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MATRIX METHODS FOR SOLVING FIELD PROBLEMS

Volume I – Matrix Techniques and Applications

Roger F. Harrington
Long-Fei Chang
Arlon T. Adams, et al

Syracuse University

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FOREWORD

This report was prepared by Syracuse University, Electrical Engineering Department, Syracuse, New York under Contract No. AF30(602)-3724, Project No. 4519; Task No. 451901, covering the period March 1965 through March 1966.

RADC Project Engineer was John J. Patti (EMCRR):

Chapters I, II and III, on Basic Concepts, Formulation of Electromagnetic Problems, and Wire Antennas and Scatterers of Arbitrary Shape, were written by Roger F. Harrington. Chapter IV, on Calculations for Linear Wires, was written by Joseph Mautz and R. F. Harrington. Chapter V, on Scattering by Conducting Cylinders, was written by Robert Wallenberg, and Chapter VI, on Scattering by Dielectric Cylinders, was written by Arlon Adams. The next chapter on Scattering by Bodies of Revolution is by Thomas Bristol, and the final chapter on Matrix Inversion is by Long-Fei Chang. The material in each chapter is not necessarily the sole work of the author, since the project was a coordinated effort of all persons working on it.

This technical report has been reviewed and is approved.

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ABSTRACT

This report presents general procedures for solving field problems of engineering interest using digital computer techniques. The basic concept is to represent a boundary-value problem by a superposition integral, approximate the integral equation by a matrix equation, and invert the matrix for a solution. The theory is described in terms of the method of moments, which is equivalent to the variational method.

For electromagnetic antenna and scattering problems, the method gives a matrix whose elements can be interpreted as generalized impedances. These impedances are closely related to those used in the theory of loaded antennas and scatterers, and hence such loaded structures can also be treated. A solution for wire antennas and scatterers of arbitrary shape is formulated in detail, and calculations for linear wire antennas and scatterers, both loaded and unloaded, have been made.

Additional problems treated by these procedures are two-dimensional scattering by conducting cylinders and by dielectric cylinders, and three-dimensional scattering by bodies of revolution. These problems are used to show the effect of various approximations in the solution, in an attempt to draw some general conclusions as to the best approximations. Special procedures for inverting matrices have also been considered, to take into account any symmetry properties present in the matrices. A considerable saving in computation time can often be made by properly utilizing these symmetries.

CONTENTS

EVALUATION

The objective of this effort was to develop practical techniques for solving a wide class of electromagnetic scattering problems.

A unified and general approach is taken to electromagnetic field problems in terms of numerical analysis. The approach results in matrix equations representing the solution of the integro-differential equations of the problem being analyzed. The accuracy of the solution obtained is very good for most engineering purposes. In some cases, specifically that case of the linear wire antenna, it appears that the solution is more accurate than any previously obtained. The unified viewpoint developed in this effort contributes additional insight into the solution of electromagnetic scattering problems, and establishes a general framework with which to conduct the analysis. Techniques and some applications of the method of approach are described in Volume I of the final report.

The results of this effort are directly applicable to the problem of the linear wire antenna, in that computer programs and complete numerical results are presented in Volume II of the final report.

John J. Patti

JOHN J. PATTI
Project Engineer

MATRIX METHODS FOR SOLVING FIELD PROBLEMS

I. BASIC CONCEPTS

A. Introduction. The object of this project* is to develop practical techniques for solving electromagnetic scattering problems using digital computers. The general approach is to approximate the integro-differential equations describing the boundary value problem by matrix equations and to invert the matrix equations.

The theory is conveniently discussed using the concepts of linear spaces. The equations of interest are of the general type

$$L(f) = g \qquad (1\text{-}1)$$

where L is a known linear operator, g is a known function, and f is the unknown to be determined. The general procedure will be to take an equation involving functions, such as (1-1), and to approximate it by a matrix equation. The solution to an inhomogeneous matrix equation is found by matrix inversion. Most computer libraries have subroutines for matrix inversion, and hence a problem will be considered solved once the elements of a well conditioned matrix are evaluated.

B. Formulation of Problems. General procedures for solutions will be discussed in the language of linear spaces and operators. Hence, the problems to be considered should be formulated in this notation. Given a deterministic problem, we wish to put it into the form $L(f) = g$, identifying the operator L, its domain (the functions f on which it operates), and its range (the functions g resulting from the operation).

Furthermore, we usually need an inner product $\langle f,g\rangle$, defined to satisfy[1]

$$\langle f,g\rangle = \langle g,f\rangle \qquad (1\text{-}2)$$

$$\langle \alpha f + \beta g,h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle \qquad (1\text{-}3)$$

$$\begin{aligned} \langle f^*,f\rangle &> 0 \qquad \text{if } f \neq 0 \\ &= 0 \qquad \text{if } f = 0 \end{aligned} \qquad (1\text{-}4)$$

where α and β are scalars, and * denotes complex conjugate. We sometimes need the adjoint operator L^a and its domain, defined by

$$\langle Lf,g\rangle = \langle f,L^a g\rangle \qquad (1\text{-}5)$$

for all f in the domain of L. An operator is self adjoint if $L^a = L$ and the domain of L^a is that of L.

Properties of the solution depend upon properties of the operator. An operator is real if Lf is real whenever f is real. An operator is positive definite if

$$\langle f^*,Lf\rangle > 0 \qquad (1\text{-}6)$$

for all $f \neq 0$ in its domain. It is positive semidefinite if > is replaced by $\geq$ in (1-6), negative definite if > is replaced by < in (1-6), etc. We shall identify other properties of operators as we need them.

If the solution to $L(f) = g$ exists and is unique for all g, then the inverse operator L^{-1} exists such that

$$f = L^{-1}(g) \qquad (1\text{-}7)$$

If g is known, then (1-7) represents the solution to the original problem. However, (1-7) is itself an inhomogeneous equation for g if f is known, and its solution is $L(f) = g$.

[1] The usual definition of inner product in Hilbert space corresponds to $\langle f^*,g\rangle$ in our notation. For this report it is more convenient to show the conjugate operation explicitly wherever it occurs, and to define the adjoint operator without conjugation.

In general, it is desirable to think of L and L^{-1} as a pair of operators, each one of which is the inverse of the other. The choice in any particular problem is one of convenience.

C. Example: Electrostatics. Consider a volume density of charge $\rho(x,y,z)$ in unbounded Euclidean space of constant permittivity ϵ, and its associated electrostatic potential $\phi(x,y,z)$. The differential equation for the problem is Poisson's equation

$$-\epsilon \nabla^2 \phi = \rho \tag{1-8}$$

subject to the boundary condition that $r\phi \to C_1$ as $r \to \infty$ for every ρ of finite extent and where C_1 is a constant. In operator notation, (1-8) is

$$L\phi = \rho \tag{1-9}$$

where

$$L = -\epsilon \nabla^2 \tag{1-10}$$

subject to the above boundary condition. The well-known solution to (1-8) is

$$\phi(x,y,z) = \iiint \frac{\rho(x',y',z')}{4\pi\epsilon R} \, dx' \, dy' \, dz' \tag{1-11}$$

where $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ is the distance from a source point (x',y',z') to a field point (x,y,z). Hence, the inverse operator is

$$L^{-1} = \iiint dx' \, dy' \, dz' \, \frac{1}{4\pi\epsilon R} \tag{1-12}$$

It is important to realize that (1-12) is inverse to (1-10) only for the stated boundary conditions. The boundary conditions of a differential operator should be considered an integral part of the operator, and if they are changed the operator is changed. Hence, L^{-1} will also be different if the boundary conditions of L are changed.

A suitable inner product for electrostatic problems is

$$\langle \phi_1, \phi_2 \rangle = \iiint \phi_1(x,y,z)\ \phi_2(x,y,z)\ dx\ dy\ dz \qquad (1\text{-}13)$$

That (1-13) satisfies (1-2), (1-3), and (1-4) is easily verified. The choice of inner product is not unique. For example, the integrand of (1-13) could be multiplied by an arbitrary positive function (weighting function) $w(x, y, z) > 0$ and it would still be an acceptable scalar product. However, the particular choice (1-13) makes the operators L and L^{-1} self adjoint, as we shall now show.

Let ϕ_1 and ϕ_2 represent arbitrary functions in the domain of L, and form the left-hand side of (1-5)

$$\langle L\phi_1, \phi_2 \rangle = \iiint (-\epsilon \nabla^2 \phi_1)\ \phi_2\ d\tau \qquad (1\text{-}14)$$

where $d\tau = dx\ dy\ dz$. Green's identity is

$$\iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\ d\tau = \oiint_S (\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n})\ ds \qquad (1\text{-}15)$$

where S is the surface bounding the volume V. We consider S to be a sphere of radius r, such that in the limit $r \to \infty$ the volume V includes all space. Let $\phi = \phi_1$ and $\psi = \phi_2$. The boundary condition is $r\phi \to C_1$ as $r \to \infty$, which requires $r^2 \partial\phi/\partial n \to C_I$ as $r \to \infty$. Since ds varies only as r^2, the surface integral in (1-15) vanishes, and we have

$$\iiint \phi_2 \nabla^2 \phi_1\ d\tau = \iiint \phi_1 \nabla^2 \phi_2\ d\tau \qquad (1\text{-}16)$$

Since ϵ is constant, (1-14) is symmetrical in ϕ_1 and ϕ_2, and

$$\langle L\phi_1, \phi_2 \rangle = \langle \phi_1, L\phi_2 \rangle \tag{1-17}$$

We required ϕ_2 to satisfy the same boundary conditions as ϕ_1, hence L is self adjoint. If L is self adjoint, so is L^{-1}, since

$$\langle L\phi_1, \phi_2 \rangle = \langle \rho_1, L^{-1}\rho_2 \rangle \tag{1-18}$$

The mathematical concept of self adjointness of an operator corresponds to the physical principle of reciprocity.

It is evident from (1-10) and (1-12) that L and L^{-1} are real operators. We shall show that they are also positive definite, that is, they satisfy (1-6). Again we need to show it only for L or L^{-1}, since (1-18) is valid for $\phi_1 = \phi_2$. (The conjugate operation is not needed since L is real.) Form

$$\langle \phi, L\phi \rangle = \iiint \phi(-\epsilon \nabla^2\phi) d\tau \tag{1-19}$$

and use the vector identity

$$\phi\nabla^2\phi = \nabla \cdot \phi\nabla\phi - \nabla\phi \cdot \nabla\phi \tag{1-20}$$

and the divergence theorem. The result is

$$\langle \phi, L\phi \rangle = \iiint_V \epsilon \nabla\phi \cdot \nabla\phi \, d\tau - \oiint_S \epsilon \phi\nabla\phi \cdot ds \tag{1-21}$$

where S bounds V. Again take S a sphere of infinite radius, and the last term of (1-21) vanishes because of the boundary condition $r\phi \to C_1$ as $r \to \infty$. Now

$$\langle \phi, L\phi \rangle = \iiint \epsilon(\nabla\phi)^2 \, d\tau \tag{1-22}$$

and, since $\epsilon > 0$, L is positive definite. The mathematical concept of positive definiteness is often related to the physical concepts of work or energy. In (1-19), the right-hand side is proportional to the electrostatic energy.

D. Method of Moments. We now discuss a general procedure for solving linear equations, called the method of moments. Consider the inhomogeneous equation

$$L(f) = g \tag{1-23}$$

where L is a linear operator, g is known and f is to be determined. Let f be expanded in a series of functions $f_1, f_2, f_3, \ldots$ in the domain of L, as

$$f = \sum_i \alpha_i f_i \tag{1-24}$$

where the α_i are constants. We shall call the f_i expansion functions or basis functions. For exact solutions, (1-24) is normally an infinite summation and the f_i form a complete set of basis functions. For approximate solutions, (1-24) is a finite summation. Substituting (1-24) into (1-23), and using the linearity of L, we have

$$\sum_i \alpha_i \, L(f_i) = g \tag{1-25}$$

It is assumed that a suitable inner product $\langle f,g \rangle$ has been determined for the problem. Now define a set of weighting functions, or testing functions, $w_1, w_2, w_3, \ldots$ in the range of L, and take the inner product of (1-25) with each w_j. The result is

$$\sum_i \alpha_i \langle w_j, Lf_i \rangle = \langle w_j, g \rangle \tag{1-26}$$

$j = 1, 2, 3, \ldots$ This set of equations can be written in matrix form as

$$[\ell_{ji}]\ [\alpha_i] = [g_j] \tag{1-27}$$

where

$$[\ell_{ji}] = \begin{bmatrix} \langle w_1, Lf_1 \rangle & \langle w_1, Lf_2 \rangle & \cdots \\ \langle w_2, Lf_1 \rangle & \langle w_2, Lf_2 \rangle & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{1-28}$$

$$[\alpha_i] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} \qquad [g_j] = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \vdots \end{bmatrix} . \tag{1-29}$$

If the matrix $[\ell]$ is nonsingular its inverse $[\ell^{-1}]$ exists. The α_i are then given by

$$[\alpha_i] = [\ell_{ij}^{-1}]\ [g_j] \tag{1-30}$$

and the solution for f is given by (1-24). For concise expression of this result, define the matrix of functions

$$[\tilde{f}_i] = [f_1\ f_2\ f_3\ \ldots] \tag{1-31}$$

and write

$$f = [\tilde{f}_i]\ [\alpha_i] = [\tilde{f}_i]\ [\ell_{ij}^{-1}]\ [g_j] \tag{1-32}$$

This solution may be exact or approximate, depending upon the choice of the f_i and w_i.

If the matrix $[\ell]$ is of infinite order, it can be inverted only in special cases, for example, if it is diagonal. The classical eigenfunction method leads to a diagonal matrix, and can be thought of as a special case of the method of moments. If the sets f_i and w_i are finite, the matrix is of finite order and can be inverted by known methods.

A principal task in the solution of any particular problem is the choice of f_i and w_i. The f_i should be chosen so that they are relatively independent functions and so that some superposition (1-24) can approximate f reasonably well. The w_i should be chosen so that they are also relatively independent and so that the products $\langle w_i, g\rangle$ test relatively independent properties of g. We shall say more about this in Section I-F when we discuss the stationary nature of the solution.

Some additional factors which affect the choice of f_i and w_i are (1) the accuracy of solution desired, (2) the ease of evaluation of the matrix elements, (3) the size of the matrix that can be inverted, and (4) the realization of a well conditioned matrix [ℓ].

E. Example: Charged Conducting Plate.[1] Figure 1-1 represents a square conducting plate, 2a meters on a side, lying in the plane z = 0 with center at the origin. Let $\sigma(x,y)$ represent the surface charge density on the plate, assumed infinitely thin. The electrostatic potential at any point in space is

$$\phi(x,y,z) = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \, \frac{\sigma(x',y')}{4\pi\epsilon R} \qquad (1\text{-}33)$$

where $R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$. The boundary condition is $\phi = V$ (constant) on the plate. The integral equation for the problem is therefore

$$V = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \, \frac{\sigma(x', y')}{4\pi\epsilon \sqrt{(x-x')^2 + (y-y')^2}} \qquad (1\text{-}34)$$

$|x| < a$, $|y| < a$. The charge density $\sigma(x,y)$ is the unknown to be determined. A parameter of interest is the capacitance

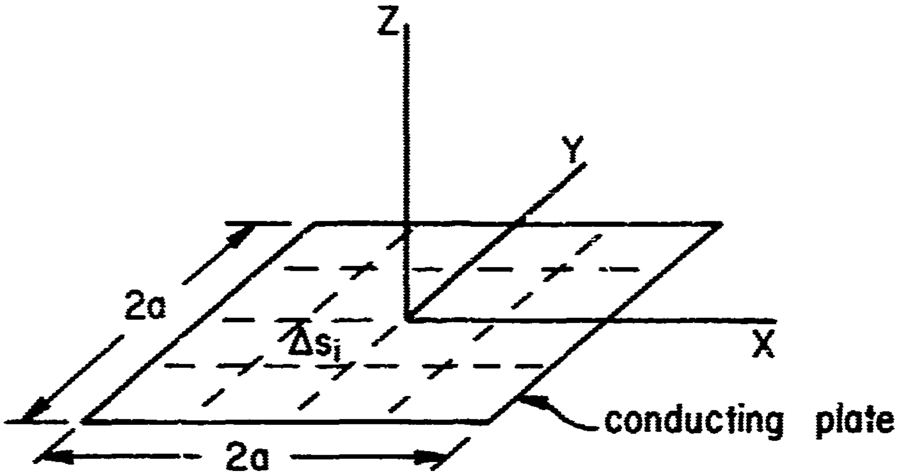

Fig. I-1 A square conducting plate.

$$C = \frac{q}{V} = \frac{1}{V} \int_{-a}^{a} dx \int_{-a}^{a} dy \ \sigma(x,y) \qquad (1\text{-}35)$$

which can be calculated once $\sigma(x,y)$ is found.

For an approximate solution, consider the plate to be divided into N square subareas, as shown in Fig. 1-1. Let the subareas be denoted $\Delta s_1, \Delta s_2, \ldots, \Delta s_N$. Define functions

$$f_i = \begin{cases} 1 & \text{on } \Delta s_i \\ 0 & \text{on all other } \Delta s_j \end{cases} \qquad (1\text{-}36)$$

and let

$$\sigma(x,y) \approx \sum_{i=1}^{N} \alpha_i \ f_i \qquad (1\text{-}37)$$

where α_i are constants to be evaluated. In other words, we are approximating σ by a constant over each subarea. If (1-37) is substituted into the integral equation (1-34) and the resultant equation satisfied at the center (x_j,y_j) of each Δs_j, we obtain the matrix of equations

$$V = \sum_{i=1}^{N} \ell_{ji}\alpha_i \qquad j = 1, 2, \ldots, N \qquad (1\text{-}38)$$

where

$$\ell_{ji} = \int_{\Delta x_i} dx' \int_{\Delta y_i} dy' \ \frac{1}{4\pi\epsilon\sqrt{(x_j-x')^2 + (y_j-y')^2}} \qquad (1\text{-}39)$$

Note that ℓ_{ji} is the potential at the center of Δs_j due to a unit charge density over Δs_i. A solution of the simultaneous equations (1-38) gives an approximation to the charge density on the plate according to (1-37). The corresponding approximation to (1-35)

$$C \approx \frac{1}{V} \sum_{i=1}^{N} \alpha_i \, \Delta s_i \tag{1-40}$$

gives the capacitance of the plate.

To put the above solution into the notation of the method of moments, let

$$f(x,y) = \sigma(x,y) \tag{1-41}$$

$$g(x,y) = V \tag{1-42}$$

$$L(f) = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \; \frac{f(x',y')}{4\pi\epsilon \sqrt{(x-x')^2 + (y-y')^2 + z^2}} \tag{1-43}$$

Then (1-23) is equivalent to (1-34). Define the inner product as

$$\langle f,g \rangle = \int_{-a}^{a} dx \int_{-a}^{a} dy \; f(x,y) \; g(x,y) \tag{1-44}$$

The unknown $f = \sigma$ is approximated by (1-37) where the f_i are defined by (1-36). The weighting functions are defined by

$$\langle w_j,f \rangle = \int_{-a}^{a} dx \int_{-a}^{a} dy \; w_j(x,y) \; f(x,y) = f(x_j,y_j) \tag{1-45}$$

These w_j do not exist as ordinary functions, but are symbolic functions. In particular,

$$w_j(x,y) = \delta(x-x_j) \; \delta(y-y_j) \tag{1-46}$$

where $\delta(x)$ is the Dirac delta function. Now the elements of the matrix (1-28) are given by (1-39), and those of (1-29) are

$$[g_j] = \begin{bmatrix} V \\ V \\ \vdots \\ V \end{bmatrix} \tag{1-47}$$

Hence, (1-27) is equivalent to (1-38), and (1-30) represents the solution.

For numerical results, the ℓ_{ji} of (1-39) must be evaluated. Let $2b = 2a/\sqrt{N}$ denote the side length of each Δs_i. The potential at the center of Δs_i due to unit charge density over its own surface is

$$\ell_{ii} = \int_{-b}^{b} dx \int_{-b}^{b} dy \frac{1}{4\pi\epsilon\sqrt{x^2 + y^2}}$$

$$= \frac{2b}{\pi\epsilon} \ln (1 + \sqrt{2}) = \frac{2b}{\pi\epsilon} (0.88137) \qquad (1\text{-}48)$$

This derivation used Dwight[2] 200.01 and 731.2. The potential at the center of Δs_j due to unit charge density over Δs_i can be similarly evaluated, but for most purposes need not be. Usually it is sufficiently accurate to consider the charge on Δs_i as a point charge and use

$$\ell_{ji} \approx \frac{q_i}{4\pi\epsilon R_{ij}} = \frac{b^2}{\pi\epsilon\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}} \qquad (1\text{-}49)$$

This approximation is 3.8 percent in error for adjacent subareas, and has even less error for nonadjacent ones. Table 1-1 shows calculations solving (1-38) and (1-39) for various numbers of subareas. The second column uses approximation (1-49), the third column evaluates ℓ_{ji} more precisely. A good estimate of the exact capacitance is $C/2a = 4000$ micromicrofarads. Figure 1-2 shows a plot of the charge density along the subareas nearest the center line of the square plate. Note that σ exhibits the well-known square root singularity at the edges of the plate.

This example illustrates two simple but useful approximations, (1) the use of expansion functions f_i each of which exists over only a

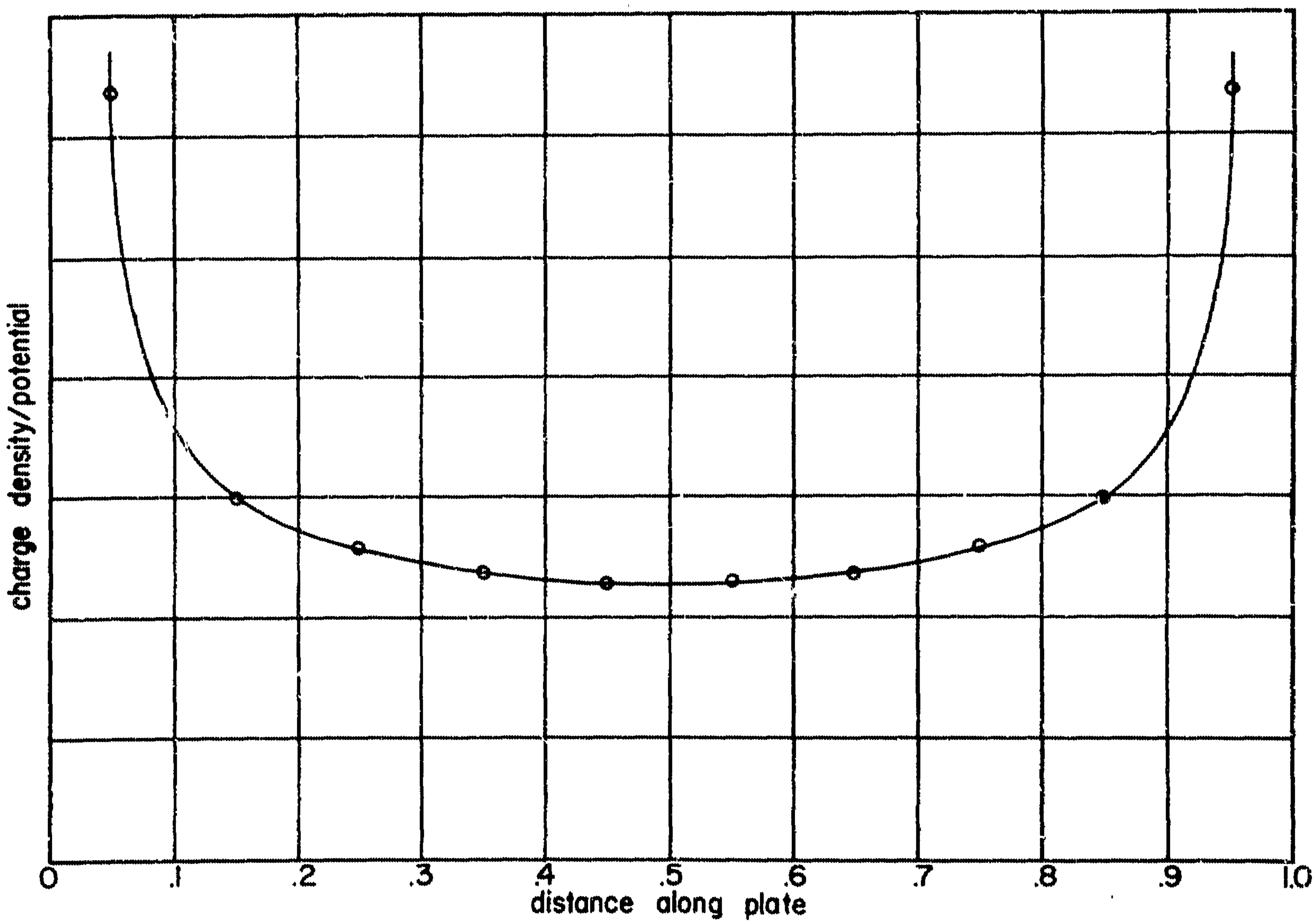

Fig. I-2 Approximate charge density on subareas closest to the centerline of a square conducting plate.

Table 1-1. Capacitance of a unit square plate (micromicrofarad/meter)

No. of subareas	C/2a (approx. ℓ_{ji})	C/2a (exact ℓ_{ji})
1	3150	3150
9	3730	3680
16	3820	3770
36	3920	3870
100		3950

restricted region of space (method of subsections), and (2) the satisfaction of the approximate equation at specific points in the space (point matching method). These approximations, though crude, give acceptable results for many purposes.

F. Variational Interpretation. The special case of the method of moments for which the approximating functions f_i are equal to the weighting functions w_i is known as Galerkin's method. That Galerkin's method is equivalent to the Rayleigh-Ritz variational method is well known.[3,4] That the general method of moments is also a variational method is usually not noted, but the proof is essentially the same.

We first interpret the method of moments according to the concepts of linear spaces. Let (Lf) denote the range of L, (Lf_i) denote the space spanned by the Lf_i, and (w_i) denote the space spanned by the w_i. The method of moments, Eqs. (1-26), equates the projection of Lf on (w_i) to the projection of the approximate Lf on (w_i). Figure 1-3 represents this pictorally (f_a denotes approximation to f). In the special case of Galerkin's method, $(w_i) = (f_i)$. Because the process

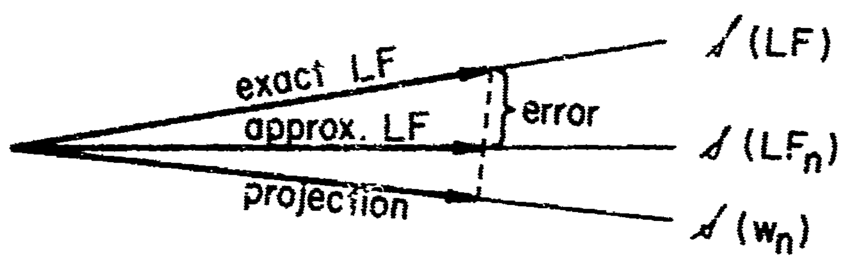

Fig. I–3 Pictoral representation of the method of moments in function space.

of obtaining projections minimizes the error, the method of moments is an error minimizing procedure. Because the error term is orthogonal to the projections, it is of second order. This same conclusion is obtained by means of the calculus of variations.[4] The derivation will not be given here, but we shall summarize the results.

Given an operator equation $Lf = g$, it is desired to determine a functional of f

$$\rho(f) = \langle f,g' \rangle \tag{1-50}$$

This functional may be f itself if g' is an impulse function. Let L^a be the adjoint operator to L, and define an associated function f' by

$$L^a f' = g' \tag{1-51}$$

It can then be shown[4] that

$$\rho = \frac{\langle f,g' \rangle \langle f',g \rangle}{\langle Lf,f' \rangle} \tag{1-52}$$

is a variational formula with stationary value (1-50) when f is a solution to $Lf = g$ and f' a solution to (1-51). For an approximate evaluation of (1-50), let

$$f = \sum_i \alpha_i f_i \qquad f' = \sum_i \beta_i w_i \tag{1-53}$$

and substitute into (1-52). It can then be shown[4] that the necessary and sufficient equations for ρ to be stationary are those of (1-26), that is, the method of moments. This variational procedure is known as the Rayleigh-Ritz method, and hence the method of moments is identical to it.

The second equation of (1-53) gives us some additional insight into how to choose the weighting functions. For good results they should be chosen so that some linear combination can closely represent the associated field f', that is, the field of whatever g' appears in (1-50). For example, if we want f itself, then g' is an impulse function, and f' is the Green's function. This is a poorly behaved function, difficult to approximate by a simple set of w_i. Hence, we should expect calculations of the field itself to converge less slowly than calculations of other parameters which are associated with a well-behaved f'.

G. References.

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II. FORMULATION OF ELECTROMAGNETIC PROBLEMS

<u>A. Introduction.</u> Most engineering problems in electromagnetic theory consist of a source, some matter, and a measurement to be performed. Antenna system problems and radar scattering problems are typical examples. Basically, the solution involves a determination of the interaction of each element of matter with every other element, with the source, and with the measuring device. The equation describing the problem is an inhomogeneous operator equation of the form of equation (1-1). As shown in Chapter I, the operator equation can be transformed into a matrix equation. The interaction between elements of matter is characterized by the matrix operator, the coupling of the source to the matter is characterized by an excitation matrix, and the coupling of the matter to the measuring device is characterized by a measurement matrix. An exact solution for arbitrary excitation usually requires matrices of infinite order. Approximate solutions can be obtained using matrices of finite order. The solution to the problem is represented by the inverse of the interaction matrix. Parameters of engineering interest, such as antenna impedances and radar cross sections, are typically found from the inverse interaction matrix by postmultiplication with the excitation matrix and premultiplication with the measurement matrix.

The interaction matrices can be interpreted as generalized network matrices for the matter. A conducting body is characterized by a network matrix which depends on the shape of the body and on the wavelength. This matrix is a generalization of the one used for the analysis of loaded scatterers[1] and can be used to extend the loaded scatterer concept to continuously loaded bodies.* Electromagnetic excitation of the body

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*Examples of continuously loaded bodies are dielectric coated conductors, magnetic coated conductors, and objects constructed of imperfect conductors.

corresponds to voltage excitation of the network, and measurement of a field quantity corresponds to a weighted combination of the network currents. Dielectric bodies are characterized by two generalized network matrices, one independent of the dielectric permittivity of the body, and the other dependent on the permittivity. Electromagnetic excitation of the body corresponds to voltage excitation of the two networks connected in series, and a field measurement again corresponds to a weighted combination of the terminal currents. Magnetic bodies are the dual problem to dielectric bodies, and have a dual representation in terms of generalized networks. Hence, they also are characterized by two matrices, one independent of the permeability and the other dependent on the permeability. Electromagnetic excitation of the body corresponds to current excitation of the two networks connected in shunt, and a field measurement corresponds to a weighted combination of the terminal voltages. Bodies having both dielectric and magnetic properties are characterized by three network matrices, one of which is independent of the permittivity and permeability, one dependent on the permittivity, and one dependent on the permeability. Field excitation of the body corresponds to voltage excitation of some ports connected in series, and current excitation of other ports connected in shunt. Field measurement corresponds to a weighted combination of the currents at those terminals excited by voltage sources, and of the voltages at those terminals excited by current sources.

The classical eigenfunction method of solution corresponds to obtaining a network matrix having only diagonal elements. The inverse matrix is also diagonal, with elements equal to the reciprocal of the elements of the original matrix. Matrix inversion is then trivial, and the infinite order of the matrix unimportant. However, the eigenfunctions are known for only a few special body shapes, and are often difficult to calculate. If the network matrix can be approximated by a well-conditioned matrix of finite order, the inversion is readily accomplished by known algorithms. For example, a high speed computer can invert a matrix of the order one hundred by one hundred in a few minutes. The realization of a well-behaved network matrix of reasonable order therefore constitutes an engineering solution to the problem. This Chapter deals primarily with the formulation of the desired matrix equations and their network interpretation.

B. Conducting Bodies. Figure 2-1 represents the general problem of a material body in the presence of an electromagnetic source. For this section, it is assumed that the body is a perfect electric conductor. Let $\underline{E}^i$ be the impressed electric field, that is, the field produced by the source when the body is absent. Let $\underline{E}^s$ be the scattered field, that is, the field produced by currents on the body. The scattered field is related to the conduction current $\underline{J}$ on the body S according to

$$\underline{E}^s = L_e(\underline{J}) = \oiint_S \underline{J}(\underline{r}')\, \Gamma_e(\underline{r},\underline{r}')\, ds' \qquad (2\text{-}1)$$

where $\Gamma_e(\underline{r},\underline{r}')$ is the tensor Green's function relating a current element at $\underline{r}'$ to its electric field at $\underline{r}$. The total field at any point in space is $\underline{E}^i + \underline{E}^s$ and the boundary condition is that the tangential components $\underline{u} \times (\underline{E}^i + \underline{E}^s) = 0$ on S, where $\underline{u}$ is the unit normal to S. Hence, the appropriate equation for determining $\underline{J}$ is

$$\underline{u} \times L_e(\underline{J}) = -\underline{u} \times \underline{E}^i \qquad \text{on S} \qquad (2\text{-}2)$$

where L_e is the operator defined by (2-1). A suitable inner product satisfying (1-2), (1-3), and (1-4) is

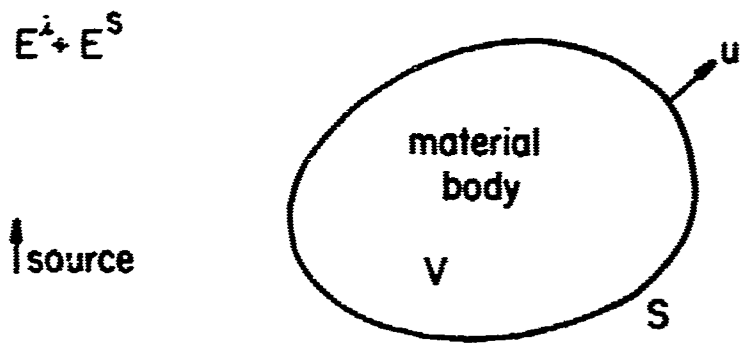

Fig. 2-1 A material body excited by an external source.

$$\langle J,E\rangle = \oiint_S \mathbf{J} \cdot \mathbf{E}\, ds \tag{2-3}$$

which is the quantity defined as reaction.[2,3] Note that (2-3) involves only the tangential components of $\mathbf{E}$, since $\mathbf{J}$ must be tangential to S.

The method of moments is now applied to (2-2). Let the current $\mathbf{J}$ be expanded in s series of functions $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$, ..., defined on S, as

$$\mathbf{J} = \sum_n I_n \mathbf{J}_n \tag{2-4}$$

where the I_n are complex constants. A substitution of (2-4) into (2-2), and an application of the linearity of L, yields

$$\mathbf{u} \times \sum_n I_n L_e(\mathbf{J}_n) = -\mathbf{u} \times \mathbf{E}^i \tag{2-5}$$

Now define a set of testing functions $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, ..., which are vectors tangential to S, i.e., they are current-type vectors. The method of moments requires that (2-5) be valid for the inner product with each $\mathbf{u} \times \mathbf{W}_m$, that is

$$\sum_n I_n \langle W_m, L_e J_n\rangle = - \langle W_m, E^i\rangle \tag{2-6}$$

for all m. The unit normal $\mathbf{u}$ has been dropped from (2-6) since $\mathbf{W}_m$ and $\mathbf{J}_n$ are all tangential to S.

Now define the following matrices

$$[I_n] = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \end{bmatrix} \qquad [V_m] = \begin{bmatrix} -\langle W_1, E^i\rangle \\ -\langle W_2, E^i\rangle \\ \vdots \end{bmatrix} \tag{2-7}$$

$$[Z_{mn}] = \begin{bmatrix} \langle W_1, L_e J_1\rangle & \langle W_1, L_e J_2\rangle & \cdots \\ \langle W_2, L_e J_1\rangle & \langle W_2, L_e J_2\rangle & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{2-8}$$

and rewrite (2-6) in matrix form as

$$[Z_{mn}] \, [I_n] = [V_m] \tag{2-9}$$

The solution for the expansion coefficients is

$$[I_n] = [Y_{nm}] \, [V_m] \tag{2-10}$$

where

$$[Y_{nm}] = [Z_{mn}^{-1}] \tag{2-11}$$

is the matrix inverse to $[Z_{mn}]$. Define the matrix of current functions

$$[\tilde{J}_n] = [J_1 \quad J_2 \quad J_3 \quad \ldots] \tag{2-12}$$

and the solution for the current itself can be written in matrix form as

$$J = [\tilde{J}_n] \, [I_n] = [\tilde{J}_n] \, [Y_{nm}] \, [V_m] \tag{2-13}$$

This equation is approximate or exact depending upon the choice of J_n and W_n.

The designation of the matrices in the above solution as [V], [I], [Z], and [Y] is, of course, intended to call attention to an analogy with the corresponding voltage, current, impedance, and admittance matrices of N-port network theory. This analogy is summarized by Fig. 2-2. The conducting body is characterized by a generalized impedance matrix [Z], or its inverse, a generalized admittance matrix [Y]. The network matrices basically depend only on the geometry of the conductor and the wavelength, and not on the impressed field to which the body is subjected. The impressed electric field determines the voltage excitation of the network analogy according to (2-7). The resultant input currents of the network analogy then correspond to the expansion coefficients for the current according to (2-4).

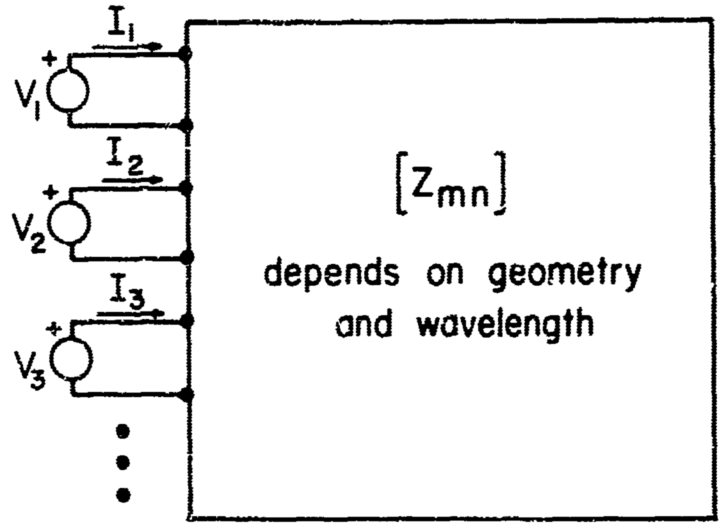

Fig. 2-2 Network analogue for a conducting body in an arbitrary impressed field.

C. Dielectric Bodies. Now let the body of Fig. 2-1 be a dielectric of permittivity ϵ, which may be a function of position or even a tensor. Again the impressed electric field is denoted by $\mathbf{E}^i$, and the scattered field by $\mathbf{E}^s$. The scattered field is produced by the polarization currents J over the body according to

$$\mathbf{E}^s = L_e(\mathbf{J}) = \iiint_V \mathbf{J}(\mathbf{r}') \, \Gamma_e(\mathbf{r},\mathbf{r}') \, d\tau' \tag{2-14}$$

which differs from (2-1) only in that the integral is over a volume V instead of the surface S. The total field at any point is given by $\mathbf{E}^i + \mathbf{E}^s$, and within V the polarization current is given by

$$\mathbf{J} = j\omega(\epsilon - \epsilon_o)(\mathbf{E}^i + \mathbf{E}^s) \tag{2-15}$$

where ϵ_o is the permittivity of free space. Using (2-14) for $\mathbf{E}^s$, one can rearrange (2-15) into the form

$$L_e(\mathbf{J}) - \frac{\mathbf{J}}{j\omega\Delta\epsilon} = -\mathbf{E}^i \qquad \text{in V} \tag{2-16}$$

where $\Delta\epsilon = \epsilon - \epsilon_o$. This is the appropriate equation for determining $\mathbf{J}$. The left-hand side of (2-16) could be redefined as a single operation on $\mathbf{J}$, but it is more convenient to consider it as the sum of two operations. A suitable inner product for this problem is

$$\langle J, E\rangle = \iiint_V J \cdot E \, d\tau \tag{2-17}$$

which is again a reaction.

For the method of moments, let the polarization current $\mathbf{J}$ be expanded in a series of functions $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \ldots$, defined over V, as

$$\mathbf{J} = \sum_n I_n \mathbf{J}_n \tag{2-18}$$

where the I_n are complex constants. Substituting (2-18) into (2-16), and using the linearity of L, one obtains

$$\sum_n I_n \left\{ L_e(\underline{J}_n) - \frac{\underline{J}_n}{j\omega\Delta\epsilon} \right\} = -\underline{E}^i \tag{2-19}$$

Define a set of testing functions $\underline{W}_1$, $\underline{W}_2$, $\underline{W}_3$, ..., over V, and require the equality (2-19) to hold for the inner products with each $\underline{W}_m$. Hence

$$\sum_n I_n \left\{ \langle W_m, L_e J_n \rangle - \frac{1}{j\omega} \langle W_m, \frac{J_n}{\Delta\epsilon} \rangle \right\} = - \langle W_m, E^i \rangle \tag{2-20}$$

for all m. Now define matrices $[I_n]$ and $[V_m]$ as in (2-7), and two impedance matrices, one $[Z_{mn}]$ as in (2-8), and the other

$$[\hat{Z}_{mn}] = \frac{-1}{j\omega} \begin{bmatrix} \langle W_1, J_1/\Delta\epsilon \rangle & \langle W_1, J_2/\Delta\epsilon \rangle & \cdots \\ \langle W_2, J_1/\Delta\epsilon \rangle & \langle W_2, J_2/\Delta\epsilon \rangle & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{2-21}$$

Equation (2-20) can then be written in matrix form as

$$[Z_{mn} + \hat{Z}_{mn}] \, [I_n] = [V_m] \tag{2-22}$$

The solution for the expansion coefficients I_n is again given by (2-10) where

$$[Y_{nm}] = [(Z_{nm} + \hat{Z}_{nm})^{-1}] \tag{2-23}$$

In terms of the matrix (2-12) of expansion functions, the polarization current is given by (2-13). Again the solution is approximate or exact, depending upon the $\underline{J}_n$ and $\underline{W}_n$.

In terms of generalized network parameters, one can interpret this solution as two impedance matrices connected in series as shown in Fig. 2-3. One matrix, [Z], depends only on the geometry of the body and the wavelength,

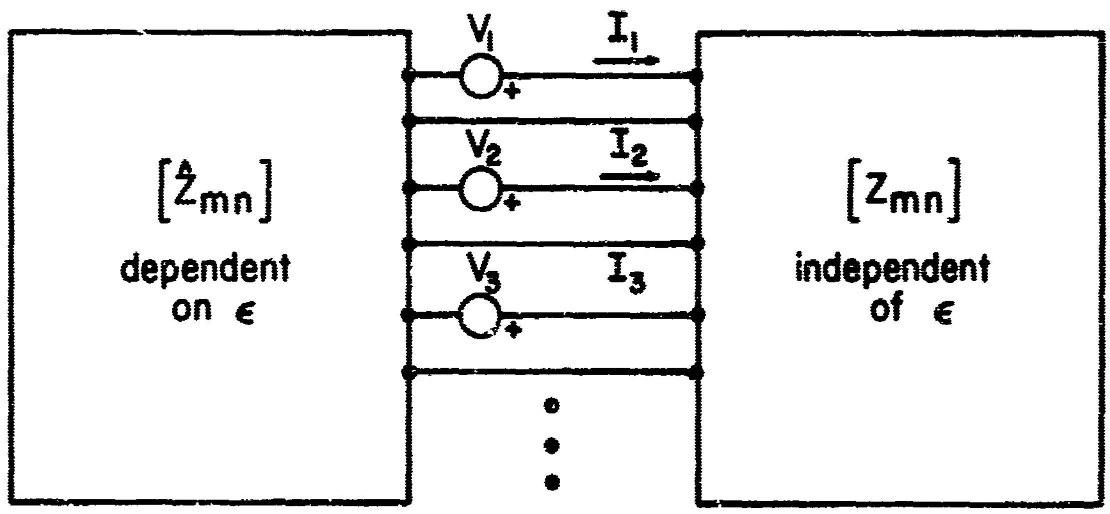

Fig. 2-3 Network analogue for a dielectric body in an arbitrary impressed field.

and the other matrix, $[\hat{Z}]$, depends also on the dielectric permittivity. The impressed electric field determines the voltage excitation of the networks connected in series. The resultant terminal currents then determine the coefficients I_n of the polarization current. Note that as $\epsilon \to \infty$, the network analogy of Fig. 2-3 reduces to that for the conducting body, Fig. 2-2.

D. Magnetic Bodies. If the body of Fig. 2-1 is magnetic but not dielectric, the problem is dual to that for the dielectric body, discussed above. To be specific, let $\underline{H}^i$ denote the impressed magnetic field, and $\underline{H}^s$ the scattered field, produced by the magnetization current $\underline{M}$ over the body. The relationship of $\underline{H}^s$ to $\underline{M}$ is the dual to (2-14) for the dielectric case, that is,

$$\underline{H}^s = L_m(\underline{M}) = \iiint_V \underline{M}(\underline{r}')\ \Gamma_m(\underline{r},\underline{r}')\ d\tau' \tag{2-24}$$

Let μ denote the permeability of the body, and μ_o that of free space. The magnetic current of magnetization is then dual to (2-15), or

$$\underline{M} = j\omega(\mu - \mu_o)(\underline{H}^i + \underline{H}^s) \tag{2-25}$$

Finally, combining (2-24) and (2-25), one has the equation for determining M

$$L_m(\underline{M}) - \frac{\underline{M}}{j\omega\Delta\mu} = -\underline{H}^i \qquad \text{in V} \tag{2-26}$$

which is dual to (2-16). Define an inner product

$$\langle M,H\rangle = -\iiint_V \underline{M} \cdot \underline{H}\ d\tau \tag{2-27}$$

The minus sign is certainly not necessary in (2-27), but is used so that the inner product corresponds to that defined in the reaction concept.

