FTD-TT-65-1698

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

STRUCTURAL MECHANICS OF AIRFRAME STRUCTURES

By

A. F. Feofanov

FOREIGN TECHNOLOGY DIVISION

This translation was made to provide the users with the basic essentials of the original document in the shortest possible time. It has not been edited to refine or improve the grammatical accuracy, syntax or technical terminology.

This volume is devoted to problems of structural mechanics and mathematical design methods for modern statically determinate and statically indeterminate airframe structures consisting of rods, thin walls and shells. Variational and finite-difference methods are examined as they apply to calculations for structures of this type. Engineering methods are presented for the design of solid wings and fuselages for aircraft with large cutouts. A special section is devoted to calculation techniques for structures operating under conditions of high temperature as a result of kinetic heating.

A table of conversions between the units of the MKGSS [meter, kilogram-force, second] and SI [International] systems for the references is given at the end of the book.

The book is intended for design engineers working in other fields of mechanical engineering as well as in aviation, and may also be helpful to students in the higher educational institutions for aeronautics.

Reviewer: Chair of the Moscow Aviation Institute Named for Sergo Ordzhonikidze

Editor: Candidate of Technical Sciences A.I. Sverdlov

FTD-TT- 65-1698/1+2

UNEDITED ROUGH DRAFT TRANSLATION

STRUCTURAL MECHANICS OF AIRFRAME STRUCTURES

BY: A. F. Feofanov

English Pages: 307

TM5002561

THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION.	PREPARED BY: TRANSLATION DIVISION FOREIGN TECHNOLOGY DIVISION WP-AFB, OHIO.

FTD-TT- 65-1698/1+2

Date 26 March 19 66

AFLC-WPAFB-JUN 66 74

S

Ministerstvo
Vysshego i Srednego Spetsial'nogo Obrazovaniya RSFSR

Moskovskiy Ordena Lenina Aviatsionnyy Institut
Imeni Sergo Ordzhonikidze

STROITEL'NAYA MEKHANIKA AVIATSIONNYKH KONSTRUKTSIY

Trudy Instituta

Vyiusk 160

Izdatel'stvo Mashinostroyeniye

Moskva 1964

Pages: 1-284

FTD-TT-65-1698/1+2

TABLE OF CONTENTS

BLANK PAGE

FOREWORD

The present volume represents a further development of the author's books "Raschety tonkostennykh konstruktsiy [Calculations for Thin-Walled Structures] (Oborongiz [State Publishing House for the Defense Industry], 1953) and "Stroitel'naya mekhanika tonkostennykh konstruktsiy" [Structural Mechanics of Thin-Walled Structures] (Oborongiz, 1958) and stands as the author's creative contribution to the development and systematization of modern structural-mechanical methods for aviation structures.

The book begins with an examination of the analogies between rod and thin-walled structures with a skin that works only in shear, and examines the general theorems of the structural mechanics of rod and thin-walled structures.

In setting forth the variational methods, the author has succeeded in clarifying the physical significance of some of the complex mathematical relationships encountered, with the aid of a number of judiciously selected examples.

The book considers the stressed and strained state of various low-aspect ratio wing designs, stress concentration at points where there are abrupt changes in the transverse sections of cylindrical shells, thermal stresses, strains and other problems in the strength of airframe structures.

The current importance of the problems examined, the engineering approach to their solution, with invocation of the variational methods and equations in finite differences are a credit to the great experience of the author, who was able to set forth the complex problems of contemporary airframe structural mechanics in simple and accessible form.

Prof. Doctor of Technical Sciences I.F. Obraztsov

AUTHOR'S PREFACE

The present volume, "Structural Mechanics of Airframe Structures" represents a further development of our work "Structural Mechanics of Thin-Walled Structures" (Oborongiz, 1958).

The main content of the volume breaks down into four parts.

Part One treats rod systems, thin-walled structures with skin working only in shear, and composite structures – thin-walled + rod systems. The theory of rod systems is examined here only to the extent that it is necessary to show the intimate relationship between thin-walled structures and rod systems.

In Part Two, variational methods of solving problems in structural mechanics are examined on the basis of the principle of possible displacements. To supplement the approximate methods that are set forth in the work mentioned above, the Ritz-Timoshenko, Bubnov-Galerkin and Kantorovich-Vlasov methods and the method of finite differences are analyzed and illustrated with examples.

In Part Three, the Kantorovich-Vlasov method and equations in finite differences are used to make separate investigations of the stressed and strained states of aircraft-wing and aircraft-fuselage-type shells, and the method of displacements is examined as it applies to the design of the wing.

Part Four is concerned with thermal stresses. It begins with calculations for systems with a thin wall, and thereafter the principal attention is devoted to determination of thermal stresses resulting from aerodynamic heating of solid and hollow wings. The thermal-stress calculations are made using the method of forces, the Kantorovich-Vlasov method and equations in finite differences. At the end of this Part, we examine the effect of thermal stresses in which they lower the torsional rigidity of wings.

The author expresses his gratitude to I.F. Obraztsov, V.M. Strigunov, S.G. Yelenevskiy, A.A. Belous, V.I. Figurovskiy and V.I. Klimov for the advice and recommendations that they rendered during preparation of the manuscript, to P.D. Grushin, L.I. Balabukh, I.A. Sverdlov and S. Ya. Makarov for the valuable comments made during the review, and to A.I. Sverdlov for his meticulous editing of the manuscript.

A. Feofanov

Part One

THEORY AND DESIGN OF ROD SYSTEMS AND THIN-WALLED STRUCTURES WITH SKIN WORKING ONLY IN SHEAR

Chapter 1

FORMATION OF SYSTEMS AND INVESTIGATION OF THEIR GEOMETRICAL INVARIABILITY

INTRODUCTION

The mathematical schemes of the structures considered in Part One may be represented as systems having Y hinged nodes connected to one another either by rods only (rod systems) or by rods and walls (thin-walled structures). Each rod and each wall represents a single coupling – in the form of a rod axial force or the tangential force of the wall.

As a point, each link has two degrees of freedom in a plane and three degrees of freedom in space, since two of its coordinates (for example, the cartesian coordinates (x, y) may vary independently during free motion in a plane, while three (x, y, z) may do so in space.

Consequently, if the nodes were not connected to one another by the couplings, the degree of freedom of a system consisting of Y free nodes would be W = 2Y in the plane and W = 3Y in space. But the couplings reduce the freedom of the nodes by as many units as there are couplings C. Hence for a system consisting of Y nodes and C couplings, the degree of freedom is:

in a plane

$$W=2Y-C, \tag{1.1}$$

in space

$$W=3Y-C. \tag{1.2}$$

If the system is secured to a support and has the necessary minimum number of couplings $\bar{C}$ to confer geometrical invariability upon the system, its degree of freedom W = 0. According to Formulas (1.1) and (1.2), we obtain the smallest necessary number of couplings for this case:

for two-dimensional systems

$$\bar{C}=2У, \tag{1.3}$$

for three-dimensional systems

$$\bar{C}=3У. \tag{1.4}$$

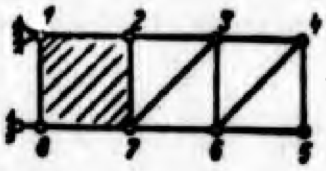


Fig. 1.1

If the system is free (a geometrically invariant system separated from its supporting couplings), its degree of freedom W will be 3 in the plane (only three coordinates - x, y and angle - can be varied independently) and 6 in space (x, y, z and three angles). Consequently, we obtain the number of necessary couplings for free systems from Formulas (1.1) and (1.2):

for two-dimensional systems

$$\bar{C}=2У-3; \tag{1.5}$$

for three-dimensional systems

$$\bar{C}=3У-6. \tag{1.6}$$

It will be shown below that Formulas (1.3)-(1.6) express only necessary and not sufficient conditions for geometrical invariability, by which we mean the ability of a system not to permit relative displacement of its parts without deformation; here, the deformations of the system must be commensurable with the elastic deformations of its component elements (rods, walls). Thus, for example, if we dispense with rod 3-7 (Fig. 1.1), the system will become geometrically variable, i.e., in this case its rods 3-2, 3-4, 3-6, 4-6, 4-5, 5-6, and 6-7 may displace kinematically with respect to one another due to transformatio of the rod rectangle 2-3-6-7 into a parallelogram.

1. FORMATION OF SIMPLE FREE SYSTEMS

Formation of Simple Two-Dimensional Systems

Let us take a rod (Fig. 1.2a) as the starting element in the formation of two-dimensional free systems. Two rods are required to attach one node to it as a base, according to Formula (1.3). This requirement is necessary but not sufficient to ensure geometrical invariability of the attached node. Indeed, if the two rods securing the node lie on the same line (Fig. 1.2b), we obtain a geometrically variable system in which large displacements of the attached node correspond to small deformations of the rods (see dashed lines on Fig. 1.2b). Such systems are known as instantaneously variable systems. These systems are not rigid and are not used in engineering, since finite forces applied to the nodes produce infinitely large or altogether indeterminate forces in the rods of the system.

Fig. 1.2. 1) Main rod; 2) attached node.

Thus, Formulas (1.3) and (1.5) express necessary but insufficient conditions for geometrical invariability of two-dimensional systems. These conditions are sufficient only provided that each node is connected by two couplings forming a finite angle α (Fig. 1.2c).

Consequently, to secure the node geometrically invariably in the plane, two rods may not lie on the same straight line. By connecting each subsequent node with two rods not in the same straight line in the plane of the triangle formed, we obtain the so-called simple free truss (Fig. 1.3a).

A simple truss can be disassembled (broken up) by removing, one after another, the links and the two rods connecting them. The disassembly must begin with the node that is connected by two rods. The re-

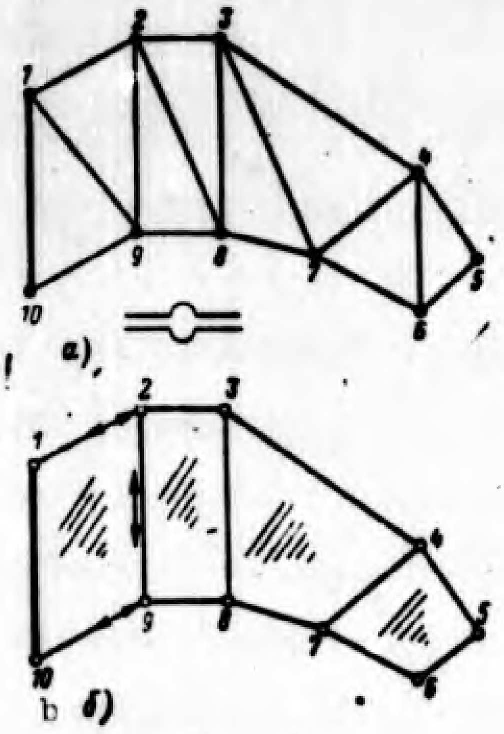


Fig. 1.3

mainder will be a rod, i.e., a free but geometrically invariable system.

Simple trusses possess the smallest number of couplings necessary for geometrical invariability. Actually, to connect nodes 1 and 10, a single coupling is expended, while two couplings are required to attach the remaining $Y - 2$ nodes. Consequently, the system as a whole a number of couplings equal to $2(Y - 2) + 1 = 2Y = 3$, which, according to Formula (1.5), corresponds to the minimum necessary number of couplings. Further, by definition, each node is connected by two rods not lying on the same straight line. Consequently, the secondary condition is also satisfied, and hence the simple truss is a geometrically invariable system.

Figure 1.3b shows a two dimensional system in which the diagonal members have been replaced by walls. Thus, the number of couplings corresponds to the necessary minimum. The secondary condition is also satisfied. Actually, we connect to the original rod 1-10 not node 9, but rod 2-9, which, as a system consisting of two nodes and one coupling, has three degrees of freedom ($W = 2Y - C = 2 \cdot 2 - 1 = 3$). This means that a minimum of three couplings are required to secure the rod geometrically invariably in a plane. These couplings deprive rods 2-9 of three degrees of freedom only in the event that the restraining couplings do not intersect at a finitely or infinitely distant point (the point will be infinitely remote if the couplings are parallel to one another); otherwise, the rod will be able to rotate about the point of intersection as an instantaneous center of rotation.

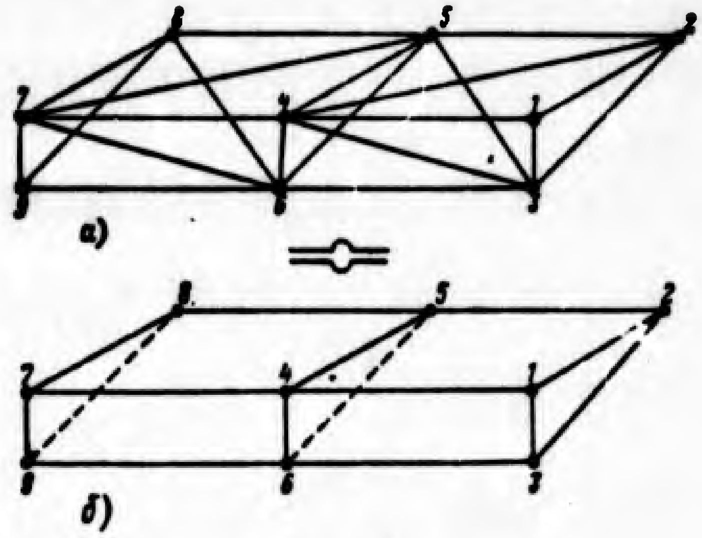


Fig. 1.4

As will be seen from Fig. 1.3b, rod 2-9 is attached to the main rod 1-10 by three couplings. Here the secondary condition is again satisfied, since the directions of the couplings in the form of the axial forces of rods 1-2 and 10-9 and the tangential force 2-9 of wall 1-2-9-10 do not intersect at the same point (see arrows on Fig. 1.3b). Consequently, rod 2-9 is secured geometrically invariably. By connecting each rod (3-8, 4-7, 5-6) successively by three couplings that do not intersect at the same point to the geometrically invariable wall (disk) 1-2-9-10, we obtain a thin-walled system (Fig. 1.3b) that is equivalent on the basis of geometric invariability criteria to the rod system shown in Fig. 1.3a, which is indicated by the symbol .

<u>Formation of Simple Three-Dimensional Systems</u>

A rod triangle, e.g., 1-2-3 (Fig. 1.4a) is usually taken as the starting element in construction of a simple three-dimensional system. The remaining nodes (4, 5, 6..., etc.) are connected successively by three rods. This formation ensures the minimum necessary number of couplings, which is expressed by Formula (1.6). Actually, to connect the first three nodes, three couplings are expended, while three couplings are used for each of the remaining Y − 3 nodes. Thus, the total number of couplings is equal to $C = 3(Y - 3) + 3 = 3Y - 6$ which corresponds to Formula (1.6). However, both Formula (1.6) and Formula (1.4) express necessary but insufficient conditions for geometrical invariability. These conditions are sufficient only provided that each node is attached by three rods not lying in the same plane.

2. INVESTIGATION OF GEOMETRICAL INVARIABILITY OF SIMPLE FREE SYSTEMS BY THE DESTRUCTION METHOD

In investigating the geometrical invariability of simple systems, recourse is taken frequently to the method of destruction, which consists in successive removal of individual nodes or parts of the system that are definitely known to be geometrically invariable. Thus, if we succeed in "destroying" the entire system down to a single rod in the case of two-dimensional systems or a rod triangle in the case of three dimensional systems, the system is simple (separable) and geometrically invariable and has the minimum number of couplings.

Let us employ the method of destruction to study the geometrical invariability of the free system shown in Fig. 4a. Dissassembly of the three dimensional system must begin at the node at which three rods meet. In the system under consideration, node 1 is such a node. Having satisfied ourselves that this node is secured by three rods not lying in the same plane, we remove it together with the three connecting rods. Then, in exactly the same way, we may remove nodes 2, 3, 4, 5, and 6 one after the other. The remainder will be the rigid triangle 7-8-9, which has six degrees of freedom ($W = 3 \cdot 3 - 3 = 6$), which corresponds to the number of degrees of freedom of a geometrically invariable free system in space.

From this it is evident that the system is separable and, consequently, simple and geometrically invariable, since we have satisfied ourselves that each node is attached by three rods not lying in the same plane and, as we ascertained earlier, it has the minimal number of couplings. The thin-walled system composed of rods and walls (Fig. 1.4b) also satisfies all of these criteria.

3. FORMATION OF COMPLEX SYSTEMS

Formation of Complex Systems by Connecting Two or More Invariable Two-Dimensional Systems

If the system has the minimal number of couplings, but they are arranged in such a way that any node of a two-dimensional system is connected by more than two couplings, or by more than three in the case of a three dimensional system, the system is said to be complex, since it cannot be dissassembled (destroyed) by removing one node after another with the two rods connecting each of them in the case of a two dimensional system or with the three rods in the case of the three dimensional system.

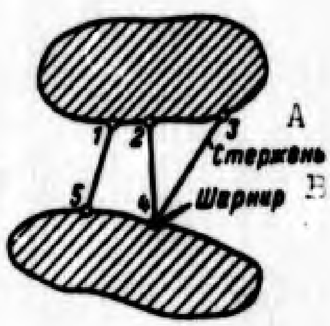


Fig. 1.5. A) Rod; B) hinge.

Fig. 1.6. 1) Wall; 2) hinge.

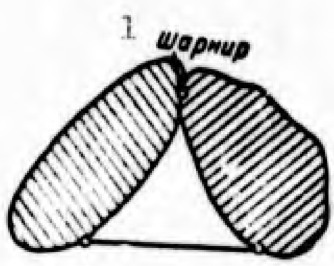


Fig. 1.7. 1) Hinge.

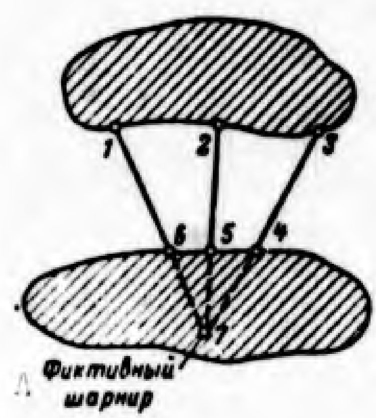


Fig. 1.8. A) Fictitious hinge.

Several definitely invariable systems are joined together by means of rods, walls and hinges (Figs. 1.5-1.8). A point (node) of intersection of rods in structural mechanics is also known as a hinge, which is assumed to be ideal, i.e., to work without friction. Figure 1.5 shows a real hinge (point 4), while Fig. 1.8 shows a fictitious one (point 7). A hinge in a plane is equivalent to two couplings, and in space to three couplings. Indeed, if an invariable system is connected with a fixed hinge axis (Fig. 1.9), it will lose two degrees of freedom in a plane, since only the angle α remains as an independent parameter of the motion, while a three dimensional system attached to a ball joint will lose three of the six degrees of freedom (in the directions of the three coordinate axes x, y, z).

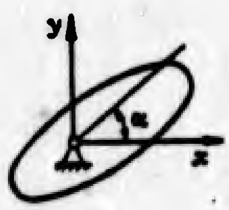


Fig. 1.9

Suppose that D invariable two dimensional systems, which will be referred to as disks, are connected with one another by C couplings. Let us determine the number of degrees of freedom of such a system. If the disks were not connected to one another by couplings, then the D disks would possess a degree of freedom W = 3D in a plane. But the couplings lower the degree of freedom by C, so that the degree of freedom of a system of disks connected into a single system by C couplings is

$$W = 3\text{Д} - C. \qquad (1.7)$$

If the disk system is attached to a support and has only the minimum necessary number of connecting couplings $\bar{C}$ that endows it with geometrical invariability, then W = 0. From Formula (1.7), we obtain the minimum necessary number of connecting couplings for this case:

$$\bar{C} = 3\text{Д}. \qquad (1.8)$$

A free, but invariable system of disks has W = 3, i.e., two degrees of linear displacement and one degree of rotation. We obtain the minimum necessary number of couplings for free systems from Formula (1.7) for this case:

$$C=3\text{Д}-3. \tag{1.9}$$

Formulas (1.8) and (1.9) express conditions that are only necessary and not sufficient for geometrical invariability of two dimensional systems consisting of D disks. These conditions are sufficient only provided that when two disks are connected, the directions of the connecting couplings do not intersect at the same finitely or infinitely remote point. If the directions of the three rods intersect at the same point, the disk may rotate about this point of intersection as about an instantaneous center of rotation (see Fig. 1.8); if the directions of all rods are parallel to one another (Fig. 1.10), they intersect at a single infinitely remote point. On the other hand, in the case in which three disks are joined together, the three hinges (real or fictitious) may not lie on the same line; otherwise, the system will be found to be instantaneously variable, in the same way as that considered earlier (see Fig. 1.2b). By way of example, Figs. 1.11 and 1.12 show three-hinge instantaneously variable two dimensional systems, and Figs. 1.13 and 1.14 geometrically invariable systems.

<u>Example</u>.

Investigate the geometrical instability of the two dimensional system shown in Fig. 1.15.

We isolate the three definitely geometrically invariable disks A, B and C (Fig. 1.16). The number of rods connecting these disks is six. According to Formula (1.9), the smallest necessary number of rods

$$C=3\text{Д}-3=6.$$

Consequently, the condition regarding the minimum number of rods necessary for geometrical invariability is satisfied. The secondary condition is also satisfied, since the three hinges 1, 2 and 3 do not

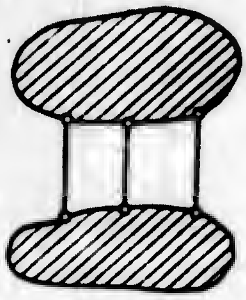


Fig. 1.10.

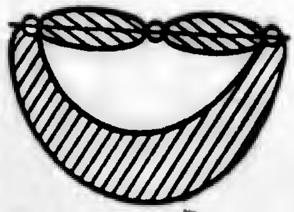


Fig. 1.11.

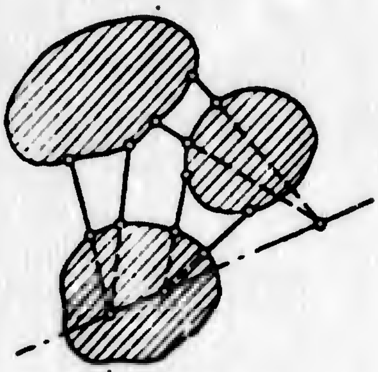


Fig. 1.12.

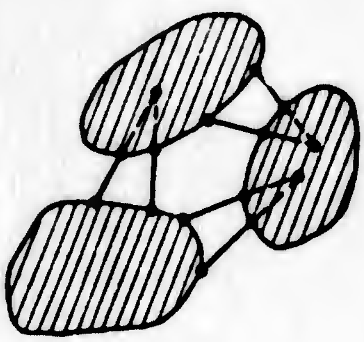


Fig. 1.13.

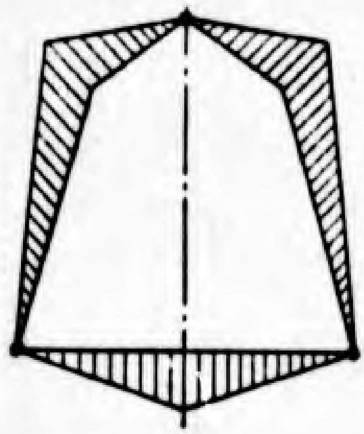


Fig. 1.14.

Fig. 1.15.

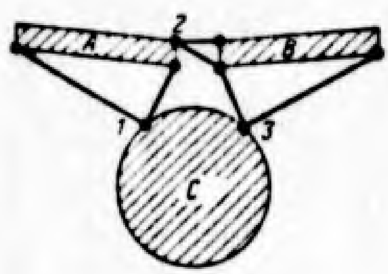


Fig. 1.16

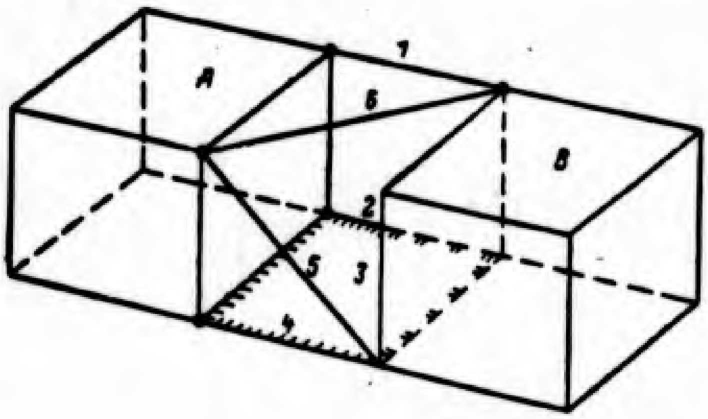


Fig. 1.17.

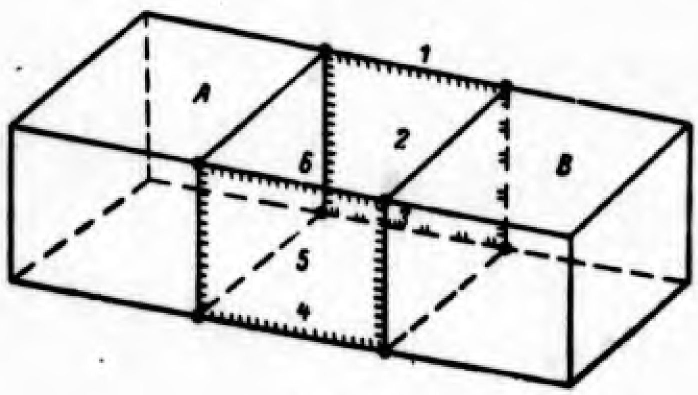


Fig. 1.18.

