SERVOMECHANISMS, EXPONENTIAL SMOOTHING, AND A MULTIECHELON INVENTORY PROBLEM

DECISION studies group

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1. Servomechanisms

1.1 Review of Previous Results

In a recent report [1] the problem of designing an inventory system from a servomechanism point of view was examined. The inventory model adopted was a periodic review system with fixed order points, taken as the nonnegative integers for the sake of convenience. The t^{th} demand period is measured by real time between $t-1$ and t for $t = 1,2,3,\ldots$. Adopting a fixed time lag T for delivery, an order θ_t is to be placed at the end of each demand period in anticipation of future demands.

It is natural under the above circumstances, to suppose that the demand during the t^{th} demand period is described by a discrete parameter stochastic process X_t, $t = 0,1,2,\ldots$, where $X_0 \equiv 0$. Furthermore, for $t \geq 1$, it is assumed that $X_t = m(t) + \epsilon_t$, which describes the process in terms of its mean value function $m(t)$ and an error term. The errors, $\{\epsilon_t\}$, are considered to be uncorrelated with $E[\epsilon_t] \equiv 0$ and $V[\epsilon_t] \equiv \sigma^2$. If we define the inventory level over a safety level at the end of the t^{th} period to be I_t, then, allowing back orders against future deliveries, a simple recursion relation exists relating inventories to demands and orders, namely,

$$(1.1) \qquad I_t = I_{t-1} + \theta_{t-1-T} - X_t \qquad t = 1,2,3,\ldots$$

In this relationship, $I_o \equiv 0$ and $\theta_m = 0$ for $m \leq 0$. It is seen that, admitting no control over demand, the inventory level is automatically controlled once an ordering rule θ_t is specified. It is this automatic control feature, employing the feedback of previous inventory levels, that naturally suggests an analysis from a servomechanism point of view.

Adopting as a minimum requirement that an order rule should be chosen in such a way as to control the inventory about a safety level, order rules were further restricted to those which could be represented as linear combinations of past demands and inventory levels. Thus, for some sequences $\{A_o, A_1, A_2, \ldots\}$ and $\{B_o, B_1, B_2, \ldots\}$, we require

$$(1.2) \qquad \theta_t = \sum_{j=0}^{t} A_j X_{t-j} + \sum_{j=0}^{t} B_j I_{t-j} \qquad t = 1,2,3,\ldots$$

Next, recognizing that if X_t is random then so is I_t, the concept of minimizing inventory level was replaced by that of minimizing the expected value of the inventory level denoted $E_t = E[I_t]$, $t = 0,1,2,\ldots$. Because of the amenability of the form of equations (1.1) and (1.2) to z-transforms, equivalent expressions were then derived to yield, respectively,

$$(1.3) \qquad I(z) = zI(z) + z^{T+1}\theta(z) - X(z) \quad \text{and}$$

$$(1.4) \qquad \theta(z) = A(z)X(z) + B(z)I(z) \quad .$$

In the above notation, if $\{V_o, V_1, V_2, \ldots\}$ is a sequence, its z-transform $\sum_{k=0}^{\infty} V_k z^k$ will always be denoted $V(z)$. Solving (1.3) and (1.4) simultaneously, it is possible to express the basic relationship between inventory and demand as,

$$(1.5) \qquad I(z) = S(z)X(z), \quad \text{where} \quad S(z) = \frac{z^{T+1}A(z)-1}{1-z-z^{T+1}B(z)} .$$

$S(z)$ is often called the transfer function and, if preferred, a similar relationship exists between the expected inventory level and the mean demand, namely,

$$(1.6) \qquad E(z) = S(z)m(z) .$$

$S(z)$ is the same transfer function as defined in (1.5).

The problem is thus reduced to determining an order rule by means of specifying the sequences $\{A_o, A_1, A_2, \ldots\}$ and $\{B_o, B_1, B_2, \ldots\}$ or, equivelently, their transforms $A(z)$ and $B(z)$. Now for $m(z) \equiv 1$, i.e.,

$$m(t) = \begin{cases} 1 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t > 0 \end{cases} ,$$

E_t would be the response in expected inventory to a one-time impulse at $t = 0$. In this case, $E(z) = S(z)$, and the requirement that the response eventually dampen, i.e., $\lim_{t \to \infty} E_t = 0$ or $\lim_{t \to \infty} S_t = 0$ is equivalent in the transform domain to the requirement $\lim_{z \to 1^-} (1-z)S(z) = 0$. Since this in turn will be true if the poles of $S(z)$ are outside the unit circle in the z-plane, we follow Vassian [2] and let

$$B(z) = \frac{-(1-z)}{1-z^{T+1}} \quad .$$

In that case, $S(z)$ has no finite poles and, with this determination of $B(z)$, all that remains is to choose $A(z)$. Mathematically it turns out to be easier to select a certain function of $A(z)$ namely

$$A^*(z) = z^{T+1} A(z) \frac{(1-z^{T+1})}{1-z} X(z) \quad .$$

After some algebraic simplification and inversion from the transform domain, it is possible to express $\{A_t^*\}$ as the sequence,

$$(1.7) \qquad A^*_{t+1+T} = I_{t+1+T} + \sum_{j=t+1}^{t+1+T} X_j \ , \qquad t = 0,1,2,\ldots \quad \text{and}$$

$$A_j^* = 0 \quad \text{for} \quad j = 0,1,2,\ldots,T \quad .$$

Taking expectations,

$$(1.8) \qquad E[A^*_{t+1+T}] = E_{t+1+T} + \sum_{j=t+1}^{t+1+T} m(j) \quad .$$

In view of the basic role played by A_j^* in (1.7), it is quite suitable that A^*_{t+1+T} be called the <u>forecast</u> of demand from period $t+1$ through period $t+1+T$. Moreover, the requirement that $\lim_{t \to \infty} E_t = 0$ forces A^*_{t+1+T} to be chosen in such a way that $E(A^*_{t+1+T}) - \sum_{j=t+1}^{t+1+T} m(j) \to 0$ as $t \to \infty$. This condition is referred to as <u>asymptotically accurate forecasting</u>.

Finally, once an asymptotically accurate forecaster is selected, we can easily express the order rule in terms of such a forecast by means of the relation,

$$\theta_t = A^*_{t+1+T} - \sum_{j=1}^{T} \theta_{t-j} - I_t \quad \text{for} \quad t \geq T \ . \tag{1.9}$$

In terms of servomechanism diagrams, the inventory system can be visualized quite easily as in Figures 1.1 and 1.2 below, first as a simple input-output system and, secondly, as a more complicated system with a feedback loop clearly displayed.