To apply the method of moments, let the magnetization current $\underline{M}$ be expanded in a series of functions $\underline{M}_1$, $\underline{M}_2$, $\underline{M}_3$, ..., defined over V, as

$$\underline{M} = \sum_n V_n \underline{M}_n \tag{2-28}$$

where the V_n are complex constants. From (2-28) and (2-26), it follows that

$$\sum_n V_n \left\{ L_m(\underline{M}_n) - \frac{\underline{M}_n}{j\omega\Delta\mu} \right\} = -\underline{H}^i \tag{2-29}$$

Again define a set of testing functions $\underline{W}_1$, $\underline{W}_2$, $\underline{W}_3$, ..., over V, and take inner products of (2-29) with each W_m. Then, dual to (2-20), one has

$$\sum_n V_n \left\{ \langle W_m, L_m M_n \rangle - \frac{1}{j\omega} \langle W_m, \frac{M_n}{\Delta\mu} \rangle \right\} = - \langle W_m, H^i \rangle \tag{2-30}$$

for all m. The following generalized network matrices are now defined

$$[V_n] = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \end{bmatrix} \qquad [I_n] = \begin{bmatrix} -\langle W_1, H^i \rangle \\ -\langle W_2, H^i \rangle \\ \vdots \end{bmatrix} \tag{2-31}$$

$$[Y_{mn}] = \begin{bmatrix} \langle W_1, L_m M_1 \rangle & \langle W_1, L_m M_2 \rangle & \cdots \\ \langle W_2, L_m M_1 \rangle & \langle W_2, L_m M_2 \rangle & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{2-32}$$

$$[\hat{Y}_{mn}] = \frac{-1}{j\omega} \begin{bmatrix} \langle W_1, M_1/\Delta\mu \rangle & \langle W_1, M_2/\Delta\mu \rangle & \cdots \\ \langle W_2, M_1/\Delta\mu \rangle & \langle W_2, M_2/\Delta\mu \rangle & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{2-33}$$

The matrix equation for (2-30) is then

$$[Y_{mn} + \hat{Y}_{mn}] [V_n] = [I_m] \tag{2-34}$$

which is dual to (2-22). The solution for the $[V_n]$ is the inverse of (2-34). The magnetic current of the magnetization is given by (2-28) once the I_n are found.

The network representation of this solution shown in Fig. 2-4 is the dual case of Fig. 2-3. The matrix $[Y_{mn}]$ depends only on the geometry of the body and the wavelength, and the matrix $[\hat{Y}_{mn}]$ depends on the permeability. The impressed magnetic field determines the current excitation of the networks connected in shunt. The resultant terminal voltages then correspond to the expansion coefficients V_n for the magnetic current.

E. Bodies both Dielectric and Magnetic. If the body has both ϵ and μ different from their free-space values, a combination of the preceding two analyses must be used. For this purpose, in addition to the relationship between electric current and electric field, (2-14), and magnetic current and magnetic field, (2-24), one needs the relationship between electric current and magnetic field

$$\underline{H} = N(\underline{J}) = \iiint_V \underline{J}(\underline{r}') \, \hat{\Gamma}(\underline{r},\underline{r}') \, d\tau' \qquad (2\text{-}35)$$

where $\hat{\Gamma}(\underline{r},\underline{r}')$ is the tensor Green's function which relates an element of $\underline{J}$ at $\underline{r}'$ to its magnetic field at $\underline{r}$, and the corresponding relationship between magnetic current and electric field

$$\underline{E} = -N(\underline{M}) \qquad (2\text{-}36)$$

where N is the same operator defined in (2-35). The minus sign difference between (2-35) and (2-36) reflects the minus sign difference between the two curl equations of Maxwell. The integral equation for the problem is now a matrix of (2-16) and (2-26), with the appropriate interaction terms added. To be explicit

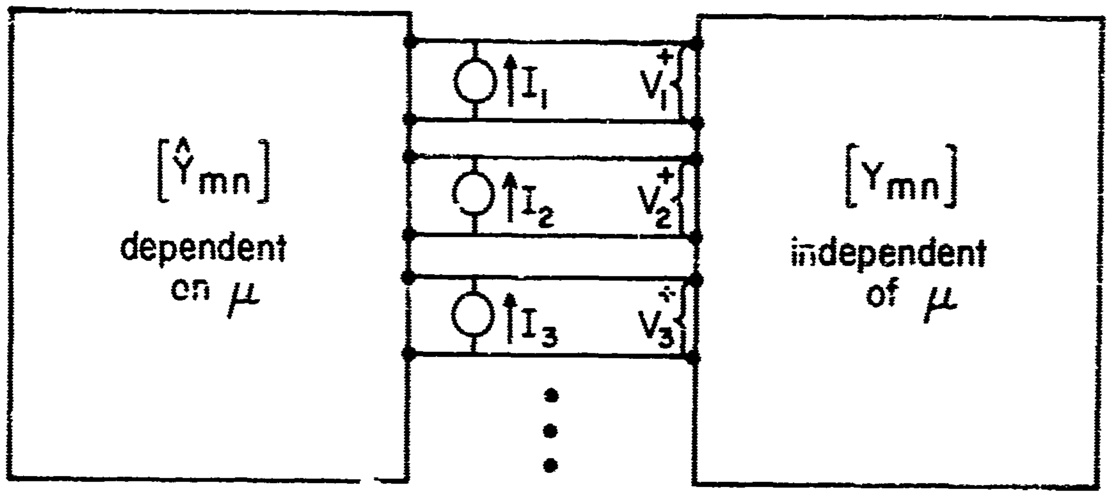

Fig. 2-4 Network analogue for a magnetic body in an arbitrary impressed field.

$$\begin{bmatrix} L_e & N \\ N & -L_m \end{bmatrix} \begin{bmatrix} J \\ -M \end{bmatrix} = \frac{1}{j\omega} \begin{bmatrix} J/\Delta\epsilon \\ M/\Delta\mu \end{bmatrix} = \begin{bmatrix} -E^i \\ -H^i \end{bmatrix} \tag{2-37}$$

is the equation to be solved for J and M when the body is both dielectric and magnetic.

Equation (2-37) is just a more complicated case of an operator equation. This is evident if the following matrices are defined

$$f = \begin{bmatrix} J \\ -M \end{bmatrix} \qquad g = \begin{bmatrix} E \\ H \end{bmatrix} \tag{2-38}$$

$$\mathcal{L} = \begin{bmatrix} L_e & N \\ N & -L_m \end{bmatrix} \qquad \mathcal{M} = \begin{bmatrix} \frac{-1}{j\omega\Delta\epsilon} & 0 \\ 0 & \frac{1}{j\omega\Delta\mu} \end{bmatrix} \tag{2-39}$$

and (2-37) rewritten as

$$\mathcal{L}(f) + \mathcal{M}(f) = -g^i \tag{2-40}$$

Of course, $\mathcal{L}$ and $\mathcal{M}$ could be combined into a single operator, and (2-40) written as (1-1). The appropriate inner product for this problem is

$$\langle f,g\rangle = \iiint_V \tilde{f}\, g \, d\tau = \iiint_V (J \cdot E - M \cdot H)\, d\tau \tag{2-41}$$

which is precisely the general definition of reaction.[2,3]

A solution by the method of moments is obtained as follows. Define "electric" expansion and weighting functions as

$$f_n^e = \begin{bmatrix} J_n \\ 0 \end{bmatrix} \qquad w_n^e = \begin{bmatrix} W_n^e \\ 0 \end{bmatrix} \tag{2-42}$$

and "magnetic" expansion and weighting functions as

$$f_n^m = \begin{bmatrix} 0 \\ -M_n \end{bmatrix} \qquad w_n^m = \begin{bmatrix} 0 \\ -W_n^m \end{bmatrix} \tag{2-43}$$

The expansion for f is then of the form

$$f = \sum_n (I_n f_n^e + V_n f_n^m) \tag{2-44}$$

where the I_n and V_n correspond to the α_n of (1-24). Following the method of moments, one obtains the matrix equation

$$\begin{bmatrix} [Z_{mn}] & [B_{mn}] \\ [C_{mn}] & [Y_{mn}] \end{bmatrix} \begin{bmatrix} [I_n] \\ [V_n] \end{bmatrix} + \begin{bmatrix} [\hat{Z}_{mn}] & [I_n] \\ [\hat{Y}_{mn}] & [V_n] \end{bmatrix} = \begin{bmatrix} [V_m^i] \\ [I_m^i] \end{bmatrix} \tag{2-45}$$

Here the various matrices have the same definitions as in sections II-C and II-D. The additional matrices $[B_{mn}]$ and $[C_{mn}]$ describe the interaction between electric and magnetic currents, and the superscript i has been added to the source terms (right hand side) to distinguish them fromt the response terms.

The generalized network representation of Eq. (2-45) is shown in Fig. 2-5. The network denoted $[L_{mn}]$ again depends only on geometry and wavelength, not on ϵ or μ. The network $[\hat{Z}_{mn}]$ in series with the voltage sources depends on ϵ, and the network $[\hat{Y}_{mn}]$ in shunt with the current sources depends on μ. The impressed electric field determines the voltage sources according to (2-7), and the impressed magnetic field determines the current sources according to (2-31).

F. Measurement. Figure 2-6 represents a general problem of electromagnetic engineering, one consisting of a source, a receiver, and material bodies. Explicitly, Fig. 2-6 shows a source connected to a transmitting

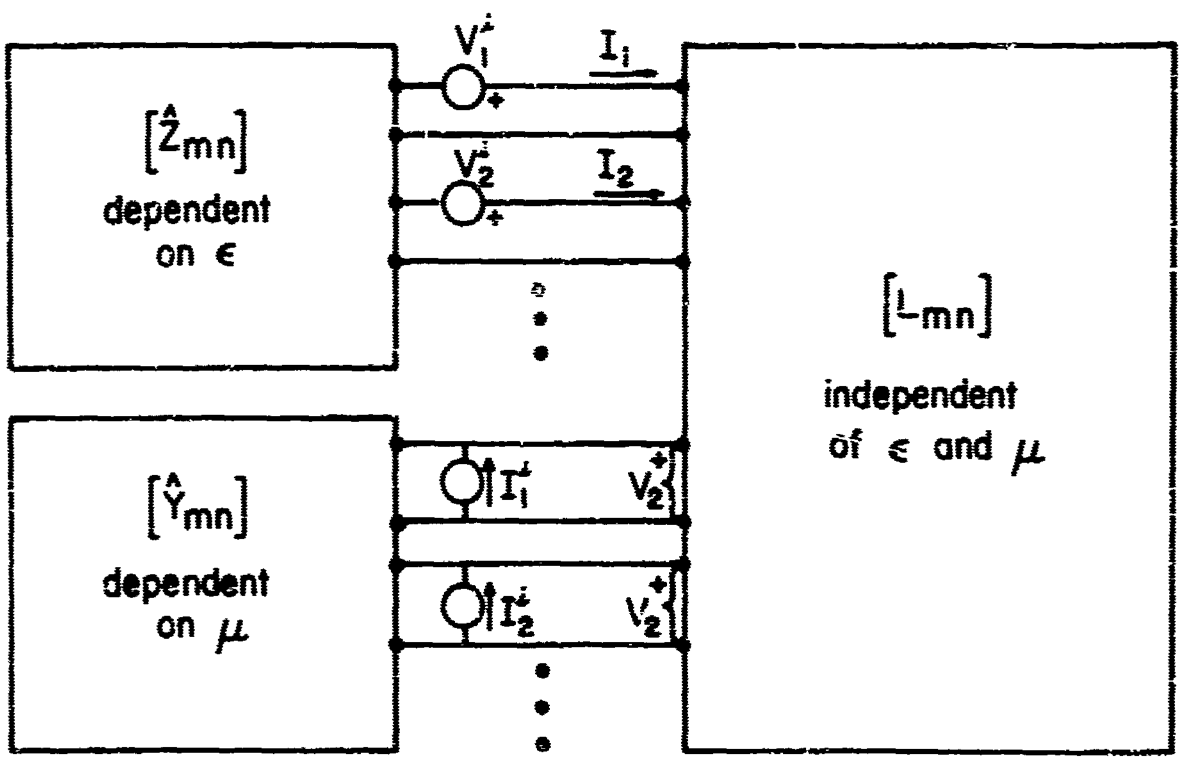

Fig. 2-5 Network analogue for a body having both dielectric and magnetic properties.

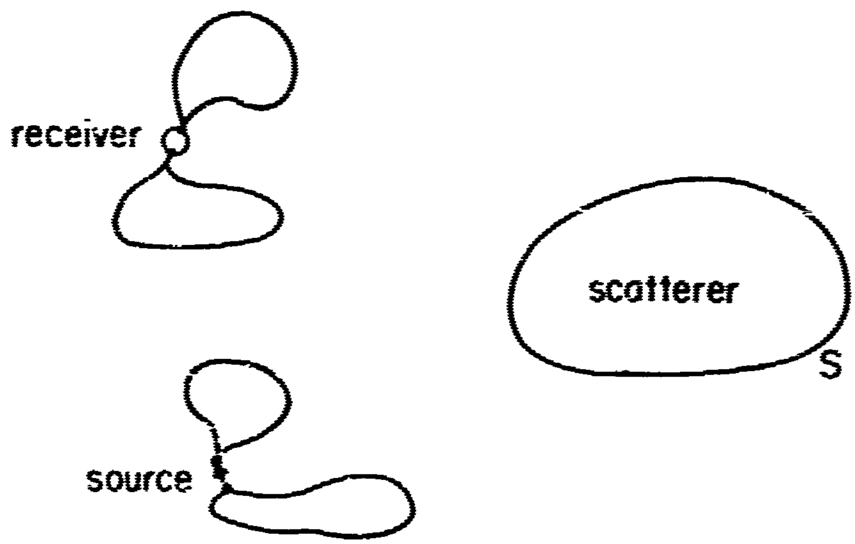

Fig. 2-6 A typical electromagnetic engineering problem.

antenna, a measurement device connected to a receiving antenna, and a scattering object. However, any of the material bodies may be absent; hence a general problem includes the transmitting antenna, the receiving antenna, and free-space scattering as special cases. The source and receiver may, of course, be at the same point in space, as, for example, in an input impedance problem.

The voltage at the receiver terminals due to a current source at the transmitter terminals can be expressed in terms of the transfer impedance Z_{rt} between the two sets of terminals.[1,3] In terms of reaction, this transfer impedance is given by

$$Z_{rt} = \frac{-1}{I_r I_t} \iiint (E^t \cdot J^r - H^t \cdot M^r) \, d\tau \qquad (2\text{-}46)$$

where E^t, H^t is the field produced when I_t is applied to the transmitter terminals, and J^r, M^r is the current resulting when I_r is applied to the receiver terminals. If ε and μ are scalars, or even symmetric tensors, the usual reciprocity theorem applies, and (2-46) can be rewritten as

$$Z_{rt} = \frac{-1}{I_t I_r} \iiint (E^r \cdot J^t - H^r \cdot M^t) \, d\tau \qquad (2\text{-}47)$$

E^r, H^r is the field produced by I_r applied to the receiver terminals, and J^t, M^t is the current produced by I_t applied to the transmitter terminals. The important fact to note is that the measurement is of the form of an inner product of the current caused by an excitation of the transmitter with a field caused by an excitation of the receiver. Hence, in the general notation of the method of moments,

$$\text{measurement} = \langle f, g^r \rangle \qquad (2\text{-}48)$$

where g^r is function depending on the measurement to be performed. For example, g^r is a E^r, H^r in (2-47). Equation (2-48) is the general expression for a functional of f. In terms of the solution obtained by the method of moments, equation (1-32),

$$\text{measurement} = \langle f, g^r \rangle = [\tilde{g}_n^r]\,[\ell_{nm}^{-1}]\,[g_m] \tag{2-49}$$

where

$$[g_n^r] = \begin{bmatrix} \langle f_1, g^r \rangle \\ \langle f_2, g^r \rangle \\ \vdots \end{bmatrix} \tag{2-50}$$

This measurement matrix is of the same form as the excitation matrix $[g_m]$ of (1-29), except that the inner products are with the f_n instead of the w_n.

As an example of a measurement in a field problem, consider a conducting body in a plane-wave incident field, and let the measurement be the bistatic radar echo at some distant receiver. By definition, let Echo be the field quantity whose magnitude squared is the conventional echo area, that is

$$\sigma = |\text{Echo}|^2 \tag{2-51}$$

A formula for Echo is obtained by letting both the transmitter and receiver of Fig. 2-6 recede to infinity according to an appropriate limiting procedure.[1] The result is

$$\text{Echo} = \frac{-\eta k}{2\sqrt{\pi}} \oint\!\!\oint_S \mathbf{E}_o^r \cdot \mathbf{J}^t \, ds \tag{2-52}$$

where $\eta = \sqrt{\mu/\varepsilon}$, $k = 2\pi/\lambda$, $\mathbf{E}_o^r$ is a normalized (unit amplitude) plane wave from the receiver, and $\mathbf{J}^t$ is the current on the conducting scatterer when excited by a normalized plane wave from the transmitter. The current on the scatterer has already been determined by the method of moments in section II-B, the result being (2-13). The remaining integration in (2-52) gives

$$\text{Echo} = \frac{\eta k}{2\sqrt{\pi}} [\tilde{V}_n^r] [Y_{nm}] [V_m] \tag{2-53}$$

where $[V_n^r]$ is given by

$$[V_n^r] = \begin{bmatrix} -\langle J_1, E_o^r \rangle \\ -\langle J_2, E_o^r \rangle \\ \vdots \end{bmatrix} \tag{2-54}$$

This is the same form as the voltage excitation matrix (2-7), except that J_n replace the W_n. In the Galerkin method ($J_n = W_n$) the two voltage matrices $[V_n^r]$ and $[V_n]$ are just different excitations of the same generalized network. In the method of moments, $[V_n^r]$ may be interpreted as the excitation of an adjoint generalized network, defined as the transpose of $[Z_{mn}]$.

G. Discussion. The method of moments is so general that almost any solution can be interpreted as an application of it. The use of eigenfunctions for the expansion functions and adjoint eigenfunctions for the weighting functions is the classical eigenfunction method. Point-matching procedures, that is, satisfaction of the operator equation at specific points on a body, are equivalent to using impulse functions as weighting functions in the method of moments. Methods which approximate the operator, for example, the method of nets, are interpretable as special cases of the general method. Hence, one should view the method of moments as a unifying concept rather than as a particular technique of solution. In applying the method to engineering problems, one is confronted with many possible choices and approximations. The particular technique of approximation called the method of subsections, introduced in Section I-E, has been found quite useful for digital computation.

Basically, the method of subsections can be thought of as dividing the object into a number of pieces, and calculating the interactions according to the method of moments. One of the first applications to time-harmonic electromagnetic fields was the calculation of scattering by square conducting cylinders.[4] General computer programs for basically the same method, but with somewhat better approximations, have been used for cylinders of arbitrary shape.[5] Solutions of this type have also been obtained for dielectric cylinders,[6] for conducting bodies of revolution,[7] and for wire antennas of arbitrary shape.[8] An analysis of wire antennas and scatterers of arbitrary shape, both loaded and unloaded, are formulated using the methods of this section in the next chapter.

In terms of the method of moments, the method of subsections involves using expansion functions which each exist over separate sections of the body, and using weighting functions which test the field in each section (often by point matching). In the case of conducting bodies, this is equivalent to obtaining the self-impedance of each subsection of current, and the mutual impedance of each element with every other element of current. In the case of dielectric (or magnetic) bodies, the interpretation is similar, except that the self impedance (or admittance in the magnetic case) can be divided into two components, one dependent on ϵ (or μ) and the other independent of ϵ (or μ). The mutual terms are always independent of ϵ (or μ). Hence, the careted networks (left-hand ones) of Figures 2-3, 2-4, and 2-5 become noninteracting elements, that is, expressible as diagonal matrices. For example, the network representation of Fig. 2-5 becomes that of Fig. 2-7 when the method of subsections is used. The effect of ϵ

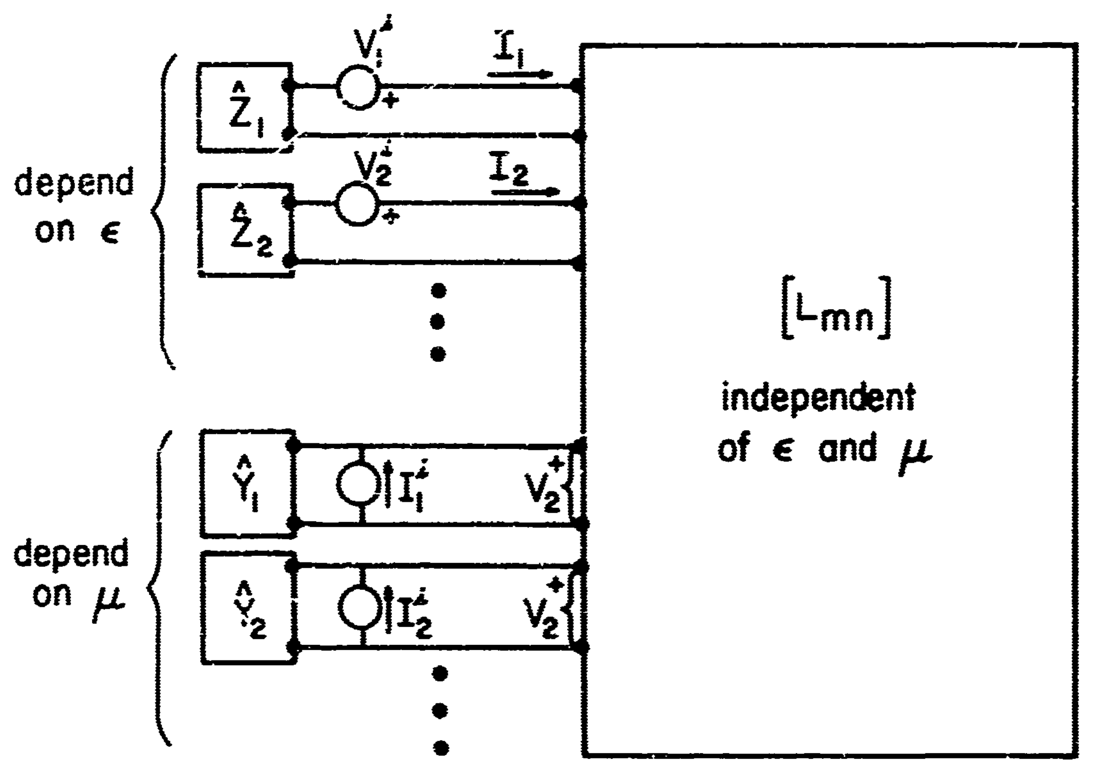

Fig. 2-7 Network analogue for an arbitrary body when the method of subsections is used.

and μ is then just a loading of the appropriate terminals of the generalized network for the body.

The use of these techniques for computation gives rise to a number of practical difficulties, which can be overcome by careful analysis. For example, the field of a discontinuous distribution of current has singularities, and the order of differentiation and integration cannot always be interchanged. The usual tensor Green's function often cannot be used, but must be interpreted as symbolic of the more fundamental vector and scalar potential formulas.[9] The accuracy of a solution and the ease of computation depend to a large extent on the ingenuity and care used in the formulation of the problem. The method of subsections is basically a small body technique, since the matrices involved become too big for large bodies. The treatment of large bodies requires great care in the choice of expansion functions and testing functions, and even then the interaction matrix usually is valid for only a subclass of excitations. Perhaps the best approach to the large bodies is a perturbation one, that is, the use of an approximate solution as one of the expansion functions. The method of moments then gives a correction to this approximate solution. A disadvantage to this approach is that the generality of the solution is lost, being valid for only one excitation of the body. Much more work remains to be done on the large body problem.

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III. WIRE ANTENNAS AND SCATTERERS OF ARBITRARY SHAPE

A. Introduction. A general analysis of objects constructed of thin conducting wires and situated in an arbitrary impressed electromagnetic field is given in this chapter. The wires are assumed to be perfectly conducting and to have diameters much less than a wavelength. The distinction between antennas and scatterers is primarily that of the location of the source. If the source is at the object it is viewed as an antenna, if the source is distant from the object it is viewed as a scatterer.

The method of solution is basically that of numerically solving an integral equation. Similar techniques have been used for scattering by conducting cylinders, for scattering by bodies of revolution, and for analyzing thin wire antennas, as discussed in Chapter II. However, for any particular problem there are many different ways to obtain a solution. The conventional retarded potential formulation is chosen here because of its conceptual simplicity. The resultant equation differs from classical integral equations in that derivatives of the unknown current appear in the integrand. This requires approximation of the derivative of the current (charge) as well as the current itself. It has been found that this can be done conveniently by replacing all derivatives by difference approximations.

A particularly descriptive exposition of the solution can be made in terms of network parameters, as discussed in Chapter II. To effect a solution, the wire is considered as N short segments connected together. The end points of each segment define a pair of terminals in space. These N pairs of terminals can be thought of as forming an N port network, and the wire object is obtained by short circuiting all ports of the network. One can determine the impedance matrix for the N port network by applying a current source to each port in turn, and calculating the open circuit voltages at all ports. This procedure involves only current elements in empty space. The admittance matrix is the inverse of the impedance matrix. Once the admittance matrix is known, the port currents (current distribution on the wire) are found for any particular voltage excitation (applied field) by matrix multiplication.

B. Formulation of the Problem. An integral equation for the charge density σ_s and current $\underset{\sim}{J}_s$ on a conducting body S in a known impressed field $\underset{\sim}{E}^i$ is obtained as follows. The scattered field $\underset{\sim}{E}^s$, produced by σ_s and $\underset{\sim}{J}_s$, is expressed in terms of retarded potential integrals, and the boundary condition $\underset{\sim}{n} \times (\underset{\sim}{E}^i + \underset{\sim}{E}^s) = 0$ on S is applied. This is summarized by

$$\underset{\sim}{E}^s = -j\omega \underset{\sim}{A} - \nabla\phi \qquad (3\text{-}1)$$

$$\underset{\sim}{A} = \mu \iint_S \underset{\sim}{J}_s \frac{e^{-jkR}}{4\pi R}\, dS \qquad (3\text{-}2)$$

$$\phi = \frac{1}{\epsilon} \iint_S \sigma_s \frac{e^{-jkR}}{4\pi R}\, dS \qquad (3\text{-}3)$$

$$\sigma_s = \frac{-1}{j\omega} \nabla_s \cdot \underset{\sim}{J}_s \qquad (3\text{-}4)$$

$$\underset{\sim}{n} \times \underset{\sim}{E}^s = -\underset{\sim}{n} \times \underset{\sim}{E}^i \quad \text{on S} \qquad (3\text{-}5)$$

Figure 3-1a represents an arbitrary thin-wire scatterer, for which the following approximations are made. (1) The current is assumed to flow only in the direction of the wire axis. (2) The current and charge densities are approximated by filaments of current $\underset{\sim}{I}$ and charge σ on

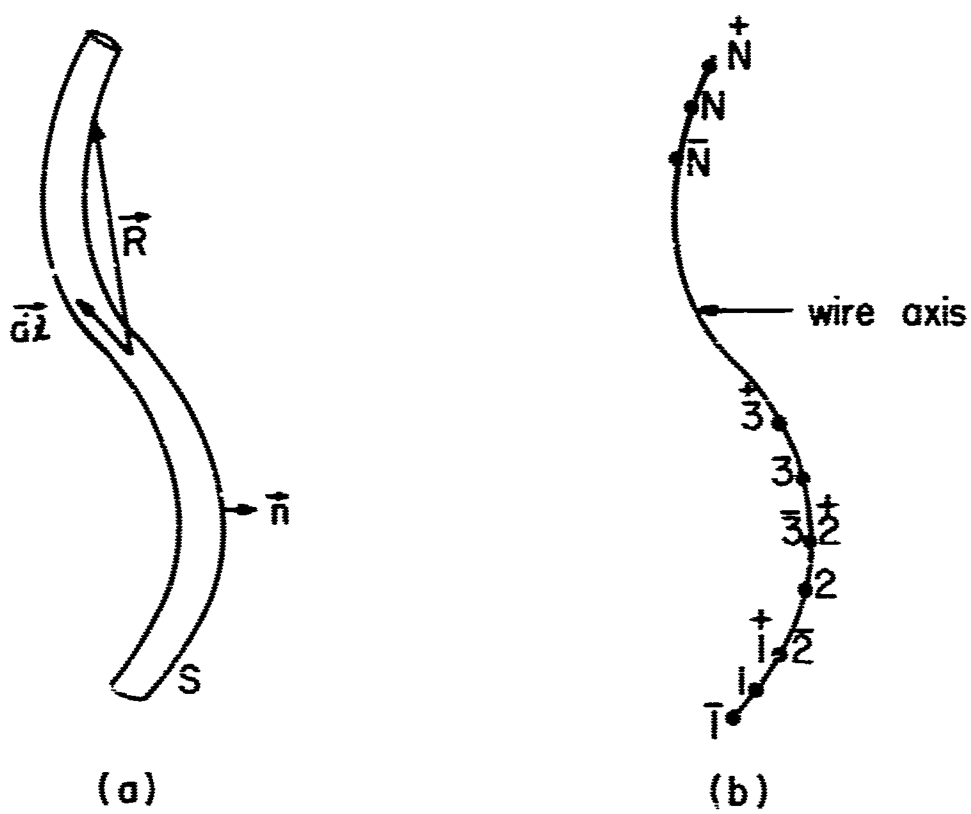


Fig. 3-1 (a) A wire scatterer. (b) The wire axis divided into N segments.

the wire axis. (3) The boundary condition (3-5) is applied to the axial component of $\underline{E}$ at the wire surface. To this approximation, Eqs. (3-1) to (3-5) become

$$-E^i_\ell = -j\omega A_\ell - \frac{\partial\phi}{\partial\ell} \qquad \text{on } S \tag{3-6}$$

$$\underline{A} = \mu \int_{\text{axis}} \underline{I}(\ell) \frac{e^{-jkR}}{4\pi R} d\ell \tag{3-7}$$

$$\phi = \frac{1}{\epsilon} \int_{\text{axis}} \sigma(\ell) \frac{e^{-jkR}}{4\pi R} d\ell \tag{3-8}$$

$$\sigma = \frac{-1}{j\omega} \frac{dI}{d\ell} \tag{3-9}$$

where ℓ is the length variable along the wire axis, and R is measured from a source point on the axis to a field point on the wire surface.

A numerical solution to the above equations is obtained as follows. Integrals are approximated by the sum of integrals over N small segments, obtained by treating I and σ as constant over each segment. Derivatives are approximated by finite differences over the small intervals used for integration. Figure 3-1b illustrates the division of the wire axis into N segments, and defines the notation. The n-th segment is identified by its starting point $\bar{n}$, its midpoint n, and its termination $\overset{+}{n}$. An increment $\Delta\ell_n$ denotes that between $\bar{n}$ and $\overset{+}{n}$, $\Delta\ell_{\bar{n}}$ and $\Delta\ell_{\overset{+}{n}}$ denote increments shifted 1/2 segment minus or plus along ℓ. The desired approximations for (3-6) to (3-9) are then

$$-E^i_\ell(m) \approx -j\omega A_\ell(m) - \frac{\phi(\overset{+}{m}) - \phi(\bar{m})}{\Delta\ell_m} \tag{3-10}$$

$$A(m) \approx \mu \sum_n I(n) \int_{\Delta \ell_n} \frac{e^{-jkR}}{4\pi R} \, d\ell \qquad (3\text{-}11)$$

$$\phi(\overset{+}{m}) \approx \frac{1}{\epsilon} \sum_n \sigma(\overset{+}{n}) \int_{\Delta \ell_n^+} \frac{e^{-jkR}}{4\pi R} \, d\ell \qquad (3\text{-}12)$$

$$\sigma(\overset{+}{n}) \approx \frac{-1}{j\omega} \; \frac{I(n+1) - I(n)}{\Delta \ell_n^+} \qquad (3\text{-}13)$$

with equations similar to (3-12) and (3-13) for $\phi(\overline{m})$ and $\sigma(\overline{n})$.

The σ's are given in terms of the I's by (3-13), and hence (3-10) can be written in terms of the $I(n)$ only. One can view the N equations represented by (3-10) as the equations for an N port network with terminal pairs $(\overset{+}{n}, \overline{n})$. The voltage applied to each port is approximately $E^i \cdot \Delta \ell_n$. Hence, by defining matrices

$$[I] = \begin{bmatrix} I(1) \\ I(2) \\ \vdots \\ I(N) \end{bmatrix} \qquad [V] = \begin{bmatrix} E^i(1) \cdot \Delta \ell_1 \\ E^i(2) \cdot \Delta \ell_2 \\ \vdots \\ E^i(N) \cdot \Delta \ell_N \end{bmatrix} \qquad (3\text{-}14)$$

one can rewrite (3-10) in matrix form as

$$[V] = [Z] \, [I] \qquad (3\text{-}15)$$

The elements of the matrix [Z] can be obtained by substituting (3-11) to (3-13) into (3-10) and rearranging into the form of (3-15). Alternatively, one can apply (3-10) to (3-13) to two isolated elements and obtain the impedance elements directly. This last procedure will be used because it is somewhat easier to follow.

Consider two representative elements of the wire scatterer, as shown in Fig. 3-2. The integrals in (3-11) and (3-12) are of the same form, and are denoted by

$$\psi(n,m) = \frac{1}{\Delta\ell_n} \int_{\Delta\ell_n} \frac{e^{-jkR_{mn}}}{4\pi R_{mn}} d\ell_n \qquad (3\text{-}16)$$

Symbols + and - are used over m and n when appropriate. Evaluation of the ψ in general is considered in Section III-F. Let element n of Fig. 3-2 consist of a current filament I(n), and two charge filaments of net charge

$$q(\overset{+}{n}) = \frac{1}{j\omega} I(n) \qquad q(\overset{-}{n}) = \frac{-1}{j\omega} I(n) \qquad (3\text{-}17)$$

where $q = \sigma\Delta\ell$. The vector potential at m due to I(n) is, by (3-11),

$$\underline{A} = \mu \underline{I}(n)\ \Delta\ell_n\ \psi(n,m) \qquad (3\text{-}18)$$

The scalar potentials at $\overset{+}{m}$ and $\overset{-}{m}$ due to the charges (3-17) are, by (3-12),

$$\begin{aligned} \phi(\overset{+}{m}) &= \frac{1}{j\omega\epsilon}\ [I(n)\ \psi(\overset{+}{n},\overset{+}{m}) - I(n)\ \psi(\overset{-}{n},\overset{+}{m})] \\ \phi(\overset{-}{m}) &= \frac{1}{j\omega\epsilon}\ [I(n)\ \psi(\overset{+}{n},\overset{-}{m}) - I(n)\ \psi(\overset{-}{n},\overset{-}{m})] \end{aligned} \qquad (3\text{-}19)$$

Substituting from (3-18) and (3-19) into (3-10), and forming $Z_{mn} = \underline{E}^i(m) \cdot \Delta\underline{\ell}_m / I(n)$, one obtains

$$Z_{mn} = j\omega\mu\Delta\underline{\ell}_n \cdot \Delta\underline{\ell}_m\ \psi(n,m) + \frac{1}{j\omega\epsilon}\ [\psi(\overset{+}{n},\overset{+}{m}) - \psi(\overset{-}{n},\overset{+}{m}) - \psi(\overset{+}{n},\overset{-}{m}) + \psi(\overset{-}{n},\overset{-}{m})] \qquad (3\text{-}20)$$

This result applies for self impedances (m = n) as well as for mutual impedances.

The wire object is completely characterized by its impedance matrix, subject, of course to the approximations involved. The object is defined by

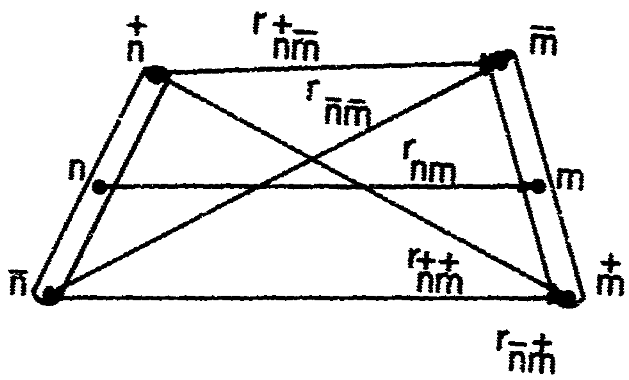


Fig. 3-2 Two segments of a wire scatterer.

2N+1 points on the wire axis, plus the wire radius. The impedance elements are calculated by (3-20), and the voltage matrix is determined by the impressed field, according to (3-14). The current at N points on the scatterer is then given by the current matrix, obtained from the inversion of (3-15) as

$$[I] = [Y]\,[V] \qquad\qquad [Y] = [Z]^{-1} \qquad\qquad (3\text{-}21)$$

Once the current distribution is known, parameters of interest such as field patterns, input impedances, echo areas, etc., can be calculated by numerically evaluating the conventional formulas.

<u>C. Wire Antennas.</u> A wire antenna is obtained when the wire is excited by a voltage source at one or more points along its length. Hence, for an antenna excited in the n-th interval, the applied voltage matrix (3-14) is

$$[V^s] = \begin{bmatrix} 0 \\ \vdots \\ V_n \\ \vdots \\ 0 \end{bmatrix} \qquad\qquad (3\text{-}22)$$

i.e., all elements zero except the n-th, which is equal to the source voltage. The current distribution is given by (3-21), which for the [V] of (3-22) becomes

$$[I] = V_n \begin{bmatrix} Y_{1n} \\ Y_{2n} \\ \vdots \\ Y_{Nn} \end{bmatrix} \qquad\qquad (3\text{-}23)$$

Hence, the n-th column of the admittance matrix is the current distribution for a unit voltage source applied in the n-th interval. Inversion of the impedance matrix therefore gives simultaneously the current distributions when the antenna is excited in any arbitrary interval along its length. The diagonal elements Y_{nn} of the admittance matrix are the input admittances of the wire object fed in the n-th interval, and the Y_{mn} are the transfer admittances between a port in the m-th interval and one in the n-th interval.

The radiation pattern of a wire antenna is obtained by treating the antenna as an array of N current elements $I(n)\Delta\ell_n$. By standard formulas, the far-zone vector potential is given by

$$\underset{\sim}{A} = \frac{\mu e^{-jkr_o}}{4\pi r_o} \sum_n I(n)\, \Delta\underset{\sim}{\ell}_n\, e^{jkr_n \cos \xi_n} \tag{3-24}$$

where $\underset{\sim}{r}_o$ and $\underset{\sim}{r}_n$ are the radius vectors to the distant field point and to the source points, respectively, and ξ_n is the angle between $\underset{\sim}{r}_o$ and $\underset{\sim}{r}_n$. The far-zone field components are

$$E_\theta = -j\omega A_\theta \qquad E_\phi = -j\omega A_\phi \tag{3-25}$$

where θ and ϕ are the conventional spherical coordinate angles.

An alternative formula for the radiation pattern, more convenient for computation, can be obtained by reciprocity. Figure 3-3 represents a distant current element $I\underset{\sim}{\ell}_r$ (subscripts r denote "receiver"), adjusted to product the unit plane wave

$$\underset{\sim}{E}^r = \underset{\sim}{u}_r\, e^{-j\underset{\sim}{k}_r \cdot \underset{\sim}{r}_n} \tag{3-26}$$

in the vicinity of the antenna. Here $\underset{\sim}{u}_r$ is a unit vector specifying the polarization of the wave, $\underset{\sim}{k}_r$ is a wave number vector pointing in the direction of travel of the wave, and $\underset{\sim}{r}_n$ is the radius vector to a point n on the antenna. By reciprocity,[1]

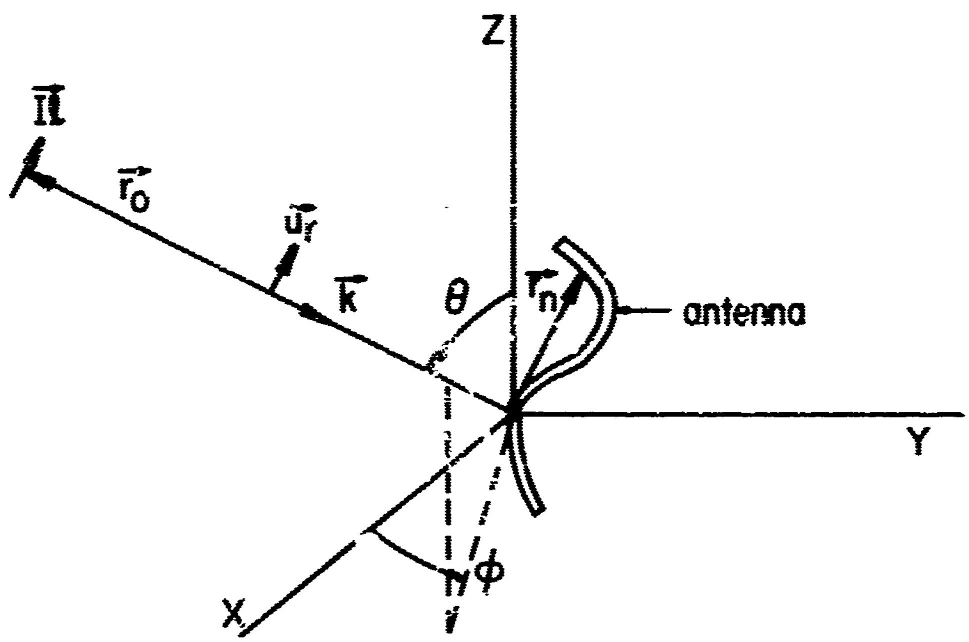


Fig. 3-3 A wire antenna and distant dipole.

$$E_r = \frac{1}{I\ell} \int_{\text{antenna}} \underline{E}^r \cdot I \, d\underline{\ell} \qquad (3\text{-}27)$$

where E_r is the $\underline{u}_r$ component of $\underline{E}$ from the antenna, and I is the current on the antenna. The constant $1/I\ell$ is that needed to produce a plane wave of unit amplitude at the origin, which is

$$\frac{1}{I\ell} = \frac{\omega \mu e^{-jkr_o}}{j\,4\pi r_o} \qquad (3\text{-}28)$$

A numerical approximation to (3-27) is obtained by defining a voltage matrix

$$[V^r] = \begin{bmatrix} \underline{E}^r(1) \cdot \Delta\underline{\ell}_1 \\ \underline{E}^r(2) \cdot \Delta\underline{\ell}_2 \\ \vdots \\ \underline{E}^r(N) \cdot \Delta\underline{\ell}_N \end{bmatrix} \qquad (3\text{-}29)$$

where $\underline{E}^r$ is given by (3-26), and expressing (3-27) as the matrix product

$$E_r = \frac{\omega \mu e^{-jkr_o}}{j4\pi r_o} [\tilde{V}^r][I] = \frac{\omega \mu e^{-jkr_o}}{j4\pi r_o} [\tilde{V}^r][Y][V^s] \qquad (3\text{-}30)$$

where $[\tilde{V}]$ denotes the transpose of $[V]$.

Note that $[V^r]$ is the same matrix as (3-14), that is, it is the voltage matrix for plane-wave excitation of the wire. Equation (3-30) remains valid for an arbitrary excitation $[V^s]$; it is not restricted to the single source excitation (3-22).

The power gain pattern for the $\underline{u}_r$ component of the radiation field is given by

$$g(\theta,\phi) = \frac{4\pi r_o^2}{\eta} \frac{|E_r(\theta,\phi)|^2}{P_{in}} \qquad (3\text{-}31)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of space, and P_{in} is the power input to the antenna (* denotes conjugate)

$$P_{in} = \text{Re}\left\{[\widetilde{V}^s]\ [I^*]\right\} = \text{Re}\left\{[\widetilde{V}^s]\ [Y^*]\ [V^{s*}]\right\} \tag{3-32}$$

For the special case (3-22) of a single source, P_{in} becomes simply $\text{Re}(|V_n|^2\ Y_{nn})$. Using (3-30) and (3-32) in (3-31), one has

$$g(\theta,\phi) = \frac{\eta k^2}{4\pi}\ \frac{|[\widetilde{V}^r(\theta,\phi)]\ [Y]\ [V^s]|^2}{\text{Re}\left\{[\widetilde{V}^s]\ [Y^*]\ [V^{s*}]\right\}} \tag{3-33}$$

where $[V^r(\theta,\phi)]$ is given by (3-29) for various angles of incidence θ,ϕ. If the total power gain pattern is desired, the g's for two orthogonal polarizations may be added together.

D. Wire Scatterers. Consider now the field scattered by a wire object in a plane wave incident field. Figure 3-4 represents a scatterer and two distant current elements, $I\underline{\ell}_t$ at the transmitting point $\underline{r}_t$, and $I\underline{\ell}_r$ at the receiving point $\underline{r}_r$. The $I\underline{\ell}_t$ is adjusted to produce a unit plane wave at the scatterer

$$\underline{E}^t = \underline{u}_t\ e^{-j\underline{k}_t \cdot \underline{r}_n} \tag{3-34}$$

where the notation is analogous to that of (3-26). The voltage excitation matrix (3-14) is then

$$[V^t] = \begin{bmatrix} \underline{E}^t(1) \cdot \Delta\underline{\ell}_1 \\ \underline{E}^t(2) \cdot \Delta\underline{\ell}_2 \\ \vdots \\ \underline{E}^t(N) \cdot \Delta\underline{\ell}_N \end{bmatrix} \tag{3-35}$$

and the current $[I]$ is given by (3-21) with $[V] = [V^t]$. The field produced by $[I]$ can then be found by conventional techniques.