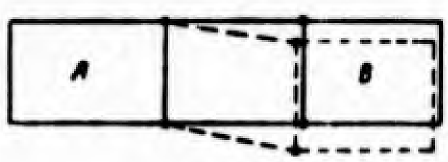


Fig. 1.19.

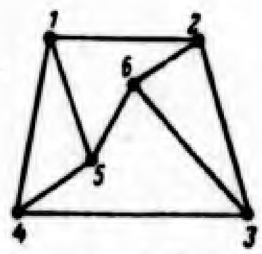


Fig. 1.20.

lie on the same line.

Thus, the system shown in Fig. 1.15 is geometrically invariable.

If a system of three dimensional bodies T is attached to a support and has only the minimum necessary number of couplings $\bar{C}$ that endows it with geometrical invariability, it becomes obvious that

$$\bar{C}=6T, \tag{1.10}$$

while for a free but invariable system

$$\bar{C}=6T-6. \tag{1.11}$$

Formulas (1.10) and (1.11) express necessary but not sufficient conditions for geometrical invariability of complex systems. These conditions are sufficient only provided that the directions of the connecting rods do not intersect the same (finitely or infinitely remote) axis; otherwise this axis will be an instantaneous axis of rotation.

Figures 1.17 and 1.18 show examples of the connection of two three dimensional geometrically invariable systems A and B. According to Formula (1.11), the minimum necessary number of couplings is

$$\bar{C}=6T-6=6\cdot2-6=6.$$

In Fig. 1.17, the six couplings connecting systems A and B (5 rods and one wall) do not intersect the same axis and, consequently, the three dimensional systems A and B are connected geometrically invariably.

In Fig. 1.18, six couplings (four rods and two walls) intersect the same axis at infinity, and, consequently, systems A and B are connected geometrically variably. Figure 1.19 shows the top view of the system shown in Fig. 1.18 and the relative displacement of the connected systems A and B that is possible in this case.

Formation of Complex Systems by Replacement of Couplings

Examples of complex systems obtained by replacement of couplings are shown in Figs. 1.20 and 1.21.

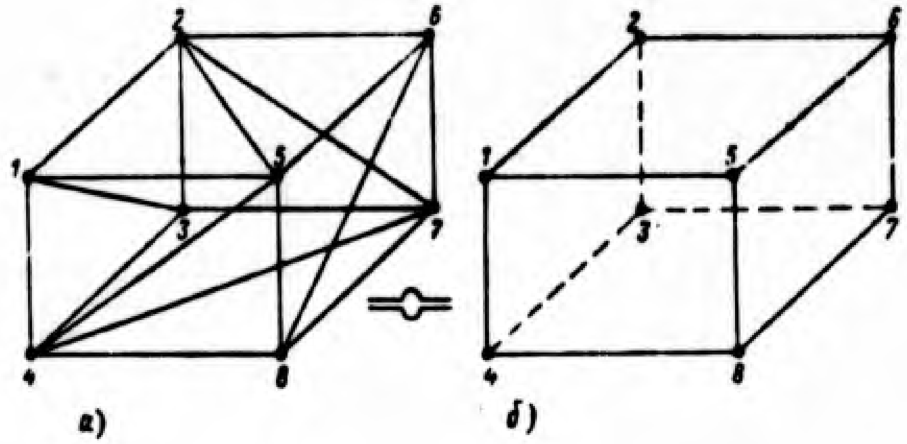


Fig. 1.21

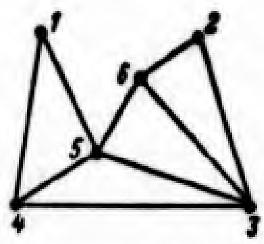


Fig. 1.22

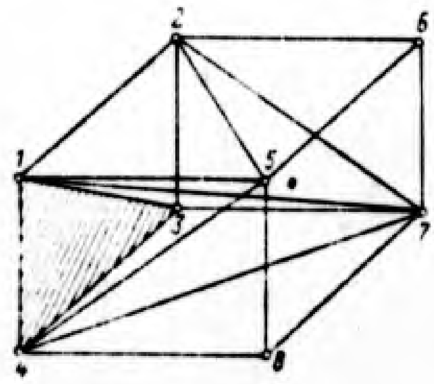


Fig. 1.23

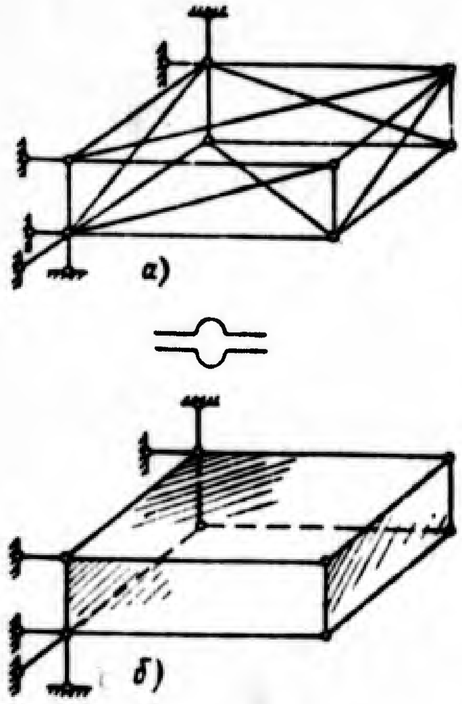
The complex two dimensional truss of Fig. 1.20 is obtained from the simple truss of Fig. 1.22 by substituting rod 1-2 for rod 3-5.

The complex truss of Fig. 1.21a is obtained from the simple truss of Fig. 1.23 by substituting rod 6-8 for rod 1-7.

More detailed investigations of the formation and geometrical invariability of rod systems may be found, for example, in the work of A.Yu. Romashevskiy and V.I. Klimov [20].

4. ATTACHMENT TO SUPPORTS

All structures are attached to supports of some sort. Thus, for example, the support for the stabilizer and fin is the tail section of the airplane's fuselage; for an engine nacelle or undercarriage mounted on the wing, the support is the wing; the support of the wing is the fuselage, and so forth.



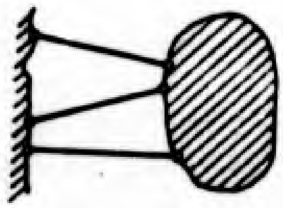
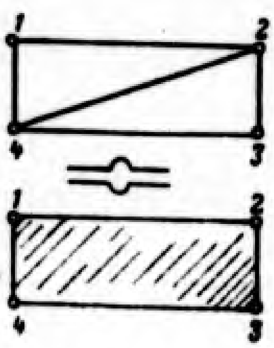


Fig. 1.24.

Fig. 1.25.



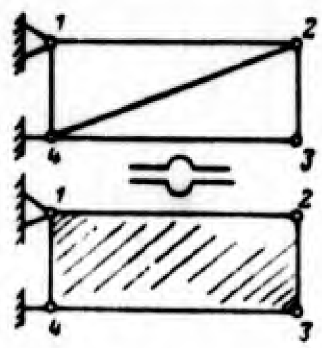


Fig. 1.26.

Fig. 1.27.

To attach a structure to its supports immovably, it is necessary to use couplings to deprive it of all degrees of freedom that it possesses in free motion.

It is known from the above exposition that a minimum of two supporting rods are required to ensure immovable attachment of a node in

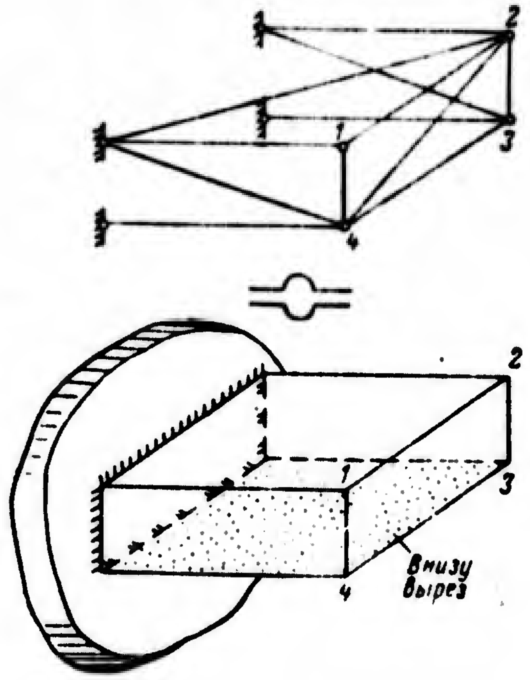


Fig. 1.28. A) Cutout at bottom.

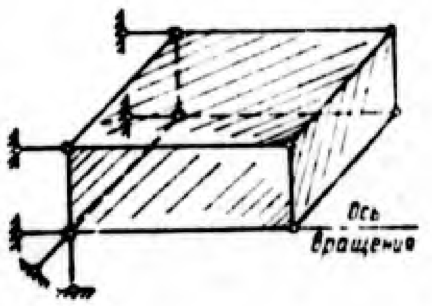


Fig. 1.29. 1) Axis of rotation.

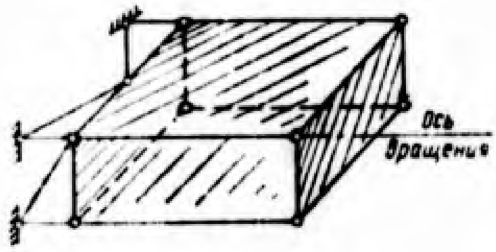


Fig. 1.30. 1) Axis of rotation.

a plane, and three in space. The geometrically invariable system has three degrees of freedom in the plane and six in space; consequently, a minimum of three supporting rods are required to fix it in a plane (Fig. 1.24) and six in space (Fig. 1.25).

The two dimensional system of Fig. 1.26 has three degrees of freedom in the plane ($W = 2 \cdot 4 - 5 = 3$), and seven in space* ($W = 3 \cdot 4 - 5 = 7$); consequently, a minimum of three couplings is required to secure it to a support in a plane and seven in space (Figs. 1.27 and 1.28).

The placement of the necessary minimum of couplings must be such that they actually do deprive the system of all degrees of freedom, i.e., the secondary requirement regarding the placement of the couplings must be satisfied. Figures 1.24, 1.27 and 1.28 show one of the correct arrangements of the supporting couplings, while Figs. 1.29 and 1.30 show incorrect placement (the supporting rods intersect the same infinitely remote axis).

Manuscript Page No.	[Footnotes]
19	This system has six degrees of freedom in space, as a rigid body, and an additional degree of freedom is provided by the ability of the system to change its configuration freely from plane 1-2-3-4 (to fold on axis 2-4).

Manuscript Page No.	[Transliterated Symbols]
12	Д = D = disk = disk

Chapter 2

INTRODUCTION TO CALCULATIONS FOR STATICALLY DETERMINATE SYSTEMS

1. GENERAL PROPERTIES OF STATICALLY DETERMINATE SYSTEMS

A system is said to be statically determinate if the forces that arise in its elements on application of a given load are finite and can be determined on the basis of equilibrium conditions assuming an absolutely rigid system. It follows from this definition that the forces do not depend on the cross-sectional dimensions of the elements or the material of the structure.

On cutting out nodes of a system secured to a support, we may write for each node: two static equations in the case of a two dimensional system and three static equations in the case of a three dimensional system; these express the equilibrium conditions for the node under the action of known external forces and unknown [internal] forces. There will be a total of 2Y such equations for two dimensional systems and 3Y for three dimensional systems. For a free system, i.e., a system separated from its support, the numbers of equations will be 2Y – 3 for two dimensional systems and 3Y – 6 for three dimensional systems, since in the absence of the excessive supporting couplings when two dimensional systems are released from their supports, three equations are written for determination of the forces in the supporting couplings, and six in the case of three dimensions.

In accordance with Formulas (1.3)-(1.6), we state that the number of equations obtained from the equilibrium conditions to determine the unknown forces in the couplings is equal to the smallest necessary number of couplings that confers geometrical invariability upon the system. It follows from this that a system that is geometrically invariable and does not have excessive couplings is always statically determinate and, conversely, a statically determinate system is geometrically invariable and has no excessive couplings.

If one coupling is removed from a statically determinate system, the system becomes variable and the panel that was connected by the discarded coupling may freely change its configuration. Consequently, if the discarded coupling is replaced by a coupling with new dimensions, no forces will arise in the system. It follows from this that no internal self-equilibrating stresses (for example, thermal stresses) can arise in statically determinate systems. And once this is so, external forces are compensated by a uniquely possible distribution of internal forces.

2. CALCULATIONS FOR SIMPLE TWO DIMENSIONAL STATICALLY DETERMINATE SYSTEMS

Calculations for simple two dimensional statically determinate rod systems are treated in courses on theoretical mechanics and the strength of materials, so that we shall indicate here only a calculation for a specific two-dimensional system in which one span has, instead of a diagonal member, a wall that works only in shear from the rods framing it (Fig. 2.1a).

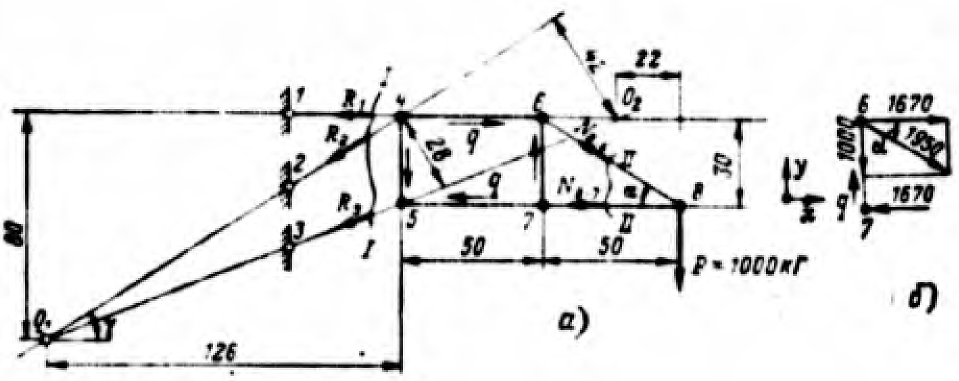


Fig. 2.1. A) P = 1000 kgf.

It is clear from Fig. 2.1a that the system under consideration has the minimum number of couplings, is geometrically invariable, and, consequently, statically determinate. Let us analyze it in greater detail on the basis of the conclusions drawn in the preceding section. Node 8 is secured geometrically invariably, since it is connected by two rods that do not lie on the same straight line; rod 6-7 is also secured geometrically invariably by three couplings that do not intersect at the same point. Consequently, system 4-6-8-7-5 may be regarded as a disk secured by three supporting rods 1-4, 2-4 and 3-5 that do not intersect at the same point. It follows from this that the system has the smallest necessary number of couplings and is geometrically invariable.

We begin the calculation with determination of the support reactions (forces in the supporting rods).

We pass section I-I. Considering the equilibrium of the right-hand part and assuming that the unknown forces in the rods are tensile, we obtain

$$\sum M_{01} = +P226 - R_1 80 = 0; \quad R_1 = \frac{1000 \cdot 226}{80} = 2830 \ \kappa\Gamma;$$
$$\sum M_{02} = P22 - R_2 42 = 0; \quad R_2 = \frac{1000 \cdot 22}{42} = 523 \ \kappa\Gamma;$$
$$\sum M_4 = P100 + R_3 28 = 0; \quad P_3 = -\frac{1000 \cdot 100}{28} = -3570 \ \kappa\Gamma.$$

Here we shall use the method of moment points, which will enable us to obtain each equation with a single unknown.

We pass to determination of the forces in rods 8-6 and 8-7 and the elements of the disk 4-6-7-5.

We pass section II-II. We denote the forces N acting on nodes 8 by the subscripts 8-6 and 8-7; the first figure of the subscript to N indicates the node in question, while the second indicates the node to which the tensile force

$$\sum M_6 = P50 + N_{8-7} 30 = 0; \quad N_{8-7} = -\frac{1000 \cdot 50}{30} = -1670 \ \kappa\Gamma$$

is directed.

From the equilibrium conditions for node 8, we obtain

$$\sum X = -N_{8-7} - N_{8-6} \cos\alpha = 0; \quad \cos\alpha = \frac{50}{\sqrt{30^2 + 50^2}} = 0{,}86.$$

Consequently:

$$1670 - N_{8-6} 0{,}86 = 0; \quad N_{8-6} = \frac{1670}{0{,}86} = 1950 \text{ кГ}.$$

A force of 1950 kgf acts upon node 6 from the direction of node 8. The components of this force on the x- and y-axes are equal respectively to

$$N_{6-4} = 1950 \cdot 0{,}85 = 1670 \text{ кГ}; \quad N_{6-7} = -1950 \sin\alpha = -1000 \text{ кГ}.$$

A force N_{7-8} = - 1670 kgf acts upon node 7 from node 8 (Fig. 2.1b).

Assuming the action of a tangential-force flux PKS q from the skin on rod 6-7, from bottom to top, we obtain from the equilibrium conditions for this rod

$$q\,30 - 1000 = 0; \quad q = +\frac{1000}{30} \simeq +33 \text{ кГ/см}.$$

The plus sign in front of q indicates that the assumption made regarding its direction was justified. In accordance with the law of pairs for PKS, we also apply the disturbance q to Fig. 21b for the remaining three rods.

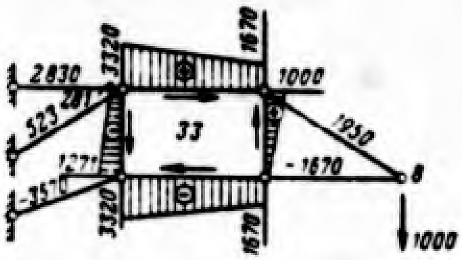


Fig. 2.2

Then it will not be difficult to determine the forces N_{4-6} and N_{5-7} as well, followed by N_{4-5} and N_{5-4}:

$$N_{4-6} = 1670 + q50 = 1670 + 33 \cdot 50 = 3320 \text{ кГ};$$
$$N_{5-7} = -1670 - q50 = -3320 \text{ кГ};$$
$$N_{4-5} = -523 \sin\gamma.$$

Remembering that

$$\sin\gamma = \frac{80}{\sqrt{80^2 + 126^2}} = \frac{80}{149} = 0{,}537,$$

we obtain

$$N_{4-5} = -281\ \kappa\Gamma.$$

Consequently,

$$N_{5-4} = -281 - 33 \cdot 30 = -1271\ \kappa\Gamma.$$

A check of the equilibrium of node 4 serves as a control:

$$\sum X = 3320 - 2830 - 523 \cos\gamma = 48.$$

The error of ≈1.5% is within the limits of measurement error; consequently, the calculation has been performed correctly.

The stressed state of the system is shown in Fig. 2.2.

3. CALCULATION FOR SIMPLE THREE DIMENSIONAL STATICALLY DETERMINATE SYSTEMS

In the design of rod systems, particularly in three dimensions, it is helpful to remember the following propositions (lemmas) that proceed from the equilibrium conditions.

1. If no external force is applied to a two-dimensional hinged node at which two rods converge, or to a three-dimensional hinged node at which three rods converge, the forces in these rods are zero (Figs. 2.3 and 2.4).

2. If three rods of which two are situated on the same line converge at a two dimensional hinge node, or no external force is applied to a three dimensional hinge node to which rods lying, except for one, all in the same plane, the forces are equal to zero in the rods standing alone (rod 1 in Figs. 2.5 and 2.6).

On the basis of the first lemma, it is easy to prove, for example, that the forces in the rods of the truss shown in Fig. 2.7 will all be "zero," except for rods 4-1', 4-4' and 4-3'.

In determining the support reactions of either three dimensional or two dimensional systems, the action of the supporting rods is replaced by the reactions corresponding to the initial assumption, made

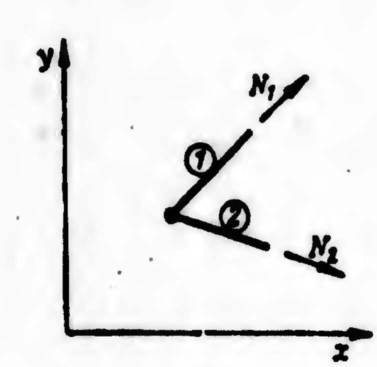


Fig. 2.3.

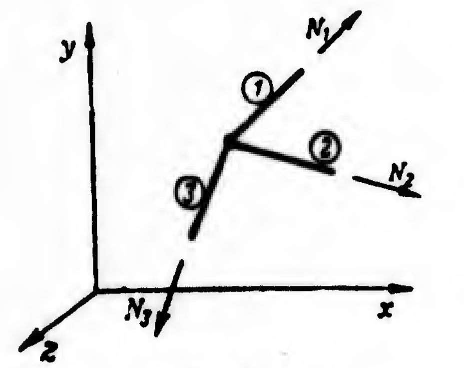


Fig. 2.4.

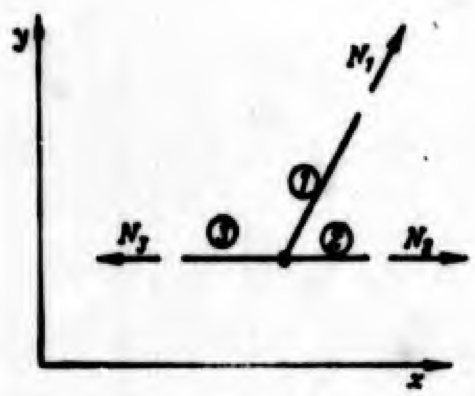


Fig. 2.5.

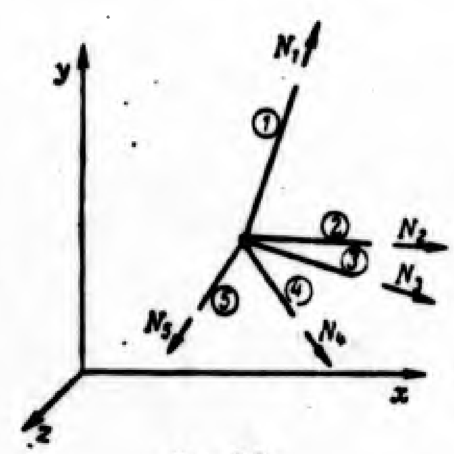


Fig. 2.6.

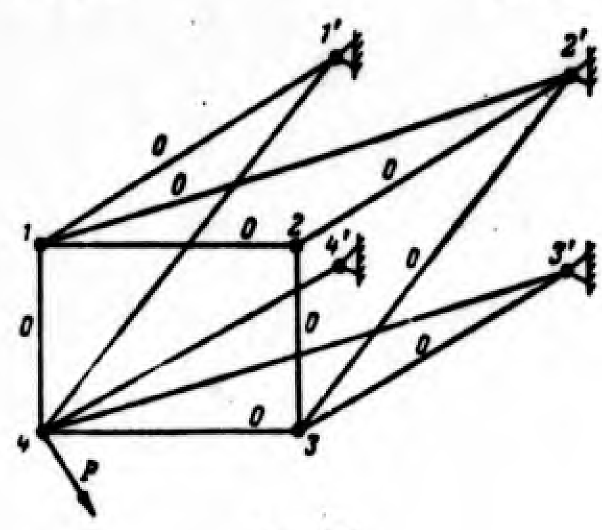


Fig. 2.7.

in writing the static equations, that all rods exert tensile forces (Fig. 2.8). The magnitudes of the six unknown support reactions R_1, R_2, R_3, R_4, R_5 and R_6 can be determined from six static equations:

$$\sum X=0; \quad \sum Y=0; \quad \sum Z=0, \quad \sum M_x=0; \quad \sum M_y=0; \quad \sum M_z=0.$$

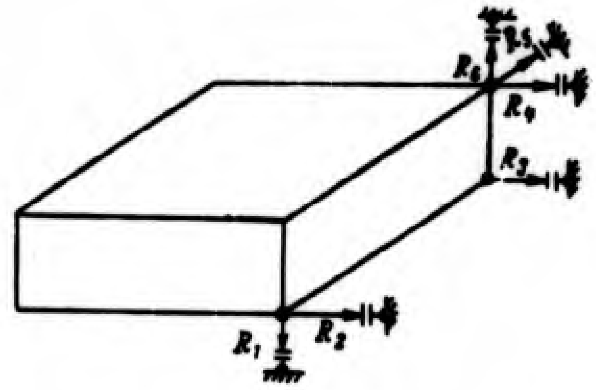


Fig. 2.8

Usually, however, the solution is obtained in a simpler manner, writing the equations of the moments about axes intersecting the largest number of rods.

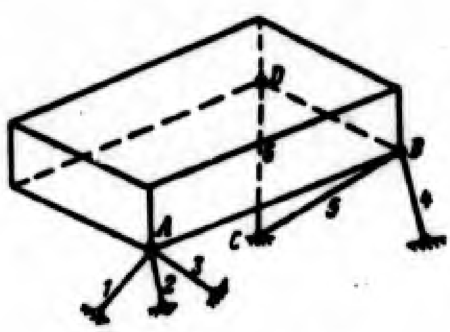


Fig. 2.9

Let us determine the support reactions R_4 and R_5 of the system shown in Fig. 2.9. The axis A-B intersects five directions. With respect to this axis, the sum of moments $\Sigma M_{A-B}=0$ will contain only a single unknown, R_6. The axis A-C also intersects five directions. From the equation $\Sigma M_{A-C}=0$ we find R_4. Axis A-D intersects four directions. We find R_5 from the equation $\Sigma M_{A-D}=0$ (since R_4 has already been found). However, cases may be encountered in which one or another reaction can also be determined simply from the projection equation.

Let us consider a method for determining the forces in three dimensional rod systems (method of cutting out nodes).