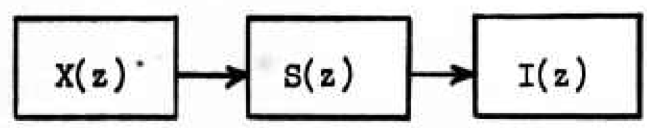


Figure 1.1: Input-Output System

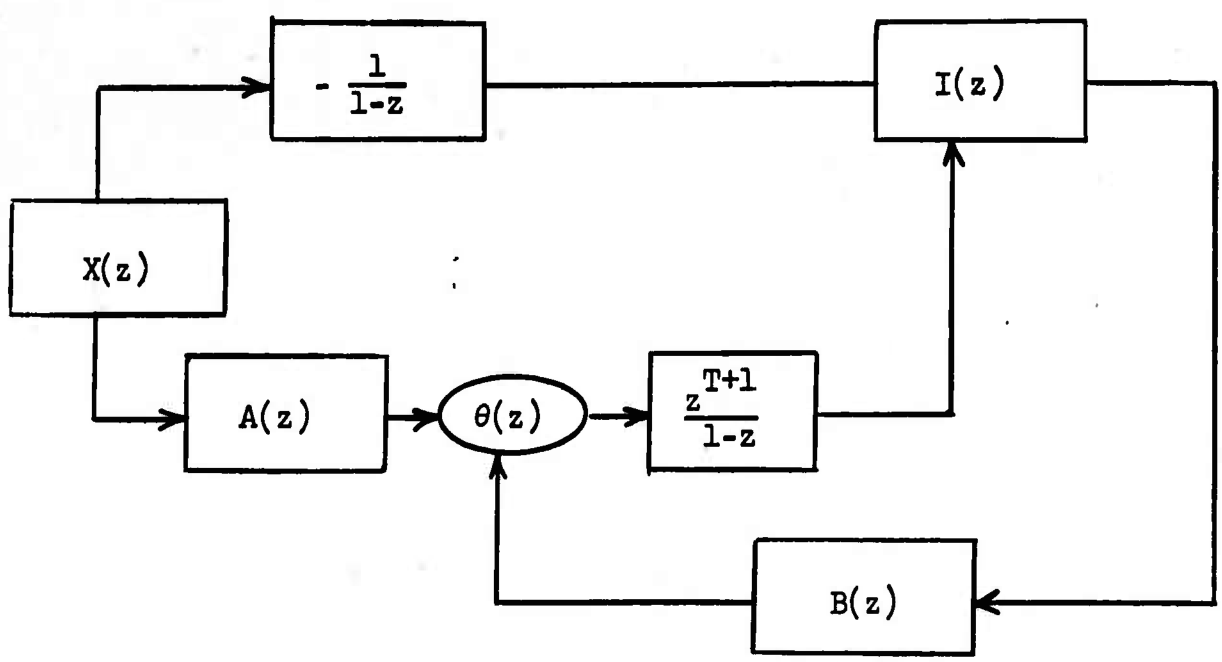


Figure 1.2: The Inventory System

1.2 Additional Results for the Case of Constant Mean Demand

Whenever the demand process is assumed to have a constant mean, it is possible to obtain several explicit results which in turn provide further insight into the inventory system just described. To this end, we assume henceforth that $m(t) \equiv a$ where $a > 0$ so that, in transform notation, $m(z) = \frac{a}{1-z}$. Given the demand history $X_1, \ldots, X_t$ up to time $t \geq T$, we first use exponential smoothing to estimate mean demand thereby obtaining

$$\hat{a}_t = \alpha \sum_{k=0}^{t} \beta^k X_{t-k} \tag{1.10}$$

where

$$\beta = 1 - \alpha, \text{ and } 0 \leq \alpha \leq 1 .$$

The constant α is called the <u>smoothing constant</u>. It is then natural to define the forecast of demand over the following $T+1$ time periods by

$$A^*_{t+1+T} = (T+1)\hat{a}_t$$

thereby obtaining

$$E[A^*_{t+1+T}] = (T+1)E[\hat{a}_t] = a(T+1)(1-\beta^t) .$$

In this case,

$$E[A^*_{t+1+T}] - \sum_{j=t+1}^{t+1+T} m(j) = -a(T+1)\beta^t \to 0 \quad \text{as} \quad t \to \infty$$

as required for asymptotic accuracy.

Having selected an appropriate forecast one can then find an explicit formula for $A(z)$. Indeed, since

$$\hat{a}(z) = \frac{\alpha X(z)}{1-\beta z}$$

it follows that

$$A^*(z) = (T+1)z^{T+1}\hat{a}(z) = \frac{\alpha(T+1)z^{T+1}X(z)}{1-\beta z} .$$

But also,

$$A^*(z) = \frac{z^{T+1}(1-z^{T+1})}{1-z} A(z)X(z)$$

so that, upon substitution,

$$A(z) = \frac{\alpha(T+1)(1-z)}{(1-\beta z)(1-z^{T+1})} . \tag{1.11}$$

As a casual observation, with the above explicit formula for $A(z)$, we can express the transfer function explicitly as

$$S(z) = \frac{\alpha(T+1)z^{T+1}}{1-\beta z} - \sum_{k=0}^{T} z^k .$$

Since

$$E(z) = S(z)m(z) = \frac{aS(z)}{1-z} ,$$

it follows that

$$\lim_{z \to 1^-} (1-z)E(z) = a \lim_{z \to 1^-} S(z) = a[(T+1)-(T+1)] = 0$$

as required.

With the present form for the transfer function it is also possible to express the inventory level strictly in terms of past demands. Since

$$I(z) = S(z)X(z) = \frac{\alpha(T+1)z^{T+1}}{1-\beta z} X(z) - \frac{1-z^{T+1}}{1-z} X(z) ,$$

it follows from inversion that

$$I_t = \alpha(T+1) \sum_{k=0}^{t-T-1} \beta^k X_{t-T-1-k} - \sum_{k=0}^{T} X_{t-k} \quad \text{for} \quad t \geq T . \tag{1.12}$$

Also,

$$\theta_t = \alpha(T+1)[X_t - \hat{a}_{t-1}] + X_t \quad \text{for} \quad t \geq 1 . \tag{1.13}$$

The above formulas will be useful in the ensuing discussion.

1.3 Variance of Inventory Level

While the requirement of asymptotic accuracy has a great deal of intuitive appeal and represents a minimum requirement in adopting a forecasting technique, it is, after all, a mean value property. Even though the expected inventory level is effectively zero after a certain time, it does not of course follow that the actual inventory level is even near zero at that time. This fact was dramatically demonstrated recently by DeWinter [3]. Simulating the above model, random demands were generated and the corresponding random inventory levels were observed for many time periods.

In one example treated by DeWinter, the expected inventory level was effectively zero after 83 time periods. This was a case where

$a = 16$ and $T = 4$. Exponential smoothing, with a constant of $\alpha = 0.2$ was used. The system was allowed to run for 1,000 time periods using a random input corresponding to a Poisson demand with mean 16. Even after 900 periods, the actual inventory level was, at times, as low as -33 units and as high as +27 units. Other examples treated by DeWinter showed a similar behavior in the inventory fluctuations.

The above results are in a sense not too surprising since the variance of the inventory level had never been considered explicitly. With the formula (1.12) derived above, it is now possible to analyze the inventory variance explicitly. Recall that in the present model we have assumed $X_t = m(t) + \epsilon_t$ where the random variables $\{\epsilon_t\}$ are uncorrelated with constant variance. In that case (1.12) may be used to compute the variance, V_t, of I_t as follows. From (1.12) we may write

$$I_t = \sum_{j=0}^{t} \gamma_j X_j$$

where, for $t > T$

$$\gamma_j = \begin{cases} \alpha(T+1)\beta^{t-T-1-j} & \text{for} \quad 0 \leq j \leq t - T - 1 \\ -1 & \text{for} \quad t - T \leq j \leq t \quad . \end{cases}$$

Since $X_0 \equiv 0$,

$$I_t = \sum_{j=1}^{t} \gamma_j X_j$$

and consequently,

$$V_t = \sigma^2 \sum_{j=1}^{t} \gamma_j^2 = \sigma^2\alpha^2(T+1)^2 \sum_{j=1}^{t-T-1} \beta^{2(t-T-1-j)} + (T+1)\sigma^2 \ .$$

After some simplification,

$$(1.14) \qquad V_t = \sigma^2(T+1)[1 + \frac{\alpha(T+1)}{1+\beta}(1-\beta^{2(t-T-1)})] .$$

Now, for positive K, the function

$$f(x) = \frac{1-x}{1+x}(1-x^{2K})$$

defined for $0 \leq x \leq 1$ has a derivative equal to

$$f'(x) = \frac{2[(x^{2K}-1) + Kx^{2K-1}(x^2-1)]}{(1+x)^2} .$$

Since $x^2 \leq 1$ and $x^{2K} \leq 1$, it follows that $f'(x) \leq 0$ so that f is monotone decreasing in x. This means that V_t is monotone decreasing in β or monotone increasing in α. Moreover, for $\alpha = 0$, $V_t = \sigma^2(T+1)$ while for $\alpha = 1$, $V_t = \sigma^2(T+1)(T+2)$. Consequently we have found finite bounds for the inventory variance, valid for all $t > T$

$$(1.15) \qquad \sigma^2(T+1) \leq V_t \leq \sigma^2(T+1)(T+2) .$$

Allowing t to increase beyond bound in (1.14) the limiting variance, V, of the inventory level is found to be

$$(1.16) \qquad V = \sigma^2(T+1)[1 + \frac{\alpha(T+1)}{1+\beta}] .$$

Of course V is monotone decreasing in β as is V_t. As a matter of curiosity, taking the constants used by DeWinter in the above example, i.e., $T = 4$, $\alpha = .2$ and $a = 16$, we find a limiting variance of $V = 125$. Now three standard deviations would be roughly 33 units. Consequently for a value of t sufficiently large, an inventory level of -33 (c.f. p. 8) would be compatible with the asymptotic variance. The results of the simulations are thus quite in keeping with the theory. Moreover, since $E_t = E[I_t] = -a(T+1)\beta^t < 0$ we would expect the actual inventory levels to tend toward the negative side of the zero mean. Such appears to be the case, not only in this example, but in others examined.

2. Exponential Smoothing

2.1 Biasedness

For the simple problem described in Section 1.2, the exponentially smoothed estimate of a is given by equation (1.10). The estimate $\hat{a}_t$ can be represented recursively by,

$$\hat{a}_t = \alpha X_t + \beta \hat{a}_{t-1} \qquad (t = 1,2,\ldots) \quad , \tag{2.1}$$

which exhibits $\hat{a}_t$ as a linear combination of $\hat{a}_{t-1}$ and X_t with the present observation receiving "weight" α.