The distant scattered field can also be evaluated by reciprocity, the same as in the antenna case. A dipole $I\underline{\ell}_r$ at the receiving point is

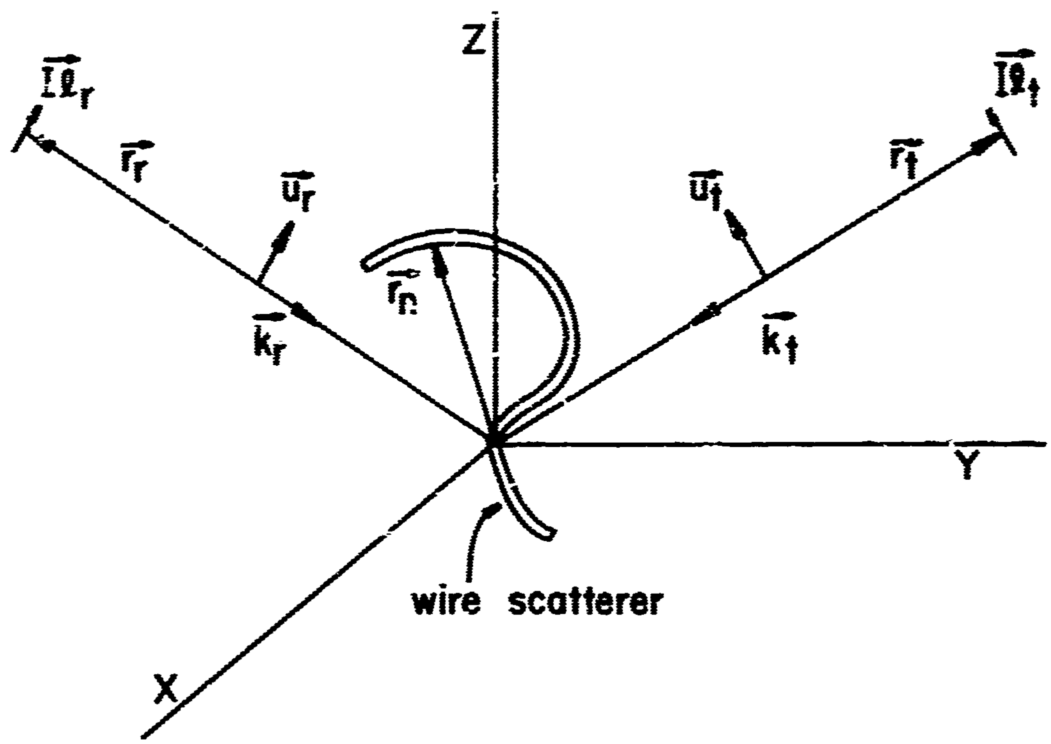


Fig. 3–4 Definitions for plane-wave scattering.

adjusted to produce the unit plane wave (3-26) at the scatterer. The scattered field is then given by (3-30) with $[V^s]$ replaced by $[V^t]$, that is,

$$E_r = \frac{\omega \mu e^{-jkr_r}}{j 4\pi r_r} [\tilde{V}^r] [Y] [V^t] \tag{3-36}$$

A parameter of interest is the bistatic scattering cross section σ, defined as that area for which the incident wave contains sufficient power to produce the field E_r by omnidirectional radiation. In equation form, this is

$$\sigma = 4\pi r_r^2 |E_r|^2$$

$$= \frac{\eta^2 k^2}{4\pi} |[\tilde{V}^r] [Y] [V^t]|^2 \tag{3-37}$$

For the monostatic cross section, set $[V^r] = [V^t]$ in (3-37). The cross section depends on the polarization of the incident wave and of the receiver. A better description of the scatterer can be made in terms of a scattering matrix.[2]

Another parameter of interest is the total scattering cross section σ_t, defined as the ratio of the total scattered power to the power density of the incident wave. The total power radiated by [I] is given by (3-32) for any excitation; hence the scattered power is given by (3-32) with $[V^s]$ replaced by $[V^t]$. The incident power density is $1/\eta$, so

$$\sigma_t = \eta \,\mathrm{Re}\, [\tilde{V}^t] [Y^*] [V^{t*}] \tag{3-38}$$

Note that σ_t is dependent on the polarization of the incident wave.

E. Discussion. The solution of this chapter is a first order solution to the approximate integral equation (3-6). In the present solution (1) the vector potential was evaluated using a step function approximation to the current, (2) the scalar potential was evaluated using a step function approximation to the charge, and (3) segments of integration were approximated by straight line paths. Higher order solutions can be obtained by more accurate evaluation of (1), (2), and (3) above. For example, a linear approximation to the current and charge could be used, or each segment of integration could be approximated by a circular arc. However, increasing the order of solution greatly increases the complexity of the solution. If a higher order solution is desired, a numerical procedure, obtained by dividing each segment n into a number of subintervals, appears more practicable than an analytic increase of the order.

As the order of solution is increased, much of the complication comes from the treatment of singularities. The derivative of the current (i.e., charge) is discontinuous at wire ends and at any voltage source along the wire. In the first order solution this problem has been more or less ignored and computations appear to justify that this procedure is permissible. For example, at the end of a wire the solution (3-20) treats the charge as an equivalent line segment extending 1/2 interval beyond the end of the wire. The actual charge is singular (or almost so), and could be treated by a special subroutine. While this modification is simple, a similar modification for voltage sources along the wire is not practicable for a general program. This is because the impedance matrix would then depend on the location of the source instead of being a characteristic of the wire object alone. On the basis of experience, it appears that a first order solution with no special treatment of singularities is adequate for most engineering purposes. This is particularly true for far-zone quantities, such as radiation patterns and echo areas, which are relatively insensitive to small variations in the current distribution.

The solution is in the form of an impedance or admittance representation of the scatterer. Hence, the general procedures for determining scattering from loaded bodies, available in the literature,[2] can be used directly. This means that the behavior of wire antennas and scatterers loaded by lumped impedance elements in any of the N intervals can be calculated from the present solution by matrix manipulations. Furthermore, the effect of finite conductivity of the wire, or, more generally, arbitrary boundary conditions at the wire surface, can be analyzed by considering the wire to be appropriately loaded in all intervals along its length.

F. Evaluation of ψ. An accurate evaluation of the scalar ψ function of (3-16) is desired. Let the coordinate origin be located at the point m, and the path of integration lie along the z axis. Then

$$\psi(m,n) = \frac{1}{8\pi\alpha} \int_{-\alpha}^{\alpha} \frac{e^{-jkR_{mn}}}{R_{mn}} \, dz' \qquad (3\text{-}39)$$

where

$$2\alpha = \Delta\ell_n \qquad (3\text{-}40)$$

$$R_{mn} = \begin{cases} \sqrt{\rho^2 + (z - z')^2} & m \neq n \\ \sqrt{a^2 + (z')^2} & m = n \end{cases} \qquad (3\text{-}41)$$

and a = wire radius. The geometry for these formulas is given in Fig. 3-5.

One approximation to the ψ's can be obtained by expanding the exponential in a Maclaurin series, giving

$$\psi = \frac{1}{8\pi\alpha} \int_{-\alpha}^{\alpha} \left(\frac{1}{R_{mn}} - jk - \frac{k^2}{2} R_{mn} + \ldots\right) dz' \qquad (3\text{-}42)$$

The first term is identical with the static potential of a filament of charge. The second term is independent of R_{mn}. Hence, a two term

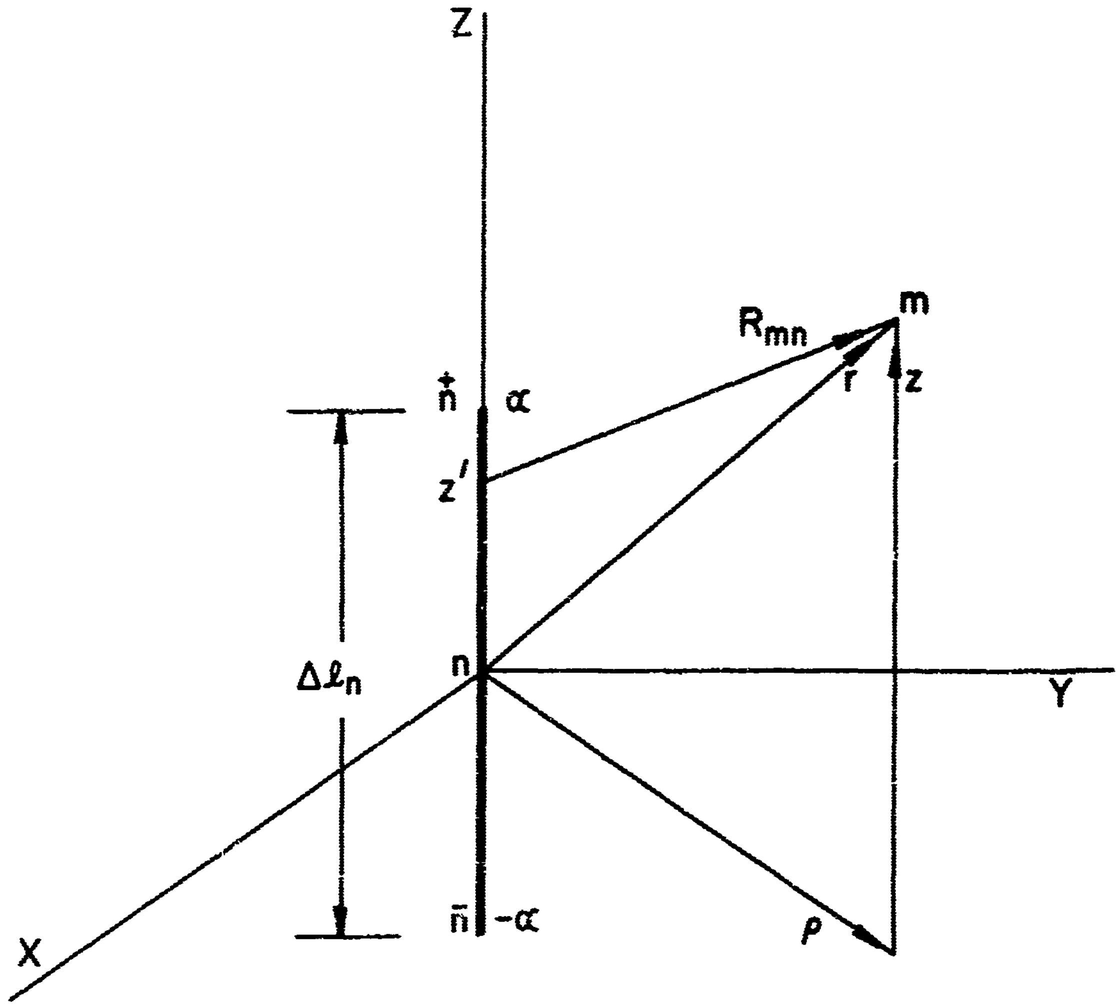


Fig. 3-5 Geometry for evaluating $\psi(m,n)$.

approximation to (3-39) is

$$\psi(m,n) \approx \frac{1}{8\pi\alpha} \log \left[\frac{z + \alpha + \sqrt{\rho^2 + (z+\alpha)^2}}{z - \alpha + \sqrt{\rho^2 + (z-\alpha)^2}}\right] - \frac{jk}{4\pi} \tag{3-43}$$

If $r = \sqrt{\rho^2 + z^2}$ is large, then

$$\psi(m,n) \approx \frac{e^{-jkr}}{4\pi r} \tag{3-44}$$

For a first order solution, one can take (3-43) as applying for small r, say $r \leq 2\alpha/3$, and (3-44) for large r, say $r > 2\alpha/3$.

For higher order approximations, more rapid convergence can be obtained by taking a phase term e^{-jkr} out of the integrand. Then

$$\psi = \frac{e^{-jkr}}{8\pi\alpha} \int_{-\alpha}^{\alpha} \frac{e^{-jk(R_{mn} - r)}}{R_{mn}} dz'$$

$$= \frac{e^{-jkr}}{8\pi\alpha} \int_{-\alpha}^{\alpha} \left(\frac{1}{R_{mn}} - \frac{jk(R_{mn}-r)}{R_{mn}} - \frac{k^2(R_{mn}-r)^2}{2R_{mn}} + \ldots\right)dz' \tag{3-45}$$

Term by term integration gives

$$\psi(m,n) = \frac{e^{-jkr}}{8\pi\alpha} [I_1 - jk(I_2 - rI_1)$$

$$- \frac{k^2}{2} (I_3 - 2rI_2 + r^2I_1)$$

$$+ j \frac{k^3}{6} (I_4 - 3rI_3 + 3r^2I_2 - r^3I_1) + \ldots] \tag{3-46}$$

where

$$I_1 = \log \left[\frac{z + \alpha + \sqrt{\rho^2 + (z + \alpha)^2}}{z - \alpha + \sqrt{\rho^2 + (z - \alpha)^2}}\right] \tag{3-47}$$

$$I_2 = 2\alpha \tag{3-48}$$

$$I_3 = \frac{\alpha+z}{2}\sqrt{\rho^2 + (\alpha+z)^2} + \frac{\alpha-z}{2}\sqrt{\rho^2 + (z-\alpha)^2} + \frac{\rho^2}{2} I_1 \tag{3-49}$$

$$I_4 = 2\alpha\rho^2 + \frac{2\alpha^3 + 6\alpha z^2}{3} \tag{3-50}$$

An expansion of the type (3-46) is theoretically valid for all r, but it fails numerically for large r because it involves subtractions of numbers almost equal in magnitude. For $\rho < a$, one should set $\rho = a$ in the expansion.

An expression suitable for large r is obtained by expanding (3-39) in a Maclaurin series in z' as

$$\psi = \frac{1}{8\pi\alpha}\int_{-\alpha}^{\alpha} [f(0) + f'(0)z' + \frac{1}{2!} f''(0)\,(z')^2 + \ldots]\, dz' \tag{3-51}$$

where

$$f(z') = \frac{e^{-jk\sqrt{\rho^2 + (z - z')^2}}}{\sqrt{\rho^2 + (z - z')^2}}$$

When a five term expansion of (3-51) is integrated term by term, there results

$$\psi = \frac{e^{-jkr}}{4\pi r}\,[A_o + jkaA_1 + (ka)^2A_2 + j(ka)^3A_3 + (ka)^4A_4] \tag{3-52}$$

where

$$A_o = 1 + \frac{1}{6}\left(\frac{\alpha}{r}\right)^2 [-1 + 3\left(\frac{z}{r}\right)^2] + \frac{1}{40}\left(\frac{\alpha}{r}\right)^4 [3 - 30\left(\frac{z}{r}\right)^2 + 35\left(\frac{z}{r}\right)^4]$$

$$A_1 = \frac{1}{6}\left(\frac{\alpha}{r}\right) [-1 + 3\left(\frac{z}{r}\right)^2] + \frac{1}{40}\left(\frac{\alpha}{r}\right)^3 [3 - 30\left(\frac{z}{r}\right)^2 + 35\left(\frac{z}{r}\right)^4] \tag{3-53}$$

$$A_2 = -\frac{1}{6}\left(\frac{z}{r}\right)^2 - \frac{1}{40}\left(\frac{\alpha}{r}\right)^2 \left[1 - 12\left(\frac{z}{r}\right)^2 + 15\left(\frac{z}{r}\right)^4\right]$$

$$A_3 = \frac{1}{60}\left(\frac{\alpha}{r}\right) \left[+3\left(\frac{z}{r}\right)^2 - 5\left(\frac{z}{r}\right)^4\right] \qquad (3\text{-}53)$$

$$A_4 = \frac{1}{120}\left(\frac{z}{r}\right)^4$$

For accuracy of better than one percent, one can use (3-46) for $r < 10\alpha$ and (3-52) for $r \geq 10\alpha$.

An alternative derivation of the type of (3-52) can be obtained as follows. For $r > z'$, one has the expansion

$$\frac{e^{-jkR_{mn}}}{-jkR_{mn}} = \sum_{n=0}^{\infty} (2n+1)\ j_n(kz')\ h_n^{(2)}(kr)\ P_n\left(\frac{z}{r}\right) \qquad (3\text{-}54)$$

where j_n are the spherical Bessel functions of the first kind, $h_n^{(2)}(kr)$ are the spherical Hankel functions of the second kind, and $P_n(z/r)$ are the Legendre polynomials. If (3-54) is substituted into (3-39) and integrated term by term, there results

$$\psi(m,n) = \frac{1}{4\pi j} \sum_{n=0}^{\infty} b_n h_n^{(2)}(kr)\ P_n\left(\frac{z}{r}\right) \qquad (3\text{-}55)$$

where

$$b_n = \frac{2n+1}{2\alpha} \int_{-k\alpha}^{k\alpha} j_n(x)\ dx \qquad (3\text{-}56)$$

Equation (3-55) can be rearranged into the form of (3-52), although the recurrence formulas for $h_n^{(2)}$ and P_n make computation directly from (3-55) almost as easy.

G. References.

1. R. F. Harrington, "Time-Harmonic Electromagnetic Fields," McGraw-Hill Book Co., New York, 1961, Chaps. 3 and 7.

2. R. F. Harrington, "Theory of Loaded Scatterers," Proc. IEE (London), vol. 111, no. 4, April 1964, pp. 617-623.

IV. CALCULATIONS FOR LINEAR WIRES

A. Piecewise-linear Current, Galerkin's Method. A linear wire antenna or scatterer is one constructed of a straight piece of wire, or of several colinear pieces of wire. Again the wire cross section is assumed circular, and its diameter small compared to wavelength. This is a special case of the problem considered in Chapter III, and some calculations were made using that theory. However, to obtain faster convergence, the final calculations were made using a piecewise-linear approximation to the current. This is equivalent to using "triangle" functions as expansion functions in the method of moments. For this section, the weighting functions were taken to be the same triangle functions, and hence the solution corresponds to Galerkin's method.

Figure 1 shows a straight section of wire, and defines the coordinate system. The wire extends from $z = 0$ to $z = L$ along the z axis, and is of radius a. It is assumed that only the axial component of the current on the wire is significant, and it is expressed in terms of the net current $I(z)$ at any point z. As discussed in Chapter III, the problem is represented by the operator equation

$$LI(z) = E^i(z) \qquad (4\text{-}1)$$

where E^i is the z component of the impressed electric field at the wire surface. The operator L is determined by the usual vector potential method, and is given by

$$LI = \frac{j}{2a\omega\epsilon}\left(\frac{d^2}{dz^2} + k^2\right)\int_0^L G\left(\frac{z-z'}{2a}\right) I(z')\, dz' \qquad (4\text{-}2)$$

where

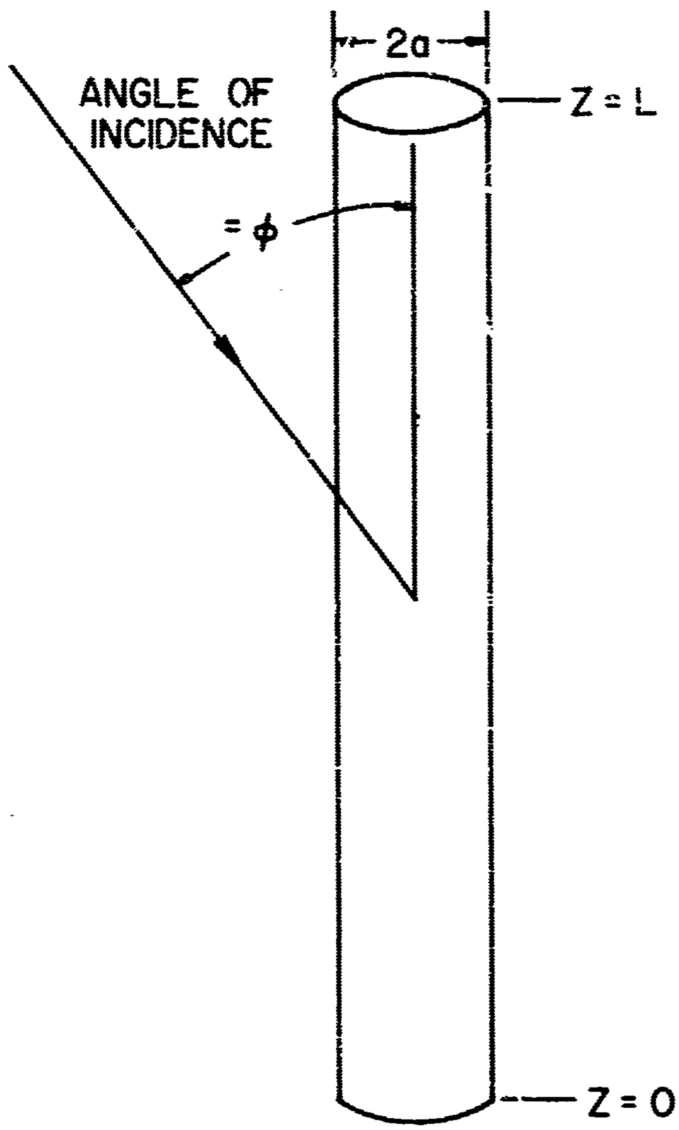


Fig. 4—1 LINEAR ANTENNA

$$G(x) = \frac{1}{2\pi^2} \int_0^{\pi/2} \frac{e^{-j2ak\sqrt{x^2 + \sin^2\theta}}}{\sqrt{x^2 + \sin^2\theta}} \, d\theta \qquad (4\text{-}3)$$

The boundary conditions for the current are

$$I(0) = I(L) = 0 \qquad (4\text{-}4)$$

since no charge can accumulate at the ends of the wire.

For convenience, consider the wire to be $(n+1)$ meters long. Let $I(z)$ be approximated by a series of n triangle functions

$$I(z) = \sum_{i=1}^{n} I_i \, T(z - i) \qquad (4\text{-}5)$$

where

$$T(z) = \begin{cases} 1 - |z|, & |z| < 1 \\ 0, & |z| > 1 \end{cases} \qquad (4\text{-}6)$$

Note that (4-5) satisfies the boundary conditions (4-4). The approximate current (4-5) has the value $I(i) = I_i$, and is linear on each interval defined by successive integers. Following Galerkin's method (the method of moments with $w_i = f_i$), one obtains the usual matrix equation

$$\sum_{i=1}^{n} Z_{ji} \, I_i = V_j \qquad (4\text{-}7)$$

where

$$Z_{ji} = \int_{j-1}^{j+1} T(z-j) \, LT(z-i) \, dz \qquad (4\text{-}8)$$

$$V_j = \int_{j-1}^{j+1} T(z-j) E^i(z) \, dz \qquad (4\text{-}9)$$

The integrals (4-8) were evaluated by procedures part analytical and part numerical. The inversion of the matrix equation (4-7) gives an approximate solution for the current on the wire according to (4-5). As discussed in Section I-F, this solution is also a variational solution to the problem.

B. Point Matching and Finite Differences. The solution of Chapter III was obtained using point-matching and finite-difference approximations. In this section the linear wire is again treated in this manner, but the current expansion functions are taken as triangle functions instead of the pulse functions of Chapter III. It was found that solutions of this type (piecewise-linear current) converged significantly faster than those of the step-function type. In fact, for approximately the same accuracy, the number of antenna subsections required in the piecewise-linear solution was about one-half that needed in the step-function solution.

A direct attempt at a point-matching solution using the expansion functions (4-6) and the operator (4-2) fails because the electric field is infinite at points where the derivative of the current is discontinuous. These points are the integers $i = 1, 2, \ldots, n+1$ along the wire. Even if the field is matched midway between these points the solution fails. Basically, this is due to the fact that the field due to triangle current functions is a rapidly varying field, and no one point is sufficient to test it.

However, if one replaces all derivatives by finite difference approximations, these difficulties are circumvented and a meaningful solution is obtained.

The derivation of the solution is analogous to that of Section III-B except that the vector potential alone was used, instead of the vector and scalar potential. The operator (4-2) operating on a triangle function is

$$LT(z) = \frac{j}{2a\omega\epsilon} \left(\frac{d^2}{dz^2} + k^2\right) C(z) \tag{4-10}$$

where

$$C(z) = \int_{-1}^{1} T(z') \, G\left(\frac{z'-z}{2a}\right) dz' \tag{4-11}$$

The increments along the wire have been scaled to unity, hence the finite difference approximation to the second derivative is

$$\frac{d^2C}{dz^2} \approx C(z+1) - 2C(z) + C(z-1) \tag{4-12}$$

The difference operator which approximates (4-10) is therefore

$$L'T(z) = \frac{j}{2a\omega\epsilon} [C(z+1) + (k^2-2) \, C(z) + C(z-1)] \tag{4-13}$$

The approximate equation for the problem is now the operator equation with I given by (4-5) and L replaced by L', that is,

$$\sum_{i=1}^{n} I_i \, L'T(z - i) = E^i(z) \tag{4-14}$$

The point-matching method requires that (4-14) be satisfied at each point $z = j$ along the wire. Hence, the matrix equation for this solution is

$$\sum_{i=1}^{n} Z'_{ji} \, I_i = V_j \tag{4-15}$$

$j = 1, 2, \ldots, n$, where the $V_j = E^i(j)$, and from (4-13),

$$Z'_{ji} = \frac{j}{2a\omega\epsilon} [C(j-i-1) + (k^2-2) \, C(j-i) + C(j-i+1)] \tag{4-16}$$

The inversion of (4-15) gives an approximate solution for the current according to (4-5).

It was found that when the wire segments were less than $\lambda/10$ in length there was no significant difference in the solution using (4-16) from that using (4-8). Since the computation of (4-16) is somewhat simpler than (4-8), most actual computations were made using (4-16). However, the computer program was written so that either solution could be used as desired.

C. Linear Antennas with Arbitrary Feed. Computations for linear antennas fed by a voltage source at an arbitrary point along the wire have been made for length to diameter ratios ranging from $L/2a = 10$ to 2000, and for length to wavelength ratios up to $L/\lambda = 2.1$. To illustrate the results, some plots of the calculations for the case $L/2a = 74.2$ [corresponding to $\Omega = 2\ln(L/a) = 10$] are given here.

Figure 4-2 shows the input susceptance $Y = G + jB$ to the antenna as the position of the feed is varied along the antenna. It clearly shows how the coupling to the resonances changes as the feed position is changed. Figure 4-3 shows the input impedance $Z = R+jX$ for the same cases as Fig. 4-2. Note that the concept of varying coupling by varying the feed position is less evident from the impedance curves. This is because the equivalent circuit is basically a set of resonators in parallel, which is more naturally treated by the admittance concept. Figure 4-4 shows a comparison of the present solution with that of King[1] and that of Hu.[2] There is apparently little difference in the three solutions for $L/\lambda < 0.6$, but there are differences in the vicinity of $L/\lambda = 1$. The present solution changed less than 5 percent as the number of antenna subsections was increased from 32 to 64, hence it is felt to be reasonably accurate. The previous solutions apparently are less accurate than the present one in the range $L/\lambda > 0.6$.

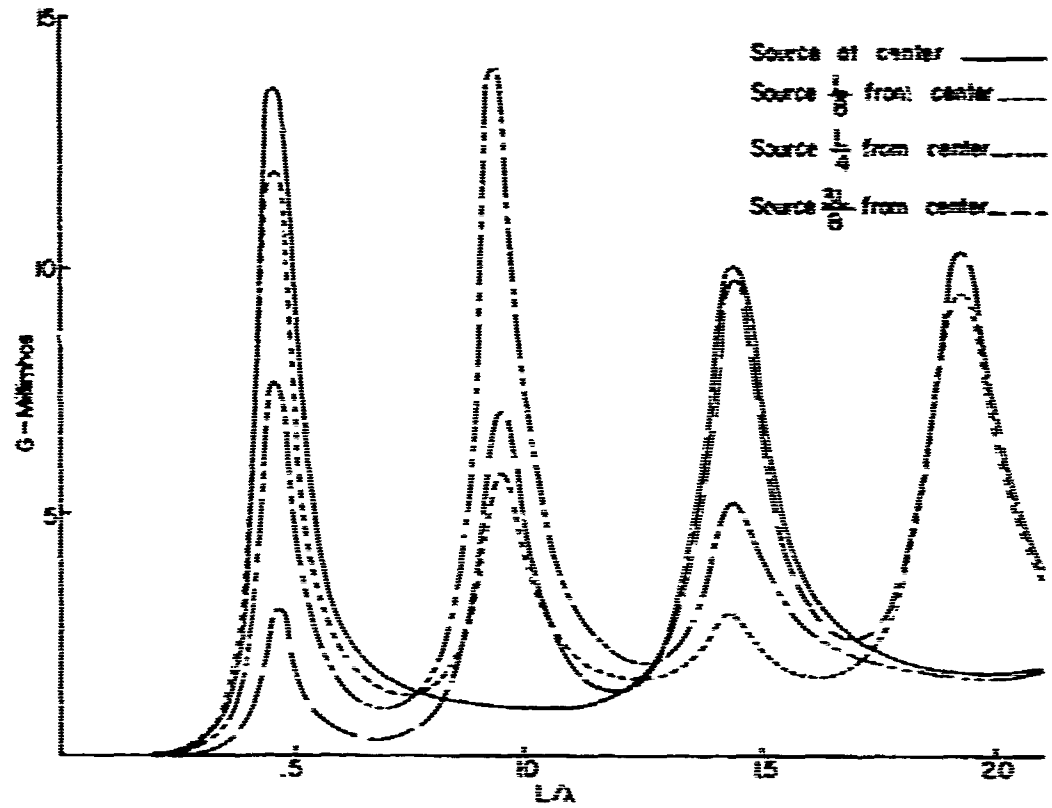


Fig 4-2a Input conductance for a linear antenna, L/2a=742, as the feed point varies

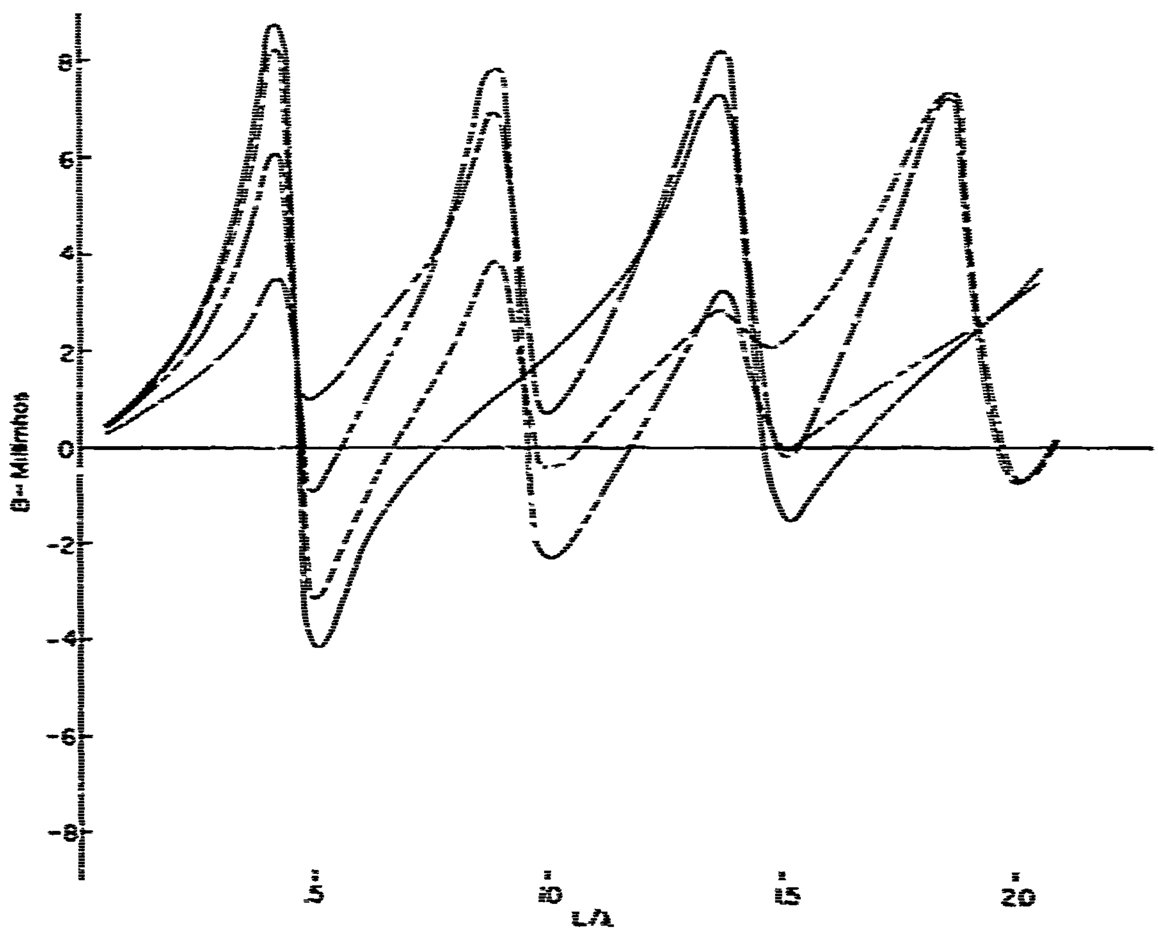


Fig. 4-2b Input susceptance for a linear antenna, L/2a=742, as the feed point varies

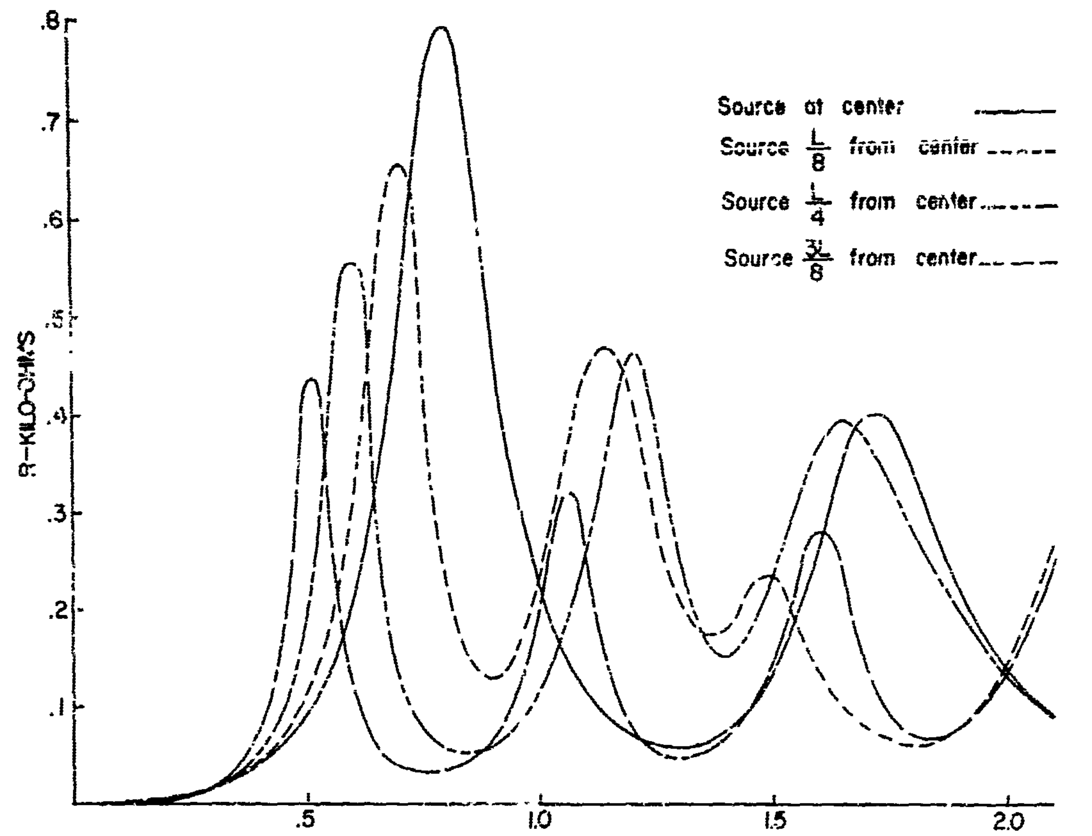


Fig. 4-3a Input resistance for a linear antenna, L/2a = 74.2, as the feed point varies.

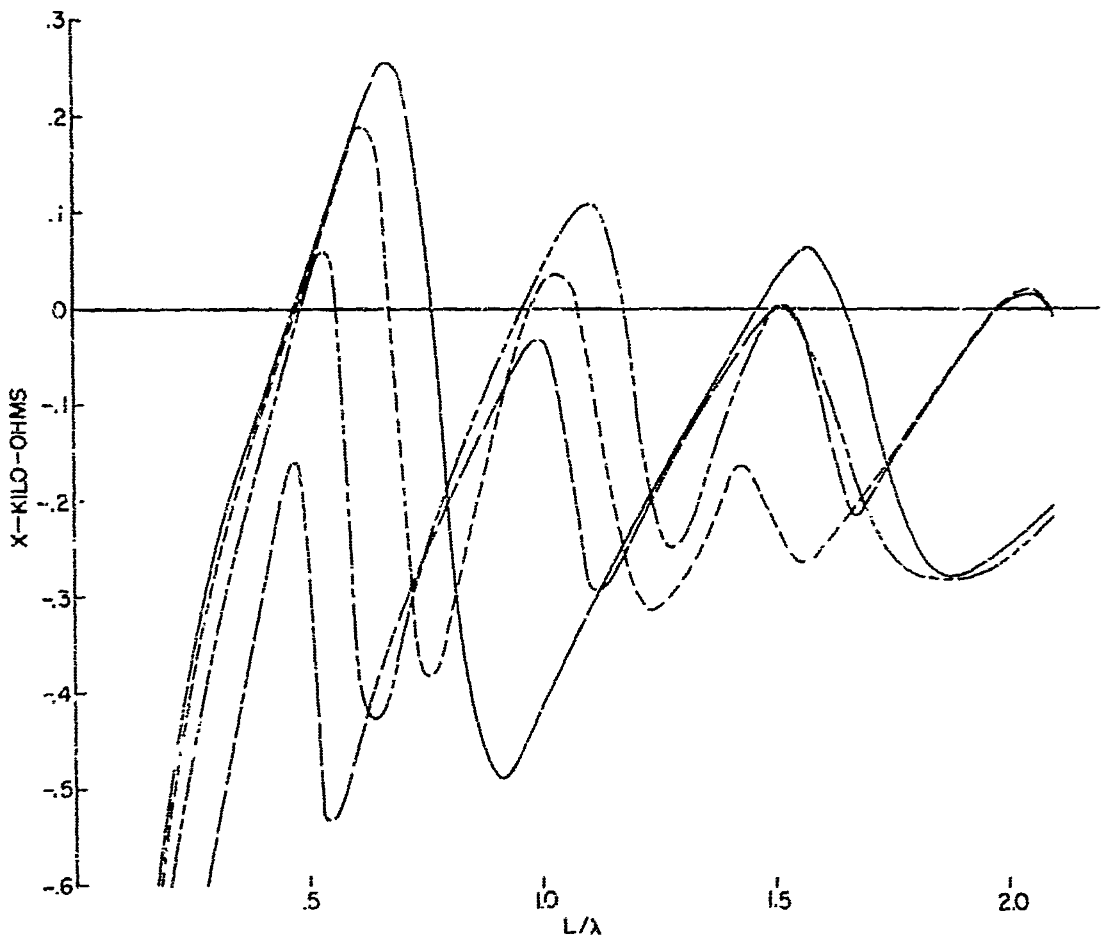
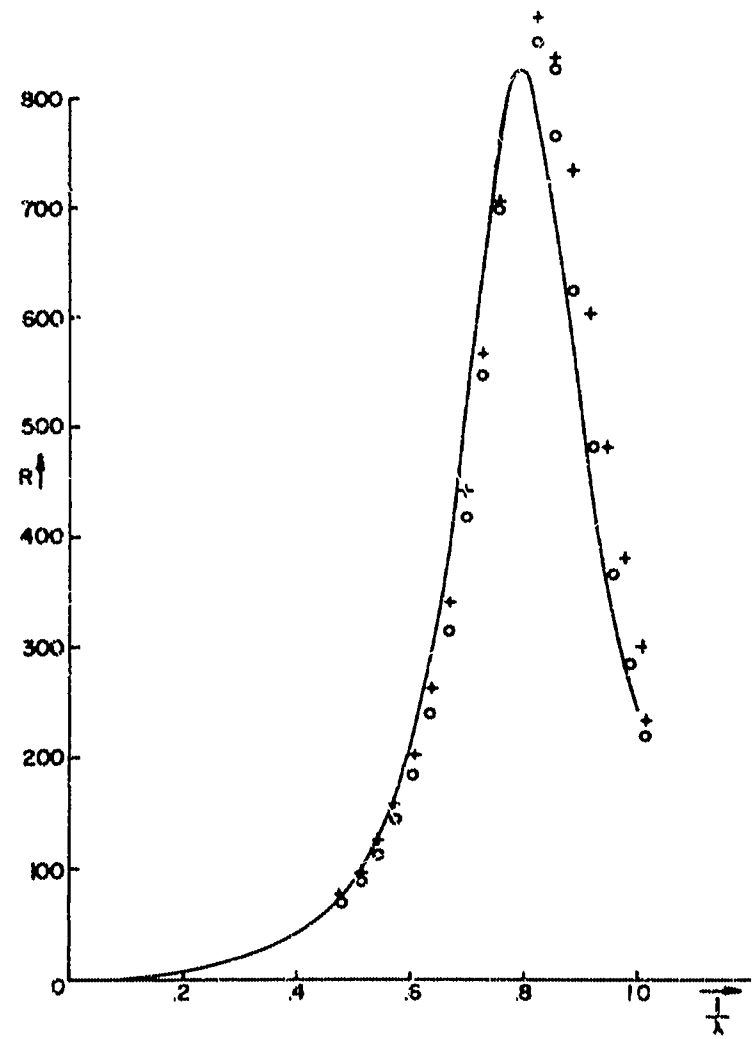
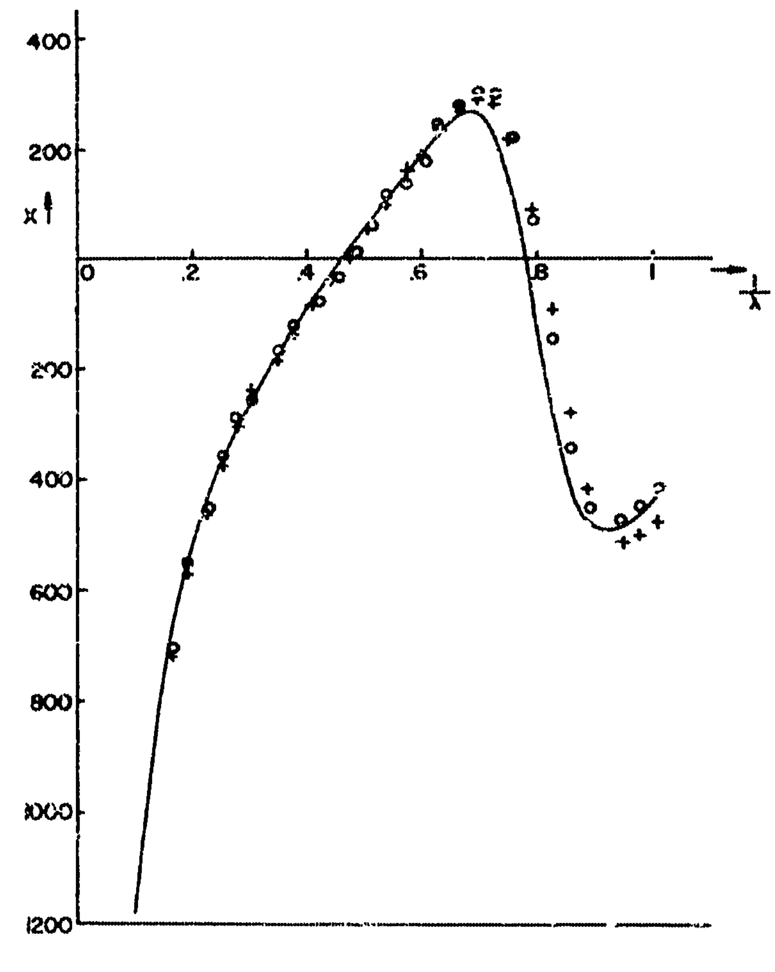


Fig. 4-3b Input reactance for a linear antenna, L/2a = 74.2, as the feed point varies.



(a) input resistance



(b) input reactance

Fig. 4-4

Comparison of the input impedance of a center-fed linear antenna, $L/2a = 74.2$

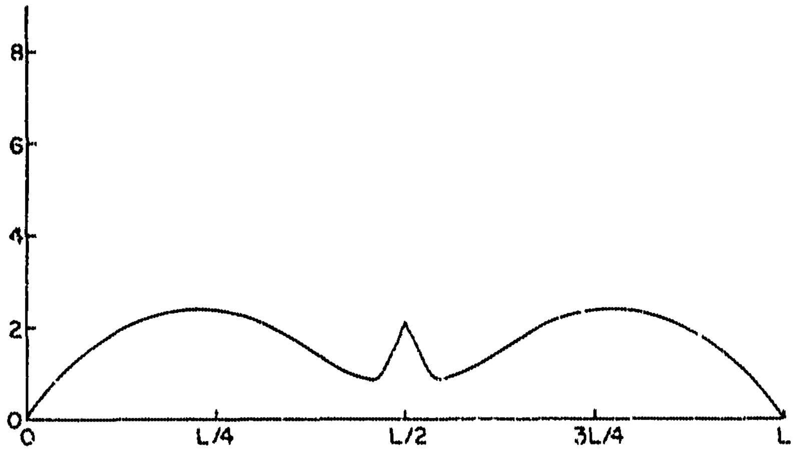
Present theory ——, King's solution ooo, Hu's solutions +++.

Figures 4-5 to 4-7 illustrate how the current distribution on the antenna changes as the feed point changes. Only the magnitude is plotted, but the real and imaginary parts were also calculated. In each case the source strength was one volt. Figure 4-5 is for $L = \lambda$, Figure 4-6 is for $L = 1.5\lambda$, and Figure 4-7 is for $L = 2\lambda$ The case $L = \lambda/2$ showed little change in the shape of the current distribution as the source was moved, and hence was not plotted.

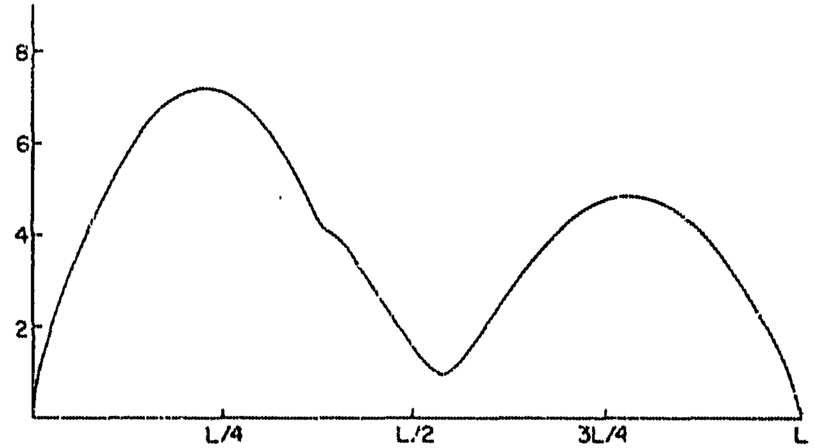
Figures 4-8 to 4-10 show how the gain pattern for the linear antenna changes as the feed point changes. These curves were normalized so that they represent power gain over an isotropic radiator. Figure 4-8 is for $L = \lambda$, Figure 4-9 is for $L = 1.5\lambda$, and Figure 4-10 is for $L = 2\lambda$. The case $L = \lambda/2$ was not plotted since the gain pattern is relatively insensitive to feed position. Note how the antenna tends to behave as a traveling wave antenna as the feed is moved to one end. This is particularly evident in the $L = 2\lambda$ case.