In determining the support reactions, the entire system is separated from the support, while in determining the forces in two dimensional trusses, part of the truss is separated – it does not matter which part, as long as no more than three rods the forces in which are of interest to us are cut. The cutting out of nodes is based on the method of sections. It is characterized by the fact that a definite part of the truss – a node of a three dimensional truss – is partitioned off. Needless to say, this method can also be used for two dimensional trusses, but it is frequently found more cumbersome in this case than the method of moment points. For three dimensional trusses, however, this method is one of the basic ones.

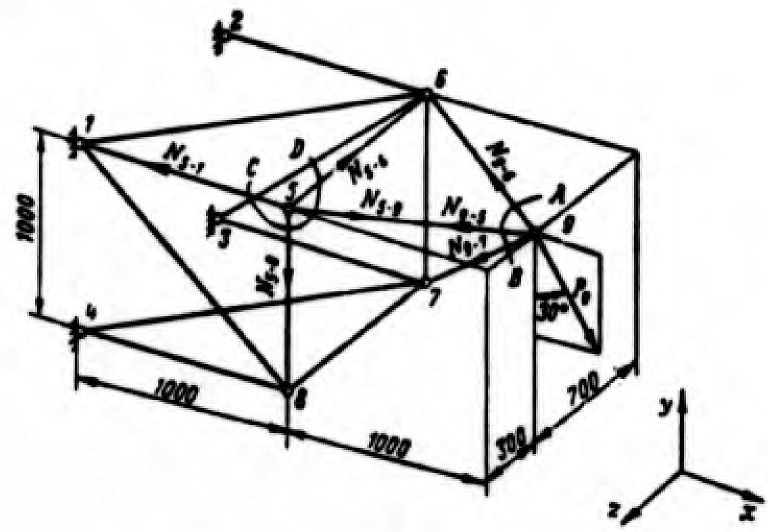


Fig. 2.10

If a three dimensional girder is geometrically invariable and has no superfluous couplings, an attached one has 3Y rods and a free one 3Y – 6 rods.

Cutting out nodes from the three dimensional attached truss, we may write three static equations for each node:

$$\Sigma X=0;\ \Sigma Y=0;\ \Sigma Z=0,$$

expressing the equilibrium conditions for the node under the action of the known external forces and the unknown forces in the rods. We shall have a total of 3Y such equations. If, on the other hand, the truss is free, the number of equations is reduced to 3Y – 6, since when the three dimensional truss is released from its supports, six equations serve to define the forces in the supporting rods. Thus, we may once again satisfy ourselves that there are as many independent static equations as there are rods and, consequently, as many independent forces to be determined.

Let us use a concrete example to explain the calculation procedure used in the node-cutting method. Suppose we are given the system shown in Fig. 2.10. Using the "destruction" method, we satisfy ourselves that the system is geometrically invariable and has no superfluous couplings, since, by discarding nodes 9, 5, 8, 7 and 6 in succession, we find that each node is secured by three rods not lying in the same plane.

In determining the forces in the rods by the method of cutting out nodes, it is necessary to pass from node to node in the same order in which the nodes are discarded in investigating geometrical invariability by the destruction method. Otherwise it will be necessary to solve more than three equations simultaneously.

Thus, we begin with node 9. Passing the section A-B and removing the left part of the truss, we replace the action of the truss on node 9 by tensile forces. The first digit of the subscript to N indicates the node being considered, and the second indicates the node to which the tensile force is directed. Then we write the conditions of equilibrium for node 9 on the x, y and z-axes. These conditions will take the form

$$\sum X = N_{9-5}\cos(9-5, x)+N_{9-6}\cos(9-6, x)+N_{9-7}\cos(9-7, x)+ +P_9\cos(P_9, x)=0; \quad (2.1)$$

$$\sum Y = N_{9-5}\cos(9-5, y)+N_{9-6}\cos(9-6, y)+N_{9-7}\cos(9-7, y)+ +P_9\cos(P_9, y)=0; \quad (2.2)$$

$$\sum Z = N_{9-5}\cos(9-5, z)+N_{9-6}\cos(9-6, z)+N_{9-7}\cos(9-7, z)+ +P_9\cos(P_9, z)=0. \quad (2.3)$$

The cosine of the inclination of the rod (force) to any coordinate axis is equal in magnitude to the projection of the rod onto this axis divided by the length of the rod; the signs, on the other hand, vary, depending on the node being considered. Thus, for example, cos (9-5, z) is a positive quantity, but cos(5-9, z) is negative, as becomes clear from examination of the z-axis directions of the projection of force N_{9-5} at node 9 and the projection of force N_{5-9} at node 5 (see Fig. 2.10).

It is convenient to calculate the direction cosines in the form known as the geometrical proportions table. In compiling this table, the individual coordinate axes are thought of in each case as having been transferred to the node under consideration at the moment. The values of the geometrical proportions for the rods going to nodes 9 and 5 are shown in Table 2.1.

Let us determine the forces in the rods of the truss due to a force P_9 = 1000 kgf, the situation of which is shown in Fig. 2.10.

Substituting values of the cosines into Eqs. (2.1), (2.2) and (2.3), we obtain

$$\sum X = -0{,}955N_{9-5} - 0{,}82N_{9-6} - 0{,}632N_{9-7} + 0{,}5 \cdot 1000 = 0; \quad (1)$$

$$\sum Y = -0{,}632N_{9-7} - 0{,}865 \cdot 1000 = 0; \quad (2)$$

$$\sum Z = +0{,}275N_{9-5} - 0{,}574N_{9-6} - 0{,}443N_{9-7} = 0. \quad (3)$$

TABLE 2.1

A) Стержни	B) Длина проекции стержней			C) Квадраты длины проекций			D) Квадраты длины стержней	E) Длина стержней	F) Косинусы углов		
	x	y	z	x^2	y^2	z^2	l^2	l	x/l	y/l	z/l
9—5	−1,0	0	0,3	1,0	0	0,09	1,09	1,045	−0,955	0	+0,275
9—6	−1,0	0	−0,7	1,0	0	0,49	1,49	1,22	−0,82	0	−0,574
9—7	−1,0	−1,0	−0,7	1,0	1,0	0,49	2,49	1,58	−0,632	−0,632	−0,443
5—1	−1,0	0	0	1,0	0	0	1,00	1,00	−1,00	0	0
5—6	0	0	−1,0	0	0	1,0	1,00	1,00	0	0	−1,0
5—8	0	−1,0	0	0	1,0	0	1,00	1,00	0	−1,0	0
G) и т. д.											

A) Rod; B) length of projection of rod; C) square of projection length; D) square of rod length; E) rod length; F) cosines of angles; G) etc.

From Eq. (2)

$$N_{9-7} = -\frac{865}{0,632} = -1370 \ \kappa\Gamma.$$

Substituting the value of the force N_{9-7} in Eqs. (1) and (3), and noting the sign of the force, we obtain

$$-0,955N_{9-5} - 0,82N_{9-6} + 865 + 500 = 0. \qquad (1')$$

$$+0,275N_{9-5} - 0,574N_{9-6} + 606 = 0. \qquad (3')$$

From Eqs. (1') and (3'), we obtain

$$N_{9-5} = +370 \ \kappa\Gamma;$$
$$N_{9-6} = +1235 \ \kappa\Gamma.$$

Let us go on to node 5. We cut it out and replace the action of the discarded part of the truss by tensile forces N_{5-1}, N_{5-6}, N_{5-8} and N_{5-9}. Then we write the three static equations for node 5. Remembering that $N_{5-8} = 0$ (rod 5-8 stands alone), we obtain

$$\sum X = -1,0N_{5-1} + 0,955N_{5-9} = 0;$$
$$\sum Y = 0;$$
$$\sum Z = N_{5-6} - 0,275N_{5-9} = 0.$$

From the first equation,

$$N_{5-1} = +0,955 \cdot 370 = 354 \ \kappa\Gamma.$$

From the third equation,

$$N_{5-6} = -0,275 \cdot 370 = -104 \ \kappa\Gamma.$$

We can also determine the forces in the other rods in a similar manner; here, if walls replace the diagonal members, the PKS q in these walls can be determined by the calculating procedure indicated in the preceding example for determination of the PKS q of a wall.

4. CALCULATIONS FOR COMPLEX STATICALLY DETERMINATE SYSTEMS

Two Dimensional Systems

In calculations for complex rod systems, the method usually employed is that of replacing couplings, in which we proceed as follows. The com-

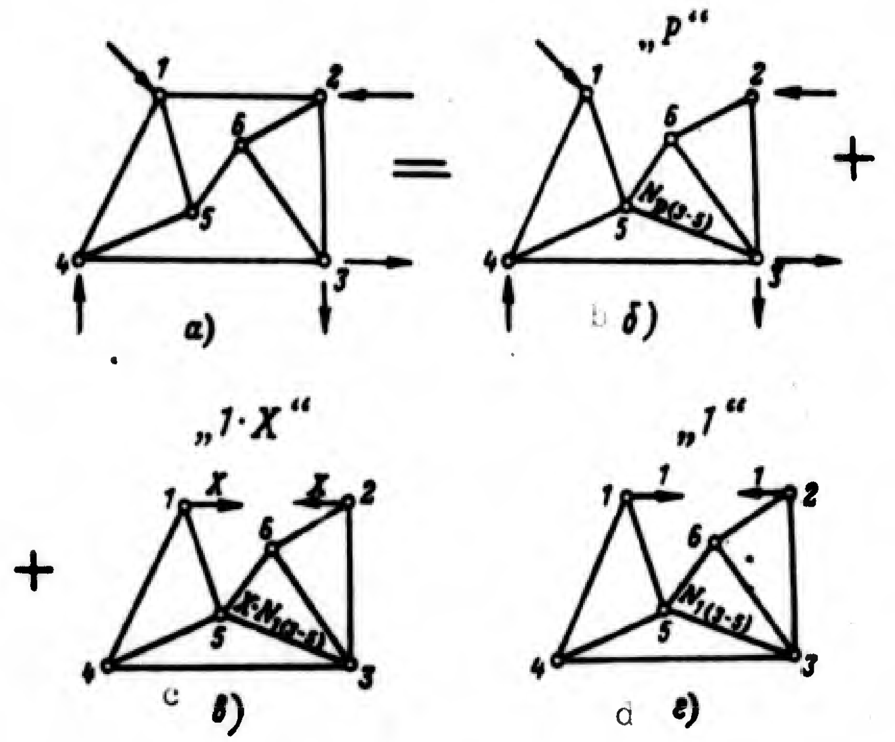


Fig. 2.11

plex system is transformed to a simple one by replacement of couplings. Thus, in the case of the complex system of Fig. 2.11a, rod 1-2 may be replaced by rod 3-5. As a result of this substitution, as is easily seen, we obtain the simple system of Fig. 2.11b, in which the forces $N_{P(k-n)}$ due to external loads are easily determined. The subscript P on N will be used here arbitrarily to denote forces due to external loads. In stage "P," the force in the substituent rod is equal to $N_{P(3-5)}$.

In state "1" (unit state) of the transformed system (Fig. 2.11d), forces equal to 1 kgf are applied in the direction of rod 1-2 toward nodes 1 and 2. The forces in the rods due to the action of these forces will be denoted by $N_{1(k-n)}$. In this state, the force in the substituting rod is $N_{1(3-5)}$.

Rod 3-5 does not occur in the given complex system; consequently, the force in it must be equal to zero. On this basis, we multiply all loads in state "1" (see Fig. 2.11d) by a coefficient X that satisfies the equation

$$N_{P(3-5)}+XN_{1(3-5)}=0.$$

From this equation, we obtain the value of the unknown coefficient X:

$$X=-\frac{N_{P(3-5)}}{N_{1(3-5)}}.$$

Multiplying all unit forces of state "1" by X, we obtain the state "1·X" (Fig. 2.11c).

Adding the forces of states "P" and "1·X" (see Figs. 2.11b and 2.11c), we find the forces N_{k-n} in all rods of the given complex system, i.e.,

$$N_{k-n}=N_{P(k-n)}+XN_{1(k-n)}.$$

Specifically, for rod 1-2,

$$N_{1-2}=0+1\cdot X=X\ \text{кГ}.$$

Thus, the calculation by the coupling-substitution method proceeds as follows:

1) by replacing couplings, we transform the complex system into a simple one;

2) in the simple system, we determine $N_{P(k-n)}$ due to external forces (state "P");

3) in the simple system, we determine the forces $N_{1(k-n)}$ due to a unit load applied to the nodes in which position and direction of the substituted rod;

4) we determine the value of the coefficient X;

5) adding, we determine the true forces N_{k-n}.

The coupling-substitution method is applied in calculations for complex rod systems despite the fact that it is extremely laborious, since the system must be calculated twice. However, the basic deficiency of the method consists in the fact that it is limited to rod systems only. In the case of complex thin-walled structures (see, for example, Fig. 2.13b), it is not possible to use this method. Consequently, it is necessary to find a method for calculation of complex systems that will be simple and at the same time suitable for design of both complex rod systems and complex thin-walled structures. The following procedure may be recommended for the design of complex systems.

The difficulties encountered in calculations for complex systems consist in the fact that there is no node in a two dimensional system that is connected by two rods, and no node in a three dimensional system connected by three rods; consequently, there is no node from which the calculation can be begun immediately. But this difficulty can be circumvented if one of the forces is assigned at the outset.

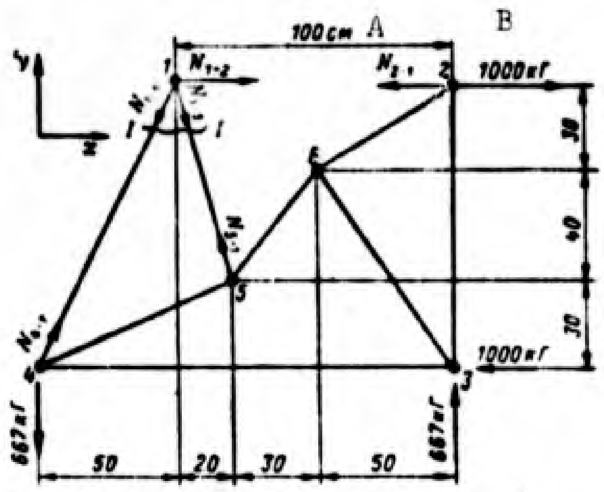


Fig. 2.12. A) 100 cm; B) 1000 kgf.

Thus, for example, we denote the as yet unknown force in rod 1-2 by N_{1-2} (Fig. 2.11a). Then, from the equilibrium conditions for node 1, we can determine the forces N_{1-4} and N_{1-5} as functions of the external force and N_{1-2}. Then we use the equilibrium conditions for node 4 to determine the forces in rods N_{4-5} and N_{4-3} as functions of the external forces and the force N_{1-2}. Then we go on to node 5, to which are directed the unknown force N_{5-6} and the two forces N_{5-1} and N_{5-4}, expressed as functions of N_{1-2} and the external forces. Consequently, we have here two unknowns N_{1-2} and N_{5-6} that can be determined readily in explicit form from the equilibrium conditions for node 5. After the force N_{1-2} has been determined, no difficulty is encountered in determination of the forces in the remaining rods.

Let us follow this calculation procedure in a numerical example. Suppose that we are given the complex girder of Fig. 2.11. We begin the calculation from node 1. We separate it from the entire system by section I-I and replace the action of the discarded rods 1-4 and 1-5 on the node in question by tensile forces N_{1-4} and N_{1-5} (Fig. 2.12). Then we write the equilibrium conditions for node 1 on the x- and y-axes. In the general case, these conditions will obviously take the form

$$\sum X = N_{1-2}\cos(1-2, x) + N_{1-4}\cos(1-4, x) + N_{1-5}\cos(1-5, x) + P_1\cos(P_1, x) = 0; \quad \text{(a)}$$

$$\sum Y = N_{1-2}\cos(1-2, y) + N_{1-4}\cos(1-4, y) + N_{1-5}\cos(1-5, y) + P_1\cos(P_1, y) = 0. \quad \text{(b)}$$

The values of the direction cosines are given in Table 2.2.

TABLE 2.2

A Стержни	B Длина проекции стержней			C Квадраты длины проекций			D Квадраты длин стержней l^2	E Длина стержней l	F Косинусы углов		
	x	y	z	x^2	y^2	z^2			x/l	y/l	z/l
1–2	100	0	0	10 000	0	0	10 000	100	1,000	0	0
1–5	20	–70	0	400	4 900	0	5 300	72,8	0,275	–0,960	0
1–4	–50	–100	0	2 500	10 000	0	12 500	112	–0,447	–0,892	0
2–6	–50	–30	0	2 500	900	0	3 400	58,3	–0,857	–0,514	0
2–3	0	–100	0	0	10 000	0	10 000	100	0	–1,000	0
3–6	–50	+70	0	2 500	4 900	0	7 400	86	–0,582	+0,815	0
3–4	–150	0	0	22 500	0	0	22 500	150	–1,000	0	0
5–4	–70	–30	0	4 900	900	0	5 800	76	–0,922	–0,395	0
5–6	+30	+40	0	900	1 600	0	2 500	50	+0,600	+0,800	0

A) Rod; B) length of rod projection; C) square of projection length; D) square of rod length; E) rod length; F) cosines of angles.

Substituting the values of the cosines in Eqs. (a) and (b) for

$P_1 = 0$, we obtain the equilibrium conditions for node 1 in the form

$$\sum X = N_{1-2} - 0{,}447 N_{1-4} + 0{,}275 N_{1-5} = 0; \quad \text{(a)}$$

$$\sum Y = -0{,}892 N_{1-4} - 0{,}96 N_{1-5} = 0. \quad \text{(b)}$$

From the solution of these equations we obtain

$$N_{1-4} = 1{,}425 N_{1-2};$$
$$N_{1-5} = -1{,}325 N_{1-2}.$$

We pass on to node 4.

$$\sum X = 0{,}447 N_{4-1} + 0{,}922 N_{4-5} + 1 N_{4-3} = 0; \quad \text{(a)}$$

$$\sum Y = 0{,}892 N_{4-1} + 0{,}395 N_{4-5} - 667 = 0. \quad \text{(b)}$$

From the solution of these equations, we obtain

$$N_{4-3} = 2{,}354 N_{1-2} - 1158;$$
$$N_{4-5} = -3{,}22 N_{1-2} + 1{,}688.$$

We pass to node 5.

$$\sum X = -0{,}275 N_{5-1} + 0{,}600 N_{5-6} - 0{,}922 N_{5-4} = 0; \quad \text{(a)}$$

$$\sum Y = 0{,}960 N_{5-1} + 0{,}800 N_{5-6} - 0{,}395 N_{5-4} = 0. \quad \text{(b)}$$

Substituting in these equations

$$N_{5-1} = N_{1-5} = -1{,}325 N_{1-2} \text{ and } N_{5-4} = N_{4-5} = -3{,}22 N_{1-2} + 1688,$$

we obtain on solving

$$N_{5-6} = 830 \ \kappa\Gamma,$$
$$N_{1-2} = 315 \ \kappa\Gamma.$$

Consequently:

$$N_{1-5} = -1{,}325 \cdot 315 = -418 \ \kappa\Gamma;$$
$$N_{1-4} = 1{,}425 \cdot 315 = 449 \ \kappa\Gamma;$$
$$N_{4-3} = 2{,}354 \cdot 315 - 1558 = -817 \ \kappa\Gamma;$$
$$N_{4-5} = -3{,}22 \cdot 315 + 1688 = 675 \ \kappa\Gamma.$$

We pass to node 6.

$$\sum X = -0{,}600 \cdot 830 + 0{,}582 N_{6-3} + 857 N_{6-2} = 0; \quad \text{(a)}$$

$$\sum Y = -0{,}800 \cdot 830 - 0{,}815 N_{6-3} + 0{,}514 N_{6-2} = 0. \quad \text{(b)}$$

From the solution of these equations we obtain

$$N_{6-2}=795\ кГ,$$
$$N_{6-3}=-315\ кГ.$$

From the equilibrium conditions for node 2 on the y-axis, we obtain

$$\sum Y=-0{,}514\cdot 795-1N_{2-3}=0$$

or

$$N_{2-3}=-408\ кГ.$$

As a check, we verify the equilibrium of node 3.

$$\sum X=0;\quad 817+0{,}592\cdot 315-1000=1004-1000\simeq 0;$$
$$\sum Y=0;\quad -815\cdot 315-408+667=-665+667\simeq 0.$$

Three Dimensional Systems

In exactly the same manner, we can make the calculations for complex three dimensional systems. Thus, Figs. 2.13, a and b, show a complex system in the form of a wing compartment attached geometrically invariably to a support by the smallest necessary number of rods. Now determine the stressed state of this system under the influence of a force Q = 1000 kgf.

In the calculation for the rod system of Fig. 2.13a, we assign the force $N_{1'-4}$. Then from the equilibrium conditions for rod 1-1',

$$\sum X=-N_{1'-4}\frac{100}{107{,}7}+2500+N_{1+2'}\frac{100}{141{,}4}=0.$$

From this

$$N_{1-2'}=N_{1'-4}\frac{141{,}4}{107{,}7}-2500\frac{141{,}4}{100}.$$

From the equilibrium conditions for rib 1'-2'-3'-4', we determine the value of $N_{1'-4}$.

The sum of the moments of all forces acting on the rib about axis 3-3' (see Fig. 2.14a) is

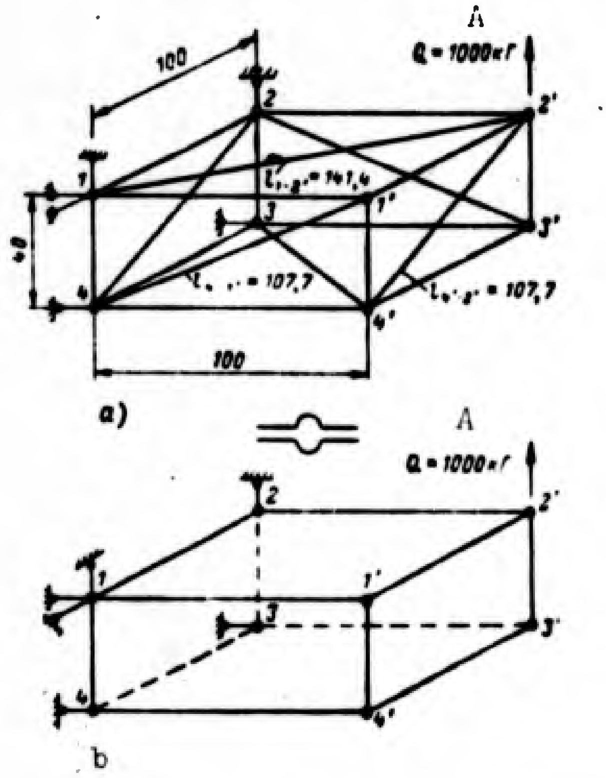


Fig. 2.13. A) Q = 1000 kgf.

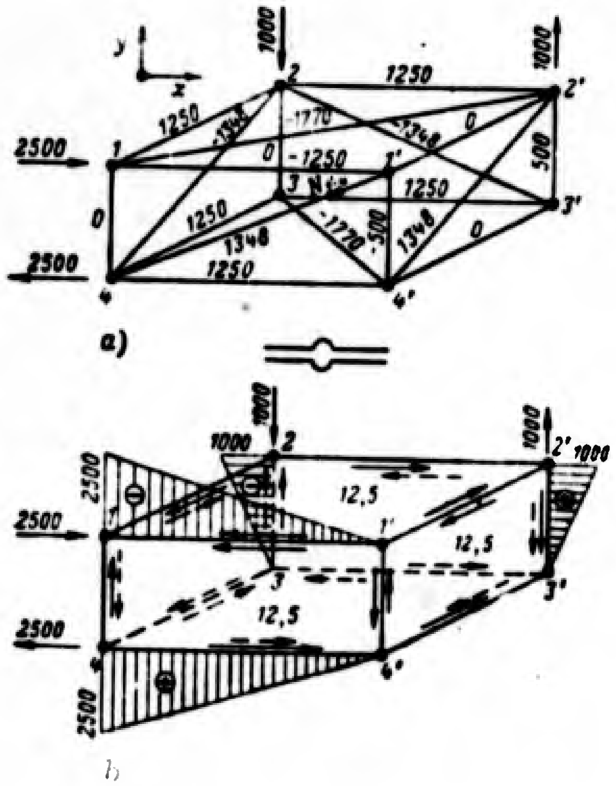


Fig. 2.14

$$\sum M_{3-3'} = N_{1'-4}\frac{40}{107,7}\cdot 100 + N_{1-2'}\frac{100}{141,4}40 = 0.$$

Substituting the value of $N_{1-2'}$ as a function of $N_{1'-4}$ into this equation, we obtain

$$2N_{1'-4}\frac{100}{107{,}7}-2500=0.$$

From this

$$N_{1'-4}=1348 \text{ кГ}.$$

Consequently,

$$N_{1-2'}=1348\,\frac{141{,}4}{107{,}7}-2500\,\frac{141{,}4}{100}=-1770 \text{ кГ}.$$

Given the value of $N_{1'-4}$ and the fact that rods 1-4, 2-3, 1'-2' and 4'-3' stand separately (are zero rods), determination of the forces at the remaining nodes gives rise to no difficulty. The magnitudes of the forces on all rods are shown in Fig. 2.14a. In determining the force factors in the compartment of the thin-walled structure (Fig. 2.14b), we assign the PKS $q_{1-1'-4'-4}$ by analogy with the calculation for the rod-bounded compartment. Then we have from the equilibrium conditions for rod 1-1'

$$2500-q_{1-1'-4'-4}100-q_{1-2-2'-1'}100=0.$$

From this

$$q_{1-2-2'-1'}=-q_{1-1'-4'-4}+\frac{2500}{100}. \qquad \text{(a)}$$

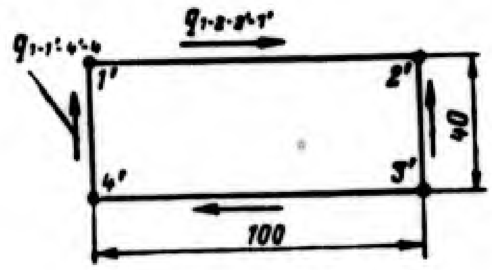


Fig. 2.15

From the equilibrium conditions for rib 1'-2'-3'-4' we determine the value of $q_{1-1'-4'-4}$. The sum of the moments of all forces acting o the rib about node 3 (Fig. 2.15) is

$$\sum M_{3'} = -q_{1-1'-4'-4}40\cdot 100 + q_{1-2-2'-1'}100\cdot 40 = 0$$

or

$$q_{1-1'-4'-4} = q_{1-2-2'-1}.$$

Applying Eq. (a), we obtain

$$q_{1-1'-4'-4} = +12{,}5 \ \kappa\Gamma/c\mathcal{M}.$$

It is easily proven from the equilibrium conditions for the rods that the PKS will also be 12.5 kgf/cm for the remaining walls.

