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NONSEPARABLE SOLUTIONS OF THE HELMHOLTZ WAVE EQUATION EXAMINED FOR APPLICATIONS

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July 1966

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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SUMMARY

A conventional solution of the Helmholtz or time-reduced wave equation is a simple product of functions that contain one coordinate variable in each. An unbounded set of solutions that are not separable into simple products of single-variable functions has been partially examined for applicability to vibrational problems. Applications to scalar usage have been found, and illustrations including shapes and frequencies for membranes and an acoustic cavity are reported. Efforts to make application to vector usage are described, as are numerous mathematical properties that have been discovered in the course of the work. It is concluded that vibration on or within some new shapes can now be calculated exactly with functions formed of the nonseparable solutions added to separable solutions. It is also concluded that simplifications in the mathematics and additional applications await the effort.

FOREWORD

This is a comprehensive report of an investigation which has led more deeply into mathematics than was originally planned. In order that the reader who has not the time to dwell upon the mathematical side may obtain a qualitative description of the work, the report has been organized in such a way as to give comprehensive coverage without formulas in the Summary, the Introduction, the Course of the Work Covered by Contract, the Conclusions and the Recommendations. Fundamental mathematics that is the essential groundwork for the positive findings is then given and it leads into the sections which give details of the applications. Other properties of the nonseparable solutions were discovered which do not lead directly to the reported applications, and these are detailed in Appendixes I through VII.

Principal personnel in this investigation have been Dr. D. S. Moseley as Project Leader and James M. Render. Mr. Render calculated the patterns, led in the vector effort and has contributed importantly to the present level of understanding and to the preparation and editing of this report. This report was typed by Mrs. G. J. Fowler.

The work period covered by contract was 28 June 1965 to 28 April 1966. The contract was DA 44-177-AMC-342(T) between the U. S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, and Vitro Laboratories, Silver Spring, Maryland. Title of the contract was "Application of Nonseparable Wave Equation Solutions".

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CONTENTS

ILLUSTRATIONS

SYMBOLS

a, b, d	= propagation constants associated with x, y, z, respectively
c	= velocity of phase propagation; as subscript, desired contour
f, g, h	= b + d, a + d, b - a, respectively
$\vec{i}, \vec{j}, \vec{k}$	= unit vectors along x, y, z, respectively
i	= $\sqrt{-1}$; as subscript, an index
j	= as subscript, index denoting variant
k	= ω/c; as subscript, index denoting variant
m	= integer
n	= integer denoting order; integer denoting interval in z; as subscript, normal component
p	= sound pressure; the sum x + y
q	= the difference x - y
r	= radial coordinate in r, θ, z coordinate system; as subscript, reference
r, s	= $ax \pm by$, $bx \mp ay$, respectively
t	= time
u, v, w	= $ax + \alpha$, $by + \beta$, $dz + \delta$, respectively
v	= sound particle velocity
x, y, z	= components in rectangular coordinate system
$\vec{A}$	= illustrative time-dependent vector
$\vec{B}$	= magnetic flux density
C	= numerical constant; a constant with z held constant
D	= partial derivative operator; arbitrary constant; as subscript, dimension
$\vec{E}$, E	= electric field vector, Young's modulus, respectively

$\vec{H}$, H	=	magnetic field vector, hyperbolic form of nonseparable, respectively
J_n	=	Bessel function of first kind and of order n
K	=	a constant with z held constant
L	=	side of square
M	=	ratio of resonant frequencies
O_{2D}, O_{3D}	=	generative operators in two, three dimensions, respectively
S	=	length of side of desired contour
W	=	sum of one or more variants of one or more nonseparables, $n \geq 0$
X, Y, Z	=	phase coordinates = ax, by, dz, respectively
Y	=	amplitude of flexural displacement on plate
Z	=	impedance
α, β, δ	=	phase constants added to X, Y, Z, respectively
γ	=	separation constant in reducing flexural wave equation
ϵ	=	infinitesimal translation along coordinate axis, permittivity
η	=	transverse instantaneous displacement on membrane
θ	=	circular coordinate in r, θ, z coordinate system
μ	=	permeability
ν	=	Poisson's ratio
π	=	3.14159...
ρ	=	mass density
ω	=	angular frequency = $2\pi f$
φ	=	three-dimensional time-dependent scalar function
$\vec{\nabla}$	=	del or nabla, the differential vector operator

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INTRODUCTION

VIBRATION AS WAVES

Vibration is oscillatory motion of matter in response to a transient or a periodic disturbance. Oftentimes the disturbance that is perceived by a particular portion of matter has originated at considerable distance and has been propagated to that point as a wave. In these cases, provided amplitudes are sufficiently small, the propagation through the connecting medium is described by the wave equation written for fluids and by the wave and flexural wave equations written for solids. These are partial differential equations, one or both of which permit prediction of dependent variables such as pressure, stress, stress moment, strain, displacement, particle velocity, and acceleration anywhere in the medium and at any moment of time.

For ease in handling these equations, one usually introduces the assumption that the dependent variable is a sinusoidal function of time. This changes wave and flexural wave equations into second and fourth order partial differential equations, respectively, in position coordinates only, with frequency appearing as an arbitrary parameter. The wave equation so reduced is called the Helmholtz wave equation.

SEPARABLE AND NONSEPARABLE SOLUTIONS

Functions which are solutions of the Helmholtz wave equation have been known for a century and more. They are derived by separation of variables, a method taught to engineering and science students in colleges everywhere. The method assumes that the solution is a function that is itself a simple product of functions each containing one independent variable. By this method the derivation is converted from one of solving a partial differential equation in two or three dimensions to one of solving two or three ordinary differential equations. In view of the nature of this solution and to make a distinction between it and those to follow, we have called it the separable solution of the Helmholtz wave equation.

The nonseparable solutions of the Helmholtz wave equation with which this report is concerned are mathematical functions which have appeared in the past 4 years. They arose during the course of an attempt to formulate in closed form the natural modes of an isotropic right circular solid cylinder. No useful role for them was found during that work, but they were later gathered up and submitted for publication (reference 4).

Preparation was then made for the effort that is being reported here.

PREPARATION FOR THIS CONTRACT

It was reasoned that new solutions of an equation that is used in mechanics, acoustics, radio, microwaves, and light would have utility. It was postulated that these solutions in mechanics, for example, might legitimatize experimentally observed modes which theoreticians dismiss as originating in poorly controlled homogeneity of medium and poorly known boundary conditions.

It was decided to change from circular cylindrical coordinates of the published paper to rectangular coordinates. The aptness of the two-dimensional nonseparables to membrane vibration was then apparent, since the motion being predicted was in a single direction, namely, normal to the plane of the membrane, and the dependent variables were the minimum number, namely, two, that were required for the existence of the nonseparable solution. Nodal patterns for two of the new functions were quickly generated, and certain features of these patterns were noted as being identical with certain features of Chladni figures republished by Waller (reference 7). Chladni figures are sand patterns formed on vibrating, free-edged plates.

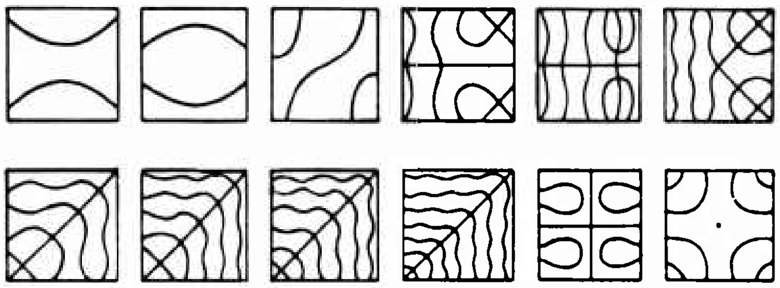

Figure 1. Selected Chladni Figures on Square Plates (reference 7).

A program was then outlined that would successively investigate applications in four models, namely, membranes, acoustic cavities, electromagnetic cavities, and elastic solids. These were so ordered because it was felt that this was their order of difficulty and because it was thought unwise to undertake only the first and last without the experience to be gained in executing the intermediate ones. First attention in each was to be given to finding if new modes for shapes already known would be given by the nonseparables. For each model there were guides proposed as to number of frequencies, number of regular shapes, highest order of solution to be considered, etc.

Between proposal and contract, a factor of considerable importance was discovered. There was found a generative operator* by which the simplest nonseparable could be obtained from the known separable, and this operator proved capable of generating ever higher orders of nonseparable through successive application.

*Credit for recognizing this operator belongs to Dr. B. R. Levy, Mathematics Branch, Office of Naval Research, Washington, D. C.

COURSE OF THE WORK COVERED BY CONTRACT

THE SEARCH FOR APPLICATIONS

Under Contract DA 44-177-AMC-342(T) entitled "Applications of Nonseparable Wave Equation Solutions", the U. S. Army Aviation Materiel Laboratories has been supporting a research effort that began as an effort to discover if applications exist. The initial model was a membrane.

It was soon apparent that nonseparable solutions for membranes were not producing new modes in old shapes, for calculated nodal lines on membranes bore no resemblance to regularly bounded patterns. Therefore, that goal was shelved in favor of achieving a nodal pattern of any shape that would demonstrate one or more closed areas. Nodal lines that do not close on an infinite membrane are interesting but not practical.

When the schedule called for it, the program moved from membranes to acoustical cavities, that is, from two to three dimensions. More patterns were evaluated. At length an x-y pattern in a z-plane did demonstrate closed areas, and study of the solution which had produced them revealed the secret: in effect the plot was of a function very similar to a membrane function compounded of a nonseparable added to a strong proportion of separable. A return to membranes was made, and closed areas were immediately demonstrated. Applied to accustical cavities, the principle of using a nonseparable to perturb a separable solution was successful in generating closed volumes.

In preparation for the third model, examination was made of nonseparable solutions as scalar potentials from which vectors could be derived. These were then studied in relation to boundary conditions on the cavity walls, and this included a return to acoustical cavities with normal particle velocity to be zero on the wall. The mathematical meaning of two boundary conditions, electric and magnetic, needing to be met on the same wall was investigated, and electric and magnetic potentials were constructed.

Attention then turned to elastic solids where two conditions are required on each surface or edge. Extensional and flexural vibration in thin plates was examined.

SUMMARY OF RESULTS

Application to vibrating membranes has been found. With the aid of the new solutions, frequency and amplitude distribution on shapes never before analyzed have been found exactly and without recourse to representation by Fourier series. It appears that the exact solution of any shape that is a continuous distortion of a square or rectangle is within grasp.

Application to an acoustic cavity with pressure-node walls has been demonstrated. Frequency and amplitude distribution of pressure within the volume can now be calculated. Wall shape in this demonstration is a smoothly distorted cube.

Application to an electromagnetic cavity of perfectly conducting walls was discovered during the preparation of this report and has not been well confirmed. It appears that nonseparables of first order can serve as electric and magnetic potentials from which are obtained electric and magnetic fields that predict fields within a hollow rectangular parallelepiped. Dimensions of the cavity must be proportional to any set of three integers.

There has been no success with the acoustic cavity under the condition of a rigid wall and only the foregoing success with the electromagnetic cavity. These models have in common the boundary conditions that tangential and/or normal components of vector field quantities are specified, and much effort has been expended in trying to use nonseparable solutions to represent suitable vector fields.

The fourth model is the elastic solid. Two modes of motion in a thin rectangular plate were examined in the last weeks of the contract. The first was extensional vibration, which is motion parallel to the surface of the plate. Nonseparables appear unable to satisfy the pair of stress conditions that apply at each edge. It may be noted that separables do not satisfy them either. The second was flexural vibration, or motion normal to the plane of the plate. This examination was cut short by schedule, but it appears that nonseparables in trigonometric and hyperbolic forms can be taken together to satisfy boundary conditions along a plate edge. The prediction is that the edge shape will in general be a distorted rectangle, just as the edge shape of the membrane has been found to be.

MATHEMATICAL FOUNDATIONS

THE HELMHOLTZ WAVE EQUATION

For amplitudes sufficiently small, the vector differential equation for wave motion is

$$\nabla^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad , \tag{1}$$

where the vector $\vec{A}$ is the dependent variable, $\vec{\nabla}$ is the vector differential operator, c is the phase propagation velocity, and t is time. The $\vec{\nabla}$ operator taken twice is the Laplacian, which in rectangular coordinates is

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$

The velocity c is that velocity which is appropriate to wave type and medium, examples being velocities of sound pressure in a gas, dilatational or shear stress and strain waves in a solid, and electromagnetic wave components in a vacuum.

The Helmholtz wave equation follows immediately from Equation (1) if $\vec{A}$ is assumed to be a vector function which varies sinusoidally in time. Thus, let

$$\vec{A} = \vec{A}_o e^{i\omega t} \tag{2}$$

in which $\vec{A}_o$ is the amplitude of vector $\vec{A}$, i is $\sqrt{-1}$, and ω is the angular frequency of the time variation. Then Equation (1) becomes

$$(\nabla^2 + k^2)\vec{A}_o = 0 \quad , \tag{3}$$

in which $k = \omega/c$.

Since the amplitude $\vec{A}_o$ is a vector, in component form it is

$$\vec{A}_o = \vec{i}A_x + \vec{j}A_y + \vec{k}A_z \quad ,$$

and upon substitution in Equation (3) there arise three independent scalar Helmholtz equations which are

$$(\nabla^2 + k^2)A_x = 0$$

$$(\nabla^2 + k^2)A_y = 0$$

$$(\nabla^2 + k^2)A_z = 0 \quad . \tag{4}$$

SEPARABLE SOLUTIONS

Separable solutions of scalar Helmholtz wave equation are those which consist of single, simple products of functions containing one variable each. For example, the separable solutions of the last of Equation (4) are, in three dimensions,

$$A_z = \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} \begin{Bmatrix} \sin by \\ \cos by \end{Bmatrix} \begin{Bmatrix} \sin dz \\ \cos dz \end{Bmatrix}, \tag{5}$$

where

$$k^2 = a^2 + b^2 + d^2. \tag{6}$$

Each brace contains two functions which may be summed in any proportion but which depend upon one coordinate variable only. The function A_z is the product of the three braces.

An alternative way of writing Equation (5) is with the use of arbitrary phase constants, as

$$A_z = \sin (ax + \alpha) \sin (by + \beta) \sin (dz + \delta). \tag{7}$$

NONSEPARABLE SOLUTIONS IN TWO DIMENSIONS

Let us consider the scalar Helmholtz wave equation

$$(\nabla^2 + k^2)W = 0, \tag{8}$$

in which W is a function of x and y and not of z.