D. Linear Wire Scatterers. Computations for linear wire scatterers excited by plane waves at various angles of incidence have also been made for $L/2a$ ratios varying from 10 to 2000, and L/λ values up to 2.1. The following results for the case $L/2a = 74.2$ ($\Omega = 10$) are given here to illustrate the general behavior of wire scatterers.

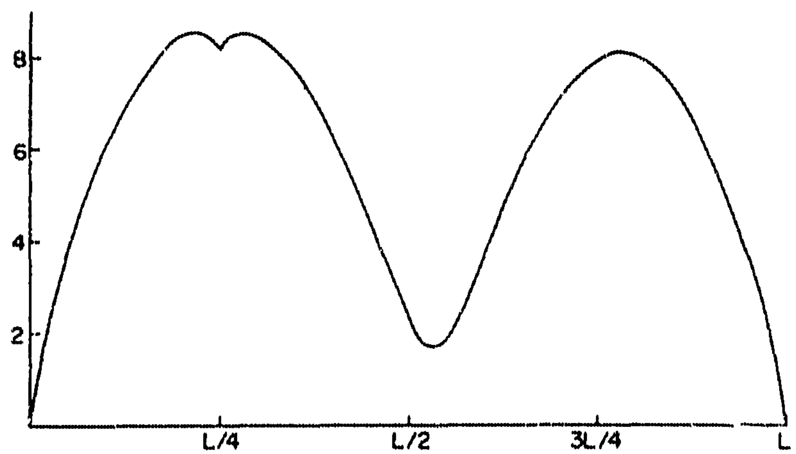
Figure 4-11 shows echo area/λ^2 vs. L/λ and compares the result with the solution of Y. Y. Hu.[2] Professor Hu's solution is actually an impedance type solution using Galerkin's method, with the wire divided into two sections. On each section a constant plus a sinusoid was used for expansion functions. As can be seen from Fig. 4-11, the accuracy of her solution is good over the range of L/λ considered. This indicates that long wires probably can be accurately treated if divided into segments of length up



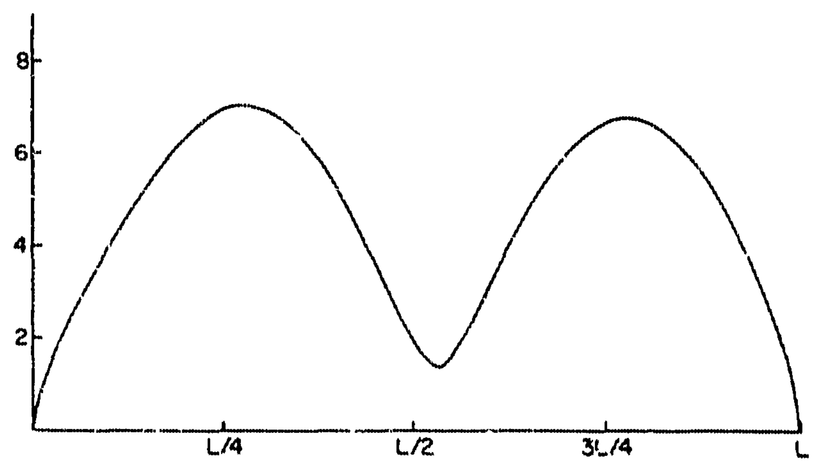
(a) Source at $z = \frac{L}{2}$



(b) Source at $z = \frac{3L}{8}$

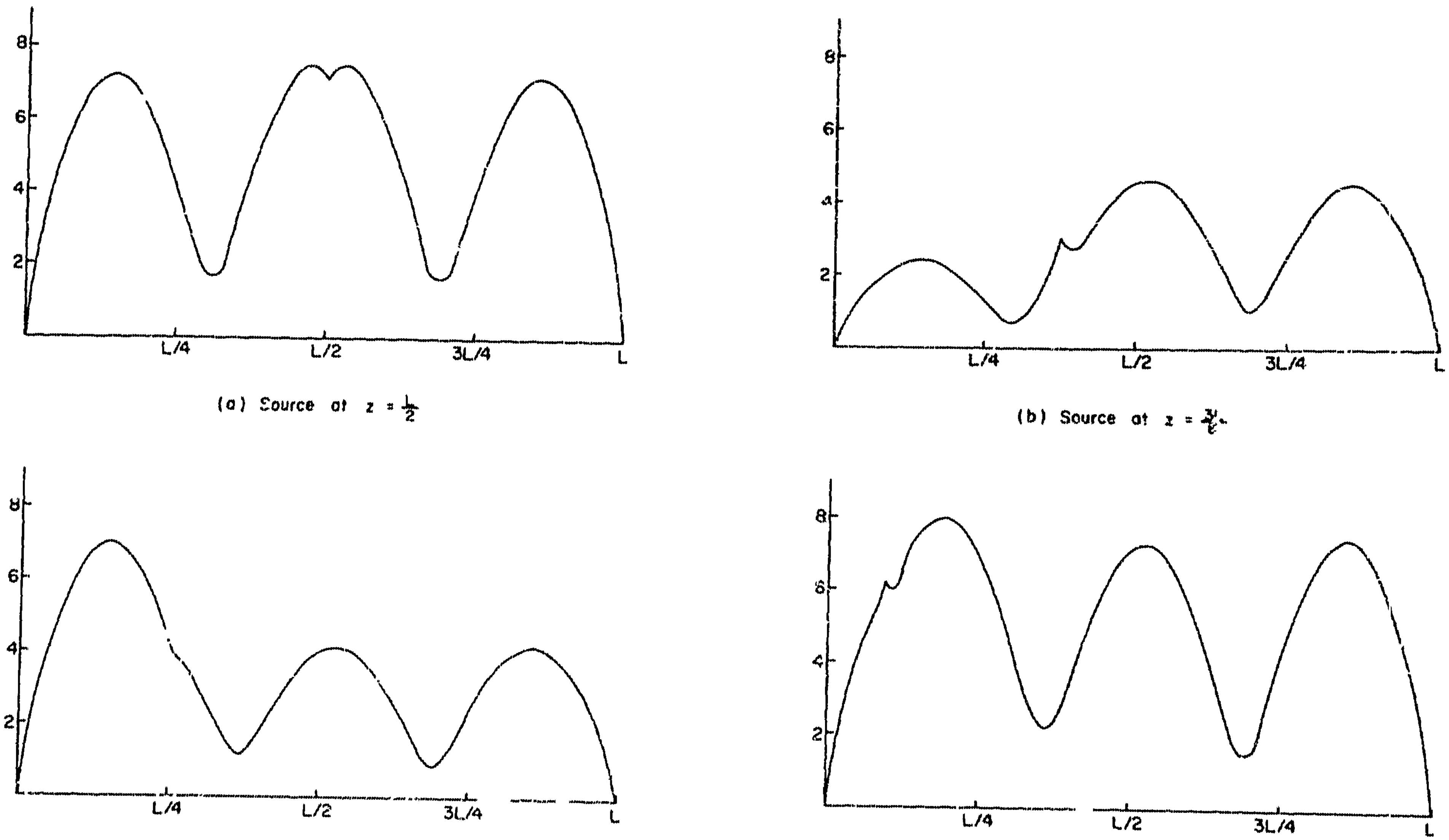


(c) Source at $z = \frac{L}{4}$



(d) Source at $z = \frac{L}{8}$

Fig 4-5 Current (magnitude) on a linear antenna, $L/2a = 74.2$, $L = \lambda$, as the feed point varies.



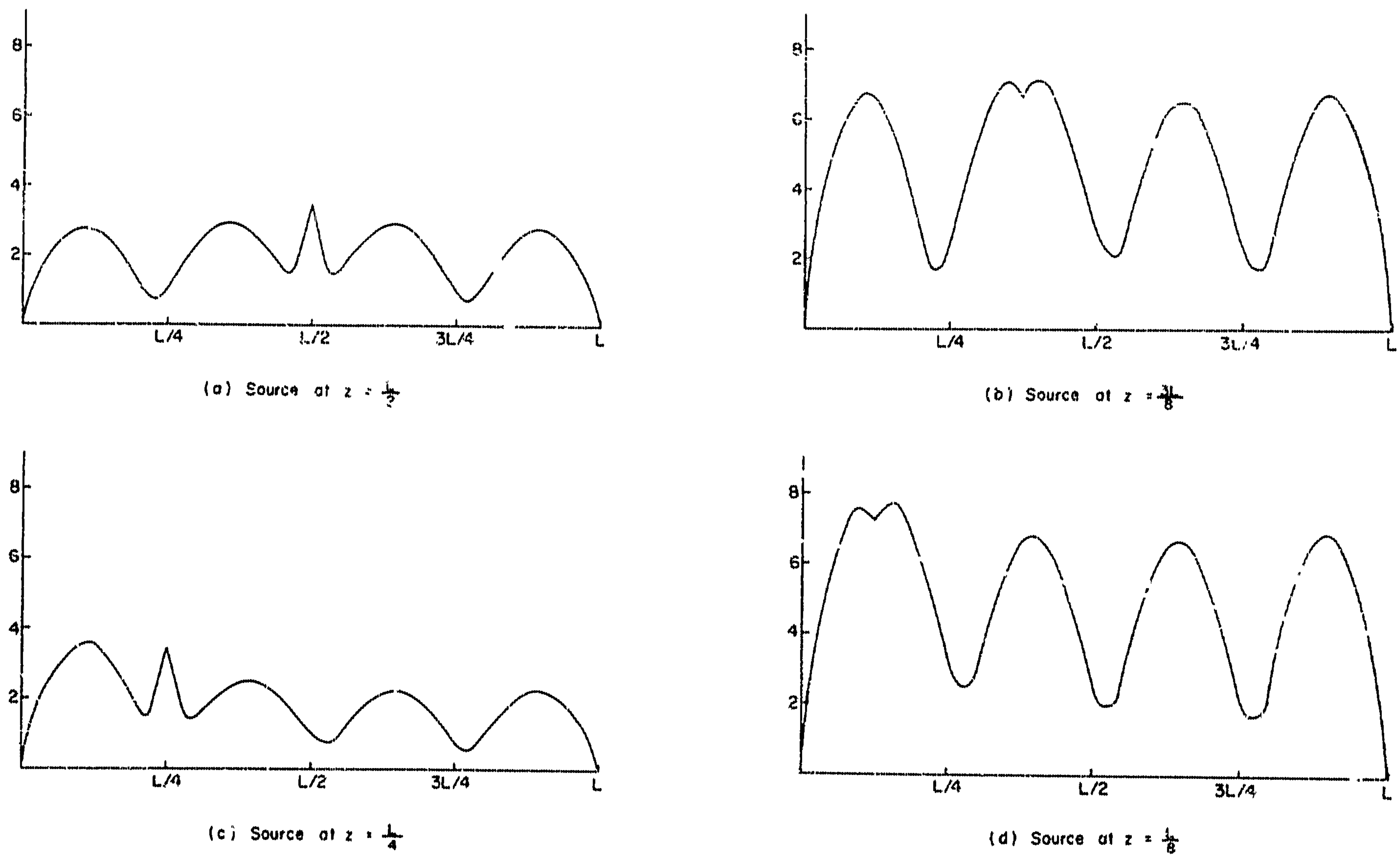
(a) Source at $z = \frac{L}{2}$

(b) Source at $z = \frac{3L}{8}$

(c) Source at $z = \frac{L}{4}$

(d) Source at $z = \frac{L}{8}$

Fig. 4-6 Current (magnitude) on a linear antenna, $L/2a = 74.2$, $L = 1.5\lambda$, as the feed point varies.



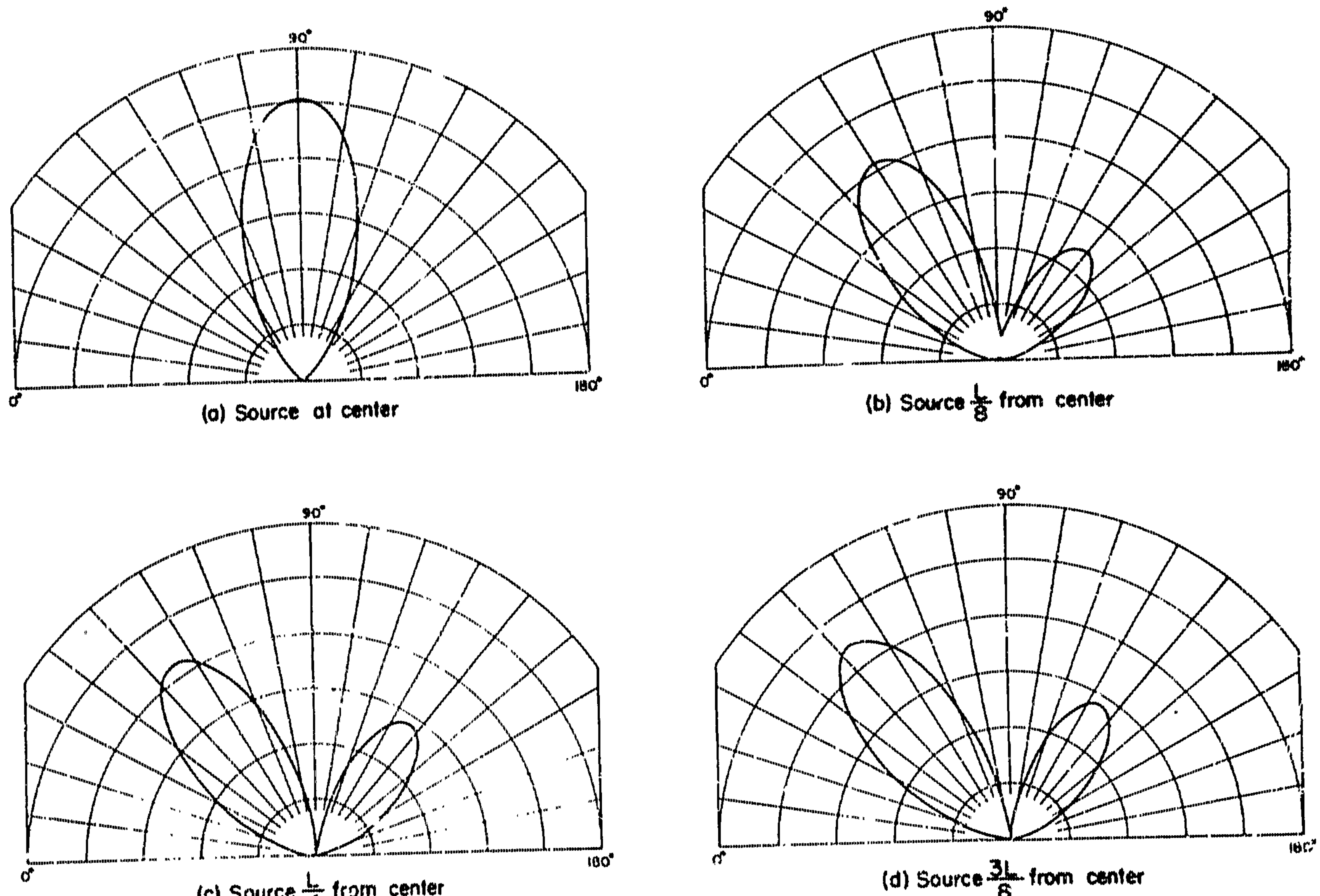
(a) Source at $z = \frac{L}{2}$

(b) Source at $z = \frac{3L}{8}$

(c) Source at $z = \frac{L}{4}$

(d) Source at $z = \frac{L}{8}$

Fig. 4-7 Current (magnitude) on a linear antenna, $L/2a = 74.2$, $L = 2\lambda$, as the feed point varies.



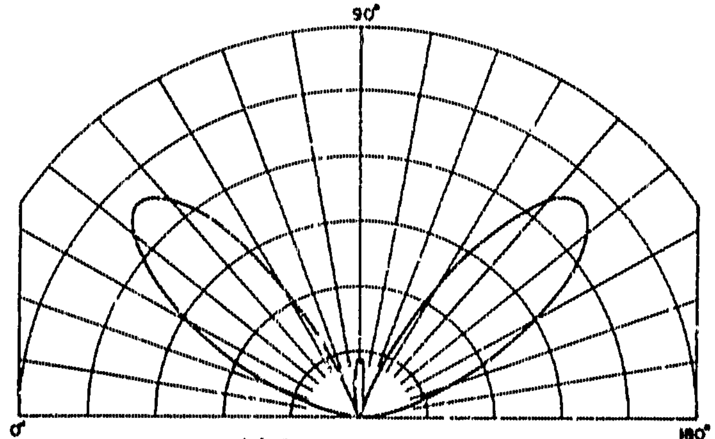
(a) Source at center

(b) Source $\frac{L}{8}$ from center

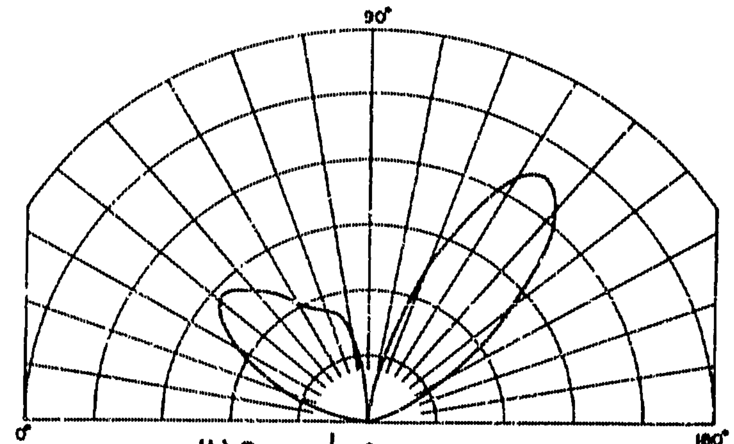
(c) Source $\frac{L}{4}$ from center

(d) Source $\frac{3L}{8}$ from center

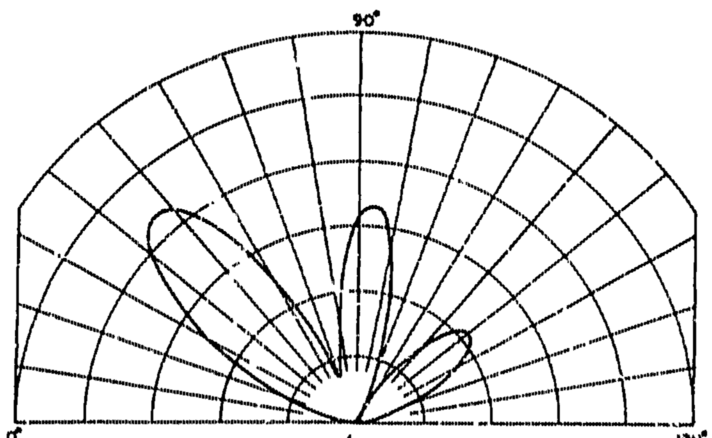
Fig. 4-8 Power gain pattern for a linear antenna, $\frac{L}{2a} = 74.2$, $L = \lambda$ as the feed point varies.



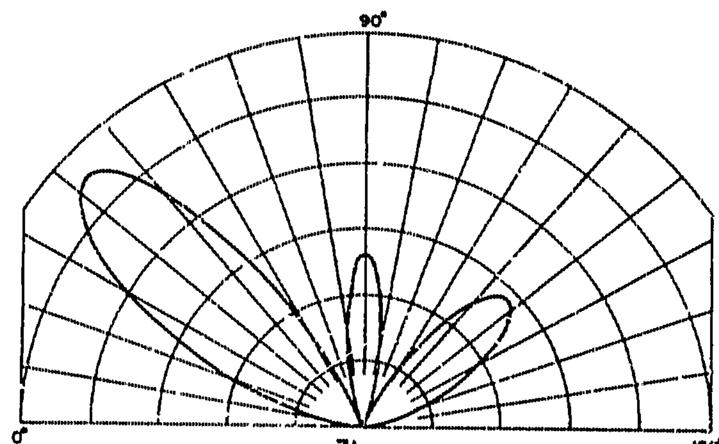
(a) Source at center



(b) Source $\frac{L}{8}$ from center



(c) Source $\frac{L}{4}$ from center



(d) Source $\frac{3L}{8}$ from center

Fig. 4-9 Power gain pattern for a linear antenna, $\frac{L}{2a} = 74.2$, $L = 1.5\lambda$ as the feed point varies.

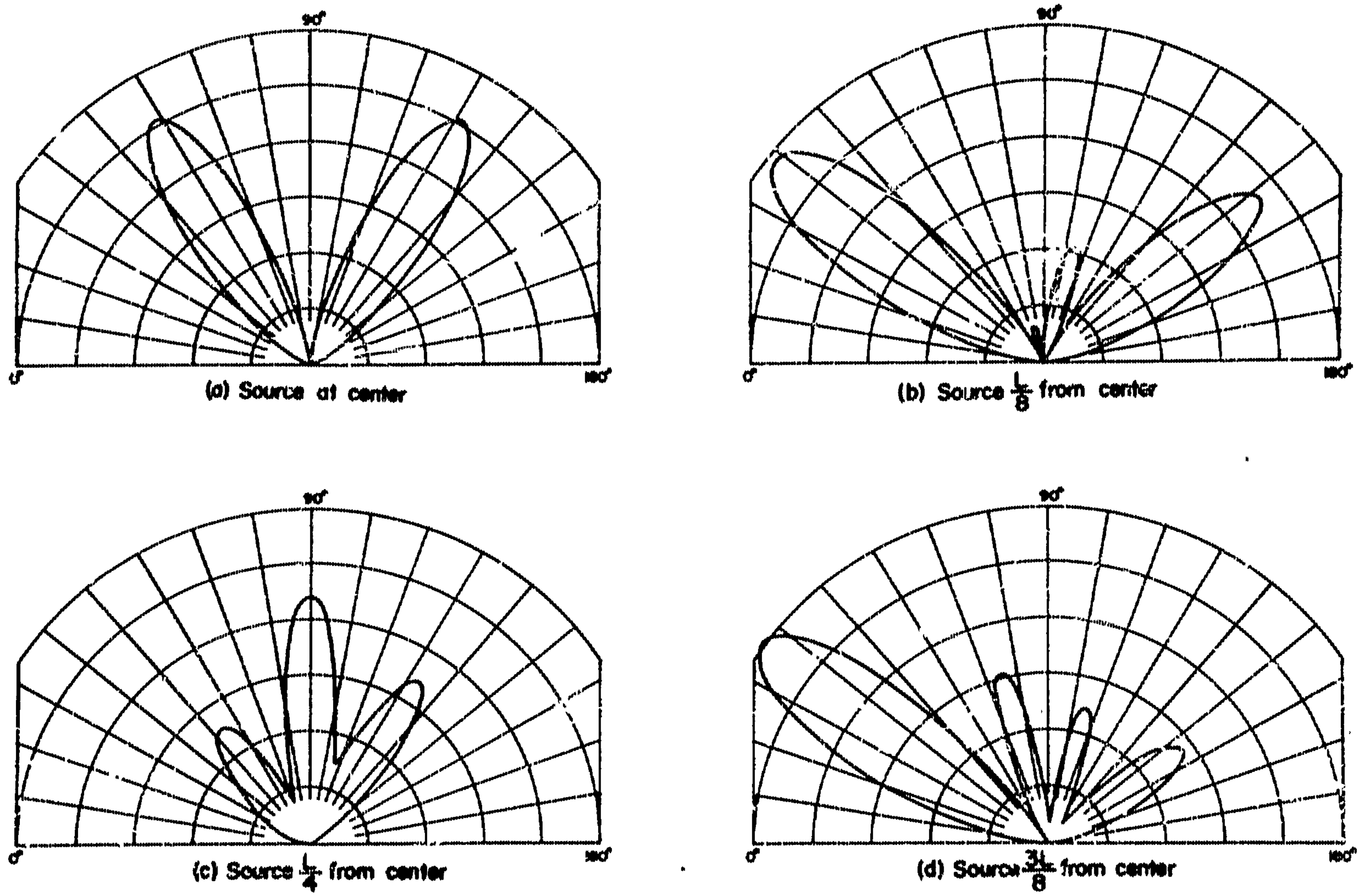


Fig. 4-10 Power gain pattern for a linear antenna, $\frac{L}{2a} = 74.2$, $L = 2\lambda$ as the feed point varies.

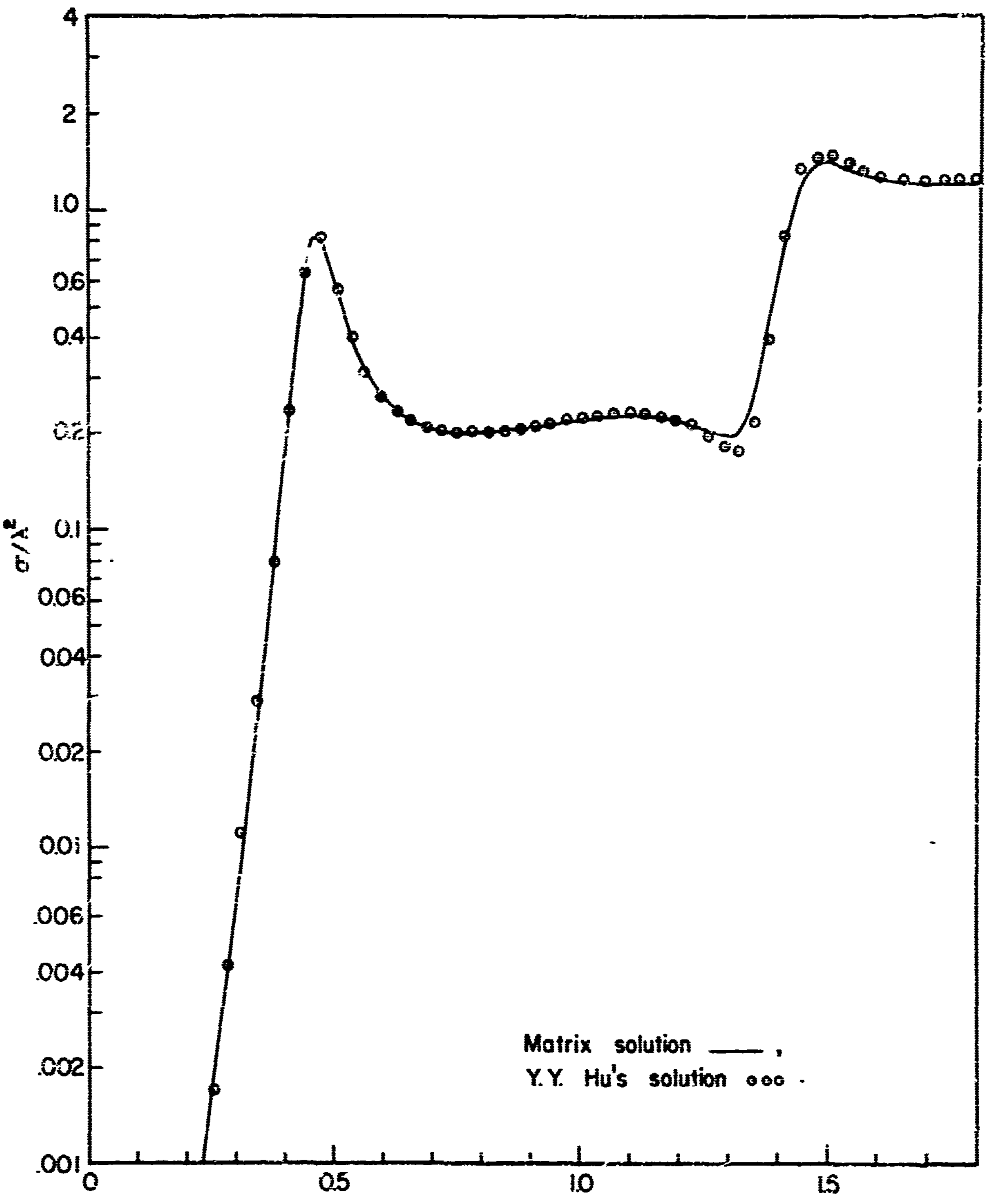


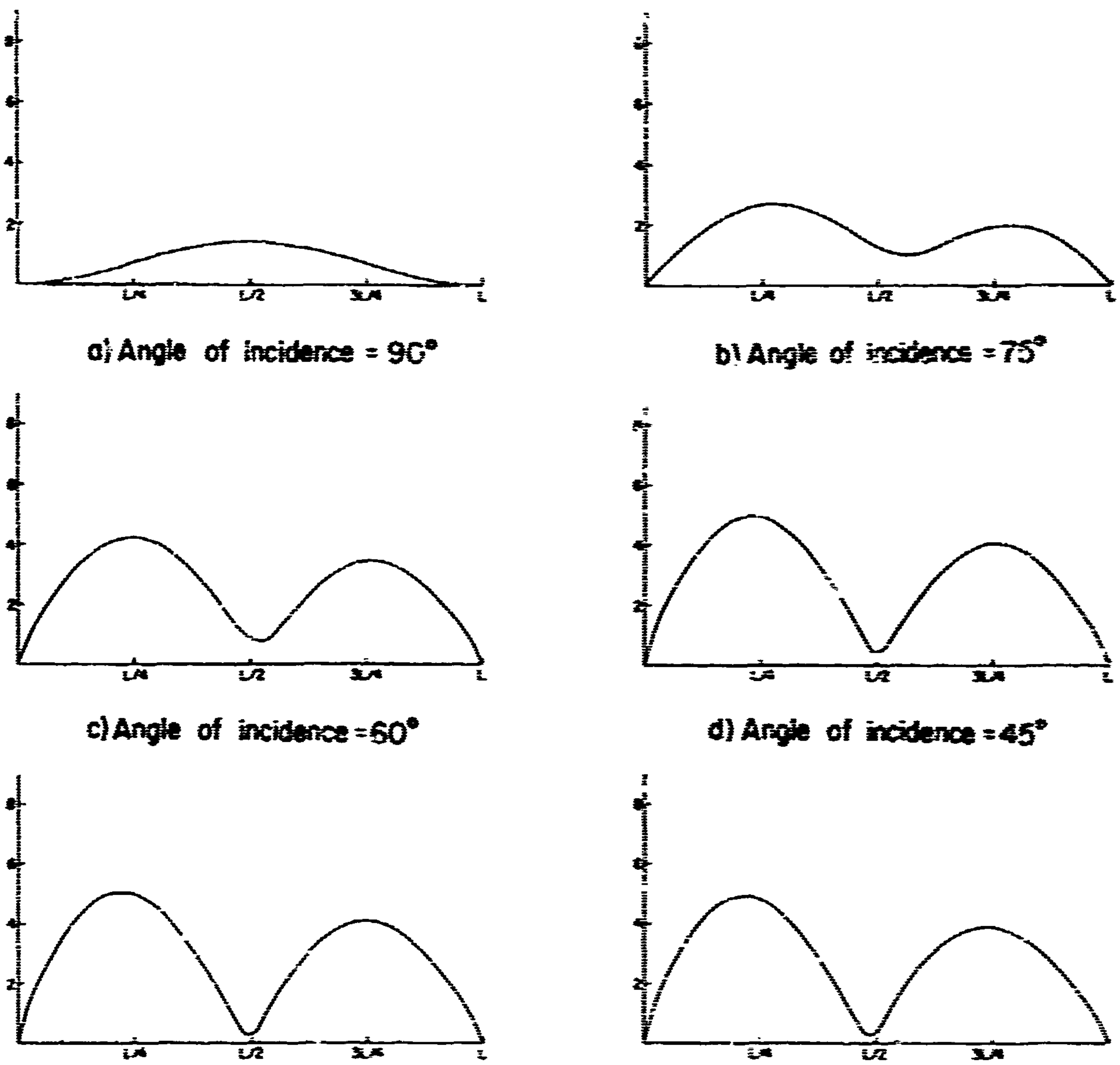
Fig. 4-11 Echo area of a wire scatterer, length to diameter ratio 74.2.

to a wavelength, if constants plus sinusoids are used as expansion functions. However, such a procedure gives rather difficult integrals to evaluate, and the solution of this paper is easier to compute if the wire is not too long.

Figures 4-12 to 4-14 show how the current distribution on the wire changes as the angle of incidence of the plane wave excitation changes. For these curves the magnitude of the component of E parallel to the wire was one volt per wavelength. Only the magnitude of the current is shown, but the real and imaginary components were also calculated. Figure 4-12 is for the case $L = \lambda$, with angles of incidence varying from 90° to 15° in steps of 15°. Figure 4-13 is the corresponding set for $L = 1.5\lambda$, and Figure 4-14 is for $L = 2\lambda$. The current on a wire $L = \lambda/2$ was relatively insensitive to angle of incidence, and is not shown.

Figures 4-15 to 4-17 show the bistatic radar cross section patterns for the same cases as the current was shown in Figures 4-12 to 4-14. The angle of incidence is shown in each case by an arrow. Note that there is a large lobe at an angle of scatter equal to the angle of incidence, but on the other side of the direction normal to the wire. This corresponds to specular diffraction from the wire, and is more pronounced as the wire becomes longer. Figure 4-15 is for wires of length $L = \lambda$, Figure 4-16 for $L = 1.5\lambda$, and Figure 4-17 for $L = 2\lambda$. Again the case $L = \lambda/2$ is not shown because the shape of the cross section patterns is relatively insensitive to the angle of incidence.

E. Loaded Wire Scatterers. The impedance elements characterizing a linear wire are basically the same parameters as the impedance elements used in the theory of loaded scatterers.[3] Hence, it is relatively easy to calculate the behavior of a wire with lumped impedances at points along



a) Angle of incidence = 90°

b) Angle of incidence = 75°

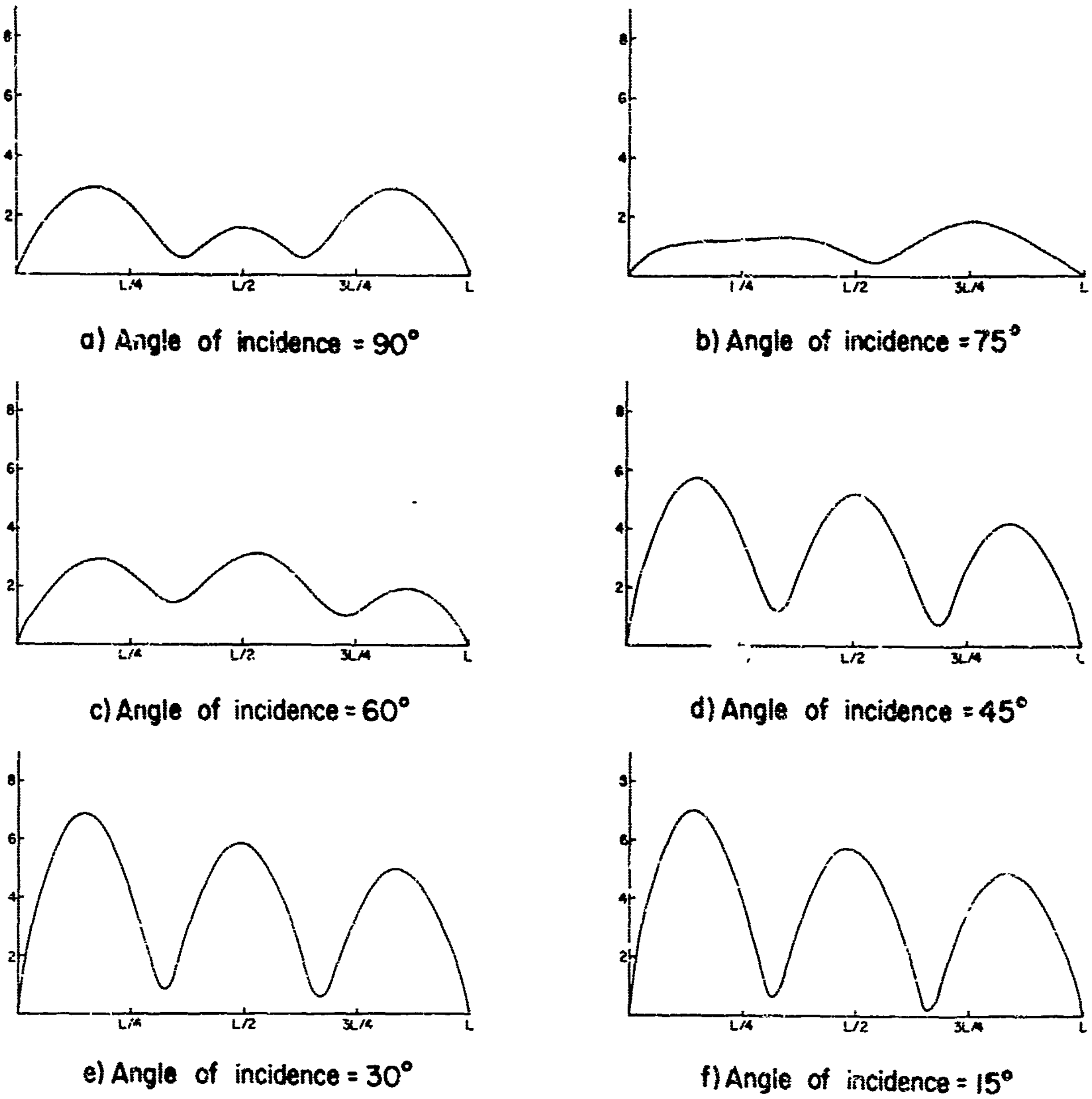
c) Angle of incidence = 60°

d) Angle of incidence = 45°

e) Angle of incidence = 30°

f) Angle of incidence = 15°

Fig. 4-12 Current (magnitude) on a linear antenna $\frac{L}{2a} = 74.2$, $L = \lambda$, for plane wave excitation ($|E_z|$ = one volt per wavelength)



a) Angle of incidence = 90°

b) Angle of incidence = 75°

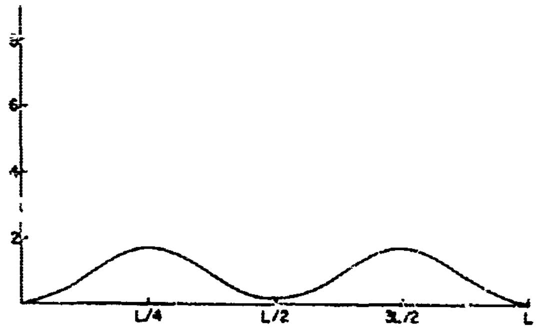
c) Angle of incidence = 60°

d) Angle of incidence = 45°

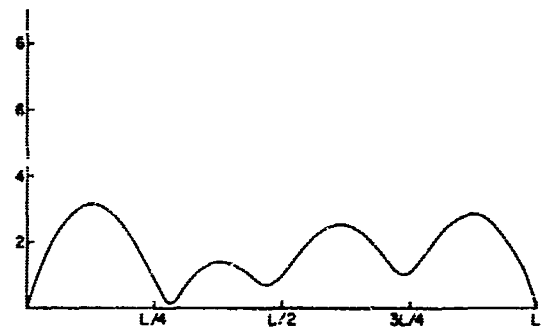
e) Angle of incidence = 30°

f) Angle of incidence = 15°

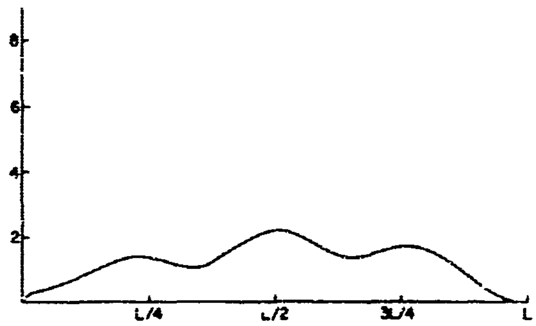
Fig. 4-15 Current (magnitude) on a linear antenna $\frac{L}{2a} = 74.2$, $L = 1.5\lambda$, for plane wave excitation ($|E_z|$ = one volt per wavelength)



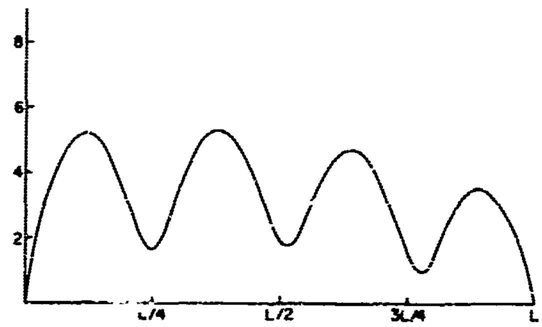
a) Angle of incidence = 90°



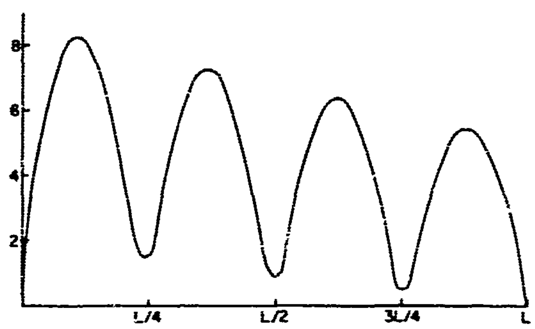
b) Angle of incidence = 75°



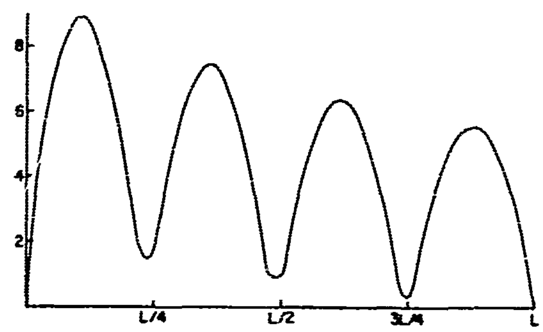
c) Angle of incidence = 60°



d) Angle of incidence = 45°



e) Angle of incidence = 30°



f) Angle of incidence = 15°

Fig. 4-14 Current (magnitude) on a linear antenna $\frac{L}{2a} = 74.2$, $L = 2\lambda$, for plane wave excitation ($|E_z|$ = one volt per wavelength)

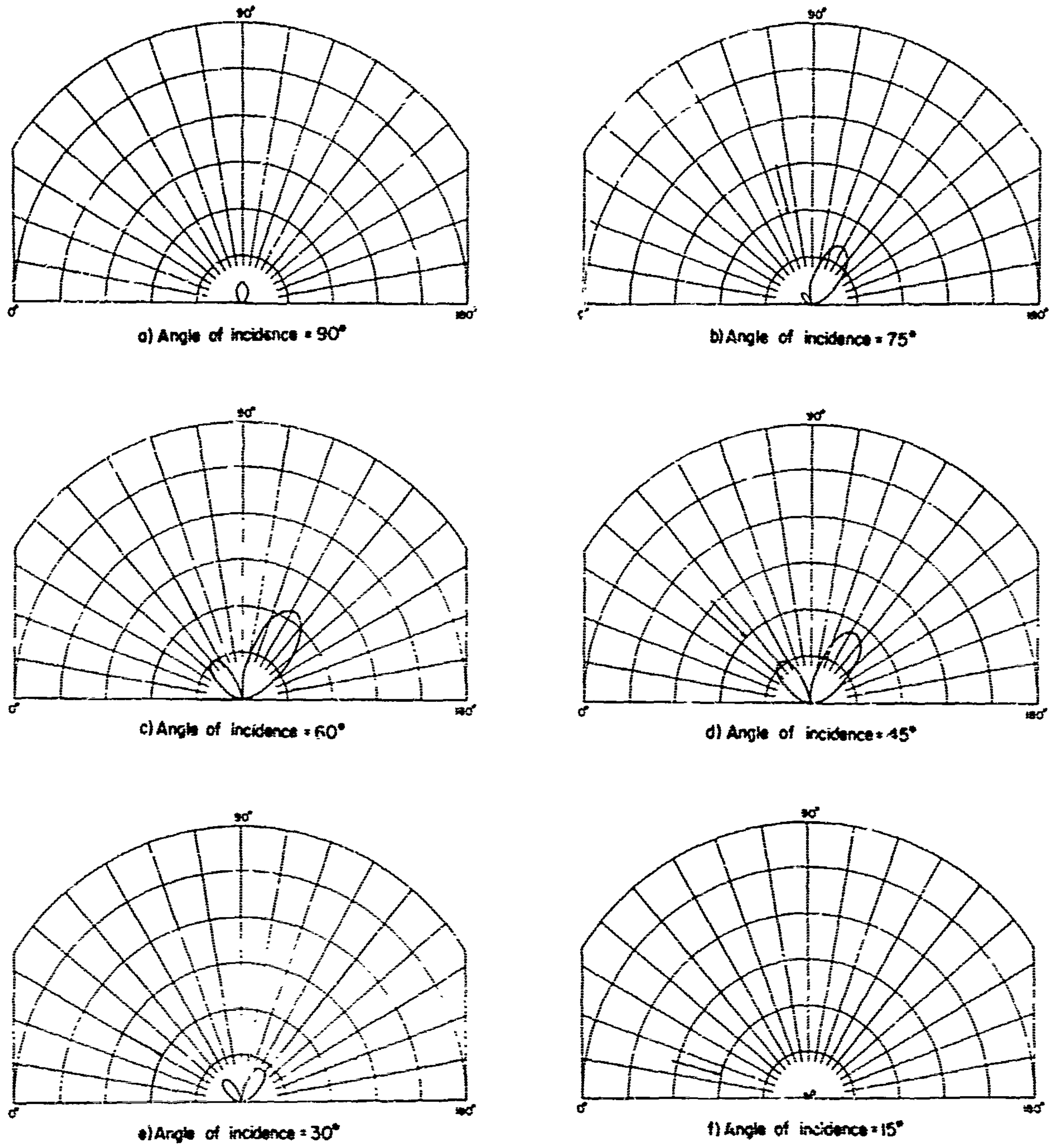


Fig. 4-15 Bistatic radar cross section patterns for a linear antenna, $\frac{L}{2a} = 74.2$, $L = \lambda$, as the angle of incidence varies.