5. DETERMINATION OF DISPLACEMENTS OF STATICALLY DETERMINATE SYSTEMS

Principle of Possible Displacements

If external forces are applied to an elastic system, each point of the system must necessarily be displaced. Suppose, for example, that node 1 of the structure shown in Fig. 2.16, which had the coordinates x, y, z, has been moved to point 1' with the coordinates x', y' and z' after application of the force P.

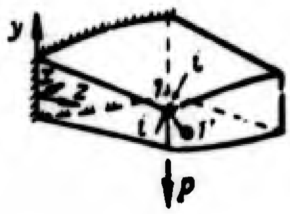


Fig. 2.16

The increments in the coordinates, which we shall call displacements, are so small within the limits of elastic deformations that they may be regarded as very small quantities compared to any of the system's linear dimensions.

To determine the magnitude and direction of the total displacement, it is necessary to know its three projections onto the coordinate axes, but it is most often necessary to determine only the projection of the total displacement onto a specified direction, for example i-i.

To determine the displacements, we shall apply the principle of possible displacements: if a system is at equilibrium, then the sum of the work done by external and internal forces to accomplish any possible small displacements is equal to zero. Here the forces (and displacements) must be regarded as generalized, i.e., they may represent various groups of forces or moments. For generalized forces having the dimensions kgf, kgf·cm, kgf/cm (force, moment, PKS), the corresponding generalized displacements must have the dimensions cm, rad, cm^2 (deflection, angle of turn, area).

Determining Displacements of Statically Determinate Systems

Suppose that, as a result of action of external forces P_1, P_2,..., P_k on the system as a whole, internal forces q_P and N_P have appeared in the elements of the system. This state of the external and internal forces will be referred to as the "P" state.

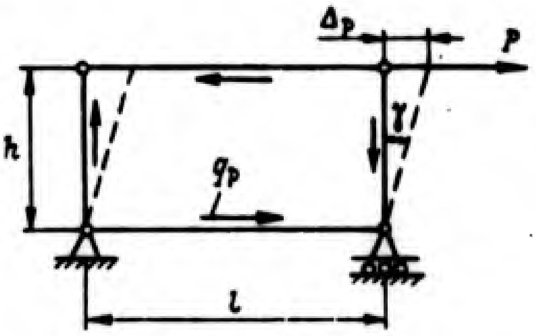


Fig. 2.17

The work ΔA_P of a certain external force P to accomplish the displacement Δ_P corresponding to it will be expressed as

$$\Delta A_P = P\Delta_P.$$

The force P is not halved in the right member of this equation because the possible displacements are very small and, consequently, the force P remains constant over their path.

The work of the internal forces is, by virtue of their physical significance, always negative, since the internal forces oppose the displacements caused by the external forces. Thus, for example, in accomplishing the displacements due to the PKS q_P, the external force P (Fig. 2.17) performs a positive work

$$\Delta A_P = P\Delta_P = P\gamma h,$$

while the PKS q_P performs the negative work

$$-q_P l\gamma h = -\frac{q_P l\tau_P h}{G} = -\frac{q_P^2 F}{G\delta},$$

where F is the plan area of the wall.

The work of the rod internal force when the rod is connected to the skin is

$$-\int_0^l \frac{N_p^2 dl}{Ef},$$

where l is the length of the rod and $\underline{f}$ is the cross-sectional area of the rod.

The work of the internal force in a rod not attached to the skin is

$$-\frac{N_p^2 l}{Ef}.$$

According to the principle of possible displacements, the work done by the external and internal forces of state "P" in accomplishing the corresponding displacements of the same state is equal to zero: consequently

$$\sum \Delta A_p - \sum \frac{q_p^2 F}{G\delta} - \sum \int_0^l \frac{N_p^2 dl}{Ef} - \sum \frac{N_p^2 l}{Ef} = 0.$$

Here the summation extends over all elements forming the structure, including even the support rods.

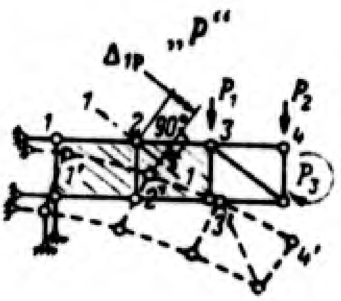


Fig. 2.18

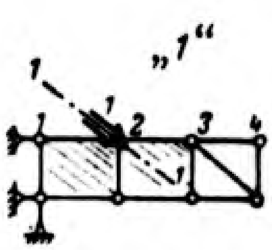


Fig. 2.19

However, according to the principle of possible displacements, the sum of the works of the state-"P" forces is equal to zero not only for displacements of the same state, but also for any other possible displacements, so that we can determine the displacements as follows.

Let one of the states of the given structure be the true state of equilibrium under the action of a system of external generalized forces P_1, P_2, ..., P_n (Fig. 2.18) and the internal forces q_p and N_p produced by them, and determine for this state the displacement Δ_{1p} of a certain node, for example 2, in the direction 1-1. We have called this state of the system the "P" state.

As another possible state of this structure, we take the state of equilibrium (Fig. 2.19) under the action of a unity external force (a force coinciding in location and direction with the unknown generalized displacement Δ_{1P}) and the corresponding internal forces q_1 and N_1. The subscript 1 indicates that the internal forces result from the action of a unit force (the unit state or state "1").

Expressing the work of the state-"1" forces in accomplishing the displacements of the "P" state, we obtain

$$1\cdot\Delta_{1P}-\sum\frac{q_1q_pF}{G\delta}-\sum\int_0^l\frac{N_1N_p\,dl}{Ef}-\sum\frac{N_1N_pl}{Ef}=0.$$

From this, the unknown displacement is

$$\Delta_{1P}=\sum\frac{q_1q_pF}{G\delta}+\sum\int_0^l\frac{N_1N_p\,dl}{Ef}+\sum\frac{N_1N_pl}{Ef}. \tag{2.4}$$

The subscripts on the Δ indicate that the displacement is determined from the external forces of state "P" at the point 1 of force application (see Fig. 2.19) in the direction of the same force.

The value of the integrals $\int_0^l\frac{N_1N_p\,dl}{Ef}$ may be determined from Vereshchagin's rule [30].

Example 1.

In Section 2 of the present chapter, we determined the stressed state of a system due to a force P = 1000 kgf applied to node 8 (see Fig. 2.1). Let us determine the vertical displacement of node 6 of this system for the following data: cross-sectional area of any rod $\underline{f}$, skin thickness δ, supporting rods 1-4, 2-4 and 3-5 absolutely rigid.

To employ Expression (2.4), it is necessary to determine q_1 and N_1 from a force equal to 1 kgf and applied vertically to node 6. Under the influence of this force from above, we obtain (see Fig. 2.1)

$$\sum M_{01}=1\cdot176-R_1 80=0;\quad R_1=\frac{176}{80}=2{,}20;$$

$$\sum M_{02}=-1\cdot28-R_2 42=0;\quad R_2=-\frac{28}{42}=-0{,}667;$$

$$\sum M_4=1\cdot50+R_3 28=0;\quad R_3=-\frac{50}{28}=-1{,}78.$$

From the equilibrium conditions of rod 6-7, we determine the PKS of the skin:

$$q_1\cdot30-1=0;\quad q_1=\frac{1}{30}.$$

Let us determine the forces in the rods framing the skin:

$$N_{6-4}=0;\quad N_{4-6}=\frac{50}{30}=1{,}67;$$

$$N_{7-6}=0;\quad N_{6-7}=-1;$$

$$N_{7-5}=0;\qquad N_{5-7}=-1{,}67;$$

$$N_{4-5}=0{,}667\sin\gamma=0{,}667\cdot0{,}537=0{,}357;$$

$$N_{5-4}=0{,}357-\frac{1}{30}30=-0{,}643.$$

The stressed state of the disk (panel) 4-6-7-5 due to a 1-kgf force is shown in Fig. 2.20.

Applying Formula (2.4) and Vereshchagin's rule, we obtain the unknown displacement

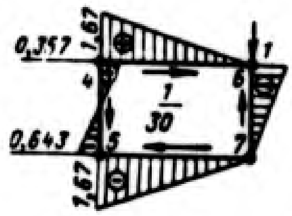
$$f_6=\frac{1}{30}\,\frac{33\cdot30\cdot50}{G\delta}+2\,\frac{1}{6}\,\frac{1{,}67\,(2\cdot3320+1670)\,50}{Ef}+$$

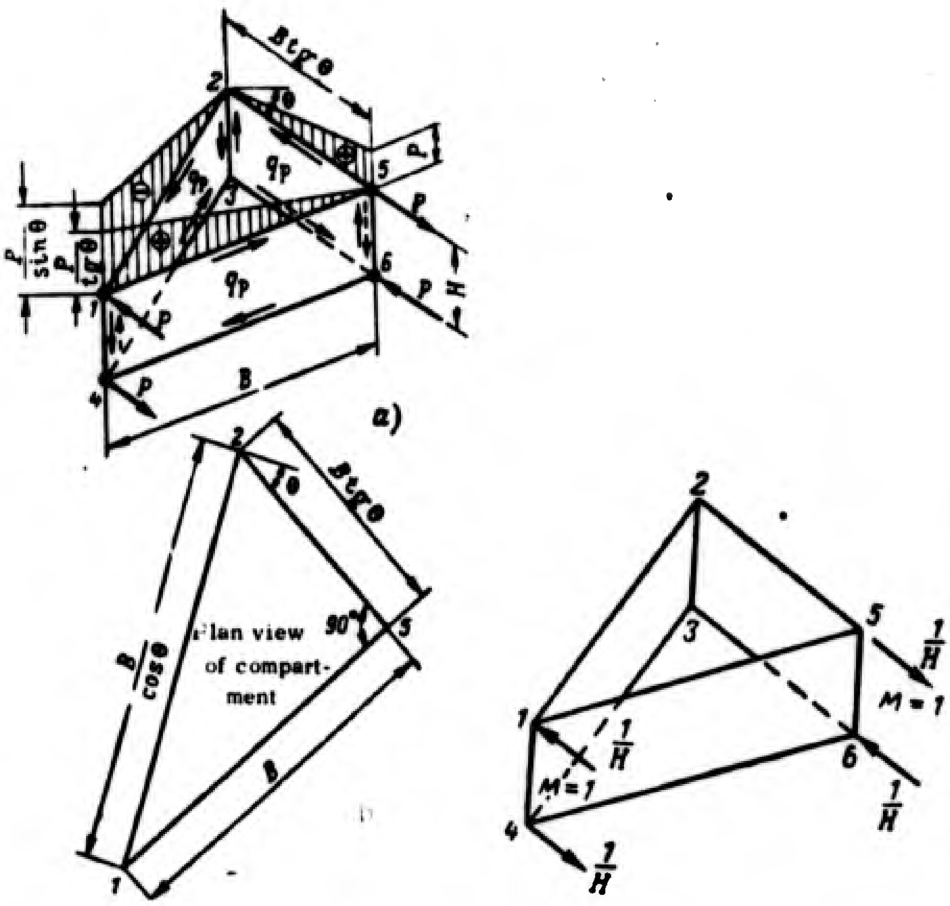
$$+\frac{1}{3}\,\frac{1000\cdot30}{Ef}+\frac{1}{6}\,[0{,}643\,(2\cdot1271+281)-0{,}357\,(2\cdot281+1271)]\,\frac{30}{Ef}=$$

$$=\frac{1650}{Gf}+\frac{238785}{Ef}.$$

<u>Example 2</u>.

There is a triangular compartment in the structure of the root section of a pictureframe sweptback wing with ribs perpendicular to the spars. Let us consider this compartment in isolation (Fig. 2.21, <u>a</u> and <u>b</u>) under the action of a group of self-compensating forces (known as a bimoment) applied to the section 1-5-6-4. Determine the angle of de-





planation* of section 1-5-6-4 (the relative turn angle of struts 1-4 and 5-6).

In the "P" state, we determine from the equilibrium conditions of rod 2-5

$$q^P_{2-5-6-3} = \frac{P}{B\,\mathrm{tg}\,\theta},$$

and from the equilibrium conditions of strut 5-6 we find the PKS in wall 1-5-6-4:

$$q^P_{1-5-6-4}=\frac{P}{B\,\mathrm{tg}\,\theta}.$$

Examining the equilibrium of strut 2-3, we find the PKS in wall 1-2-3-4:

$$q^P_{1-2-3-4}=\frac{P}{B\,\mathrm{tg}\,\theta}.$$

Thus, in the walls of the compartment, the PKS is

$$q_P=\frac{P}{B\,\mathrm{tg}\,\theta}.$$

Further, we determine the longitudinal forces N_P, examining the equilibrium of the rods. The results of the calculation are shown in Fig. 2.21a.

To determine the "1" state corresponding to the definition of the deplanation angle of section 1-5-6-4, a force group 1/H in the form of a bimoment (Fig. 2.22), i.e., two moments each of which is equal to 1, in this section instead of the forces P. In this state, therefore,

$$N_{1\,(5-7)}=\frac{1}{H};\quad N_{1\,(1-5)}=\frac{1}{H\,\mathrm{tg}\,\theta};\quad N_{1\,(1-2)}=\frac{1}{H\sin\theta};$$

$$q_1=\frac{1}{HB\,\mathrm{tg}\,\theta}.$$

If the wall skin thickness is uniform and the cross-sectional areas of the lower framing members are equal to those of the upper ones, we apply Formula (2.4) to obtain the sought deplanation angle in the form

$$\gamma=\frac{P}{B\,\mathrm{tg}\,\theta}\,\frac{H}{HB\,\mathrm{tg}\,\theta G\delta}\left(B\,\mathrm{tg}\,\theta+B+\frac{B}{\cos\theta}\right)+$$
$$+\frac{2}{E}\left(\frac{1}{3}P\frac{1}{H}\,\frac{B\,\mathrm{tg}\,\theta}{f_{2-3}}+\frac{1}{3}\,\frac{P}{\mathrm{tg}\,\theta}\,\frac{1}{H\,\mathrm{tg}\,\theta}\,\frac{B}{f_{1-5}}+\right.$$
$$\left.+\frac{P}{\sin\theta}\,\frac{1}{H\sin\theta}\,\frac{B}{\cos\theta f_{1-2}}\right)$$

or

$$\gamma=\frac{P}{\mathrm{tg}^2\,\theta}\left[\frac{1}{B\delta G}\left(\mathrm{tg}\,\theta+1+\frac{1}{\cos\theta}\right)+\right.$$

$$+\frac{2}{3}\,\frac{B}{HE}\left(\frac{\mathrm{tg}^3\theta}{f_{2-6}}+\frac{1}{f_{1-6}}+\frac{1}{f_{1-2}\cos^3\theta}\right)\Big].$$

Manuscript Page No.	[Footnotes]
45	It will be recalled that deplanation is the deviation of points in the system's cross section from their plane.

Manuscript Page No.	[Transliterated Symbols]
24	ПКС = PKS = potok kasatel'nykh sil = tangential-force flux

Chapter 3

INTRODUCTION TO CALCULATION FOR STATICALLY INDETERMINATE SYSTEMS

1. DEGREE OF STATIC INDETERMINACY

If a system has a number of couplings C larger than the necessary minimum $\bar{C}$ that confers invariability upon it, such a system is said to be statically indeterminate, since the number of unknown force factors in the C couplings of such systems is larger than the number of static equations.

The number of excess couplings above the necessary minimum,

$$K=C-\bar{C} \qquad (3.1)$$

is known as the degree of static indeterminacy.

Substituting the expression for $\bar{C}$ in (3.1) [see Formulas (1.3)-(1.6)], we obtain the values of the degree of static indeterminacy: for two dimensional attached systems

$$K=C-2У; \qquad (3.2)$$

for three dimensional attached systems

$$K=C-3У; \qquad (3.3)$$

for two dimensional free systems

$$K=C-2У+3; \qquad (3.4)$$

for three dimensional free systems

$$K=C-3У+6. \qquad (3.5)$$

The number C in attached systems also includes supporting couplings.*

2. CANONICAL EQUATIONS OF THE METHOD OF FORCES

Any statically indeterminate (S.N.) system can be reduced to a statically determinate (S.O.) system. To do this, it is sufficient to count the internal forces X in the excessive couplings as external forces. For example, let a monocoque consisting of two compartments with all rods attached to the skin except for the rib diagonal members (Fig. 3.1) be at equilibrium under the action of external forces P_1, $\dot{M}$ and the unknown internal forces N and $\underline{q}$. This structure is doubly statically indeterminate (K = 2).

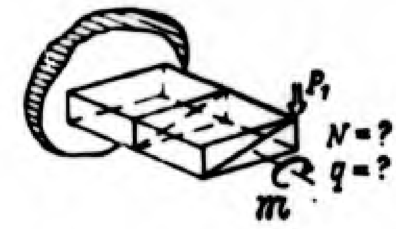


Fig. 3.1

To reduce this system to a S.O. system, it is necessary to eliminate the two excessive couplings. For this purpose, we might pass sections in the top flange of the forward spar, as indicated on Figs. 3.2 and 3.3. The S.O. system obtained in this manner will be referred to as the basic system.

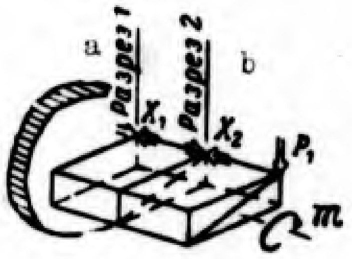


Fig.3.2. a) Section 1; b) section 2.

Let us apply a given load P_1, $\dot{M}$ to the basic system. Obviously, gaps will appear in the sections, and we shall denote them by Δ_{1P} and Δ_{2P} (see Fig. 3.3). Now we apply forces X_1 and X_2 (see Fig. 3.2) to the flange in the sections such that the gaps vanish; thus we obtain the original system.

The conditions for the absence of gaps may be written as follows:

for the gap in section 1:

for the gap in section 2:

$$\left.\begin{aligned} X_1\delta_{11}+X_2\delta_{12}+\Delta_{1P}&=0;\\ X_1\delta_{21}+X_2\delta_{22}+\Delta_{2P}&=0.\end{aligned}\right\} \qquad (a)$$

Here δ_{11} is the change in the gap in section 1 due to a force X_1 equal to unity (for example, 1 kgf, Fig. 3.4). Hence the product $X_1\delta_{11}$ is equal to the change in the gap in section 1 due to the total force X_1. The change in the gap in section 1 caused by the force $X_2 = 1$ (see Fig.

3.2) is denoted by δ_{12}, and so forth. The first subscript on Δ and δ indicates in each case the position (and direction) of the displacement, while the second indicates the cause of the displacement.

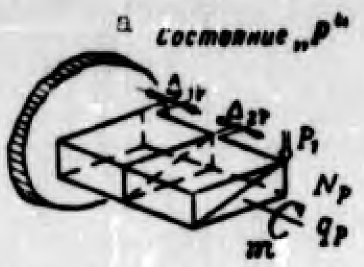


Fig. 3.3. a) State "P."

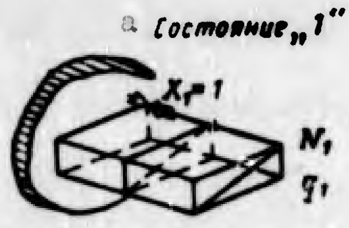


Fig. 3.4. a) State "1."

The directions of the forces X_1 and X_2 are taken arbitrarily. If, as a result of the solution, any of the forces is found to be negative, this will signify that the direction of this force is actually opposite to that assumed.

Equations (a) are the so-called canonical equations of the method of forces, from which we can find the unknown internal forces X_1 and X_2; in this case, they are the flange forces at the places of section 1 and section 2 (see Fig. 3.2).

The displacements Δ and δ are determined by the formula

$$\Delta_{ik}=\sum\frac{q_iq_kF}{G\delta}+\sum\int_0^l\frac{N_iN_k\,dl}{Ef}+\sum\frac{N_iN_kl}{Ef}, \tag{3.6}$$

where the subscript i = 1, 2,..., n pertains to unit states and the subscript k = 1, 2,..., n, p applies to both unit states and the P state. The summation extends over all elements forming the structure, even including the supporting rods (see further Example 1, Section 4). In the presence of elements working in flexure in the calculation scheme, the right member of Eq. (3.6) has the additional term $\sum\int_0^l\frac{M_iM_p\,dl}{EJ}$.

For systems with two superfluous (excessive) couplings

$$\Delta_{1P}=\sum\frac{q_1q_pF}{G\delta}+\sum\int_0^l\frac{N_1N_p\,dl}{Ef}+\sum\frac{N_1N_pl}{Ef};$$

$$\Delta_{2P}=\sum\frac{q_2q_pF}{G\delta}+\sum\int_0^l\frac{N_2N_p\,dl}{Ef}+\sum\frac{N_2N_pl}{Ef};$$

$$\delta_{11}=\sum\frac{q_1^2F}{G\delta}+\sum\int_0^l\frac{N_1^2\,dl}{Ef}+\sum\frac{N_1^2\,l}{Ef};$$

$$\delta_{22}=\sum\frac{q_2^2F}{G\delta}+\sum\int_0^l\frac{N_2^2\,dl}{Ef}+\sum\frac{N_2^2\,l}{Ef};$$

$$\delta_{12}=\delta_{21}=\sum\frac{q_1q_2F}{G\delta}+\sum\int_0^l\frac{N_1N_2\,dl}{Ef}+\sum\frac{N_1N_2\,l}{Ef}.$$

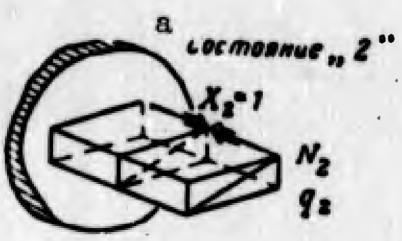


Fig. 3.5. a) State "2."

Here N_P and q_P are forces produced in the basic system by the forces P_1 and M (see Fig. 3.3). N_1, q_1 and N_2, q_2 are the forces of the basic system in the unit states "1" and "2," respectively (see Figs. 3.4 and 3.5).

Summing the internal forces due to the separate disturbances on the basis of the principle of independent force effects, we find the total internal forces:

$$N=N_P+N_1X_1+N_2X_2;$$

$$q=q_P+q_1X_1+q_2X_2.$$

We have considered the case of a doubly statically indeterminate system (K = 2).

In the general case with k = n, the canonical equations assume the form

$$\left.\begin{array}{l} X_1\delta_{11}+X_2\delta_{12}+X_3\delta_{13}+\ldots+X_n\delta_{1n}+\Delta_{1P}=0,\\ X_1\delta_{21}+X_2\delta_{22}+X_3\delta_{23}+\ldots+X_n\delta_{2n}+\Delta_{2P}=0,\\ X_1\delta_{31}+X_2\delta_{32}+X_3\delta_{33}+\ldots+X_n\delta_{3n}+\Delta_{3P}=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ X_1\delta_{n1}+X_2\delta_{n2}+X_3\delta_{n3}+\ldots+X_n\delta_{nn}+\Delta_{nP}=0. \end{array}\right\} \quad (3.7)$$

Here $\delta_{12}=\delta_{21}$, $\delta_{13}=\delta_{31}$, $\delta_{23}=\delta_{32}$ and so forth.

The forces X_1, X_2, etc., are known as excessive unknowns, since they are the reactions of excessive couplings. They are generalized forces, i.e., they may also be moments and even whole groups of forces.

3. SIMPLIFYING THE CALCULATION IN THE CASE OF SYMMETRY

Given geometrical and elastic symmetry in the system, we may, by fortunate selection of the basic system, appreciably reduce the number of unknowns in the canonical equations and, consequently, the number of equations.