This will be recognized as the wave equation for motion on a membrane when W is the amplitude of transverse displacement (or velocity or acceleration) everywhere on the membrane.

The separable solution of Equation (8) is

$$W^{(0)} = \sin (ax + \alpha) \sin (by + \beta), \tag{9}$$

for which the frequency equation is

$$k^2 = a^2 + b^2. \tag{10}$$

In addition to the separable solution, there is a set, infinite in number, of nonseparable solutions of Equation (8), for each of which the frequency equation is the same as Equation (10). Each nonseparable contains the coordinate variables as explicit multiplying factors, and since the highest exponent or power that is present on such factor is a unique characteristic of that solution, it has been used to declare the order of the solution.

Nonseparable solutions of the first three orders in two dimensions are

$$W^{(1)} = bx \cos (ax + \alpha) \sin (by + \beta) - ay \sin (ax + \alpha) \cos (by + \beta), \tag{11}$$

$$W^{(2)} = -\Big\{\left[(bx)^2 + (ay)^2\right] \sin (ax + \alpha) \sin (by + \beta) + by \sin (ax + \alpha) \cos (by + \beta) + ax \cos (ax + \alpha) \sin (by + \beta) + 2abxy \cos (ax + \alpha) \cos (by + \beta)\Big\}, \tag{12}$$

$$W^{(3)} = \left[3(bx)^2 + (ay)^2 + 1\right] ay \sin (ax + \alpha) \cos (by + \beta) + 3ab (x^2 - y^2) \sin (ax + \alpha) \sin (by + \beta) + 3xy (a^2 - b^2) \cos (ax + \alpha) \cos (by + \beta) - \left[3(ay)^2 + (bx)^2 + 1\right] bx \cos (ax + \alpha) \sin (by + \beta). \tag{13}$$

A nonseparable solution of order n + 1 can be obtained from that of order n by employing the generative operator O_{2D}. The relationship is

$$W^{(n+1)} = O_{2D} W^{(n)}, \qquad n \geq 0 \tag{14}$$

where

$$O_{2D} = b\frac{\partial}{\partial a} - a\frac{\partial}{\partial b}. \tag{15}$$

It is easily verified that the separable solution may be considered a non-separable solution of order zero and hence a member of the set. This is also clear in Appendix I and Appendix II, which present exponential and variant forms of two-dimensional solutions.

APPLICATION TO MEMBRANES

Motion at right angles to the plane of a membrane is the simplest two-dimensional application of the Helmholtz wave equation. This is true because (1) the motion can be treated as a scalar, and (2) a single condition of zero displacement at the boundary completes the problem. Separable solutions of Equation (8) in appropriate coordinates have long been known to satisfy this condition for simple shapes such as rectangles, circles, ellipses, etc., yielding a spectrum of resonant frequencies for each shape and the amplitude distribution for each frequency.

Out of respect for this background, nonseparable solutions ($W^{(n)}$ in rectangular coordinates with n = 1, 2, 3) were first studied in relation to membranes. This has resulted in three findings, of which two are negative and one is positive. The negative findings are (1) that nonseparable solutions have yielded no new modes for the rectangular shape that is already solved by the separable solution, and (2) that nonseparable solutions taken singly do not describe the vibration of a membrane of any finite shape. By the latter we are saying that nodal patterns of nonseparables of first and higher order contain no closed areas. A collection of nodal patterns for various nonseparables having a = b and taken singly is presented in Appendix III.

DISCOVERY OF CLOSED AREAS

The positive finding is that a nonseparable of second order can be added to a separable to yield frequency and amplitude within a closed boundary that is a smooth distortion of the boundary given by the separable alone. If η be defined as the oscillatory displacement of any point of the membrane, then we may write

$$\eta = \left[W^{(0)} + CW^{(2)}\right] \cos \omega t, \qquad (16)$$

where C is a constant and the bracket is the amplitude of the displacement. The amplitude is thus given for all values of x and y. Zero amplitude occurs at all points (x_i, y_i) which solve the two-dimensional transcendental equation

$$W^{(0)} + CW^{(2)} = 0. \qquad (17)$$

Plot of all points (x_i, y_i) in x-y space is the nodal pattern of the displacement η. As would be expected, each nodal line separates displacements that differ from each other by 180° in phase.

Closed areas have been found in three evaluations of Equation (17) using $\alpha = \beta = 0$, and a = b = 1, and $C = (12.5\pi^2)^{-1} = .0081$, $C = (6.5\pi^2)^{-1} = .0156$, and $C = (2.5\pi^2)^{-1} = .0405$. These patterns together with the one for C = 0 are presented as Figure 2.

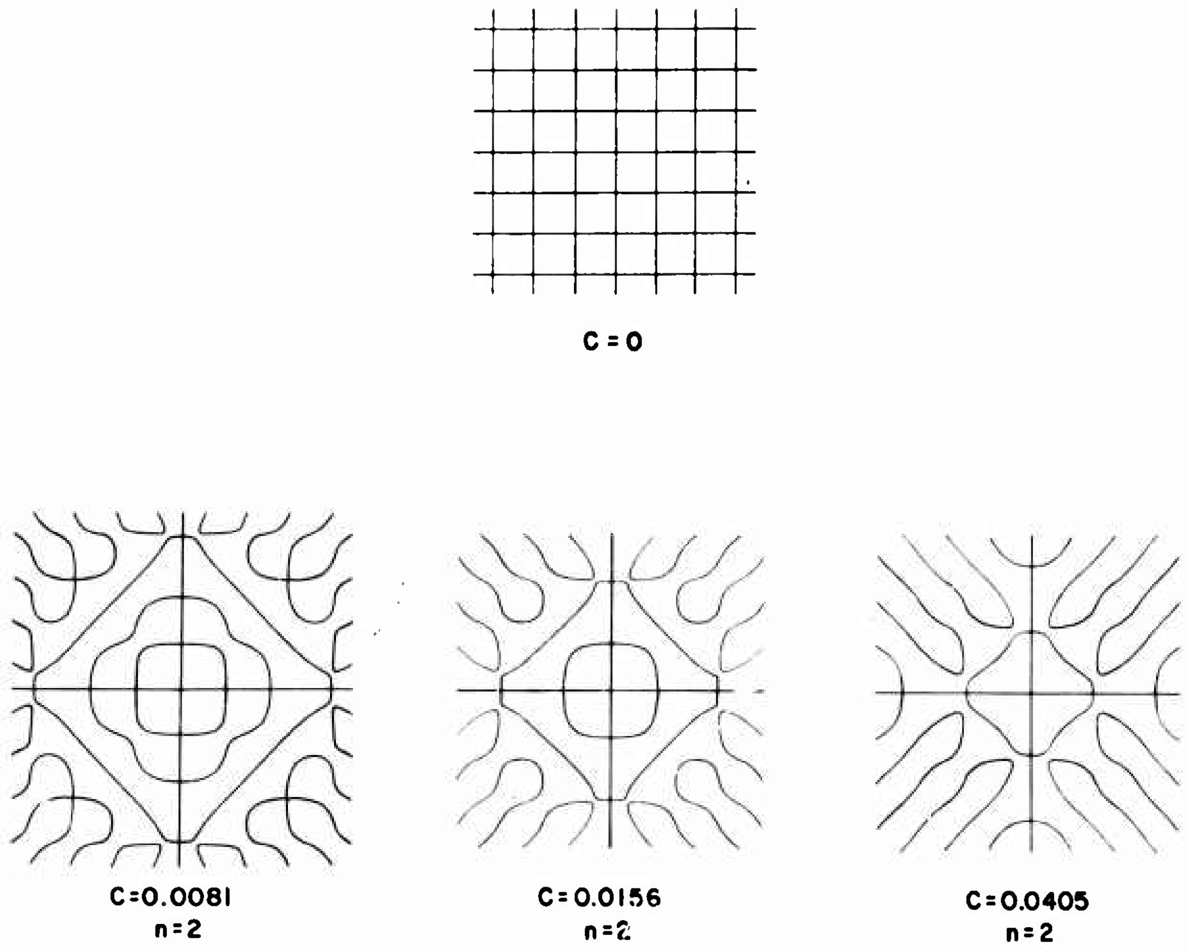

Figure 2. Nodal Patterns on Infinite Membrane for Sums of Separable and Nonseparable Solutions. (Note Closed Areas. Function is $W^{(0)} + CW^{(2)}$ with $a = b = 1$ and $\alpha = \beta = 0$.)

Several features of these patterns may be noted. The outermost closed boundaries are not similar in the geometric sense of the word. The number of closed areas decreases as the proportion of nonseparable to separable increases, until it is zero at a value of C little larger than C = .0405. The distortion of the square that is the building block for C = 0 increases with distance from the origin in each of the lower patterns. The distortion of the square that corners at the origin increases as C increases. A characteristic of the three patterns formed by the addition of nonseparable solution to separable solution is that most of the nodal lines of the pattern do not close on themselves to enclose finite areas of membrane.

PRACTICAL USE OF THESE PATTERNS

Any nodal line in the patterns of Figure 2 can become the edge of a membrane by supposing that a clamp conforming to the line is impressed there. If a closed contour is selected, it can serve as the boundary of a membrane of finite extent. Displacement at any instant within and on that boundary is given by Equation (16), and frequency is given by

$$\omega^2 = c^2(a^2 + b^2) = 2c^2, \tag{18}$$

for every pattern since a = b = 1 throughout Figure 2.

The following procedure, illustrated by Figure 3, may be used to compute frequency of membrane vibration within any closed contour shown in Figure 2. First, we remove the requirement that the propagation constants are equal to unity. Second, the nodal patterns are considered to be points (ax_i, ay_i) plotted in X-Y space, where X = ax and Y = ay. Third, the pattern for C = 0 is superposed upon the one containing the selected pattern by causing the axes to coincide. Frequencies of the two patterns are identical in accordance with Equation (18), and that of the square pattern is the reference ω_r given by

$$\omega_r^2 = \frac{2c^2\pi^2}{L^2}, \tag{19}$$

where L is the side of the square. Fourth, with aL the X-side of the square, let aS be the X-side of the desired contour. Note that aL = π and aS = Mπ. Fifth, convert axes from X-Y to x-y and convert square of side L into square of side S. Propagation constant a = π/L then becomes $a' = \pi/S = \pi/ML$. Thus frequency of the desired contour is given by

$$\omega_c^2 = \frac{2c^2\pi^2M^2}{S^2} = M^2\omega_r^2. \tag{20}$$

Figure 3 has been prepared to illustrate this procedure. The upper and lower left diagrams are superpositions of the square pattern upon the desired contour in X-Y space. The upper and lower right diagrams show the square converted to side S.

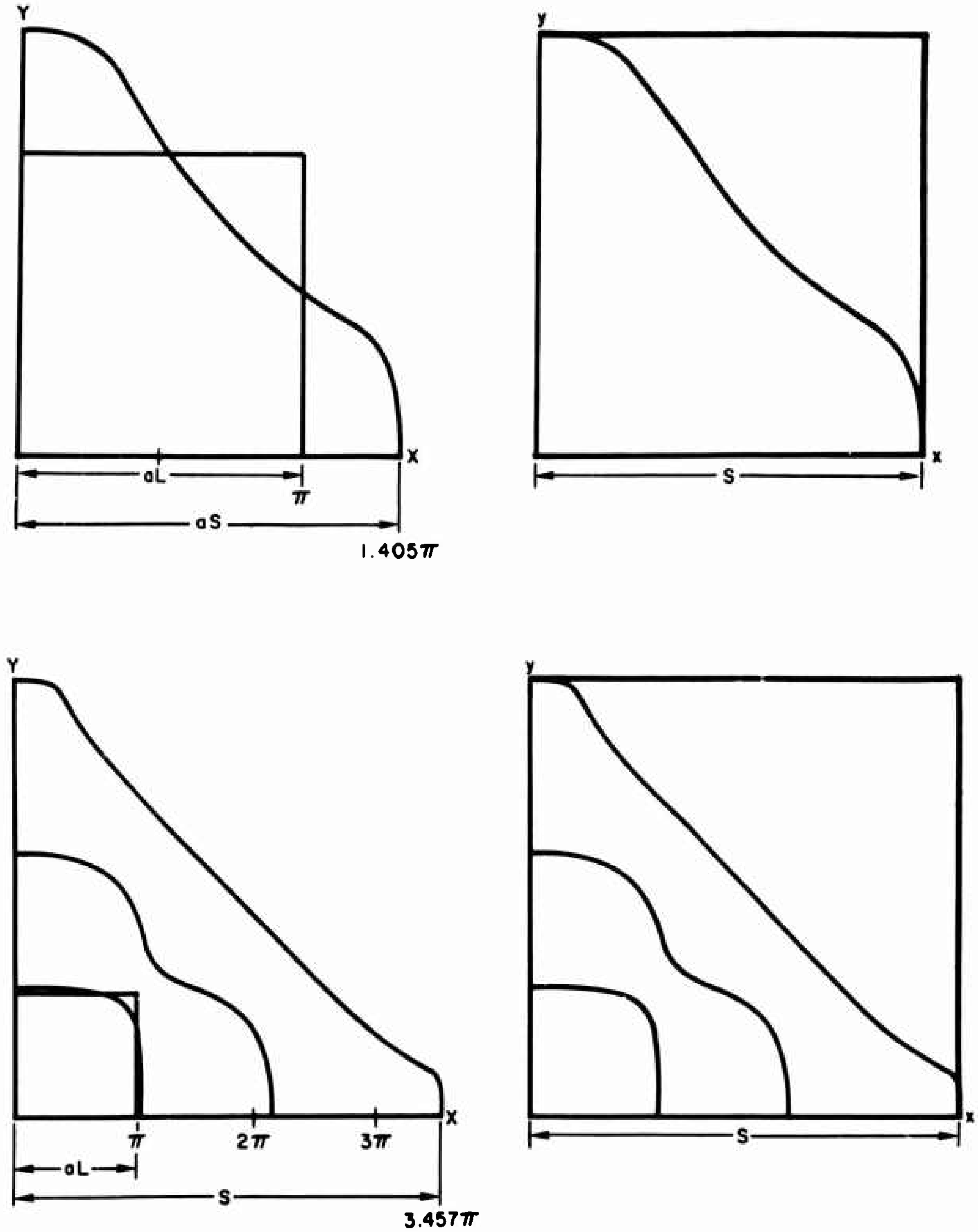

Figure 3. Examples of Construction Used To Compute Frequency of Contoured Membrane.

