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A NOTE ON NON-LINEAR APPROXIMATION THEORY

by

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### ABSTRACT

Recently J. R. Rice [1] initiated a geometrical study of non-linear approximations. In what follows below we offer a small contribution to certain analytical aspects of mean-square non-linear approximations. One result is to exhibit a family of non-linear topological subspaces of the space  $C[a,b]$  with the mean-square metric which has a local unique best approximation property.

Recently J. R. Rice [1] initiated a geometrical study of non-linear approximations. In what follows below we offer a small contribution to certain analytical aspects of mean-square non-linear approximations. One result is to exhibit a family of non-linear topological subspaces of the space  $C[a,b]$  with the mean-square metric which has a local unique best approximation property.

Let  $S$  be a subset of a normed linear space  $E$ . Let  $\hat{S}$  be any open set containing  $H(S)$ , the convex hull of  $S$ . Let  $A$  denote a map from  $\hat{S}$  into a real inner product space  $F$ . Assume the range of  $A$  is not dense in  $F$ . Assume that  $A$  is twice Gateaux (G) differentiable on  $H(S)$ . Assume, moreover, that for each point  $x \in H(S)$  these G-derivatives are bounded linear operators, that is  $A'(x)$  is a bounded linear operator from  $E$  to  $F$  and  $A''(x)$  is a bounded linear operator from  $E$  to the space of bounded linear operators from  $E$  to  $F$ . Let  $p$  be a point in  $F$  which is not in the closure of the range of  $A$ . Set  $h(x) = \|A(x) - p\|^2$ ,  $B = \sup\{\|A''(x)\| : x \in H(S)\}$ ,  $\mu = \inf\{\|A'(x)k\|/\|k\| : x \in H(S) \text{ and } x+k \in S\}$ ,  $\gamma = \inf\{\|A(z+k) - A(z)\|/\|k\| : z \text{ and } z+k \in S\}$  and  $C = \sup\{\|A'(x)\| : x \in H(S)\}$ . The notations  $A'(x)k$  and  $(A''(x)k)(u)$  will be employed for first and second differentials, respectively. For simplicity the norms in  $E$  and  $F$  are denoted by the same symbol  $\|\cdot\|$ .

We show first that if  $z \in S$  is stationary point of  $h$ , that is  $h'(z)k = 0$  for  $z+k \in S$ , and  $h(z)$  is sufficiently small, then  $z$  is global minimizer for  $h$ . State otherwise,  $A(z)$  is a best approximation to  $p$  out of the range of  $A$ . Moreover,  $z$  is unique.

Lemma 1. If  $A$  is twice differentiable on  $H(S)$ , then so is  $h$ . Let  $\theta = z + \sigma k$  for any  $\sigma \in (0,1)$ , where  $z$  and  $z+k$  are in  $S$ . Then

$$\begin{aligned} \frac{1}{2}(h''(\theta)k)(k) &= [A(z)-p, (A''(\theta)k)(k)] + [A(\theta)-A(z), (A''(\theta)k)(k)] \\ &+ [A'(\theta)k, A'(\theta)k]. \end{aligned}$$

Proof. If  $f : E \rightarrow F$  is Frechet differentiable and  $g : F \rightarrow R$  is Gateaux differentiable, the composition  $gf$  satisfies the chain-rule  $h'(z)k = g'(f(z))f'(z)k$ . (See e.g. [2, p. 659]). If  $f(\theta) = A(\theta) - p$  and  $g(x) = \|x\|^2$ , then  $h'(\theta)k = 2[A(\theta)-p, A'(\theta)k]$ . By writing  $\frac{1}{2t} [h'(\theta+tk)k - h'(\theta)k] = \frac{1}{t} \{ [A(\theta+tk)-A(\theta)+A(\theta)-p, (A'(\theta+tk)-A'(\theta))k + A'(\theta)k] - [A(\theta)-p, A'(\theta)k] \}$ , expanding, and passing to the limit as  $t \rightarrow 0$ , we get:

$$\frac{1}{2}(h''(\theta)k)(k) = [A(\theta)-p, (A''(\theta)k)(k)] + [A'(\theta)k, A'(\theta)k].$$

Whence the lemma.

Theorem 1. Assume that  $C$  and  $B$  are finite and that  $\mu$  and  $\gamma$  are positive. Assume  $z$  is a stationary point for  $h$ . If  $h(z) < \mu^2/B(1 + \frac{2C}{\gamma})$ , then  $h(z) < h(x)$  for all  $x \neq z$  in  $S$ .

Proof. Suppose that  $z$  is not a unique best approximation. Then for some  $k \neq 0$  and  $z+k \in S$ ,  $\|A(z+k)-p\| \leq h(z)$ . Since  $\|A(z)-p\| = h(z)$  it follows that  $2h(z) \geq \|A(z+k)-A(z)\| \geq \|k\| \gamma$ ; whence  $\|k\| \leq 2h(z)/\gamma$ .