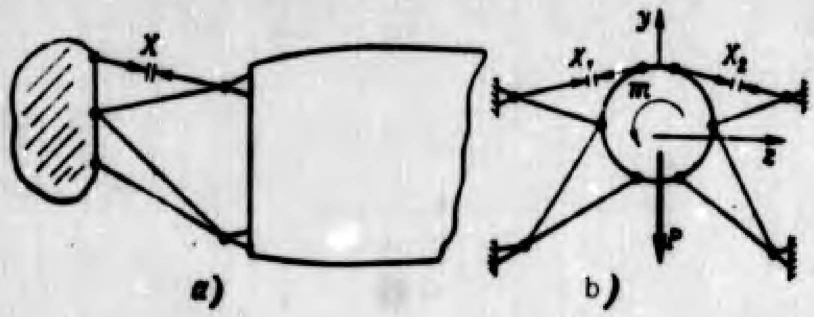


Fig. 3.6

For example, the engine mount shown in Fig. 3.6, a and b, is doubly statically indeterminate, since two geometrically invariable bodies (fuselage and engine) are attached by 8 couplings, while the minimum necessary number of couplings

$$C=6T-6=6.$$

Consequently, the number of excessive couplings

$$K=8-6=2.$$

If, however, the structure of the engine mount and the external loads are symmetrical with respect to the y-axis (Fig. 3.6b), then $X_1 = X_2$. If, on the other hand, the structure is symmetrical but the external forces are antisymmetrical (for example, a torsional moment M) then $X_1 = X_2$. Thus, symmetry of the structure enables us in this case to solve the system as a singly statically indeterminate case.

4. EXAMPLES OF CALCULATION FOR STATICALLY INDETERMINATE SYSTEMS BY THE METHOD OF FORCES

Example 1.

Let a torsional moment in the form of a force couple (Fig. 3.7) be applied to the root section of a pictureframe a monocoque having a uniform thickness δ of of the spar webs and skin and uniform rod areas $\underline{f}$. Now determine the axial forces, assuming the ribs to be absolutely rigid in their plane.

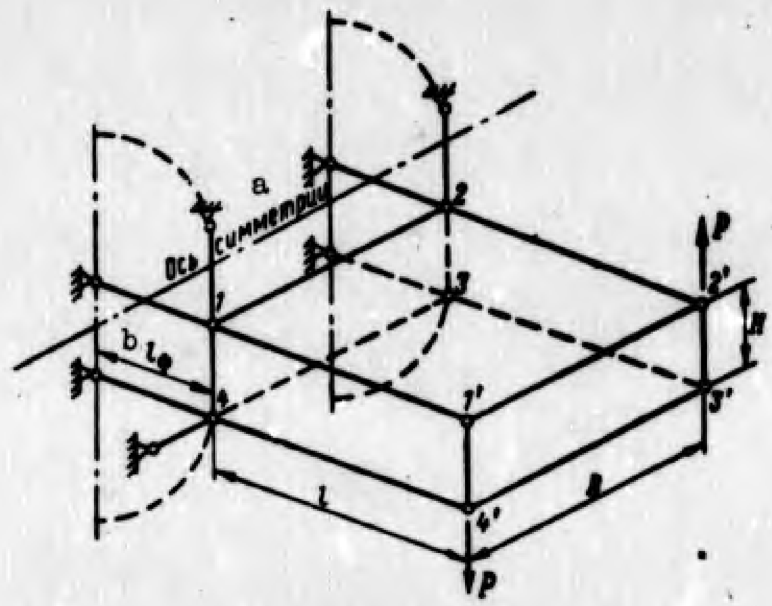


Fig. 3.7. a) Axis of symmetry; b) ℓ_f.

According to Formula (3.3), the degree of static indeterminacy

$$K=(7+18)-3\cdot 8=1.$$

On releasing node 1 from the horizontal supporting rod, we obtain the statically determinate basic system.

In state "P" (Fig. 3.8), we obtain from the equilibrium condition of rib 1'-2'-3'-4' with respect to axis 3-3'

$$-P\cdot B+q_p^{II}H\cdot B+q_p^{III}B\cdot H=0. \qquad \text{(a)}$$

From the equilibrium conditions of rod 1-1', the PKS $q_p^{II} = q_p^{III}$. Consequently, we obtain from Expression (a):

$$q^{II}=q^{III}=\frac{P}{2H}.$$

Remembering that node 1 has been released from its support and only the torsional moment $M = P\cdot B$ acts on the compartment, we can easily prove that the axial forces are equal to zero in nodes 2, 3 and 4 as well. Thus, from the equilibrium conditions for rod 2-2', we obtain $q_P^{I} = q_P^{III}$, and from the equilibrium conditions of rod 3-3', we obtain $q_P^{IV} = q_P^{I}$. Here, q_P^{IV} is the PKS of the lower skin. Finally, from the equilibrium conditions for rod 1'-2', the PKS of the butt rib $q_p^T = \frac{P}{2H}$, and from those for rod 1-2, the PKS of the root rib $q_p^k = \frac{P}{2H}$.

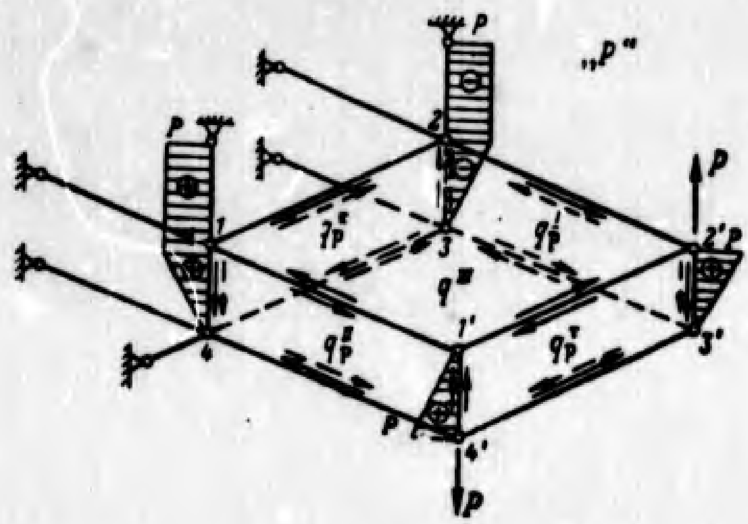


Fig. 3.8

Consequently

$$q_p^{I} = q_p^{II} = q_p^{III} = q_p^{IV} = q_p^{T} = q_p^{k} = \frac{P}{2H}$$

or

$$q_p^{I} = q_p^{II} = q_p^{III} = q_p^{IV} = q_p^{T} = q_p^{k} = \frac{P \cdot B}{2H \cdot B} = \frac{\mathfrak{M}}{\Omega},$$

where $P \cdot B = \dot{M}$ is the torsional moment and Ω is twice the area of the compartment's cross-section outline.

Check. From the equilibrium conditions for rod 1'-4',

$$q_p^{II} H + q_p^{T} H = P$$

or

$$\frac{P}{2} + \frac{P}{2} = P.$$

The axial forces and the PKS of state "P" are shown in Fig. 3.8.

In state "1" (Fig. 3.9), from the equilibrium conditions for spar 1-1'-4-4', the PKS

$$q_1^{VI}=q_1^{III}=q_1^{IV}=q_1^{V}=-\frac{1\cdot H}{2lH}=-\frac{1}{2l}.$$

From the equilibrium conditions for struts 1'-4' and 2'-3'

$$q_1^{II}=\frac{1}{2l} \text{ and } q_1^{I}=\frac{1}{2l}.$$

The force factors of state "1" are shown in Fig. 3.9.

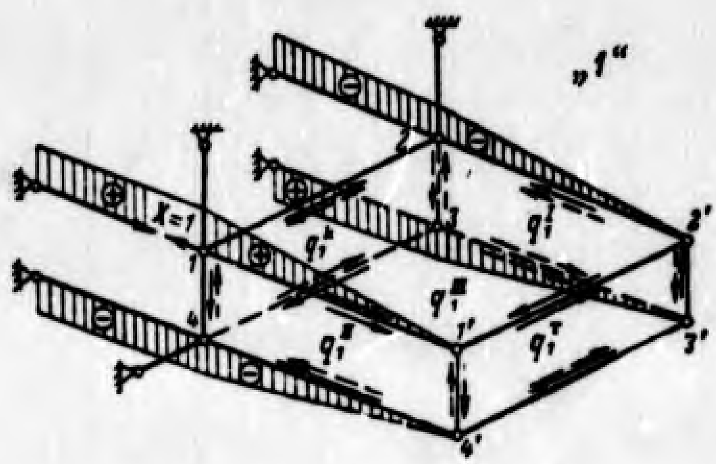


Fig. 3.9

We determine the unknown value of X from the canonical equation

$$X\delta_{11}+\Delta_{1P}=0. \qquad \text{(b)}$$

We determine the values of the coefficients Δ_{1P} and δ_{11} for absolutely rigid ribs and G = const:

$$\Delta_{1P}=\sum\frac{q_1 q_P F}{G\delta}+\sum\int_0^l\frac{N_1 N_P\,dl}{Ef}+\sum\frac{N_1 N_P l}{Ef}=$$

$$=-2\frac{1}{2l}\,\frac{P}{2H}\,\frac{Bl}{G\delta}+2\frac{1}{2l}\,\frac{P}{2H}\,\frac{Hl}{G\delta}=-\frac{P(B-H)}{2HG\delta}=-\frac{\mathfrak{M}\left(1-\frac{H}{B}\right)}{2HG\delta};$$

$$\delta_{11}=\sum\frac{q_1^2 F}{G\delta}+\sum\int\frac{N_1^2\,dl}{Ef}+\sum\frac{N_1^2 l}{Ef}=$$

$$=2\frac{1}{4l^2}\,\frac{Bl}{G\delta}+2\frac{1}{4l^2}\,\frac{Hl}{G\delta}+4\,\frac{1}{3}\,\frac{l}{Ef}+4\frac{l_\phi}{Ef}=$$

$$=\frac{1}{2lG\delta}(B+H)+\frac{4}{Ef}\left(\frac{l}{3}+l_\phi\right).$$

From Eq. (b), we obtain

$$X=\frac{\mathfrak{M}\left(1-\frac{H}{B}\right)}{H\left[\frac{B+H}{l}+\frac{8G\delta}{Ef}\left(\frac{l}{3}+l_\phi\right)\right]}.$$

As will be seen from this last expression, X = 0 when H = B.

To obtain the actual state, it is sufficient to multiply state "1" by X and add it to state "P." The actual state "D" is shown in Fig. 3.10, where

$$q^{*}=q^{III}=q^{IV}=q^{V}=\frac{\mathfrak{M}}{\Omega}-\frac{X}{2l}$$

and

$$q^{I}=q^{II}=\frac{\mathfrak{M}}{\Omega}+\frac{X}{2l}.$$

Let us determine the value of X for the following data: B = 100 cm; H = 20 cm; l_f = 50 cm; l = 100 cm; f = 5 cm^2, δ = 0.1 cm. The material is duralumin; E/G = 2.6; M = 300,000 kgf·cm.

$$X=\frac{3\cdot10^5\left(1-\frac{20}{100}\right)}{20\left[\frac{120}{100}+\frac{8\cdot0,1}{2,6\cdot5}\left(\frac{100}{3}+50\right)\right]}=\frac{0,8\cdot3\cdot10^5}{20\,(1,2+2,05+3,08)}=1900.$$

If the supporting rods are regarded as absolutely rigid,

$$X=\frac{0,8\cdot3\cdot10^5}{20\,(1\cdot2+2,05)}=3700.$$

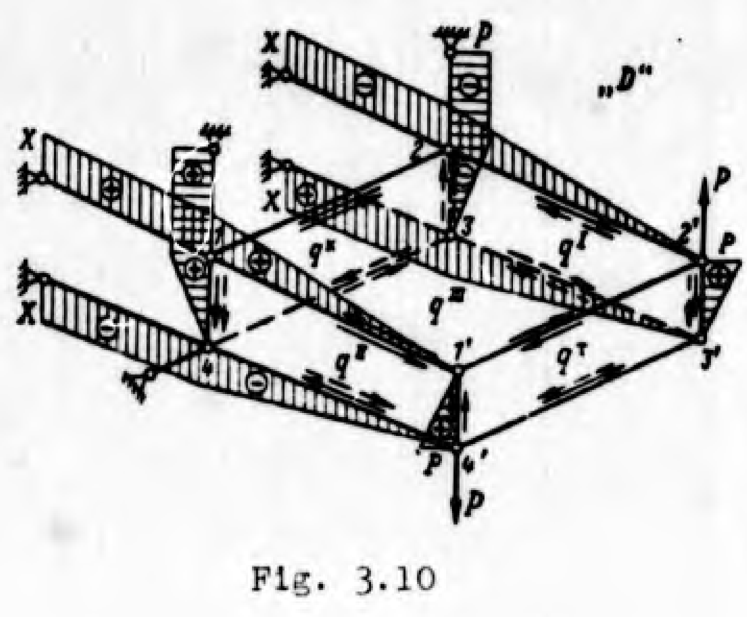


Fig. 3.10

Example 2.

Determine the internal forces in a bulkhead due to the action of a pair of forces P = 1000 kg applied to nodes 1 and 2. The working diagram of the bulkhead is shown in Fig. 3.11. The flux of tangential forces transmitted to the bulkhead by the fuselage skin is regarded as constant.

First we determine the PKS acting on the bulkhead from the fuselage skin.,

From the familiar formula for singly closed sections (Bredt formula),

$$q=\frac{\mathfrak{M}}{\Omega}.$$

Since

$$\mathfrak{M}=1000\cdot 25=25\,000\ \kappa\Gamma\cdot c\mu$$

and

$$\Omega=2(72\cdot 57-2\cdot 16\cdot 16)=7184\ c\mu^2,$$

$$q=\frac{25\,000}{7184}=3{,}48\ \kappa\Gamma/c\mu.$$

Due to the fact that the external loads are antisymmetrical with respect to the bulkhead's vertical axis of symmetry, the internal forces will also be antisymmetrical with respect to the same axis. On this basis, we conclude that the axial forces at the points of intersection of rods 1-2, 9-10, 12-11 and 6-5 with the bulkhead axis of symmetry have zero values. Thus, passing a section through the bulkhead along the axis of symmetry at the top or at the bottom (i.e., through panel 1-2-10-9 or 11-5-6-12), we obtain the basic system. As the superfluous unknown, we select the PKS $q_{1-2-10-9}$ in wall 1-2-10-9.

We determine the force factors in state "P" (Fig. 3.12). As was shown earlier, $N_{0-2} = 0$. It will be recalled that the first subscript indicates the point at which the force N is applied, while the second indicates the direction in which it acts. Taking point 0 of rod 1-2

($N_{0-2} = 0$) as the origin from which axial forces are reckoned, we go on to determine the forces in the rods entering nodes 2,10, 3, 4, 11 and 5.

Node 2.

$$N_{2-0}=3{,}48\cdot 12{,}5=43{,}5;$$

$$\sum M_{10}=-43{,}5\cdot 16+N_{2-3}\cdot 11{,}3=0;\ N_{2-3}=\frac{43{,}5\cdot 16}{11{,}3}=61{,}3;$$

$$\sum Y=-1000-61{,}3\cdot 0{,}707-N_{2-10}=0;$$

$$N_{2-10}=-1000-43{,}5=-1043{,}5.$$

Node 10.

$$N_{10-2}=-1043{,}5.$$

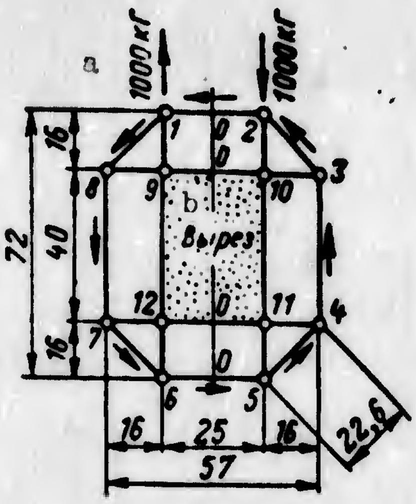


Fig. 3.11. a) 1000 kgf; b) cutout.

As was shown earlier, $N_{0-10} = 0$ and, consequently, we also have $N_{10-3} = 0$.

Node 3.

$$N_{3-2}=N_{2-3}+3{,}48\cdot 22{,}6=61{,}3+3{,}48\cdot 22{,}6=140;$$
$$N_{3-4}=140\cdot 0{,}707=99;$$
$$N_{3-10}=-140\cdot 0{,}707=-99.$$

We find the PKS of wall 10-3-4-11 from the equilibrium conditions for rod 10-3:

$$q_{10-3-4-11}=\frac{99}{16}=6{,}18\ \text{кГ/см}.$$

The direction in which this PKS acts on the rods is shown in Fig. 3.12.

After determining $q_{10-3-4-11}$, we may go on to determine the forces in the rods coming into nodes 4, 11 and 5.

The axial forces and PKS of state "P" are shown in Fig. 3.12.

The force factors in state "1" are even easier to determine (Fig. 3.13).

The unknown value of the PKS X_1 is determined from the canonical equation

$$X_1\delta_{11}+\Delta_{1P}=0.$$

Here X_1 is an unknown abstract number whose magnitude is equal to

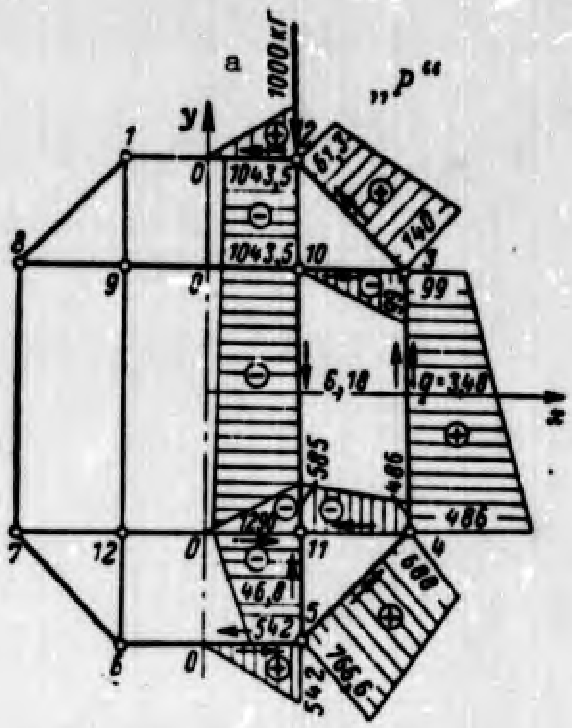


Fig. 3.12. a) 1000 kgf.

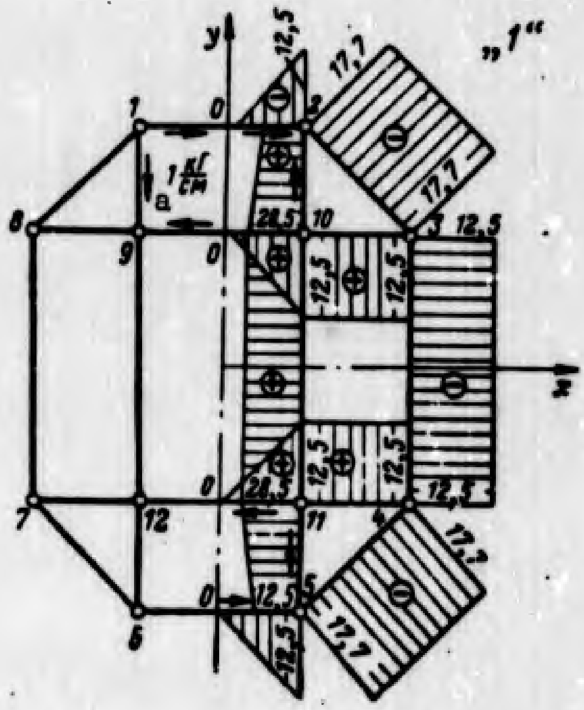


Fig. 3.13. a) 1 kgf/cm.

$q_{1\text{-}2\text{-}10\text{-}9}$.

The values of

$$\Delta_{1P}=\sum \frac{q_1 q_P F}{G\delta}+\sum\int_0^l \frac{N_1 N_P\,dx}{Ef}$$

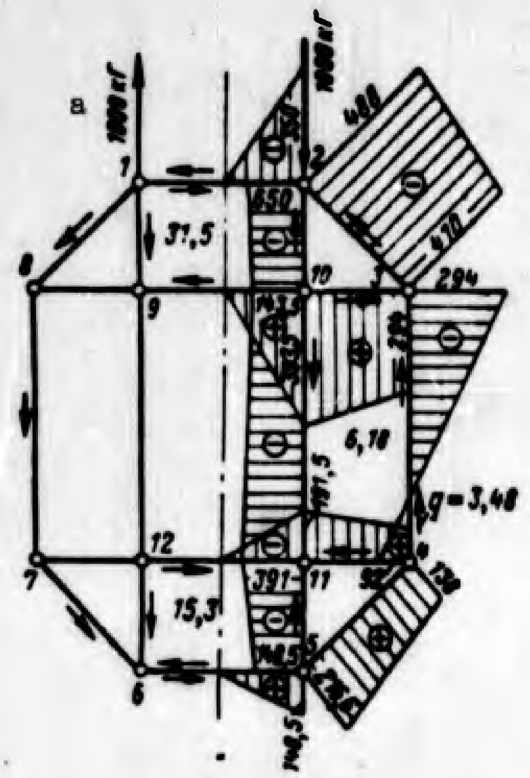


Fig. 3.14. a) 1000 kgf.

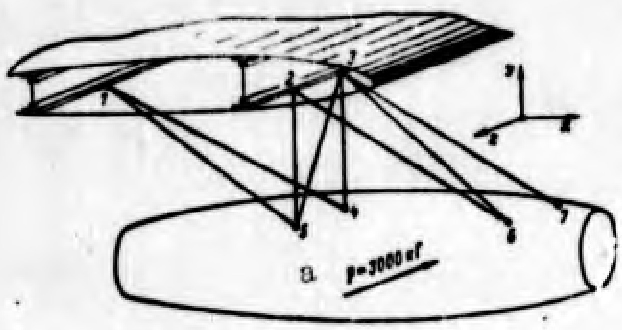


Fig. 3.15. a) P = 3000 kgf.

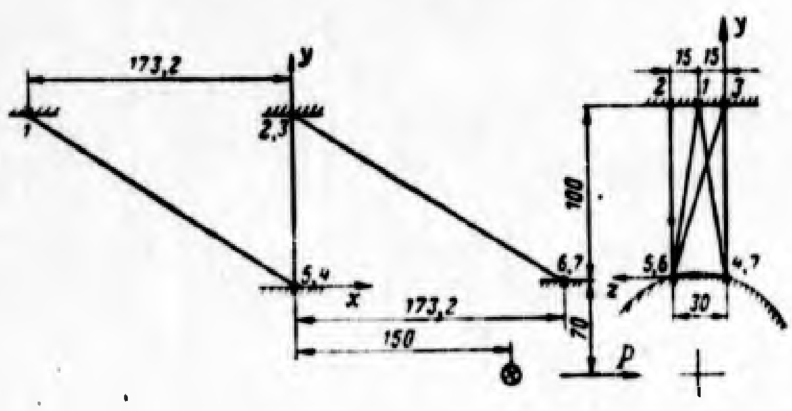


Fig. 3.16

and

$$\delta_{11}=\sum\frac{q_1^2F}{G\delta}+\sum\int_0^l\frac{N_1^2dx}{Ef}$$

are listed in Table 3.1.

From Table 3.1

$$E\Delta_{1P}=-1890{,}57;\; E\delta_{11}=60{,}08.$$

Consequently

$$X_1=\frac{1890{,}57}{60{,}08}=31{,}5.$$

Knowing X_1, we determine the actual values of N and q (see last column of Table 3.1) and construct diagrams of N (Fig. 3.14, which also indicates numerical values of q).

Example 3.

Determine the forces in the rods of an engine-nacelle pylon (Fig. 3.15) under a lateral load P = 3000 kgf. The cross-sectional areas of the duralumin rods are $f_{2-6} = f_{3-7} = 5\ cm^2$. The areas of the other rods are assumed equal to $3\ cm^2$. Figure 3.16 shows the projections of the eight rods suspending the pylon (dimensions in cm).

In determining the forces in the rods connecting the wing and the engine nacelle, both the wing and the gondola may be considered as absolutely rigid bodies.

According to Formula (1.11), the minimum necessary number of couplings is

$$C=6T-6=6\cdot2-6=6.$$

Hence the system is doubly statically indeterminate.

The basic system is taken with sections of rods 1-5 and 3-6 (Fig. 3.17).* The geometrical proportions are given in Table 3.2.

TABLE 3.1

A Стержни обшивки	B l см F см²	C f см² δ см	$\frac{l}{f}$ $2{,}6\frac{F}{\delta}$	D N_p кГ q_p кГ/см	E N_1 кГ q_1 кГ/см	$E\Delta_{1p}$	$E\delta_{11}$	F N_1X_1 кГ q_1X_1 кГ/см	G $N=N_p+$ $+N_1X_1$ кГ $q=q_p+$ $+q_1X_1$ кГ/см
0 2	$\frac{25}{2}$	1	12,5	0 +43,5	0 −12,5	−2,27·10³	0,65·10³	0 −393,5	0 −350
2 3	22,6	1	22,6	+61,3 +140	−17,5 −17,5	−39,8·10³	6,92·10³	−550 −550	−488 410
3 4	40	1	40	+99 +486	−12,5 −12,5	−146·10³	5,0·10³	−393,5 −393,5	−294,5 +92,5
4 5	22,6	1	22,6	+688 +766,6	−17,5 −17,5	−288·10³	6,92·10³	−550 −550	+138 +216,6
5 0	$\frac{25}{2}$	1	12,5	+542 0	−12,5 0	−282·10³	0,65·10³	−393,5 0	+148,5 0
5 11	16	2	8	−542 −1291	+12,5 +28,5	−158·10³	3,53·10³	+393,5 +900	−148,5 391
11 0	$\frac{25}{2}$	1	12,5	−585 0	+12,5 0	−30,4·10³	0,65·10³	+393,5 0	−191,5 0
11 4	16	1	16	−585 −486	+12,5 +12,5	107·10³	2,49·10³	+393,5 +393,5	−191,5 92,5
11 10	40	2	20	−1291 −1043,5	+28,5 +28,5	−666·10³	16,2·10³	+900 +900	−391 −143,5
10 0	$\frac{25}{2}$	1	12,5	0 0	+12,5 0	0	0,65·10³	+393,5 0	+393,5 0
10 3	16	1	16	0 −99	+12,5 +12,5	−9,9·10³	2,49·10³	+393,5 +393,5	+393,5 +294,5
10 2	16	2	8	−1043,5 −1043,5	+28,5 +12,5	−171·10³	3,53·10³	+900 +393,5	−143,5 650
0—2—10—0	12,5·16=200	0,1	5200	0	+1	0	5,2·10³	+31,5	+31,5
10—3—4—11	40·16=640	0,1	16640	+6,18	0	0	0	0	+6,18
0—11—5—0	12,5·16=200	0,1	5200	+46,8	−1	−244·10³	5,2·10³	31,5	+15,3
						−1890,57	60,08		

A) Rods, skin panels; B) ℓ in cm, F in cm^2; C) f in cm^2, δ in cm; D) N_P in kgf, q_P in kgf/cm; E) N_1 in kgf, q_1 in kg/cm; F) N_1X_1 in kgf, q_1X_1 in kgf/cm; G) $N = N_P + N_1X_1$ in kg, $q = q_P + q_1X_1$ in kgf/cm.