Frequency ratio M has been computed for all patterns of Figure 2. These are displayed in Figure 4, where side lengths of contours chosen from Figure 2 have been made equal to clarify the presentation.

Also included in Figure 4 are six contours which are predicted for an isosceles right triangle, solutions for which have long been known (reference 5, page 80, and reference 3, page 755) and require no nonseparable solutions. They were chosen for presentation because of their resemblance to patterns generated in this program. Values of M have also been calculated for them, and these are presented beneath each triangle. The multiple values beneath the triangles correspond to the different contours in each and are to be compared as directed by the arrows.

It should also be noted that the M-ratios for the in-triangle contours are greater in every case than those of the corresponding contours derived from nonseparables. However, the difference is perhaps too small to be of engineering significance, which suggests that the triangle model with a correction factor might be useful where applicable in design work.

It is expected that patterns within modified rectangles, i.e., for $a \neq b$, can be evaluated with the aid of nonseparables, whereas the only triangle solution of which we know corresponds to $a = b$.

Note should also be taken of the fact that none of the higher frequencies found for patterns of this contract may properly be called overtones since the outermost contours are not similar in the geometric sense to those of the lowest frequency. How to make them similar remains a challenge.

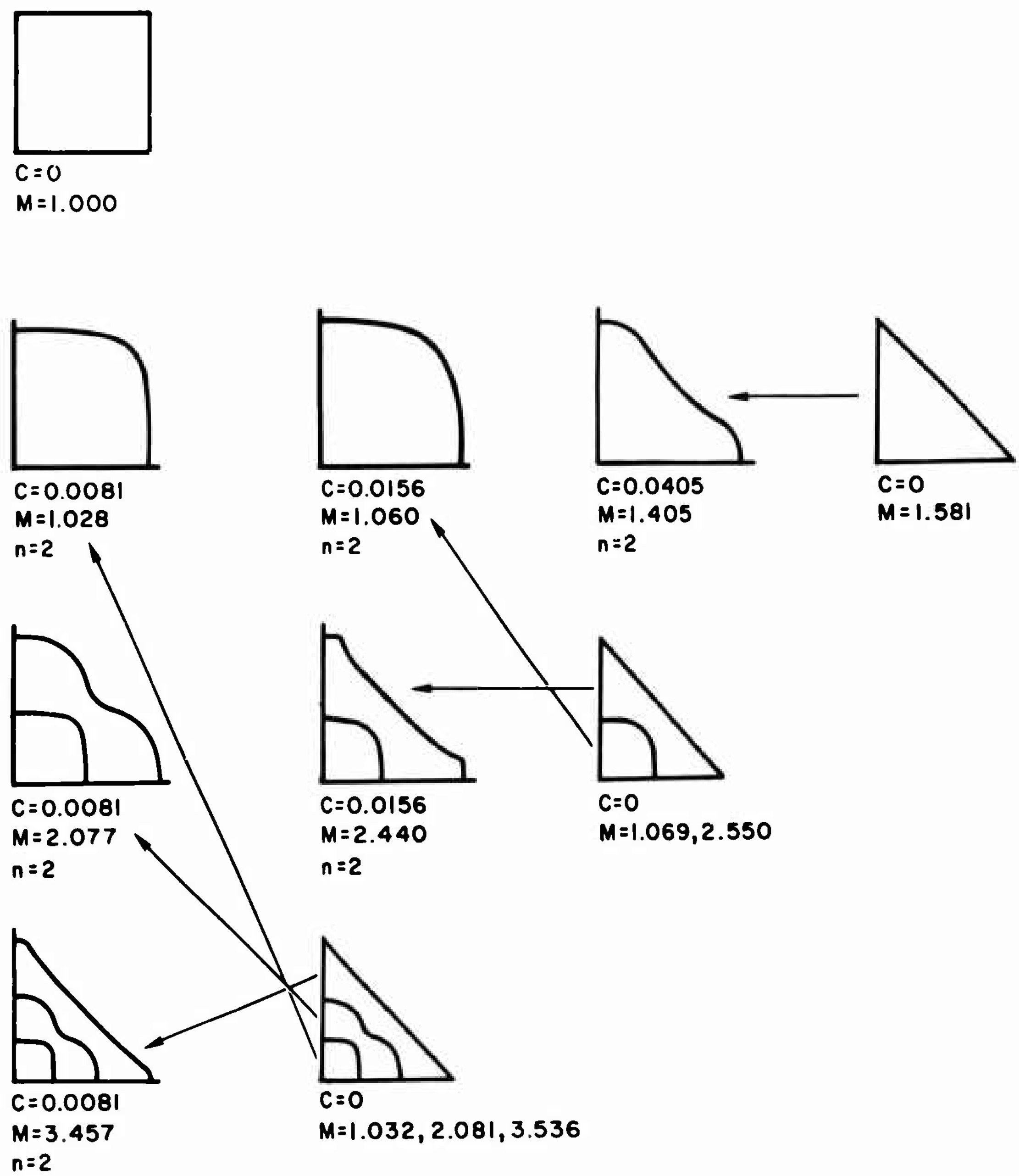

Figure 4. Closed Contours and Their Frequencies in Ratio to That of Square of Same Side Length.

APPLICATION TO ACOUSTICAL CAVITIES

DISCOVERY OF CLOSED VOLUMES

The ability of nonseparable solutions to generate nodal surfaces which enclosed finite volumes was in doubt for some time. It was at length demonstrated that the addition of a small portion of nonseparable to a separable was at least one way to produce closed nodal surfaces, just as had been found for the two-dimensional case of the membrane.

Contributing to the perplexity was the discovery that in three dimensions more than one nonseparable solution of each order exists. At least three appear to have been isolated. Via a three-dimensional generative operator (see Appendix IV) discovered early in the contract, nonseparable solutions of first and second order can be written. A symmetric nonseparable of second order was synthesized after study of results with the operator, yet it appears distinct from its progenitor. The third was a nonseparable of first order which was synthesized directly from study of properties that a function must have to be a solution of the Helmholtz wave equation, and it seems unrelated to the first two.

The first and third of these are presented in Appendix IV, and nodal patterns of the former are presented in Appendix V. The second was chosen for addition to zero order and is given here.

The chosen function is a symmetric function when $a = b = d = 1$ and when $\alpha = \beta = \delta = 0$, since it then transforms into itself when its variables are interchanged in pairs. It is

$$W_{3D}^{(2)} = (x^2 + y^2 + z^2) \sin x \sin y \sin z + x \cos x \sin y \sin z + y \sin x \cos y \sin z + z \sin x \sin y \cos z + yz \sin x \cos y \cos z + xz \cos x \sin y \cos z + xy \cos x \cos y \sin z. \qquad (21)$$

Closed nodal surfaces were discovered to exist in the function

$$\varphi = \left[W_{3D}^{(0)} + CW_{3D}^{(2)}\right] \cos \omega t \qquad (22)$$

when $C = (18.75\pi^2)^{-1} = .00540$, and $W^{(0)}$ and $W^{(2)}$ are given by Equations (7) and (21), respectively, with $a = b = d = 1$ and $\alpha = \beta = \delta = 0$.

These surfaces are formed by the three planes which contain the coordinate axes and by curved surfaces which meet these planes at right angles. The curved surfaces are those non-zero points (x_i, y_i, z_i) which make the bracket of Equation (22) equal to zero. The intersection of curved surfaces with the planes was found by evaluating points (x_i, y_i) for $z = \epsilon$ with ϵ

allowed to approach zero closely enough to justify the approximations $\sin \epsilon = \epsilon$ and $\cos \epsilon = 1$. The nodal pattern in the x-y plane given by $z = \epsilon$ is presented in Figure 5(a). The pattern which is Figure 5(b) lies in the plane $z = \pi$ and is repeated for $z = \pm m\pi$ with $m \neq 0$.

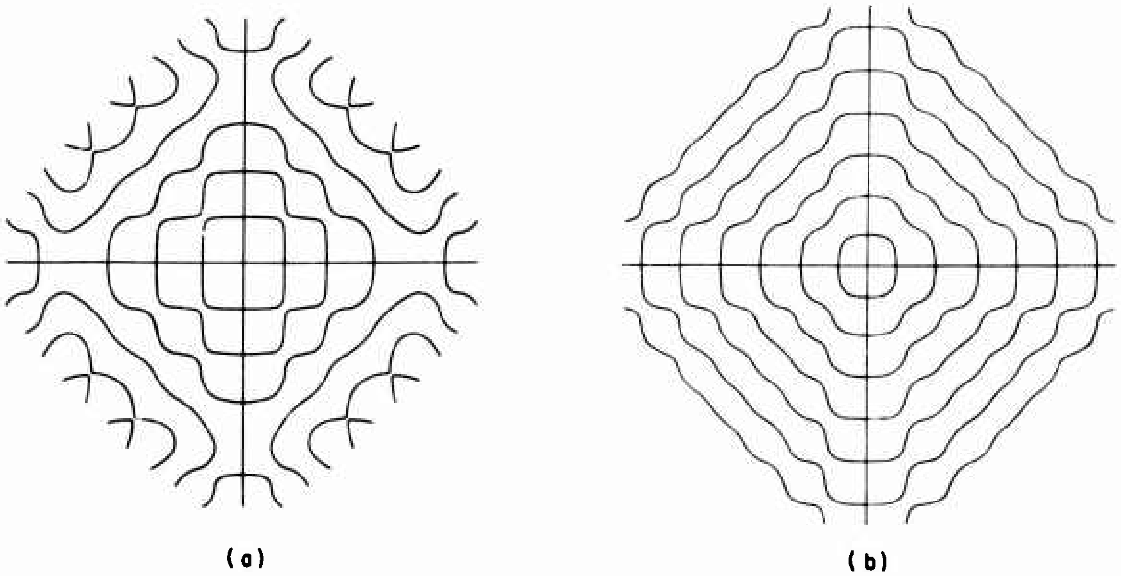

Figure 5. Nodal Patterns in z-Planes Passed Through Nodal Surfaces. (a) $Z = \epsilon$ with $\epsilon \to 0$; (b) $z = \pm m\pi$ with $m \neq 0$. Function is $W_{3D}^{(0)} + .00540W_{3D}^{(2)} = 0$.

Since $W^{(0)}$ and $W^{(2)}$ are symmetrical to cyclic rotation of coordinates, the patterns of Figure 5 are equally applicable with coordinates cycled. Based upon this property, a three-dimensional model has been constructed of these patterns as an aid to visualization of the volumes enclosed by the nodal surfaces. Three photographs of this model are presented. Figure 6(a) represents the smallest enclosed volume. Figure 6(b) represents the volume of intermediate size enclosing the first. Figure 6(c) allows one to visualize all three nodal surfaces and their enclosed volumes. Other surfaces, apparent from Figure 5, are not closed and only their intersections with the principal planes are shown.

SIGNIFICANCE OF DISCOVERY OF CLOSED VOLUMES

In the theory of acoustics the boundary condition of an acoustic cavity is a specified wall impedance, given by rms sound pressure in ratio to rms particle velocity normal to the wall. In symbols this is

$$Z = \frac{p_{rms}}{(v_n)_{rms}}, \tag{23}$$

where p is a scalar and v_n is one component of the vector particle velocity.

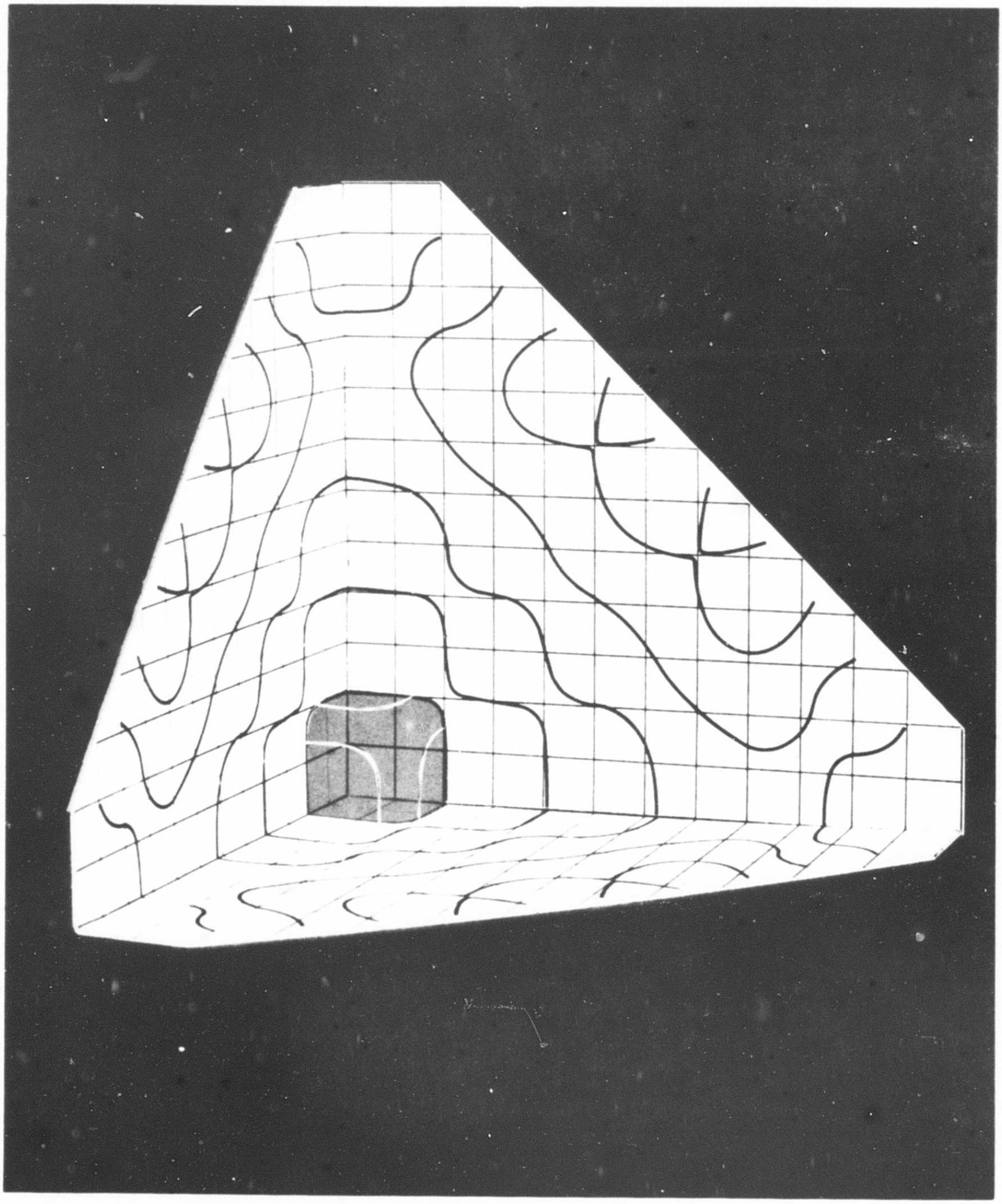

Figure 6. Three-Dimensional Model Constructed from Patterns of Figure 5.
(a) Innermost Surface Represented