Note from (2.1) that computation of $\hat{a}_t$ does not require knowledge of the entire demand history but only $\hat{a}_{t-1}$ and X_t. As a result, if computations are performed on an electronic computer, memory is conserved. This feature along with the intuitive appeal of weighting the present more than the past and the computational simplicity accounts for a large part of the popularity of the procedure as an estimating technique.

It is interesting to note however that

$$E[\hat{a}_t] = (1-\beta^t)a \quad \text{for} \quad t = 1,2,\ldots \quad .$$

That is to say, the exponentially smoothed estimator $\hat{a}_t$ underestimates a for all finite t, although it is unbiased in the limit. The amount of bias is greatest for small t and in particular

$$E[\hat{a}_1] = \alpha a \quad .$$

When α is chosen between .1 and .3, as is often suggested, the initial error can be substantial.

Based on the first observation, our forecast of demand during the next $T+1$ periods is

$$A^*_{t+T+1} = (T+1)\alpha X_1 \quad ,$$

which, on the average, will underestimate future demand. The resulting order would then be inadequate to satisfy demands occuring during the lead time and an unfavorable inventory position will occur.

Figure 2.1 shows I_t for a simulated facility. Demand at the facility in period t was generated according to a Poission distribution with $m(t) \equiv 16$. Lead time T was assumed to be 4 periods. Using procedures described in Zehna [1],

$$A^*_{t+1+T} = (T+1)\alpha \sum_{k=0}^{t} \beta^k X_{t-k} \qquad (t = 0,1,2,\ldots)$$

and

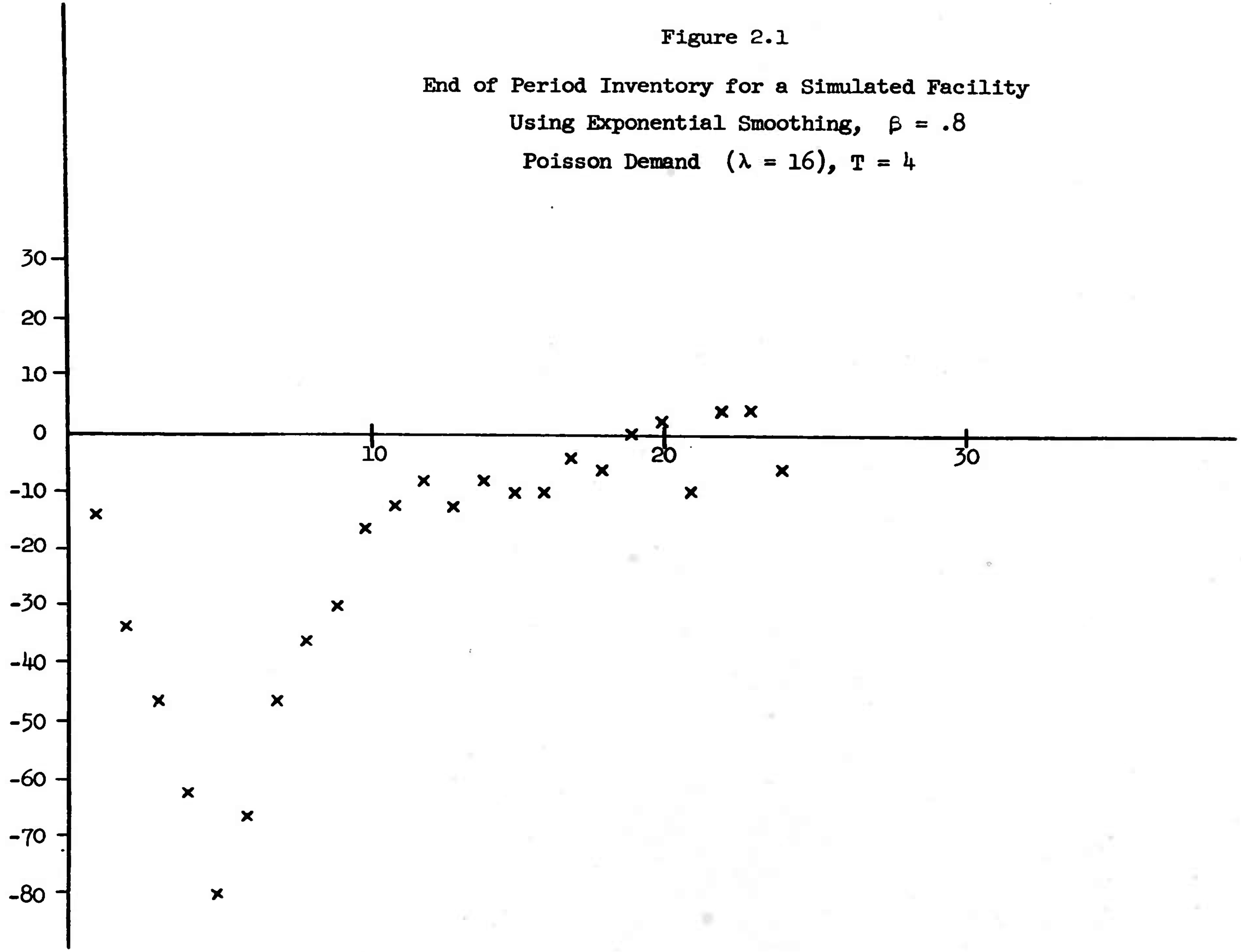
$$\theta_t = A^*_{t+1+T} - \sum_{j=1}^{T} \theta_{t-j} - I_t \qquad (t = 0,1,2,\ldots) \quad .$$

Figure 2.1 shows that, after starting with $I_o = 0$, I_t falls sharply until period 6 when the first order is received. Unfortunately, the biased character of the forcast resulted in this order being inadequate.

Subsequent orders are similarly inadequate and recovery from an unfavorable position (-80 units) is seen to be slower than would be desirable.

Figure 2.1

End of Period Inventory for a Simulated Facility
Using Exponential Smoothing, $\beta = .8$
Poisson Demand $(\lambda = 16)$, $T = 4$



2.2 Finite Exponential Smoothing*

In order to remove the bias inherent in exponential smoothing, a modification is proposed.

Let $X_1, X_2, \ldots, X_t$ denote a sequence of observations from a process having mean value function $m(t) \equiv a$. Define the finite exponentially smoothed estimate of a by

$$\tilde{a}_t = \alpha_t \sum_{k=0}^{t} \beta^k X_{t-k} \tag{2.2}$$

where

$$X_0 \equiv 0, \quad \alpha_t = \frac{\alpha}{1-\beta^t}, \quad 0 \leq \alpha \leq 1, \quad \text{and} \quad \beta = 1 - \alpha \ .$$

It follows that $\tilde{a}_t$ has the desirable property that for all t

$$E[\tilde{a}_t] = a \ .$$

That is, $\tilde{a}_t$ is unbiased and, a fortiori, asymptotically unbiased.

Analagous to expression (2.1) is the recursion relation

$$\tilde{a}_t = \alpha_t \, [X_t + (\beta/\alpha_{t-1})\tilde{a}_{t-1}] \quad (t = 1,2,\ldots) \tag{2.3}$$

which follows from the expression (2.2). Since $1 + \beta/\alpha_{t-1} = 1/\alpha_t$, equation (2.3) shows that $\tilde{a}_t$, like $\hat{a}_t$, is a weighted average of the present demand observation and the past estimate of a. As with $\hat{a}_t$, $\tilde{a}_t$ is simple to compute and does not require that the demand history

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* It will be convenient to abbreviate "exponential smoothing" by e.s. and "finite exponential smoothing" by f.e.s.

$X_1, X_2, \ldots, X_t$ remain in computer memory if computations are computerized.

From the definitions of e.s. and f.e.s. it follows immediately that

$$\lim_{t \to \infty} [\hat{a}_t] = \lim_{t \to \infty} E[\tilde{a}_t] = a \tag{2.4}$$

2.3 An Order Rule for Finite Exponential Smoothing

One of the significant consequences of the automatic control feature of the model presently being examined is the fact that once a forecaster is given, the order rule is automatically determined and so then is the inventory level. This is not to say, however, that an explicit formula for θ_t or I_t is easy to obtain.