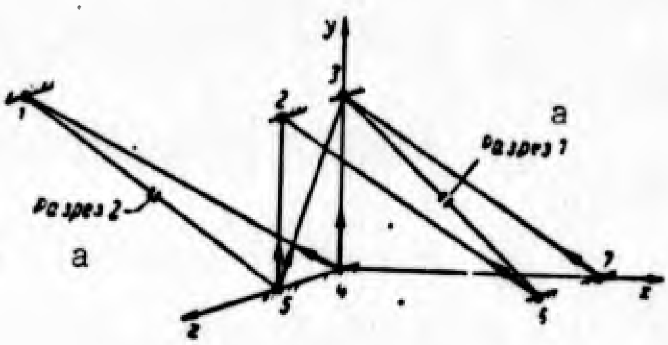


Fig. 3.17. a) Section.

TABLE 3.2

A Стержни	l	f	B Проекции стержней x	y	z	C Косинусы углов l/x	l/y	l/z
6—2	200	5,0	−1,73	1,0	0	−0,866	0,5	0
6—3	202	3,0	−1,73	1,0	−0,3	−0,856	0,495	−0,1485
7—3	200	5,0	−1,73	1,0	0	−0,866	0,5	0
4—3	100	3,0	0	1,0	0	0	1,0	0
4—1	200,5	3,0	−1,73	1,0	+0,15	−0,864	0,5	0,075
5—3	104,4	3,0	0	1,0	−0,3	0	0,96	−0,287
5—2	100	3,0	0	1,0	0	0	1,0	0
5—1	200,5	3,0	−1,73	1,0	−0,15	−0,864	0,5	−0,075

A) Rod; B) rod projections; C) cosines of angles.

Determination of Forces N_P in Rods of Basic System Under an External Load P = 3000 kgf

We pass a section through each rod and replace the action of the rods on the pylon by tensile forces (see Fig. 3.17). Then we determine these forces from the equilibrium conditions for the pylon in space as a solid body.

1) From the condition $\sum M_{2-3}=0$ we obtain $N_{4-1} = 0$.

2) $\sum Z = N_{5-3}(-0{,}287) - 3000 = 0,$ whence

$$N_{5-3} = -10\,450\ \text{кГ}.$$

3) $\sum M_{3-4} = N_{6-2}0{,}866\cdot 30 - 3000\cdot 150 = 0,$ whence

$$N_{6-2} = +17\,300\ \text{кГ}.$$

4) $\sum M_{2-5} = -N_{7-3}0{,}866\cdot 30 - 3000\cdot 150 = 0,$ whence

$$N_{7-3} = -17\,300\ \text{кГ}.$$

$$5)\ \sum M_{4-7}=N_{5-2}30+N_{6-2}0{,}5\cdot 30+$$
$$+N_{5-3}0{,}96\cdot 30-3000\cdot 70=0, \text{ whence } N_{5-2}=8330\,\kappa\Gamma.$$
$$6)\ \sum Y=N_{5-2}+N_{4-3}+N_{5-3}0{,}96+$$
$$+N_{6-2}0{,}5+N_{7-3}0{,}5=0, \text{ whence } N_{4-3}=1670\,\kappa\Gamma.$$

Determination of Forces N_1

We apply a unit load $X_1 = 1$ kgf in section 1 (see Fig. 3.17) and determine the forces in the rods of the basic system from this unit load.

From the static equations, which we also used in determining N_p, we obtain

$$N_{4-1}=0;\quad N_{5-3}=-0{,}517;\quad N_{6-2}=0;$$
$$N_{7-3}=-0{,}988;\quad N_{5-2}=0;\quad N_{4-3}=0{,}495;\quad N_{6-3}=1.$$

Determination of Forces N_2

We apply a unit load $X_2 = 1$ kgf in section 2 (see Fig. 3.17) and determine the forces in the rods of the basic system due to this load. In this case, the forces in the rods are

$$N_{4-1}=-1;\ N_{5-3}=-0{,}522;\ N_{6-2}=-0{,}998;$$
$$N_{7-3}=0{,}998;\ N_{5-2}=0{,}5;\ N_{4-3}=0{,}02;\ N_{5-1}=1.$$

We determine the unknowns X_1 and X_2 from the canonical equations:

$$X_1\delta_{11}+X_2\delta_{12}+\Delta_{1P}=0; \qquad \text{(a)}$$
$$X_1\delta_{21}+X_2\delta_{22}+\Delta_{2P}=0. \qquad \text{(b)}$$

Values of the displacements δ and Δ are listed in Table 3.3.

We find from Table 3.3

$$\sum\delta_{11}=124{,}0;\quad \sum\delta_{12}=E\delta_{21}=-29{,}8;\quad \sum\delta_{22}=231{,}0;$$
$$\sum\Delta_{1P}=898{,}5\cdot 10^3;\quad E\Delta_{2P}=-1051\cdot 10^3.$$

From simultaneous solution of Eqs. (a) and (b), we find

$$X_1=-6360;$$
$$X_2=3370.$$

The actual forces in the rods

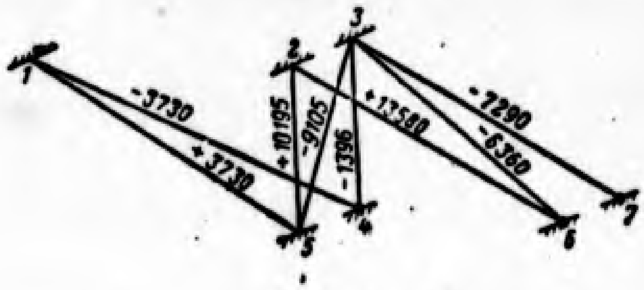


Fig. 3.18

TABLE 3.3

Стержни (A)	l, см (B)	f, см	l/f	N_P	N_1	N_2	$E\delta_{11}$	$E\delta_{12}$	$E\delta_{22}$
6—2	200	5	40	+17 300	0	−0,998	0	0	+39,8
6—3	202	3	67,4	0	+1,0	0	67,4	0	0
7—3	200	5	40	−17 300	−0,988	+0,998	39,0	−39,5	+39,8
4—3	100	3	33,3	+1670	+0,495	+0,02	8,15	+0,33	+0,01
4—1	200,5	3	66,8	0	0	−1,0	0	0	+66,8
5—3	104,4	3	34,8	−10 450	−0,517	−0,522	9,3	+9,4	+9,45
5—2	100	3	33,3	+8330	0	+0,5	0	0	+8,4
5—1	200,5	3	66,8	0	0	+1,0	0	0	+66,8
							+124,0	−29,8	+231,0

Стержни (A)	$E\Delta_{1P}$	$E\Delta_{2P}$	N_1X_1	N_2X_2	N	Контроль (C): $N_1N\, l/f$	Контроль (C): $N_2N\, l/f$
6—2	0	$-690\cdot10^{8}$	0	−3720	+13580	0	$-540\cdot10^{8}$
6—3	0	0	−6360	0	−6360	$-430\cdot10^{3}$	0
7—3	$+683\cdot10^{3}$	$-690\cdot10^{8}$	+6290	+3720	−7290	$+289\cdot10^{3}$	$-290\cdot10^{3}$
4—3	$+27{,}5\cdot10^{3}$	+110,0	−3140	+74,5	−1396	$-23\cdot10^{3}$	−1000
4—1	0	0	0	−3730	−3730	0	$+249\cdot10^{3}$
5—3	$+188\cdot10^{3}$	$+189{,}5\cdot10^{3}$	+3290	−1945	−9105	$+163{,}5\cdot10^{3}$	$+165{,}5\cdot10^{3}$
5—2	0	$+139\cdot10^{3}$	0	+1866	+10195	0	$+167\cdot10^{3}$
5—1	0	0	0	+3730	+3730	0	$+249\cdot10^{3}$
	$+898{,}5\cdot10^{3}$	$-1051\cdot10^{3}$				$-453\cdot10^{3}$ $+452{,}5\cdot10^{3}$	$-831\cdot10^{3}$ $+830{,}5\cdot10^{3}$

A) Rod; B) cm; C) check.

$$N = N_P + N_1X_1 + N_2X_2$$

are listed in Table 3.3 and indicated on Fig. 3.18.

A check based on the conditions of zero gaps in sections 1 and 2 indicates that the forces in the rods have been determined correctly (see last two columns at the bottom right of Table 3.3).

5. DETERMINING DISPLACEMENTS OF STATICALLY INDETERMINATE SYSTEMS

Formula (2.4) may be used to determine the displacements of statically indeterminate systems, only in this case q_P, N_P and q_1, N_1 relate to a given statically indeterminate system. However, since q_1 and N_1 pertain to a statically indeterminate system, determination of a displacement requires solving the staticall indeterminate problem of a unit force applied in the position and direction of the displacement sought.

We may satisfy ourselves by simple reasoning that in determining the displacements of statically indeterminate systems by Formula (2.4), we shall obtain the correct result even if we substitute values of q_1 and N_1 that pertain not to the statically indeterminate system, but to the basic statically determinate system into this formula. Indeed, as soon as the forces due to the external loads in all elements of the statically indeterminate system have been determined, the forces in the excessive couplings may be included among the external forces and the given system may then be regarded as a statically determinate structure. Since there are no external forces in the absence of the unit force (this also applies to sections through the superfluous couplings), the force factors q_1 and N_1 will belong to the statically determinate system.

Thus, to determine the displacements in a given statically indeterminate system, we first evaluate the static indeterminacy and find the unknown force factors. Then a force $P_k = 1$ is applied to the basic statically indeterminate system in the position and direction in which it is necessary to find the displacement. Then, substituting the values of q and N from the solution of the statically indeterminate system and the values of q_1^0 and N_1^0 from the solution of the basic statically determinate system into Eq. (2.4), we find the magnitude of the displacement.

Hence the following formula may be used to determine the displacements in statically indeterminate structures:

$$\Delta_{1P}=\sum\frac{q_1^0 qF}{G\delta}+\sum\int_0^l\frac{N_1^0 N\,dl}{Ef}+\sum\frac{N_1^0 Nl}{Ef}. \qquad (3.8)$$

Example.

Determine the displacement of the point of application of the force P (see Fig. 3.15) in the direction of this force.

Applying the unit force in the direction and position of application of force P, we obtain the forces in the rods:

$$N_1^0=\frac{N_P}{3000}.$$

Applying Formula (3.8) and Table 3.3, we obtain the unknown displacement:

$$f=\sum\frac{N_1^0 Nl}{Ef}=\frac{1}{3000\,E}\left(\frac{17\,300\cdot 13\,560\cdot 200}{5}-\right.$$
$$-\frac{17\,300\cdot 3720\cdot 200}{5}-\frac{1670\cdot 1396\cdot 100}{3}+$$
$$\left.+\frac{10\,450\cdot 9105\cdot 104,4}{3}+\frac{8330\cdot 10\,195\cdot 100}{3}\right)=$$
$$=\frac{128,7\cdot 10^8}{3000\cdot 7\cdot 2\cdot 10^6}=0,6 \text{ cm}.$$

Manuscript Page No. [Footnotes]

48 Determination of the degree of static indeterminacy for a large number of couplings in aircraft structures is considered in [29].

61 In calculations for statically indeterminate systems, a consistent basic system is usually selected for various design cases. We have also adopted such a basic system, since in determining the forces in the rods of the pylon, we have provided a series of theoretical cases of loading for the engine nacelle. If, on the other hand, we orient ourselves only to the particular case of a lateral load, the force at node 1

acting in the direction of the z-axis might be taken as the unknown.

Manuscript Page No.	[Transliterated Symbols]
48	C.H. = S.N. = staticheski neopredelennyy = statically indeterminate
49	C.O. = S.O. = staticheski opredelennyy = statically determinate
53	Ф = f = fyuzelyazh = fuselage
54	ПКС = PKS = potok kasatel'nykh sil = tangential-force flux
54	т = t = tortsevoy = butt
54	к = k = kornevoy = root

Part Two

VARIATIONAL METHODS OF STRUCTURAL MECHANICS

Methods based on the fact that for any possible deflection from an equilibrium position, the variation of the system's total energy is equal to zero are used effectively for solving problems in structural mechanics.

The value of the potential energy can be expressed either in displacements or in stresses. In the former case, the variation of the system's potential energy gives equations of equilibrium, and in the latter case, equations of consistency.

These equations may be either algebraic, or ordinary differential, or partial differential, depending on whether the potential energy is a function of unknown parameters or a functional of unknown functions that depend on one or more coordinates.

However, the principle of virtual displacements can also be used directly to solve problems in structural mechanics; here, if this principle is applied to a strip of infinitesimally small length, we naturally obtain differential equations [4], while if the strip has finite length we obtain equations in finite differences.

If integration of the differential equations encounters great mathematical difficulty, we may limit ourselves to an approximate solution instead of solving them exactly, approximating the unknown function in the form of a series with the unknown parameters. Variation of the parameters yields a system of consistent algebraic equations.

Another way of approximate solution consists in substituting the approximating series directly into the potential-energy expression and minimizing it to obtain a system of equations from which the unknowns may be determined.

For approximate solution of the two dimensional problem, the solution of the unknown function is represented in the form of a series in which the functions are assigned in one direction and unknowns in the other. Thus, the two dimensional problem is reduced to a one dimensional problem.

The questions touched upon here will be treated in greater detail in the following chapters of the present division.

Chapter 4

ENERGY PRINCIPLES.

EULER EQUATIONS OF THE VARIATIONAL PROBLEM

1. BASIC EQUATIONS OF ELASTICITY THEORY

Before turning our attention to the variational principles, let us recall the basic formulas of elasticity theory [24].

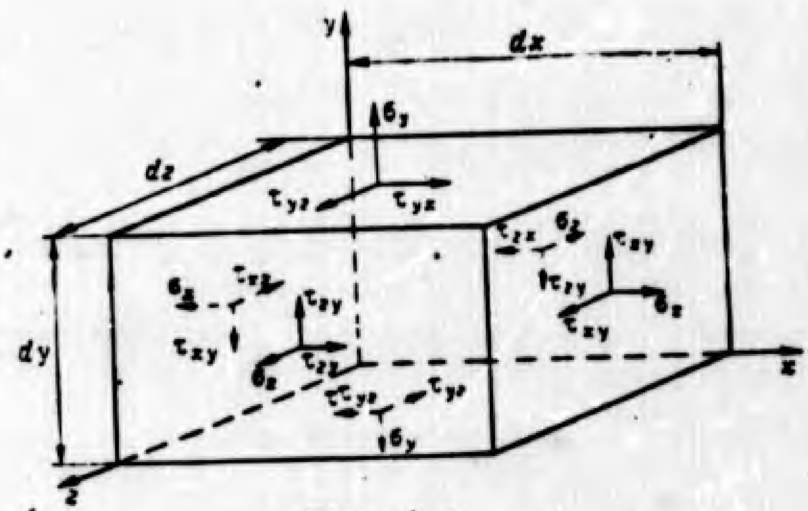


Fig. 4.1

1. Stress components. Let us imagine an infinitesimally small rectangular parallelepiped of volume $dV = dx\, dy\, dz$ to have been cut out of the body. We replace the action of the removed parts of the body on each face by normal stresses σ and tangential stresses τ. To indicate the direction in which and the parallelepiped face on which the stresses act, a single subscript indicating the direction of the normal to the face on which they act is sufficient for normal stresses. Two subscripts are necessary for tangential stresses: the first indicates the direction of the normal to the face on which the tangential stresses act and the second indicates the axis parallel to which these stresses are

pointed. In accordance with this rule, Fig. 4.1 shows all stresses acting on the isolated parallelepiped.

To represent the stresses acting on all six faces of the parallelepiped, it would be necessary to have three symbols for σ and six for τ. However, bearing in mind the law of pairs for tangential stresses, we can obtain only six independent stress components at a given point:

$$\sigma_x,\ \sigma_y,\ \sigma_z;\ \tau_{xy}=\tau_{yx};\ \tau_{yz}=\tau_{zy};\ \tau_{zx}=\tau_{xz}.$$

2. Differential equations of internal equilibrium. Before turning to the differential equations of equilibrium, let us recall that it is customary in elasticity theory to distinguish between two types of external forces: surface forces and volume forces.* The components of the surface force referred to a unit area will be denoted by $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, while the components of the volume force, which are referred to the unit mass, will be denoted by X, Y and Z.

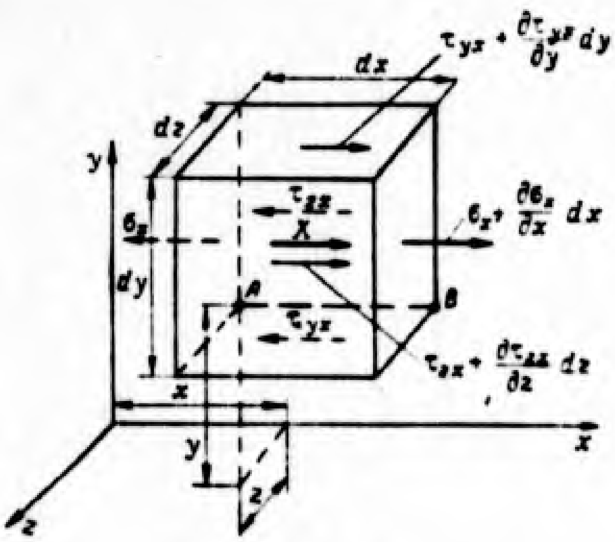


Fig. 4.2

Since the element dx dy dz is inside the body, it is subject to the volume forces X, Y and Z and stress components which, as continuous functions of the coordinates x, y and z, have the corresponding increments along these axes.

Concentrating our attention on the forces acting along the x-axis

(Fig. 4.2), we obtain

$$\left(\sigma_x-\frac{\partial\sigma_x}{\partial x}dx\right)dy\,dz-\sigma_x dy\,dz+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}dy\right)dx\,dz-\tau_{yx}dx\,dz+$$
$$+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\right)dx\,dy-\tau_{zx}dx\,dy+X dx\,dy\,dz=0.$$

After rearranging and canceling the dx dy dz, we obtain the first of the following three equations (the other two are found in the same way on projection of the forces onto the directions of the y- and z-axes):

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+X=0,\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Y=0,\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0.\end{aligned}\right\} \qquad (4.1)$$

Equations (4.1) are the basic differential equations of internal equilibrium in cartesian coordinates.

3. Equilibrium conditions on the surface of the body. If the stress components $\sigma_x, \sigma_y, \sigma_z; \tau_{xy}=\tau_{yx}; \tau_{yz}=\tau_{zy}; \tau_{zx}=\tau_{xz}$ are known for a given point, the stress components at this point may be calculated on an arbitrary area with a normal N. These stress components along the x, y and z axes are equal respectively to

$$\left.\begin{aligned}\bar{x}=l\sigma_x+m\tau_{yx}+n\tau_{zx},\\ \bar{y}=l\tau_{xy}+m\sigma_y+n\tau_{zy},\\ \bar{z}=l\tau_{xz}+m\tau_{yz}+n\sigma_z,\end{aligned}\right\} \qquad (4.2)$$

where $l=\cos(N, x)$; $m=\cos(N, y)$; $n=\cos(N, z)$.

The stress components vary throughout the volume of the body and must be compensated on its external surface by forces $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ applied to the surface.

Hence on the basis of Formulas (4.2), the equilibrium conditions on the surface of the body will take the form

$$\left.\begin{aligned}\bar{X}=l\sigma_x+m\tau_{yx}+n\tau_{zx},\\ \bar{Y}=l\tau_{xy}+m\sigma_y+n\tau_{zy},\\ \bar{Z}=l\tau_{xz}+m\tau_{yz}+n\sigma_z.\end{aligned}\right\} \qquad (4.3)$$

4. Strain components. If a body is subject to strain and the displacements u, v and w (parallel, respectively, to the coordinates x, y and z) are the components of the total displacement of point A (see Fig. 4.2) and continuous functions of the coordinates, then a segment AB of length dx will acquire an increment $\frac{\partial u}{\partial x}dx$ and, consequently, a relative elongation in the direction of the x-axis.

$$\varepsilon_x=\frac{\partial u}{\partial x}.$$

The formulas for ε_y and ε_z are analogous.

Prior to deformation, the two segments dx and dy were at a right angle to one another. After deformation, this angle has changed by an amount γ_{xy}. Since the end point of segment dx will have moved in the direction of the y-axis by

$$\frac{\partial v}{\partial x}dx,$$

and the end point of segment dy will have moved parallel to the x-axis by

$$\frac{\partial u}{\partial y}dy,$$

we obtain on the whole

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}.$$

the formulas are analogous for γ_{yz} and γ_{zx}.

Thus, the strain components $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}$ and γ_{zx} are expressed as follows in terms of the displacements:

$$\left.\begin{array}{l}\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_z=\frac{\partial w}{\partial z},\\ \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}.\end{array}\right\} \qquad (4.4)$$

5. Strain consistency equations. The strain and displacement components, which are related to one another by Eqs. (4.4), are functions of the coordinates x, y and z. It follows from this that the strains are interdependent as functions of the coordinates. To ascertain this relationship, it is necessary to eliminate u, v and w from Eqs. (4.4).

Following this elimination, we obtain the following six strain consistency equations:

$$
\left.
\begin{aligned}
&\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\,\partial y}; \quad 2\frac{\partial^2\varepsilon_x}{\partial y\,\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right),\\
&\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\,\partial z}; \quad 2\frac{\partial^2\varepsilon_y}{\partial z\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right),\\
&\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{zx}}{\partial z\,\partial x}; \quad 2\frac{\partial^2\varepsilon_z}{\partial x\,\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right).
\end{aligned}
\right\} \quad (4.5)
$$

If the strain components satisfy Eqs. (4.5), the displacements u, v and w, which are related to the strains by Eqs. (4.4), will be single-values continuous functions of the coordinates x, y and z.

6. Relationships between stresses and strains. On the basis of Hook's law the following relationships obtain between strains and stresses:

$$
\left.
\begin{aligned}
&\varepsilon_x=\frac{1}{E}[\sigma_x-\mu(\sigma_y+\sigma_z)], \quad \gamma_{xy}=\frac{\tau_{xy}}{G},\\
&\varepsilon_y=\frac{1}{E}[\sigma_y-\mu(\sigma_z+\sigma_x)], \quad \gamma_{yz}=\frac{\tau_{yz}}{G},\\
&\varepsilon_z=\frac{1}{E}[\sigma_z-\mu(\sigma_x+\sigma_y)], \quad \gamma_{zx}=\frac{\tau_{zx}}{G}
\end{aligned}
\right| \quad (4.6)
$$

and the inverse relationships

$$
\left.
\begin{aligned}
&\sigma_x=\lambda\Delta+2G\varepsilon_x, \quad \tau_{xy}=\gamma_{xy}G,\\
&\sigma_y=\lambda\Delta+2G\varepsilon_y, \quad \tau_{yz}=\gamma_{yz}G,\\
&\sigma_z=\lambda\Delta+2G\varepsilon_z, \quad \tau_{zx}=\gamma_{zx}G,
\end{aligned}
\right\} \quad (4.7)
$$

where

$$\lambda=\frac{\mu E}{(1+\mu)(1-2\mu)};$$

$$\Delta=\varepsilon_x+\varepsilon_y+\varepsilon_z;$$

E is Young's modulus, $G=\frac{E}{2(1+\mu)}$ is the shear modulus and μ is Poisson's ratio.

7. General remarks on the basic equations of elasticity theory. In solving problems of elasticity theory in stresses or displacements, the necessary conditions for continuity of the material will be ensured only provided that the displacements u, v and w are continuous functions of the coordinates.

In solving elasticity-theory problems in stresses, this condition will be satisfied if the strain consistency equations are satisfied. For this purpose, to determine the six unknown stress components, the six equations of the system (4.5) are transformed by application of Eqs. (1.1) and (4.6) into differential relationships between only the stress components and the volume force. Together with the boundary conditions (4.3), the resulting six second-order partial differential equations fully determine the six stress components and continuity of the displacements u, v and w.