Since  $h'(z)k = 0$  for all  $k \in E$  we have by Taylor's Theorem that  $h(z+k) - h(z) = (h''(\theta)k)(k)/2$  for some  $\theta$  on the open line segment joining  $z$  and  $z+k$ . It would be a contradiction to show  $h(z+k)-h(z) > 0$ .

By the above lemma we have:

$$\frac{1}{2}(h''(\theta)k)(k) > -Bh(z)\|k\|^2 - CB\|k\|^3 + \mu^2\|k\|^2,$$

where the inequality  $\|A(\theta)-A(z)\| \leq C\|k\|$  has been employed [2, p. 659]. Since  $\|k\| \leq 2h(z)/\gamma$ , it follows that  $(h''(\theta)k)(k)$  is positive whenever  $h(z) < \mu^2/B(1 + \frac{2C}{\gamma})$ . Q.E.D.

Remark 1. The hypothesis that  $\gamma$  is positive may be replaced by the hypothesis that  $S$  is bounded by a sphere of radius  $R < \mu^2/2CB$ . Then if  $h(z) < \mu^2/2B$ , the above theorem holds. Observe also that if  $A$  is a linear operator,  $B = 0$  and  $z$  is a best approximation independently of the value of  $h(z)$ .

Remark 2. Assume the hypothesis of Theorem 1. Assume  $E$  is finite dimensional and  $A$  is defined on  $E$ . Assume that for some  $x_0 \in S$  the level set  $D = \{x \in E : h(x) \leq h(x_0)\}$  is a proper subset of  $S$ . Then  $p$  has a best approximation out of the range of  $A$ .

Proof. Take  $x$  in  $D$ . Then  $h(x_0) \geq \mu \|x - x_0\| - h(x_0)$ . Hence  $D$  is bounded. By the continuity of  $h$ ,  $D$  is closed. Clearly in seeking the minimum of  $h$  we may confine our attention to  $D$ . Thus  $h$  achieves a minimum on  $D$ . Q.E.D.

Remark 3. Observe that if we assume the hypotheses of Remark 2, and the additional hypotheses that  $x_0$  itself is not a minimizing point for  $h$ , then for every  $p \in F$  there exists stationary points  $x$  for  $h$ , that is points  $x \in S$  such that  $[A(x) - p, A'(x)k] = 0$  for all  $x + k \in S$ . To see this observe that because  $h$  is continuous,  $D$  has interior, and therefore a necessary condition that a point  $z$  minimize  $h$  is that  $h'(z) = 0$ .

Remark 4. If  $S$  is complete and  $\mu$  and  $\gamma$  are positive, then the ranges of  $A$ , and  $A'(x)$  are closed for all  $x \in H(S)$ .

Proof. Let  $\{A(x^k)\}$  be any Cauchy sequence in the range of  $A$ . Since  $\|A(x^k) - A(x^s)\| \geq \gamma \|x^k - x^s\|$ , the sequence  $\{x^k\}$  is also Cauchy with limit, say,  $y$ . But  $\|A(x^k) - A(y)\| \leq C \|x^k - y\|$ , showing that  $A$  is closed. The same proof works for  $A'(x)$ .

We now turn our attention to the "reverse problem" of approximation theory: Given a point  $y$  on the range of  $A$ , does there exist a point  $p \neq y$  whose best approximation is  $y$ ?

Theorem 2. Assume the hypotheses of Theorem 1. Assume that  $S = E$  and  $E$  is complete. Fix  $x \in S$ . There exists a point  $p \in F \sim \text{range } A$  such that  $A(x)$  is a unique best approximation to  $p$  if and only if  $A'(x)$  is not onto  $F$ .

Proof. We prove first that the not onto property for  $A'(x)$  is necessary. Otherwise,  $\|A(x)-p\| < \|A(y)-p\|$  for all  $y \neq x$  but the range of  $A'(x)$  is onto. Since  $[A(x)-p, A'(x)k] = 0$  for all  $k \in E$ , and we may choose  $\bar{k}$  such that  $A'(x)\bar{k} = A(x)-p \neq 0$ , we get  $[A(x)-p, A'(x)\bar{k}] > 0$ , a contradiction. Conversely, if  $A'(x)$  is not onto, then since the range of  $A'(x)$  is closed, it follows the deficiency of the range of  $A'(x)$  exceeds 0. Let  $O(x)$  denote the orthogonal complement of the range of  $A'(x)$ . Clearly if  $p \in O(x) + A(x)$ ,  $x$  is a stationary point. By Theorem 1, moreover, if  $\|A(x)-p\| < \mu^2/B(1 + \frac{2C}{\gamma})$ , then  $A(x)$  is a unique point on the range of  $A$  closest to  $p$ .