We have just seen that if finite smoothing is used for estimating the constant mean, a, then the estimator $\tilde{a}_t$ is unbiased for all t. However it should be noted that in the form $\tilde{a}_t = \alpha_t \sum_{k=0}^{t} \beta^k X_{t-k}$, $\tilde{a}_t$ is expressed as the ordinary product of the t^{th} term of the two sequences $\{\alpha_t\}$ and $\{\sum_{k=0}^{t} \beta^k X_{t-k}\}$. In this form, an explicit expression for the z-transform, $\tilde{a}(z)$, is not easy to obtain. In fact, the z-transform of $\alpha_t = \frac{\alpha}{1-\beta^t}$ is itself not a recognizable one in closed form. Also, while it is still quite natural to forecast by means of the formula

$$\tilde{X}^*_{t+1+T} = (T+1)\,\tilde{a}_t \quad , \tag{2.5}$$

there is no convenient or even recognizable form for the z-transform, $A^*(z)$. Looking over the development of Section 1.2, it is apparent that, without such an explicit formula for $A^*(z)$, neither $A(z)$ nor $S(z)$ can be expected in closed form by the same technique. In turn, this prevents an explicit formula for the inventory level I_t from being derived by means of inverting transforms.

It is however possible to obtain an order rule analagous to (1.13) directly from the recursive relations.

Letting

$$\varphi_t = \tilde{X}^*_{t+1+T} + \sum_{k=1}^{t} X_k \qquad t = 1,2,\ldots \quad , \tag{2.6}$$

so that $\tilde{\theta}_1 = \varphi_1$, it can be shown by induction that

$$\tilde{\theta}_t = \varphi_t - \varphi_{t-1} \quad \text{for} \quad t \geq 2 \quad . \tag{2.7}$$

Upon substituting expressions (2.3), (2.5) and (2.6) into (2.7) we obtain, for $t \geq 1$,

$$\tilde{\theta}_t = \alpha_t(T+1)(X_t-\tilde{a}_{t-1}) + X_t \quad , \tag{2.8}$$

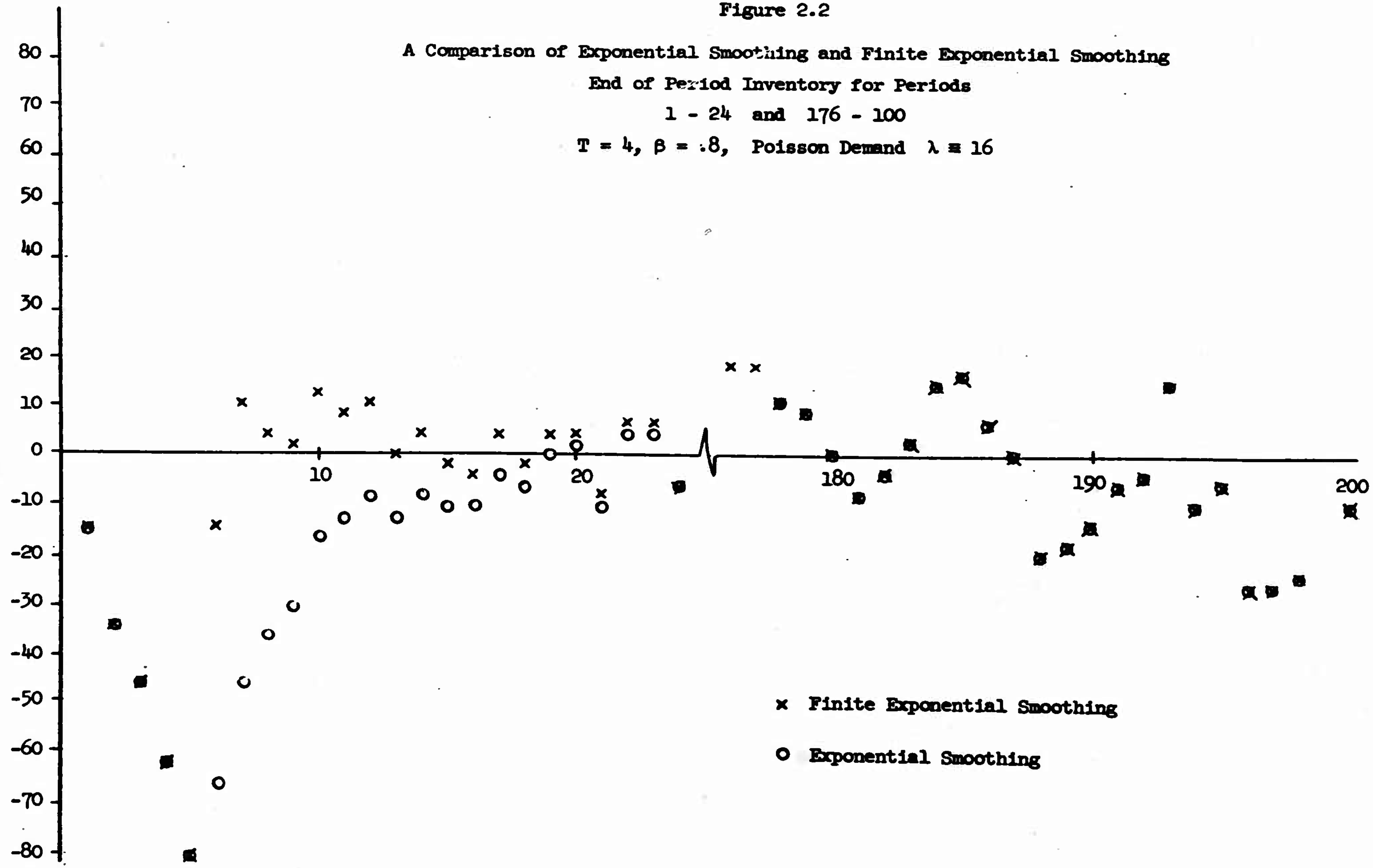
the analogue of expression (1.13). Since $\tilde{a}_t$ and $\hat{a}_t$ are asymptotically equivalent, it follows that $\tilde{\theta}_t$ and θ_t are similarly related. Hence, for large values of t, the order quantities, based on equivalent forecasting techniques, will be the same as with e.s., and consequently, the inventory levels will also be the same.

2.4 Use of Finite Exponential Smoothing

In Figure 2.2 the performance of the simulated facility when using e.s. is compared with that using f.e.s. The simulation operated identically except for the different smoothing procedures. The random

Figure 2.2

A Comparison of Exponential Smoothing and Finite Exponential Smoothing
End of Period Inventory for Periods
1 - 24 and 176 - 100
T = 4, β = .8, Poisson Demand λ = 16



sequence of demands was the same for both problems in order to facilitate a comparison of results.

If we denote by $\hat{I}_t$ the inventory at the end of period t when e.s. is employed as a forecast proceudre, and by $\tilde{I}_t$ the corresponding variable when f.e.s. is employed, we note that

$$\hat{I}_t = \tilde{I}_t \qquad (t = 1,2,\ldots,5)$$

At the end of period 6, however, our inventory position is substantially better using f.e.s. This improvement is a result of the unbiased character of the f.e.s. forecast. Periods 1 through 24 vividly show the rapid recovery obtained with f.e.s.

There are suggestions in the literature that when a change in the process generating the data is suspected one should disregard data which originated prior to the change, since it is no longer pertinent to forecasting the future. If this is done one must employ f.e.s. to avoid the bias that would result from starting anew and using e.s. as the forecast procedure.

Figure 2.2 also shows that for periods 176 through 200,

$$\hat{I}_t = \tilde{I}_t \quad .$$

This result is expected for t sufficiently large in view of expression (2.4).

The inventory at the end of period t is

$$\tilde{I}_t = \sum_{k=1}^{t-T-1} \tilde{\theta}_k - \sum_{k=0}^{t} X_k \tag{2.9}$$

Substituting expression (2.7) into (2.9) and simplifying the summation we note that

$$\tilde{I}_t = \varphi_{t-T-1} - \sum_{k=0}^{t} X_k$$

Substitution for φ and simplification yields

$$(2.10) \qquad \tilde{I}_t = (T+1)\alpha_{t-T-1} \sum_{k=0}^{t-T-1} \beta^k X_{t-T-1-k} - \sum_{k=0}^{T} X_{t-k} \; .$$

Equation (2.10) is the f.e.s. analogue of equation (1.12).