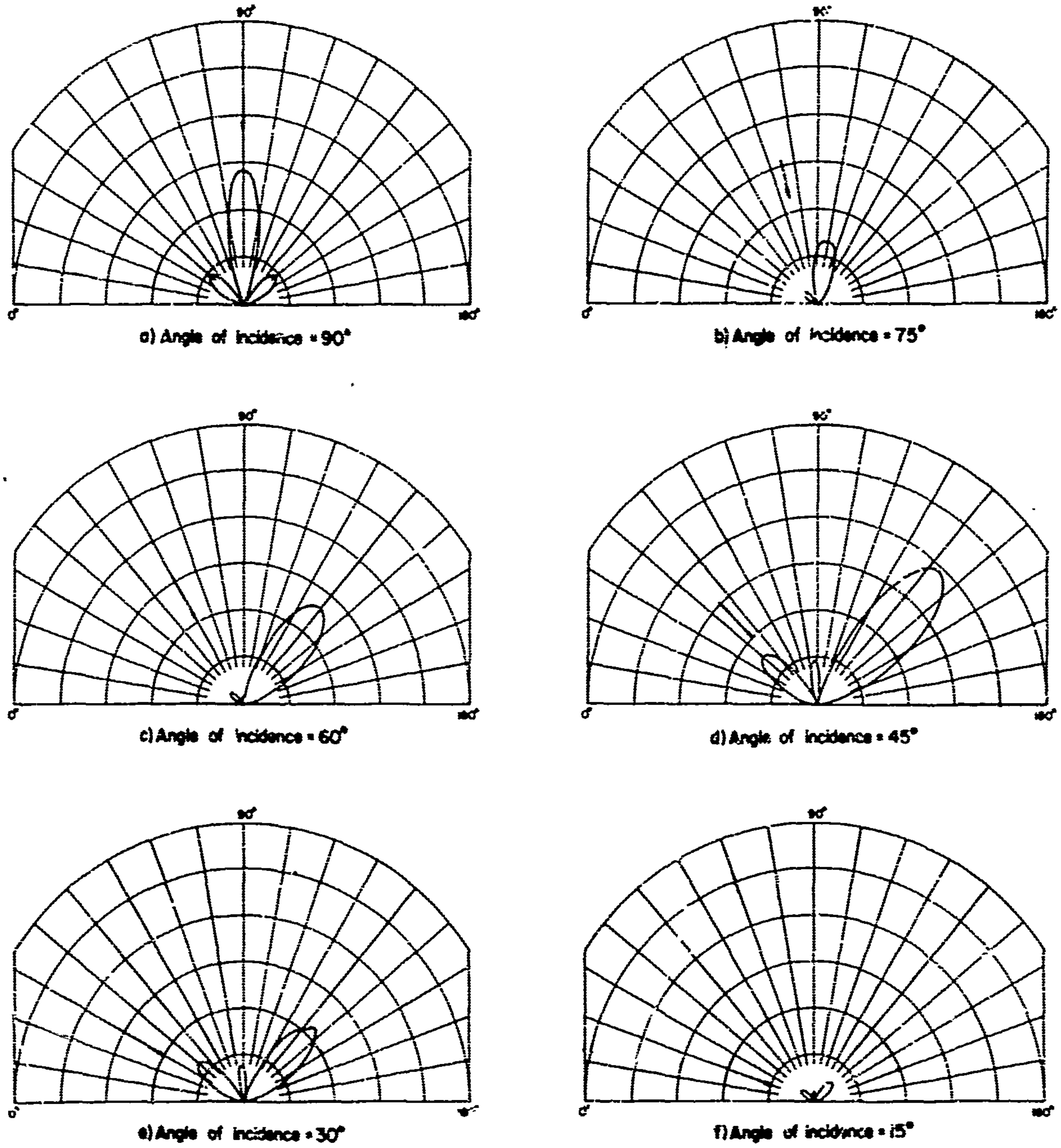
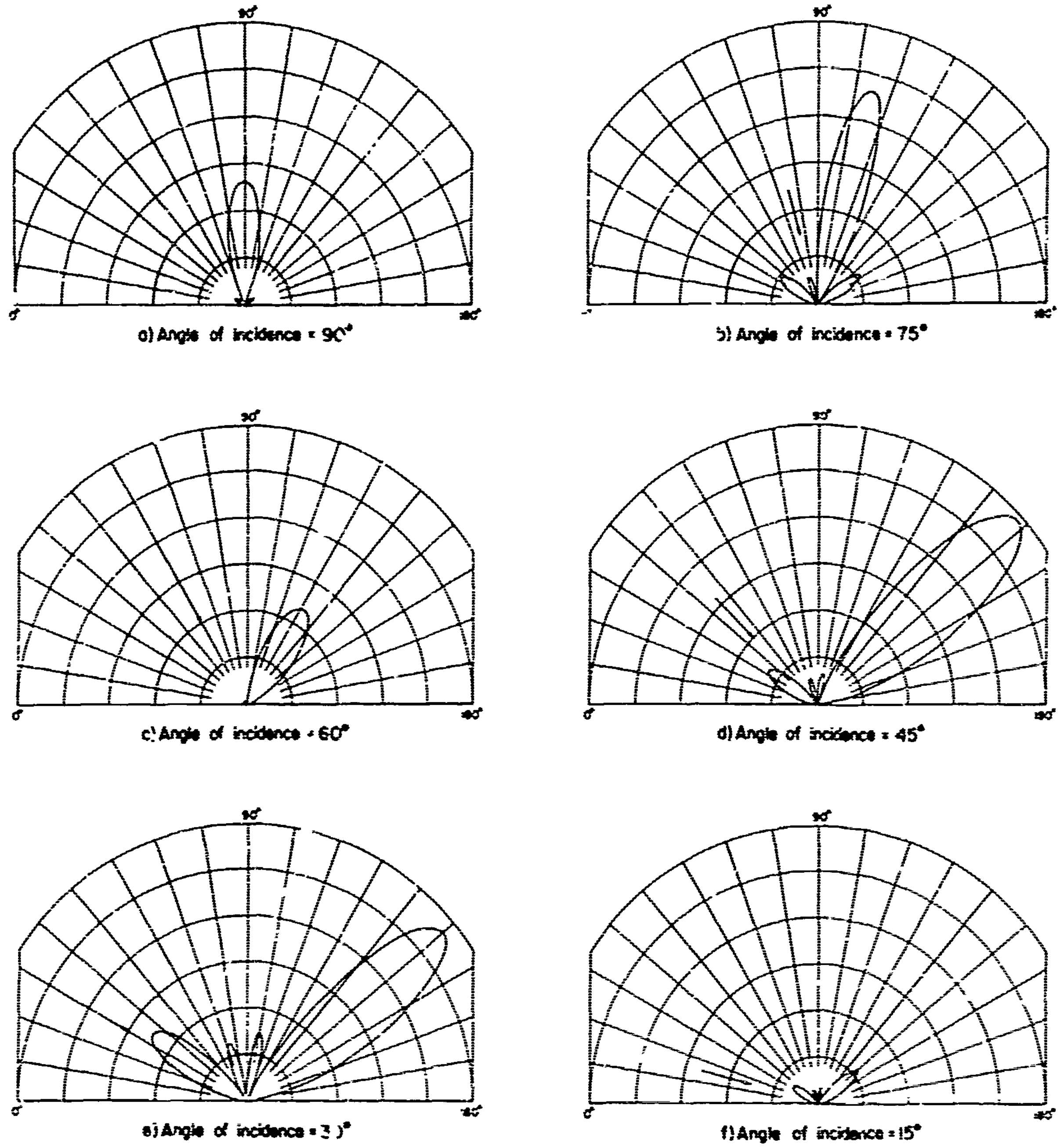


Fig.4-16 Bistatic radar cross section patterns for a linear antenna, $\frac{L}{2a} = 74.2$, $L = 1.5\lambda$, as the angle of incidence varies.



a) Angle of incidence = 90°

b) Angle of incidence = 75°

c) Angle of incidence = 60°

d) Angle of incidence = 45°

e) Angle of incidence = 30°

f) Angle of incidence = 15°

Fig. 4-17 Bistatic radar cross section patterns for a linear antenna, $\frac{L}{2a} = 74.2$, $L = 2\lambda$, as the angle of incidence varies.

its length. Again calculations for $L/2a$ ratios of from 10 to 2000, and for L/λ ratios up to 2.1, were made. The following results for the case $L/2a = 74.2$ ($\Omega = 10$) are representative of the general behavior of loaded scatterers.

Figure 4-18 shows the echo area/λ^2 for a center loaded dipole with various loads Z_L, chosen to correspond to those used by Professor Hu for her calculations.[2] Again Professor Hu's results are good, but apparently become worse as $Z_L \rightarrow \infty$. However, this discrepancy is not as bad as it seems, since the echo area of an open-circuited dipole is very sensitive to the capacitance across the gap. For an infinitesimal gap, the capacitance is infinite, just as it is in linear antenna theory. In any approximate solution, the gap capacitance depends on the approximations made. In Professor Hu's solution, no function capable of representing a singularity in charge distribution is used, hence her results give a low gap capacitance. In the method of subsections used for the calculations, the gap capacitance increases with the number of subsections chosen. This is because the function expansion (4-5) can come closer to representing a singularity as the number of subsections increases. Actually, a very small adjustment of the gap capacitance in Professor Hu's results would bring them into close agreement with the results of this report, even for the case $Z_L \rightarrow \infty$.

Figure 4-19 shows the echo area/λ^2 for a center loaded dipole resonated by reactances at various values of L/λ. The general theory of resonant scatterers is available in the literature.[4] By definition, a resonant loaded scatterer is one for which the load impedance is a reactance equal to the negative of the input reactance to the scatterer when fed as an antenna. This definition gives a maximum echo area when the open-circuit echo from the scatterer is much smaller than the short-

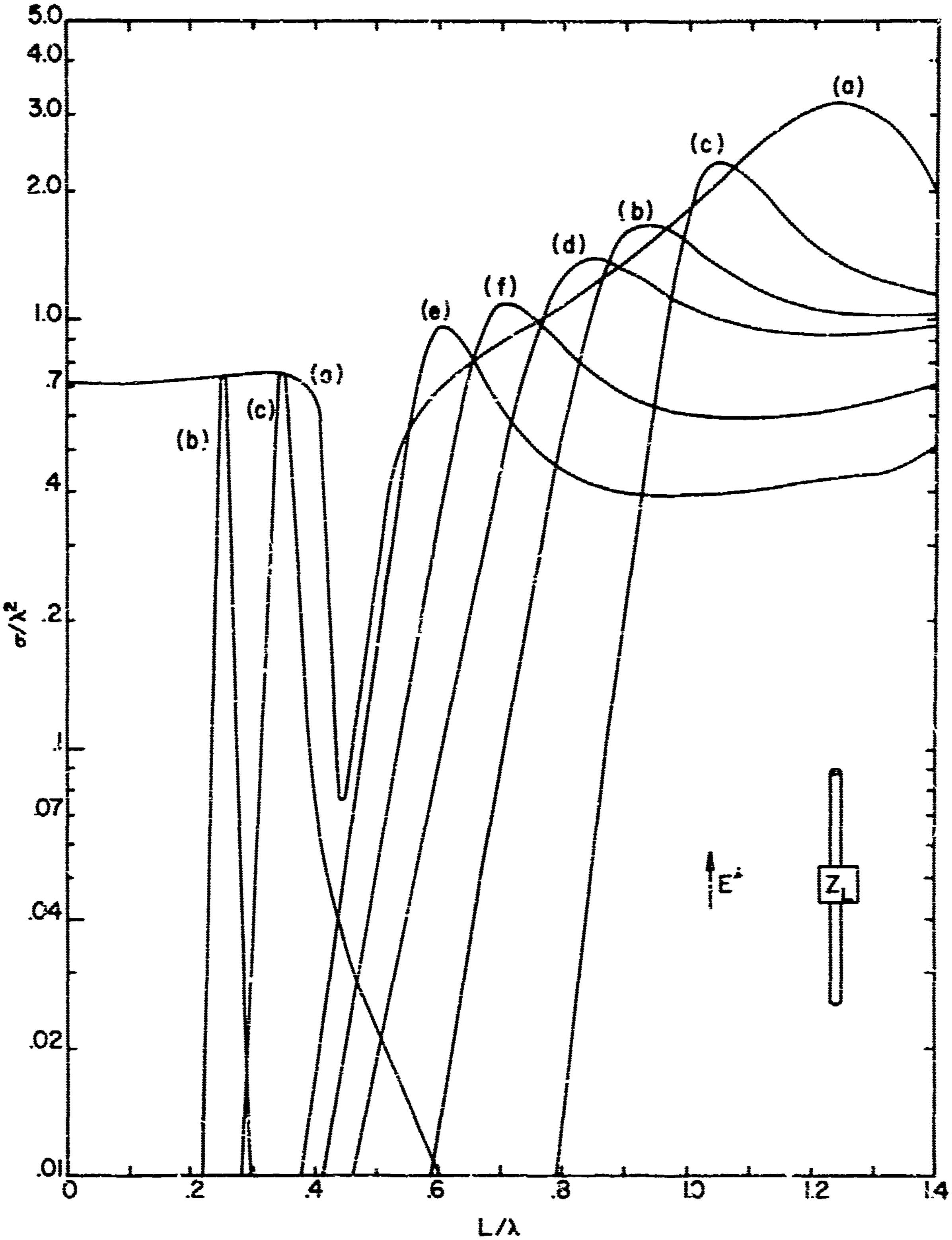


Fig. 4-19 Echo area/λ^2 for center loaded dipole.

circuit echo. Such a condition is usually met for small scatterers, but not for ones of dimensions $\lambda/2$ or greater. Figure 4-19 illustrates this in the following way. Curve (a) is for a center-loaded dipole continually tuned to resonance. Curves (b) and (c) are for the dipole tuned to resonance by an inductor at $L/\lambda = 0.25$ and 0.35, respectively. Curves (e) and (f) are for the dipole tuned to resonance by a capacitor at $L/\lambda = 0.55$ and 0.65. The fact that curves (e) and (f) overshoot curve (a) indicates that resonance defined in terms of input impedance does not necessarily give maximum echo area. Curve (d) is for an open-circuited dipole. The curve for a short-circuited dipole would lie in between curves (c) and (e).

F. References

1. R. W. P. King, "The Theory of Linear Antennas," Harvard University Press, Cambridge, Massachusetts, 1956.

2. Y. Y. Hu, "Back-scattering Cross Sections of a Center-loaded Cylindrical Antenna," IRE Trans., vol. AP-6, January, 1958.

3. R. F. Harrington, "Theory of Loaded Scatterers," Proc. IEE (London), vol. 111, no. 4, April, 1964.

4. R. F. Harrington, "Small Resonant Scatterers and Their Use for Field Measurements," IRE Trans., vol. MTT-10, no. 3, May 1962.

V. SCATTERING BY CONDUCTING CYLINDERS OF ARBITRARY SHAPE

A. Introduction. This chapter is intended to illustrate a variety of mathematical techniques which may be used for calculating scattering from conducting cylinders of arbitrary cross section. Various approximations have been tried and the results will be illustrated.

An integral equation formulation of the solution has been applied to a number of examples by K. K. Mei[1] and M. G. Andreasen.[2] Andreasen's examples involve bodies with curving surfaces, corners, and even two symmetrically placed bodies. The fine detail given for the current density indicates a large number of points were considered on each scatterer. An estimate of his published curves will be given for comparison.

A differential equation approach has been formulated by R. F. Harrington[3] and has been found to give fairly good results. However, it was felt that the integral equation formulation was most promising so emphasis was placed on it.

By the usual vector potential method the integral equation for TM polarization ($H_z = 0$) is

$$\frac{4}{\eta} E_z^i(\rho) = \int_{\text{contour}} J_s(\rho') \, H_0^{(2)}(k|\rho - \rho'|) \cdot dk\underline{s}' \qquad (5\text{-}1)$$

$$|\rho - \rho'| = \sqrt{(x - x')^2 + (y - y')^2}$$

where $J_s(\rho')$ is the surface current density on the conductor and $E_z^i(\rho)$ is the incident field.

When E_z^s is the field due to J_s, then the bistatic width is

$$L_e(\phi) = \lim_{\rho \to \infty} 2\pi\rho \left| \frac{E_z^s}{E_z^i} \right|^2 \qquad (5\text{-}2)$$

and is computed using the large argument formula for the Hankel function. Since $\rho >> \rho'$ we approximate

$$|\rho - \rho'| = \rho - \rho' \cos(\phi - \phi')$$

where ϕ and ϕ' are defined in Fig. 5-1. Then

$$L_e(\phi) = \frac{\lambda}{8\pi} \left| \int_{\text{contour}} J_s(\rho') \, e^{jk\rho' \cos(\phi - \phi')} \cdot dk\underline{s}' \right|^2 \qquad (5\text{-}3)$$

for an incident electric field of strength η. The scattering width may also be calculated using reciprocity as illustrated in Section III-D for wires of arbitrary shape.

B. Pulse Function Approximation to Current. The simplest solution to (5-1) is one in which the integral over the surface is approximated as a sum of integrals over N small intervals. In each interval the current is assumed uniform and the field is matched at one point within each interval.

Figure 5-1 illustrates the division of a symmetrical body into N intervals and defines the notation. The i-th segment is given by the coordinates of ρ_i, at which point the integral equation is satisfied. The starting and end points of the interval are defined by the coordinates of $\bar{\rho}_i$ and $\overset{+}{\rho}_i$. The length of the interval is defined as

$$\Delta\ell_i = \Delta^-\ell_i + \Delta^+\ell_i \qquad (5\text{-}4)$$

where $\Delta^-\ell_i = |\rho_i - \bar{\rho}_i|$ and $\Delta^+\ell_i = |\overset{+}{\rho}_i - \rho_i|$.

In terms of the language of the method of moments the expansion functions are pulses and the weighting functions are delta functions. Thus,

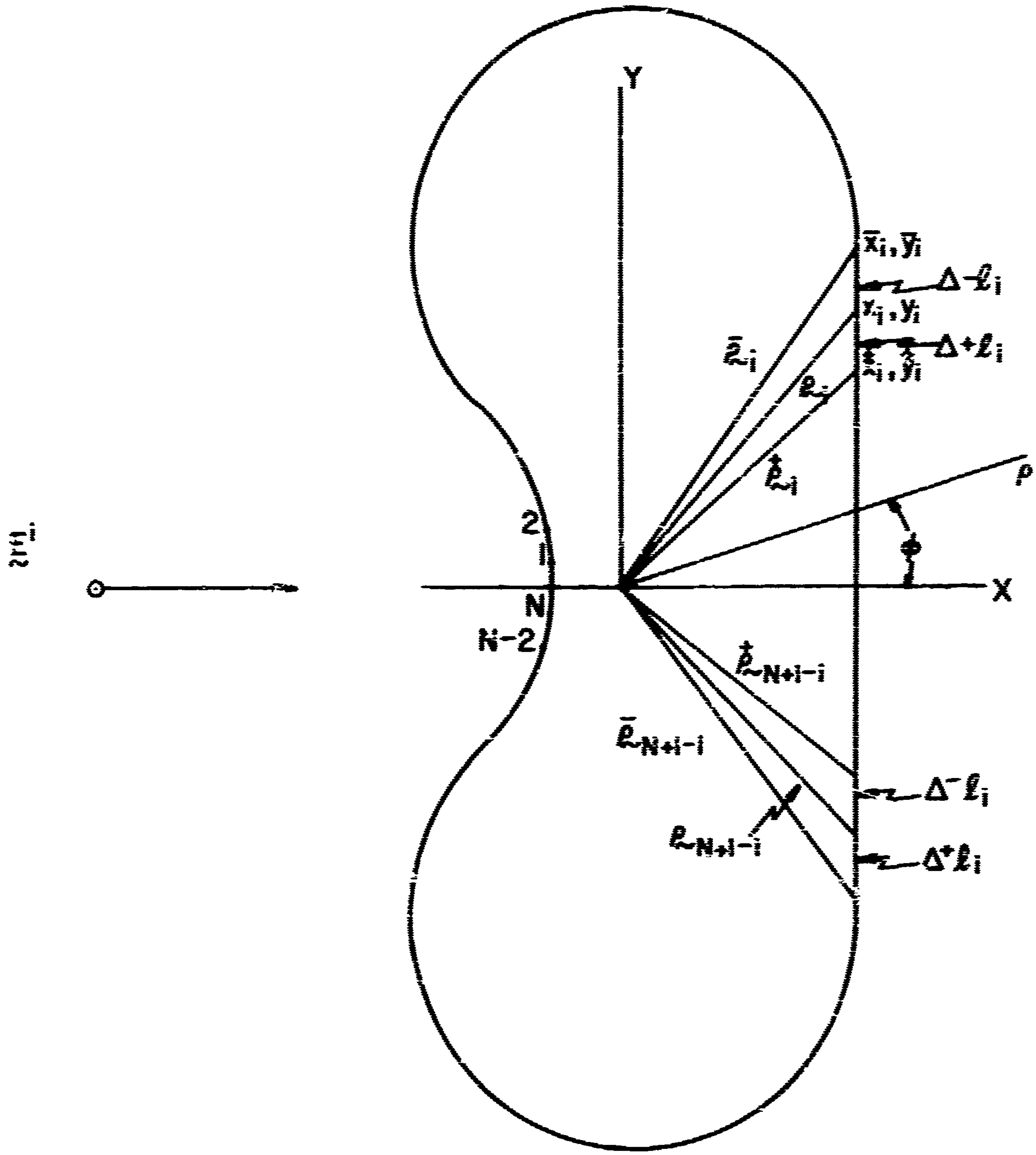


Fig. 5-1 Symmetrical Scatterer.

$$L\,J_s(\rho') = \frac{4}{\eta} E_z^i(\rho) \tag{5-5}$$

$$J_s = \sum_{i=1}^{N} \alpha_i P_i \tag{5-6}$$

$$P_i = \begin{cases} 1 & \text{within } \Delta\ell_i \\ 0 & \text{outside } \Delta\ell_i \end{cases}$$

$$L(P_i) = \int_{\Delta\ell_i} P_i H_o^{(2)}(k|\rho - \rho'|)\, dks_i \tag{5-7}$$

The resulting set of equations, assuming an incident plane wave of strength η from $x = -\infty$ is

$$\sum_{i=1}^{N} \ell_{ji}\, \alpha_i = g_j \qquad j = 1, 2, \ldots, N \tag{5-8}$$

where

$$\ell_{ji} = \int_{\Delta\ell_i} H_o^{(2)}(k|\rho_j - \rho_i'|)\, dks_i \tag{5-9}$$

and

$$g_j = 4e^{-jkx_j} \tag{5-10}$$

The diagonal matrix elements ℓ_{ii} must be integrated analytically since the integrand is singular. This integration is illustrated in Section V-E.1. When the field and source points are different ($j \neq i$) no singularities are involved and to a crude approximation

$$\ell_{ji} = H_o^{(2)}(k|\rho_j - \rho_i|)\, k\Delta\ell_i \qquad (5\text{-}11)$$

These are the approximations it is believed K. K. Mei made in his solution.

For symmetric bodies with normal incidence

$$\alpha_i = \alpha_{N+1-i} \qquad \rho = \rho_{N+1-i} \qquad \Delta\ell_i = \Delta\ell_{N+1-i} \qquad (5\text{-}12)$$

so that the matrix equation (5-8) may be reduced to

$$\sum_{i=1}^{N/2} \ell'_{ji}\, \alpha_i = 4e^{-jkx_j} \qquad j = 1, 2, \ldots, N/2 \qquad (5\text{-}13)$$

where

$$\ell'_{ii} = \ell_{ii} + \ell_{i,N+1-i}$$

$$= \ell_{ii} + H_o^{(2)}(k|\rho_i - \rho_{N+1-i}|)\, k\Delta\ell_i \qquad (5\text{-}14)$$

and

$$\ell'_{ji} = \ell_{ji} + \ell_{j,N+1-i}$$

$$= \left[H_o^{(2)}(k|\rho_j - \rho_i|) + H_o^{(2)}(k|\rho_j - \rho_{N+1-i}|)\right] k\Delta\ell_i \qquad (5\text{-}15)$$

Note there are the same number of points on both halves of the scatterer so that N is even.

Inversion of the matrix equations (5-8) or (5-13) gives the α_i which represent the current density within the i-th interval. This current may also be found by computing the magnetic field outside the conductor. Hence

$$J_s(\rho_j) = \frac{-1}{j\omega\mu}\, \underline{n}_j \times \left[\nabla_j \times E_z^i(\rho_j) + \nabla_j \times E_z^s(\rho_j)\right] \qquad (5\text{-}16)$$

where $\underline{n}_j$ is the normal to the conductor at the j-th point. This method requires calculating two matrices, the column matrix $\left[H^i\right]$ whose elements are

$$H_j^i = \frac{j}{k} \underline{n}_j \times \nabla_j \times \hat{z}\, e^{-jkx_j} \qquad (5\text{-}17)$$

and the square matrix H_{ji}^s whose elements are

$$H_{ji}^s = \frac{1}{4jk} \underline{n}_j \times \nabla_j \times \hat{z} H_o^{(2)}(k \,|\underline{\rho}_j - \underline{\rho}_i|)\, k\Delta\ell_i \qquad (5\text{-}18)$$

where $\hat{z}$ is the unit vector in the z direction. The matrix equation for $\underline{J}_s$ is then

$$[J_s] = [H^i] + [H^s]\,[\alpha] \qquad (5\text{-}19)$$

The singular matrix elements H_{ii}^s are evaluated in Section V-E.2, using an analytic integration. Equation (5-18) is in itself a very crude approximation. However, it is easily evaluated using analytic differentiation of the argument since no singularities are involved. The rectangular components of the normal are easily approximated as

$$\begin{aligned} n_x &= -(\overset{+}{y}_i - \overline{y}_i)/\Delta\ell_i \\ n_y &= (\overset{+}{x}_i - \overline{x}_i)/\Delta\ell_i \end{aligned} \qquad (5\text{-}20)$$

The solution was first programmed using (5-43) and (5-11) for the matrix elements. The results for the current density and echo area are shown in Figures 5-2a and 5-2b. Note that even with such crude approximations, the echo width as calculated using the α_i's is in good agreement with Andreasen's. The current density as calculated by the α_i's is considerably in error. However, as calculated by $\underline{n} \times \underline{H}$, $\underline{J}_s$ is in good agreement, being in error only in the region of unequal intervals and rapid curvature of the ellipse. No more current calculations were made using this method. The reason for the more accurate current evaluation is probably related to the fact that this procedure is somewhat of an

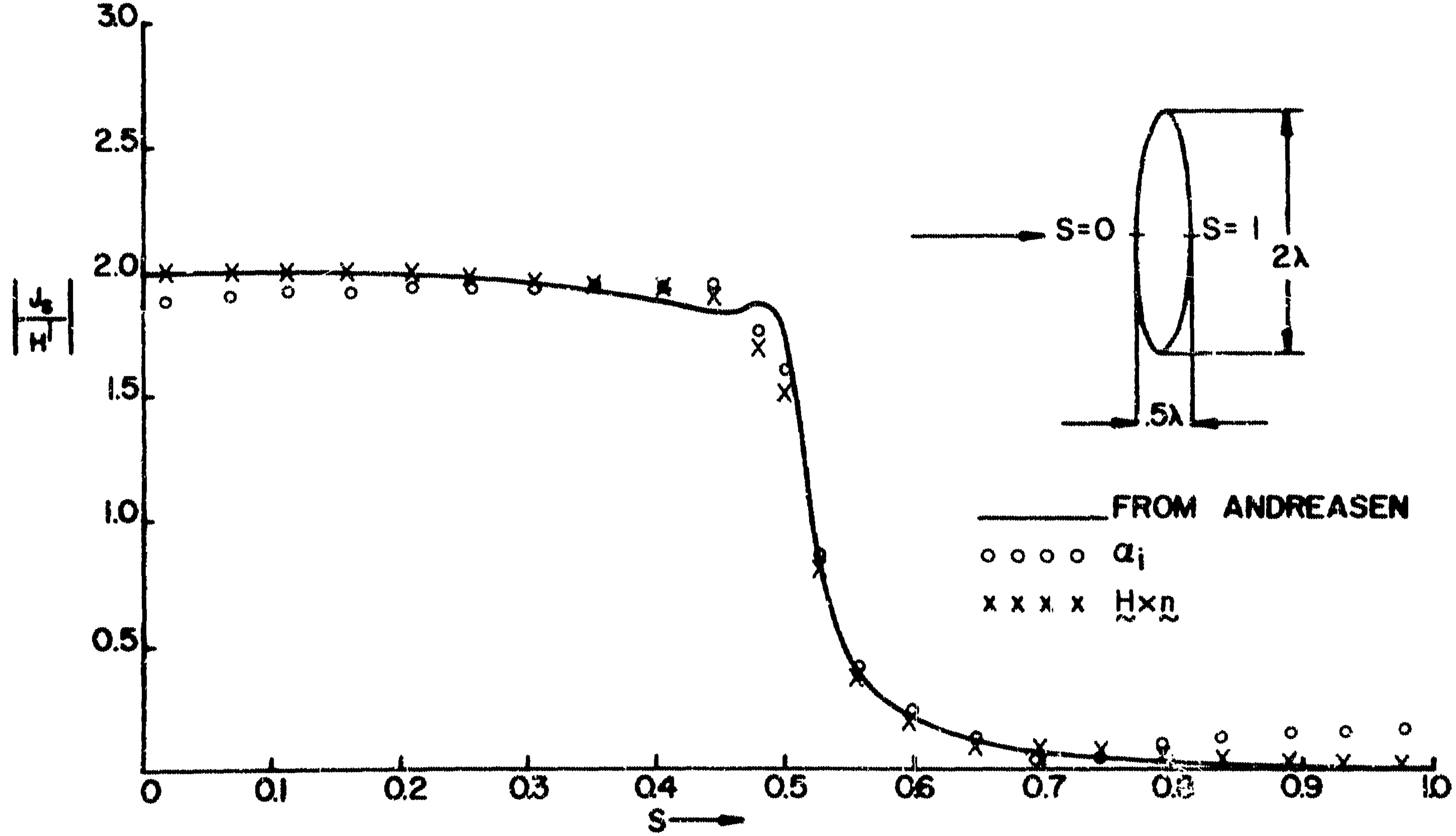


Fig. 5-2a Current density on elliptic cylinder.

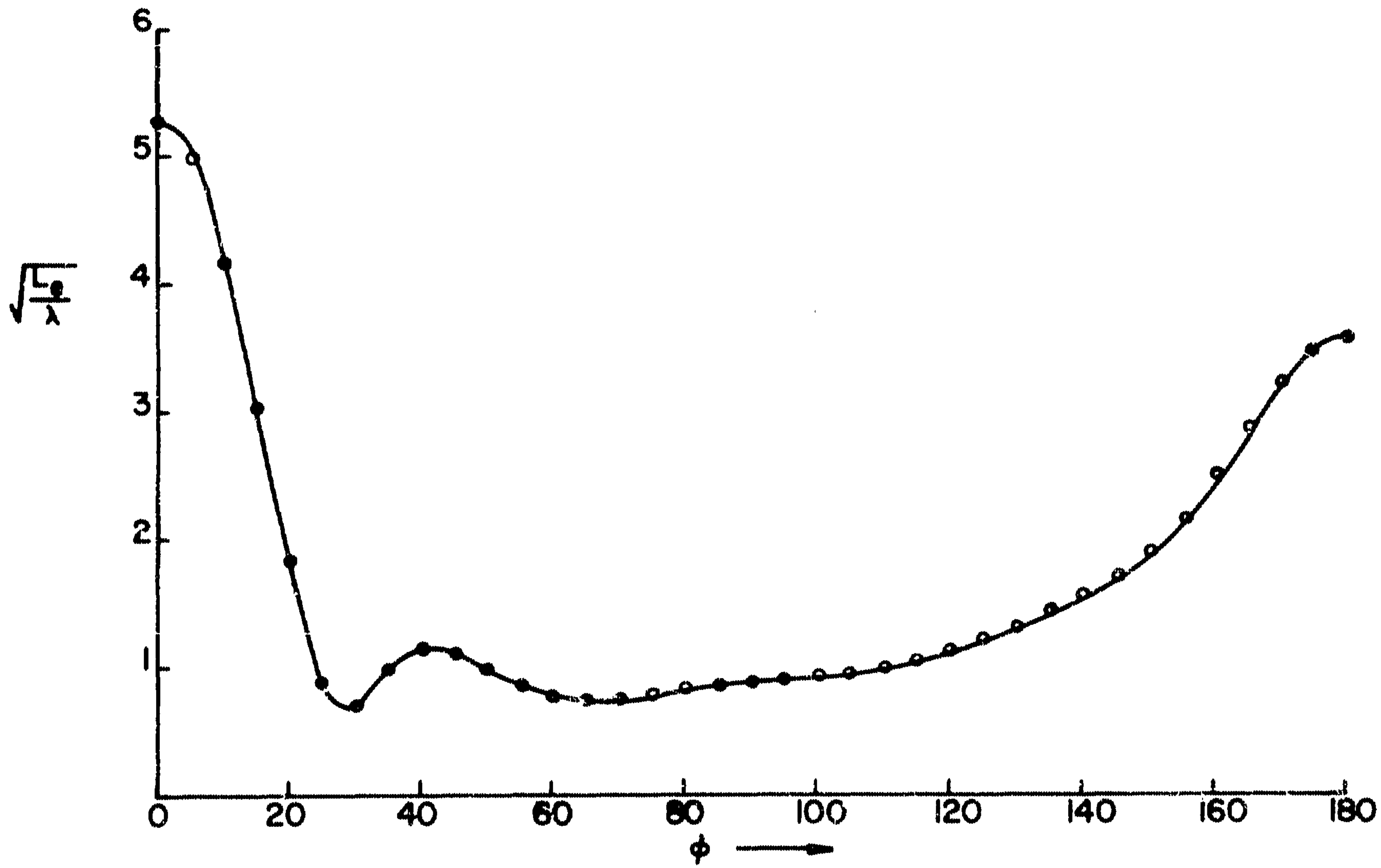


Fig. 5-2b Scattering pattern for eliptic cylinder.

iteration process where only an approximation to L has been made.

For a better evaluation of the ℓ_{ji} $(j \neq i)$ of (5-9), the addition theorem for Hankel functions[4]

$$H_o^{(2)}(k|\underline{\rho}_j - \underline{\rho}'|) = \sum_{n=0}^{\infty} \epsilon_m J_m(kt)\, H_m^{(2)}(k|\underline{\rho}|) \cos m\phi' \qquad (5\text{-}21)$$

$$\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

is used to expand the kernel of the integral equation around each $\underline{\rho}_i$ in terms of the variable of integration t' along the scatterer contour. Figure 5-3 defines the notation where ϕ' is now the angle between $\underline{t}'$ and $\underline{\rho} = \underline{\rho}_j - \underline{\rho}_i$. Each interval is taken to be flat so that

$$\cos\phi' = \frac{\underline{t}' \cdot \underline{\rho}}{|\underline{t}'| \cdot |\underline{\rho}|} \qquad \text{and} \qquad \underline{t}' = \overset{+}{\underline{\rho}}_i - \overline{\underline{\rho}}_i \qquad (5\text{-}22)$$

The result of the integration for the ℓ_{ji} is given in Section V-E.3. An upper limit of four seemed sufficient for convergence of the integrated series (5-49).

It should be noted that the elements ℓ_{ji} and ℓ_{ij} are most efficiently computed at the same time since they involve the same values of $H_m^{(2)}(k|\underline{\rho}_j - \underline{\rho}_i|)$. The symmetry of the scatterer may also be taken into account using (5-12) and (5-15).

When the scatterer surface curves rapidly, the matrix element (5-49) may be altered by defining angles $\overline{\phi}'$ and $\overset{+}{\phi}'$ as shown in Fig. 5-4. The diagonal elements remain as in (5-43) but for $j \neq i$, the matrix elements become

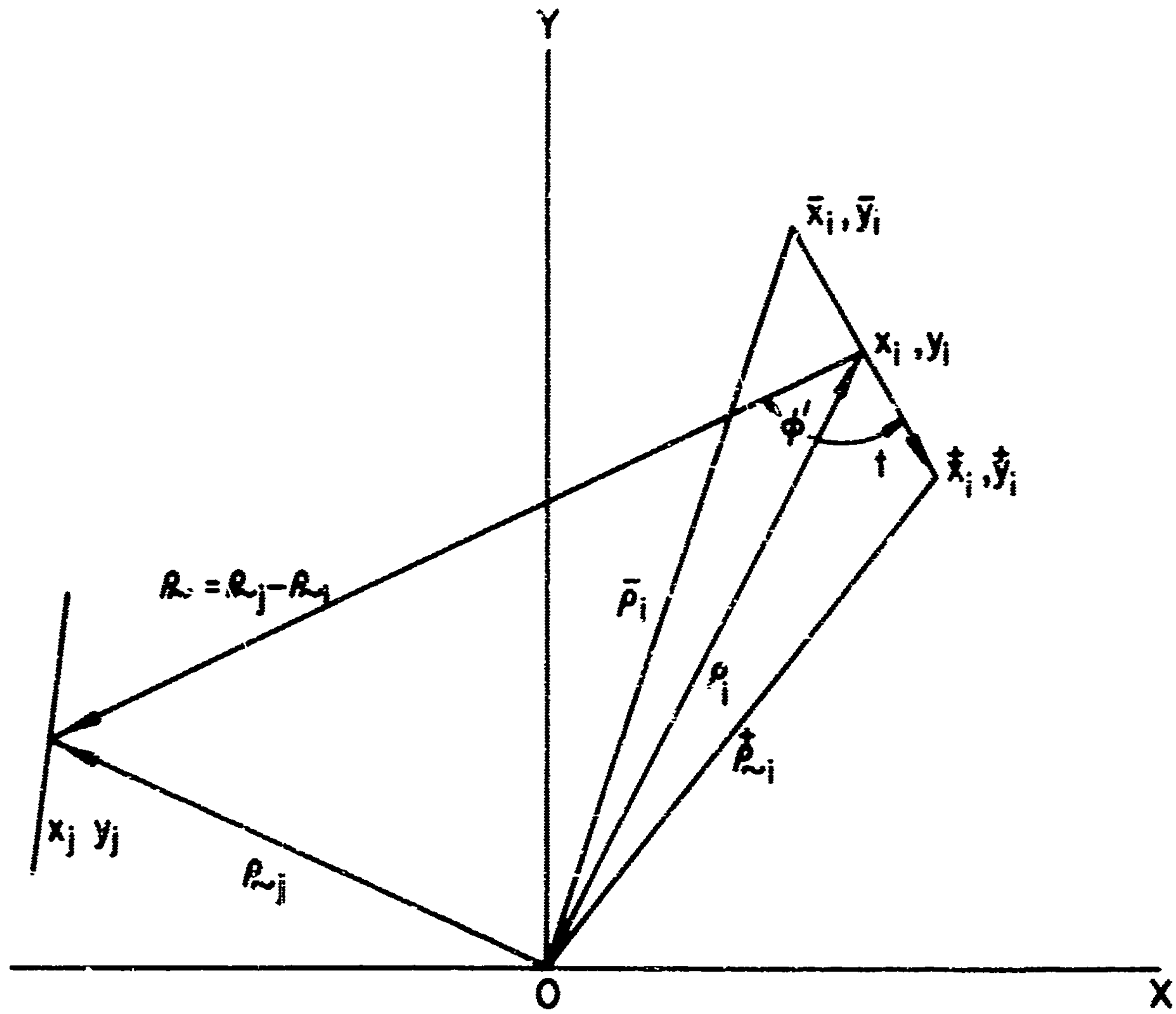


Fig. 5-3 Definition of variables t' and ϕ' on scatterer.

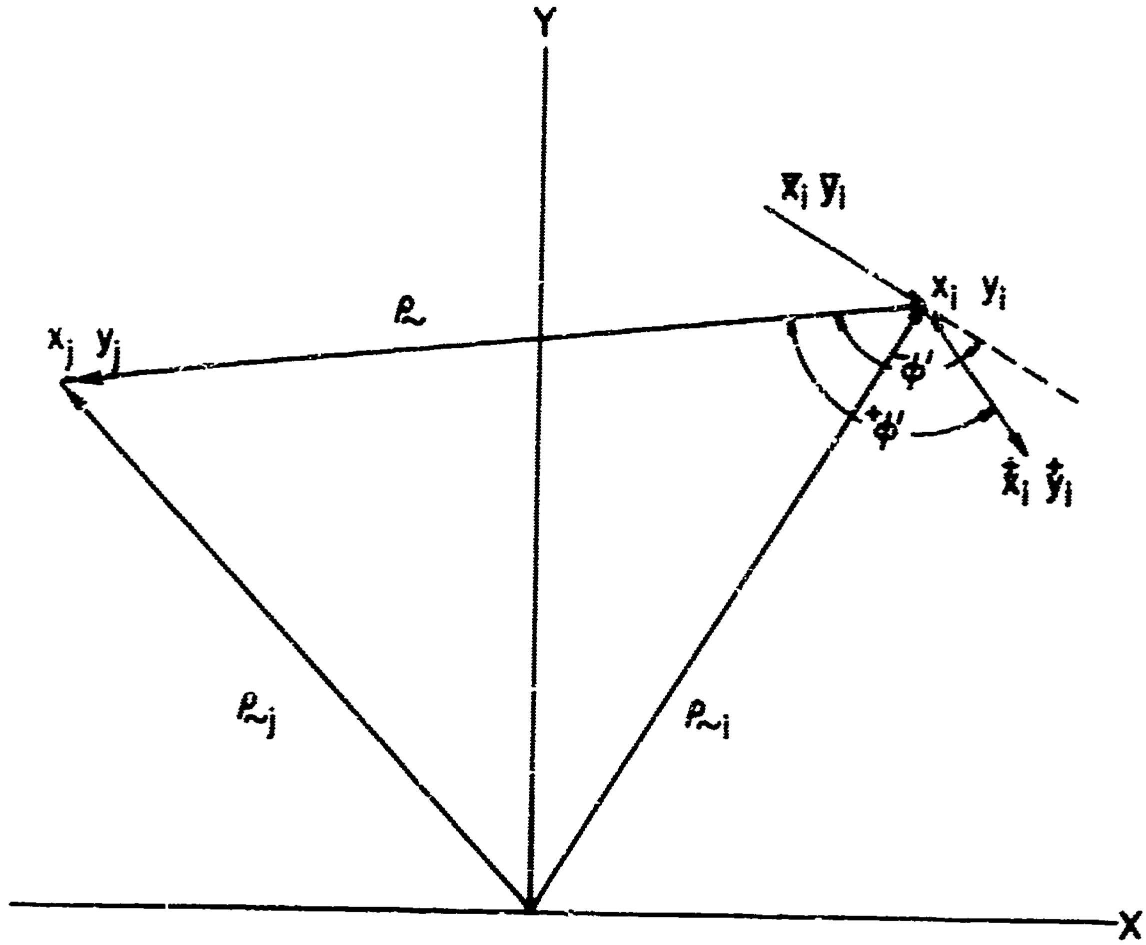


Fig. 5-4 Illustration defining angles $\overset{+}{\phi}'$ and $\bar{\phi}'$ on scatterer.

$$\ell_{ji} = 2\sum_{m=0}^{4} (-1)^m \frac{\epsilon_m}{m!} \left(\frac{k\bar{\Delta}\ell_i}{2}\right)^{m+1} \left[\frac{1}{m+1} - \frac{1}{(m+1)(m+3)}\left(\frac{k\bar{\Delta}\ell_i}{2}\right)^2\right] H_m^{(2)}(k|\rho|) \cos m\bar{\phi}'$$

$$+2\sum_{m=0}^{4} \frac{\epsilon_m}{m!} \left(\frac{k\overset{+}{\Delta}\ell_i}{2}\right)^{m+1} \left[\frac{1}{m+1} - \frac{1}{(m+1)(m+3)}\left(\frac{k\overset{+}{\Delta}\ell_i}{2}\right)^2\right] H_m^{(2)}(k|\rho|) \cos m\overset{+}{\phi}'$$

(5-23)

where

$$\cos \bar{\phi}' = \frac{\bar{\xi}' \cdot \rho}{\bar{\Delta}\ell_i|\rho|} \qquad \bar{\xi}' = \rho_i - \bar{\rho}_i \qquad (5\text{-}24)$$

and a similar equation for $\cos \overset{+}{\phi}'$.

Figure 5-5 shows the solution for the α_i's for three different cases. In all three, (5-43) is used as the singular point element. The solution given by dots used (5-49) for the off diagonal elements as did the solution given by circles. In the latter, more points were added about s = .5 where the ellipse curves rapidly. The solution given by crosses used (5-23) for off diagonal elements. In all three solutions, a "jump" occurs where the spacing between points changes drastically. This problem remained throughout the work and is probably due to an inadequate integration procedure. Further work will be done to eliminate this problem. Echo area has not been shown since the results plotted as in Fig. 5-2b.

A further calculation was done where (5-43) was used for the singular matrix elements and (5-11) for the ℓ_{ji} when $k|\rho| > 1.3$. For the ℓ_{ji} where $k|\rho| < 1.3$, the better approximation (5-23) was used. The results plot identically with Fig. 5-4, illustrating that the current density at a point is highly dependent on the field near that point and not very sensitive to the field far away.