Example. Let  $C[a,b]$  denote the space of continuous function on  $[a,b]$ . In  $C[a,b]$  define an inner product  $\langle f, g \rangle = \int_a^b f(t)g(t)dt$ . In  $E_n$  denote by  $[x, y]$  the inner product of  $n$ -tuples  $x$  and  $y$ . Let  $\|\cdot\|_2$  and  $\|\cdot\|$ , respectively, be the norms in  $C[a,b]$  and  $E_n$  which arise from these inner products. Let  $v(t) = (v_1(t), \dots, v_n(t))$ ,  $v_i \in C[a,b]$ ,  $(i=1, 2, \dots, n)$ , be a Haar family on  $[a,b]$ . This means that for any subset of distinct points  $t_i$ ,  $(1 \leq i \leq n)$ , the matrix  $\{v_j(t_i) : 1 \leq i \leq n, 1 \leq j \leq n\}$  has rank  $n$ . Thus if multiple roots are counted only once, the generalized polynomial  $p(x, t) = [v(t), x]$  has at most  $n - 1$  roots. Let  $P = \|p(x, \cdot)\|$ , i.e.

$$\|[v(\cdot), x]\|_2 \leq P\|x\| \quad \text{for all } x \in E_n.$$

We now specify the mapping  $A$  discussed above as follows:  
 $A(x) = f([v(\cdot), x])$ , where  $f$  is the continuous real-valued function specified below. Thus  $A : E_n \rightarrow C[a,b]$ .



Specifically, let  $f$  be a differentiable, real-valued function defined everywhere on the reals,  $R$ . Assume that  $f'(t) \geq \alpha > 0$  for all  $t \in R$ , and that  $f''$  exists and is bounded above on  $R$ , say by  $N$ .

Lemma 2. The mapping  $A$  satisfies the hypotheses of the above Theorems 1 and 2.

Proof. Because  $x \neq 0$  implies  $[v(\cdot), x]$  has at most  $n - 1$  roots,  $\|[v(\cdot), x]\|_2 > 0$  for all  $x \neq 0$ . Thus  $\|[v(\cdot), x]\|_2 \geq \beta \|x\|$  for some  $\beta > 0$  and all  $x \in E_n$ . We shall denote the point  $A(x)$  in  $C[a, b]$  by  $A(x, \cdot)$  when the pointwise values  $A(x, t)$  are singled out for attention.

Since  $A'(x, t)k = f'([v(t), x])[v(t), k]$ , we get  $\|A'(x, \cdot)k\|_2 = \|f'([v(\cdot), x])[v(\cdot), k]\|_2 \geq \alpha \| [v(\cdot), k] \|_2 \geq \alpha \beta \|k\| = \mu \|k\|$ . Similarly using the ordinary mean-value theorem on the real-valued functions  $f([v(t), \cdot])$  defined on the ray  $\{z = x + \theta k : 0 < \theta < 1\}$  we get  $\|A(x+k, \cdot) - A(x, \cdot)\|_2 / \|k\| = \|f([v(\cdot), x+k]) - f([v(\cdot), x])\|_2 / \|k\| = \|f'([v(\cdot), x+\theta k])[v(\cdot), \frac{k}{\|k\|}]\|_2 \geq \alpha \| [v(\cdot), \frac{k}{\|k\|} ] \|_2 = \mu$ . Thus  $\gamma = \mu$  in the above Theorem 1.

By Remarks 2 and 3 there exists at least one point  $z \in S$  such that  $[A(z) - p, A'(z)k] = 0$  for all  $k \in E_n$ .

We calculate that  $\|(A''(x)k)(k)\| = \|f''([v(\cdot), x])[v(\cdot), k][v(\cdot), k]\|_2 \leq NP^2 \|k\|^2$ , whence  $NP^2 = B$ . Let  $M = \max\{f'(r) : r \in R_0\}$ . Then  $\|A'(x)\| \leq MP = C$ .

It remains only to check that  $A$  and  $A'(x)$  are not onto  $F$ . The former follows because  $f$  is strictly monotone and continuous. Thus the range of  $A$  is a homeomorphic image of the span of  $\{v_1, \dots, v_n\}$ , which is a finite dimensional subspace of  $C[a,b]$ . For the latter we observe that the range of  $A'(x)$  for each  $x$  is spanned by the functions  $f'[(v(\cdot), x)]v_j$ ,  $1 \leq j \leq n$ . Thus  $A'(x)$  is a finite dimensional linear subspace. Q.E.D.

Remark 5. We have constructed for each function  $f$  satisfying the above hypotheses a topological subspace of  $C[a,b]$  with the mean-square metric which is homeomorphic to a Haar subspace. This subspace has a local unique approximation property--each point sufficiently close to the subspace has a unique best approximation in the subspace. Moreover, every point on the subspace is the unique best approximation to some nearby point which is not in the subspace.

REFERENCES

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- [2] L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Linear Spaces. MacMillan, New York, 1964.