Following the procedure described earlier for $\hat{I}_t$, we can obtain $\tilde{V}_t$ the variance of $\tilde{I}_t$. Thus,

$$\tilde{I}_t = \sum_{j=1}^{t} \gamma_j X_j$$

where

$$\gamma_j = \begin{cases} (T+1)\alpha_{t-T-1} \, \beta^{t-T-1-j} & 1 \leq j \leq t - T - 1 \\ -1 & t - T \leq j \leq t \end{cases}$$

Hence

$$\tilde{V}_t^2 = [(T+1) + (T+1)^2 \alpha_{t-T-1}^2 \sum_{j=1}^{t-T-1} \beta^{2(t-T-1-j)}]\sigma^2$$

which upon simplification yields

$$(2.11) \qquad \tilde{V}_t^2 = \sigma^2 (T+1)[1 + (T+1)(\frac{1-\beta}{(1-\beta^{t-T-1})^2})(\frac{1-\beta^{2(t-T-1)}}{1+\beta})] \; .$$

For $K \geq 1$, the function

$$f(x) = \left(\frac{1-x}{1-x^K}\right)\left(\frac{1+x^K}{1+x}\right)$$

defined for $0 \leq x < 1$ has derivative

$$f'(x) = \frac{-2 + 2Kx^{K-1} - 2Kx^{K+1} + 2x^{2K}}{(1-x^K)^2(1+x)^2} .$$

To show that $f'(x) \leq 0$ it is sufficient to show

$$1-x^{2K} \geq Kx^{K-1}(1-x^2)$$

or, equivalently,

$$1 + x + \cdots + x^{2K-1} \geq Kx^{K-1}(1+x) .$$

Noting that x^K is a convex function of K, it follows that, for $1 \leq r \leq K$,

$$x^K - x^{r-1} \leq x^{K+(K-r)} - x^{r-1+(K-r)} \quad \text{or}$$

$$x^{r-1} + x^{2K-r} \geq x^{K-1} + x^K \;\dagger .$$

Summing over r,

$$\sum_{r=1}^{2K} x^{r-1} \geq Kx^{K-1}(1+x) \quad \text{as required} .$$

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† This follows from the fact that if f is convex, $y_1 \leq y_2$ and $h \geq 0$, $f(y_2) - f(y_1) \leq f(y_2+h) - f(y_1+h)$.

Thus, as would be expected, $\tilde{V}_t^2$ is monotonically decreasing in β. For $\beta = 0$, $\tilde{V}_t^2 = \sigma^2(T+1)(T+2)$ while, for $\beta = 1$, L'Hospitals rule is employed to yield $\tilde{V}_t^2 = \sigma^2(T+1)[1 + (T+1)(\frac{1}{t-T-1})]$. Hence for all $t > T$, $\tilde{V}_t^2$ has finite bounds

$$(2.12) \qquad \sigma^2(T+1)[1 + (T+1)(\frac{1}{t-T-1})] \leq \tilde{V}_t^2 \leq \sigma^2(T+1)(T+2) \quad .$$

Allowing t to approach infinity in expression (2.11) yields the limiting variance

$$(2.13) \qquad \tilde{V} = \sigma^2(T+1)[1 + \frac{\alpha(T+1)}{1+\beta}] \quad .$$

Denoting by $\hat{V}_t^2$ the variance of $\hat{I}_t$ and recalling that

$$\alpha_t = \frac{\alpha}{1-\beta^t} \quad ,$$

it follows that for $t > T$

$$(2.14) \qquad \tilde{V}_t^2 > \hat{V}_t^2 \quad .$$

Letting t increase without bound,

$$\lim_{t \to \infty} \tilde{V}_t^2 = \lim_{t \to \infty} \hat{V}_t^2$$

Expression (2.14) would seem to be in conflict with the simulation results, which indicated that the fluctuations about zero were smaller for f.e.s. than e.s. The findings are however compatible since the deviation about zero for $\hat{I}_t$ includes the effect of a bias term. That is, if $\hat{E}_t = E[\hat{I}_t]$,

$$E[\hat{I}_t^{\,2}] = E[\hat{I}_t - \hat{E}_t + \hat{E}_t]^2$$

$$= \hat{V}_t^{\,2} + \hat{E}_t^{\,2}$$

while

$$E[\tilde{I}_t^2] = \tilde{V}_t^2 \quad .$$

3. Multiechelon Supply System

3.1 A Servomechanism Model of the System

The system just described may be generalized to a multiechelon system by casting it into the framework of a multiple input-output servomechanism. As an example, consider a system composed of two tenders supplied from a common depot. Here there are two inputs, namely, the demands $X_t^{(1)}$ and $X_t^{(2)}$ from the two tenders and there are three outputs of interest; these are the respective inventory levels $I_t^{(1)}$ and $I_t^{(2)}$ at the tenders along with the inventory level $I_t^{(3)}$ at the depot. Of course, the demand at the depot level is the sum of the two order rules $\theta_t^{(1)}$ and $\theta_t^{(2)}$ initiated at the two tenders.

Considering each of the tenders to be operating according to the model outlined in the previous section each of the tender inventory levels may be expressed by means of z-transforms in terms of separate transfer functions. Thus,

$$I^{(1)}(z) = S^{(1)}(z)X^{(1)}(z) \text{ and} \tag{3.1}$$

$$I^{(2)}(z) = S^{(2)}(z)X^{(2)}(z) \quad .$$

Now thinking of the depot as constituting still another version of the same model with an input of $\theta_t^{(1)} + \theta_t^{(2)}$, there is a transfer function $S^{(3)}(z)$ such that

$$I^{(3)}(z) = S^{(3)}(z)(\theta^{(1)}(z) + \theta^{(2)}(z)) \quad . \tag{3.2}$$

Because of the relationship

$$\theta^{(i)}(z) = A^{(i)}(z)X^{(i)}(z) + B^{(i)}(z)I^{(i)}(z) \qquad i = 1,2$$

it is possible to express $I^{(3)}(z)$ further in terms of the demands at the tenders. In fact, since

$$I^{(i)}(z) = S^{(i)}(z)X^{(i)}(z) \quad ,$$

it follows that for $i = 1,2$

$$\begin{aligned}\theta^{(i)}(z) &= A^{(i)}(z)X^{(i)}(z) + B^{(i)}(z)S^{(i)}(z)X^{(i)}(z) \\ &= [A^{(i)}(z) + B^{(i)}(z)S^{(i)}(z)]\ X^{(i)}(z) \quad .\end{aligned}$$

This result allows us to express all of the input-output relationships simultaneously in matrix form. Following the notation of Howard [4], we obtain,

$$(3.3) \qquad \begin{bmatrix} I^{(1)}(z) \\ I^{(2)}(z) \\ I^{(3)}(z) \end{bmatrix} = \begin{bmatrix} h_{11}(z) & 0 \\ 0 & h_{22}(z) \\ h_{31}(z) & h_{32}(z) \end{bmatrix} \begin{bmatrix} X^{(1)}(z) \\ X^{(2)}(z) \end{bmatrix}$$

where

$$h_{11}(z) = S^{(1)}(z) \quad ,$$

$$h_{22}(z) = S^{(2)}(z) \quad ,$$

$$h_{31}(z) = S^{(3)}(z)[A^{(1)}(z) + B^{(1)}(z)S^{(1)}(z)] \quad \text{and}$$

$$h_{32}(z) = S^{(3)}(z)[A^{(2)}(z) + B^{(2)}(z)S^{(2)}(z)] \quad .$$

For visual aid, a flow chart of this multiechelon system is given in Figure 3.1. We have assumed that T_i is a delivery lag time $i = 1,2,3$, and in general T_1, T_2 and T_3 are distinct. Observe that the order rule at the depot level $\theta^{(3)}(z)$, necessary to maintain its own inventory level, is a linear combination of its input (the sum of the tender orders) and its invenotry level. In fact,

$$\theta^{(3)}(z) = A^{(3)}(z)[\theta^{(1)}(z) + \theta^{(2)}(z)] + B^{(3)}(z)I^{(3)}(z)$$

as in the single echelong model.

3.2 Reduction to Single Echelon Model

The final remark made in the previous section suggests the possibility of analyzing the inventory at the depot level as simply another single echelon model. This can in fact be accomplished but not without a little difficulty. Assume that the demands at the tenders have constant means so that there exist constants a_1 and a_2 such that $a_i \equiv m_i(t) = E[X_t^{(i)}]$ for $i = 1,2$, whence $m^{(1)}(z) = \frac{a_1}{1-z}$ and $m^{(2)}(z) = \frac{a_2}{1-z}$ are the respective transforms of the mean value functions. Assume further that exponential smoothing is used to forecast demands at each of the facilities. Without loss of generality we suppose that each facility is required to use the same smoothing constant α.