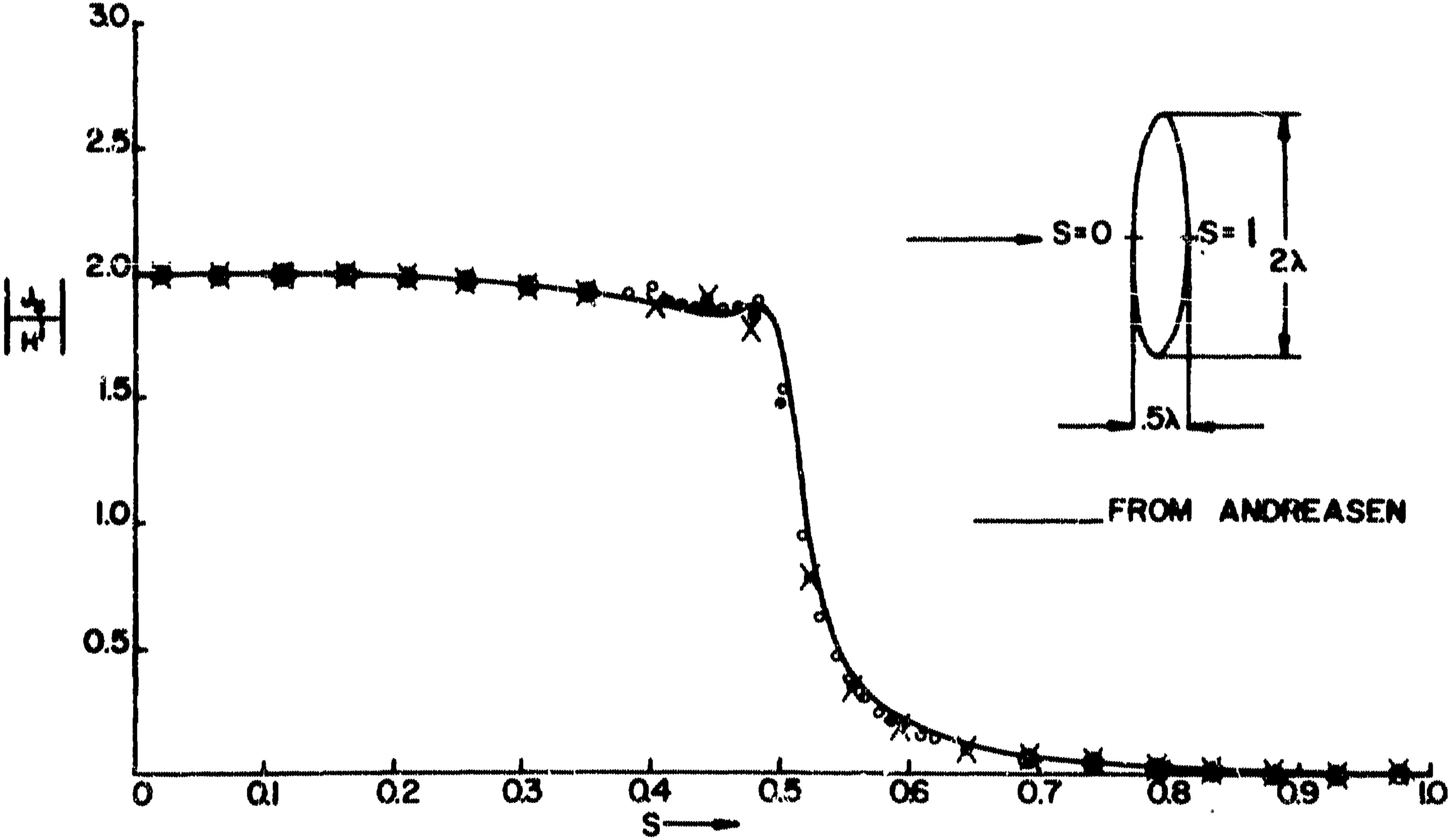


Fig. 5-5 Pulse approximation to current density on ellipse.

C. Higher Approximations to Current Density. A first possible improvement in the form of the assumed current is to constrain the α_i's in every other interval to be linearly related to the α_i's in the two adjoining intervals. If one begins with (5-13) for a symmetrical body, the number of unknown α_i's is reduced from N/2 to approximately half as many and the integral equation is then satisfied at only every other point. From Fig. 5-6 we see

$$\alpha_i = \frac{\alpha_{i+1}(\Delta^+\ell_{i-1} + \Delta^-\ell_i) + \alpha_{i-1}\,(\Delta^+\ell_i + \Delta^-\ell_{i+1})}{(\Delta^+\ell_{i-1} + \Delta\ell_i + \Delta^-\ell_{i+1})}$$

$$= \frac{\alpha_{i+1}\,\Delta^-k_i + \alpha_{i-1}\,\Delta^+k_i}{\Delta k_i} \qquad (5\text{-}25)$$

where the definitions of the Δk's are obvious. The column matrix of the α_i's becomes

$$[\alpha] = \begin{bmatrix} \alpha_1 \\ \dfrac{\alpha_3\Delta^-k_2 + \alpha_1\Delta^+k_2}{\Delta k_2} \\ \alpha_3 \\ \vdots \\ \dfrac{\alpha_{i+1}\Delta^-k_i + \alpha_{i-1}\Delta^+k_i}{\Delta k_i} \\ \vdots \end{bmatrix} \qquad (5\text{-}26)$$

The resulting matrix equations to be solved are

$$[\ell'^R_{mn}]\,[\alpha^R_m] = \frac{1}{\eta}\,[E^i_z(\rho_m)] \qquad (5\text{-}27)$$

where $[\alpha^R_m]$ and $[E^i_z(\rho_m)]$ are M x 1 column matrices and $[\ell'^R_{mn}]$ is a M x M square matrix where M = (N+2)/4, which is the case considered below.

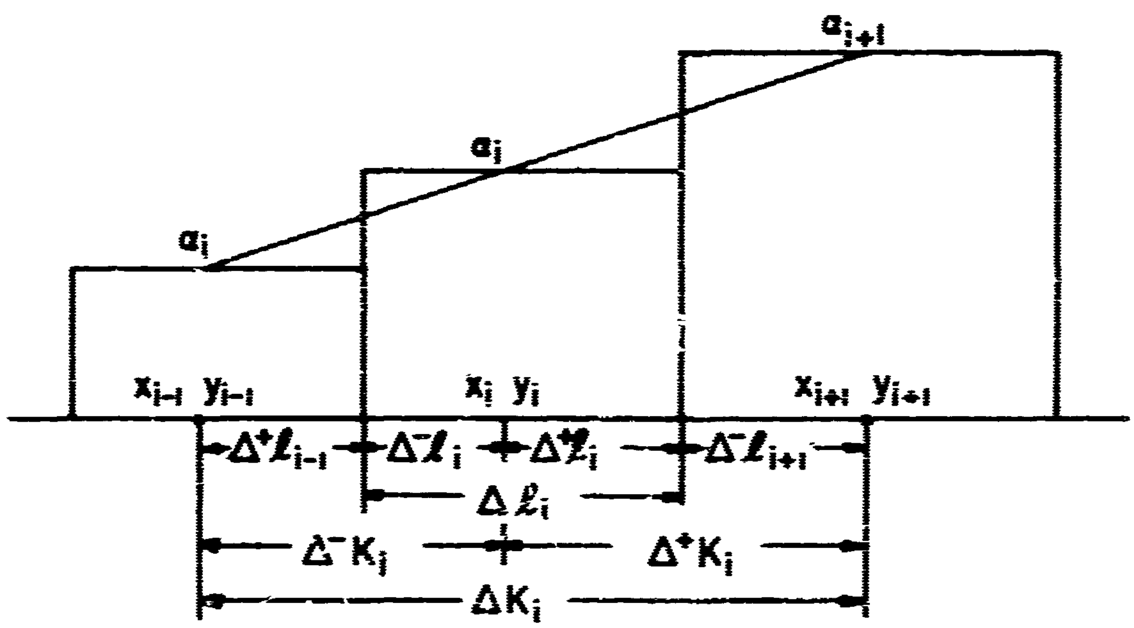


Fig. 5-6 Pictorial Representation of Linear Constraint on a_i.

The elements of the reduced matrix may be calculated as

$$\ell'^{R}_{mn} = \frac{\ell'_{2m-1,2n-2}\Delta^{-}k_{2n-2}}{\Delta k_{2n-2}} + \ell'_{2m-1,2n-1} + \frac{\ell'_{2m-1,2n}\Delta^{+}k_{2n}}{\Delta k_{2n}}$$

$m = 1, \ldots, M \quad n = 2, \ldots, M-1$

$$\ell'^{R}_{mn} = \ell'_{2m-1,1} + \frac{\ell'_{2m-1,2}\Delta^{+}k_{2}}{\Delta k_{2}} \qquad n = 1 \quad m = 1, \ldots, M$$

$$\ell'^{R}_{mM} = \ell_{2m-1,(N/2)} + \frac{\ell'_{2m-1,(N/2)-1}\Delta^{-}k_{(N/2)-1}}{\Delta k_{(N/2)-1}}$$

$n = M \quad m = 1, \ldots, M$

(5-28)

and those of $[E_z^i]$ are

$$E_z^i = \eta e^{-jkx_{2m-1}} \qquad m = 1, \ldots, M \qquad (5\text{-}29)$$

for a plane wave from $x = -\infty$.

The result of using (5-43) and (5-23) for the ℓ_{ji} matrix elements and (5-28) and (5-29) for the reduced matrices is shown in Fig. 5-7. The field has now only been matched at 19 points but the results are quite good

It is also possible to expand the current density in a Taylor series about each ρ_i in terms of the body surface coordinate t. Keeping only the linear term in t

$$J_s(\rho) = J_s(\rho_i) + \frac{\partial J_s(\rho_i)}{\partial t}\, t = \alpha_i + \frac{\alpha_{i+1} - \alpha_{i-1}}{\Delta k_i}\, t \qquad (5\text{-}30)$$

where the derivative term is approximated as shown. Integration over one interval then gives a contribution to the i - 1, i, and i + 1 elements

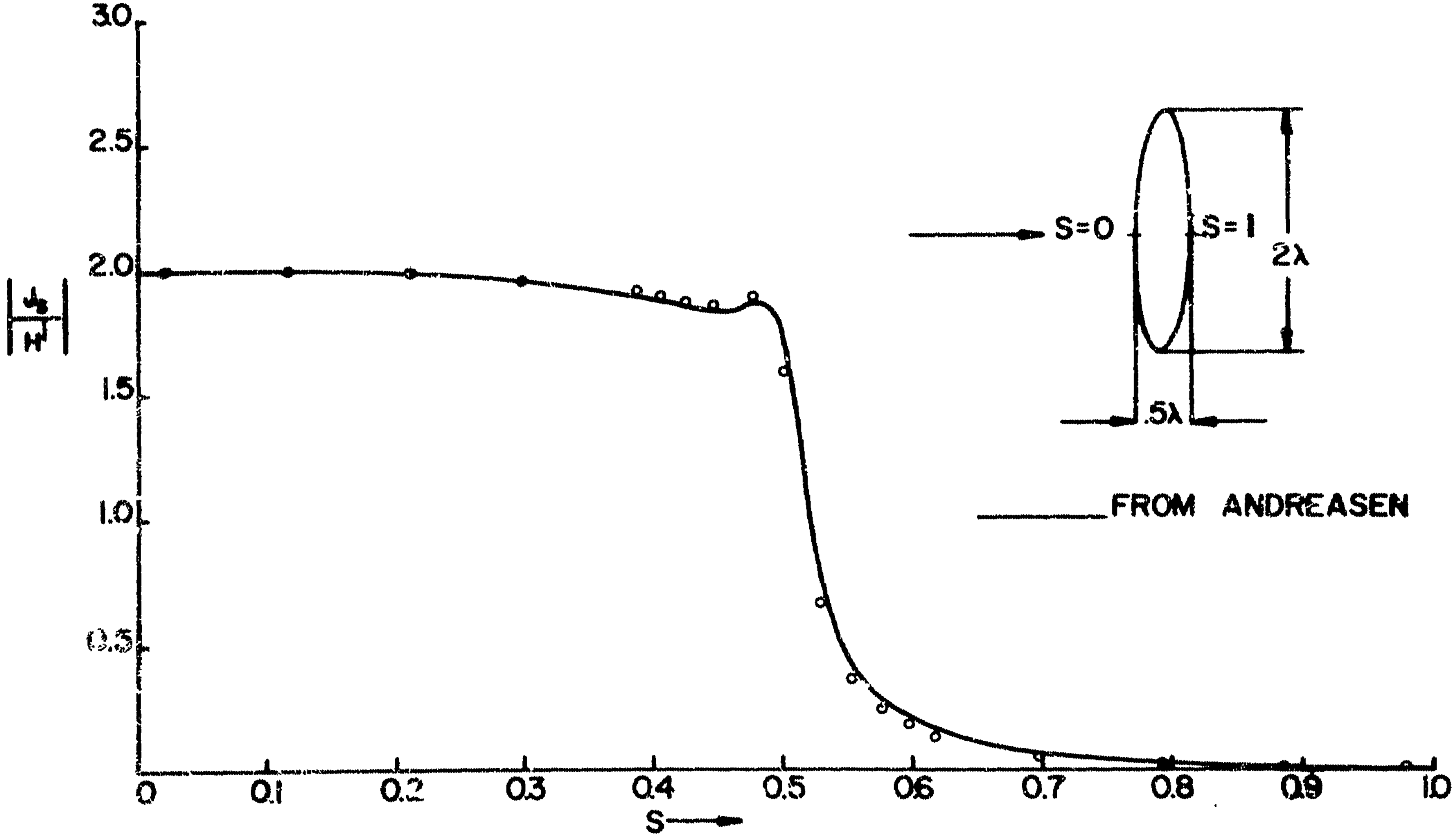


Fig. 5-7 Linearly constrained current density on ellipse.

of a row of the matrix. A matrix element is not computed directly but is obtained as a result of integrating over three intervals on the scatterer surface and summing the proper contributions. The contributions to the matrix elements of the j-th row as a result of integrating over the i-th interval (column) is

$$\alpha_i \int_{\Delta\ell_i} H_o^{(2)}(k|\rho_j-\rho_i'|)\, dkt_i' + \frac{\alpha_{i+1} - \alpha_{i-1}}{\Delta k_i} \int_{\Delta\ell_i} t'\, H_o^{(2)}(k|\rho_j-\rho_i'|)\, dkt_i' \tag{5-31}$$

where Δk_i has been defined in (5-25). The first integral has been computed previously for both $j = i$ and $j \neq i$ and the results given by (5-43) and (5-23). The second integral is evaluated in Section V-E.4.

A piecewise linear approximation to the current density may also be made. In this case, triangles are chosen as expansion functions in the method of moments and the weighting functions are again delta functions Thus

$$J_s = \sum_{i=1}^{N} \alpha_i T_i \tag{5-32}$$

where

$$T_i = \begin{cases} 1 + \dfrac{t}{\Delta^- k_i} & -\Delta^- k_i < t < 0 \\[2ex] 1 - \dfrac{t}{\Delta^+ k_i} & 0 < t < \Delta^+ k_i \end{cases} \tag{5-33}$$

Integration over the i-th interval then contributes to the matrix elements of the j-th row as

$$\alpha_i \int_{\Delta\ell_i} H_o^{(2)}(k|\rho_j - \rho_i'|)dkt_i' + \frac{\alpha_i - \alpha_{i-1}}{\Delta^- k_i} \int_{-\Delta^-\ell_i}^{0} t' H_o^{(2)}(k|\rho_j-\rho_i'|)\, dkt_i'$$

$$+ \frac{\alpha_{i+1} - \alpha_i}{\Delta^+ k_i} \int_0^{\Delta^+\ell_i} t' H_o^{(2)}(k|\rho_j - \rho_i'|)\, dkt_i' \tag{5-34}$$

The integrals here have all been computed when doing a Taylor series approximation to the current density. In this case, however, the matrix elements have been constructed from somewhat different contributions. For symmetric scatterers use is again made of (5-15).

The dots in Fig. 5-8 give the result for the Taylor series approximation and the circles are the result for the piecewise linear approximation.

D. Solution Using Galerkin's Method. A solution for the TM polarization was also attempted using the expansion and weighting functions to be the same in the method of moments.

Choosing a pulse approximation to the current

$$w_i = f_i = P_i = \begin{cases} 1 & \text{within } \Delta\ell_i \\ 0 & \text{outside } \Delta\ell_i \end{cases} \tag{5-35}$$

and

$$\sum_{i=1}^{N} Z_{ji}\alpha_i = \frac{4}{\eta}\int_{\Delta\ell_j} P_j(t'')\, E_z^i(\rho_j'')\, dkt'' \qquad j = 1, \ldots, N \tag{5-36}$$

where

$$Z_{ji} = \int_{\Delta\ell_j} P_j(t'')\, LP_i(t')\, dkt'' \tag{5-37}$$

with $LP_i(t')$ given by (5-7). Here primed variables denote "source" coordinates and double primed variables denote field coordinates. For integration purposes the Hankel function with $j \neq i$ is expanded in both "source" and "field" variables in Section V-E.5. The interval is considered "flat" as in Fig. 5-3. In the resulting expansion, integration of the small argument Bessel function yields terms identical to those

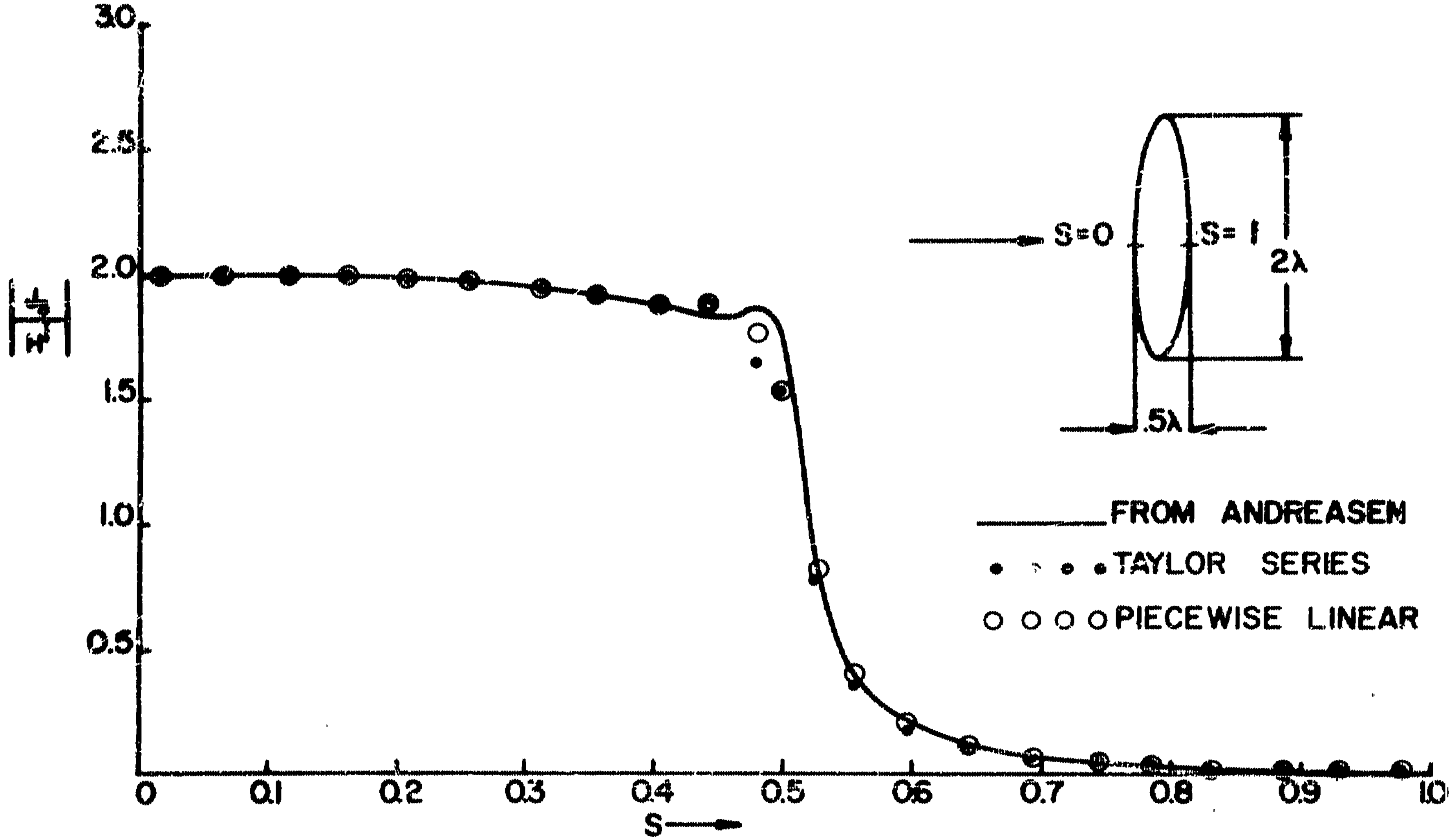


Fig. 5-8 Taylor series and piecewise linear approximations to current density on ellipse.

within the brackets of (5-49). The complete result is omitted because of its length. The singular point integration is straightforward and is done in Section V-E.6.

The terms of the column matrix on the right hand side of (5-36) must be carefully evaluated. Thus the interval is divided into two straight line segments as in Fig. 5-5. The result of the integration is

$$\int_{\bar{x}_j}^{\overset{+}{x}_j} e^{-jkx} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ (\mathrm{SGN})\ dkx$$

$$= j\ \mathrm{SGN} \left[\sqrt{1 + \left(\frac{d\bar{y}}{d\bar{x}}\right)^2} \left(e^{-jkx_j} - e^{-jk\bar{x}_j}\right) + \sqrt{1 + \left(\frac{d\overset{+}{y}}{d\overset{+}{x}}\right)^2} \left(e^{-jk\overset{+}{x}_j} - e^{-jkx_j}\right) \right] \qquad \mathrm{SGN} = \begin{cases} 1 & |\overset{+}{x}| > |\bar{x}| \\ -1 & |\overset{+}{x}| < |\bar{x}| \end{cases} \tag{5-38}$$

where $d\bar{y}/d\bar{x}$ and $d\overset{+}{y}/d\overset{+}{x}$ are regarded as constant over the half intervals and are approximated as

$$\frac{d\bar{y}}{d\bar{x}} = (y_j - \bar{y}_j)/(x_j - \bar{x}_j) \qquad \frac{d\overset{+}{y}}{d\overset{+}{x}} = (\overset{+}{y}_j - y_j)/(\overset{+}{x}_j - x_j) \tag{5-39}$$

The N equations of Galerkin's method may easily be reduced to N/2 equations when the scatterer is symmetrical. The procedure is similar to that used previously.

This method has been programmed and the result of a set of computations is shown in Fig. 5-9.

It is also possible to expand the Hankel function in terms of the x and y coordinates of the body. In vector form

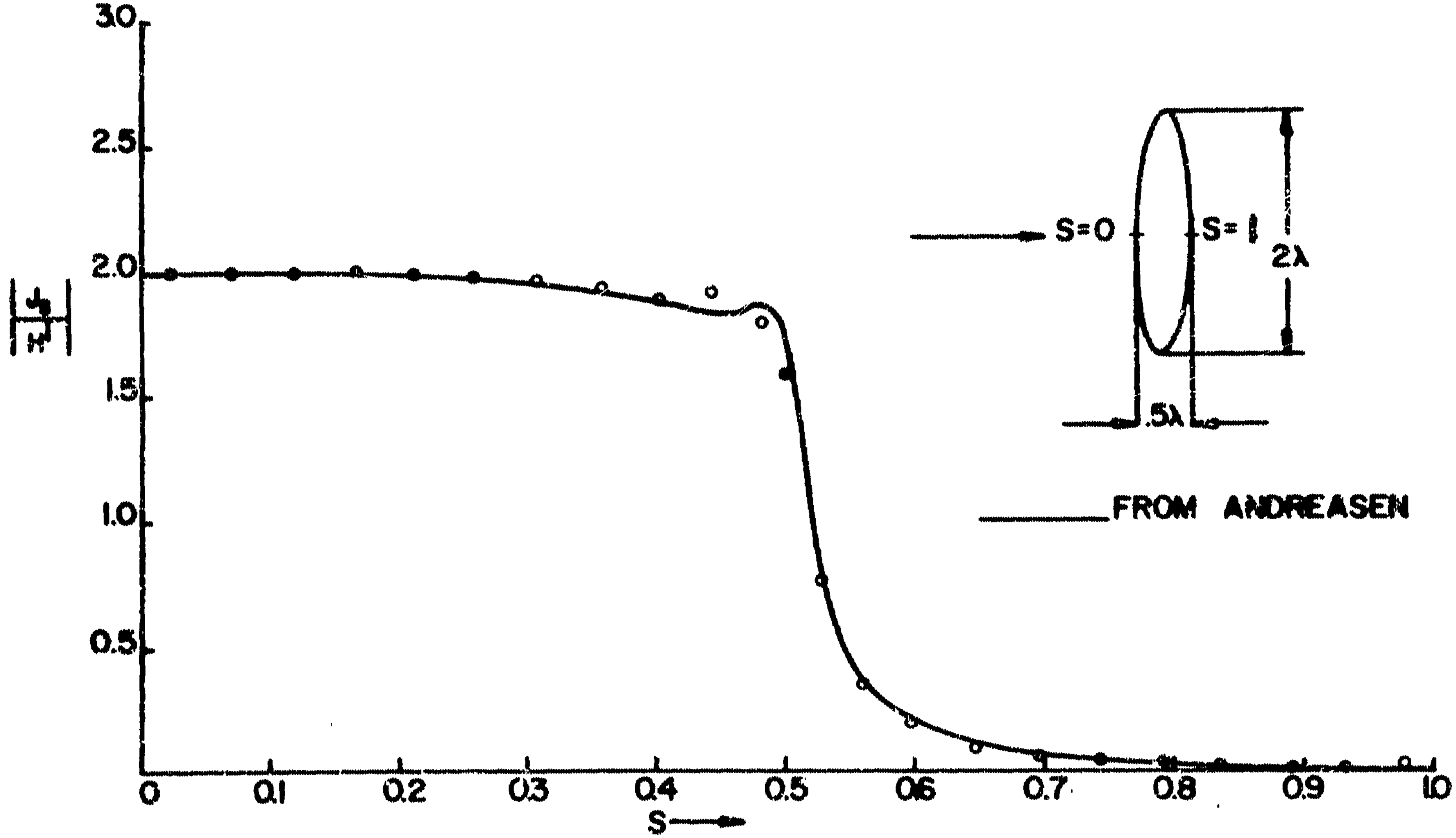


Fig. 5-9 Pulse approximation to current density on ellipse using Galerkin's method.

$$H_0^{(2)}(k|\rho_j-\rho_i|) = H_0^{(2)} + (\rho \cdot \nabla)\, H_0^{(2)} + (\rho' \cdot \nabla')\, H_0^{(2)} + \frac{1}{2}(\rho \cdot \nabla)(\rho \cdot \nabla)\, H_0^{(2)}$$

$$+ \frac{1}{2}(\rho' \cdot \nabla')(\rho' \cdot \nabla')\, H_0^{(2)} + (\rho \cdot \nabla)(\rho' \cdot \nabla')\, H_0^{(2)} \tag{5-40}$$

where

$$(\rho \cdot \nabla) = (x - x_i)\frac{\partial}{\partial x} + (y - y_i)\frac{\partial}{\partial y}$$

and the Hankel functions are evaluated at $x' = x_i$, $y' = y_i$, $x = x_j$, $y = y_j$. This is exactly a Taylor series in four variables including second derivative terms. The integrations involve powers of $(x - x_i)$ and $(y - y_i)$ and are performed in a fashion similar to that used to evaluate (5-38).

The result of programming this method plots very close to the solution given in Fig. 5-9. There are several shortcomings to this method. The expanded equations contain many terms, although most are easily programmed. However, a piecewise linear approximation to the current density requires terms in both variables x and y. Inspection of Galerkin's method indicates that the resulting number of terms in the integration would be very formidable.

E. Evaluation of Integrals.

1. Evaluation of $[\ell_{ii}]$ with Pulse Approximation to Current. The integration is over a small interval $\Delta\ell_i$ so that the small argument approximation

$$H_0^{(2)}(kt') = 1 - \left(\frac{kt'}{2}\right)^2 - j\,\frac{2}{\pi}\left[\left(1 - \left(\frac{kt'}{2}\right)^2\right)\log \gamma\,\frac{kt'}{2} + \left(\frac{kt'}{2}\right)^2\right] \tag{5-41}$$

may be used. Now t' is the variable of integration along the i-th segment of the scatterer with origin at the point defined by ρ_i. Hence (5-9) becomes

$$\ell_{ii} = \int_{-k\Delta^-\ell_i}^{0} H_o^{(2)}(-kt')\, dkt' + \int_{0}^{k\Delta^+\ell_i} H_o^{(2)}(kt')\, dkt' \qquad (5\text{-}42)$$

Integrating

$$\ell_{ii} = 2\left(\frac{k\Delta^-\ell_i}{2}\right)\left(1 - \frac{1}{3}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right) + 2\left(\frac{k\Delta^+\ell_i}{2}\right)\left(1 - \frac{1}{3}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right)$$

$$-j\,\frac{4}{\pi}\left(\frac{k\Delta^-\ell_i}{2}\right)\left[\log\left(\gamma\,\frac{k\Delta^-\ell_i}{2}\right)\left(1 - \frac{1}{3}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right) - 1 + \frac{4}{9}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right]$$

$$-j\,\frac{4}{\pi}\left(\frac{k\Delta^+\ell_i}{2}\right)\left[\log\left(\gamma\,\frac{k\Delta^+\ell_i}{2}\right)\left(1 - \frac{1}{3}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right) - 1 + \frac{4}{9}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right]$$

(5-43)

where

$$\gamma \approx 1.781072$$

2. <u>Evaluation of $[H^s_{ii}]$ for Pulse Approximation to Current.</u> To evaluate H^s_{ii} a small argument approximation to the Hankel function may again be used. However since the singularity is nonintegrable, the integral is first expressed with the field point off the surface of the conductor a small distance n. Thus

$$H^s_{ii} = \frac{1}{4jk}\ \underline{n} \times \int_{-k\Delta^-\ell_i}^{k\Delta^+\ell_i} \nabla \times \hat{z}\, H_o^{(2)}(k\sqrt{n^2 + t'^2})\, dkt' \qquad (5\text{-}44)$$

where

$$k\sqrt{n^2 + t'^2} << 1$$

Using

$$\nabla \times \hat{z}\, H_o^{(2)}(k\sqrt{n^2 + t'^2}) = \underline{t}'\,\frac{\partial}{\partial n} H_o^{(2)}(k\sqrt{n^2 + t'^2})$$

and

$$H_1^{(2)}(k\sqrt{n^2 + t'^2}) \cong \frac{2j}{\pi k\sqrt{n^2 + t'^2}}$$

The integral then becomes

$$H_{ii}^s = \frac{-1}{4j}\, \underline{n} \times \underline{t}'\, \frac{2j}{\pi}\left[\tan^{-1}\frac{k\Delta^+\ell_i}{n} - \tan^{-1}\frac{-k\Delta^-\ell_i}{n}\right] \qquad (5\text{-}45)$$

Letting $n \rightarrow 0$

$$H_{ii}^s = \hat{z}\,\frac{1}{2} \qquad (5\text{-}46)$$

which could also have been obtained directly using Ampere's law.

5. Evaluation of $[\ell_{ji}]$ for a Pulse Approximation to Current. Since the integration is over a small segment $\Delta\ell_i$, a small argument approximation to the m-th order Bessel function may be made when using the addition theorem expansion. Thus

$$\ell_{ji} = \int_{-k\Delta^-\ell_i}^{0} \sum_{m=0}^{\infty} \epsilon_m J_m(-kt')\, H_m^{(2)}(k|\rho|) \cos m(\phi' - \pi)\, dkt'$$

$$+ \int_{0}^{+k\Delta^+\ell_i} \sum_{m=0}^{\infty} \epsilon_m J_m(kt')\, H_m^{(2)}(k|\rho|) \cos m\phi'\, dkt' \qquad (5\text{-}47)$$

$$\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

where

$$J_m(kt') = \frac{1}{m!}\left(\frac{kt'}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{kt'}{2}\right)^{m+2} \qquad (5\text{-}48)$$

On integration

$$\ell_{ji} = 2\sum_{m=1}^{\infty} \frac{\epsilon_m}{m!}\left[\left(\frac{k\Delta^-\ell_i}{2}\right)^{m+1}(-1)^m\left[\frac{1}{m+1} - \frac{1}{(m+1)(m+3)}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right]\right.$$

$$\left.+\left(\frac{k\Delta^+\ell_i}{2}\right)^{m+1}\left[\frac{1}{m+1} - \frac{1}{(m+1)(m+3)}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right]\right] H_m^{(2)}(k|\rho|)\cos m\phi' \tag{5-49}$$

4. Evaluation of Linear Term in Expansion for Current. The integration is done exactly as in Sections V-E.1 and V-E.3. For the singular terms

$$\int_{k\Delta\ell_i} t' \, H_o^{(2)}(k|\rho_i - \rho_i'|)dkt'$$

$$= -\frac{2}{k}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\left(1 - \frac{1}{2}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right) + \frac{2}{k}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\left(1 - \frac{1}{2}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right)$$

$$+j\frac{4}{k\pi}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\left[\log\left(\frac{\gamma k\Delta^-\ell_i}{2}\right)\left(1 - \frac{1}{2}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right) - \frac{1}{2} + \frac{5}{8}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right]$$

$$-j\frac{4}{k\pi}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\left[\log\left(\frac{\gamma k\Delta^+\ell_i}{2}\right)\left(1 - \frac{1}{2}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right) - \frac{1}{2} + \frac{5}{8}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right] \tag{5-50}$$

For integration over nonsingular intervals

$$\int_{k\Delta\ell_i} t' \, H_o^{(2)}(k|\rho_j - \rho_i|)\, dkt_i'$$

$$= -\frac{4}{k}\sum_{m=0}^{4}(-1)^m\frac{\epsilon_m}{m!}\left(\frac{k\Delta^-\ell_i}{2}\right)^{m+2}\left[\frac{1}{m+2} - \frac{1}{(m+1)(m+4)}\left(\frac{k\Delta^-\ell_i}{2}\right)^2\right]H_m^{(2)}(k|\rho|)\cos m\bar{\phi}'$$

$$+\frac{4}{k}\sum_{m=0}^{4}\frac{\epsilon_m}{m!}\left(\frac{k\Delta^+\ell_i}{2}\right)^{m+2}\left[\frac{1}{m+2} - \frac{1}{(m+1)(m+4)}\left(\frac{k\Delta^+\ell_i}{2}\right)^2\right]H_m^{(2)}(k|\rho|)\cos m\overset{+}{\phi}' \tag{5-51}$$

5. Expansion of Hankel Function in Both Source and Field Variables. One method of expanding the Hankel function is to use the addition theorem in its most general form[5]

$$H_n^{(2)}(k|\rho|) = \sum_{m=-\infty}^{\infty} J_m(kt'') H_{n+m}^{(2)}(k\rho)\, e^{jm\phi''} \qquad (5\text{-}52)$$

along with the special case used previously

$$H_0^{(2)}(k|\rho - \rho'|) = \sum_{m=0}^{\infty} \epsilon_m J_m(kt') H_m^{(2)}(k\rho) \cos m\phi' \qquad (5\text{-}53)$$

Let primed variables denote source coordinates and double primed variables denote field coordinates with the distance between source and field points denoted by ρ. To obtain an expansion which is symmetrical in both t' and t'' form

$$H_0^{(2)}(k|\rho-\rho'|) = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{\infty} \epsilon_m J_m(kt') J_p(kt'') H_{m+p}(k\rho) \cos m\phi' \, e^{jp\phi''}$$

$$+ \frac{1}{2}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{\infty} \epsilon_m J_m(kt'') J_p(kt') H_{m+p}(k\rho) \cos m\phi'' \, e^{jp\phi'} \qquad (5\text{-}54)$$

An upper limit of $m = p = 2$ is used and only the significant terms are retained. Using the recurrence relationships for Hankel functions, the sums may be put in a form where many of the terms combine or are negligible. The complex exponential terms need be retained only when the source and field intervals are close together.

6. Evaluation of the $[\ell_{ij}]$ in Galerkin's Method for a Pulse Approximation to Current Density. An integration over both source and field coordinates is now required. For this purpose, the small argument Hankel

function is now written as

$$H_o^{(2)}(|t-t'|) = 1 - (\frac{t-t'}{2})^2 - j\frac{2}{\pi}\left[(1 - (\frac{t-t'}{2})^2) \log \frac{\gamma(t-t')}{2} + (\frac{t-t'}{2})^2\right]$$

for $t > t'$ and with t and t' interchanged for $t < t'$. The integration to be carried out is then

$$\ell_{ii} = \int_{-k\Delta^-\ell_i}^{k\Delta^+\ell_i} \left[\int_{-k\Delta^-\ell_i}^{kt} H_o^{(2)}\left[k(t-t')\right] dkt' + \int_{kt}^{k\Delta^+\ell_i} H_o^{(2)}\left[k(t'-t)\right] dkt'\right] dkt \tag{5-56}$$

After the first integration is carried out the second may be reduced considerably. The final result is

$$\ell_{ii} = (k\Delta\ell_i)^2 (1 - \frac{1}{24}(k\Delta\ell_i)^2)$$
$$+ j\frac{1}{\pi}(k\Delta\ell_i)^2 (3 - 2\log(\gamma \frac{k\Delta\ell_i}{2})(1 - \frac{1}{24}(k\Delta\ell_i)^2) - \frac{19}{144}(k\Delta\ell_i)^2) \tag{5-57}$$

F. References

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4. R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Co., New York, 1961, p. 232.

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VI. SCATTERING BY DIELECTRIC CYLINDERS

A. Introduction. In this section, the problem of scattering by dielectric cylinders is discussed and a particular solution is formulated in terms of electric and magnetic surface currents on the obstacle. The fields inside and outside are represented in terms of the field radiated by electric and magnetic surface currents distributed over the surface of the obstacle.

The two dimensional curve representing the surface of the obstacle is approximated with straight line segments and currents are assumed to be constant over each straight line segment or to be delta functions located at the center of the strip. The fields due to infinite strips of electric or magnetic current, with $\underline{J}_s$ or $\underline{M}_s$ directed either in the z direction or along the straight line segment are calculated. A dual formulation is used to extend the results to the TE polarization.

B. Formulation of the Problem (TM to z). Consider the problem indicated in Fig. 6-1. A plane wave with electric field in the $\hat{z}$ direction (TM to z) is incident upon an infinite dielectric cylinder of arbitrary cross section, permittivity ϵ, and permeability μ.

We can represent the fields in terms of incident and scattered fields

$$\begin{aligned} E_z &= E_z^s + E_z^{inc} \\ H_\phi &= H_\phi^s + H_\phi^{inc} \\ H_\rho &= H_\rho^s + H_\rho^{inc} \end{aligned} \tag{6-1}$$

For purposes of calculating the fields external to the cylinder we can use the equivalence principle,[1] replacing the induced polarization

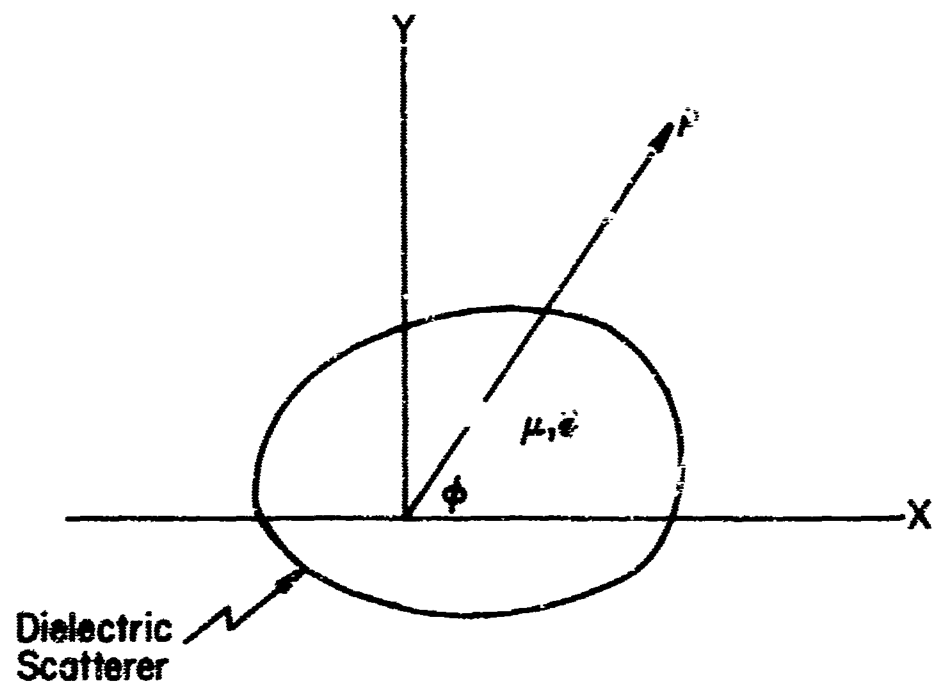


Fig. 6–1 Cylinderical Scatterer and Coordinate System

and magnetization sources $\partial \underset{\sim}{P}/\partial t$ and $\nabla \times \underset{\sim}{M}$ with surface currents $\underset{\sim}{J}_s = \underset{\sim}{n} \times \underset{\sim}{H}$ and $\underset{\sim}{M}_s = \underset{\sim}{E} \times \underset{\sim}{n}$ distributed over the surface of the obstacle. These sources are such as to produce zero fields inside (if the currents are exact). Since these equivalent sources produce zero fields inside, we can replace the cylinder with any material and the external fields will not be affected. If we choose the material as free space, then the problem becomes a homogeneous one of calculating the fields due to $\underset{\sim}{J}_s$, $\underset{\sim}{M}_s$ in infinite homogeneous free space.

Similarly, for purposes of calculating the internal fields, we can replace the effects of the incident field with surface currents $-\underset{\sim}{J}_s = +\underset{\sim}{n} \times \underset{\sim}{H}$ and $-\underset{\sim}{M}_s = +\underset{\sim}{E} \times \underset{\sim}{n}$ distributed over the surface of the cylinder. These sources are such as to produce zero fields outside the cylinder (if the currents are exact), so that we can replace the free space outside the cylinder with any material and the internal fields will not be affected. If we choose the material to be of permittivity ϵ and permeability μ, then the problem becomes the homogeneous one of calculating the fields due to $-\underset{\sim}{J}_s$, $-\underset{\sim}{M}_s$ in infinite homogeneous space of characteristics μ, ϵ.

Vector potential functions can be calculated for these two homogeneous problems. Denoting potentials outside by subscripts 1, then

$$A_1 = \frac{1}{4j} \oint_{\text{contour}} \underset{\sim}{J}_s(\ell') \, H_0^{(2)}(k_0 \rho) \cdot d\underset{\sim}{\ell}' \tag{6-2}$$

$$= \frac{1}{4j} \oint_{\text{contour}} \underset{\sim}{M}_s(\ell') \, H_0^{(2)}(k_0 \rho) \cdot d\underset{\sim}{\ell}' \tag{6-3}$$

where

$$\rho = \sqrt{(x - x')^2 + (y - y')^2}$$

$d\underset{\sim}{\ell}'$ = differential element of length around contour of cylinder

$k_0 = \omega\sqrt{\epsilon_0 \mu_0}$

The fields outside are given by

$$
\begin{aligned}
E_1 &= E^i - \nabla \times F_1 + \frac{1}{j\omega\epsilon_o} \nabla \times \nabla \times A_1 \\
H_1 &= H^i + \nabla \times A_1 + \frac{1}{j\omega\mu_o} \nabla \times \nabla \times F_1
\end{aligned}
\qquad (6\text{-}4)
$$

The vector potentials for the fields inside are similar except the material wave number and material constants appear. Thus

$$
\begin{aligned}
A_2 &= \frac{1}{4j} \oint - J_s(\ell') \, H_o^{(2)}(k_o \rho) \, d\ell' \\
F_2 &= \frac{1}{4\pi} \oint - M_s(\ell') \, H_o^{(2)}(k\rho) \, d\ell' \\
E_2 &= - \nabla \times F_2 + \frac{1}{j\omega\epsilon} \nabla \times \nabla \times A_2 \\
H_2 &= \nabla \times A_2 + \frac{1}{j\omega\mu} \nabla \times \nabla \times F_2
\end{aligned}
\qquad (6\text{-}5)
$$

where $k = \omega\sqrt{\mu\epsilon}$

Since there are no surface currents in the actual problem, the tangential E and H fields must be continuous at the surface of the cylinder. On the surface of the cylinder

$$
\begin{aligned}
n \times E_1 &= n \times E_2 \\
n \times H_1 &= n \times H_2
\end{aligned}
\qquad (6\text{-}6)
$$

Equations (6-6) may be expressed using (6-4) and (6-5) in terms of two integro-differential operator equations

$$
n \times \left[- \frac{1}{j\omega\epsilon_o} \nabla \times \nabla \times A_1 + \frac{1}{j\omega\epsilon} \nabla \times \nabla \times A_2 + \nabla \times F_1 - \nabla \times F_2 \right] = n \times E^{inc}
$$

$$
n \times \left[-\nabla \times A_1 + \nabla \times A_2 - \frac{1}{j\omega\mu_c} \nabla \times \nabla \times F_1 + \frac{1}{j\omega\mu} \nabla \times \nabla \times F_2 \right] = n \times H^{inc}
$$

(6-7)

which are of the form:

$$L_{11}(f_1) + L_{12}(f_2) = g_1$$
$$L_{21}(f_1) + L_{22}(f_2) = g_2 \qquad (6\text{-}8)$$

where L_{11}, L_{12}, L_{21}, L_{22} are integro-differential operators, f_1, f_2 are the unknowns $\underline{J}_s$, $\underline{M}_s$ and g_1, g_2 are the known functions $\underline{n} \times \underline{E}^{inc}$ and $\underline{n} \times \underline{H}^{inc}$.

Equations (6-7) may be rewritten in the form

$$\underline{n} \times \left[\underline{E}_2^J - \underline{E}_1^J + \underline{E}_2^M - \underline{E}_1^M\right] = \underline{n} \times \underline{E}^{inc}$$
$$\underline{n} \times \left[\underline{H}_2^J + \underline{H}_1^J + \underline{H}_2^M - \underline{H}_1^M\right] = \underline{n} \times \underline{H}^{inc} \qquad (6\text{-}9)$$

where $\underline{E}_2^J$ is the field inside due to the $-\underline{J}_s$ source in infinite homogeneous media with constants μ and ϵ and $\underline{E}_1^J$ is the field outside due to the $\underline{J}_s$ source in free space. Like definitions hold for the $\underline{H}$ fields.

