which makes use of Hamming's technique to numerically integrate Gilmore's differential equations which describe the collapse.

Included is a computer program which will perform the necessary calculations on a IBM 7090/1401 digital computer. Results obtained are in good agreement with those of Hickling and Plesset, whose work was unknown to the present author when he undertook the study.

It may be inferred that the peak shock wave pressure is significantly reduced by a decrease in ambient pressure, an increase in internal pressure, and/or a variation of the specific heat ratio by proper selection of the gas. Control of the last two parameters can be investigated as a possible means of protecting glass spheres against sympathetic implosion in multiple sphere buoyancy systems.

ADMINISTRATIVE INFORMATION

INTRODUCTION

PURPOSE

Because of the excessive weight-displacement ratios obtained with tough metals such as steel or aluminum, designers are turning toward nonductile materials for use in buoyancy systems for all depth vehicles. Spherical glass shells are among the components for such systems.[1,2] In a system which contains a number of buoyancy spheres, it is essential to know the effect that the collapse of one sphere will have on neighboring spheres in order to prevent catastrophic failure. A two-part investigation has been initiated:

1. The definition of the free-field pressure-time history due to the implosion of a single sphere.

2. Determination of the loading and response of a sphere to the pressure field generated by the implosion of a neighboring sphere.

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[1]References are listed on page 36.

This report deals with the analytical determination of Part 1, based on the assumptions that the spherical shell has negligible weight and thickness and that it contains air at arbitrary pressure.

BACKGROUND

The need for more complete understanding of the hydromechanical problem of cavitation and the gas bubble phenomena of underwater explosions has encouraged more and more detailed investigations into the pulsations of underwater gas bubbles. One of the earliest of these investigations was made by Rayleigh in 1917.[3] A more refined treatment was successfully completed by Herring in 1941.[4] Sometime later (1952), Gilmore[5] took a different approach and postulated equations to describe the growth or collapse of a spherical bubble in a viscous compressible liquid. Gilmore's description is presented in the next section.

THEORY

GILMORE'S BUBBLE WALL EQUATION

On the basis of the Kirkwood-Bethe hypothesis,[6] Gilmore has derived an equation (w'ich he calls a "second order" approximation) which accurately describes the (nonmigratory) oscillations of a spherical gas-filled cavity in an infinite compressible liquid. If R is the radius of the sphere, H the specific enthalpy of the surrounding liquid, and C the isentropic sound speed in the liquid, then Gilmore's equation is:

$$\ddot{R}R\left(1 - \frac{\dot{R}}{C}\right) + \frac{3}{2}\dot{R}^2\left(1 - \frac{\dot{R}}{3C}\right) = H\left(1 + \frac{\dot{R}}{C}\right) + \frac{R\dot{H}}{C}\left(1 - \frac{\dot{R}}{C}\right) \quad [1.1]$$

$$R(0) = R_0, \qquad \dot{R}(0) = 0$$

where

$$C = c_\infty\left(\frac{P + B}{p_\infty + B}\right)^{\frac{n-1}{2n}} \quad [1.2]$$

and

$$H = \int_{p_\infty}^{p} \left(\frac{P + B}{p_\infty + B}\right)^{-1/n} \frac{dp}{\rho_\infty} = \frac{n(p_\infty + B)}{(n-1)\rho_\infty}\left[\left(\frac{P + B}{p_\infty + B}\right)^{\frac{n-1}{n}} - 1\right] \qquad [1.3]$$

Here c_∞, p_∞, and ρ_∞ are respectively sound speed, pressure, and density in the undisturbed liquid. B and n are constants which characterize the adiabatic compression of the liquid (for water, $B = 3000$ atm, $n = 7$). P is the pressure in the liquid at the bubble wall. If viscosity and surface tension are neglected and the pressure p inside the bubble is uniform, then pressure is continuous across the boundary of the sphere, i.e., $P = p$ except at time $t = 0$ when the pressure in the fluid is artificially and discontinuously reduced from $P = p_\infty$ to $P = p(0) = p_0$. The gas can be assumed to undergo an adiabatic expansion (or compression). From thermodynamics, for an ideal gas,

$$p_0 v_0^\gamma = p v^\gamma \qquad [1.4a]$$

where γ is a constant (the specific heat ratio), v is specific volume, and the subscript 0 refers to some initial state. Since the volume of a sphere is proportional to its radius

$$\frac{v_0}{v} = \left(\frac{R_0}{R}\right)^3 \qquad [1.4b]$$

Elimination of volume between Equation [1.4a] and Equation [1.4b] yields

$$P = \begin{cases} p_\infty & t = 0 \\ \\ p = p_0 \left(\dfrac{R_0}{R}\right)^{3\gamma} & t > 0 \end{cases} \qquad [1.4c]$$

For air ($\gamma = 4/3$), the exponent 3γ becomes 4. Usually the value of γ is taken to be 1.4 for air. This value represents the behavior of air fairly accurately, but since γ decreases with increasing pressure, 4/3 represents a rough average. It will be seen later that the use of a constant value of γ leads to a deficiency in the model.

Combining Equations [1.1], [1.2], [1.3], and [1.4c] and taking $n = 7$ yields the follo$\cdot$ ordinary differential equation for R:

$$
R\ddot{R}\left[1 - \dot{R}\left(\frac{p_\infty + B}{p_0\left(\dfrac{R_0}{R}\right)^4 + B}\right)^{3/7}\right] + \frac{7}{6}\left(\frac{p_\infty + B}{\rho_\infty}\right)\left[1 - \left(\frac{p_0\left(\dfrac{R_0}{R}\right)^4 + B}{p_\infty + B}\right)^{6/7}\right]
$$

$$
+ \frac{\dot{R}}{c_\infty}\left(\frac{p_\infty + B}{p_0\left(\dfrac{R_0}{R}\right)^4 + B}\right)^{3/7} - \frac{\dot{R}}{c_\infty}\left(\frac{p_0\left(\dfrac{R_0}{R}\right)^4 + B}{p_\infty + B}\right)^{3/7}\right] + \frac{3}{2}\dot{R}^2
$$

$$
- \frac{\dot{R}^3}{2c_\infty}\left(\frac{p_\infty + B}{p_0\left(\dfrac{R_0}{R}\right)^4 + B}\right)^{3/7} + \frac{4p_0}{\rho_\infty c_\infty}\dot{R}\left(\frac{R_0}{R}\right)^4\left[\left(\frac{p_0\left(\dfrac{R_0}{R}\right)^4 + B}{p_\infty + B}\right)^{4/7}\right.
$$

$$
\left. - \frac{\dot{R}}{c_\infty}\left(\frac{p_\infty + B}{p_0\left(\dfrac{R_0}{R}\right)^4 + B}\right)\right] = 0 \tag{1.5}
$$

$$
R(0) = R_0, \qquad \dot{R}(0) = 0
$$

Once B, R_0, p_0, p_∞, and c_∞ are specified, it is possible to find a numerical solution for $R(t)$, $\dot{R}(t)$, $\ddot{R}(t)$, and $p(t)$.

The initial velocity $\dot{R}(0)$ is taken throughout this paper to be zero. With the help of Equation [1.5], Gilmore has pointed out that near $t = 0$, there is a small finite jump in velocity during an infinitesimally small interval of time, i.e.,

$$
\dot{R}(0_+) = \frac{p_0 - p_\infty}{\rho_\infty c_\infty}
$$

Hickling and Plesset[7] give a good physical explanation of this jump in terms of the initial pressure discontinuity between p_0 and p_∞. This velocity jump may lead one to choose $R(0_+) = (p_0 - p_\infty)/\rho_\infty c_\infty$ as the initial condition on the velocity. Since the difference

between $(p_0 - p_\infty)/\rho_\infty c_\infty$ and zero is small compared to the magnitude of the velocities of interest, the question as to whether to start the solution at $\dot{R}(0) = 0$ or at $\dot{R}(0_+) = (p_0 - p_\infty)/\rho_\infty c_\infty$ is somewhat academic. The plots of bubble wall velocity do not show this initial jump because the writer's solution of Gilmore's equation was actually carried out from $t = 0_+$ to avoid an infinite initial acceleration. (One of the terms appearing in the expression for the initial acceleration is the derivative with respect to time of P as given in Equation [1.4c]; this derivative is infinite at $t = 0$). The approximation $(p_0 - p_\infty)/\rho_\infty c_\infty = 0$ was made to simplify the calculations of the initial values of R and $\dot{R}$ which are tedious even with such an approximation.

It should be emphasized that one of the inherent assumptions upon which Equation [1.5] is based is that the cavity remain spherical throughout the collapse and subsequent oscillations. In some bubble collapse experiments,[8] however, the single bubble has occasionally been observed to dissociate into many smaller bubbles at the end of the first collapse.

THE INTEGRATION: HAMMING'S METHOD

An ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \qquad [2.1]$$

can be integrated numerically by one of various finite difference methods. One of these, the Hamming method, which is particularly well suited for the solution of Equation [1.5] can be found in Ralston and Wilf.[9] It is outlined briefly here:

1. The x-axis is equally divided into a large number of small intervals. The value of y at the end of the nth interval (i.e., the interval between x_n and x_{n-1}) is denoted by y_n.

2. Knowing the values of y_i and y_i' at previous intervals x_i up to and including the nth $(i = n)$, it is possible to calculate p_{n+1}, a first approximation to the $(n + 1)^{st}$ value of y_{n+1} at x_{n+1}, by means of

$$p_{n+1} = y_{n-3} + \frac{4h}{3} \left(2y_n' - y_{n-1}' + 2y_{n-2}'\right)$$

where h is the width of the interval; p_{n+1} is called the predictor.

3. Since the prediction p_{n+1} is based on the value of a series expansion of y, error due to truncation is incurred. Most of the difference between the true value of y and the estimated value of y is taken into account by the modifier (denoted by m_{n+1})

$$m_{n+1} = p_{n+1} - \frac{112}{121} (p_n - c_n)$$

where c_n is as defined in the next paragraph and the derivative of m_{n+1} is $m'_{n+1} = f(x_{n+1}, m_{n+1})$.