Now the results of Section 1 apply to each of the tender levels, and we obtain as in that section,

$$\theta_t^{(i)} = \alpha(T_i+1)[X_t^{(i)}-\hat{a}_{t-1}^{(i)}] + X_t^{(i)} \qquad i = 1,2 \tag{3.4}$$

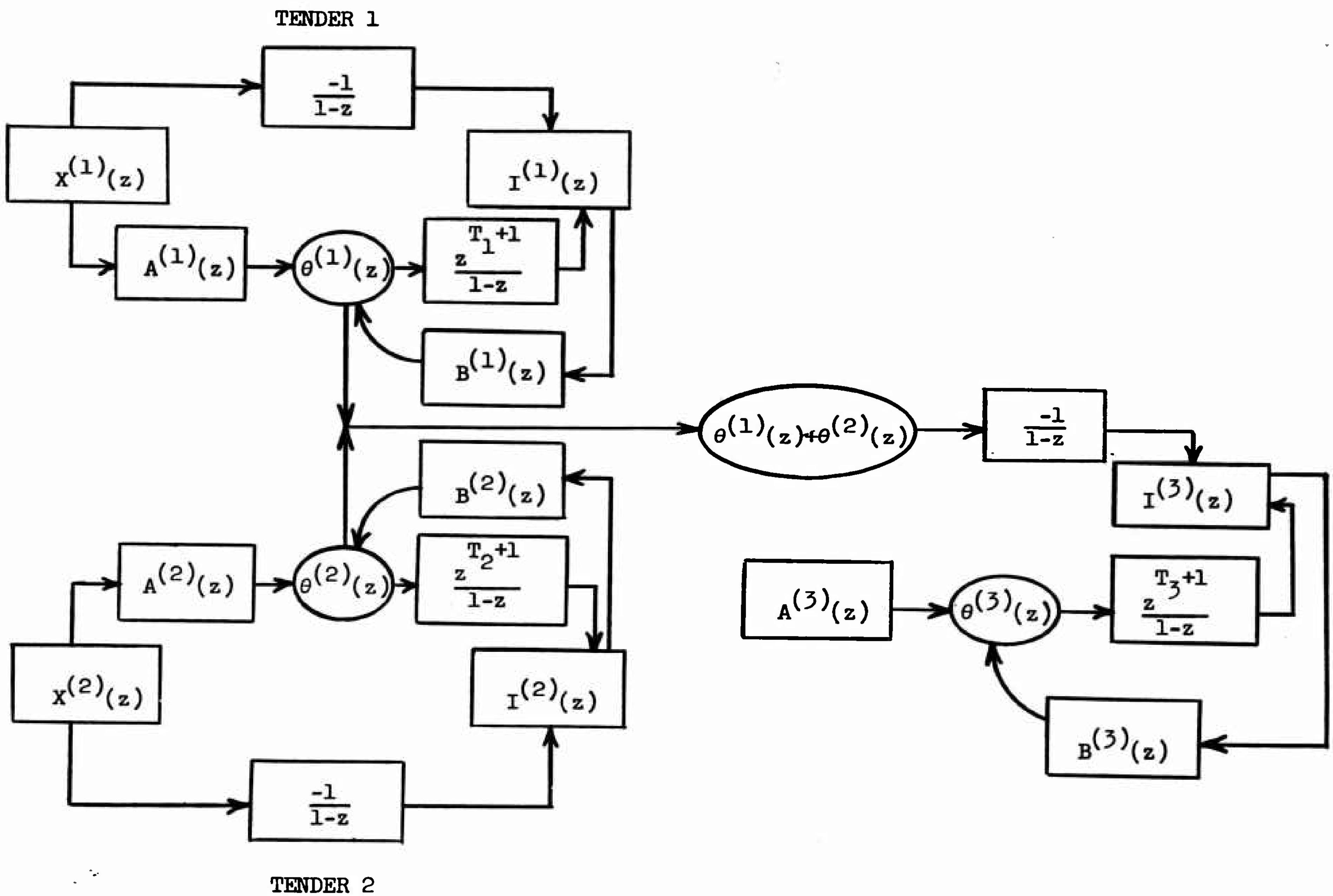


Figure 3.1 Multiechelon Inventory System

where

$$\hat{a}_t^{(i)} = \alpha \sum_{k=0}^{t} \beta^k X_{t-k}^{(i)}$$

is the smoothed estimate of the constant mean a_i. From this it follows immediately that

$$E[\theta_t^{(i)}] = a_i + a_i\alpha(T_i+1)\beta^{t-1} \qquad i = 1,2. \tag{3.5}$$

Consequently, $E[\theta_t^{(1)} + \theta_t^{(2)}] = (a_1+a_2) + \alpha\beta^{t-1}[a_1(T_1+1) + a_2(T_2+1)]$, a quantity which depends upon t. This is unfortunate since $\theta_t^{(1)} + \theta_t^{(2)}$ is the input at the depot level. Thus, even though the simple model of constant mean demand is imposed at the tender levels, it does not follow that the same model applies at the depot level. Because of the nature of the input at the depot, namely, the expected value being an exponential function of time, the analysis in terms of z-transforms is not particularly suitable.

On the other hand, if finite exponential smoothing were used at the tender level then the expected orders would be constant and the simple model would then apply. Accordingly we require that the i^{th} tender place orders according to the rule,

$$\tilde{\theta}_t^{(i)} = \alpha_t(T_i+1)[X_t^{(i)}-\tilde{a}_{t-1}^{(i)}] + X_t^{(i)}$$

where $\tilde{a}_{t-1}^{(i)}$ is the finite exponentially smoothed estimate of mean demand a_i given by $\tilde{a}_{t-1}^{(i)} = \alpha_{t-1} \sum_{k=0}^{t-1} \beta^k X_{t-1-k}^{(i)}$. As before, $\alpha_t = \dfrac{\alpha}{1-\beta^t}$. In that case, we have

$$E[\vartheta_t^{(i)}] = \alpha_t(T_i+1)[a_i-a_i] + a_i = a_i \ , \qquad i = 1,2$$

so that

$$E[\vartheta_t^{(1)} + \vartheta_t^{(2)}] = a_1 + a_2 \ ,$$

a constant independent of time.

Letting $X_t^{(3)} = \vartheta_t^{(1)} + \vartheta_t^{(2)}$ denote the input to the depot, the latter may now operate as a single echelon model with constant mean demand $a_1 + a_2$. The depot order rule $\theta_t^{(3)}$ will then be determined once the sequence $A_t^{(3)}$ or, alternatively, the depot demand forecast $A_{t+1+T_3}^{(3)*}$ is determined. In this regard, the depot now has a choice between finite and ordinary exponential smoothing as a forecasting technique and it should be observed that in either case the resulting order rule $\theta_t^{(3)}$ is a function of second order exponential smoothing applied to the tender demands $X_t^{(1)}$ and $X_t^{(2)}$. This is because the depot input $X_t^{(3)}$ is already a weighted sum of tender demands.

The investigation of the variance of the depot inventory level is not so simple. For one thing the single echelon model does not quite fit in this case for, even though the expected demand is constant, the variance is not. In fact, if f.e.s. is used at the depot level then, as previously derived,

$$\vartheta_t^{(i)} = \alpha_t(T_i+1)[X_t^{(i)}-\tilde{a}_{t-1}^{(i)}] + X_t^{(i)} \qquad \text{for} \quad i = 1,2$$

or,

$$\vartheta_t^{(i)} = [\alpha_t(T_i+1)+1]X_t^{(i)} - \alpha_t(T_i+1)\alpha_{t-1}\sum_{k=0}^{t-1}\beta^k X_{t-1-k}$$

in which case,

$$V[\vartheta_t^{(i)}] = [\alpha_t(T_i+1)+1]^2\sigma^2 + \alpha_t^2\alpha_{t-1}^2(T_i+1)^2 \sum_{k=0}^{t-1} \beta^{2k}\sigma^2$$

$$= \sigma^2[\alpha_t^2(T_i+1)^2 + 2\alpha_t(T_i+1) + 1 + \alpha_t^2\alpha_{t-1}^2(T_i+1)^2 \frac{(1-\beta^{2t})}{1-\beta^2}] \text{ for } i = 1,2.$$

Now it is quite reasonable to suppose that the order rules $\vartheta_t^{(1)}$ and $\vartheta_t^{(2)}$ from the two tenders are independent. In that case,

$$V[X_t^{(3)}] = V[\vartheta_t^{(1)}] + V[\vartheta_t^{(2)}] = \sigma^2[\alpha_t^2(T_1+1)^2 + 2\alpha_t(T_1+1) + \alpha_t^2(T_2+1) + 2\alpha_t(T_2+1)$$

$$+ 2 + \frac{(1-\beta^{2t})}{(1-\beta^2)} \alpha_t^2\alpha_{t-1}^2 ((T_1+1)^2 + (T_2+1)^2)]$$

which is not a constant although $\lim_{t\to\infty} V[X_t^{(3)}]$ is constant. What is worse, since both $\vartheta_t^{(1)}$ and $\vartheta_t^{(2)}$ are linear combinations of their respective demand histories up to time t it is not evident that the members of the sequence $\{X_t^{(3)}\}$ are uncorrelated. Hence, the computation of the variance of the inventory level

$$I_t^{(3)} = \alpha(T_3+1) \sum_{k=0}^{t-T_3-1} \beta^k X_{t-T_3-1-k}^{(3)} - \sum_{k=0}^{T_3} X_{t-k}^{(3)}$$

(for $t > T_3$) is not at all straightforward. Thus, simulation techniques will have to be adopted to estimate the variance of the depot inventory level.