The equilibrium conditions (4.1) must be satisfied in solving problems in displacements. For this purpose, Relationships (4.7) are substituted in Eqs. (4.1) and then the strain components are replaced by their expressions (4.4). As a result, we obtain three differential equations with three unknown functions of the displacements u, v and w. These equations, together with the boundary conditions (4.3), expressed in terms of u, v and w, fully determine the displacements u, v and w. Here the consistency equations will be satisfied automatically, since solution of the differential equations in the displacements ensures single-valuedness and continuity of the displacements u, v and w.

Thus, elasticity theory yields a number of equations equal to the number of unknowns. However, integration of the system of partial differential equations encounters great mathematical difficulty and is feasible only in simple cases. In determining the stressed and strained state of engineering structures, therefore, extensive use is made of approximate methods that are based on elasticity theory but embody assumptions that simplify the partial differential equation system appreciably, reducing it to a system of ordinary differential equations or even to a system of algebraic equations.

2. PRINCIPLE OF POSSIBLE DISPLACEMENTS

Let us dwell on the principle of possible displacements in greater detail than was the case in Section 5 of Chapter 2.

As we know from theoretical mechanics, this principle, set forth for a material point, consists in the fact that if a point is at equilibrium, the sum of the works done by all forces acting on the point in accomplishing any small real or nearly imagined kinematically possible displacement is equal to zero.

The remark concerning the smallness of the displacements is accounted for by the fact that when the principle of possible displacements is employed, the external forces acting on the point are regarded as constant in both magnitude and direction. In general, however, this assumption is justified only when the displacements are assumed to be so small that the forces cannot vary during the time in which they operate to produce the corresponding displacements.

Since a free point may move unimpeded in any direction, all displacements are possible for it. If, however, the motion of the point is subject to certain conditions that restrict its motion in some direction, the possible displacements in this case are only those that are consistent with the assigned conditions. For example, if the material point M (Fig. 4.3) can move only on the plane curve AB because of the couplings imposed upon it, it is this curve that will describe the real displacements of point M. The possible displacement of the point M, however, is an infinitesimally small displacement δs along the tangent $\underline{t}$. Hence, according to the principle of possible displacements, verifying whether point M is at equilibrium requires that all forces P_i acting upon it be projected onto the tangent $\underline{t}$; then the resulting projections of the forces $P_i \cos(P_i, t)$ are multiplied by δs and added. If this sum is equal to zero, i.e.,

$$\sum_{i=1}^{n} P_i \cos(P_i, t)\delta s = 0, \qquad (a)$$

then point M is at equilibrium.

Bearing in mind that $\delta s \neq 0$, we obtain from Eq. (a) the familiar equilibrium equation for forces applied at a point:

$$\sum_{i=1}^{n} P_i \cos(P_i, t) = 0. \qquad (b)$$

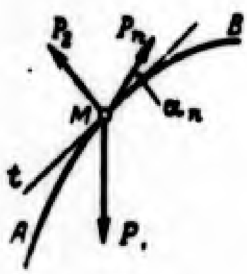


Fig. 4.3

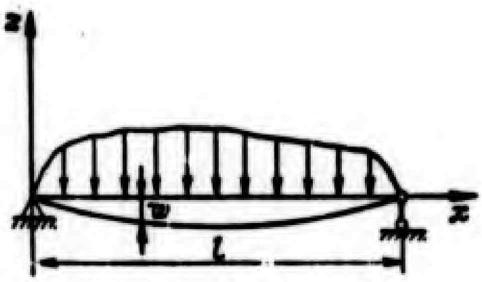


Fig. 4.4

Now let us turn to the elastic body. Let us assume that the particles of the elastic body completely fill its volume. This means that possible displacements within the body must be consistent with continuity of the material, while those at its surface must be consistent with the assigned boundary conditions. Thus, in the case of bending of a freely supported beam (Fig. 4.4), the boundary conditions require that the transverse displacements be equal to zero at both ends of the beam; in this case, therefore, the possible displacements must be taken equal to zero. The condition regarding continuity of the material will also be satisfied if the possible displacements are expressed by means of continuous functions. The selection of possible displacements is not restricted in any other respect, so that there is a broad range of options regarding these displacements. Consequently, returning to the example of bending of a freely supported beam, we may state that the variation δw (possible infinitesimally small change in the real displacement w) has an arbitrary value as x varies from $x = 0$ to $x = \ell$, and is determinate and equal to $\delta w = 0$ at points $x = 0$ and $x = \ell$. Thus, where the displacements are assigned, their variations are equal to zero.

Suppose that the body is in a state of rest under the action of external forces. If from this true state the elastic body is brought into a neighboring infinitesimally nearby state by means of possible displacements, the internal and external forces will perform a work that is equal to zero on the basis of the principle of possible displacements.

Let us denote the work of the internal forces of the entire elastic body, that performed by these forces in accomplishing the possible displacements, by δA, and the work of the external forces by δW. Then, on the basis of the principle of possible displacements

$$\delta A + \delta W = 0. \tag{4.8}$$

Equation (4.8) expresses the Lagrange principle of possible displacements.

Thus, if the possible displacements are continuous functions of the coordinates and satisfy the boundary conditions, satisfaction of Eq. (4.8) will be a necessary and sufficient criterion for equilibrium of the elastic body.

3. POTENTIAL ENERGY OF DEFORMATION

As we know, the potential energy of deformation is the energy stored within an elastic body as a result of its deformation under a static load.

To determine the expression for the potential energy of deformation, we isolate an elementary parallelepiped of volume $dV = dx\,dy\,dz$ (Fig. 4.5) from the body and impart, for example to the displacement u, an increment δu. Then one of the faces $dy\,dz$ will be displaced by δu, while the other face $dy\,dz$ which is at a distance dx from the former, will be displaced by $\delta u + \frac{\partial \delta u}{\partial x} dx$. Forces $\sigma_x\,dy\,dz$ which are external for the isolated element, correspond to these displacements. Consequently, denoting by U the potential energy of deformation, we obtain

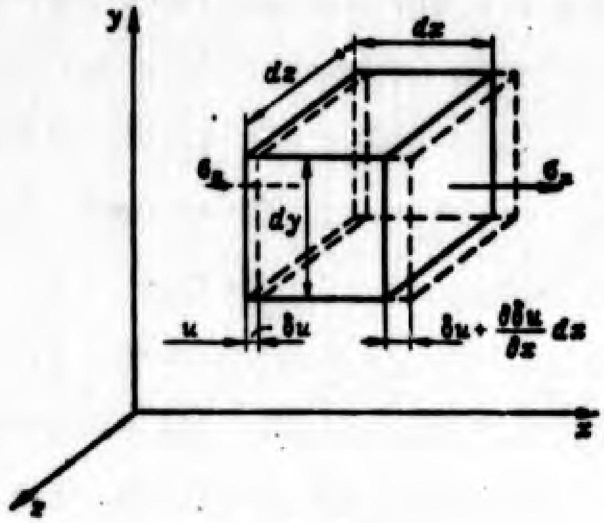


Fig. 4.5

$$\delta(dU_x)=-\sigma_x\,dy\,dz\,\delta u+\sigma_x\,dy\,dz\left(\delta u+\frac{\partial\delta u}{\partial x}dx\right)$$

or

$$\delta(dU_x)=\sigma_x\frac{\partial\delta u}{\partial x}\,dx\,dy\,dz.$$

Let $U_0=\frac{dU}{dx\,dy\,dz}$ denote the potential energy od deformation related to the unit of volume. Then

$$\delta U_{0x}=\sigma_x\frac{\partial\delta u}{\partial x}$$

or

$$\delta U_{0x}=\sigma_x\delta\varepsilon_x. \qquad (4.9)$$

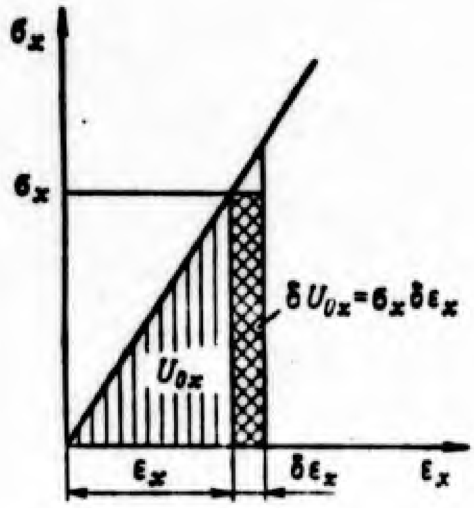


Fig. 4.6

For the linear deformable body in Fig. 4.6, the value of δU_{0x} is represented in the form of a rectangle accurate to quantities of high order of smallness.

The work of the forces from all stress components in accomplishing the possible displacements $\delta\varepsilon_x$, $\delta\varepsilon_y$, $\delta\varepsilon_z$, $\delta\gamma_{xy}$, $\delta\gamma_{yz}$, $\delta\gamma_{zx}$, referred to the volume unit, is equal to the total variation of the specific potential energy of deformation:

$$\delta U_0=\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\sigma_z\delta\varepsilon_z+ +\tau_{xy}\delta\gamma_{xy}+\tau_{yz}\delta\gamma_{yz}+\tau_{zx}\delta\gamma_{zx}. \qquad (4.10)$$

Since the stress components are considered to be constant during variation of the deformations, the total differential of the specific potential energy as a function of the strain components will be

$$\delta U_0=\frac{\partial U_0}{\partial\varepsilon_x}\delta\varepsilon_x+\frac{\partial U_0}{\partial\varepsilon_y}\delta\varepsilon_y+ +\frac{\partial U_0}{\partial\varepsilon_z}\delta\varepsilon_z+\frac{\partial U_0}{\partial\gamma_{xy}}\delta\gamma_{xy}+ +\frac{\partial U_0}{\partial\gamma_{yz}}\delta\gamma_{yz}+\frac{\partial U_0}{\partial\gamma_{zx}}\delta\gamma_{zx}. \qquad (4.11)$$

On comparing this equation with Eq. (4.10), we note that the partial derivative of specific potential energy with respect to the strain component is equal to the corresponding stress component:

$$\begin{aligned}&\frac{\partial U_0}{\partial \varepsilon_x}=\sigma_x;\quad \frac{\partial U_0}{\partial \varepsilon_y}=\sigma_y,\quad \frac{\partial U_0}{\partial \varepsilon_z}=\sigma_z;\\ &\frac{\partial U_0}{\partial \gamma_{xy}}=\tau_{xy}.\quad \frac{\partial U_0}{\partial \gamma_{yz}}=\tau_{yz};\quad \frac{\partial U_0}{\partial \gamma_{zx}}=\tau_{zx}.\end{aligned}\tag{4.12}$$

The variation of the energy of deformation for the entire volume of the body

$$\delta U=\int_V \delta U_0 dV.\tag{4.13}$$

On the basis of Formula (4.9) and Hooke's law $\sigma_x = \varepsilon_x E$, we obtain

$$U_{0x}=\int_0^{\varepsilon_x}\varepsilon_x E\delta\varepsilon_x.$$

The sign δ applied to the possible change has the same significance in integration and differentiation as the sign d. Therefore,

$$U_{0x}=\frac{E\varepsilon_x^2}{2},$$

or

$$U_{0x}=\frac{\sigma_x\varepsilon_x}{2}.$$

Hence the energy U_{0x} is equal to the area of the triangle shown in Fig. 4.6.

Taking account of the work of all stress components, we find

$$U_0=\frac{1}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}).\tag{4.14}$$

The potential energy of deformation of the entire elastic body is obtained by integrating $U_0 dV$ over the entire volume V:

$$\begin{aligned}U&=\int_V U_0 d\cdot V=\\ &=\frac{1}{2}\int_V(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx})dV.\end{aligned}\tag{4.15}$$

Substituting the values of the stress components from (4.7) into this expression, we obtain, after rearrangement, U_0 as a function only of the strain components:

$$U_0 = \frac{1}{2}\lambda\Delta^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2). \qquad (4.16)$$

To present U_0 as a function only of the stress components, we introduce the values of the deformation components from (4.6) into (4.15). Then we obtain

$$\overline{U}_0 = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \\ + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2). \qquad (4.17)$$

4. LAGRANGE VARIATIONAL EQUATION - PRINCIPLE OF POTENTIAL ENERGY

To obtain the Lagrange variational equation, it is necessary to find the expressions for the terms appearing in the equation of the possible-displacements principle:

$$\delta A + \delta W = 0.$$

It became clear in the foregoing Section 3 that δU is the total work of the forces of interaction between elementary volumes of the elastic body. These forces, being external with respect to each elementary volume, accomplish deformation. During the deformation, the external forces are at all times compensated by internal forces, but the directions of the internal forces are opposite to those of the deformation taking place. This means that the work of the internal forces δA is equal in magnitude and opposite in sign to δU, i.e.,

$$\delta A = -\delta U. \qquad (4.18)$$

The variation of the work of the external forces

$$\delta W = \int_V (X\delta u + Y\delta v + Z\delta w)\,dV + \\ + \int_F (\overline{X}\delta u + \overline{Y}\delta v + \overline{Z}\delta w)\,dF. \qquad (4.19)$$

Since the variations δu, δv, δw are equal to zero on those parts of the surface where the displacements are assigned, integration extends only over those parts of the surface on which the surface forces are assigned.

Substituting the expressions for δA and δW in Eq. (4.8), we obtain

$$\int_V (X\delta u+Y\delta v+Z\delta w)dV+$$
$$+\int_F (\bar{X}\delta u+\bar{Y}\delta v+\bar{Z}\delta w)dF-\delta U=0. \qquad (4.20)$$

This is the Lagrange variational equation, which expresses the equilibrium condition of the elastic body.

Let us show that the equilibrium equations of an element isolated from the body and the static boundary conditions on most parts of the surface where displacements are not assigned (see [13]) can be obtained by using the Lagrange variational equation.

In order fully to clarify the content of the Lagrange variational equation (4.20), it must be rearranged to some degree.

Since the problem is being solved for assigned external forces and, consequently, they are not varied, the variation sign δ in Expression (4.20) may be taken outside the integrand. Then

$$\delta\int_V (Xu+Yv+Zw)dV+\delta\int_F (\bar{X}u+\bar{Y}v+\bar{Z}w)dF-\delta U=0$$

or

$$\delta(U-W)=0. \qquad (4.21)$$

Here

$$U=\int U_0 dV,$$

where U_0 is defined by Formula (4.16);

$$W=\int_V (Xu+Yv+Zw)dV+\int_F (\bar{X}u+\bar{Y}v+\bar{Z}w)dF. \qquad (4.22)$$

Thus

$$\Pi=U-W$$

represents the potential energy of the system of external and internal forces, expressed as a function of u, v and w. The potential energy of the system of external and internal forces is the work that can be obtained in the reverse process, i.e., when the elastic body is relieved of load. Hence the potential energy is equal in magnitude and opposite in sign to the work of the internal and external forces, i.e.,

$$\Pi = -A - W,$$

where $-A = U$ is the potential energy of deformation and $-W$ is the potential energy of the external forces.

The equality

$$\delta\Pi = \delta(U - W) = 0 \qquad (4.23)$$

indicates that if the elastic body is taken from the actual strained state that prevailed at equilibrium, in which it had a potential energy $\Pi = U - W$, to a neighboring state, its potential energy will not change, i.e., the potential energy has a stationary value at the equilibrium position of the body.

Taking the second variation of Π, it can be shown [13] that it is essentially positive, i.e.,

$$\delta^2\Pi > 0.$$

This means that the potential energy of the system of external and internal forces has a minimum value in the equilibrium position.

Thus, among all of the displacements corresponding to the assigned boundary conditions and consistent with continuity of the material, the actual displacements that also satisfy the equilibrium conditions impart the minimum value to potential energy. Herein consists the principle of system potential energy.

To illustrate the application of the Lagrange variational equation (4.23), let us determine the deflection $w(x)$ of a cantilever beam being acted upon by a uniformly distributed linear load q (Fig. 4.7).

Disregarding the shear energy, we find from (4.15)

$$U = \frac{1}{2}\int_V \sigma_x \varepsilon_x \, dV = \frac{E}{2}\iiint \varepsilon_x^2 \, dx\, dy\, dz.$$

Remembering that $\varepsilon_x = Z\frac{1}{\varrho} = Z\frac{d^2w}{dx^2}$, we find

$$U = \frac{E}{2}\iint Z^2 dy\, dz \int\left(\frac{d^2w}{dx^2}\right)^2 dx$$

or

$$U = \frac{EJ}{2}\int_0^l (w'')^2 dx.$$

The potential energy of the external load

$$-W = -\int_0^l qw\, dx.$$

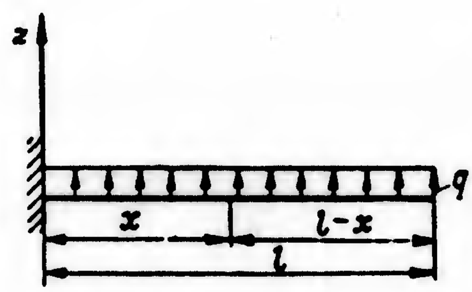


Fig. 4.7

Consequently, the total potential energy of the system

$$\Pi = U - W = \frac{EJ}{2}\int_0^l (w'')^2 dx - \int_0^l qw\, dx. \qquad \text{(a)}$$

The variation of the expression for Π is

$$\delta\Pi = \frac{EJ}{2}\int_0^l 2w''\delta w''\, dx - \int_0^l q\delta w\, dx. \qquad \text{(b)}$$

Integrating the first term of the expression for δΠ by parts, we find

$$EJ\int_0^l w''\delta w''\, dx = \left[EJw''\delta w'\right]_{x=0}^{x=l} - EJ\int w'''\delta w'\, dx =$$

$$= \left[EJw''\delta w'\right]_{x=0}^{x=l} - \left[EJw'''\delta w\right]_{x=0}^{x=l} + \int_0^l EJw^{IV}\delta w\, dx.$$

Thus, the condition $\delta\Pi = 0$ assumes the form

$$[EJw''\delta w']_{x=0}^{x=l} - [EJw'''\delta w]_{x=0}^{x=l} + \int_0^l (EJw^{IV} - q)\delta w\, dx = 0, \qquad (c)$$

or

$$[EJw''\delta w']_{x=0} - [EJw'''\delta w]_{x=0} + [EJw''\delta w']_{x=l} - [EJw'''\delta w]_{x=l} +$$
$$+ \int_0^l (EJw^{IV} - q)\delta w\, dx = 0. \qquad (d)$$

For the restrained end of the beam at x = 0, we have two given geometrical boundary conditions:

$$[w']_{x=0} = 0; \qquad (1)$$

$$[w]_{x=0} = 0. \qquad (2)$$

Hence, the first two terms of Eq. (d) vanish.

In the remaining sections, the variation δw is arbitrary, so that Eq. (d) can be satisfied only provided that

$$[EJw''']_{x=l} = 0; \qquad (3)$$

$$[EJw'']_{x=l} = 0; \qquad (4)$$

$$EJw^{IV} - q = 0. \qquad (e)$$

Thus, we have obtained from the Lagrange variational equation the differential equation of an elastic line, which is an equilibrium equation, and two boundary conditions for the free end of the beam, the latter indicating that the transverse force Q and the bending moment M are equal to zero here.

The boundary conditions proceeding from the energy variation for those parts of the surface where they are not assigned are known as the <u>natural boundary conditions</u>.

Let us turn to integration of Eq. (e).

$$EJw''' = -\int_l^{l-x} q\, dx + C_1 = qx + C_1.$$

Applying boundary condition (3), we obtain $C_1 = -q\ell$.

Hence

$$EJw''' = -q(l-x) = Q_x.$$

Subsequent integration using the boundary conditions (4), (1) and (2) leads us to the familiar equation of the elastic line of a uniformly loaded cantilever beam:

$$w = \frac{q}{2EJ}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right).$$

It should be noted that for simple bodies, the equations of equilibrium and the natural boundary conditions can also be derived firectly from examination of the equilibrium conditions and continuity conditions of deformation. For complex systems, however, writing the equilibrium equations and boundary conditions in differential form becomes an exceedingly difficult problem, while it is easy to obtain them by variation of the energy.

5. CASTIGLIANO'S VARIATION EQUATION - THE ADDITIONAL-ENERGY PRINCIPLE

An elastic body is a statically indeterminate system, since static investigation of the problem of elasticity theory gives only three equilibrium equations (4.1) for six unknowns $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + X = 0;$$
$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + Y = 0;$$
$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0.$$

Consequently, an infinite set of solutions, each satisfying the equations of equilibrium (4.1) and the assigned boundary conditions (4.3) on the surface, will be possible:

$$\overline{X} = l\sigma_x + m\tau_{yx} + n\tau_{zx};$$
$$\overline{Y} = l\tau_{xy} + m\sigma_y + n\tau_{zy};$$
$$\overline{Z} = l\tau_{xz} + m\tau_{yz} + n\sigma_z.$$

To answer the question that arises here as to what distinguishes the actual stressed state from this set of statically possible solutions, it is necessary to determine the variation corresponding to the transition from the actual state

$$\sigma_x, \sigma_y, \ldots\ldots\ldots$$

to a statically possible state infinitesimally removed from it:

$$\sigma_x+\delta\sigma_x, \sigma_y+\delta\sigma_y, \ldots.$$

Let us denote the deformation energy by $\bar{U}$ and the energy of the external forces by $\bar{W}$. The bar signifies that now the energy is a function of the stress components and the external forces.

Let us assume at first that there are only stresses σ_x. On variation of these stresses,

$$\delta\bar{U}_{0x}=\varepsilon_x\delta\sigma_x.$$

The value of $\delta\bar{U}_{0x}$ is represented in Fig. 4.8 in the form of a rectangle accurate to quantities of the second order of smallness. Obviously

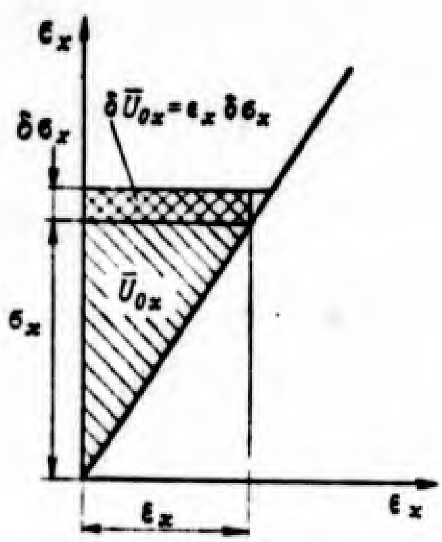


Fig. 4.8

$$\bar{U}_{0x}=\int_0^{\sigma_x}\varepsilon_x\delta\sigma_x \qquad (4.24)$$

is the area that complements the area U_{0x} to the area of a rectangle with sides σ_x and ε_x (see Figs. 4.6 and 4.8).

If the material is subject to Hooke's law, then obviously $\bar{U}_{0x} = U_{0x}$. When the body is nonlinearly elastic, $\bar{U}_{0x} \neq U_{0x}$. We shall consider this case briefly at the end of this part of the book.

On variation of all stress components, we obtain the variation of the specific deformation energy in the form

$$\delta\bar{U}_0=\varepsilon_x\delta\sigma_x+\varepsilon_y\delta\sigma_y+\varepsilon_z\delta\sigma_z+ \\ +\gamma_{xy}\delta\tau_{xy}+\gamma_{yz}\delta\tau_{yz}+\gamma_{zx}\delta\tau_{zx}. \qquad (4.25)$$

Comparing this expression with the total differential of the specific energy as a function of the stress components,

$$\delta\overline{U}_0=\frac{\partial\overline{U}_0}{\partial\sigma_x}\delta\sigma_x+\frac{\partial\overline{U}_0}{\partial\sigma_y}\delta\sigma_y+\frac{\partial\overline{U}_0}{\partial\sigma_z}\delta\sigma_z+$$

$$+\frac{\partial\overline{U}_0}{\partial\tau_{xy}}\delta\tau_{xy}+\frac{\partial\overline{U}_0}{\partial\tau_{yz}}\delta\tau_{yz}+\frac{\partial\overline{U}_0}{\partial\tau_{zx}}\delta\tau_{zx}, \tag{4.26}$$

we note that the partial derivative of the specific deformation energy with respect to the stress component is equal to the corresponding strain component:

$$\left.\begin{aligned}&\frac{\partial\overline{U}_0}{\partial\sigma_x}=\varepsilon_x; \quad \frac{\partial\overline{U}_0}{\partial\sigma_y}=\varepsilon_y; \quad \frac{\partial\overline{U}_0}{\partial\sigma_z}=\varepsilon_z;\\ &\frac{\partial\overline{U}_0}{\partial\tau_{xy}}=\gamma_{xy}; \quad \frac{\partial\overline{U}_0}{\partial\tau_{yz}}=\gamma_{yz}; \quad \frac{\partial\overline{U}_0}{\partial\tau_{zx}}=\gamma_{zx}.\end{aligned}\right\} \tag{4.27}$$

The variation of the additional energy of deformation of the entire volume of the body

$$\delta\overline{U}=\int_V(\varepsilon_x\delta\sigma_x+\varepsilon_y\delta\sigma_y+\varepsilon_z\delta\sigma_z+$$

$$+\gamma_{xy}\delta\tau_{xy}+\gamma_{yz}\delta\tau_{yz}+\gamma_{zx}\delta\tau_{zx})dV. \tag{4.28}$$

Applying the Cauchy equations (4.4) here, we obtain, for example,

$$\varepsilon_x\delta\sigma_x=\frac{\partial u}{\partial x}\delta\sigma_x$$

or

$$\varepsilon_x\delta\sigma_x=\frac{\partial(u\,\delta\sigma_x)}{\partial x}-u\frac{\partial\delta\sigma_x}{\partial x}.$$

Performing such transformations of the other terms in Expression (4.28) as well, we obtain two volume integrals as a result. After modification of the second of these integrals by the Green-Ostrogradskiy formula, we obtain

$$\delta\overline{U}=-\int_V\Big[u\left(\frac{\partial\delta\sigma_x}{\partial x}+\frac{\partial\delta\tau_{yx}}{\partial y}+\frac{\partial\delta\tau_{zx}}{\partial z}\right)+$$

$$+v\left(\frac{\partial\delta\tau_{xy}}{\partial x}+\frac{\partial\delta\sigma_y}{\partial y}+\frac{\partial\delta\tau_{zy}}{\partial z}\right)+$$

$$+w\left(\frac{\partial\delta\tau_{xz}}{\partial x}+\frac{\partial\delta\tau_{yz}}{\partial y}+\frac{\partial\delta\sigma_z}{\partial z}\right)\Big]dv+$$

$$+\int_F[u(\delta\sigma_x l+\delta\tau_{yx}m+\delta\tau_{zx}n)+$$

$$+v(\delta\tau_{xy}l+\delta\sigma_y m+\delta\tau_{zy}n)+$$
$$+w(\delta\tau_{xz}l+\delta\tau_{yz}m+\sigma_z n)]\,dF. \tag{4.29}$$

In the first integral, the expressions in parentheses are equal to the variations of the volume forces δX, δY, δZ, which are equal to zero, since the volume forces are assigned; consequently, this integral vanishes. In the second integral, the expressions in paranetheses are equal to $\delta\bar{X}$, $\delta\bar{Y}$, $\delta\bar{Z}$ in accordance with the variations of the surface forces $\bar{X}$, $\bar{Y}$, $\bar{Z}$. Consequently, Eq. (4.29) assumes the form

$$\delta\bar{U}=\int_F[u\delta\bar{X}+v\delta\bar{Y}+w\delta\bar{Z}]\,dF. \tag{4.30}$$

Since the variations $\delta\bar{X}$, $\delta\bar{Y}$, $\delta\bar{Z}$ are equal to zero on those parts of the surface where the surface forces are assigned, integration extends only over those parts of the surface where the displacements u, v, w are assigned; their values appear in the integrand.