4. The predicted value is compared with a quantity called the corrector

$$c_{n+1} = \frac{1}{8} [9y_n - y_{n-1} + 3h (m'_{n+1} + 2y'_n - y'_{n-1})]$$

5. If the predictor p_{n+1} lies close to c_{n+1} within some specified tolerance, then the final value of y_{n+1} at x_{n+1} is taken as

$$y_{n+1} = c_{n+1} + \frac{9}{121} (p_{n+1} - c_{n+1})$$

6. If the predictor p_{n+1} does not lie close enough to c_{n+1}, then either (1) a new value of p_{n+1}

$$p_{n+1} = c_{n+1} + \frac{9}{121} (p_{n+1} - c_{n+1})$$

may be calculated and an interative process carried out or (2) the interval may be halved. This is discussed in more detail in Appendix A.

In order to solve Equation [1.5] by the procedure just outlined, it is necessary to reduce the equation to a system of two simultaneous differential equations of the form of Equation [2.1]. This method is mentioned in Hildebrand.[10] Write U for $\dot{R}$; then Equation [1.5] can be written

$$\dot{R} = U \qquad\qquad [1.6a]$$

and

$$\dot{U} = \left\{ \frac{U^3}{2c_\infty} \left(\frac{p_\infty + B}{p_0\left(\frac{R_0}{R}\right)^4 + B} \right)^{3/7} - \frac{3}{2} U^2 - \frac{4U p_0}{\rho_\infty c_\infty} \left(\frac{R_0}{R}\right)^4 \left[\left(\frac{p_0\left(\frac{R_0}{R}\right)^4 + B}{p_\infty + B} \right)^{4/7} \right. \right.$$

$$\left. - \frac{U}{c_\infty} \left(\frac{p_\infty + B}{p_0\left(\frac{R_0}{R}\right)^4 + B} \right) \right] - \frac{7}{6} \left(\frac{p_\infty + B}{p_\infty} \right) \left[1 - \left(\frac{p_0\left(\frac{R_0}{R}\right)^4 + B}{p_\infty + B} \right)^{6/7} \right.$$

$$\left. \left. + \frac{U}{c_\infty} \left(\frac{p_\infty + B}{p_0\left(\frac{R_0}{R}\right)^4 + B} \right)^{3/7} - \frac{U}{c_\infty} \left(\frac{p_0\left(\frac{R_0}{R}\right)^4 + B}{p_\infty + B} \right)^{3/7} \right] \right\} \bigg/ \left[1 - U \left(\frac{p_\infty + B}{p_0\left(\frac{R_0}{R}\right)^4 + B} \right)^{3/7} \right] \qquad [1.6b]$$

Hamming's method can be applied simultaneously to Equations [1.6a] and ⌈1.6b] to yield a numerical solution. The justification for use of this particular method is discussed in Appendix B.

The function $R(t)$ (from which $U(t)$ and $p(t)$ at the bubble wall can be obtained), determined by Equation [1.6], constitutes one of the boundary conditions necessary to find the Eulerian velocity and pressure fields in the fluid outside the bubble wall.

THE EULERIAN VELOCITY AND PRESSURE FIELDS IN THE LIQUID

In his "second order" approximation, Gilmore uses the Kirkwood-Bethe hypothesis in conjunction with the method of characteristics to determine the (Eulerian) velocity and pressure fields in the liquid. If the standoff r is greater than or equal to the initial radius R_0, then the velocity u associated with the standoff will not be of the same order of magnitude as the sound speed except for the most severe implosions. Provided the approximation $u^2 \ll c^2$ is valid, the following set of equations (the expressions derived by Gilmore) are sufficient to determine the velocity and pressure fields u and $\bar{p}$ in the liquid when U and R, the bubble wall velocity and radius, are known functions of time.

$$u(r, t) = \frac{y}{c_\infty r} + \frac{K_3 y^2}{c_\infty^3 r^2} \left(1 - \frac{y}{c_\infty^2 r} + \frac{K_3^2 y^4}{2c_\infty^8 r^4} \right) \qquad [3.1]$$

where y and K_3 are given by

$$y = \frac{RU^2}{2} + \frac{R\left(p_0\left(\frac{R_0}{R}\right)^4 - p_\infty\right)}{\rho_\infty}\left(1 - \left(\frac{p_0\left(\frac{R_0}{R}\right)^4 - p_\infty}{2\rho_\infty c_\infty^2}\right)\right) \quad [3.2]$$

$$K_3 = \frac{c_\infty^3 R^2 U}{y^2}\left(1 - \frac{U^2}{2c_\infty^2}\right) - \frac{c_\infty^2 R}{y}\left(1 - \frac{U}{c_\infty}\right) \quad [3.3]$$

$$\bar{p}(r, t) = \rho_\infty\left(\frac{y}{r} - \frac{u^2}{2}\right) + \frac{\rho_\infty}{2c_\infty^2}\left(\frac{y}{r} - \frac{u^2}{2}\right)^2 \quad [3.4]$$

$$t = t_R + \left(\frac{r - R}{c_\infty}\right)\left(1 - \frac{UR}{c_\infty r}\right) \quad [3.5]$$

Here t_R is the time at which the bubble radius is R and the bubble wall velocity is U. An event which occurs at the bubble wall at time t_R requires finite time $t - t_R$ to propagate from the bubble wall to the point r in the fluid. This time lag is indicated by Equation [3.5].

If the approximation $u^2 \ll c^2$ is not made, then the numerical integration for u and $\bar{p}$ is more complicated and requires a great deal more time on the computer. Hickling and Plesset[9] avoid making this approximation. Instead of using values of U and R in Equations [3.1] through [3.5] to find $\bar{p}$ and u at discrete points, they use U and R as coefficients in a differential equation which determines $\bar{p}$ and u. Then each time U and R are determined at a single point, another differential equation must be solved to find $\bar{p}$ and u. Each solution of this second differential equation gives $\bar{p}$ and u as a function of distance from the bubble at one specific instant in time (i.e., that instant in time at which the bubble radius is that value of $R(t)$ used in this second differential equation).

RESULTS

The integration of Equation [1.6] (the equation which determines bubble radius and bubble wall velocity and acceleration) has been coded in FORTRAN for a 7090/1401 computer according to the procedure outlined. The program and some sample input and output are given in Appendix C. Plots from computer output of R, u, $\dot{U}$, and p (the bubble radius,

bubble wall velocity and acceleration, and pressure at the bubble wall, respectively) as functions of time can be found among Figures 1 through 5 for various ambient and internal pressures.

Equations [3.1] through [3.5] (those equations which determine the Eulerian velocity and pressure fields in the fluid outside the bubble wall) have been incorporated into the program. From computer output, plots of u and p (Eulerian velocity an$\ $ ressure) were obtained and are found with the corresponding plots of R, U, $\dot{U}$, and p (Figures 1 through 4).

DISCUSSION

Dynamically, the air inside the bubble behaves in a peculiar fashion which is quite evident in the more violent implosions (those at great depths). During the greater part of the collapse, the air offers insignificant resistance to the inrushing water. Just before the instant of minimum radius, however, the air violently arrests further decrease in volume, behaving very much like a rigid sphere. This is borne out especially by the curves for bubble wall acceleration. At the instant of minimum radius, the water in the immediate vicinity of the bubble "sees" a rigid sphere, but that water a little further from the wall continues to rush in since the water is compressible. The result is a spherical shock wave propagating from the wall out into the fluid.

Comparisons among Figures 1 through 5 lead to the following observations:

1. As the depth at which the implosion occurs becomes greater, the peak pressure increases, the rise time decreases, and the collapse time decreases.

2. Increasing the initial internal pressure of the gas inside the sphere has roughly the same effect as decreasing the depth of implosion. Specifically, the peak pressure can be effectively attenuated by increasing the initial internal pressure. Comparison between Figures 2e and 3e, for example, shows that the peak pressure pulse from an implosion at 1000 ft of water is reduced by almost 40 percent when the initial internal pressure is increased from 1 to 2 atm.

3. As the pressure peak propagates away from the cavity wall, it suffers an attenuation proportional to $1/r$.

4. Comparison between Figures 1, 2, and 5 indicates that for constant initial internal pressure and radius, the collapse time varies approximately inversely with the square root of the depth at which the implosion occurs.

It should be noted that the present theory does not account for the effect of migration. The model presented here is excellent for a ratio of initial internal pressure to ambient pressure which is less than perhaps one-tenth.