3.3 Generalization

The simple 3 facility system just described can be generalized to one composed of n facilities. Material is shipped between facilities according to a supply procedure specified by a set of integers $f_1, f_2, \ldots, f_n$ with $f_1 = 0$ and $1 \leq f_j < j$ for $j > 1$. The integer f_j is interpreted as the unique facility directly supplying facility j. Let $F_j = \{k \mid f_k = j\}$ $(j = 1,2,\ldots,n)$ be the set of facilities directly supplied by facility j.

It is convenient to assume that the basic time unit is the period and that all units of time can be expressed as an integral number of periods.

Each facility, j, employs a periodic review policy whereby at the end of R_j time periods it evaluates its stock position and if appropriate places an order with facility f_j. The order is delivered to facility j, T_j periods later.

Let $\tilde{X}_t^{(j)}$ denote the user demand at facility j in period t, and let

$$\tilde{X}_t = (\tilde{X}_t^{(1)}, \tilde{X}_t^{(2)}, \ldots, \tilde{X}_t^{(n)})$$

be the vector of those demands. If a particular facility has no user demand but only functions to supply other facilities then $\tilde{X}_t^m \equiv 0$. In general we assume that the user demand process can be expressed

$$\tilde{X}_t^{(j)} = m_j(t) + \epsilon_j(t), \qquad j = 1,2,\ldots,n$$

where

$m_j(t)$ is the deterministic mean value function and

$\epsilon_j(t)$ is a random variable with $E[\epsilon_j(t)] = 0.$

Let $J = \{j | F_j = \emptyset\}$. Then for $j \in J$ let

$$X_t^{(j)} = \tilde{X}_t^{(j)} ,$$

and for $j \notin J$.

$$(3.6) \qquad X_t^{(j)} = \tilde{X}_t^{(j)} + \sum_{m \in F_j} \sum_{u+K_m \in (t-R_j+1,t)} \theta_u^m \qquad j = 1,2,\ldots,n$$

where

θ_u^m represents an order from facility m made at the end of period u,

and

K_m represents the lag between when an order is made at facility m and the stock is removed at facility f_m.

Thus if $F_j \neq \emptyset$ the demand imposed on facility j is composed of the user demand and the demands which have arrived within the review period from the facilities which it supplies. In many supply systems expression (3.6) will simplify.

The previous example is a special case of this formulation in which

$$X_t^{(1)} \equiv \tilde{X}_t^{(2)}, \qquad X_t^{(2)} \equiv \tilde{X}_t^{(3)}$$

$$f_2 = f_3 = 1$$

$$F_1 = \{2,3\}, \quad F_2 = \emptyset, \quad F_3 = \emptyset \quad ,$$

$$J = \{2,3\}$$

$$R_1 = R_2 = R_3 = 1 \quad .$$

3.4 Depot Forecasting Information

The simulation described earlier was extended to the above three facility, two echelon supply system. It was assumed that $\tilde{X}_t^{(2)}$ and $\tilde{X}_t^{(3)}$ were Poisson distributed random variables with parameters $\lambda_2 = \lambda_3 = 1$, and $T_1 = T_2 = T_3 = 4$. In accordance with Section 3.2,

$$(3.7) \qquad \tilde{X}_{t+1}^{(1)} = \tilde{\theta}_t^{(2)} + \tilde{\theta}_t^{(3)} \qquad (t \geq 1) \quad .$$

Simulation was used to examine the relationship between system performance and the nature of the information transmitted from facilities 2 and 3 to facility 1.

Case I:

In this case we assume that the only information being received by facility 1 is the order information. In particular, we assume that a forecast is obtained from expression (2.3), where

$\tilde{X}_{t+1}^{(1)}$ is given by expression (3.7).

Case II:

As an alternative to this forecast procedure at facility 1, we examined the situation in which

$$\tilde{a}_1^{(1)} = \tilde{X}_t^{(2)} + \tilde{X}_t^{(3)} \quad ,$$

$$\tilde{a}_{t+1}^{(1)} = (\alpha_{t+1})[\tilde{X}_{t+1}^{(2)} + \tilde{X}_{t+1}^{(3)} + (\beta/\alpha_t)\tilde{a}_t^{(1)}] \qquad (t > 0)$$

In Case II, facility 1 makes its forecast from the demand information originating at facilities 2 and 3. We assume that this information is supplied to facility 1 without delay.

Demand information supplied without delay has two advantages. It provides a preview of the order, $\tilde{X}_t^{(j)}$ to be imposed on facility j during the next period and thereby effectively reduces T. It provides demand information whereas θ_t describes demand information "confounded" with inventory adjustments.

Because of the above advantages we might expect better performance in a system which supplies demand information to its supplier as it originates.

Two hundred periods of system operation were simulated. The performance of facility 1 as measured by the end of period inventory is shown in Figures 3.2 and 3.3 for periods 1 through 50 and 150 through 200 respectively.

Figures 3.2 and 3.3 clearly indicate reduced variation in Case II. Further investigation, both analytical and with simulation techniques is desirable to examine the nature and source of the variation.

Figure 3.2

End of Period Inventory for Facility 1 in a
3 Facility Multiechelon Supply System
Periods 1-50