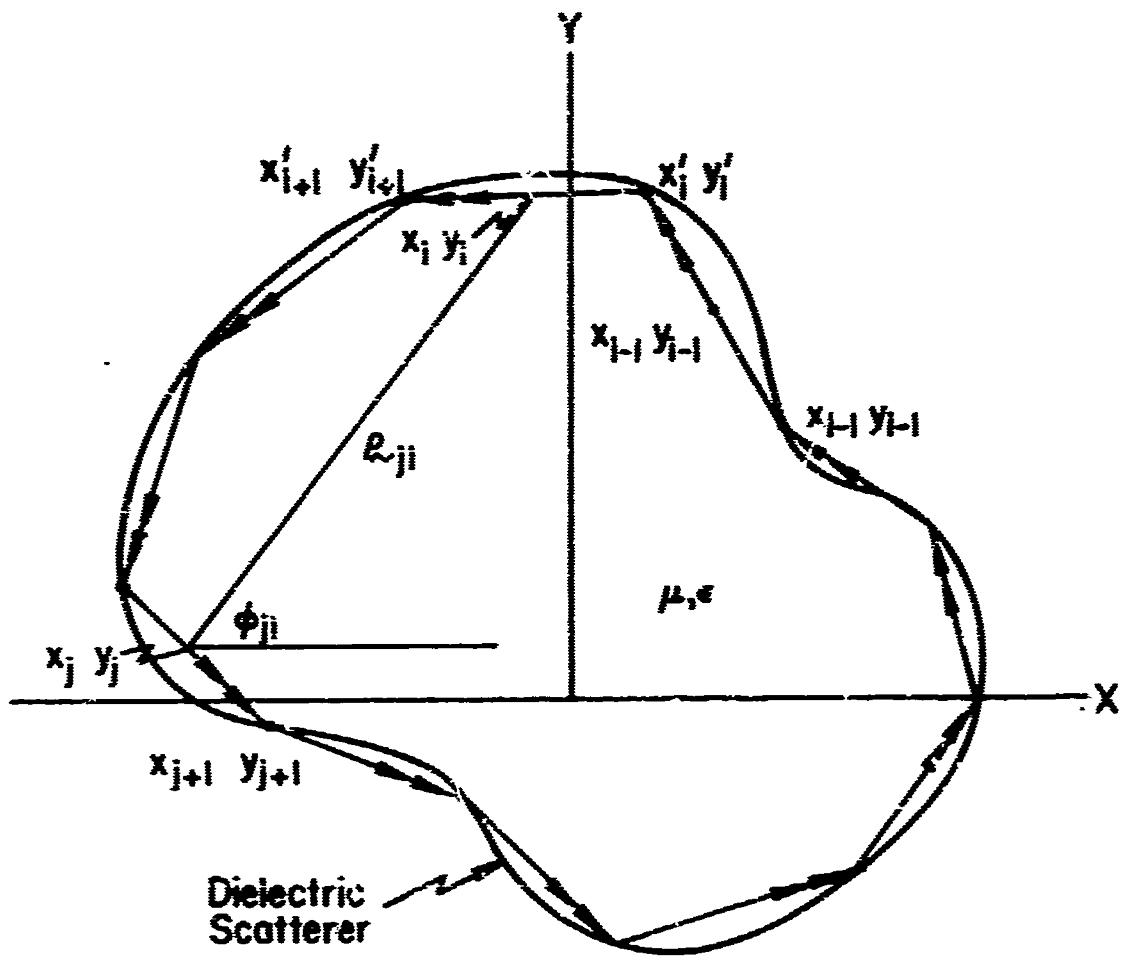
In order to convert the operator equations to a set of matrix equations the contour of the scatterer is divided into N straight line segments as in Fig. 6-2. The current is then assumed uniform within each interval so

$$\underline{J}_s = \sum_{i=1}^{N} \alpha_i \, P_i$$
$$\underline{M}_s = \sum_{i=1}^{N} \beta_i \, P_i \qquad (6\text{-}10)$$

where

$$P_i = \begin{cases} 1 & \text{within i-th segment} \\ 0 & \text{outside i-th segment} \end{cases}$$

Using (6-10) and satisfying (6-9) at the center of each segment gives two sets of matrix equations. For TM polarization



Magnetic currents are assumed counterclockwise.

Electric currents are assumed in $\hat{z}$ direction.

$$x_i = \tfrac{1}{2}(x'_i + x'_{i+1})$$

$$y_i = \tfrac{1}{2}(y'_i + y'_{i+1})$$

$$\sin \phi_{ji} = \frac{y_i - y_j}{|\rho_{ji}|}$$

$$\cos \phi_{ji} = \frac{x_i - x_j}{|\rho_{ji}|}$$

Fig. 6-2 Dielectric scatterer and notation.

$$\begin{bmatrix} [E^J] & [E^M] \\ [H^J] & [H^M] \end{bmatrix} \begin{bmatrix} J \\ \hline M \end{bmatrix} = \begin{bmatrix} E^{inc} \\ H^{inc} \end{bmatrix} \tag{6-11}$$

where

$$[J] = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad [M] = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} \tag{6-12}$$

To illustrate the form of the four square submatrices

$$[E^J] = \begin{bmatrix} E^J_{11} & E^J_{12} & \cdots \\ E^J_{21} & E^J_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} \tag{6-13}$$

where each matrix element is

$$E^J_{ji} = E^J_{2ji} - E^J_{1ji} \tag{6-14}$$

Here the partial matrix element E_{2ji} is the electric field at the center of the j-th segment due to a current $\vec{J}_s$ of magnitude unity in the i-th interval when the medium has permeability μ and permittivity ϵ. E^J_{1ji} is the same type of field in free space. Similar definitions hold for the other three submatrices and their matrix elements. A dual set of equations holds for the TE polarization as discussed in VI-E. These are obtained by substituting the dual quantities in (6-11) to (6-14). The remainder of this chapter is then devoted to finding the matrix elements for TM polarization.

C. Evaluation of Partial Matrix Elements From Magnetic Currents. We wish to calculate the fields at the midpoint of the j-th segment due to the unit magnetic current at the i-th segment of Fig. 6-2.

It is assumed that the magnetic currents are all directed counter clockwise. Then, locating ourselves at the midpoint of a segment and facing in the direction of the magnetic current arrow, the fields for the subscript 1 case (free space) will be just to the right and the fields for the subscript 2 case will be just to the left.

Consider the case i=j, that is the fields are to be evaluated at the midpoint of the j-th segment due to magnetic currents in the j-th segment. The orientation of the coordinate system is unimportant for this problem and the configuration may be simplified to that of Fig. 6-3.

The magnetic field is obtained from the vector potential $\underline{F}$, where

$$\underline{F} = \frac{1}{4j} \int_{-d/2}^{d/2} H_o^{(2)}(k\rho)\, dx' \qquad (6\text{-}15)$$

and

$$|\rho| = \sqrt{x'^2 + y^2}$$

Now in dielectric media with constants μ and ϵ

$$\underline{H} = \frac{1}{j\omega\mu} \nabla \times \nabla \times \underline{F} \qquad (6\text{-}16)$$

Since in Fig. 6-3 the tangential magnetic field H_{x2} and the partial matrix element H^M_{2jj} are identical, then

$$H_{2jj} = \frac{-1}{j\omega\mu} \frac{\partial^2}{\partial y^2} \int_{-d/2}^{d/2} H_o^{(2)}(k\rho)\, dx' \qquad (6\text{-}17)$$

This field may be evaluated in the following way. Let the field point lie on the y axis, at a point (0,y), evaluate $\int H_o^{(2)}(k\rho)\, dx'$, differentiate twice with respect to y, and then let $y \rightarrow 0$. Hence

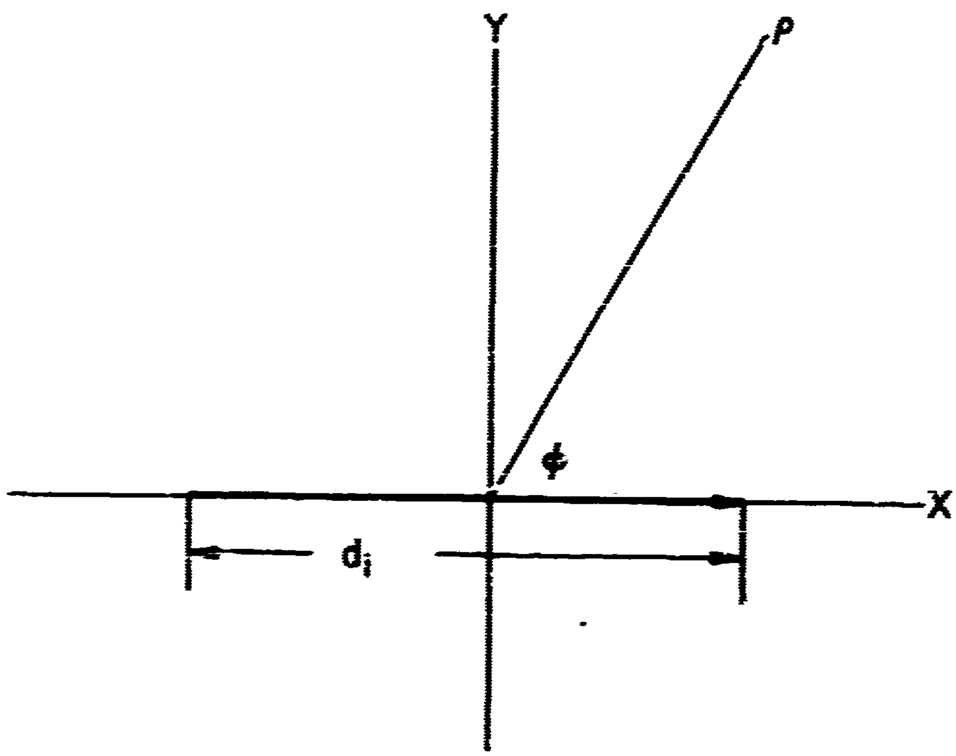


Fig. 6-3 Magnetic Current Source

$$H^M_{2jj} = H_{x2}(0,0^+) = \lim_{y \to 0} \left[-\frac{1}{j\omega\mu} \frac{\partial^2}{\partial y^2} \int_{-d/2}^{d/2} H_o^{(2)}(k\rho)\, dx' \right] \qquad (6\text{-}18)$$

Taking the first two terms of each series in the series expansion of $H_o^{(2)}(k\rho)$, using integration formulas Dwight 623 and Dwight 623.2,[2] differentiating and taking the limit as $y \to 0$, the partial matrix element becomes

$$H^M_{2jj} = -\frac{j}{2\eta_o} \sqrt{\frac{\epsilon_r}{\mu_r}}\,(kd) - \frac{8}{\eta_o \pi} \sqrt{\frac{\epsilon_r}{\mu_r}}\left(\frac{1}{kd}\right)$$

$$+ \frac{1}{\pi\eta_o} \sqrt{\frac{\epsilon_r}{\mu_r}} \left[\frac{3}{2} - \log \frac{\gamma kd}{4}\right] \qquad (6\text{-}19)$$

To calculate the tangential magnetic field $H_x(0,0^+)$ in the subscript 1 case, consider the same limiting process with the field point approaching (0,0) along the negative y axis; by symmetry the tangential $\underline{H}$ field will be identical to that obtained in the subscript 2 case, since the magnetic current source is the negative of the source in the subscript 2 case. Thus we obtain the field H_{x1} by replacing the unit magnetic current with its negative and μ, ϵ, k with μ_o, ϵ_o, k_o. Hence

$$H^M_{1jj} = -\frac{j}{2\eta_o}(k_o d) - \frac{8}{\eta_o \pi}\left(\frac{1}{k_o d}\right) + \frac{1}{\pi\eta_o}\left[\frac{3}{2} - \log \frac{\gamma k_o d}{4}\right] \qquad (6\text{-}20)$$

The tangential electric field on both sides of a strip of magnetic current of strength unity must also be obtained. From Fig. 6-3, if we assume that the electric field to the left of the midpoint is in the $\hat{z}$ direction, then from symmetry considerations, the electric field just to the right is in the negative z direction. The strength of the field is most easily obtained by using the dual of Ampere's law

$$-\int \mathbf{E} \cdot d\ell = K \tag{6-21}$$

where K is the magnetic current enclosed. The result for a unit magnetic current is easily obtained as

$$E^M_{1jj} = E^M_{2jj} = -\frac{1}{2}\ell \tag{6-22}$$

Now consider the evaluation of the fields when $j \neq i$. Replace the magnetic surface current distribution with a magnetic dipole element of strength d_i located at the midpoint of the segment. Here d_i is the length of the i-th segment. For purposes of calculation, assume a magnetic current element located at the origin and oriented in the x direction as in Fig. 6-3. Then to a crude approximation

$$\mathbf{F} = \frac{d_i}{4j} H_o^{(2)}(k\rho)$$

$$\mathbf{E} = -\nabla \times \mathbf{F} \qquad \mathbf{H} = \frac{1}{j\omega\mu} \nabla \times \nabla \times \mathbf{F} \tag{6-23}$$

Thus

$$\mathbf{E}_z = \frac{kd_i}{4j} \sin\phi \, H_o^{(2)\prime}(k\rho)$$

$$H_\rho = \frac{kd_i}{4\eta} \frac{\cos\phi}{k\rho} H_o^{(2)\prime}(k\rho) \tag{6-24}$$

$$H_\phi = -\frac{kd_i}{4\eta} \sin\phi \, H_o^{(2)\prime\prime}(k\rho)$$

Now if we locate the magnetic current element at the center of the i-th strip with an orientation corresponding to that of the i-th strip we can derive fields at the center of the j-th strip by replacing ρ with $|\rho_{ji}|$ and ϕ with $\phi_{ji} - \phi_i$. Then

$$E^M_{2ji} = \frac{-k^2 d_i}{4j} \sin(\phi_{ji} - \phi_i)\, H_1^{(2)}(k|\rho_{ji}|) \qquad (6\text{-}25)$$

$$H^M_{2ji} = H_\phi \sin(\phi_j - \phi_{ji}) + H_\rho \cos(\phi_j - \phi_{ij})$$

$$= \frac{-kd_i}{4\eta}\left[\sin(\phi_j - \phi_{ij}) \sin(\phi_{ij} - \phi_i)\, H_o^{(2)''}(k|\rho_{ji}|)\right.$$

$$\left. + \frac{\cos(\phi_j - \phi_{ij}) \cos(\phi_{ij} - \phi_i)}{k|\rho_{ij}|} H_1^{(2)}(k|\rho_{ji}|)\right] \qquad (6\text{-}26)$$

The elements E^M_{1ji} and H^M_{1ji} can be obtained by replacing k, ϵ, μ in (6-26) with k_o, ϵ_o, μ_o.

D. Evaluation of Partial Matrix Elements from Electric Currents. We wish to calculate the fields at the j-th point due to a unit electric current at the i-th segment.

First consider the case when $j = i$ as in Fig. 6-3. Then

$$A_z = \frac{1}{4j} \int H_o^{(2)}(k\rho)\, dx' \qquad (6\text{-}27)$$

so

$$E_z = \frac{-k^2}{2\omega\epsilon} \int_0^{d/2} H_o^{(2)}(kx')\, dx' \qquad (6\text{-}28)$$

Since E_z and E^J_{2jj} are identical, we have on integrating directly

$$E^J_{2jj} = \frac{-k^2}{2\omega\epsilon}\left\{\frac{d}{2} H_o^{(2)}\left(\frac{kd}{2}\right) + \frac{\pi d}{4}\left[H_1^{(2)}\left(\frac{kd}{2}\right) \mathbf{H}_o\left(\frac{kd}{2}\right) - H_o^{(2)}\left(\frac{kd}{2}\right) \mathbf{H}_1\left(\frac{kd}{2}\right)\right]\right\} \qquad (6\text{-}29)$$

where $\mathbf{H}_1$ and $\mathbf{H}_o$ are Struve functions. Alternately, if $H_o^{(2)}(kx')$ is approximated by its series expansion (retaining the first two terms of each series), and integrated directly, the following result is obtained

$$E^J_{2jj} = -\left(\frac{kd}{4}\right)\left(1 - \frac{1}{3}\left(\frac{kd}{4}\right)^2\right)$$

$$-j\,\frac{2}{\pi}\left(\frac{kd}{4}\right)\left[\log\left(\frac{\gamma kd}{4}\right)\left(1 - \frac{1}{3}\left(\frac{kd}{4}\right)^2\right) - 1 + \frac{4}{9}\left(\frac{kd}{4}\right)^2\right] \qquad (6\text{-}30)$$

The element E_{1jj} is obtained by replacing K by K_o in (6.30). The tangential magnetic field may be obtained by applying Ampere's law in a manner similar to that used in Section VI-C to calculate the tangential electric field. The result is

$$H^J_{1jj} = H^J_{2jj} = \frac{1}{2}\,\underset{\sim}{z} \qquad (6\text{-}31)$$

Now consider the fields when $j \neq i$. Replace the electric surface current $\underset{\sim}{J}_s$ with an electric filament directed in the z direction with current $I = d_i$, and located at the midpoint of segment i. Consider a filament located at the origin. Then

$$A_z = \frac{d_i}{4j}\,H_o^{(2)}(k\rho) \qquad (6\text{-}32)$$

Hence

$$E_z = \frac{-\eta k d_i}{4j}\,H_o^{(2)}(k\rho)$$

$$H_\phi = \frac{kd_i}{4j}\,H_1^{(2)}(k\rho) \qquad (6\text{-}33)$$

$$H_\rho = 0$$

The partial matrix elements for the subscript 2 case are then

$$E^J_{2ji} = -\frac{\eta k d_i}{4j}\,H_o^{(2)}(k|\rho_{ji}|)$$

$$(6\text{-}34)$$

$$H^J_{2ji} = \frac{kd_i}{4j}\,H_1^{(2)}(k|\rho_{ji}|)\,\sin\,(\phi_j - \phi_{ij})$$

The elements E^J_{1ji} and H^J_{1ji} are obtained by replacing k and η by k_o and η_o in (6-34). The matrix elements in (6-11) have thus essentially been established.

E. Duality. For incident fields TE to z, the problem may be solved by duality. In this case, all magnetic fields will be in the z direction and the electric fields will lie in the xy plane. Thus the scattered fields can be calculated by assuming magnetic current sheets in the z direction and electric currents sheets directed along the contour of the scatterer in the xy plane. Again the problem is resolved into two homogeneous problems, one involving free space and the other involving an infinite homogeneous space of characteristics μ, ϵ. Replacing μ with ϵ and $\underline{J}$ with $\underline{M}$ according to a duality principle[4] yields problems similar to those discussed in VI-A to VI-D, which can be solved using the functions developed in VI-C and VI-D.

F. References.

1. R. F. Harrington, "Time Harmonic Electromagnetic Fields," McGraw-Hill Book Co., New York, 1961, p. 106.

2. H. B. Dwight, "Tables of Integrals," Macmillan Company, New York.

3. N. W. McLachlan, "Bessel Functions for Engineers," Clarendon Press.

4. R. F. Harrington, op. cit., p. 99.

VII. SCATTERING FROM BODIES OF REVOLUTION

A. Introduction. The objective of this work is to determine matrix methods of solving for the surface currents induced on conducting bodies of revolution by incident electromagnetic waves. These currents can then be used to calculate the scattered fields.

The general problem has been solved numerically by Andreasen[1,2] for bodies with maximum cross-section perimeters of twenty wavelengths. His method consists of obtaining approximate solutions of the integral equations for the induced currents.

In this section, the general solution for the body currents will be formulated by the method of moments. From this, two specific solutions, namely, a field matching solution and a solution by Galerkin's method, will be described.

B. Description of Problem. Consider an electromagnetic wave incident on the surface of an arbitrary conducting body of revolution. The coordinates for the body and the incident wave are illustrated in Fig. 7-1 and 7-2 respectively. The incident field ($\underline{E}^i$) induces surface currents on the body which are the sources of the scattered field ($\underline{E}^s$). The boundary conditions require that the total tangential $\underline{E}$ be zero on the body surface or

$$E^s_{tang}(r) = - E^i_{tang}(r) \qquad (7\text{-}1)$$

For convenience, $(\hat{t}_1, \hat{t}_2)$ are defined as unit vectors of the local orthogonal coordinate system defined on the body surface. $\hat{t}_1$ corresponds to the $\hat{t}$ vector and $\hat{t}_2$ is $\hat{\phi}$ in Fig. 7-1. In terms of these coordinates, (7-1) becomes

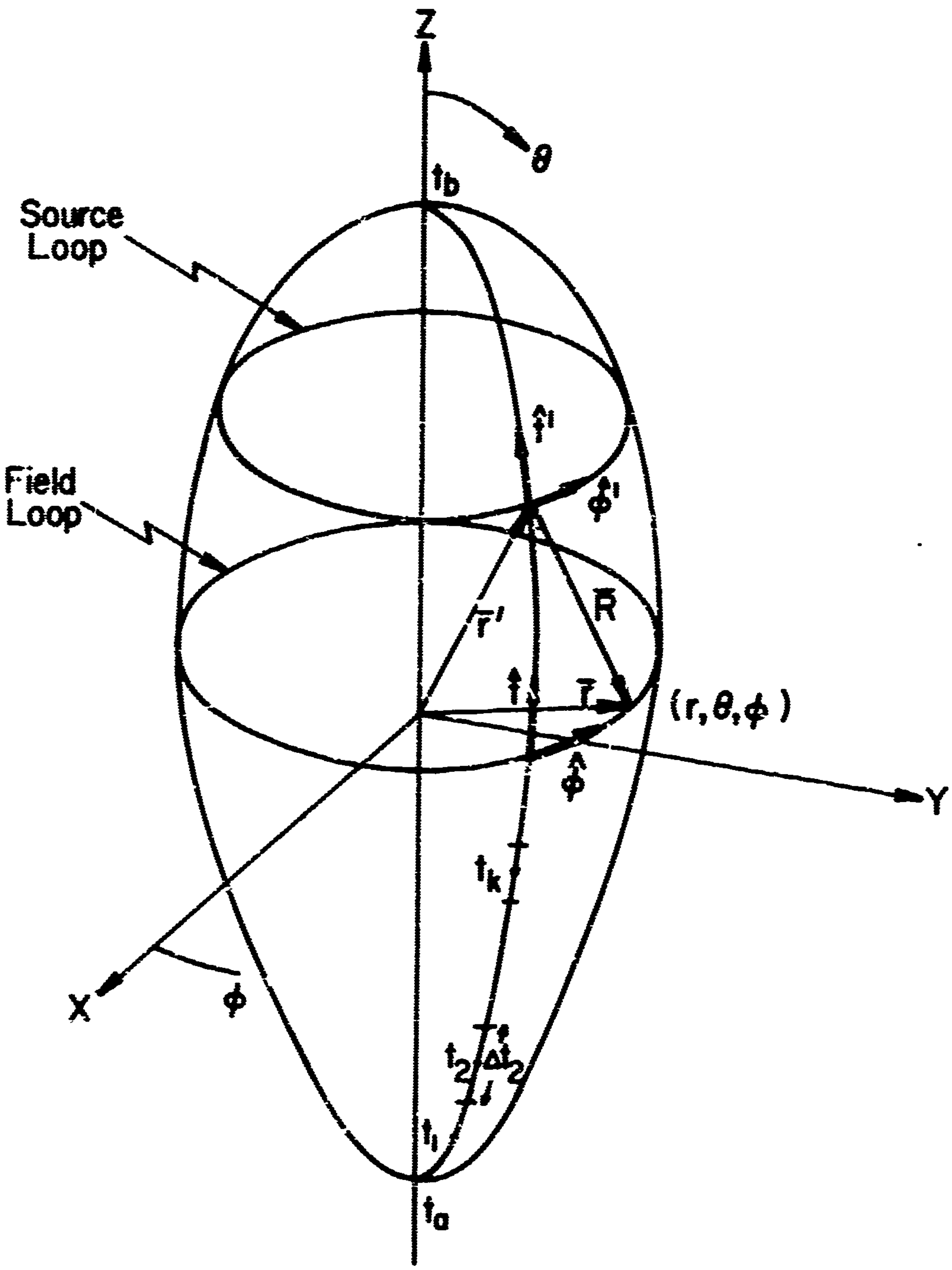


Fig. 7-1 Coordinates for Body of Revolution.

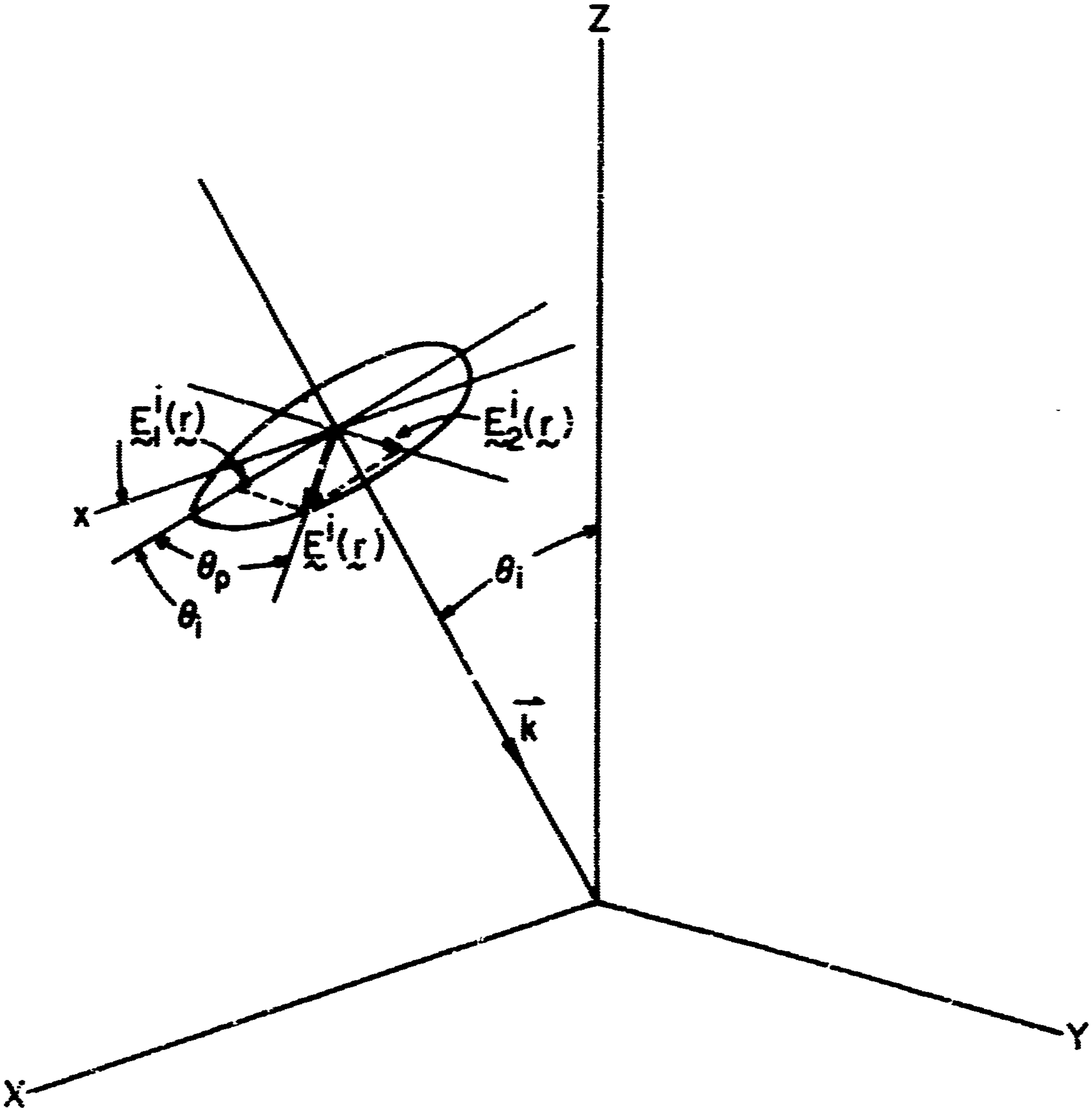


Fig. 7-2 Coordinates of Incident Plane Wave.

$$\hat{t}_n \cdot \underline{E}^s(\underline{r}) = -\hat{t}_n \cdot \underline{E}^i(\underline{r}) \qquad n = 1, 2 \qquad (7\text{-}2)$$

The scattered fields are expressed in terms of the surface currents as

$$\underline{E}^s(\underline{r}) = L\,\underline{J}_s(\underline{r}') \qquad (7\text{-}3)$$

where the operator L is defined below. Equation (7-2) becomes

$$\hat{t}_n \cdot L\,\underline{J}_s(\underline{r}') = -\hat{t}_n \cdot \underline{E}^i(\underline{r}) \qquad n = 1, 2 \qquad (7\text{-}4)$$

The problem is now reduced to solving (7-4) for $\underline{J}_s(\underline{r}')$.

To define the operator L, (7-3) is expressed in terms of the magnetic vector and scalar potentials. The vector potential for the induced surface current $\underline{J}_s(\underline{r}')$ is

$$\underline{A}^s(\underline{r}) = \frac{\mu}{4\pi} \oiint_{\text{body}} \underline{J}_s(\underline{r}')\,\frac{e^{-jkR}}{R}\,da' \qquad (7\text{-}5)$$

where in terms of the spherical coordinates of Fig. 7-1

$$R = |\underline{r} - \underline{r}'| = \left[r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'))\right]^{1/2}$$

The scalar potential is given by

$$\Psi^s(\underline{r}) = -\frac{1}{4\pi j\omega\epsilon} \oiint_{\text{body}} \left[\overline{\nabla}_{t'} \cdot \underline{J}_s(\underline{r}')\right] \frac{e^{-jkR}}{R}\,da' \qquad (7\text{-}6)$$

where $\nabla_t \cdot$ is the differential operator tangent to the body surface. In terms of the local body coordinates, this operation on an arbitrary vector $\underline{B}$ is

$$\overline{\nabla}_t \cdot \underline{B} = \frac{\partial B_t}{\partial t} + \frac{\sin v}{\rho} B_t + \frac{1}{\rho}\frac{\partial}{\partial\phi} B_\phi$$

where ρ is the cylindrical coordinate radius and v is the counterclockwise angle between $\hat{t}$ and the z axis.

Now the scattered field is expressed in terms of the induced current as

$$\underline{E}^s(\underline{r}) = L\, \underline{J}_s(\underline{r}') = -\, j\omega\, \underline{A}^s(\underline{r}) - \overline{\nabla}_t\, \psi(\underline{r}) \qquad (7\text{-}7)$$

where $\overline{\nabla}_t$ is the gradient operator on the body surface

$$\overline{\nabla}_t = \hat{t}\,\frac{\partial}{\partial t} + \frac{\hat{\phi}}{\rho}\,\frac{\partial}{\partial \phi}$$

The definition of L is implied by (7-7)

C. Expansion of the Incident Field and Surface Current. An arbitrary plane wave can be expanded in a discrete set of cylindrical modes. Each mode has a different azimuthal wave number and each propagates with the same velocity along the $\hat{z}$-axis of Fig. 7-2. The advantage of this expansion is that the current induced by each mode can be computed separately and the total current can then be found by superimposing all of the significant mode currents. This procedure is necessary for handling large bodies on available computers.

The details of the expansion are given by Andreasen.[2] The propagation vector $\underline{k}$ of the incident wave is chosen to be in the x-z plane of Fig. 7-2. Then the incident wave is separated into a wave polarized parallel to the x-z plane and a wave polarized perpendicular to this plane.

The expansion for the tangential electric field component of the parallel polarized wave $\underline{E}_1^i$ at the surface of the body is[2]

$$\frac{\underline{E}_1^i}{\eta} = \hat{t} \sum_{m=0}^{\infty} E_{t1}(t) \cos(m\phi) + \hat{\phi} \sum_{m=1}^{\infty} E_{\phi 1}(t) \sin(m\phi) \qquad (7\text{-}8)$$

where

$$E_{t1}(t) = \cos\theta_p\, e^{jkz\cos\theta_i}\, \epsilon_m\, j^{m+1} \Big[j \sin\theta_i \cos v\, J_m(k\rho \sin\theta_i)$$

$$- \cos\theta_i \sin v\, J_m'(k\rho \sin\theta_i) \Big]$$

$$E_{\phi 1}(t) = \cos\theta_p \cos\theta_i\, e^{jkz\cos\theta_i}\, \epsilon_m\, j^{m+1}\, m\, \frac{J_m(k\rho \sin\theta_i)}{k\rho \sin\theta_i}$$

η is the impedance of free space, ϵ_m is Neumann's number, J_m is the Bessel function of order m and the prime indicates the differentiation with respect to the argument.

For the perpendicular polarized wave, the tangential field $\underline{E}_2^i$ is expanded as[2]

$$\frac{\underline{E}_2^i}{\eta} = \hat{t} \sum_{m=1}^{\infty} E_{t2}(t) \sin m\phi - \hat{\phi} \sum_{m=0}^{\infty} E_{\phi 2}(t) \cos m\phi \qquad (7\text{-}9)$$

where

$$E_{t2}(t) = -\sin\theta_p \sin v\, e^{jkz\cos\theta_i}\, \epsilon_m\, j^{m+1}\, m\, \frac{J_m(k\rho \sin\theta_i)}{k\rho \sin\theta_i}$$

$$E_{\phi 2}(t) = \sin\theta_p\, e^{jkz\cos\theta_i}\, \epsilon_m\, j^{m+1}\, J_m'(k\rho \sin\theta_i)$$

Combining (7-8) and (7-9), the incident field is

$$\frac{\underline{E}^i}{\eta}(r') = \sum_{m=0}^{\infty} \Big\{ \hat{t} \left[E_{t1}(t) \cos m\phi + E_{t2}(t) \sin m\phi \right]$$

$$+ \hat{\phi} \left[E_{\phi 1}(t) \sin m\phi - E_{\phi 2}(t) \cos m\phi \right] \Big\} \qquad (7\text{-}10)$$

The total current induced by this incident field is then of the form

$$J_s^{total}(r) = \sum_{m=0}^{\infty} \left\{ \hat{t} \left[J_{t1}(t) \cos m\phi + J_{t2}(t) \sin m\phi \right] \right.$$

$$\left. + \hat{\phi} \left[J_{\phi 1}(t) \sin m\phi - J_{\phi 2}(t) \cos m\phi \right] \right\} \qquad (7\text{-}11)$$

For each incident mode m, the currents induced by either polarization are

$$J_{s1}^m(r) = \hat{t}\, J_{t1}^m(t) \cos m\phi + \hat{\phi}\, J_{\phi 1}^m(t) \sin m\phi \qquad (7\text{-}12)$$

for the parallel polarized case and

$$J_{s2}^m(r) = \hat{t}\, J_{t2}^m(t) \sin m\phi - \hat{\phi}\, J_{\phi 2}^m(t) \cos m\phi \qquad (7\text{-}13)$$

for the perpendicular polarization. The methods of solution discussed below are valid for solution for either (7-12) or (7-13). It is then convenient to express (7-12) and (7-13) in the general form

$$J_s^m(r) = \hat{t}\, J_t^m(t)\, h_t(m\phi) + \hat{\phi}\, J_\phi^m(t)\, h_\phi(m\phi) \qquad (7\text{-}14)$$

The advantage of the above expansions is now evident since a single method of solution can be used to separately solve for the currents induced by each mode of either polarization.

D. Solutions by Method of Moments. The general formulation for the current solutions will be obtained by the application of the method of moments (Section I-D) to (7-4). Suitable choices of approximating functions for the current and weighting functions give the field matching solution and the solution by Galerkin's method which are discussed below.

The coordinates of the body currents are called the "source coordinates" and are the primed coordinates in Fig. 7-1. The coordinates of the incident and scattered fields are denoted by the unprimed "field coordinates." From (7-14), for a given mode, say the m-th mode, the current induced by either incident field polarization is of the form

$$\underline{\mathcal{J}}_s^m(\underline{r}') = \hat{t}'\ J_{t'}^m(t')\ h_t(m\phi') + \hat{\phi}'\ J_{\phi'}^m(t')\ h_\phi(m\phi') \tag{7-15}$$

A procedure analogous to the development of (7-14) is used to express the incident field corresponding to (7-15) as

$$\frac{\underline{E}^i(\underline{r})}{\eta} = \hat{t}\ E_t^m(t)\ h_t(m\phi) + \hat{\phi}\ E_\phi^m(t)\ h_\phi(m\phi) \tag{7-16}$$

where $h_t(m\phi)$ and $h_\phi(m\phi)$ are the respective trigonometric functions in (7-12) or (7-13) for whichever mode and polarization is being considered. The scattered fields due to $\underline{\mathcal{J}}_s$

$$\underline{E}^s(\underline{r}) = L\ \underline{\mathcal{J}}_s^m(\underline{r}') = -\frac{j\eta}{4\pi} \oiint_{\text{body}} da' \left[k\ \frac{e^{-jkR}}{R} + \frac{\nabla_t}{k}\ \frac{e^{-jkR}}{R}\ \overline{\nabla}_{t'}\ \cdot\right] \underline{\mathcal{J}}_s^m(\underline{r}') \tag{7-17}$$

The t and ϕ scattered field components are found by taking the inner product of (7-17) with the t and ϕ unit vectors at the field points. The result for either polarization is

$$\begin{aligned}\hat{t}\cdot L\ \underline{\mathcal{J}}_s^m(\underline{r}') = -\frac{j\eta}{4\pi}\, h_t(m\phi) \Bigg\{ k^2 \int_{t_a}^{t_b} \Bigg\{ \Big[&\sin v \sin v'\ (G_{m-1} + G_{m+1}) \\ & 2\cos v \cos v'\ G_m\Big]\ J_{t'}^m(t') - \sin v\ (G_{m-1} - G_{m+1}) \\ & J_{\phi'}^m(t')\Bigg\} \rho'\ dt' + 2\int_{t_a}^{t_b} \left[\frac{\partial J_{t'}^m(t')}{\partial t'} + \frac{\sin v'}{\rho'}\ J_{t'}^m(t')\right. \\ & \left. + \frac{m}{\rho'}\ J_{\phi'}^m(t')\right]\ \frac{\partial G_m}{\partial t}\ \rho'\ dt' \Bigg\}\end{aligned} \tag{7-18}$$

$$\hat{\phi} \cdot L\, \mathcal{J}_s^m(\mathbf{r}') = -\frac{j\eta}{4\pi}\, h_\phi(m\phi) \left\{ k^2 \int_{t_a}^{t_b} \left[-\sin v' \, (G_{m-1} - G_{m+1})\, J_{t'}^m(t') \right.\right.$$

$$\left. + (G_{m-1} + G_{m+1})\, J_\phi^m(t') \right] \rho\, dt'$$

$$\left. - \frac{2m}{\rho} \int_{t_a}^{t_b} \left[\frac{\partial J_{t'}^m(t')}{\partial t'} + \frac{\sin v'}{\rho'} J_{t'}^m(t') + \frac{m}{\rho'} J_\phi^m(t') \right] G_m \rho'\, dt' \right\} \tag{7-19}$$

where

$$G_m = \int_0^\pi \frac{e^{-jkR_o}}{kR_o} \cos m\phi'\, d\phi' \tag{7-20}$$

and

$$R_o = R(\phi = 0)$$

To apply the method of moments to either polarization of the m-th mode, the mode current terms are approximated as

$$J_t^m(t') \approx \sum_{i=1}^{N} \alpha_i\, T_i(t')$$

$$J_\phi^m(t') \approx \sum_{i=1}^{N} \beta_i\, \Phi_i(t') \tag{7-21}$$

where $T_i(t')$ and $\Phi_i(t')$ are the approximating functions in the method of moments. A suitable set of weighting functions $(\omega_1, \omega_2, \ldots, \omega_N)$ and an inner product are defined. The approximate currents (7-21) are substituted in (7-18) and (7-19) and the inner products of (7-4) with the weighting functions are formed as

$$\begin{bmatrix} \langle \hat{t} \cdot L\, \underset{\sim}{J}_s(\underset{\sim}{r}'),\ \omega \rangle \\ \langle \hat{\phi} \cdot L\, \underset{\sim}{J}_s(\underset{\sim}{r}'),\ \omega \rangle \end{bmatrix} = -\begin{bmatrix} \langle \hat{t} \cdot \underset{\sim}{E}^i(\underset{\sim}{r}'),\ \omega \rangle \\ \langle \hat{\phi} \cdot \underset{\sim}{E}^i(\underset{\sim}{r}'),\ \omega \rangle \end{bmatrix} \tag{7-22}$$

where the current and field terms are defined in (7-15) and (7-16). Both of the matrices of (7-22) are complex. The left sides of the respective real and imaginary matrices of (7-22) can be rearranged as

$$\left[\begin{array}{c|c} [\ell_{ij}^{tt'}] & [\ell_{ij}^{t\phi'}] \\ \hline [\ell_{ij}^{\phi t'}] & [\ell_{ij}^{\phi\phi'}] \end{array}\right] \left[\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_N \\ \hline \beta_1 \\ \vdots \\ \beta_N \end{array}\right] = \left[\begin{array}{c} \langle \hat{t} \cdot \underset{\sim}{E}^i(\underset{\sim}{r}),\ \omega_1 \rangle \\ \vdots \\ \langle \hat{t} \cdot \underset{\sim}{E}^i(\underset{\sim}{r}),\ \omega_N \rangle \\ \hline \langle \hat{\phi} \cdot \underset{\sim}{E}^i(\underset{\sim}{r}),\ \omega_1 \rangle \\ \vdots \\ \langle \hat{\phi} \cdot \underset{\sim}{E}^i(\underset{\sim}{r}),\ \omega_N \rangle \end{array}\right] \qquad \begin{array}{l} i = 1,2,\ldots,N \\ j = 1,2,\ldots,N \end{array} \tag{7-23}$$

The solutions for the complex currents, α_i and β_i are found by inverting the $[\ell]$ matrices. Thus

$$\begin{bmatrix} \alpha \\ \hdashline \beta \end{bmatrix} = [\ell]^{-1} \begin{bmatrix} e_t \\ e_\phi \end{bmatrix} \tag{7-24}$$

where the definitions of the matrices in (7-24) are implied from (7-23).

<u>E. Field Matching Solution.</u> The field matching solution amounts to satisfying (7-1) at N points along the body perimeter from t_a to t_b (see Fig. 7-1) for each polarization of each mode. The perimeter length from t_a to t_b is divided into N segments. The length of the k-th segment is Δt_k and the center point is t_k as illustrated in Fig. 7-1. Since in (7-22) the ϕ variations in the scattered and incident fields are the same, the $h(m\phi)$ terms can be cancelled by choosing the weighting functions to be

$$\omega_k = \delta(t - t_k) \qquad k = 1, 2, \ldots, N$$

Then the inner product is defined as

$$\langle A,B \rangle = \int_{t_a}^{t_b} \tilde{A}\, B \, dt \tag{7-25}$$

where A and B are matrices and $\tilde{A}$ is the transpose of A. Now (7-22) is seen to be equivalent to (7-1) at the N points along the body perimeter. The current approximating functions in (7-21) can be chosen to be

$$T_k(t') = \Phi_k(t') = \begin{cases} 1 & \text{on } \Delta t_k \\ 0 & \text{elsewhere} \end{cases}$$

This amounts to assuming that the currents are of constant amplitude on each interval. Other current expansions, such as a piecewise linear approximation, are possible but the computations are more cumbersome.

Substitution of the approximate currents into (7-18) and (7-19) give approximations to the scattered fields at the sampling points. These are substituted in (7-22) and the approximate current amplitudes are computed from (7-24).

The accuracy of the results improves as the number of sampling points is increased. The scattered field at each field point on the body is calculated as the sum of the contributions to the currents on each of the source intervals. For each contribution, (7-18) and (7-19) must be evaluated. These calculations will usually be approximate. When the field point (t_i) is several segments away from the source sampling point (t'_j), sufficient accuracy may be obtained if the integrands of (7-18) and (7-19) are assumed constant over the sampled source intervals and equal to their respective values at the center of each source interval. In general form, the integrals are then approximated as

$$\int_{\Delta t'_k} I(t')\, dt' \approx I(t_k)\, \Delta t'_k$$

When the field and source segments are within a few segments of each other, it may be necessary to subdivide each source interval into subintervals. The contributions to the scattered field from the currents on each subinterval can be determined by evaluating the integrands at the center of each subinterval and evaluating the respective integrals as

$$\int_{\Delta t'_k} I(t')\, dt' \approx \sum_{j=1}^{M} I(t'_{kj})\, \Delta t'_{kj}$$

where M is the number of subintervals of the k-th source interval. When $t_i = t'_j$ the G_k terms (7-20) have singularities since $R_o = 0$. The contribution from the currents outside of the small region

$$\phi' - \epsilon \leq \phi' \leq \phi' + \epsilon$$

can be determined as in the above case. The contributions from the currents within the remaining small surface around the field point can be determined by an approximate analytic procedure. This is discussed in section VII-G.

The current derivative terms in (7-18) and (7-19) can be approximated by finite difference approximations. For example, the derivative of the t-component of current at the k-th sampling point is

$$\frac{\partial J_t(t')}{\partial t'} \approx \frac{\alpha_{k+1} - \alpha_{k-1}}{\frac{\Delta t'_{k+1}}{2} + \Delta t'_k + \frac{\Delta t'_{k-1}}{2}}$$

Higher order approximations could be used if necessary.

The evaluations of the G_k terms defined in (7-20) and $\partial G_k/\partial t$ are discussed in Section VII-G.

F. Solution by Galerkin's Method. The proposed solution by Galerkin's method is equivalent to requiring

$$\langle L\, \underset{\sim}{J}_s(\underset{\sim}{r}'),\ \underset{\sim}{J}_s^k(\underset{\sim}{r})\rangle = -\langle \underset{\sim}{E}^i(\underset{\sim}{r}),\ \underset{\sim}{J}_s^k(\underset{\sim}{r})\rangle \qquad (7\text{-}26)$$

where $\underset{\sim}{J}_s^k(\underset{\sim}{r})$ is the approximation to the current on the k-th body segment. To obtain this formulation from the method of moments, the weighting functions ω_k are equal to the current approximating functions. For example let

$$\omega_k = T_k(t') = \Phi_k(t') = \begin{cases} 1 & \text{on } \Delta t'_k \\ 0 & \text{elsewhere.} \end{cases}$$

The inner product is defined by (7-25). Equations (7-22) are now the equations of Galerkin's method.

The constraint imposed on the solution by (7-26) is more stringent than that of the field matching solution. Therefore it may be possible to obtain a higher degree of accuracy in the variational solution with the same number of body intervals as in the field matching method. However, the calculations of the matrix terms in (7-23) are more cumbersome. Many numerical techniques which are analogous to those described in the previous section can be used to simplify the computations.

G. Expansion of e^{-jkR}/R^n. In either the field matching or variational solution, one of the major computations is the evaluation of G_m and $\partial G_m/\partial t$ in (7-18) and (7-19). G_m is defined in (7-20) as

$$G_m = \int_0^\pi \frac{e^{-jkR_o}}{kR_o} \cos m\phi' \, d\phi' \tag{7-20}$$

where

$$R_o = \sqrt{r^2 + r'^2 - 2rr' (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \phi')} \tag{7-27}$$

Using the derivative chain rule

$$\frac{\partial G_m}{\partial t} = \frac{\partial G_m}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial G_m}{\partial \theta} \frac{\partial \theta}{\partial t} \tag{7-28}$$

Applying (7-20), (7-28) is expressed as

$$\frac{\partial G_m}{\partial t} = \int_0^\pi \frac{e^{-jkR_o}}{R_o^2} (f_1 + f_2 \cos \phi') \cos m\phi' \, d\phi'$$

$$+ \int_0^\pi \frac{e^{-jkR_o}}{R_o^3} (f_3 + f_4 \cos \phi') \cos m\phi' \, d\phi' \tag{7-29}$$

where the f_i's are the appropriate functions of (r, r', θ, θ') which result from (7-28).