Figure 1a — Bubble Radius

Figure 1b — Bubble Wall Velocity
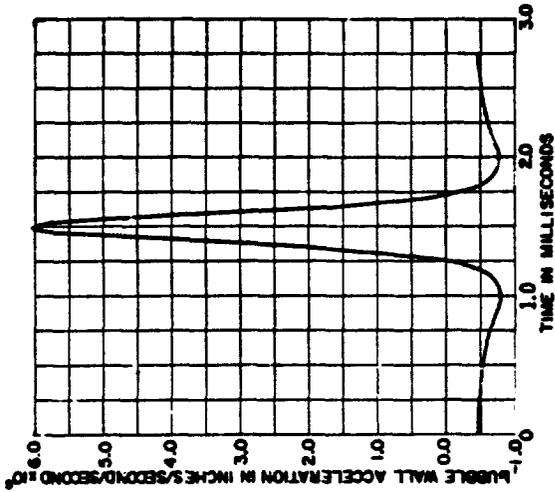
Figure 1c — Bubble Wall Acceleration

Figure 1d — Pressure at the Bubble Wall
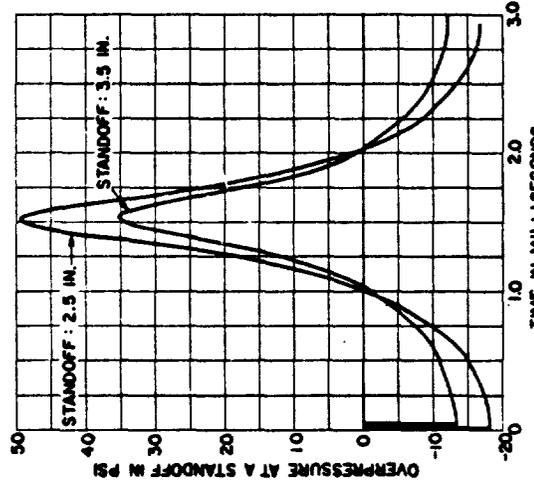
Figure 1e — Overpressures

Figure 1f — (Eulerian) Velocity

Figure 1 — Spherical Collapse as a Function of Time for a Water Depth of 100 Feet,
an Initial Radius of 1 Inch, and an Initial Internal Pressure of 1 Atmosphere

Figure 2c — Bubble Wall Acceleration

Figure 2f — (Eulerian) Velocity

Figure 2b — Bubble Wall Velocity

Figure 2e — Overpressures

Figure 2a — Bubble Radius

Figure 2d — Pressure at the Bubble Wall

Figure 2 — Spherical Collapse as a Function of Time for a Water Depth of 1000 Feet,
an Initial Radius of 1 Inch, and an Initial Internal Pressure of 1 Atmosphere

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Figure 3 — Spherical Collapse as a Function of Time for a
Water Depth of 1000 Feet, an Initial Radius of 1 Inch,
and an Initial Internal Pressure of 2 Atmospheres
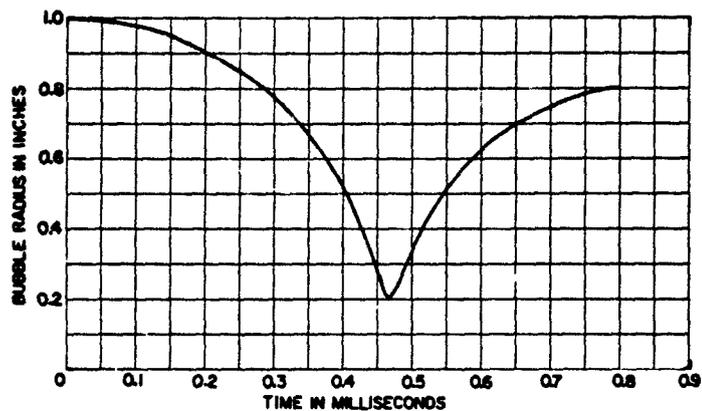
Figure 3a — Bubble Radius
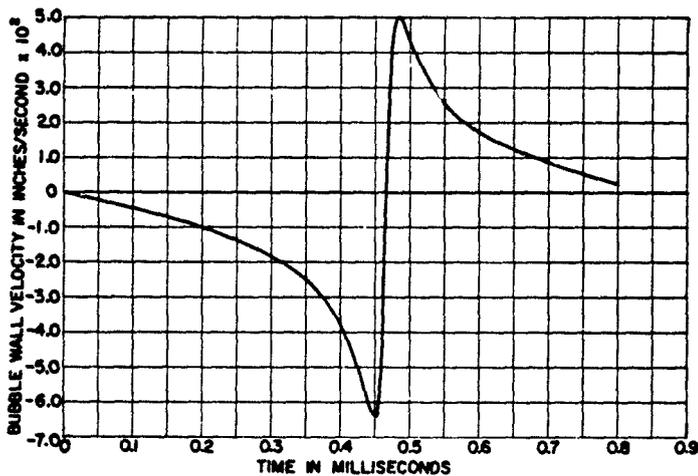
Figure 3b — Bubble Wall Velocity
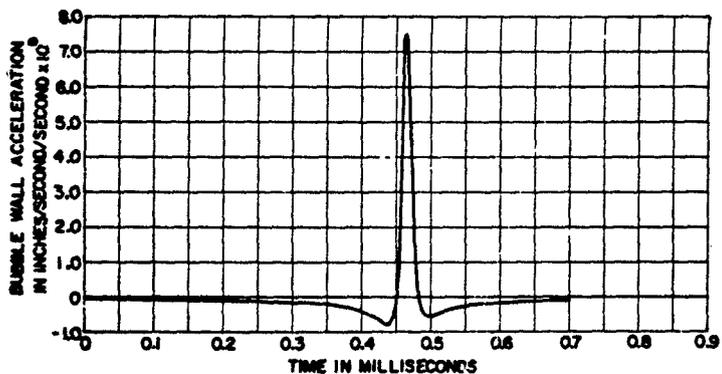
Figure 3c — Bubble Wall Acceleration
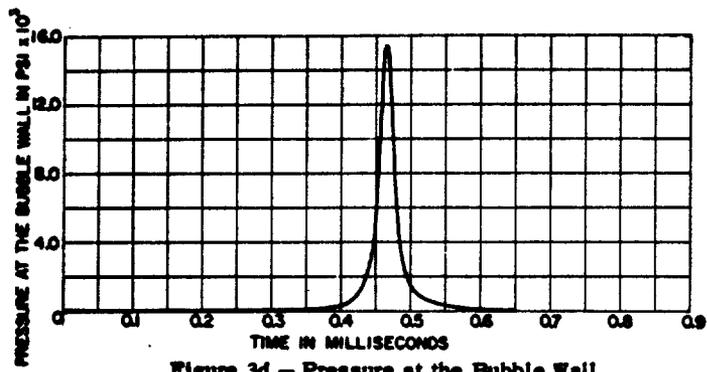
Figure 3d — Pressure at the Bubble Wall
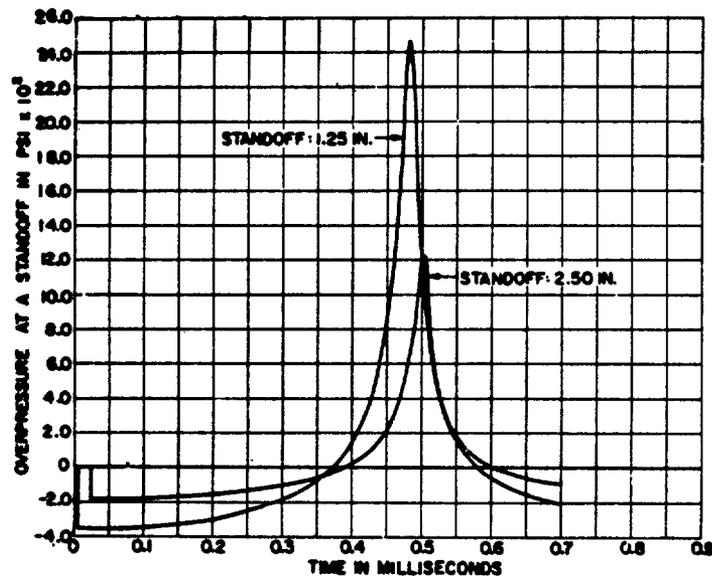
Figure 3e — Overpressures
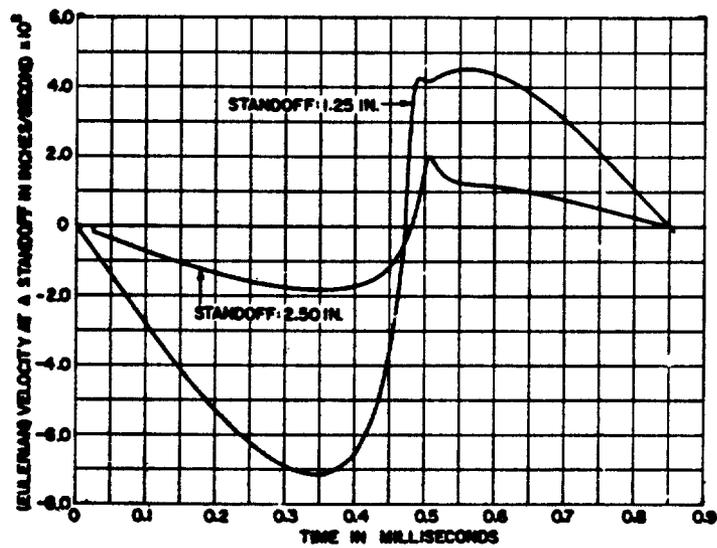
Figure 3f — (Eulerian) Velocity