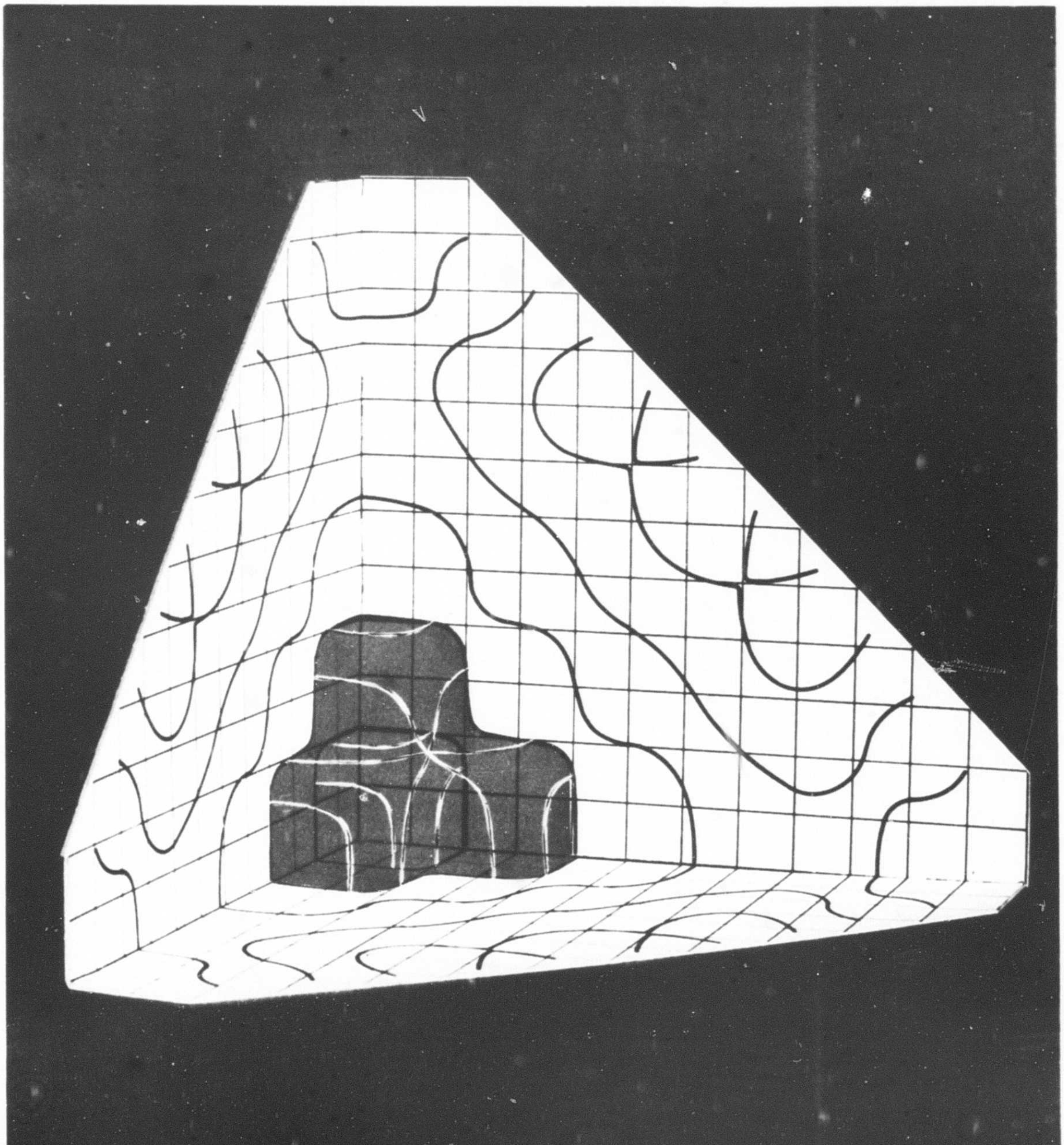

Figure 6(b). Inner Two Surfaces Represented.

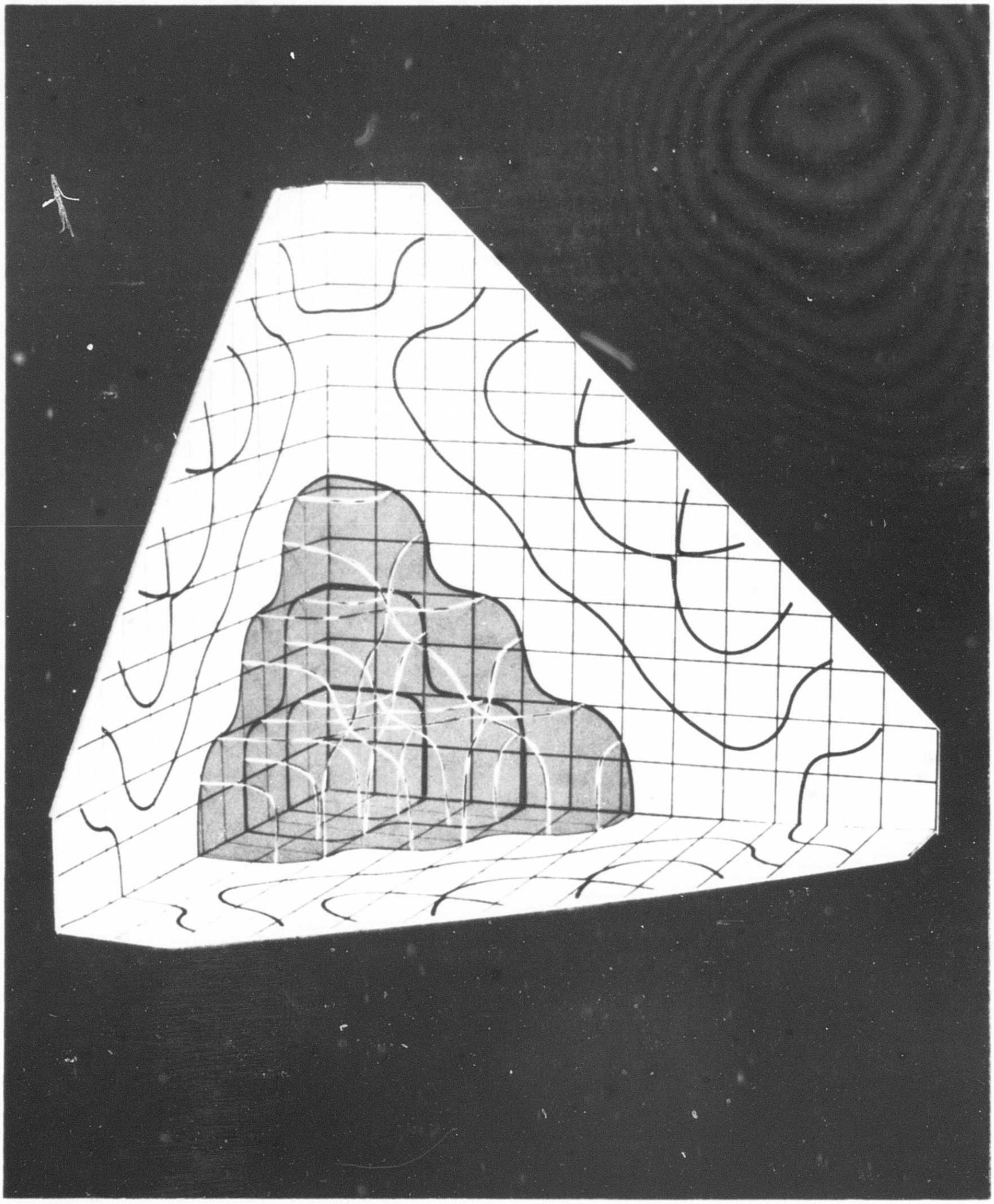

Figure 6(c). All Closed Surfaces Represented.

Most frequently the assumed condition is that Z is infinite through the vanishing of the normal component of particle velocity upon the wall. The other two of the three mutually orthogonal components of particle velocity are not necessarily zero at the wall and are not determined by that portion of wall.

It has not proved possible during the time of this contract to discover a nonseparable velocity potential from which to obtain $v_n = 0$ on a closed surface. There is, however, a mathematical meaning for the alternate condition of zero impedance upon the walls. It is that the pressure there is zero. Thus we can say from Equation (22) that $\varphi = p$, or that φ is itself a velocity potential from which we may derive p according to

$$p = -\rho\frac{\partial\varphi}{\partial t}, \tag{24}$$

where ρ is the mean mass density of the medium within the cavity.

One may compute frequency of resonance of the volumes of Figure 6 by referring those volumes to the cubes which would enclose them. The method is essentially that described for membrane patterns. If the frequencies of the reference cube and of the distorted cube be ω_r and ω_c, respectively, the ratio $M = \omega_c/\omega_r$ is as follows: for innermost volume, M = 0.983; inner two volumes, M = 2.042; all three volumes, M = 3.103. In each case, the reference cube is one whose side length is equal to the side length of the desired volume.

While enclosing a volume with walls of zero impedance is mathematically justified, it is nonetheless physically impossible. To produce a pressure-release boundary would require an enclosing wall of no mass or stiffness, yet a wall which would be capable of confining a liquid or gas without escape of matter.

Therefore it is acknowledged that the significance of the discovery of closed volumes is its contribution to progress in understanding the problem of applying nonseparables to acoustic cavities.

APPLICATION TO ELECTROMAGNETIC CAVITIES

An electromagnetic field is characterized by field vectors $\bar{E}$ and $\bar{H}$. Within a lossless cavity these vectors must individually satisfy the vector Helmholtz wave equation which is Equation (1), and on the walls these vectors must satisfy specific conditions. The conditions considered in this work were that the tangential components of the electric field $\bar{E}$ and the normal component of the magnetic field $\bar{H}$ be zero.

Auxiliary functions known as the electric scalar potential and the magnetic vector potential are related to $\bar{E}$ and $\bar{H}$ everywhere through the relationships (reference 6, page 86)

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}, \tag{25}$$

$$\vec{H} = \frac{1}{\mu}\vec{\nabla} \times \vec{A}, \tag{26}$$

and

$$\vec{\nabla} \cdot \vec{A} + \mu\epsilon\frac{\partial\varphi}{\partial t} = 0, \tag{27}$$

where

- φ is the electric potential
- $\vec{A}$ is the magnetic potential
- μ,ϵ are permittivity and permeability, respectively, within the cavity.

These potentials satisfy the scalar and vector Helmholtz wave equations, respectively, wherein

$$k^2 = \frac{\omega^2}{c^2} = \omega^2\mu\epsilon. \tag{28}$$

Equation (27) relates the electric potential and all the components of $\bar{A}$. In an exercise detailed in Appendix VI, the electric potential was equated to a nonseparable, after which the three terms of the divergence of $\bar{A}$ were assumed to be equal to one another. Components of $\bar{A}$ were then found by integration, whereupon field vectors $\bar{E}$ and $\bar{H}$ follow from Equations (25) and (26).

In a specific case the electric potential was chosen to be

$$\varphi = 2aV_o(x \cos ax \sin ay - y \sin ax \cos ay) \sin az\, e^{i\omega t}, \tag{29}$$

where aV_o has the dimensions of volts per meter. This leads to $\vec{E}$ and $\vec{H}$ components which are

$$E_x = -4aV_o \cos ax \sin ay \sin az\, e^{i\omega t},$$

$$E_y = 4aV_o \sin ax \cos ay \sin az\, e^{i\omega t},$$

$$E_z = 0, \tag{30}$$

and

$$H_x = -\frac{i4k^2}{3\omega\mu} V_o \sin ax \cos ay \cos az\, e^{i\omega t},$$

$$H_y = -\frac{i4k^2}{3\omega\mu} V_o \cos ax \sin ay \cos az\, e^{i\omega t},$$

$$H_z = \frac{i8k^2}{3\omega\mu} V_o \cos ax \cos ay \sin az\, e^{i\omega t}, \tag{31}$$

where

$$k^2 = \frac{\omega^2}{\mu\epsilon}3a^2 .$$

These satisfy the reduced Maxwell equations for the region within the cavity, namely,

$$\vec{\nabla} \cdot \vec{H} = 0,$$

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{32}$$

and they satisfy the boundary conditions on a cavity having sides L, L' and L'' which obey the relationship

$$L : L' : L'' = n : n' : n'' \tag{33}$$

for n-values that are any set of three positive integers.

Frequency is given by

$$\omega^2 = \frac{3}{\mu\epsilon}\left(\frac{n\pi}{L}\right)^2 . \tag{34}$$

STUDY OF ELASTIC SOLIDS

Investigation of application to elastic solids has grappled with two examples of two boundary conditions at each edge. In an effort to simplify the elastic problem as much as possible, an example of plane stress or plane strain was first sought.

Consideration was given to extensional vibration in a thin rectangular plate, a problem which is posed and formulated by Love (reference 1, page 497). One shear and one normal stress are specified along every edge. Study soon showed, however, that this problem is mathematically similar to the problem of the right circular solid cylinder with symmetry about the axis (reference 4). Nonseparable solutions had already proved of no avail in solving the cylinder problem, and a review confirmed the previous finding. The conclusion drawn from consideration of the extensional vibration in thin rectangular plates is that the inability of nonseparable solutions to meet the required pair of conditions is other than a deficiency of nonseparable solutions, since separable solutions do not meet them either.