For the integration techniques discussed in Section VII-E, (7-20) and (7-29) are integrals over the source loop at the middle of the source interval (see Fig. 7-1).

The integrals can now be expressed in a general form as

$$I_n = \int_0^\pi \frac{e^{-jkR_o}}{R_o^n} \cos p\phi' \cos m\phi' \, d\phi' \qquad \begin{array}{l} m = 0, 1, 2, \ldots \\ p = 0, 1 \\ n = 1, 2, 3 \end{array} \tag{7-30}$$

when the integrands of (7-30) have singularities at $R_o = 0$, approximate analytic integration over the singularities is necessary. The details of this method are given in the references.[2]

For the integrals which do not include singularities, the complexity of the integrand prohibits exact analytic evaluation. Numerical integration is possible but the computation is cumbersome. Another approach is to expand e^{-jkR}/R^n in functions of the integration variable ϕ'. Substitution of these expansions then allows approximate evaluation of (7-30) by simple analytic integrations. This is the method which is proposed for the body of revolution problem.

To illustrate the expansion, consider

$$\frac{e^{-jkR}}{R^n} = \frac{\cos kR}{R^n} - j\,\frac{\sin kR}{R^n} \tag{7-31}$$

Denoting either cos kR or sin kR as T(kR), each term in (7-31) can be expanded as

$$\frac{T(kR)}{R^n} \approx \left[A + Be^{-\alpha\phi'} + Ce^{-\beta\phi'}\right] \sum_{i=0}^{N} a_i \psi_i(\phi') \qquad 0 \le \phi' \le \pi \tag{7-32}$$

for n = 1, 2, 3 and $R > \epsilon$ which is a small positive constant to be determined below. The bracketed terms of (7-32) are chosen so that

$$\left[A + Be^{-\alpha\phi'} + Ce^{-\beta\phi'}\right] \approx \frac{1}{R^n} \qquad 0 \le \phi' \le \pi \tag{7-33}$$

Figure 7-3 is a typical curve of $1/R^n$ versus ϕ' for closely spaced loops. A, B and α are chosen so that

$$\begin{aligned} A + Be^{-\alpha\phi'} &= \frac{1}{R^n} \qquad \text{at } \phi' = \phi_3', \phi_4', \phi_5' \\ &> \frac{1}{R^n} \qquad \text{at } \phi' = \phi_1', \phi_2' \end{aligned} \tag{7-34}$$

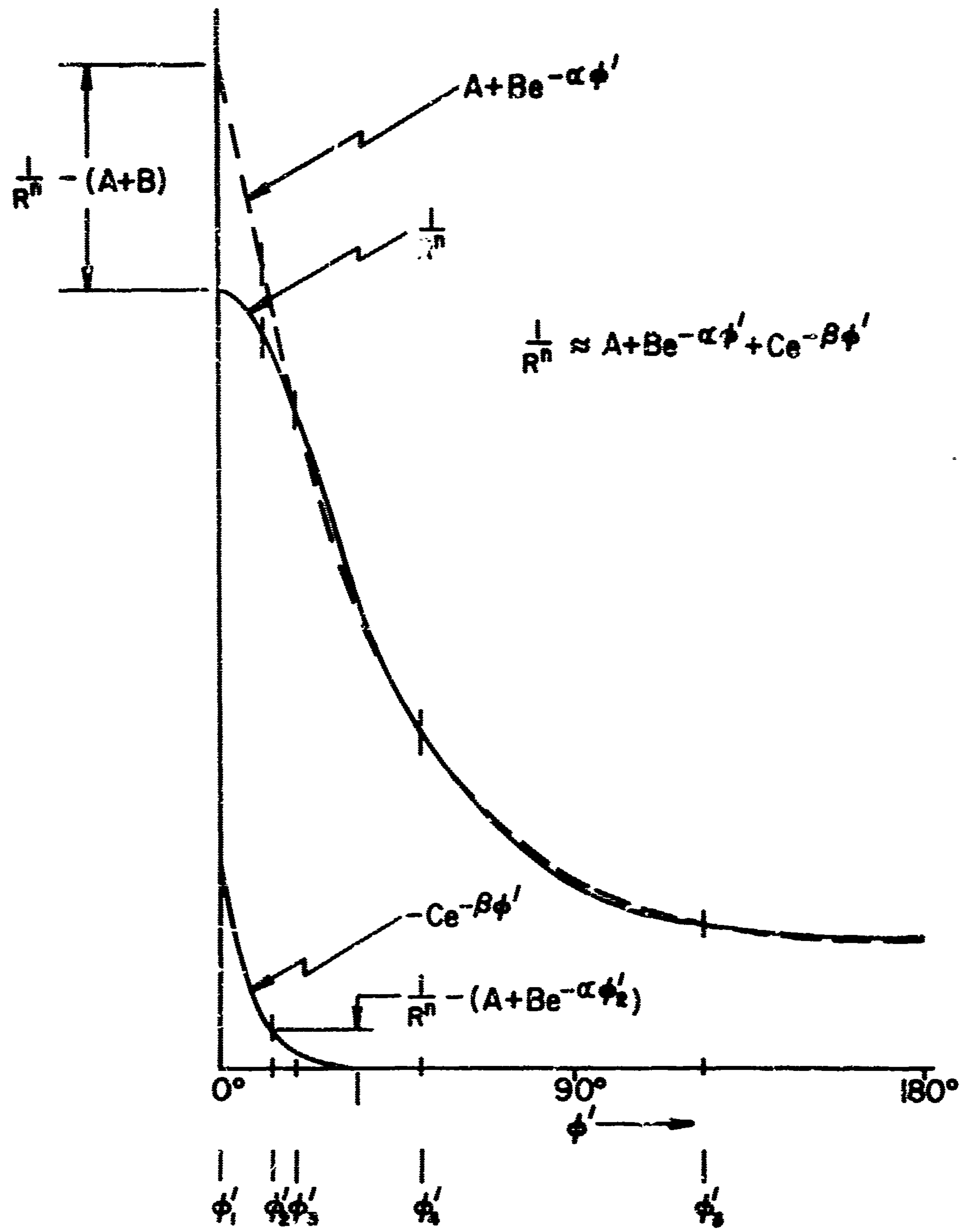


Fig. 7–3 Graphical Illustration of the Approximation of $\frac{1}{R^n}$

If $$2\phi_4' - \phi_5' = \phi_3'$$

then

$$\alpha = \frac{2}{(\phi_5' - \phi_3')} \ln \left[\frac{\frac{1}{R(\phi_4')^n} - \frac{1}{R(\phi_3')^n}}{\frac{1}{R(\phi_5')^n} - \frac{1}{R(\phi_4')^n}} \right]$$

$$B = \frac{\frac{1}{R(\phi_4')^n} - \frac{1}{R(\phi_3')^n}}{e^{-\alpha\phi_4} - e^{-\alpha\phi_3'}}$$

$$A = \frac{1}{R(\phi_3')^n} - Be^{-\alpha\phi_3'} \qquad (7\text{-}35)$$

The constants C and β are chosen so that

$$\frac{1}{R^n} - (A + Be^{-\alpha\phi'}) = Ce^{-\beta\phi'} \qquad \text{at } \phi' = \phi_1', \phi_2'$$

and

$$Ce^{-\beta\phi'} \longrightarrow 0 \qquad \text{for } \phi' > \phi_2'$$

If ϕ_2' is properly chosen, (7-33) is satisfied when

$$C = \frac{1}{R(\phi_1')^n} - (A + Be^{-\alpha\phi_1'})$$

$$\beta = \frac{1}{\phi_2'} \ln \left[\frac{C}{\frac{1}{R(\phi_2')^n} - (A + Be^{-\alpha\phi_2'})} \right]$$

To complete the expansion, (7-32) is rearranged as

$$\sum_{i=0}^{N} a_i \psi_i(\phi') = \frac{T(kR)}{R^n} \left[\frac{1}{A + Be^{-\alpha\phi'} + Ce^{-\beta\phi'}} \right] \tag{7-36}$$

$$\approx T(kR)$$

Since the T(kR) are sinusoidal functions which are even about $\phi' = \pi$, it is convenient to choose the $\psi_i(\phi')$ as trigonometric functions with the same symmetry. For example

$$\text{constant},\ \cos i\,\phi',\ \sin\left(i - \tfrac{1}{2}\right)\phi'$$

Equation (7-36) is a minimum square error approximation and the coefficients a_i are determined by satisfying (7-36) for N values of ϕ'.

Figure 7-4 gives the results of representative expansions of cos kR/R and cos kR/R^3 as

$$f_A(\phi') = \left[A + Be^{-\alpha\phi'} + Ce^{-\beta\phi'}\right] \left[a_o + a_1 \cos\phi' + a_2 \cos 2\phi' + a_3 \sin\frac{\phi'}{2} + a_4 \sin^3\frac{\phi'}{2}\right] \tag{7-37}$$

Denoting the expanded function as $f_T(\phi')$, the error at any ϕ_i' is defined as

$$E(\phi_i') = \frac{f_A(\phi_i') - f_T(\phi_i')}{f_T(\phi')_{max}}$$

From the error curves in Fig. 7-4, the average magnitude of error for either expansion is less than 0.3 percent. For these cases, R_{min} = 0.067λ. It was found that for $R_{min} > 0.067\lambda$, the average error decreased, so the cases presented give an estimate of the maximum errors that can be expected for expansions of bodies of revolution with diameters less

DATA

$R_{min} = 0.067\lambda$

$r = 0.205\lambda \quad r' = 0.206\lambda$

$\theta = 94.77^{\circ} \quad \theta' = 75.96^{\circ}$

Eq. (7-32) satisfied at

$\phi' = 0^{\circ}, 15^{\circ}, 45^{\circ}, 90^{\circ}, 150^{\circ}$

	$\frac{\cos kR}{R}$	$\frac{\cos kR}{R^3}$
$\phi_1' =$	0°	0°
$\phi_2' =$	9°	3°
$\phi_3' =$	20°	20°
$\phi_4' =$	60°	60°
$\phi_5' =$	100°	100°

EXPANSION COEFFICIENTS

	$\frac{\cos kR}{R}$	$\frac{\cos kR}{R^3}$
A	2.57	25.78
B	14.72	3941.21
C	-2.41	-667.24
α	1.85	3.75
β	18.98	39.58
a_o	-0.96	-4.86
a_1	1.82	6.11
a_2	0.05	-4.34
a_3	1.49	8.39
a_4	-0.45	-2.42

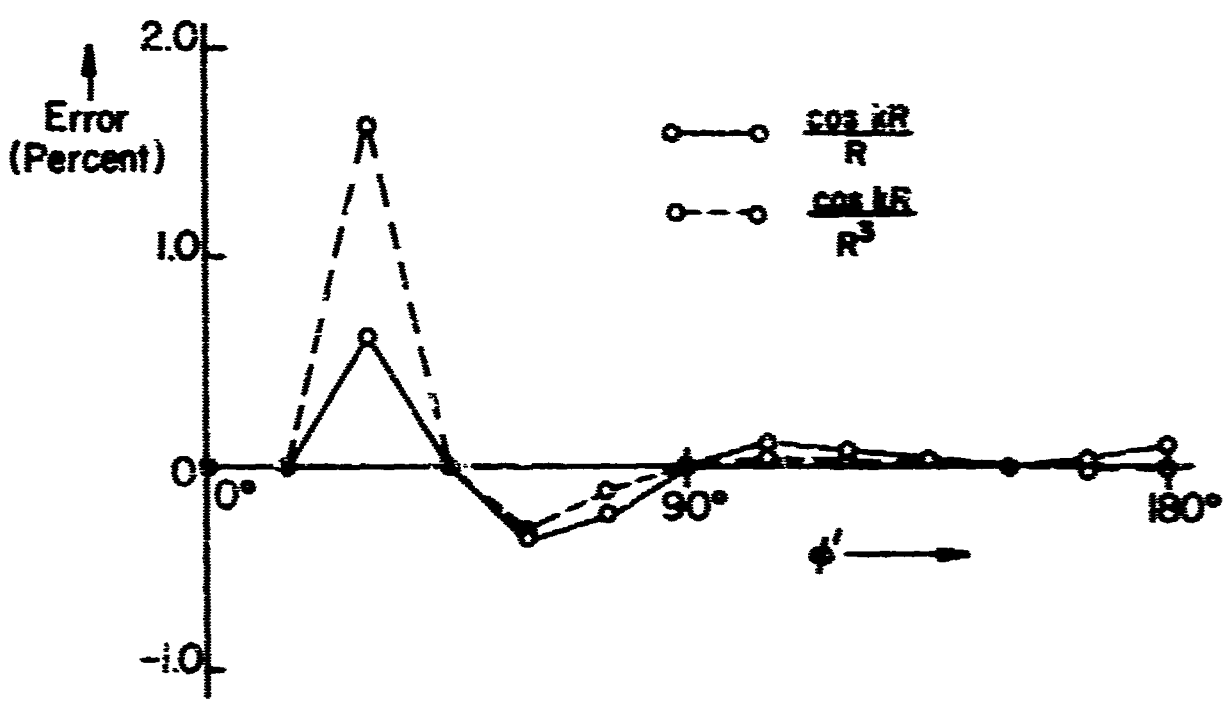


Fig. 7-4 Results of Computed Expansions

than 0.5λ and for $R_{min} \geq \epsilon = 0.067\lambda$. For larger diameters, it may be necessary to increase the number of terms in (7-37).

H. References.

1. M. G. Andreasen, "Scattering from Bodies of Revolution," IEEE Trans. on Antennas and Propagation, vol. AP-13, no. 2, March, 1965, pp. 303-310.

2. M. G. Andreasen, "Scattering from Rotationally Symmetric Metallic Bodies," Final Report, Subcontract 294, AF 19(628)-500, TRG West, Menlo Park, California, April, 1964.

VIII. INVERSION OF MATRICES

A. Introduction. The inversion of a matrix is closely related to the solution of a system of linear algebraic equations. The n linear equations with n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\
\vdots \qquad \vdots \qquad\qquad & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n
\end{aligned}
\tag{8-1}
$$

may be written as a matrix equation

$$Ax = y \tag{8-2}$$

where A is an n x n matrix, $x = (x_1, x_2, \ldots, x_n)$ an unknown vector and $y = (y_1, y_2, \ldots, y_n)$ a known vector. This matrix equation can be conveniently written as a partitioned matrix $(A|I)$ with a column vector $\left[\frac{x}{-y}\right]$

$$(A|I)\left[\frac{x}{-y}\right] = 0 \tag{8-3}$$

where I denotes an identity matrix.

The elementary row operations are defined as

a) multiply any row by a constant;

b) interchange any two rows;

c) add one row multiplied by a constant c to any other row,

or their combination.

When the elementary row operations are performed on (8-1), or equivalently, on the partitioned matrix $(A|I)$ of (8-3) they do not change the solution of the system of Eqs. (8-1) or the matrix equation (8-3).

Let L be a matrix representing some combination or product of the elementary operations described above. After these operations on (8-3), we have

$$(LA|L)\ [\frac{x}{-y}] = 0 \qquad (8\text{-}4)$$

If we carry out these elementary rowoperations so that LA becomes an identity matrix, i.e., $LA = I$, then the inverted matrix A^{-1} is found to be

$$A^{-1} = L \qquad (8\text{-}5)$$

B. Method of Gauss-Jordan Reduction. The method of Gauss-Jordan reduction is a systematic procedure designed to execute the method outlined in Section A. Let A be an n x n matrix to be inverted. The partitioned matrix of (8-3) can be written as an n x 2n matrix

$$\left[\begin{array}{ccccc|ccccc} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} & 1 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} & 0 & 0 & \cdots & 1 & 0 \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{n,n} & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \qquad (8\text{-}6)$$

Search the first column a_{i1} $(i = 1, 2, \ldots, n)$ for the largest element in magnitude, say a_{k1}

$$|a_{k1}| \geq |a_{i1}| \qquad (i = 1, 2, \ldots, n) \qquad (8\text{-}7)$$

If $a_{k1} = 0$, then the first column of A is zero, A is a singular matrix and hence A has no inverse in the usual sense. Therefore we may assume $a_{k1} \neq 0$. Interchange the first and k-th rows, and then divide the first row through by a_{k1}. If the interchange of the first and k-th rows is followed by an interchange of the first and k-th columns of the second submatrix, then the resulting matrix becomes more orderly. To make sure that the interchange of columns of the second submatrix does not disturb the matrix equation (8-3) or (8-4), it is required that y_1 and y_k be interchanged also. Finally we use the first row to eliminate all but the first term of the first column by the elementary row operations. Thus

$$(L^{(1)}A|\hat{L}^{(1)}) \left[\frac{x}{-\hat{y}^{(1)}}\right] = 0 \tag{8-8}$$

where $\hat{L}^{(1)} = L^{(1)}M^{(1)}$, $\hat{y}^{(1)} = M^{(1)}y$ [$M^{(1)}$ is a permittation matrix which transforms $y = (y_1, y_2, \ldots, y_n)$ into $\hat{y}^{(1)} = (y_k, y_2, y_3, \ldots, y_{k-1}, y_1, y_{k+1}, \ldots, y_n)$]

$$(L^{(1)}A|\hat{L}^{(1)}) = \left[\begin{array}{ccccc|ccccc} 1 & b_{12} & \cdots & b_{1,n-1} & b_{1n} & b_{11} & 0 & \cdots & 0 & 0 \\ 0 & b_{22} & \cdots & b_{2,n-1} & b_{2n} & b_{21} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & b_{n-1,2} & \cdots & b_{n-1,n-1} & b_{n-1,n} & b_{n-1,1} & 0 & \cdots & 1 & 0 \\ 0 & b_{n2} & \cdots & b_{n,n-1} & b_{n,n} & b_{n,1} & 0 & \cdots & 0 & 1 \end{array}\right] \tag{8-9}$$

The elements of the matrix (8-9) can be written in terms of the elements of the matrix (8-6) by inspection.

We proceed to obtain a new matrix equation

$$(L^{(2)}A|\hat{L}^{(2)}) \left[\frac{x}{-\hat{y}^{(2)}}\right] = 0 \tag{8-10}$$

from (8-8) by the elementary row operations such that the submatrix $L^{(2)}A$ has zero elements for the first and second column except the first and second diagonal terms which are normalized to unity. This can be achieved by the following steps:

1) Find an integer k such that

$$|b_{k,2}| \geq |b_{i2}| \qquad (i = 2, 3, \ldots, n-1, n)$$

The $|b_{k,2}|$ may be assumed to be non-zero otherwise the original matrix is singular.

2) interchange the second and k-th rows and also interchange the second and k-th columns of the second submatrix,

3) divide the (new) second row by b_{k2},

4) use the second row to eliminate the second column by the elementary operations except, of course, the second element of the column,

5) interchange y_2 and y_k.

It is readily seen that these steps lead to the matrix equation (8-10).

Assuming that the matrix A is nonsingular, then after repeating the process n times, we obtain

$$(L^{(n)}A|\hat{L}^{(n)}) \left[\frac{x}{-\hat{y}^{(n)}}\right] = 0 \qquad (8\text{-}11)$$

where $L^{(n)}A$ is an identity matrix, and $\hat{y}^{(n)}$ is a vector resulting from the rearrangement of the original y. Let M be a permutation matrix transforming y into $\hat{y}^{(n)}$:

$$\hat{y}^{(n)} = My \qquad (8\text{-}12)$$

Since M is a permutation matrix so that MM = I, $L^{(n)}$ and $\hat{L}^{(n)}$ are related by

$$L^{(n)} = \hat{L}^{(n)}M \qquad (8\text{-}13)$$

Equations (8-12) and (8-13) imply that if y_i and y_j are interchanged in $\hat{y}^{(n)}$, then b_{ki} and b_{kj} of $\hat{L}^{(n)}$ $(k = 1, 2, \ldots, n)$ must be interchanged.

C. Commutative Matrices. When a boundary value problem has some degree of symmetry such as rotational symmetry or reflectional symmetry, the matrix obtained by the method of approximation described in previous chapters often exhibits a special structure. This special structure of the matrix can usually be characterized by the fact that this matrix commutes with another matrix which represents the nature of the symmetry.

The product of two matrices A and B depends, in general, on the order of the factors, i.e., $AB \neq BA$. In the very special situation when the product is independent of the order of the factor i.e., $AB = BA$, then the matrices A and B are said to commute. A case of special interest to us is when a given matrix commutes with a diagonal matrix.

Any matrix that commutes with a diagonal matrix having distinct diagonal terms is necessarily a diagonal matrix.

To prove this statement let $A = [\alpha_{ij}]$ be an arbitrary n x n matrix and D be an n x n diagonal matrix with distinct diagonal terms:

$$D = [d_i \; \delta_{ij}], \; d_i \neq d_j \qquad \text{if } i \neq j$$

Since $AD = DA$, we have

$$[\alpha_{ij}][d_i \delta_{ij}] = [d_i \delta_{ij}][\alpha_{ij}]$$

which is equivalent to

$$[\sum_{k=1}^{n} \alpha_{ik} \; \delta_{kj} \; d_k] = [\sum_{k=1}^{n} d_i \; \delta_{ik} \; \alpha_{kj}]$$

This equation leads to

$$[\alpha_{ij}(d_j - d_i)] = [0]$$

where [0] denotes a zero matrix. The above equality holds only if

$$\alpha_{ij} = 0 \qquad \text{if } d_i \neq d_j \qquad \text{for } i \neq j$$

which shows that the matrix $A = [\alpha_{ij}]$ must be diagonal.

It follows from the proof given above that if a matrix A commutes, with a diagonal matrix D whose diagonal terms of equal value appear consecutively, then the matrix A must be a supermatrix with zero off-diagonal submatrices. For example, if A is a 6 x 6 matrix $[\alpha_{ij}]$ and D is a diagonal matrix:

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_3 \end{bmatrix} \qquad (8\text{-}13)$$

then AD = DA implies that

$$A = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ 0 & 0 & 0 & \alpha_{54} & \alpha_{55} & \alpha_{56} \\ 0 & 0 & 0 & \alpha_{64} & \alpha_{65} & \alpha_{66} \end{bmatrix} \qquad (8\text{-}14)$$

It is well known that a matrix equation remains invariant under a similarity transformation which includes unitary and orthogonal transformations as its special cases. For example the matrix equation

$$AC = CA \qquad (8\text{-}15)$$

which denotes the commutability of the matrices A and C, after a similarity transformation (by a non-singular matrix P) takes the same form

$$BD = DB \qquad (8\text{-}16)$$

where

$$B = P^{-1}AP \qquad \text{and} \qquad D = P^{-1}CP \tag{8-17}$$

Now suppose that C of (8-15) can be diagonalized by a similarity transformation and that the diagonalized matrix D has distinct diagonal terms. Then the matrix B of (8-16) obtained by the same similarity transformation from matrix A must be diagonal also. On the other hand, if D is a diagonal matrix with some repeated diagonal terms as shown in (8-13), then B must be a supermatrix with zero off-diagonal submatrices as shown in (8-14).

The invariant property of a matrix equation under the similarity transformation sometimes can be used to simplify the inversion of a matrix. Let A be a matrix to be inverted. Suppose we can find a matrix C which commutes with A and whose eigenvalues and the corresponding eigenelements can be easily found so that a nonsingular matrix P, which will diagonalize C by a similarity transformation, can be obtained. In addition, if the diagonalized matrix has distinct diagonal terms, then the matrix A is reduced by the same similarity transformation to a diagonal matrix which can be inverted easily. The desired inverted matrix A^{-1} can be obtained by reversing the original similarity transformation. In case the diagonal terms of the diagonalized matrix obtained by the similarity transformation are not distinct, the matrix A is reduced by the same similarity transformation to a supermatrix with zero off-diagonal submatrices. The inverted matrix A^{-1} is then obtained by first inverting these diagonal submatrices and then reversing the original similarity transformation.

The practicability of the method described above depends on whether we can find a second matrix which commutes with the matrix to be inverted and whose eigenvalues and eigenelements are known or easily obtainable so that a nonsingular matrix can be constructed to diagonalize the matrix. To find such matrix, we consider the specific configuration of a four-port network shown in Fig. 8-1. It is easily seen that when the network undergoes a rotation of $\pi/2$, π, or $3\pi/2$, the electrical structure remains unchanged. Let $e = (e_1, e_2, e_3, e_4)$ denote the set of excitations, r the set of corresponding responses in the network due to the excitation and S the matrix representation of the network parameters determined by the electrical structure of the network. Then the equation relating the excitations and responses of the network is given by

$$r = Se \tag{8-18}$$

Let R be a rotating operator signifying a rotation of $\pi/2$ in the counterclockwise direction. A rotation of the set of excitations in the counterclockwise direction by an angle of $\pi/2$, keeping the network stationary, gives a set of new responses

$$r' = SRe \tag{8-19}$$

On the other hand, the set of new responses r' is equal to the set of responses obtained by rotating the whole system, including both the excitation and network, counterclockwise by an angle of $\pi/2$:

$$r' = Rr = RSe \tag{8-20}$$

Thus by (8-19) and (8-20), we find, for every excitation e

$$SRe = RSe \tag{8-21}$$

which implies that

$$SR = RS \tag{8-22}$$

Usually, the S matrix which represents the electrical structure of the network or the dynamic properties of a boundary value problem is

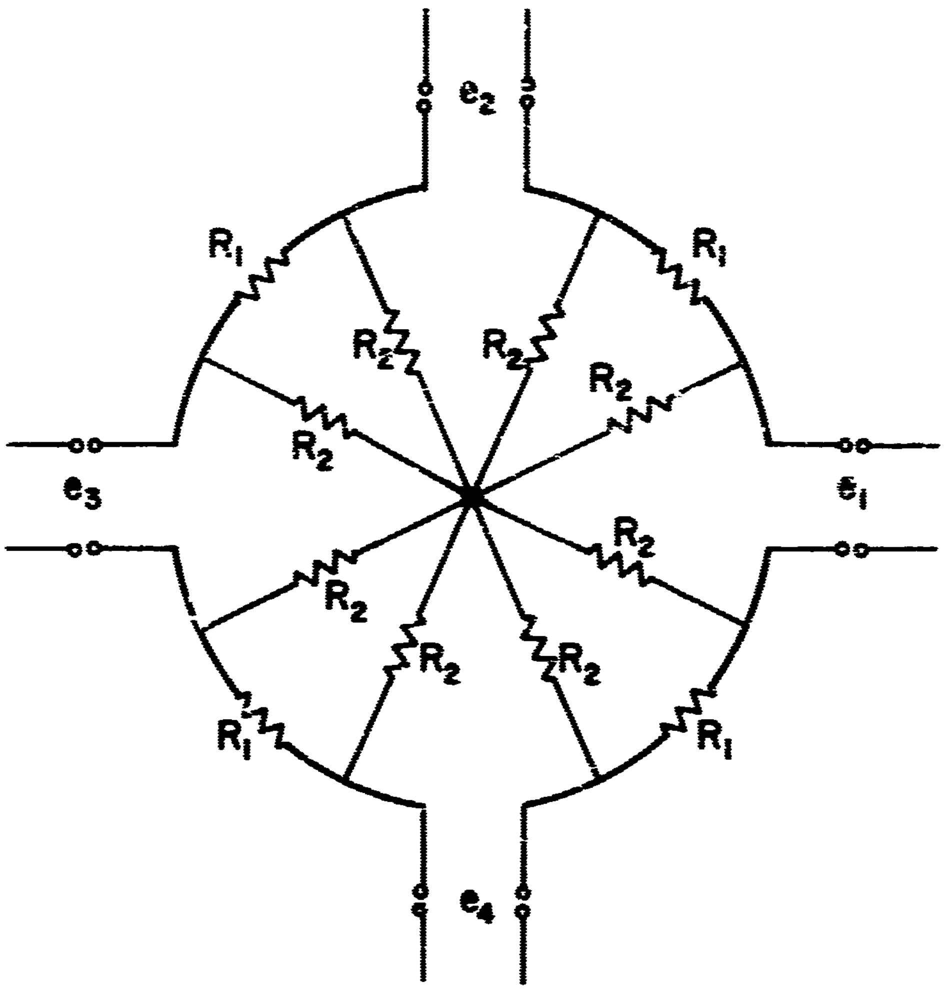


Fig. 8-1 A four port network with rotational symmetry.

rather complicated while the R matrix which characterizes the symmetry of the network or the problem is relatively simple. This feature is especially evident when the problem has a high degree of symmetry.

D. Example. Circular Loop Antenna. The boundary value problem of a circular loop antenna possesses a high degree of symmetry, namely the rotational symmetry about the center of the loop. One of the most effective approaches for obtaining an approximate solution is to divide the circular loop in N equal sections and regard the antenna as an n-port network. The matrix equation obtained by approximating the integro-differential equation describing the boundary value problem is a n x n matrix (impedance matrix)

$$A = [\alpha_{ij}] \tag{8-23}$$

The problem is to invert the matrix A to obtain an admittance matrix.

It is seen from Fig. 8-2 that the boundary value problem is invariant with respect to a rotation of the system by an angle $2\pi k/N$ ($k = 1, 2, \ldots, N$) in counterclockwise sense. Since this rotation can be represented by the C matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{8-24}$$

the matrices A and C commute with each other;

$$AC = CA \tag{8-25}$$

Next we consider the diagonalization of the C matrix. For this purpose it is necessary to obtain the eigenvalues and eigenelements of

Fig

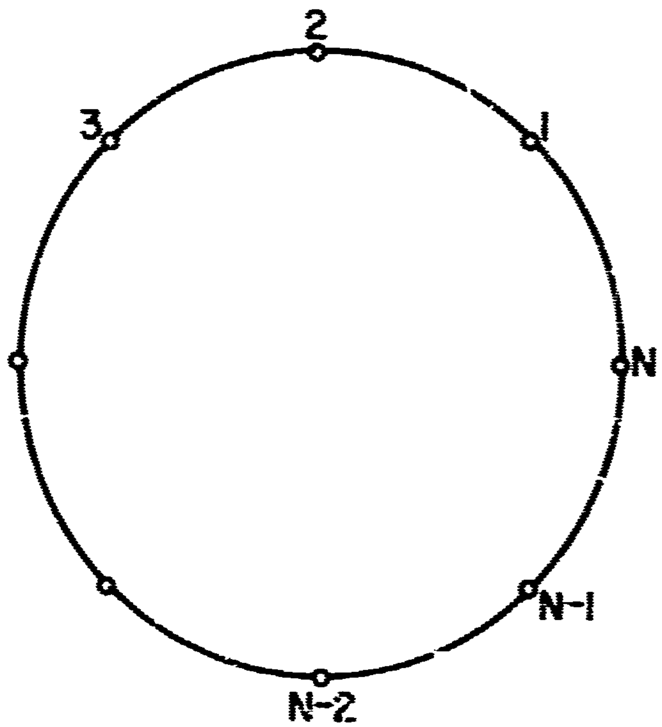


Fig. 8-2 Circular loop antenna regarded as an N port network.

C matrix. The characteristic equation

$$(C - \lambda I)x = 0 \qquad (8\text{-}26)$$

gives the N eigenvalues and the corresponding eigenelements

$$\left\{ e^{j\frac{2\pi k}{N}}, \quad k = 0, 1, 2, \ldots, N-1 \right\} \qquad (8\text{-}27)$$

$$\left\{ u_k = \sum_{\ell=0}^{N-1} \frac{1}{\sqrt{N}} e^{j\frac{2\pi k\ell}{N}} e_\ell, \quad k = 0, 1, 2, \ldots, N-1 \right\} \qquad (8\text{-}28)$$

where e_ℓ denotes a unit element in the N-dimensional Euclidean space.

The diagonalizing matrix P can be obtained from the set of normalized eigenelements by regarding them as the column elements of P:

$$P = [\gamma_{k\ell}] = \frac{1}{\sqrt{N}} [e^{j\frac{2\pi k\ell}{N}}] \quad (k,\ell = 0, 1, 2, \ldots, N-1) \qquad (8\text{-}29)$$

The inverse of P is seen to be

$$P^{-1} = P^* = \frac{1}{\sqrt{N}} [e^{-j\frac{2\pi k\ell}{N}}] \quad (k,\ell = 0, 1, 2, \ldots, N-1) \qquad (8\text{-}30)$$

That P is a unitary matrix can be seen from the following relation:

$$(P^*P)_{km} = [\frac{1}{N} \sum_{\ell=0}^{N-1} e^{-j\frac{2\pi k\ell}{N}} e^{j\frac{2\pi\ell m}{N}}]$$

$$= [\frac{1}{N} \sum_{\ell=0}^{N-1} e^{j\frac{2\pi\ell(m-k)}{N}}] = [\delta_{km}]$$

The matrix C can be diagonalized by the following similarity transformation:

$$P^{-1}CP = [e^{j\frac{2\pi k}{N}} \delta_{ki}]_{N\times N} \quad (k,i = 0, 1, 2, \ldots, N-1) \qquad (8\text{-}31)$$

which has N distinct diagonal terms equal to the eigenvalues of C. It follows that $B = P^{-1}AP$ must be a diagonal matrix also.

We now evaluate the diagonal terms of B. The elements of B, in general, can be calculated by the formula

$$\beta_{\ell m} = [\gamma_{\ell i}]^* \; [\alpha_{ik}] \; [\gamma_{km}] = \frac{1}{N} \sum_{i,k=0}^{N-1} \alpha_{ik} \; e^{j \frac{2\pi(km - \ell i)}{N}} \qquad (8\text{-}32)$$

According to the results in Section C, we see that the off diagonal terms vanish, i.e.,

$$\beta_{\ell m} = 0 \qquad \text{if } \ell \neq m$$

For the diagonal terms, we have

$$\beta_{\ell\ell} = \frac{1}{N} \sum_{i,k=0}^{N-1} \alpha_{ik} \; e^{j \frac{2\pi(k-i)\ell}{N}} \qquad (8\text{-}33)$$

This expression can be simplified by making use of the commutability of matrices A and C.

It follows from the relation AC = CA that the elements α_{ik} of A depend only on the relative positions of i and k. Let p denote the relative position of i and k and write

$$\alpha_p = \alpha_{ik} \qquad (8\text{-}34)$$

A moment of careful examination of (8-33) and Fig. 8-2 reveals that (8-33) may be regrouped as follows:

Integral number assumed by p	The partial sum of (8-33) corresponding to p
$-[\frac{N}{2}]$	$\frac{N}{2} \alpha_p e^{j \frac{2\pi \ell p}{N}}$ if N is even
	$N \alpha_p e^{j \frac{2\pi \ell p}{N}}$ if N is odd
$-[\frac{N}{2}] + 1$	$N \alpha_p e^{j \frac{2\pi \ell p}{N}}$
$\vdots$	$\vdots$
0	$N \alpha_p e^{j \frac{2\pi \ell p}{N}}$
$\vdots$	$\vdots$
$[\frac{N}{2}] - 1$	$N \alpha_p e^{j \frac{2\pi \ell p}{N}}$
$[\frac{N}{2}]$	$\frac{N}{2} \alpha_p e^{j \frac{2\pi \ell p}{N}}$ if N is even
	$N \alpha_p e^{j \frac{2\pi \ell p}{N}}$ if N is odd

Where the square bracket [Q] denotes the maximum integer of Q, e.g.,
$[2.134] = 2$.

Summation of these terms over p yields

$$\beta_{\ell\ell} = \sum_{p=-[N/2]}^{[N/2]} \left\{ 1 - \frac{1}{2} \delta(p + \frac{N}{2}) - \frac{1}{2} \delta(p - \frac{N}{2}) \right\} \alpha_p e^{j \frac{2\pi \ell p}{N}} \tag{8-35}$$

If, in addition, the reciprocity property holds as in the problem of circular loop antenna, then

$$\alpha_{ik} = \alpha_{ki} \tag{8-36}$$

or

$$\alpha_{-p} = \alpha_p \tag{8-37}$$

Consequently, (8-35) can be written as

$$\beta_{\ell\ell} = \sum_{p=0}^{[N/2]} \left\{ 2 - \delta(p) - \delta(p - \frac{N}{2}) \right\} \alpha_p \cos \frac{2\pi \ell p}{N} \tag{8-38}$$

which represent the diagonal terms of matrix B. Thus after the similarity transformation of matrix A, we obtain a diagonal matrix

$$B = P^{-1}AP = [\beta_{\ell\ell}\ \delta_{\ell m}] \tag{8-39}$$

The inverse of this matrix is easily seen to be

$$B^{-1} = P^{-1}\ A^{-1}\ P = [\frac{\delta_{\ell m}}{\beta_{\ell\ell}}] \tag{8-40}$$

Therefore, A^{-1} is given by

$$\begin{aligned} A^{-1} &= PB^{-1}P^{-1} \\ &= \frac{1}{N}\ [e^{j\frac{2\pi \ell k}{N}}]\ [\frac{\delta_{ki}}{\beta_{kk}}]\ [e^{-j\frac{2\pi im}{N}}] \\ &= \frac{1}{N}\ [\sum_{k=0}^{N-1} \frac{1}{\beta_{kk}}\ e^{j\frac{2\pi k(\ell - m)}{N}}] = [a_{\ell m}] \end{aligned} \tag{8-41}$$

Thus the elements $a_{\ell m}$ of A^{-1} can be identified as

$$a_{\ell m} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{\beta_{kk}}\ e^{j\frac{2\pi k(\ell - m)}{N}} \tag{8-42}$$

It suffices to evaluate the elements of the first row of A^{-1}, since the elements of the other rows can be constructed from these elements. For the first row, we have

$$a_{om} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{\beta_{kk}}\ e^{-j\frac{2\pi km}{N}} \tag{8-43}$$

Again for those problems which satisfy the reciprocity property, we have

$$a_{om} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{\beta_{kk}} \cos \frac{2\pi km}{N} \tag{8-44}$$

where β_{kk} are given by (8-38).

As an example, we consider the inversion of a 4 x 4 symmetric matrix

$$A = \begin{bmatrix} \alpha_o & \alpha_1 & \alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_o & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_o & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_1 & \alpha_o \end{bmatrix} \tag{8-45}$$

which commutes with a rotation matrix operator

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{8-46}$$

By (8-38), matrix A can be diagonalized by a similarity transformation to give a diagonal matrix B with the following diagonal terms:

$$\begin{aligned} \beta_{oo} &= \alpha_o + 2\alpha_1 + \alpha_2 \\ \beta_{11} &= \beta_{33} = \alpha_o - \alpha_2 \\ \beta_{22} &= \alpha_c - 2\alpha_1 + \alpha_2 \end{aligned} \tag{8-47}$$

By (8-44) the elements of the first row are found to be

$$\begin{aligned} a_{oo} &= \frac{1}{4}\left(\frac{1}{\beta_{oo}} + \frac{2}{\beta_{11}} + \frac{1}{\beta_{22}}\right) \\ a_{o1} &= a_{o3} = \frac{1}{4}\left(\frac{1}{\beta_{oo}} - \frac{1}{\beta_{22}}\right) \\ a_{o2} &= \frac{1}{4}\left(\frac{1}{\beta_{oo}} - \frac{2}{\beta_{11}} + \frac{1}{\beta_{22}}\right) \end{aligned} \tag{8-48}$$

The inverted matrix A^{-1} is found according to (8-42) with the elements of the first row given by (8-48):

$$A^{-1} = \begin{bmatrix} a_{oo} & a_{o1} & a_{o2} & a_{o3} \\ a_{o1} & a_{oo} & a_{o1} & a_{o2} \\ a_{o2} & a_{o1} & a_{oo} & a_{o1} \\ a_{o3} & a_{o2} & a_{o1} & a_{oo} \end{bmatrix} \tag{8-49}$$

E. Example. Linear Antenna. As another application, we consider the problem of a linear antenna. The electrical property of the antenna remains unchanged by rotating the antenna with respect to the mid-point by an angle π. The matrix equation intended to approximate the integro-differential equation describing the boundary value problem can be arranged by a proper choice of the approximating functions (elements) to have a matrix $A = [\alpha_{ij}]$ which commutes with the matrix representing the rotation:

$$R = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{8-50}$$

that is

$$AR = RA \tag{8-51}$$

We shall show that the inversion of matrix A can be reduced to the inversion of its diagonal submatrices. To be specific let A_{2n}, R_{2n} and I_{2n} denote matrices of order 2n. The characteristic equation for R_{2n}

$$|R_{2n} - \lambda I_{2n}| = 0 \tag{8-52}$$

gives the eigenvalues +1 and -1, each of which repeats n-times.

For the eigenvalue $\lambda = 1$ and $x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots, \xi_{2n})$, the equation

$$(R_{2n} - \lambda I_{2n})x = 0 \tag{8-53}$$

yields the following relations to be satisfied by the eigenelements:

$$\begin{aligned} &\xi_1 - \xi_{2n} = 0 \\ &\xi_2 - \xi_{2n-1} = 0 \\ &\cdots\cdots \\ &\xi_n - \xi_{n+1} = 0 \end{aligned} \tag{8-54}$$

The corresponding eigenelements, after normalization, can be chosen as

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{2}}\ (1,\ 0,\ 0,\ \ldots,\ 0,\ 0,\ 1) \\ u_2 &= \frac{1}{\sqrt{2}}\ (0,\ 1,\ 0,\ \ldots,\ 0,\ 1,\ 0) \\ &\cdots\cdots\cdots\cdots \\ u_n &= \frac{1}{\sqrt{2}}\ (0,\ \ldots,\ 0,\ 1,\ 1,\ 0,\ \ldots,\ 0) \end{aligned} \tag{8-55}$$

Similarly for the eigenvalue $\lambda = -1$, we find the n normalized eigenelements as follows:

$$\begin{aligned} u_{n+1} &= \frac{1}{\sqrt{2}}\ (0,\ \ldots,\ 0,\ 1,\ -1,\ 0,\ \ldots,\ 0) \\ u_{n+2} &= \frac{1}{\sqrt{2}}\ (0,\ \ldots,\ 1,\ 0,\ 0,\ -1,\ \ldots,\ 0) \\ &\cdots\cdots\cdots\cdots \\ u_{2n} &= \frac{1}{\sqrt{2}}\ (1,\ 0,\ \ldots,\ 0,\ 0,\ \ldots,\ 0,\ -1) \end{aligned} \tag{8-56}$$

The matrix P_{2n} which diagonalizes R_{2n} by a similarity transformation may be constructed from these eigenelements by regarding them as column elements of P_{2n}:

$$P_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \dots & 0 & -1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix} \tag{8-57}$$

which can be put into a more compact form

$$P_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & R_n \\ R_n & -I_n \end{bmatrix} \tag{8-58}$$

The inverse is seen to be

$$P_{2n}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & R_n \\ R_n & -I_n \end{bmatrix} \tag{8-59}$$

Note that $P_{2n}^{-1} = P_{2n}$, $R_{2n}^{-1} = R_{2n}$ and $R_n^{-1} = R_n$.

Putting

$$A_{2n} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{8-60}$$

$$R_{2n} = \begin{bmatrix} 0 & R_n \\ R_n & 0 \end{bmatrix} \tag{8-61}$$

and using the fact that A_{2n} commutes with R_{2n}, i.e.,

$$A_{2n}R_{2n} = R_{2n}A_{2n} \tag{8-62}$$

We find that A_{2n} can be rewritten as

$$A_{2n} = \begin{bmatrix} A_{11} & R_n A_{21} R_n \\ A_{21} & R_n A_{11} R_n \end{bmatrix} \tag{8-63}$$

The similarity transformation of A_{2n} by P_{2n} yields

$$B_{2n} = P_{2n}^{-1} A_{2n} P_{2n} = \begin{bmatrix} A_{11} + R_n A_{21} & 0 \\ 0 & R_n(A_{11} - R_n A_{21}) R_n \end{bmatrix} \tag{8-64}$$

The inverse is

$$B_{2n}^{-1} = P_{2n}^{-1} A_{2n}^{-1} P_{2n} = \begin{bmatrix} (A_{11} + R_n A_{21})^{-1} & 0 \\ 0 & R_n(A_{11} - R_n A_{21})^{-1} R_n \end{bmatrix} \tag{8-65}$$

It follows that the inverse matrix A_{2n}^{-1} is found to be

$$A_{2n}^{-1} = P_{2n} B_{2n}^{-1} P_{2n}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} (A_{11}+R_n A_{21})^{-1} + (A_{11}-R_n A_{21})^{-1} & \left[(A_{11}+R_n A_{21})^{-1} - (A_{11}-R_n A_{21})\right] R_n \\ R_n\left[(A_{11}+R_n A_{21})^{-1} - (A_{11}-R_n A_{21})^{-1}\right] & R_n\left[(A_{11}+R_n A_{21})^{-1} + (A_{11}-R_n A_{21})\right] R_n \end{bmatrix} \tag{8-66}$$

Thus, we have reduced the inversion of A_{2n} given by (8-60) to the inversion of the submatrices $(A_{11} + R_n A_{21})$ and $(A_{11} - R_n A_{21})$.

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13. ABSTRACT

This report presents general procedures for solving field problems of engineering interest using digital computer techniques. The basic concept is to represent a boundary-value problem by a superposition integral, approximate the integral equation by a matrix equation, and invert the matrix for a solution. The theory is described in terms of the method of moments, which is equivalent to the variational method.

For electromagnetic antenna and scattering problems, the method gives a matrix whose elements can be interpreted as generalized impedances. These impedances are closely related to those used in the theory of loaded antennas and scatterers, and hence such loaded structures can also be treated. A solution for wire antennas and scatterers of arbitrary shape is formulated in detail, and calculations for linear wire antennas and scatterers, both loaded and unloaded, have been made.

Additional problems treated by these procedures are two-dimensional scattering by conducting cylinders and by dielectric cylinders, and three-dimensional scattering by bodies of revolution. These problems are used to show the effect of various approximations in the solution, in an attempt to draw some general conclusions as to the best approximations. Special procedures for inverting matrices have also been considered, to take into account any symmetry properties present in the matrices. A considerable saving in computation time can often be made by properly utilizing these symmetries.

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