Figure 4 — Spherical Collapse as a Function of Time for a
Water Depth of 1000 Feet, an Initial Radius of 1 Inch,
and an initial Internal Pressure of 10 Atmospheres
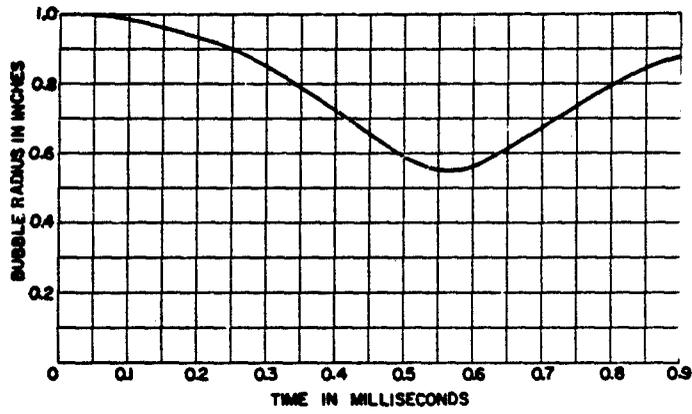
Figure 4a — Bubble Radius
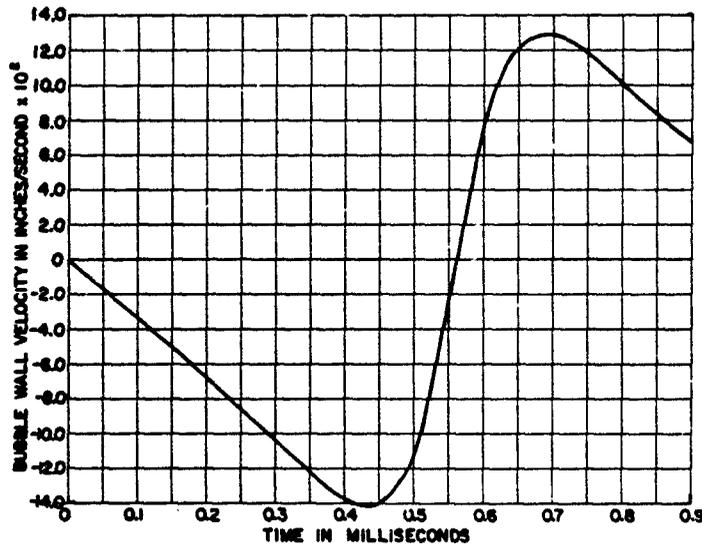
Figure 4b — Bubble Wall Velocity
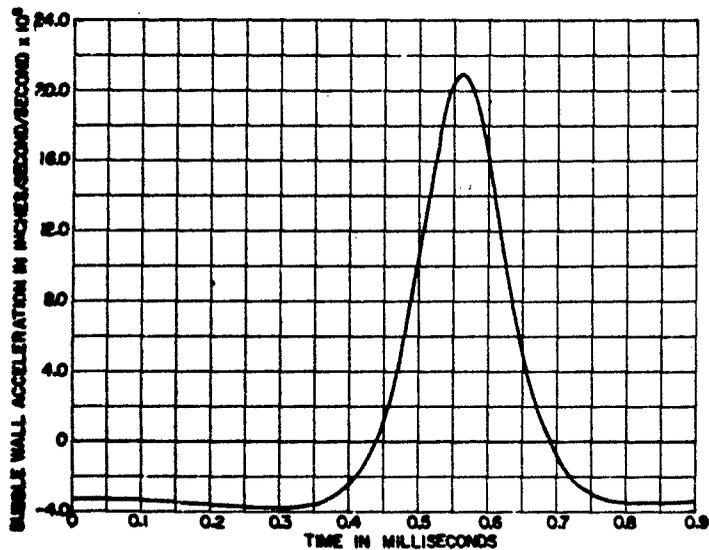
Figure 4c — Bubble Wall Acceleration
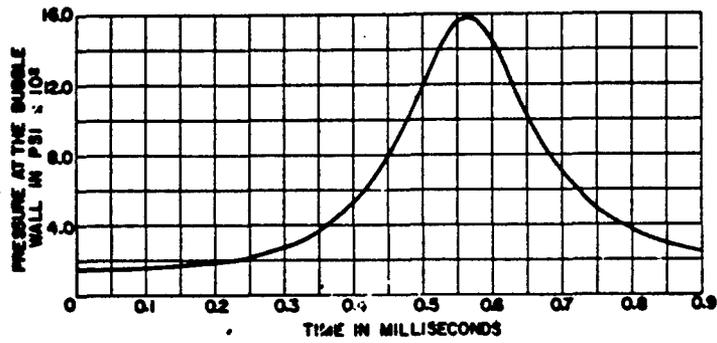
Figure 4d — Pressure at the Bubble Wall
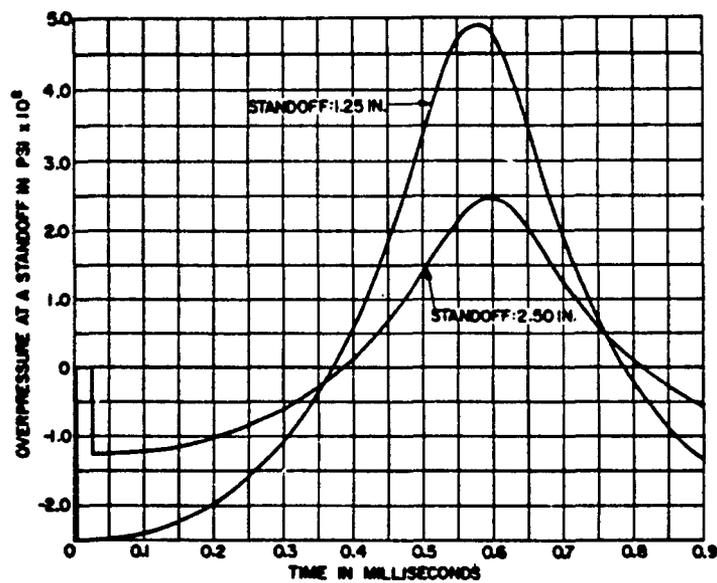
Figure 4e — Overpressures
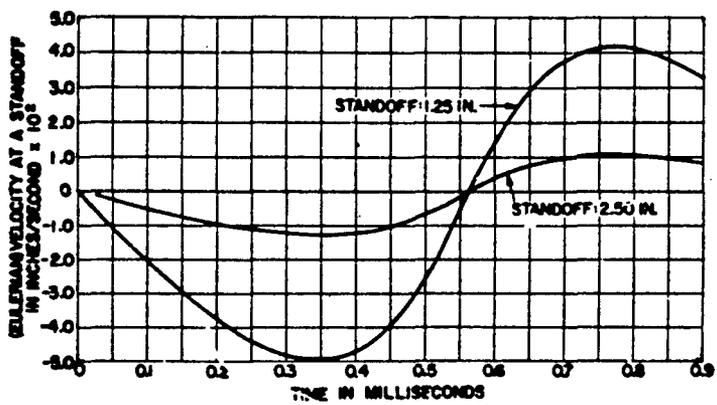
Figure 4f — (Eulerian) Velocity

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Figure 5 — Spherical Collapse as a Function of Time for a
Water Depth of 10,000 Feet, an Initial Radius of 1 Inch,
and an Initial Internal Pressure of 1 Atmosphere
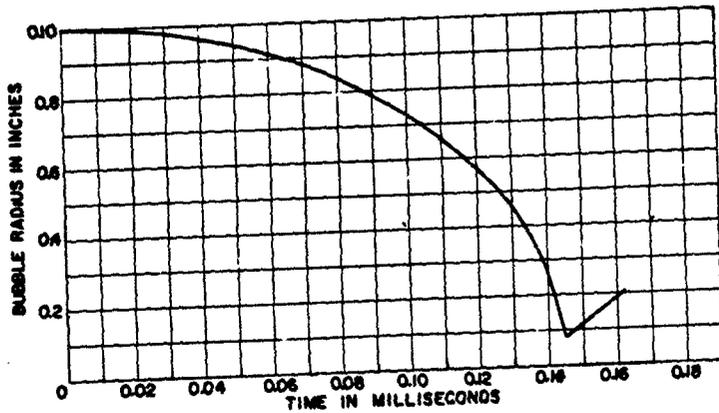
Figure 5a — Bubble Radius
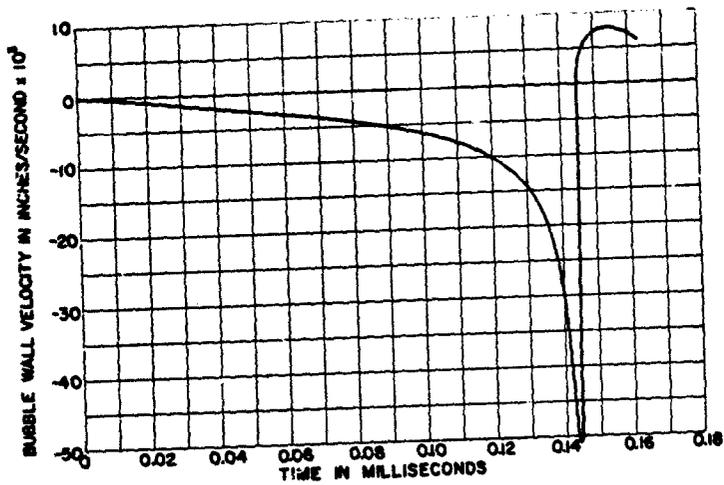
Figure 5b — Bubble Wall Velocity
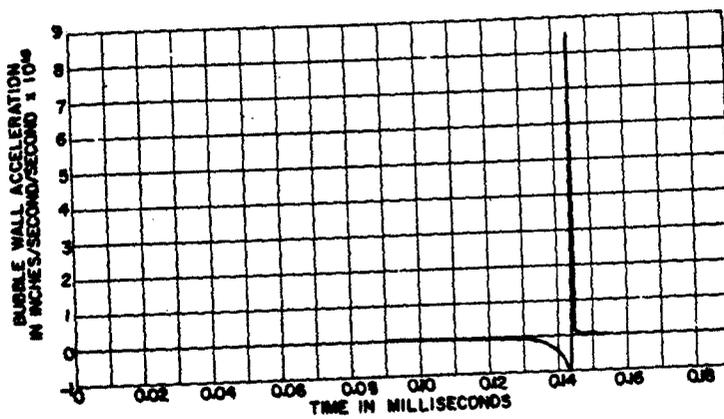
Figure 5c — Bubble Wall Acceleration
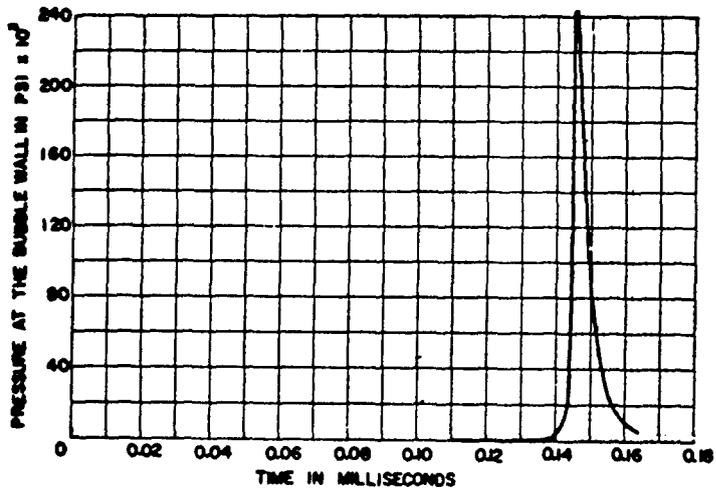
Figure 5d — Pressure at the Bubble Wall
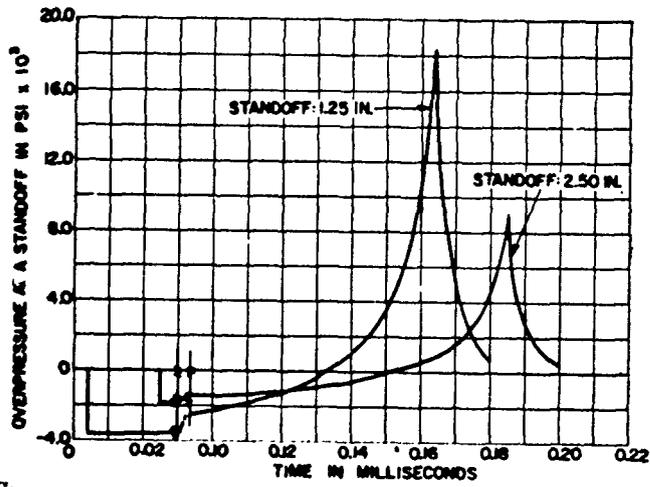
Figure 5e — Overpressures
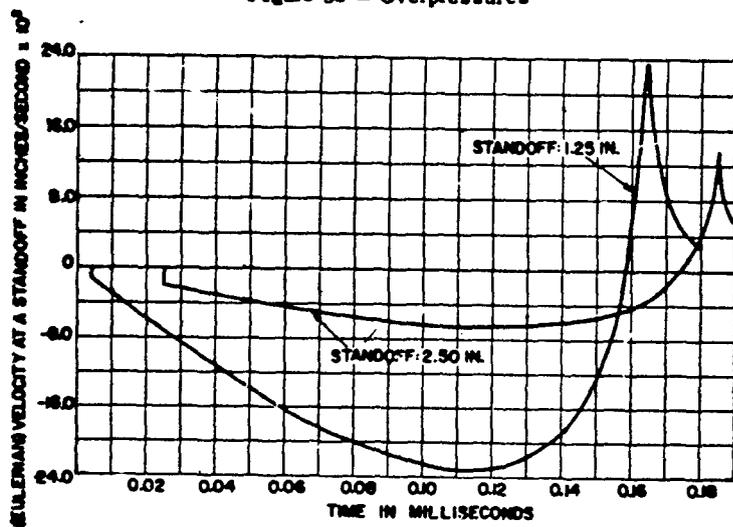
Figure 5f — (Eulerian) Velocity