Consideration was later given to flexural vibration of a thin plate. A rectangular outline was chosen because all fruitful experience under the contract had been in rectangular coordinates and because it had recently been fairly well established that one cannot have nonseparable solutions in polar coordinates r and θ (see Appendix VII). The flexural wave equation is (reference 2, page 209)

$$\nabla^4 \eta + \frac{3\rho(1 - \nu^2)}{Eh^2} \frac{\partial^2 \eta}{\partial t^2} = 0, \tag{35}$$

where the independent variables are x, y and t.

Upon assuming that

$$\eta = Ye^{-i\omega t}, \tag{36}$$

one can write the reduced flexural wave equation in factored form as

$$(\nabla^2 - \gamma^2)(\nabla^2 + \gamma^2)Y = 0, \tag{37}$$

where

$$\gamma^4 = 3\frac{\omega^2}{c^2h^2}, \qquad c^2 = \frac{E}{\rho(1 - \nu^2)},$$

and

- Y is the amplitude of displacement at right angles to the plate.
- h is the half thickness of the plate.
- E is Young's modulus.
- ρ is the density of plate material.
- ν is Poisson's ratio.

The second factor in Equation (37) is the Helmholtz wave equation for which separable and nonseparable solutions are known. The first factor is a Helmholtz wave equation but for the negative sign. Solutions of this equation are the same separable and nonseparable solutions that are already known, except that propagation constants a and b are replaced by ia and ib.

Thus the total solution of Equation (37) is

$$Y = \sum W^{(n)} + \sum H^{(n)}, \tag{38}$$

where $H^{(n)}$ are $W^{(n)}$ with each a and b replaced by ia and ib.

It is concluded that, since the reduced flexural wave equation is a scalar equation, functions of the form

$$Y = W^{(0)} + CW^{(n)} + H^{(0)} + DH^{(n)}, \qquad n = 2 \tag{39}$$

should be capable of satisfying the two conditions at every part of the edge, and closed nodal patterns should be derivable in the same manner as for the membrane.

CONCLUSIONS

Conclusions from this program are:

(1) There are applications for nonseparable solutions of the Helmholtz wave equation. They have been demonstrated for membranes, for an acoustic cavity with pressure-release walls, and for an electromagnetic cavity. They have been forecast for a thin elastic plate in flexural vibration.

(2) Vibration on or within some new shapes can now be calculated exactly with functions formed of the nonseparable solutions added to separable solutions. Separable solutions taken alone predict frequency and distribution of amplitude on squares and rectangles and in cubes. The new shapes are smooth distortions of the squares and cubes.

(3) The applicable mathematics is difficult to handle. The mathematics proceeds from assumed function to resulting shape; if the user specifies the shape and asks the resonant frequency and amplitude distribution, a cut-and-try effort must be mounted.

(4) Simplifications and additional applications await the effort. Insight gained with respect to scalar and vector fields that use nonseparables singly and with separables reveals no inherent mathematical property which would reject this conclusion.

RECOMMENDATIONS

It is recommended that this work continue with the objectives of (1) demonstrating more applications and (2) simplifying the mathematics involved.

Application to flexural vibration of plates and to resonant cavities with vector conditions on the walls is considered attainable. Application to new modes in old shapes and to prediction of overtones in new shapes should be given attention.

Simplification will result from improved organization of mathematical properties which are recorded in this report. More properties await discovery, among them being relationships which express integrals of nonseparables, conditions for orthogonality, other effects of summing variants, and generative operators in three dimensions. These new properties will also contribute to simplification.

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APPENDIX I, TWO-DIMENSIONAL SOLUTIONS IN EXPONENTIAL FORM

In two dimensions an alternative form of writing the nonseparable solutions is through use of the imaginary $i = \pm\sqrt{-1}$.

Let

$$r = ax \pm by \qquad (40)$$

and

$$s = bx \mp ay, \qquad (41)$$

whereupon the generative operator O_{2D} previously written in a and b may be rewritten in r and s, as

$$O_{2D} = b\frac{\partial}{\partial a} - a\frac{\partial}{\partial b} = s\frac{\partial}{\partial r} - r\frac{\partial}{\partial s}. \qquad (42)$$

The first five nonseparable solutions, derivable through successive application of O_{2D}, are

$$W^{(0)} = e^{ir}$$

$$W^{(1)} = ise^{ir}$$

$$W^{(2)} = [(is)^2 - ir]e^{ir}$$

$$W^{(3)} = is[(is)^3 - 3ir - 1]e^{ir}$$

$$W^{(4)} = [(is)^4 - 6ir(is)^2 + 3(ir)^2 - 4(is)^2 + ir]e^{ir}$$

$$W^{(5)} = is[(is)^4 - 10ir(is)^2 + 15(ir)^2 - 10(is)^2 + 15ir + 1]e^{ir}. \qquad (43)$$

APPENDIX II, VARIANTS

In Equation (9) the nonseparable two-dimensional solution of order zero was written in compact form by introducing phase constants α and β to represent an arbitrary combination of the factors in

$$W = \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} \begin{Bmatrix} \sin by \\ \cos by \end{Bmatrix}. \tag{44}$$

It is sometimes convenient to write out explicitly all possible forms of the solution, which in two dimensions are four variants. Using numerical subscripts to distinguish one variant from another, we may write

$$W_1^{(0)} = \sin ax \sin by$$

$$W_2^{(0)} = \cos ax \cos by$$

$$W_3^{(0)} = \sin ax \cos by$$

$$W_4^{(0)} = \cos ax \sin by. \tag{45}$$

The two-dimensional generative operator may then be applied to each of these with the result that four variants of first order are evolved. Application of O_{2D} to each of these produces four variants of second order, and so on. Variants of first, second and third orders derived from Equation (45) are

$$W^{(1)} = \begin{cases} bx \cos ax \sin by - ay \sin ax \cos by \\ (-1)(bx \sin ax \cos by - ay \cos ax \sin by) \\ bx \cos ax \cos by + ay \sin ax \sin by \\ (-1)(bx \sin ax \sin by + ay \cos ax \cos by), \end{cases} \tag{46}$$

$$W^{(2)} = (-1) \begin{cases} [(bx)^2 + (ay)^2] \sin ax \sin by + by \sin ax \cos by \\ \quad + ax \cos ax \sin by + 2abxy \cos ax \cos by \\ [(bx)^2 + (ay)^2] \cos ax \cos by - by \cos ax \sin by \\ \quad - ax \sin ax \cos by + 2abxy \sin ax \sin by \\ [(bx)^2 + (ay)^2] \sin ax \cos by - by \sin ax \sin by \\ \quad + ax \cos ax \cos by - 2abxy \cos ax \sin by \\ [(bx)^2 + (ay)^2] \cos ax \sin by + by \cos ax \cos by \\ \quad - ax \sin ax \sin by - 2abxy \sin ax \cos by, \end{cases} \tag{47}$$

$$W_1^{(3)} = [3(bx)^2 + (ay)^2 + 1]\, ay \sin ax \cos by + 3ab(x^2 - y^2) \sin ax \sin by + 3xy(a^2 - b^2) \cos ax \cos by - [3(ay)^2 + (bx)^2 + 1]\, bx \cos ax \sin by$$

$$W_2^{(3)} = [3(ay^2) + (bx)^2 + 1]\, bx \sin ax \cos by + 3ab(x^2 - y^2) \cos ax \cos by + 3xy(a^2 - b^2) \sin ax \sin by - [3(bx)^2 + (ay)^2 + 1]\, ay \cos ax \sin by$$

$$W_3^{(3)} = -[3(ay)^2 + (bx)^2 + 1]\, bx \cos ax \cos by + 3ab(x^2 - y^2) \sin ax \cos by - 3xy(a^2 - b^2) \cos ax \sin by - [3(bx)^2 + (ay)^2 + 1]\, ay \sin ax \sin by$$

$$W_4^{(3)} = [3(ay)^2 + (bx)^2 + 1]\, bx \sin ax \sin by + 3ab(x^2 - y^2) \cos ax \sin by - 3xy(a^2 - b^2) \sin ax \cos by + [3(bx)^2 + (ay)^2 + 1]\, ay \cos as \cos by. \tag{48}$$

An interesting property of the variants is that the oddness or evenness of a given variant with respect to any of the coordinates is preserved through all orders. For example, the variant $W_3^{(0)}$ is odd in x and even in y. It is found that $W_3^{(n)}$ is likewise odd in x and even in y.

It is also interesting to note that closed nodal lines have been found only with sums containing like variants, namely, for

$$W_1^{(0)} + CW_1^{(1)} = 0, \tag{49}$$

and for one experimental thrust with second variants. The last was the function

$$\left[W_2^{(0)}\right]_{(a = b = \sqrt{5/2})} + \frac{1}{2\pi}\left\{\left[W_2^{(1)}\right]_{(a = 2b = 2)} + \left[W_2^{(1)}\right]_{(b = 2a = 2)}\right\} = 0. \qquad (50)$$

APPENDIX III, OTHER TWO-DIMENSIONAL PATTERNS

This section presents a collection of nodal patterns in which closed areas are not found. One therefore must imagine a membrane of infinite extent upon which displacement η is distributed in accordance with the expression

$$\eta = W_j^{(n)} e^{i\omega t}, \qquad n = 1, 2, 3... \tag{51}$$

where j = 1, 2, 3, 4 designates the variant, and the patterns are the curves upon which $\eta = 0$ for all t. Propagation constants in all cases but the last are a = b = 1 so that for them frequency is given by the frequency equation

$$\omega^2 = c^2 k^2 = 2c^2. \tag{52}$$

Patterns of $W_j^{(n)} = 0$ are presented in Figure 7, in which variant is varied horizontally and order is varied vertically. The figure omits the pattern for j = 4, since it is that of j = 3 under a 90° rotation of axes. The same scale is used throughout.

Pattern similarities exist by row and by column, particularly in patterns of the same variant. It may also be noted that the straight-line diagonals vanish in second order, which may indicate that all even orders do not contain them.

Patterns of $W_j^{(n)} + W_k^{(n)} = 0$ are presented in Figure 8. The upper pair is for n = 1, and the lower is for n = 2. The distinctive feature of the pair is that it consists of straight lines inclined at +45° to the horizontal axis and cut by one line at -45°. The equations of the left and right patterns are, respectively,

$$W_1^{(1)} + W_2^{(1)} = -(x + y) \sin (x - y) \tag{53}$$

and

$$W_3^{(1)} - W_4^{(1)} = (x + y) \cos (x - y). \tag{54}$$

These patterns are a particularly good example of the value of changing variables in the expressions for nonseparables when a = b = 1. If we let

$$p = x + y, \qquad q = x - y,$$

the variants can be written

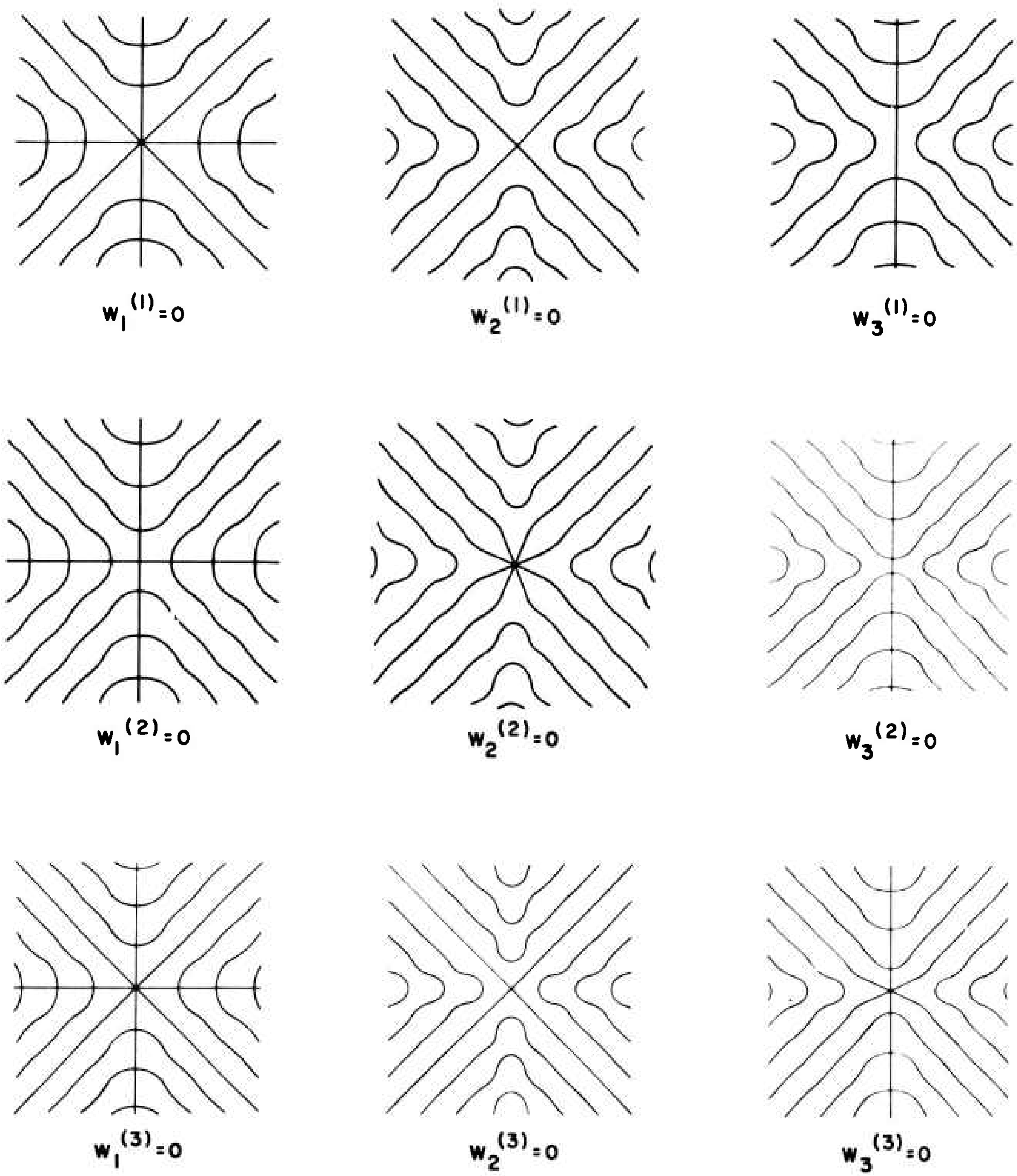


Figure 7. Patterns of $W_j^{(n)} = 0$ on an Infinite Membrane.