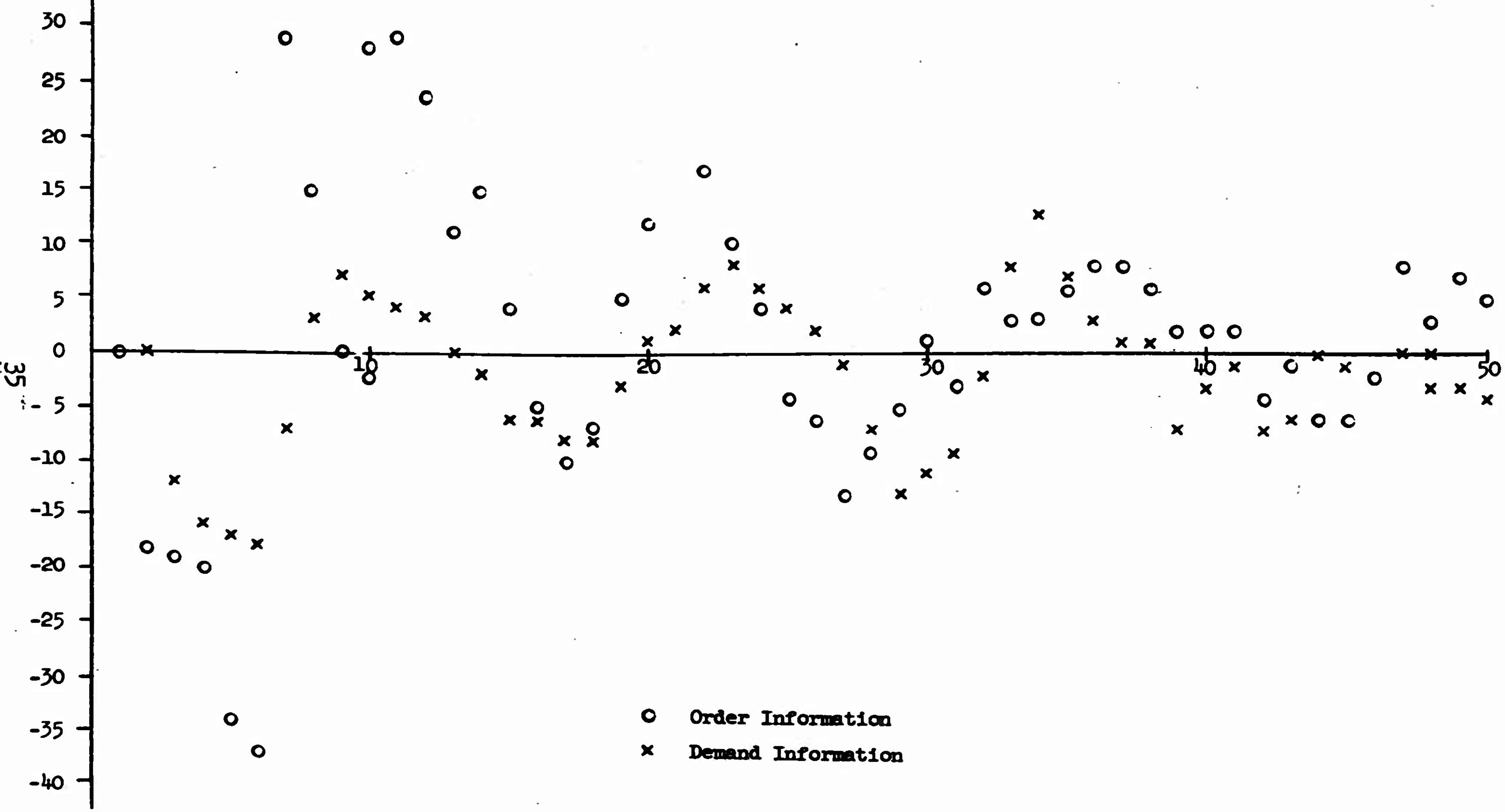
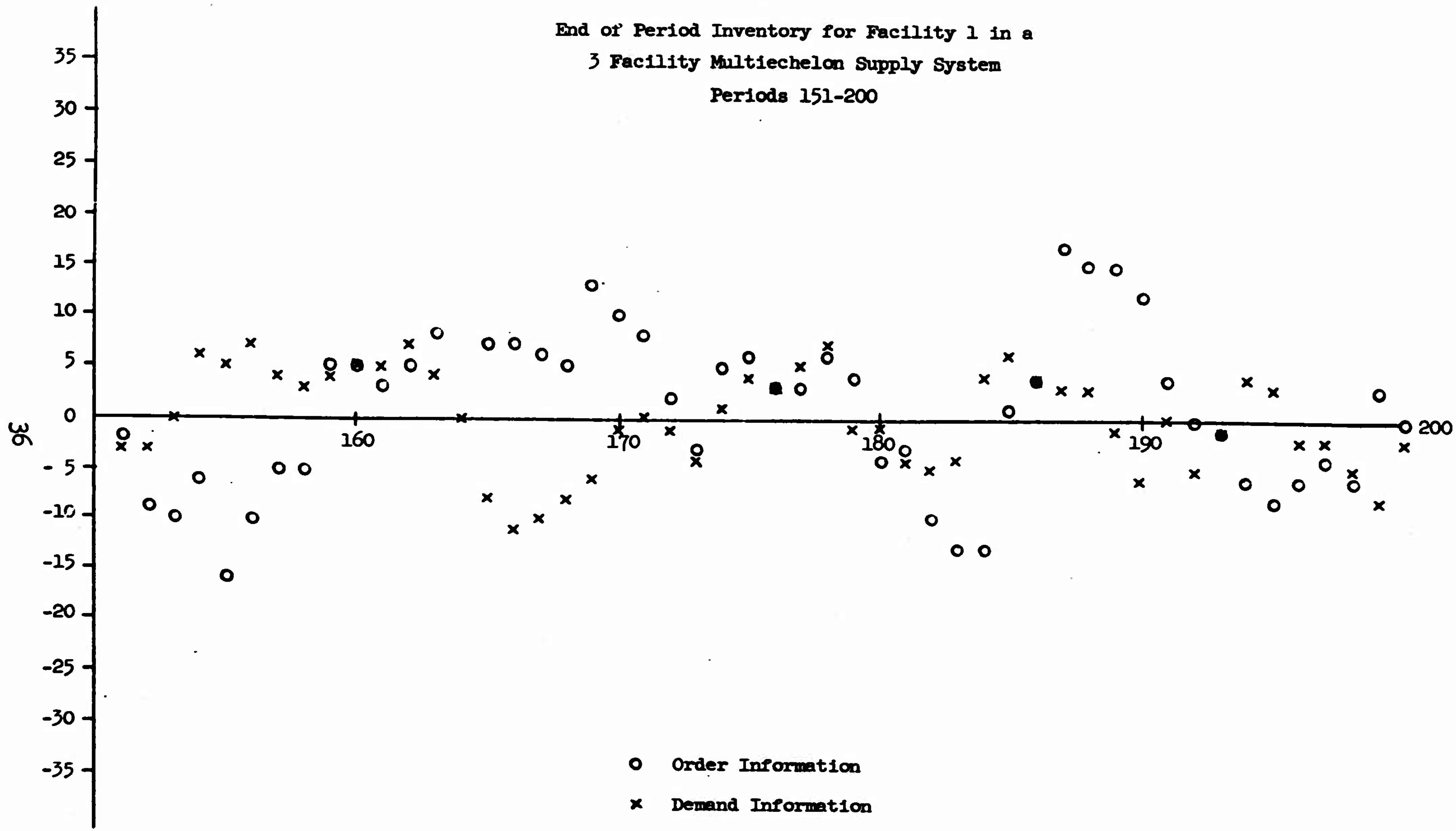


Figure 3.3

End of Period Inventory for Facility 1 in a
3 Facility Multiechelon Supply System
Periods 151-200



4. Conclusions

4.1 Servomechanism Techniques

Servomechanism techniques continue to be helpful, not only in finding explicit formulas for such quantities as order rules, but also for obtaining a much better understanding of the inventory system and the manner in which it operates. Analyzing the model treated in this paper from such a point of view has served well to discover certain weaknesses as well as strengths in the system. Certainly it has greatly emphasized the important role played by forecasting techniques. This type of analysis has also led to recognition of the trade-off in exponential smoothing between response to changes in demand patterns and the desire to minimize variation in inventory levels. Also, some progress has been made in the direction of structuring a multi-echelon system using vector-valued functions. The reduction of such a complex system to a more tractable one is another reason why this approach should be exploited in the study of inventory systems.

4.2 Finite Exponential Smoothing

The analytical results and the confirming simulation indicate the virtues of finite exponential smoothing as a forecast technique. $\tilde{a}_t$ was seen to be unbiased for all t. This meant a better inventory position during the first several periods following any initialization of the demand history.

$\tilde{a}_t$ can be computed recursively with two relations, thus requiring only little more effort or computer storage than $\hat{a}_t$.

5. Further Research

We know that by making β large $\sigma^2(\tilde{I}_t)$, $\sigma^2(\hat{I}_t)$ and consequently cyclic variation can be reduced. The penalty in so doing is however the loss of sensitivity. This occurs since the exponentially smoothed forecase $\hat{a}_t$ is a linear combination of X_t and $\hat{a}_{t-1}$ with $1-\beta$ as the weight given to the current observation. Therefore, if β is large little change occurs in $\hat{a}_t$ with X_t.

Consider a simple problem in which the mean function $m(t)$ of the demand process $\{X(t)\}$ has the following form

$$m(t) = a \qquad 1 < t < t_o$$

$$m(t) = a + c \qquad t \geq t_o$$

In a situation such as this it would be desirable to employ f.e.s. with β close to 1 throwing away all data occuring prior to the change.

Such a procedure would require a mechanism for detecting changes in the demand pattern. Such procedures have been suggested in the literature [5], [6]. The application, evaluation, and modification of such procedures to the forecasting problem represents an important area for further research.

REFERENCES

[1] Zehna, P. W., "Detecting Demand Changes," Technical Report, Office of Naval Research Contract 4457(00)(NR-347-001), Decision Studies Group, December, 1965.

[2] Vassian, H. J., "Application of Discrete Variable Servo Theory to Inventory Control," Operations Research, Vol. 3 (1965), pp. 272-282.

[3] DeWinter, R. E., "Inventory Applications of Servomechanism Models," Unpublished Thesis, U. S. Naval Postgraduate School, May 1966.

[4] Howard, R. A., "Systems Analysis of Linear Systems," Ch. 6, Multistage Inventory Models and Techniques, Scarf, Gilford and Shelly, eds., Stanford University Press, 1963.

[5] Chernoff, H. and S. Zacks, "Estimating the Current Mean of a Normal Distribution Which is Subjected to Changes in Time," Annals of Mathematical Statistics, Vol. 35 (1964), pp. 999-1018.

[6] Parent Jr., E. A., "Sequential Ranking Procedures," Technical Report No. 80, ONR Contract 225(53), Stanford University, April 1965.

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13. ABSTRACT

A periodic review inventory system is viewed as a servomechanism in which input is demand and output is inventory level. An order rule embodying a forecast procedure is determined. When demand is random with constant mean and finite exponential smoothing is used in the forecast, the variance of the inventory on hand at the end of each period is derived. An unbiased modification of exponential smoothing (finite exponential smoothing) is introduced, and the order rule employing this procedure determined. The variation in inventory at the end of the period is obtained and compared with that obtained with exponential smoothing. Bounds and asymptotic expressions for the inventory variances are derived, and the performance of the two order rules is examined in a computer simulation.

The servomechanism approach is employed to examine a multiechelon supply system. Order rules embodying forecast procedures are obtained for each facility. A system consisting of two facilities supplied by a depot facility is simulated and the effects of changing the information supplied to the depot is examined.

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