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As stated earlier, the "well behaved" spherical collapse of the model occasionally may not conform to the behavior of a real bubble in its final stage of implosion. At very high ratios of ambient to initial internal pressure, the possible dissociation of the cavity into numerous smaller bubbles represents a departure from the behavior of the model. At the standoff of interest ($r > R_0$), however, the field variables (pressure and Eulerian velocity) are thought not to deviate significantly from values obtained using the model.

All the results mentioned so far may be extended to cases for spheres of any radius. Suppose that at depth h a solution exists for a sphere with initial radius R_0 and initial internal pressure p_0. The radius, velocity, acceleration, and pressure are known functions of time at the bubble wall or at some standoff in the fluid. If the initial radius is multiplied by λ = constant, then pressure and velocity will remain the same if radius, time, and standoff are multiplied by λ and acceleration divided by λ.

After the writer's program was completed, it was discovered that Hickling and Plesset had solved the free-field implosion problem numerically in a similar but more elaborate manner without making the approximation $u^2 \ll c^2$. Their program requires 20 min of computer time for each case compared to only 2 min for the program presented here. They report two cases; the first ($p_0 = 10^{-3}$ atm, $p_\infty = 1$ atm) was solved by the program presented here, but the second ($p_0 = 10^{-4}$ atm, $p_\infty = 1$ atm) is too violent an implosion for the writer's program to be applicable because the approximation $u^2 \ll c^2$ may not be valid. (Note that this second case reported by Hickling and Plesset represents such a violent implosion that it has little in common with the type of implosion expected in a buoyancy sphere system even as deep as 30,000 feet of water).

In an attempt to verify the soundness of his approach, the writer used the initial conditions of Hickling and Plesset's first case ($p_0 = 10^{-3}$ atm, $p_\infty = 1$ atm) as input in his program. Very little discrepancy (less than 2 percent) can be found in the bubble radius and bubble wall velocity even though the Hickling and Plesset results are based on a value of $\gamma = 1.4$ and the writer's are based on a value of $\gamma = 4/3$. It can be seen from the differential (Equation [1.5]) that a small change in γ has little effect on the radius-time curve.

However, a marked difference appeared between the results of the two programs when the peak pressures inside and outside the bubble were compared. The Hickling and Plesset results showed peak pressures which were about twice those obtained in this study. This discrepancy can be readily resolved by noting the different values of γ used. The results of both programs indicate that a minimum radius $R_{MIN} = 0.0170$ will be obtained when a sphere of initial radius $R_0 = 1$ and initial internal pressure $p_0 = 10^{-3}$ atmospheres is imploded at the ambient pressure $p_\infty = 1$ atm. The pressure at the boundary, by Equation [1.4c], is

$$p = p_0 \left(\frac{R_0}{R}\right)^{3\gamma}$$

When $\gamma = 1.4$

$$p_{MAX}^{*} = p_0 \left(\frac{1}{.0170}\right)^{4.2} \tag{4.1}$$

and when $\gamma = 4/3$

$$p_{MAX} = p_0 \left(\frac{1}{.0170}\right)^{4.0} \tag{4.2}$$

By dividing Equation [4.1] by Equation [4.2], p_{MAX}^{*} is $(1/.0170)^{.2}$ or 2.25 times p_{MAX}. Had a value of $\gamma = 1.4$ been used in the writer's program, the peak pressure at the bubble wall would have compared well with that obtained by Hickling and Plesset. A similar statement is true for those peak pressures in the fluid outside the bubble wall, because the peak pressure varies inversely as the distance from the center of the bubble (i.e. as $1/r$). This verifies the validity of the writer's program and the assumption $u^2 \ll c^2$ for the range of interest ($p_\infty \leq 1000$ atm, $p_0 \leq 1$ atm).

The verification against the work of Hickling and Plesset shows that small variations in γ, the specific heat ratio, can lead to large variations in the peak pressure associated with a collapse. Such behavior suggests that the present equation of state, the ideal gas law, is a deficient description of the gas inside the cavity and that the use of a more elaborate equation of state (e.g., the Beattie-Bridgeman[11] equation of state) would give more accurate results. Use of the Beattie-Bridgeman equation could be made to investigate the differences in behavior of the collapse for different gases.(representing different values of γ). Variations of the kind of gas inside the cavity along with variations in its initial internal pressure may be used to control the characteristics of the pressure pulse emitted when a glass buoyancy sphere collapses. Such control might ultimately be used to reduce the distances between glass spheres in buoyancy sphere systems without increased risk of sympathetic implosions.

SUMMARY AND CONCLUSIONS

1. A program to integrate Gilmore's equations describing the collapse of a spherical gas filled cavity has been written (see Appendix C). The program is general enough to be used in the study of such phenomena as cavitation and underwater explosion gas bubble pulses, provided behavior is adiabatic.

2. Parameters from the program for various ambient and initial internal pressures have been plotted (see Figures 1 through 5).

3. The pressure shock wave associated with the implosion may be controllable in two different ways:

    a. Proper variation of the initial internal pressure of the gas inside the sphere.

    b. Proper variation of the specific heat ratio γ by changing the kind of gas inside the sphere.

These two effects should not be overlooked as possible means of protecting glass buoyancy spheres from sympathetic implosions.

FUTURE WORK PLANNED

1. An experimental verification will be carried out to determine how good Gilmore's model is.

2. The velocity and pressure curves can be used to synthesize analytical functions to describe the free-field implosion. This information will then form a foundation for the analysis of the effects of a single implosion in a system of buoyancy spheres. Such an analysis has, in fact, been started.

3. The present computer program is now being altered by replacing the ideal gas law by the Beattie-Bridgeman equation of state.

ACKNOWLEDGMENTS

APPENDIX A

A PROCEDURE FOR HALVING THE INTERVAL OF INTEGRATION

The description of Hamming's method pointed out that if the predictor does not lie close enough to the corrector, then one alternative is to halve the interval. According to Ralston and Wilf,[9] a suitable set of interpolation formulas for Hamming's method is:

$$y_{n-1/2} = \frac{1}{256} (80y_n + 135y_{n-1} + 40y_{n-2} + y_{n-3}) + \frac{h}{256} (-15y_n' + 90y_{n-1}' + 15y_{n-2}')$$

$$y_{n-3/2} = \frac{1}{256} (12y_n + 135y_{n-1} + 108y_{n-2} + y_{n-3}) + \frac{h}{256} (-3y_n' - 54y_{n-1}' + 27y_{n-2}')$$

where h is the original step size.

In the particular program which was written for the solution of Equation [1.5], the following procedure was adopted:

1. If p_{n+1} was not close enough to c_{n+1} then an iteration was carried out.

2. If, after the first iteration, the value of p_{n+1} was still not close enough to c_{n+1}, then the interval was halved and the entire integration process at that step was carried out from the beginning using the new half interval.

As the radius of the bubble approaches its minimum, the radius and wall velocity become more and more difficult to predict, that is, it becomes more and more likely that p_{n+1} will not fall within the desired limits of c_{n+1}. With the above procedure, more coordinates will be calculated near the minimum where the functions are changing most rapidly than are calculated in regions of small slopes.