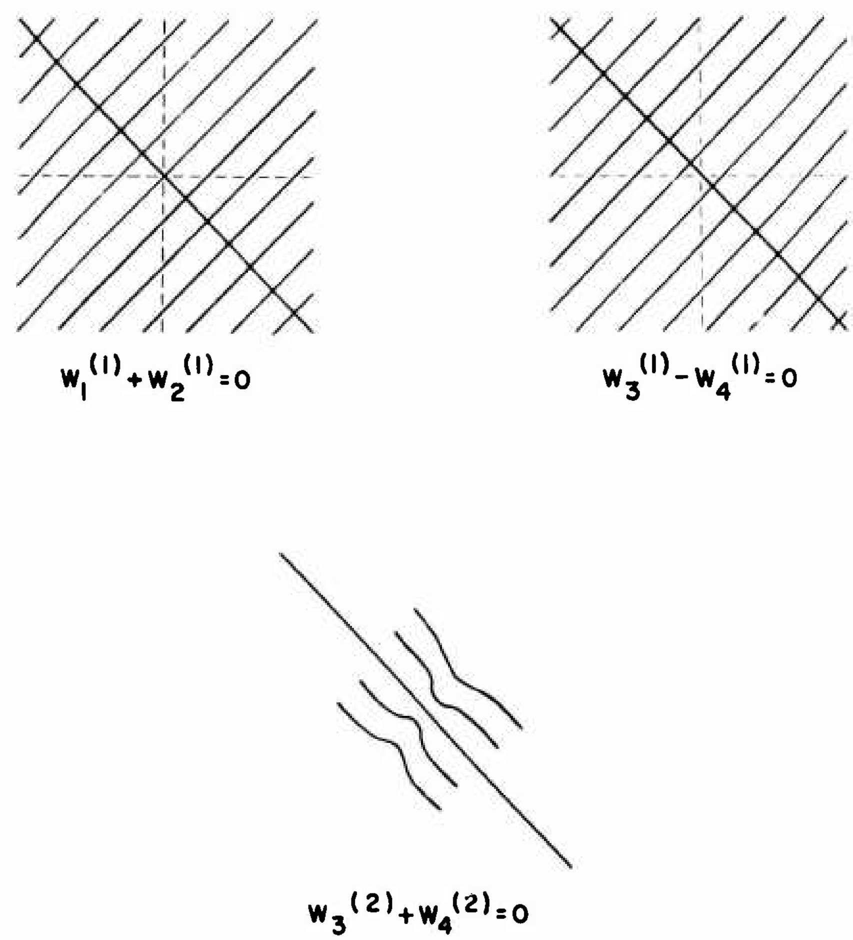


Figure 8. Patterns of $W_j^{(n)} + W_k^{(n)} = 0$ on an Infinite Membrane.

$$W_1^{(1)} = \frac{1}{2} (q \sin p - p \sin q)$$

$$W_2^{(1)} = \frac{1}{2} (q \sin p + p \sin q)$$

$$W_3^{(1)} = \frac{1}{2} (p \cos q + q \cos p$$

$$W_4^{(1)} = \frac{1}{2} (p \cos q - q \cos p), \qquad (55)$$

$$W_1^{(2)} = \frac{1}{2} [p \sin p - q \sin q + p^2 \cos q - q^2 \cos p]$$

$$W_2^{(2)} = \frac{1}{2} [q^2 \cos p + p^2 \cos q - p \sin p - q \sin q]$$

$$W_3^{(2)} = \frac{1}{2} [q^2 \sin p + p^2 \sin q + q \cos q + p \cos p]$$

$$W_4^{(2)} = \frac{1}{2} [q^2 \sin p - p^2 \sin q - q \cos q + p \cos p], \qquad (56)$$

and

$$W_1^{(3)} = - \frac{1}{2} [(q^3 + q) \sin p - (p^3 + p) \sin q + 3pq (\cos p - \cos q)]$$

$$W_2^{(3)} = \frac{1}{2} [(q^3 + q) \sin p + (p^3 + p) \sin q + 3pq (\cos p + \cos q)]$$

$$W_3^{(3)} = - \frac{1}{2} [(q^3 + q) \cos p + (p^3 + p) \cos q - 3pq (\sin p + \sin q)]$$

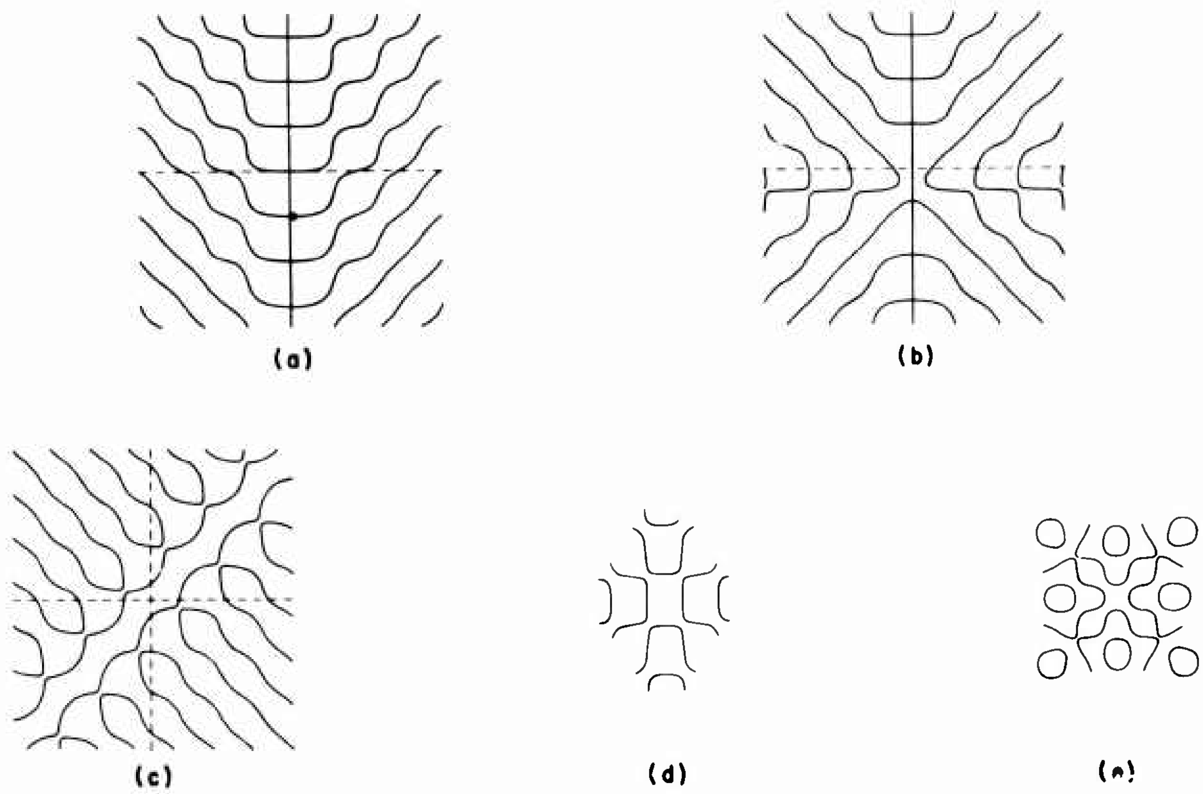
$$W_4^{(3)} = - \frac{1}{2} [(q^3 + q) \cos p - (p^3 + p) \cos q - 3pq (\sin p - \sin q)]. \qquad (57)$$

This change of variable has made easier the task of calculating patterns; however, the coordinates of x and y have been used for all patterns.

Patterns resulting from sums of first and zero order have been explored to a small extent. Closed areas were found in two cases displayed in Figure 2. Figure 9 presents five more patterns of which the upper two are of the function

$$W_1^{(0)} + CW_3^{(1)} = 0,$$

with $C = (5\pi)^{-1}$ and $C = (0.44\pi)^{-1}$ on the left and right respectively. The



(a) $W_1^{(0)} + (5.00\pi)^{-1}\, W_3^{(1)} = 0$ $\qquad a = b = 1$

(b) $W_1^{(0)} + (0.44\pi)^{-1}\, W_3^{(1)} = 0$ $\qquad a = b = 1$

(c) $W_1^{(0)} + W_2^{(0)} + (\pi)^{-1}(W_1^{(1)} - W_2^{(1)}) = 0$ $\qquad a = b = 1$

(d) $W_2^{(0)} + (2\pi)^{-1}\, W_2^{(1)} = 0$ $\qquad a = b = 1$

(e) $\left[W_2^{(0)}\right]_{a=b=\sqrt{5/2}} + \frac{1}{2\pi}\left\{\left[W_2^{(1)}\right]_{a=2b=2} + \left[W_2^{(1)}\right]_{b=2a=2}\right\} = 0$

Figure 9. Nodal Patterns on Membranes for Various Sums of First and Zero Order Functions. (Scales are the same; different portions of field have been evaluated in the five cases.)

lower three are of functions of three different compositions, one of which exhibits closed, egg-shaped areas at some distance from the origin.

Many directions for future exploration suggest themselves after study of Figure 9. One wonders, for example, if the combination of second variants - lower, center - could produce closed areas with the appropriate value of C; or again, one wonders what has been opened by the use of different propagation constants which still satisfy the frequency equation.

APPENDIX IV, OPERATORS

A distinction will be made between two types of operators encountered in this work. One type is the generative operator used to obtain solutions of higher order, and the other is the customary partial differential operator like $\partial/\partial x$, $\partial/\partial y$, etc. Of the two, the generative operator has played a greater role in the study of nonseparable solutions.

Generative Operators

These operators contain the propagation constants as variables and, when applied to an nth order nonseparable solution in rectangular coordinates, yield a solution of order n + 1. The two- and three-dimensional operators have the forms

$$O_{2D} = b\frac{\partial}{\partial a} - a\frac{\partial}{\partial b}$$

and

$$O_{3D} = f\frac{\partial}{\partial a} - g\frac{\partial}{\partial b} - h\frac{\partial}{\partial d}, \tag{58}$$

respectively,

where

$$f = b + d, \qquad g = a + d, \qquad h = a - b.$$

With the help of these operators, a one-to-one correspondence can be established between variants of different orders. If a variant number is assigned to a zero order solution, the same number can be assigned to the highest order solutions obtained by successive application of the operator. This correspondence is also preserved when phase angles or translations of coordinates are introduced.

Zero, first and second order solutions that are related through O_{3D} are, in compact form,

$$W^{(0)} = \sin u \sin v \sin w, \tag{59}$$

$$W^{(1)} = fx \cos u \sin v \sin w - gy \sin u \cos v \sin w - hz \sin u \sin v \cos w, \tag{60}$$

$$\begin{aligned} W^{(2)} = {} & [(fx)^2 + (gy)^2 + (hz)^2] \sin u \sin v \sin w + \\ & (g + h)\, x \cos u \sin v \sin w - 2ghyz \sin u \cos v \cos w + \\ & (f - h)\, y \sin u \cos v \sin w + 2fhxz \cos u \sin v \cos w + \\ & (f + g)\, z \sin u \sin v \cos w + 2fgxy \cos u \cos v \sin w, \end{aligned} \tag{61}$$

where

$$u = ax + \alpha, \qquad v = by + \beta, \qquad w = dz + \delta.$$

A three-dimensional nonseparable of first order, apparently not related to those already described, was also synthesized from study of properties that a function must have to be a solution of the Helmholtz wave equation. This is

$$W^{(1)} = x \sin ax(b \cos by \sin dz - d \sin by \cos dz) +$$
$$y \sin by(d \sin ax \cos dz - a \cos ax \sin dz) +$$
$$z \sin dz(a \cos ax \sin by - b \sin ax \cos by). \qquad (62)$$

No generative operator to derive Equation (62) from Equation (59) with $\alpha = \beta = \delta = 0$ is yet known.

Differential Operators

It was found during the work that, in all the cases tried, the result of taking the partial derivative of a nonseparable solution with respect to x, y, z, or a combination of these, is a solution of the wave equation.

Examples are given below to illustrate the effects of differential operators on various solutions. In two dimensions, the following relations hold for zero order and first order solutions:

$$D_x W_1^{(0)} = aW_4^{(0)} \qquad D_y W_1^{(0)} = bW_4^{(0)}$$

$$D_{xx} W_1^{(0)} = - a^2 W_1^{(0)} \qquad D_{yy} W_1^{(0)} = - b^2 W_1^{(0)}$$

$$D_{xy} W_1^{(0)} = abW_2^{(0)} \qquad (63)$$

$$D_x W_1^{(1)} = - aW_4^{(1)} + bW_4^{(0)} \qquad D_y W_1^{(1)} = bW_3^{(1)} - aW_3^{(0)}$$

$$D_{xx} W_1^{(1)} = - a^2 W_1^{(1)} - 2abW_1^{(0)} \qquad D_{yy} W_1^{(1)} = - b^2 W_1^{(1)} + 2abW_1^{(0)}$$

$$D_{xy} W_1^{(1)} = abW_2^{(1)} + (b^2 - a^2)W_2^{(0)}$$

$$D_{xxx} W_1^{(1)} = - a^3 W_4^{(1)} - 3a^2 bW_4^{(0)} \qquad D_{yyy} W_1^{(1)} = - b^3 W_3^{(1)} + 3ab^2 W_3^{(0)}$$

$$D_{xxy} W_1^{(1)} = - a^2 bW_3^{(1)} - (2ab^2 - a^3)W_3^{(0)}$$

$$D_{xyy}W_1^{(1)} = -ab^2W_4^{(1)} + (2a^2b - b^3)W_4^{(0)}$$

$$D_{xxxx}W_1^{(1)} = a^4W_1^{(1)} + 4a^3bW_1^{(0)} \qquad D_{yyyy}W_1^{(1)} = b^4W_1^{(1)} - 4ab^3W_1^{(0)}$$

$$D_{xxyy}W_1^{(1)} = a^2b^2W_1^{(1)} - 2ab(a^2 - b^2)W_1^{(0)} . \tag{64}$$

For the examples in three dimensions, it is convenient to drop the variant notation and to assign values to the phase angles α, β, and δ. The zero-order solution from which solutions of higher order are derived is

$$W^{(0)} = \sin(ax + \alpha) \sin(by + \beta) \sin(dz + \delta).$$

Unless otherwise indicated, the phase angles are to be assumed equal to zero in the following functions:

$$D_xW^{(0)} = \left[aW^{(0)}\right]_{\alpha = \frac{\pi}{2}} \qquad D_{xx}W^{(0)} = -a^2W^{(0)}$$