APPENDIX B

JUSTIFICATION FOR THE USE OF HAMMING'S METHOD

The choice between an elaborate procedure like Hamming's method and some other iterative routine to integrate the equations is easy to make if it is based on economy of computer time. When an iterative technique is used to converge on the correct value of the dependent variable at each step, the finite difference form of the differential equation may have to be evaluated many times because the initial prediction of the dependent variable is not likely to be very accurate. In the case of Equations [1.5], it is desirable to avoid the evaluation of the finite difference equation as often as possible because it involves so many calculations. Hamming's method, on the other hand, makes a much more accurate initial prediction of the dependent variable at each step simply because more information about past values is utilized in making such predictions. Consequently, Hamming's method yields a more rapid convergence because the finite difference form of the equation has to be evaluated only once or twice at each step to obtain an accurate value of the dependent variable there.

One disadvantage in using this technique is that it is not self-starting. Values of R, U, and N in at least four equally spaced intervals near $t = 0$ are required. Such values may be computed by expressing $R(t)$ in a Taylor Series in t about zero and using Equations [1.5] to evaluate the coefficients. This approach for solving differential equations can be found in any elementary text on the subject, for instance, Coddington.[12] Because of the difficulty of differentiating Equation [1.5], Herring's[4] equation

$$R \frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 - \frac{2}{c_\infty}\left(\frac{dR}{dt}\right)^3 - \frac{2R}{c_\infty}\left(\frac{dR}{dt}\right)\left(\frac{d^2R}{dt^2}\right)$$

$$= \frac{p(t) - p_\infty}{\rho_\infty} + \frac{R}{\rho_\infty c_\infty} \frac{dp(t)}{dt}\left(1 - \frac{\frac{dR}{dt}}{c_\infty}\right) \tag{5.1}$$

rather than Equation [1.5] was used to find the first four values of R, U, and $\dot{U}$. This equation has also been derived by Gilmore and called the "first order approximation."

As mentioned in the discussion under Equation [1.5], to find the initial acceleration of the bubble wall, Equation [5.1] should be evaluated at $t = 0_+$ rather than at $t = 0$ in order to

avoid an infinite initial acceleration.   It can be seen from Equation [1.4c] that Equation

[5.1] evaluated at $t = 0$ contains the infinite term $\left. \dfrac{dP}{dt} \right|_{t=0}$.   The result is

$$\left. \frac{d^2R}{dt^2} \right|_{t=0_+} = \frac{p_0 - p_\infty}{\rho_\infty c_\infty}$$

in agreement with acoustic theory.

APPENDIX C

THE COMPUTER PROGRAM FOR IBM 7090

The following is a list of FORTRAN IV symbols used in the program.

FORTRAN Symbol	Corresponding Symbol Used in Discussion	Explanation
A	R	Instantaneous radius of the imploding sphere.
B	B	A constant which characterizes the adiabatic nature of the liquid medium (for water, $B = 3000$ atm)
BDYP	P	Pressure in the liquid at the bubble wall
BØ	R_0	Initial radius of the imploding sphere
BØD	–	A dimensioned variable name under which the BØ, the unitial radius is read in
B2, B3, B4, B5, B6	–	Coefficients of the Taylor Series expansion of R about $t = 0$
C	c_∞	Sound speed in the undisturbed liquid medium
D	ρ_∞	Density of the undisturbed liquid medium (for water, $\rho_\infty = 2$ slugs/cu ft)
DU	$\dot{U}$	Instantaneous acceleration of the bubble wall
H	–	Depth at which the implosion takes place
HD	–	A dimensioned variable name under which H, the depth of implosion, is read in
PA	–	Atmospheric pressure (14.7 psi)
PL	p_∞	Pressure in the undisturbed liquid medium
PØ	p_0	Initial pressure of the gas inside the sphere
PØD	–	A dimensioned variable name under which PØ, the initial internal pressure, is read in
R and T	t_R	Time measured at the bubble wall
S	h	The step size, or size of the interval over which the integration is to be performed
STF1, STF2, STF(I) (I = 1, 2, 3)		The five standoffs for which Eulerian velocities and pressures are calculated; note STF(1) = STF3, etc.
STFD1, STFD2, STFD(I) (I = 1, 2, 3)		Dimensioned variable names under which the above five standoffs are read in

FORTRAN Symbol	Corresponding Symbol Used in Discussion	Explanation
STFPC1, STFCP2, STFPC(I) $I = 1, 2, 8$	$\bar{p}$	Five instantaneous pressures evaluated at the standoffs STF1, STF2, STF3, STF4 and STF5, respectively; note: STFPC(1) = STFPC3 etc.
STFUC1, STFUC2, STFUC(I) $I = 1, 2, 8$	u	Five instantaneous Eulerian velocities evaluated at the standoffs STF1, STF2, STF3, STF4, and STF5, respectively
T and R	t_R	Time measured at the bubble wall
TSTF1, TSTF2, TSTF(I) $I = 1, 2, 8$	t	The time for each of the five pressures and velocities evaluated at each of the five standoffs, respectively
U	U	Instantaneous velocity of the bubble wall
Y	y	A constant used in the expression for the Eulerian velocity (see Equations [3.1] and [3.2])
YK	K_3	A constant used in the expression for the Eulerian velocity (see Equations [3.1] and [3.3])
AP4, AP5	p_n, p_{n+1}	Predicted values of the radius, AP 5 is the predicted value being tested at the $(n+1)$th interval, and AP4 is the previous predicted value of R which was cloest to the actual value of R at the nth interval
A4MH, A3MH	$y_{n-1/2}$, $y_{n-3/2}$	When R at the $(n+1)$th interval is being predicted and the half interval routine is required, then these are the interpolated values of R between the nth and $(n-1)$th interval and between the $(n-1)$th and $(n-2)$th interval, respectively
CØD	m_{n+1}, m'_{n+1}	Modifier for U, also derivative of the modifier for R
C4, C5	c_n, c_{n+1}	Correctors for U; C5 is the corrector being tested at the $(n+1)$th interval and C4 is the corrector at the previous interval
DØD	m_{n+1}	Modifier for R
DCØD	m'_{n+1}	Derivative of the modifier for U
D4, D5	c_n, c_{n+1}	Correctors for R (see C4, C5)
UP4, UP5	p_n, p_{n+1}	Predicted values of the velocity (see AP4, AP5)
U4MH, U3MH	$y_{n-1/2}$, $y_{n-3/2}$	Half interval values of velocity (see A4MH, A3MH)

THE COMPUTER PROGRAM

The simplified flow chart is shown in Figure 6.

START

```
Read in initial radius, depth, initial internal pressure and
five values of standoff

Find a suitable size for the initial time interval on the basis
of the initial radius, initial internal pressure, and depth just
read in

Using the Taylor series expansion, calculate four initial values,
i.e., values in four intervals for each of the following: radius,
velocity, acceleration, and pressure at the bubble wall; velocity
and pressure at each of the five standoffs

Let the predictor and corrector for U at step 4 he the actual value
obtained from above for U at step 4.  Likewise for R

Predict U and R in the next interval  ←——  Carry out the procedure for halving the interval

Calculate the modifiers, derivatives of the modifiers, and the correctors for U and R.

Test the predictor against the corrector (use the predictors and correctors corresponding to U rather than R)      Guess at new predictors

Is the predictor close enough to the corrector?   NO →   Is this the first failing attempt to predict U and R at this step?   NO

YES

For this interval, use the predictors and correctors to calculate U and R; then calculate pressure and acceleration at the bubble wall and velocity and pressure at each of the five standoffs

NO

Are there enough points to determine a curve?

YES

Print out the values calculated

YES

Are there any more cases to be run?

NO

END
```

Figure 6 -- Simplified Flow Chart

DATA INPUT

The first data card is read according to the format (I2) and must have a number no greater than 50 in the first two columns (see Figure 7). This number should equal the number of data cards to follow. Each of the following data cards should contain the initial information for a single collapse. Eight pieces of information are placed on each card; the card is divided equally into eight parts, 10 spaces each. Information is read in according to the format (8F10.4). The first two pieces of information are the depth of collapse in feet of water and the initial bubble radius in inches, respectively. The next five divisions are set off for five values of standoff in inches from the bubble center at which a pressure and (Eulerian) velocity time history are desired. The last division is reserved for the initial internal pressure inside the bubble in pounds per square inch. The first eight pieces of information, corresponding to one collapse, will require no more than 2 min of running time. Each additional card, i.e., each additional collapse, requires no more than 1 min.

DATA OUTPUT

For each data card, the computer will print 2000 lines of output. The first 1000 lines of numbers are printed under headings TIME, RADIUS, VELOCITY, ACCELERATION, BDYP, TSTF1, STFPC1, STFUC1, TSTF2, STFPC2, STFUC2 (see Figure 8). The first five quantities refer to the bubble wall; BDYP is the internal pressure inside the bubble (absolute pressure, not overpressure). Each line refers to the state of the motion at one instant in time. STFPC1 and STFUC1 are the overpressure (pressure above ambient pressure) and (Eulerian) velocity as functions of the time TSTF1 for the first standoff given in the data. Similarly for STFPC2, STFUC2, and TSTF2. The next 1000 lines give similar information for the third, fourth, and fifth standoffs specified. The headings are TSTF3, STFPC3, STFUC3, TSTF4, STFPC4, STFUC4, TSTF5, STFPC5, STFUC5. All time is given in milliseconds, the radius in inches, all velocities in inches per second, the acceleration in inches per second per second, and all pressures in pounds per square inch.

Figure 7 — Data Input for Computer Program

Figure 8 — Data Output for Computer Program

Figure 8 (Continued)

COMPUTER PROGRAM

```
C  PROGRAMMER  R. LILLISTON, CODE 745, EXT 3357
C  D=CENSITY IN SLUGS/CU.FT,H=DEPTH IN FT,PO=INITIAL INTERNAL PRESSURE
C  IN LBS/SQ.IN.BO=INITIAL RADIUS IN INCHES,C=SPEED OF SOUNDIN MEDIUM IN
C  IN/SEC.PA=ATMOSPHERIC PRESSURE IN LBS/SQ.IN,STF=STANDOFF IN INCHES
C  AND TIME IS IN MILLISECONDS
C   INITIAL VALUES
      REAC(5,101)MM
  101 FORMAT(I2)
      DIMENSION TSTF(3,1000),STFPC(3,1000),STFUC(3,1000),STF(3)
      DIMENSION A(1000),U(1000),DU(1000)
      DIMENSION HO(50),BOD(50),STFO1(50),STFO2(50),STFD(3,50),POD(50)
      REAC(5,99)(HO(J),BOD(J),STFO1(J),STFO2(J),(STFD(I,J),I=1,3),
     1POD(J),J=1,MM)
   99 FORMAT(8F10.4)
      DO1 J=1,MM,1
      H=HO(J)
      BO=BOD(J)
      STF1=STFO1(J)
      STF2=STFO2(J)
      DO 9 I=1,3
    9 STF(I)=STFD(I,J)
      PC=POD(J)
C VALUES OF CONSTANTS FOR THE LIQUID - WATER
      D=2.0
      B=4.41E4
      C=6.0E4
      PA=14.7
      PL=C*H*32.2/144.0+PA
C HEADING
      WRITE(6,88)BO,H,PO,STF1,STF2,(STF(I),I=1,3)
   88 FORMAT(1H1/1H3//////42X,44H GILMORES SECOND ORDER APPROXIMATION FOR
     1 THE///45X,14HIMPLOSION OF A,F4.1,19H INCH RADIUS SPHERE///45X,14H
     2 AT A DEPTH OF,F8.1,14H FEET OF WATER///33X,51HWHEN THE INITIAL IN
     3TERNAL PRESSURE IN THE SPHERE IS,F5.1,5H PSIA///47X,33H AND THE ST
     4ANDOFFS ARE, IN INCHES///48X,1CHSTANDOFF 1,F21.2///48X,1CHSTANDOFF
     5 2,F21.2///48X,1CHSTANDOFF 3,F21.2///48X,1CHSTANDOFF 4,F21.2///
     648X,1CHSTANDOFF 5,F21.2)
C CALCULATE INITIAL VALUES OF THE RADIUS AND VELOCITY
      B2=(2.0736E4)*(PO-PL)/(BO*D*2.0)
      B3=4.C*B2**2/(3.C*C)-(2.7648E4)*PO*B2/(BO*D*C)
      B4=3.0*B2*B3/C-4.0*(B2**2)/(3.0*BO)+(2.0736E4)*(-PO*B3/(BO*D*C)
     1-PL*B2/(3.0*D*BO**2)+4.0*PO*(B2**2)/(3.0*BO*D*C**2))
      B5=2.8*(B2**3)/(C*BO)-2.9*B2*B3/BO+1.8*(B3**2)/C+3.2*B2**4/C
     1+(2.0736)*(-2.0*PL*B3/(D*BO**2)-8.0*PO*B4/(BO*C*D)
     2+24.0*PO*B3*B2/(BO*D*C**2))
      B6=16.0*(B2**2)*B3/(15.0*C*BO)+4.0*B4*B3/C+1.0*B2*B5/(3.0*C)
     1-11.2*B2**3/(9.0*BO**2)-2.9*B3**2/(2.0*BO)-9.4*B2*B4/(3.0*BO)
     2+(2.0736E4)*(1.2*PO*B3**2/(BO*D*C**2)+6.4*PO*B2*B4/(3.0*BO*D*C**2)
     3-2.0*PO*B5/(3.0*BO*D*C)-PL*B3/(1.0*D*BO**3)
     4-0.2*PL*B2**2/(C*BO**3)-4.0*PL*B4/(30.0*D*BO**2))
      R=0.0
      S=(1.44E-4)*SQRT(D)*PO**(1.0/3.0)*BO/PL**(5.0/6.0)
      WRITE(6,100)
C  HEADINGS
  100 FORMAT(123H1   TIME     RADIUS    VELOCITY    ACCELERATION      BDYP
```

```
1  TSTF1       STFPC1        STFUC1      TSTF2       STFPC2        STFU
2C2)
 DO66 I=1,4,1
 A(I)=B0+B2*R**2+B3*R**3+R**2*B4*R**2+R**2*B5*R**3+R**3*B6*R**3
 U(I)=2..*B2*R+3.0*B3*R**2+4.0*R*B4*R**2+5.0*R**2*B5*R**2+
16.0*R**3*B6*R**2
 BDYP=P0*B0**4/A(I)**4
 DU(I)=((U(I)**3)*(((PL+B)/(BDYP+B))**(3.0/7.0))/(2.0*C)
1-3.0*U(I)**2/2.0-7.0*(PL+B)/(6.0*D)*(2.0736E4)*(1.0
2-((BDYP+B)/(PL+B))**(6.0/7.0)+U(I)/C*((PL+B)/(BDYP+B))**(3.0/7.0)
3-U(I)/C*((BDYP+B)/(PL+B))**(3.0/7.0))
4-4.0*U(I)*BDYP/(D*C)*(2.0736E4)*(((BDYP+B)/(PL+B))**(4.0/7.0'
5-U(I)/C*(PL+B)/(BDYP+B)))/(A(I)-A(I)*U(I)/C*((PL+B)/(BDYP
6+B))**(3.0/7.0))
 Y=A(I)*U(I)**2/2.0+(2.0736E4)*A(I)*(BDYP-PL)/D*(1.0-
1(2.0736E4)*(BDYP-PL)/(2.0*D*C**2))
 YK=U(I)*(A(I)**2)*(C**3)*(1.0-U(I)**2/(2.0*C**2))/Y**2
1-A(I)*(C**2)*(1.0-U(I)/C)/Y
 C4STFU=Y*C/STF1+YK*Y**2/(C**3*STF1**2)*(C**2-Y/STF1
1+(YK**2/C**3)*(Y**4/C**3)/(2.0*STF1**4))
 STFUC1=C4STFU/C**2
 STFPC1=D*(Y/STF1-STFUC1**2/2.0)/(2.0736E4)
1+C*((Y/STF1-STFUC1**2/2.0)**2)/(2.0*(C**2)*(2.0736E4))
 TSTF1=((STF1-A(I))/C)*(1.0-U(I)*A(I)/(C*STF1))+R
 C4STFU=Y*C/STF2+YK*Y**2/(C**3*STF2**2)*(C**2-Y/STF2
1+(YK**2/C**3)*(Y**4/C**3)/(2.0*STF2**4))
 STFUC2=C4STFU/C**2
 STFPC2=D*(Y/STF2-STFUC2**2/2.0)/(2.0736E4)
1+C*((Y/STF2-STFUC2**2/2.0)**2)/(2.0*(C**2)*(2.0736E4))
 TSTF2=((STF2-A(I))/C)*(1.0-U(I)*A(I)/(C*STF2))+R
 DO 709 KK=1,3
 C4STFU=Y*C/STF(KK)+YK*Y**2/(C**3*STF(KK)**2)*(C**2-Y/STF(KK)
1+(YK**2/C**3)*(Y**4/C**3)/(2.0*STF(KK)**4))
 STFUC(KK,I)=C4STFU/C**2
 STFPC(KK,I)=C*(Y/STF(KK)-STFUC(KK,I)**2/2.0)/(2.0736E4)
1+C*((Y/STF(KK)-STFUC(KK,I)**2/2.0)**2)/(2.0*(C**2)*(2.0736E4))
709 TSTF(KK,I)=((STF(KK)-A(I))/C)*(1.0-U(I)*A(I)/(C*STF(KK)))+R
 R=R*1.0E3
 TSTF1=TSTF1*1.0E3
 TSTF2=TSTF2*1.0E3
 WRITE(6,110)R,A(I),U(I),DU(I),BDYP,TSTF1,STFPC1,STFUC1,TSTF2,
1STFPC2,STFUC2
110 FORMAT(F11.6,F8.5,2E13.5,E12.5,2(F9.6,2E13.5))
 R=R*1.0E-3
 TSTF1=TSTF1*1.0E-3
 TSTF2=TSTF2*1.0E-3
66 R=R+S
 T=R
C INITIAL VALUES FOR PREDICTORS AND CORRECTORS
 UP4=U(4)
 AP4=A(4)
 C4=U(4)
 D4=A(4)
 N=4
C USING THE FOUR CALCULATED INITIAL VALUES ABOVE, BEGIN INTEGRATION
 DO669 I=5,1000,1
```

```
      N=N+1
  250 L=0
  700 UP5=U(I-4)+4.0*S*(2.0*DU(I-1)-DU(I-2)+2.0*DU(I-3))/3.0
      AP5=A(I-4)+4.0*S*(2.0*U(I-1)-U(I-2)+2.0*U(I-3))/3.0
  230 COD=UP5-112.0*(UP4-C4)/121.0
      DOC=AP5-112.0*(AP4-C4)/121.0
      DCOD=((COD**3)*(((PL+B)/(PO*(BO/DOD)**4+B))**(3.0/7.0))/(2.0*C)
     1-3.0*COD**2/2.0-7.0*(PL+B)/(6.0*D)*(2.0736E4)*(1.0
     2-((PO*(BO/DOD)**4+B)/(PL+B))**(6.0/7.0)+COD/C*((PL
     3+B)/(PO*(BO/DOD)**4+B))**(3.0/7.0)-COD/C*((PO*(BO/DOD)**4+B)/(PL
     4+B))**(3.0/7.0))
     5-4.0*COD*PO*(BO/DOD)**4/(D*C)*(2.0736E4)*(((PO*(BO/DOD)**4
     6+B)/(PL+B))**(4.0/7.0)-COD/C*(PL+B)/(PO*(BO/DOD)**4+B)))/(DOD
     7-COD*COD/C*((PL+B)/(PO*(BO/DOD)**4+B))**(3.0/7.0))
      C5=(9.0*U(I-1)-U(I-3)+3.0*S*(DCOD+2.0*DU(I-1)-DU(I-2)))/8.0
      D5=(9.0*A(I-1)-A(I-3)+3.0*S*(COD+2.0*U(I-1)-U(I-2)))/8.0
      G=ABS(UP5-C5)
      IF(G-1.0)210,210,260
C HALF INTERVAL PROCEDURE
  400 S=S/2.0
      A4MH=(80.0*A(I-1)+135.0*A(I-2)+40.0*A(I-3)+A(I-4))/256.0
     4+S*(-15.0*U(I-1)+90.0*U(I-2)+15.0*U(I-3))/128.0
      A3MH=(12.0*A(I-1)+135.0*A(I-2)+108.0*A(I-3)+A(I-4))/256.0
     4+S*(-3.0*U(I-1)-54.0*U(I-2)+27.0*U(I-3))/128.0