$$D_yW^{(0)} = \left[bW^{(0)}\right]_{\beta = \frac{\pi}{2}} \qquad D_{yy}W^{(0)} = -b^2W^{(0)}$$

$$D_zW^{(0)} = \left[dW^{(0)}\right]_{\delta = \frac{\pi}{2}} \qquad D_{zz}W^{(0)} = -d^2W^{(0)}, \tag{65}$$

$$D_xW^{(1)} = \left[aW^{(1)} + (b + d)W^{(0)}\right]_{\alpha = \frac{\pi}{2}} \qquad D_{xx}W^{(1)} = -a^2W^{(1)} - 2a(b + d)W^{(0)}$$

$$D_yW^{(1)} = \left[bW^{(1)} - (a + d)W^{(0)}\right]_{\beta = \frac{\pi}{2}} \qquad D_{yy}W^{(1)} = -b^2W^{(1)} + 2b(a + d)W^{(0)}$$

$$D_zW^{(1)} = \left[dW^{(1)} - (a - b)W^{(0)}\right]_{\delta = \frac{\pi}{2}} \qquad D_{zz}W^{(1)} = -d^2W^{(1)} + 2d(a - b)W^{(0)}, \tag{66}$$

$$D_xW^{(2)} = \left[aW^{(2)} + 2(b + d)W^{(1)} - (2a - b + d)W^{(0)}\right]_{\alpha = \frac{\pi}{2}}$$

$$D_yW^{(2)} = \left[bW^{(2)} - 2(a + d)W^{(1)} - (2b + d - a)W^{(0)}\right]_{\beta = \frac{\pi}{2}}$$

$$D_z W^{(2)} = \left[dW^{(2)} - 2(a - b)W^{(1)} - (2d + a + b)W^{(0)} \right]_{\delta = \frac{\pi}{2}}$$

$$D_{xx} W^{(2)} = - a^2 W^{(2)} - 4a(b + d)W^{(1)} - \left[2(b + d)^2 - 2a(2a - b + d) \right] W^{(0)}$$

$$D_{yy} W^{(2)} = - b^2 W^{(2)} + 4b(a + d)W^{(1)} - \left[2(a + d)^2 - 2b(2b + d - a) \right] W^{(0)}$$

$$D_{zz} W^{(2)} = - d^2 W^{(2)} + 4d(a - b)W^{(1)} - \left[2(a - b)^2 - 2d(2d + a + b) \right] W^{(0)}. \quad (67)$$

Thus in general the partial differential operator of any degree will, when operating upon a nonseparable of order n, produce a sum of nonseparables which contains order n plus all lesser orders.

APPENDIX V, OTHER THREE-DIMENSIONAL PATTERNS

The function $W^{(2)}$ is derived from $W^{(0)}$ with two applications of the operator O_{3D} defined in Appendix IV. The first variant ($\alpha = \beta = \delta = 0$) with a = b = 1 has been investigated for nodal patterns in several x-y planes. When this is written in a manner to illustrate most clearly the influence of z upon the nodal pattern,

$$W^{(2)} = 2 \sin z \, [2(x^2 + y^2 + z \cot z) \sin x \sin y + x \cos x \sin y + y \sin x \cos y + 4xy \cos x \cos y] = 0. \quad (68)$$

From this it can be seen that $W^{(2)} = 0$ at $z = 0$ for all x and y and that a grid pattern of $\sin x \sin y = 0$ appears at $z = \pi$. Intermediate patterns can be found by finding the (x, y) roots which make the square bracket equal to zero for each value of z and hence of z cot z. The following table shows the substitution to be made.

VALUES OF Z COT Z FOR SELECTED VALUES OF Z

z	z cot z	z	z cot z
Approaching 0	1		
$\pm\pi/4$	$\pi/4$	$\pm 0.8\pi$	$-0.8\pi \cot 0.2\pi = -3.46$
$\pm\pi/2$	0	$\pm 0.85\pi$	$-0.85\pi \cot 0.15\pi = -5.24$
$\pm 3\pi/4$	$-3\pi/4$	$\pm 0.9\pi$	$-0.9\pi \cot 0.10\pi = -8.70$
$\pm\pi$	$\mp\infty$	$\pm 0.95\pi$	$-0.95\pi \cot 0.05\pi = -18.84$
$\pm 5\pi/4$	$5\pi/4$	$\pm 1.05\pi$	$+1.05\pi \cot 0.05\pi = +20.83$
$\pm 1.417\pi$	1	$\pm 1.1\pi$	$+1.1\pi \cot 0.10\pi = +10.64$
$\pm 3\pi/2$	0		

In approaching the selection of z values to display transition, there were few guidelines. As a consequence, the procedure was to select a value, evaluate a pattern, gauge its probable interpolative position, and choose another value. Therefore the z values actually used only partially correspond to those of the preceding table.

Nodal patterns which have been calculated are presented in Figure 10(a) and Figure 10(b). Each pattern is for $W^{(2)} = 0$ in an x-y plane at a particular value of z between 0 and π, and the x and y values of each pattern range from -3.5π to $+3.5\pi$. The x-y plane through $z = 0$ is a nodal surface, as are the y-z and z-x planes through $x = 0$ and $y = 0$, respectively. Wavy nodal surfaces rise wall-like from the floor, i.e., from the x-y plane through $z = 0$, change but little in reaching $z = 3\pi/4$, and then alter drastically in finally connecting with the grid pattern of $z = \pi$.

REPEATED PATTERNS

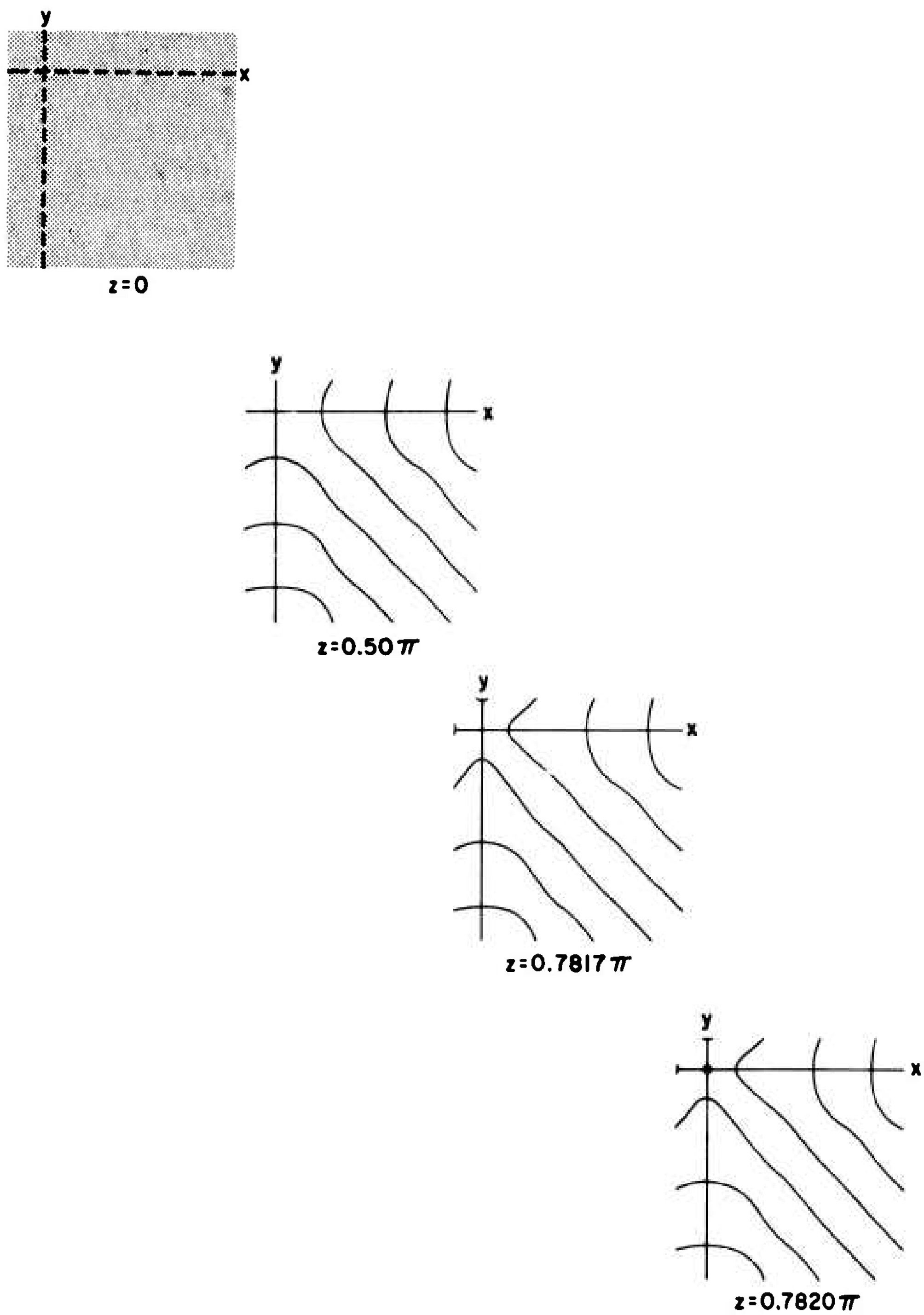
Consideration of Equation (68) and the table of values of z cot z which are inserted into Equation (68) shows that a pattern at a given value of z will be repeated for all other values of z which give the same value of z cot z. From the table, the locations of two duplicate patterns are shown to be at z approaching 0 and at $z = 1.417\pi$, where $z \cot z = 1$, and at $z = \pi/2$ and $z = 3\pi/2$, where $z \cot z = 0$.

A general curve which may be used to locate all identical patterns is presented in Figure 11. This is a plot of z cot z versus z which enables one to generalize the z-scale through the use of the parameter $n = 1, 2, 3\ldots$. For example, let the z-location near $z = 13.5 = 4.3\pi$ of a pattern identical with the pattern at $z = 4.0 = 1.27\pi$ be desired. In the organization of Figure 11, $z = 1.27\pi$ lies between $z = \pi$ and 2π so that $n = 2$, and $z = 4.3\pi$ lies between $z = 4\pi$ and 5π so that $n = 5$. It is then a simple matter to read from Figure 11 that the intersection of $z = 4.42\pi$ gives the same value of z cot z on the curve of $n = 5$ that the intersection of $z = 1.27\pi$ gives on the curve of $n = 2$.

Historically, it was the pattern of $z = .9919\pi$ in Figure 10(b) which led to the production of closed areas on membranes, for from it came recognition that Equation (68) consisted of second and zero order two-dimensional non-separables (plus a residual of secondary importance at low values of x and y) in a proportion which depended upon the value of z. Rewritten to emphasize this, Equation (68) is

$$\left[W_{3D}\right]_{z\,=\,\mathrm{const}} = K\left[\left(W_1^{(2)} + CW_1^{(0)}\right)_{2D} - \frac{1}{2}\left(y \sin x \cos y + x \cos x \sin y\right)\right] = 0 \quad (69)$$

where $K = 2 \sin z$ and $C = \frac{1}{2} z \cot z$ is a number which at $z = .99\pi$ has grown large enough to allow the pattern of $W^{(0)}$ to dominate that of the sum.



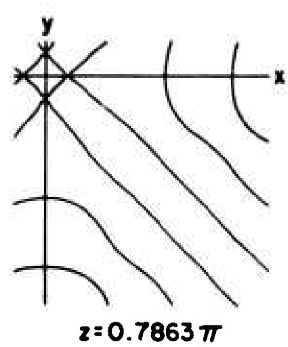


Figure 10. Nodal Patterns in Selected z-Planes for the Three-Dimensional, Operator-Generated $W^{(2)}$. (Conditions are $a = b = d = 1$ and $\alpha = \beta = \delta = 0$. The entire x-y plane for $z = 0$ is a nodal plane.)

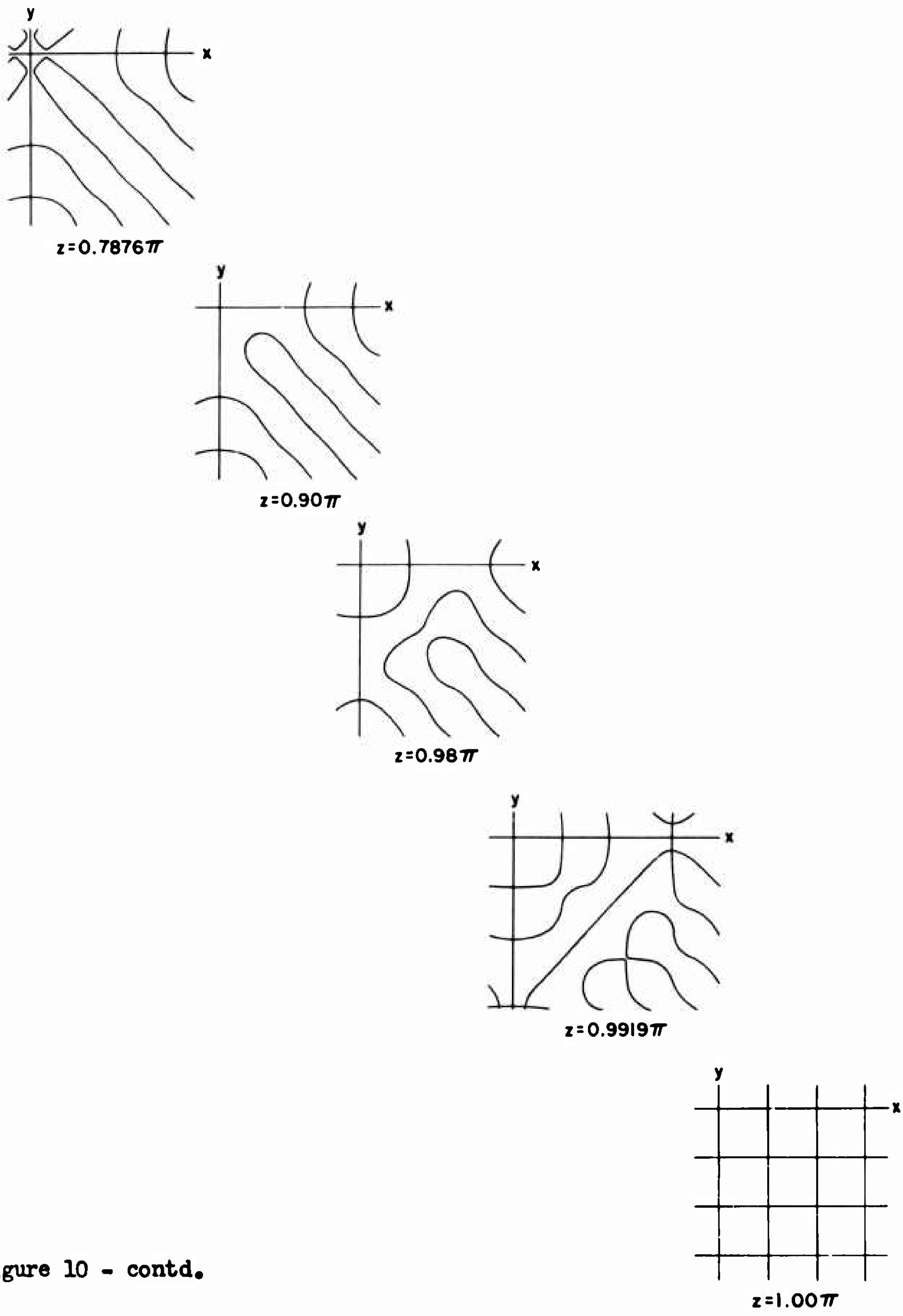


Figure 10 - contd.