      U4MH=(80.0*U(I-1)+135.0*U(I-2)+40.0*U(I-3)+U(I-4))/256.0
     4+S*(-15.0*DU(I-1)+90.0*DU(I-2)+15.0*DU(I-3))/128.0
      U3MH=(12.0*U(I-1)+135.0*U(I-2)+108.0*U(I-3)+U(I-4))/256.0
     4+S*(-3.0*DU(I-1)-54.0*DU(I-2)+27.0*DU(I-3))/128.0
      A(I-4)=A3MH
      A(I-3)=A(I-2)
      A(I-2)=A4MH
      U(I-4)=U3MH
      U(I-3)=U(I-2)
      U(I-2)=U4MH
      BDYP=PO*(BO/A(I-4))**4
      DU(I-4)=((U(I-4)**3)*(((PL+B)/(BDYP+B))**(3.0/7.0))/(2.0*C)
     1-3.0*U(I-4)**2/2.0-7.0*(PL+B)/(6.0*D)*(2.0736E4)*(1.0
     2-((BDYP+B)/(PL+B))**(6.0/7.0)+U(I-4)/C*((PL+B)/(BDYP
     3+B))**(3.0/7.0)-U(I-4)/C*((BDYP+B)/(PL+B))**(3.0/7.0))
     4-4.0*U(I-4)*BDYP/(D*C)*(2.0736E4)*(((BDYP+B)/(PL
     5+B))**(4.0/7.0)-U(I-4)/C*(PL+B)/(BDYP+B)))/(A(I-4)
     6-A(I-4)*U(I-4)/C*((PL+B)/(BDYP+B))**(3.0/7.0))
      DU(I-3)=DU(I-2)
      BDYP=PO*(BO/A(I-2))**4
      DU(I-2)=((U(I-2)**3)*(((PL+B)/(BDYP+B))**(3.0/7.0))/(2.0*C)
     1-3.0*U(I-2)**2/2.0-7.0*(PL+B)/(6.0*D)*(2.0736E4)*(1.0
     2-((BDYP+B)/(PL+B))**(6.0/7.0)+U(I-2)/C*((PL+B)/(BDYP
     3+B))**(3.0/7.0)-U(I-2)/C*((BDYP+B)/(PL+B))**(3.0/7.0))
     4-4.0*U(I-2)*BDYP/(D*C)*(2.0736E4)*(((BDYP+B)/(PL
     5+B))**(4.0/7.0)-U(I-2)/C*(PL+B)/(BDYP+B)))/(A(I-2)
     6-A(I-2)*U(I-2)/C*((PL+B)/(BDYP+B))**(3.0/7.0))
      L=L+1
      IF(L-10)700,700,71
C SET UP FOR ITERATION
  260 UP5=C5+9.0*(UP5-C5)/121.0
      AP5=D5+9.0*(AP5-D5)/121.0
```

```
      L=L+1
      IF(L-2)230,400,400
C CALCULATION OF FINAL VALUES
  210 U(I)=C5+9.0*(UP5-C5)/121.0
  500 A(I)=D5+9.0*(AP5-D5)/121.0
      BDYP=PO*B0**4/A(I)**4
      DU(I)=((U(I)**3)*(((PL+B)/(BDYP+B))**(3.0/7.0))/(2.0*C)
     1-3.0*U(I)**2/2.0-7.0*(PL+B)/(6.0*D)*(2.0736E4)*(1.0
     2-((BDYP+B)/(PL+B))**(6.0/7.0)+U(I)/C*((PL+B)/(BDYP+B))**(3.0/7.0)
     3-U(I)/C*((BDYP+B)/(PL+B))**(3.0/7.0))
     4-4.0*U(I)*BDYP/(D*C)*(2.0736E4)*(((BDYP+B)/(PL+B))**(4.0/7.0)
     5-U(I)/C*(PL+B)/(BDYP+B)))/(A(I)-A(I)*U(I)/C*((PL+B)/(BDYP
     6+B))**(3.0/7.0))
      Y=A(I)*U(I)**2/2.0+(2.0736E4)*A(I)*(BDYP-PL)/D*(1.0-
     1(2.0736E4)*(BDYP-PL)/(2.0*D*C**2))
      YK=U(I)*(A(I)**2)*(C**3)*(1.0-U(I)**2/(2.0*C**2))/Y**2
     1-A(I)*(C**2)*(1.0-U(I)/C)/Y
      C4STFU=Y*C/STF1+YK*Y**2/(C**3*STF1**2)*(C**2-Y/STF1
     1+(YK**2/C**3)*(Y**4/C**3)/(2.0*STF1**4))
      STFUC1=C4STFU/C**2
      STFPC1=D*(Y/STF1-STFUC1**2/2.0)/(2.0736E4)
     1+C*((Y/STF1-STFUC1**2/2.0)**2)/(2.0*(C**2)*(2.0736E4))
      TSTF1=((STF1-A(I))/C)*(1.0-U(I)*A(I)/(C*STF1))+T
      C4STFU=Y*C/STF2+YK*Y**2/(C**3*STF2**2)*(C**2-Y/STF2
     1+(YK**2/C**3)*(Y**4/C**3)/(2.0*STF2**4))
      STFUC2=C4STFU/C**2
      STFPC2=D*(Y/STF2-STFUC2**2/2.0)/(2.0736E4)
     1+C*((Y/STF2-STFUC2**2/2.0)**2)/(2.0*(C**2)*(2.0736E4))
      TSTF2=((STF2-A(I))/C)*(1.0-U(I)*A(I)/(C*STF2))+T
      DO 708 KK=1,3
      C4STFU=Y*C/STF(KK)+YK*Y**2/(C**3*STF(KK)**2)*(C**2-Y/STF(KK)
     1+(YK**2/C**3)*(Y**4/C**3)/(2.0*STF(KK)**4))
      STFUC(KK,I)=C4STFU/C**2
      STFPC(KK,I)=D*(Y/STF(KK)-STFUC(KK,I)**2/2.0)/(2.0736E4)
     1+D*((Y/STF(KK)-STFUC(KK,I)**2/2.0)**2)/(2.0*(C**2)*(2.0736E4))
  708 TSTF(KK,I)=((STF(KK)-A(I))/C)*(1.0-U(I)*A(I)/(C*STF(KK)))+T
      T=T*1.0E3
      TSTF1=TSTF1*1.0E3
      TSTF2=TSTF2*1.0E3
      WRITE(6,169)T,A(I),U(I),DU(I),BDYP,TSTF1,STFPC1,STFUC1,TSTF2,
     1STFPC2,STFUC2
  169 FORMAT(F11.6,F8.5,2E13.5,E12.5,2(F9.6,2E13.5))
      T=T*1.0E-3
      TSTF1=TSTF1*1.0E-3
      TSTF2=TSTF2*1.0E-3
C  RELOCATE CERTAIN QUANTITIES FOR THE NEXT STEP OF THE INTEGRATION
      C4=C5
      D4=D5
      UP4=UP5
      AP4=AP5
      IF(A(I)-A(I-1))669,669,240
  240 IF(U(I))70,669,669
  669 T=T+S
      GO TO 7
   71 WRITE(6,75)
   75 FORMAT(57HTHE PROCESS IS NOT CONVERGING QUICKLY ENOUGH AT SOME STE
```

```
   1P)
 70 IF(M-1000)7,223,223
223 WRITE(6,224)
224 FORMAT(9*H 1000 VALUES HAVE BEEN CALCULATED BUT THESE VALUES DO NO
   1T COMPLETE THE ENTIRE FIRST PERIOD)
  7 WRITE(6,107)
107 FORMAT(120H1    TSTF3        STFPC3        STFUC3        TSTF4
   1 STFPC4        STFUC4        TSTF5.        STFPC5        STFUC5    )
    DO 715 I=1,3
    DO 715 N=1,1000
715 TSTF(I,N)=TSTF(I,N)*1.0E3
    WRITE(6,701)((TSTF(I,N),STFPC(I,N),STFUC(I,N),I=1,3),N=1,1000)
701 FORMAT(3(F10.6,2E15.6))
  1 CONTINUE
222 STOP
    END
```

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This paper presents a method for calculating the instantaneous pressure, velocity, acceleration, and radius associated with the collapse of a spherical gas-filled cavity in an infinite compressible liquid. The method is an independent approach which makes use of Hamming's technique to numerically integrate Gilmore's differential equations which describe the collapse.

Included is a computer program which will perform the necessary calculations on a IBM 7090/1401 digital computer. Results obtained are in good agreement with those of Hickling and Plesset, whose work was unknown to the present author when he undertook the study.

It may be inferred that the peak shock wave pressure is significantly reduced by a decrease in ambient pressure, an increase in internal pressure, and/or a variation of the specific heat ratio by proper selection of the gas. Control of the last two parameters can be investigated as a possible means of protecting glass spheres against sympathetic implosion in multiple sphere buoyancy systems.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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It may be inferred that the peak shock wave pressure is significantly reduced by a decrease in ambient pressure, an increase in internal pressure, and/or a variation of the specific heat ratio by proper selection of the gas. Control of the last two parameters can be investigated as a possible means of protecting glass spheres against sympathetic implosion in multiple sphere buoyancy systems.