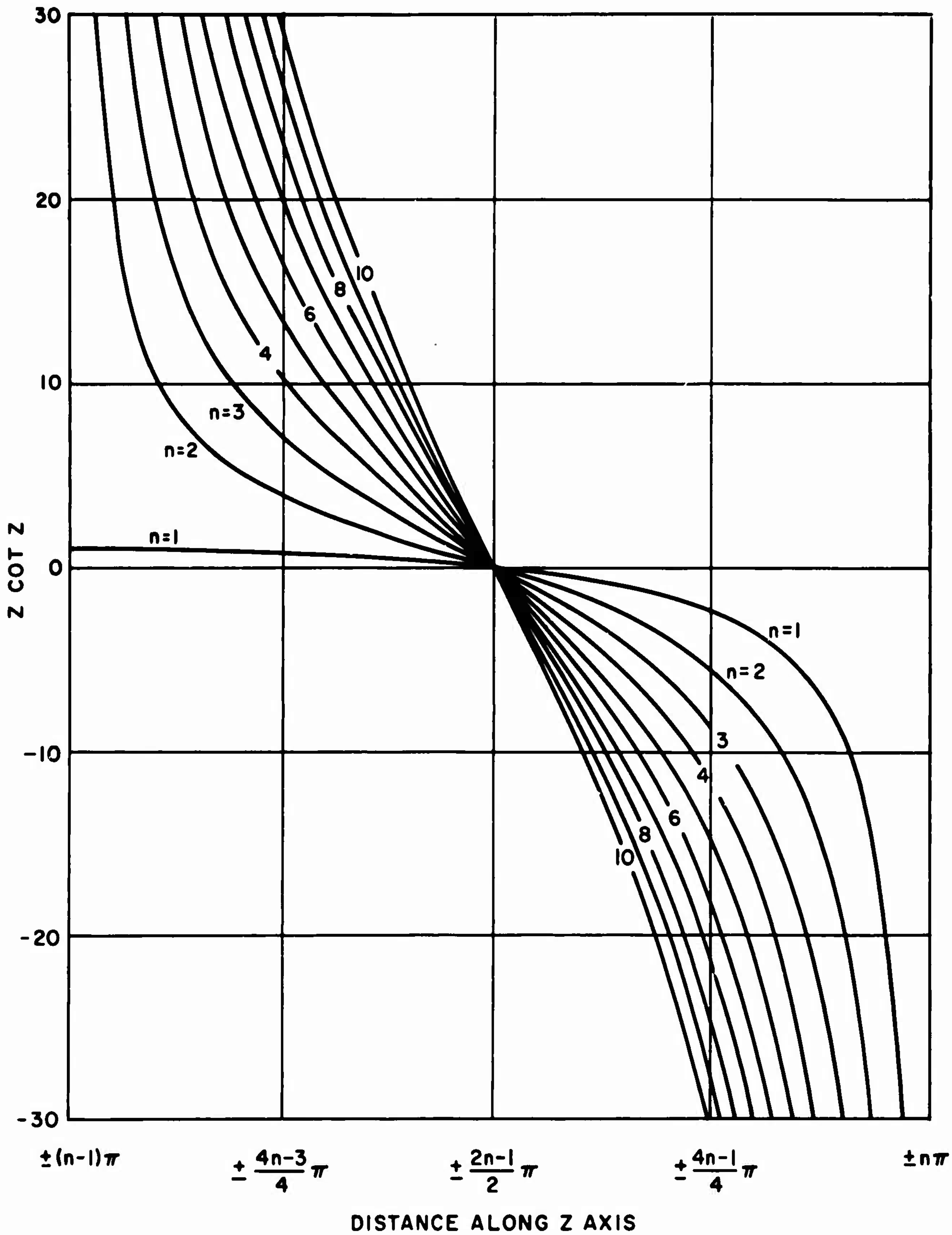


Figure 11. Generalized Graph of z cot z versus z for Values of z Between 0 and ±10π. (A horizontal line corresponding to a given value of z cot z intersects the curves at values of z where the nodal patterns are identical.)

APPENDIX VI, VECTOR EFFORTS

Boundary conditions for electromagnetic cavities are restraints upon the electric and magnetic fields which are vectors. A common boundary condition for a gas-or-liquid-filled acoustic cavity is that the particle velocity normal to the walls be zero, and particle velocity is a vector. Therefore the investigation of nonseparable solutions in respect to cavities required a study of the nonseparables in relation to vectors.

ACOUSTIC CAVITIES

This study began by treating nonseparables as velocity potentials. Particle velocity is obtained by taking the gradient of the potential. For rigid wall the requirement is that a closed surface exist upon which the gradient is zero, and hence each component of the gradient must simultaneously be zero. The x-components of particle velocity for three cases are

$$v_x = \frac{\partial W_{2D}^{(2)}(\alpha,\beta)}{\partial x} = a\left[W_{2D}^{(0)}(\alpha',\beta) - W_{2D}^{(2)}(\alpha',\beta)\right] - 2bW_{2D}^{(1)}(\alpha',\beta), \tag{70}$$

$$v_x = \frac{\partial W_{3D}^{(1)}(\alpha,\beta,\delta)}{\partial x} = aW_{3D}^{(1)}(\alpha',\beta,\delta) + (b+d)W_{3D}^{(0)}(\alpha',\beta,\delta), \tag{71}$$

$$v_x = \frac{\partial\varphi}{\partial x}. \tag{72}$$

The velocity potentials $W_{2D}^{(2)}(\alpha,\beta)$ and $W_{3D}^{(1)}(\alpha,\beta,\delta)$ are operator-derived from zero order in two and three dimensions, respectively. Application of the partial derivative changes α to $\alpha' = \alpha - \frac{\pi}{2}$. The propagation constants a, b, d are related through the frequency equation $\omega^2/c^2 = a^2 + b^2 + d^2$. (The constant d is zero in Equation (70).)

The velocity potential φ is the symmetric function given in Equation (22).

Nodal patterns of Equations (70) and (72) were evaluated, the former under the conditions that $\alpha = \beta = 0$ and $a = b = 1$, and are presented as Figures 12(a) and 12(b), respectively. In Figure 12(a) no closed areas are present. The pattern is symmetrical about the x and y axes but not about the ±45° lines through the origin. In Figure 12(b) every nodal line is closed. This x-y pattern repeats for $z = \pm m\pi$ with $m \neq 0$. At $m = 0$ the entire x-y plane is a nodal surface. The pattern is symmetrical about the x and y axes but not about the ±45° lines through the origin. Nodal patterns of v_y and v_z at $z = \pm m\pi$ differ from Figure 12(b), and no surface has been found which satisfies the requirement that the gradient be zero upon it.

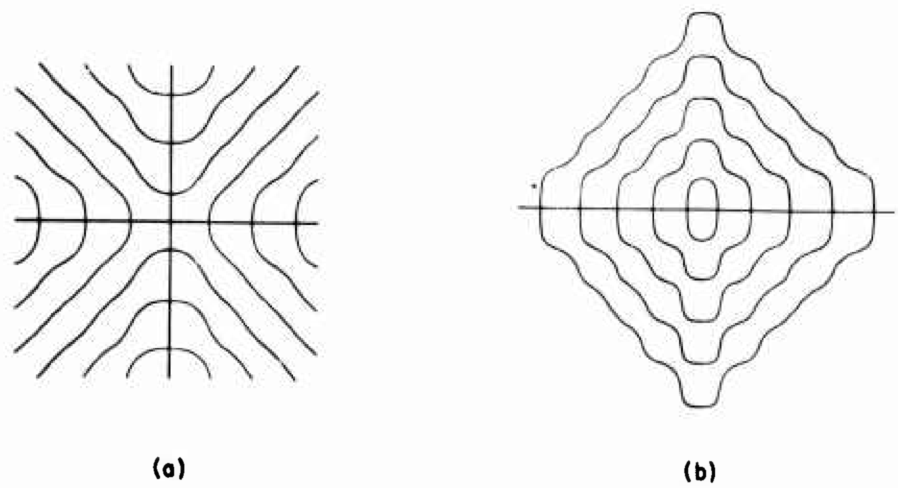


Figure 12. Nodal Patterns of x-Component of Velocity.

(a) v_x given by Equation (70);

(b) v_x given by Equation (72) in x-y planes given by $z = \pm m\pi$, $m \neq 0$.

ELECTROMAGNETIC CAVITIES

The development by which $\vec{E}$ and $\vec{H}$ field components turned out to be separable solutions when the electric potential was nonseparable is given in general form.

We let

$$\varphi = W^{(n)} e^{i\omega t}, \tag{73}$$

where $W^{(n)}$ is the operator-derived nonseparable of nth order, and choose to examine a special case, one in which

$$\frac{\partial A_x}{\partial x} = \frac{\partial A_y}{\partial y} = \frac{\partial A_z}{\partial z}. \tag{74}$$

Equation (27) may then be written

$$\frac{\partial A_x}{\partial x} = \frac{\partial A_y}{\partial y} = \frac{\partial A_z}{\partial z} = -\frac{i\omega\mu\epsilon}{3} W^{(n)} e^{i\omega t}. \tag{75}$$

Therefore,

$$A_x = -\frac{i\omega\mu\epsilon}{3} e^{i\omega t} \int W^{(n)} dx, \tag{76}$$

and similarly for A_y and A_z. (Constants of integration, including functions of variables other than the variable of integration, are dropped as incapable of satisfying the same Helmholtz wave equation that $W^{(n)}$ obeys.)

Then, by Equation (25), the electric field components are

$$E_x = -\left\{ \frac{\partial W^{(n)}}{\partial x} + \frac{k^2}{3} \int W^{(n)} dx \right\} e^{i\omega t}$$

$$E_y = -\left\{ \frac{\partial W^{(n)}}{\partial y} + \frac{k^2}{3} \int W^{(n)} dy \right\} e^{i\omega t}$$

$$E_z = -\left\{ \frac{\partial W^{(n)}}{\partial z} + \frac{k^2}{3} \int W^{(n)} dz \right\} e^{i\omega t}, \tag{77}$$

and, by Equation (26), the magnetic field components are

$$H_x = -\frac{ik^2}{3\omega\mu} \left\{ \frac{\partial}{\partial y} \int W^{(n)} dz - \frac{\partial}{\partial z} \int W^{(n)} dy \right\} e^{i\omega t}$$

$$H_y = -\frac{ik^2}{3\omega\mu} \left\{ \frac{\partial}{\partial z} \int W^{(n)} dx - \frac{\partial}{\partial x} \int W^{(n)} dz \right\} e^{i\omega t}$$

$$H_z = -\frac{ik^2}{3\omega\mu} \left\{ \frac{\partial}{\partial x} \int W^{(n)} dy - \frac{\partial}{\partial y} \int W^{(n)} dx \right\} e^{i\omega t}. \tag{78}$$

APPENDIX VII, NONSEPARABLE SOLUTIONS IN POLAR COORDINATES

Nonseparable solutions were first found in circular cylindrical coordinates during work in which symmetry about the axis was assumed. During this contract nonseparable solutions in two and three dimensions in rectangular coordinates have been explored. Since some model configurations would be best expressed in polar coordinates, e.g., thin circular plates, it was decided to seek nonseparable solutions in r and θ.

The form of the separable solution in r and θ, i.e., $\cos(m\theta)J_m(pr)$, suggests that a nonseparable solution would consist of a sum of products of functions with r to some power and θ to some power appearing explicitly in the various products.

In an exploratory effort, the two-dimensional $W_1^{(1)}$ in rectangular coordinates was transformed directly into polar coordinates. It was then of the form

$$W_1^{(1)} = br\cos\theta\cos(ar\cos\theta)\sin(br\sin\theta) - ar\sin\theta\sin(ar\cos\theta)\cos(br\sin\theta)$$

$$= br\cos\theta\left[J_o(ar) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(ar)\cos 2n\theta\right]\cdot\left[2\sum_{n=0}^{\infty}J_{2n+1}(br)\sin(2n+1)\theta\right] -$$

$$ar\sin\theta\left[2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(ar)\cos(2n+1)\theta\right]\cdot\left[J_o(br) + 2\sum_{n=1}^{\infty}J_{2n}(br)\cos 2n\theta\right]. \qquad (79)$$

This function was verified as a solution of the wave equation in polar coordinates.

However, all efforts to synthesize a polar nonseparable solution of two terms in analogy to the first order in rectangular coordinates have failed. The difficulty may stem from the fact that the functions which would possess the characteristics suggested in the second paragraph would be multivalued at every part in the r, θ plane because of the cyclical nature of θ.

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A conventional solution of the Helmholtz or time-reduced wave equation is a simple product of functions that contain one coordinate variable in each. An unbounded set of solutions that are not separable into simple products of single-variable functions has been partially examined for applicability to vibrational problems. Applications to scalar usage have been found, and illustrations including shapes and frequencies for membranes and an acoustic cavity are reported. Efforts to make application to vector usage are described, as are numerous mathematical properties that have been discovered in the course of the work. It is concluded that vibration on or within some new shapes can now be calculated exactly with functions formed of the nonseparable solutions added to separable solutions. It is also concluded that simplifications in the mathematics and additional applications await the effort.

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	ROLE	WT	ROLE	WT	ROLE	WT
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