Since the displacements u, w and w are not varied, the sign of the variation δ may be taken outside the integrand and Eq. (4.30) may be written in the form

$$\delta(\bar{U}-\bar{W})=0. \tag{4.31}$$

Here

$$\bar{U}=\int_V\bar{U}_0\,dV, \tag{4.32}$$

where $\bar{U}_0$ is determined by Formula (4.17);

$$\bar{W}=\int_F[u\bar{X}+v\bar{Y}+w\bar{Z}]\,dF. \tag{4.33}$$

Equation (4.31) is the Castigliano variational equation, which gives the equations of strain continuity [13].

Thus,

$$\bar{\Pi}=\bar{U}-\bar{W}$$

is the additional energy of the system of external and internal forces on variation of the stressed state.

Taking the second variation of $\overline{\Pi}$, we can show that it is positive; consequently, the actual stressed state imparts the minimum value to the additional energy.

Equation (4.31) expresses the principle of additional energy and may be formulated as follows. Of all statically possible systems of stresses, the actual stressed state that also satisfies the consistency equations is distinguished by the fact that the additional energy $\overline{\Pi}$ has a minimum value for it.

We note once again that the strain continuity equations are obtained from the minimization of potential energy. This becomes understandable when we remember that the statically possible stresses differ from the real ones only in that they do not satisfy the continuity equations. Consequently, to distinguish the actual stresses among them, it is sufficient that the latter satisfy the continuity equations. If the body is nonlinearly elastic,

$$\overline{U}_{0x} \neq U_{0x}.$$

as is clear from Fig. 4.9. Hence in the Lagrange equation

$$\delta(U-W)=0$$

the quantity $U=\int_V U_0 dV$ must be determined with recognition of the fact that in the general case of a nonlinearly elastic body

$$U_0=\int_0^{\epsilon_x}\sigma_x\delta\epsilon_x+\int_0^{\epsilon_y}\sigma_y\delta\epsilon_y+\int_0^{\epsilon_z}\sigma_z\delta\epsilon_z+\int_0^{\gamma_{xy}}\tau_{xy}\delta\gamma_{xy}+\int_0^{\gamma_{yz}}\tau_{yz}\delta\gamma_{yz}+\int_0^{\gamma_{zx}}\tau_{zx}\delta\gamma_{zx},$$

and in the Castigliano equation

$$\delta(\overline{U}-\overline{W})=0$$

the quantity $\overline{U}=\int_V \overline{U}_0 dV$ is determined taking account of the fact that

$$\overline{U}_0=\int_0^{\sigma_x}\epsilon_x\delta\sigma_x+\int_0^{\sigma_y}\epsilon_y\delta\sigma_y+\int_0^{\sigma_z}\epsilon_z\delta\sigma_z+\int_0^{\tau_{xy}}\gamma_{xy}\delta\tau_{xy}+\int_0^{\tau_{yz}}\gamma_{yz}\delta\tau_{yz}+\int_0^{\tau_{zx}}\gamma_{zx}\delta\tau_{zx}.$$

From this it is clear that

$$\overline{U}_0 \neq U_0.$$

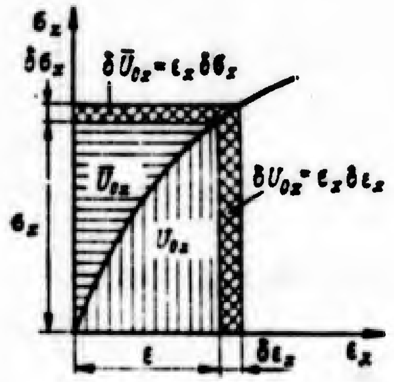


Fig. 4.9

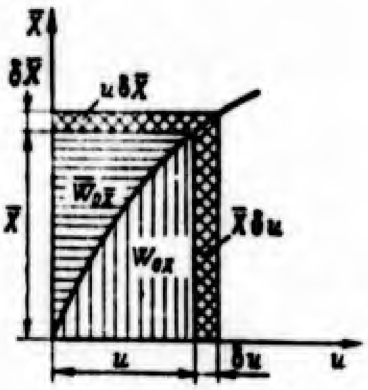


Fig. 4.10

Figure 4.10 shows areas proportional to the quantities $\overline{W}_0$ and W_0 referred to the unit volume. Hence $\overline{W}_0$ is not equal to W_0.

Professor A.Yu. Romashevskiy [deceased] [20] devoted a great deal of attention to the problem of applying the Lagrange and Castiliano variational equations to the nonlinearly elastic body. Other scientists, both in our country and abroad, have also given this problem much study (see, for example, A.A. Umanskiy, Stroitel'naya mekhanika samoleta [Structural Mechanics of the Airplane], Oborongiz [State Publishing House for the Defense Industry], 1961; Sovremennyye metody rascheta slozhnykh staticheski neopredelimykh sistem [Contemporary Methods for the Design of Complex Statically Indeterminate Systems], Sudpromgiz [State Publishing House for the Shipbuilding Industry], 1961 and others).

To illustrate the application of the Castigliano variational equation, let us examine the problem of twisting of a prismatic rod.

In applying the Castigliano variational equation, it is necessary to select in advance a stress distribution that satisfies the equilibrium conditions. This condition will be met if the stresses σ_x, σ_y, σ_z and τ_{xy} are assumed equal to zero, while the stresses τ_{zx} and τ_{zy} are expressed in terms of a stress function ϕ in the form (see [24]):

$$\tau_{zx}=\frac{\partial\phi}{\partial y} \text{ and } \tau_{yz}=-\frac{\partial\phi}{\partial x}. \qquad (4.34)$$

In accordance with Formulas (4.32) and (4.17), we obtain the additional energy of deformation per unit length of the twisted rod:

$$\overline{U}=\frac{1}{2G}\iint(\tau_{zx}^2+\tau_{zy}^2)\,dx\,dy=$$

$$= \frac{1}{2G} \iint \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] dx\, dy.$$

Now let us turn to the calculation of $\overline{W}$. Since the surface forces are assigned (equal to zero) on the lateral surface, $\overline{W} = 0$ for this part of the surface. For the end surfaces

$$\overline{W} = \left[\iint (\tau_{zx} u + \tau_{zy} v)\, dx\, dy \right]_{\substack{z=0 \\ z=1}}.$$

The displacements

$$u = -\theta Z y \text{ and } v = \theta Z x.$$

Consequently,

$$\text{for} \quad Z = 0 \quad u = v = 0;$$
$$\text{for} \quad Z = 1 \quad u = -\theta y \text{ and } v = \theta x.$$

Hence

$$\overline{W} = \theta \iint (-y \tau_{zx} + x \tau_{zy})\, dx\, dy =$$
$$= -\theta \iint \left(y \frac{\partial \phi}{\partial y} + x \frac{\partial \phi}{\partial x} \right) dx\, dy.$$

Using Green's formula, we obtain

$$\overline{W} = 2\theta \iint \phi\, dx\, dy - \theta \oint \phi [x \cos(N, x) + y \cos(N, y)]\, ds.$$

Since the function ϕ may be assumed equal to zero on the contour,

$$\overline{W} = 2\theta \iint \phi\, dx\, dy.$$

Therefore

$$\overline{\Pi} = \overline{U} - \overline{W} = \frac{1}{2G} \iint \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 - 4G\theta\phi \right] dx\, dy. \qquad (4.35)$$

According to the Castigliano variational equation (4.31),

$$\delta\overline{\Pi} = \delta \left\{ \frac{1}{2G} \iint \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 - 4G\theta\phi \right] dx\, dy \right\} = 0. \qquad (4.36)$$

To show that the strain continuity equation is obtained as a corollary of the Castigliano variational equation, we find the variation of Expression (4.36):

$$\frac{1}{2G} \iint \left[2 \frac{\partial \phi}{\partial x} \delta \frac{\partial \phi}{\partial x} + 2 \frac{\partial \phi}{\partial y} \delta \frac{\partial \phi}{\partial y} - 4G\theta\delta\phi \right] dx\, dy = 0. \qquad (a)$$

We integrate the first term of this equation by parts:

$$2\iint \frac{\partial\phi}{\partial x}\delta\frac{\partial\phi}{\partial x}\,dx\,dy = 2\int dy\int \frac{\partial\phi}{\partial x}\frac{\partial\delta\phi}{\partial x}\,dx =$$
$$= 2\int dy\left[\left(\frac{\partial\phi}{\partial x}\delta\phi\right)_{x_1}^{x_2} - \int \delta\phi\frac{\partial^2\phi}{\partial x^2}\,dx\right].$$

The values of x_1 and x_2 pertain to points on the contour, but since $\delta\phi = 0$ there, $\left(\frac{\partial\phi}{\partial x}\delta\phi\right)_{x_1}^{x_2} = 0$ and

$$2\iint \frac{\partial\phi}{\partial x}\delta\frac{\partial\phi}{\partial x}\,dx\,dy = -2\iint \frac{\partial^2\phi}{\partial x^2}\delta\phi\,dx\,dy.$$

Similarly,

$$2\iint \frac{\partial\phi}{\partial y}\delta\frac{\partial\phi}{\partial y}\,dx\,dy = -2\iint \frac{\partial^2\phi}{\partial y^2}\delta\phi\,dx\,dy.$$

Hence Eq. (a) assumes the form

$$-\iint\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + 2G\theta\right)\delta\phi\,dx\,dy = 0. \qquad (b)$$

Since the variation $\delta\phi$ is arbitrary within the cross-section contour, Eq. (b) can be satisfied only provided that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta, \qquad (4.37)$$

which is precisely the continuity equation.

6. CASTIGLIANO'S THEOREMS

The first and second Castigliano theorems have found extensive application in engineering calculations; thus it would be appropriate here to show that they are particular cases of the basic Castigliano variational equation (4.31).

<u>First theorem of Castigliano</u>.

The first theorem pertains to the case in which the work is done not by surface forces, but by lumped generalized forces $P_1, P_2 \ldots P_n$, which accomplish corresponding generalized displacements $f_1, f_2 \ldots f_n$.

Assuming the displacements $f_1, f_2 \ldots f_n$ to be given and the lumped forces $P_1, P_2 \ldots P_n$ to be independent unknown quantities, we obtain Eq. (4.31) in the form

$$\delta\overline{U} - (f_1\delta P_1 + f_2\delta P_2 + \ldots f_n\delta P_n) = 0, \qquad (1)$$

where the variation δU is the total differential of the deformation potential energy as a function of the external forces P_1, $P_2 \dots P_n$, i.e.,

$$\delta \bar{U} = \frac{\partial \bar{U}}{\partial P_1}\delta P_1 + \frac{\partial \bar{U}}{\partial P_2}\delta P_2 + \dots + \frac{\partial \bar{U}}{\partial P_n}\delta P_n. \quad (2)$$

Consequently,

$$\frac{\partial \bar{U}}{\partial P_1}\delta P_1 + \frac{\partial \bar{U}}{\partial P_2}\delta P_2 + \dots + \frac{\partial \bar{U}}{\partial P_n}\delta P_n - $$
$$-(f_1\delta P_1 + f_2\delta P_2 + \dots + f_n\delta P_n) = 0$$

or

$$\left(\frac{\partial \bar{U}}{\partial P_1} - f_1\right)\delta P_1 + \left(\frac{\partial \bar{U}}{\partial P_2} - f_2\right)\delta P_2 + \dots + \left(\frac{\partial \bar{U}}{\partial P_n} - f_n\right)\delta P_n = 0. \quad (3)$$

Since the forces P_1, $P_2 \dots P_n$ are statically independent and their variations are completely arbitrary, all variations save one may be assumed equal to zero. Then we obtain from Eq. (3)

$$\left.\begin{array}{l} \frac{\partial \bar{U}}{\partial P_1} = f_1, \\ \frac{\partial \bar{U}}{\partial P_2} = f_2, \\ \dots\dots \\ \dots\dots \\ \frac{\partial \bar{U}}{\partial P_n} = f_n. \end{array}\right\} \quad (4.38)$$

This is Castigliano's first theorem.

Thus, if the additional energy of the elastic system is expressed as a function of the statically independent generalized forces P_1, P_2, $\dots, P_n$, the partial derivative of the additional energy with respect to any of these forces is equal to the corresponding generalized displacement.

It will be recalled that the forces P_1, $P_3, \dots P_n$ will be statically independent if none of them can be obtained by linear combination of other forces from among the P_i, where $i = 1, 2, \dots n$.

Castigliano's second theorem is the theorem of least work.

If the surface forces are assigned, their variations are equal to zero and, consequently, Eq. (4.30) assumes the following form in this case:

$$\delta U = 0. \tag{4.39}$$

This is the expression of Castigliano's second theorem or the theorem of least work.

The second theorem of Castigliano gives rise to the familiar method of calculation for statically indeterminate systems based on the fact that if $X_1, X_2, \ldots, X_n$ are the unknown forces of the redundant couplings, the values of these statically indeterminate quantities can be determined from the condition that the additional deformation energy, expressed as a function of the unknowns, takes a minimum value.

Indeed, if we cut the redundant couplings to translate the unknown internal forces $X_1, X_2, \ldots X_n$ to external forces, we obtain Expression (4.39) with consideration of (2) in the form

$$\delta \bar{U} = \frac{\partial \bar{U}}{\partial X_1} \delta X_1 + \frac{\partial \bar{U}}{\partial X_2} \delta X_2 + \ldots + \frac{\partial \bar{U}}{\partial X} \delta X_n = 0.$$

From this, in view of the arbitrariness of the variations δX_1, δX_2, $\ldots, \delta X_n$, we obtain a system of n algebraic equations containing one equation for each of the unknown forces:

$$\left.\begin{aligned}
&\frac{\partial \bar{U}}{\partial X_1} = 0, && (1)\\
&\frac{\partial \bar{U}}{\partial X_2} = 0, && (2)\\
&\ldots\ldots\\
&\ldots\ldots\\
&\frac{\partial \bar{U}}{\partial X_n} = 0. && (n)
\end{aligned}\right\} \tag{4.40}$$

Thus, if the potential energy of deformation is a function of unknown forces $X_1, X_2, \ldots, X_n$ that do not depend on the coordinates, i.e.,

$$U = F(X_1, X_2, \ldots X_n),$$

then the unknown constants $X_1, X_2, \ldots, X_n$ are determined by simultaneous solution of the algebraic equations of System (4.40).

In calculations for airframe structures and other complex engineering objectives, unknowns of this type are frequently compounded by unknown functions, usually of one or two coordinates. In this case, minimization of the potential energy as the functional of the unknown functions results in a system of n differential equations, one for each unknown function. This system is most simply obtained with the aid of Euler's equations.

7. EULER'S EQUATIONS. EXAMPLES

To resolve a real displacement or stress function in the preceding sections – a function that we shall denote by v for the sake of generality – we converted to other functions $v + \delta v$ infinitesimally close to the true function v but completely arbitrary in form. It was then shown that the actual function v minimizes the potential energy. Determination of the potential-energy minimum as a functional of unknown functions is, as we have seen, a problem of variational calculus. In applying energy methods, it is convenient to have the variations of frequently encountered functionals in ready form.

For greater clarity, let us begin with the elementary problem of variational calculus in which the unknown arguments of the functional Φ are the coordinate x, varying in the interval from $x = a$ to $x = b$, the unknown continuous function $v = v(x)$ and its derivative $v'=\frac{dv}{dx}$, i.e.,

$$\Phi=\int_a^b F(x, v, v')\,dx. \qquad (4.41)$$

The variation of Functional (4.41) will be

$$\delta\Phi=\int_a^b [F(x, v+\delta v, v'+\delta v')-F(x, v, v')]\,dx=\int_a^b \delta F dx.$$

Remembering that $\delta F=\frac{\partial F}{\partial v}\delta v+\frac{\partial F}{\partial v'}\delta v'$, we obtain

$$\delta\Phi=\int_a^b (F_v\delta v+F_{v'}\delta v')\,dx,$$

where $F_v = \frac{\partial F}{\partial v}$ and $F_{v'} = \frac{\partial F}{\partial v'}$.

Integrating the second term in the integrand of this last expression by parts, we obtain

$$\delta\Phi = \int_a^b F_v \delta v\, dx + F_{v'}\delta v \Big|_{x=a}^{x=b} - \int_a^b \frac{d}{dx} F_{v'} \delta v\, dx$$

or

$$\delta\Phi = F_{v'}\delta v \Big|_{x=a}^{x=b} + \int_a^b \left(F_v - \frac{d}{dx} F_{v'}\right) \delta v\, dx.$$

The unknown function v(x) is isolated from the infinite number of possible nearby functions $v(x) + \delta v(x)$ by virtue of the fact that it causes the variation of the functional to vanish.

Consequently,

$$\delta\Phi = F_{v'}\delta v \Big|_{x=a}^{x=b} + \int_a^b \left(F_v - \frac{d}{dx} F_{v'}\right) \delta v\, dx = 0. \tag{4.42a}$$

Let us first examine the case in which the variation δv of the function v(x) is given for x = a and x = b, i.e.,

$$\delta v(a) = \delta v(b) = 0.$$

Then the term of Expression (4.42a) outside of the integrand will be zero. Consequently,

$$(F_{v'}\delta v)_{x=b} = 0 \text{ and } (F_{v'}\delta v)_{x=a} = 0.$$

Hence

$$\int_a^b \left(F_v - \frac{d}{dx} F_{v'}\right) \delta v\, dx = 0:$$

Due to the arbitrariness of δv, the last equality is valid only provided that

$$F_v - \frac{d}{dx} F_{v'} = 0. \tag{4.43}$$

The resulting differential equation is known as the Euler equation of the elementary variational problem.

If the value of δv is arbitrary at x = a and x = b, the term outside the integrand in Expression (4.42) will be equal to zero for

$$(F_{v'})_{x=a}=0 \text{ and } (F_{v'})_{x=b}=0. \qquad (4.44)$$

These conditions are known as the natural boundary conditions. Together with the two boundary conditions, Eq. (4.43), which contains two arbitrary constants, fully defines the value of the function v(x), which realizes the extreme of Functional (4.41).

We note that the two boundary conditions may occur in the form of: 1) two natural boundary conditions, if δv is arbitrary everywhere, including the points x = a and x = b; 2) two assigned boundary conditions, if δv is arbitrary everywhere except for the points x = a and x = b, where its value is assigned; 3) one natural and one assigned boundary condition.

In the case of several functions depending on a single coordinate, for example, $v_1(x)$, $v_2(x)$,......$v_n(x)$, we obtain the functional

$$\Phi=\int_a^b F(x, v_1(x), v_1'(x), v_2(x), v_2'(x) \ldots v_n(x), v_n'(x))\,dx.$$

Minimization of this functional gives a number of differential equations equal to the number of unknown functions n:

$$\left.\begin{aligned} &F_{v_1}-\frac{d}{dx}F_{v_1'}=0, &&(1)\\ &F_{v_2}-\frac{d}{dx}F_{v_2'}=0, &&(2)\\ &\ldots\ldots\ldots\\ &\ldots\ldots\ldots\\ &F_{v_n}-\frac{d}{dx}F_{v_n'}=0. &&(n) \end{aligned}\right\} \qquad (4.45)$$

Together with the boundary conditions, the equation system (4.45) fully defines the n unknown functions. If the functional also depends on the second derivative of the function v(x), i.e.,

$$\Phi=\int_a^b F(x, v, v', v'')\,dx, \qquad (4.46)$$

then $\delta\Phi = 0$ is equivalent to two conditions:

$$\int_a^b \left(F_v - \frac{d}{dx}F_{v'} + \frac{d^2}{dx^2}F_{v''}\right)\delta v\,dx = 0; \tag{4.47}$$

$$\left(F_{v'} - \frac{d}{dx}F_{v''}\right)\delta v\Big|_a^b + F_{v''}\delta v'\Big|_a^b = 0. \tag{4.48}$$

With arbitrary variations δv in the interval [a, b], these two conditions give rise to a single fourth-order differential equation and four natural boundary conditions:

$$F_v - \frac{d}{dx}F_{v'} + \frac{d^2}{dx^2}F_{v''} = 0. \tag{4.49}$$

$$\left.\begin{aligned} &\left(F_{v'} - \frac{d}{dx}F_{v''}\right)_{x=a} = 0; \quad \left(F_{v'} - \frac{d}{dx}F_{v''}\right)_{x=b} = 0, \\ &(F_{v''})_{x=a} = 0; \qquad\qquad (F_{v''})_{x=b} = 0. \end{aligned}\right\} \tag{4.50}$$

If there are several, for example n, unknown functions, they are determined, with consideration of the boundary conditions, from the following system of fourth-order differential equations:

$$\left.\begin{aligned} &F_{v_1} - \frac{d}{dx}F_{\dot{v}_1} + \frac{d^2}{dx^2}F_{\ddot{v}_1} = 0, && (1) \\ &F_{v_2} - \frac{d}{dx}F_{\dot{v}_2} + \frac{d^2}{dx^2}F_{\ddot{v}_2} = 0, && (2) \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ &F_{v_n} - \frac{d}{dx}F_{\dot{v}_n} + \frac{d^2}{dx^2}F_{\ddot{v}_n} = 0. && (n) \end{aligned}\right| \tag{4.51}$$

Let us write the functional once more for the two dimensional problem, when we have v(x, y). The functional takes the form

$$\Phi = \iint F(x, y, v, v_x, v_y, v_{xx}, v_{xy}, v_{yy})\,dx\,dy, \tag{4.52}$$

where

$$v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y} \text{ etc.}$$

It follows from the condition $\delta\Phi = 0$ that

$$\iint L(v)\,\delta v\,dx\,dy = 0, \tag{4.53}$$

where

$$L(v)=\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\frac{\partial F}{\partial v_x}-\frac{\partial}{\partial y}\frac{\partial F}{\partial v_y}+\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{xx}}+$$
$$+\frac{\partial^2}{\partial x\partial y}\frac{\partial F}{\partial v_{xy}}+\frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial v_{yy}}=0. \tag{4.54}$$

Expressions (4.42b), (4.47) and (4.53) can be combined to form a single equation, which may be written in the following form for the n unknown functions:

$$\iint\left[\sum_{i=1}^{n}L_i\delta f_i\right]dx\,dy=0, \tag{4.55}$$

where L_1 are the left members of the differential equations and δf_1 are the arbitrary variations of the various functions.

The solution of partial differential equation (4.54) encounters major mathematical difficulties and is possible only in particular cases. Hence in the design of complex structures, it is necessary to adopt reasonable assumptions reducing the two dimensional problem to a one dimensional problem described by Eqs. (4.45) or (4.51).

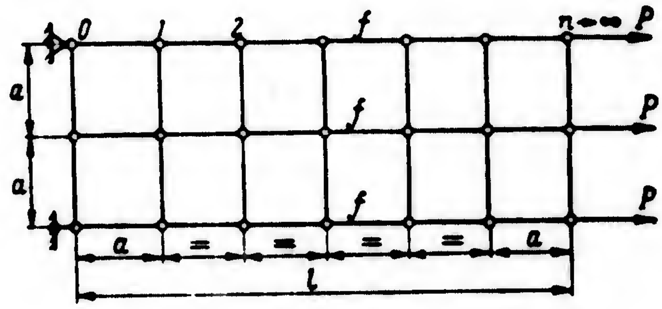


Fig. 4.11

To illustrate the application of the potential energy minimum principle, let us examine two examples.

Example 1.

Figure 4.11 shows a two dimensional system consisting of three longitudinal strips, closely spaced vertical struts and a thin wall. The strips have a cross-sectional area f and the sheet material has a thickness δ. Determine the stressed state of this system under the action of three forces P applied at the free end of the system.

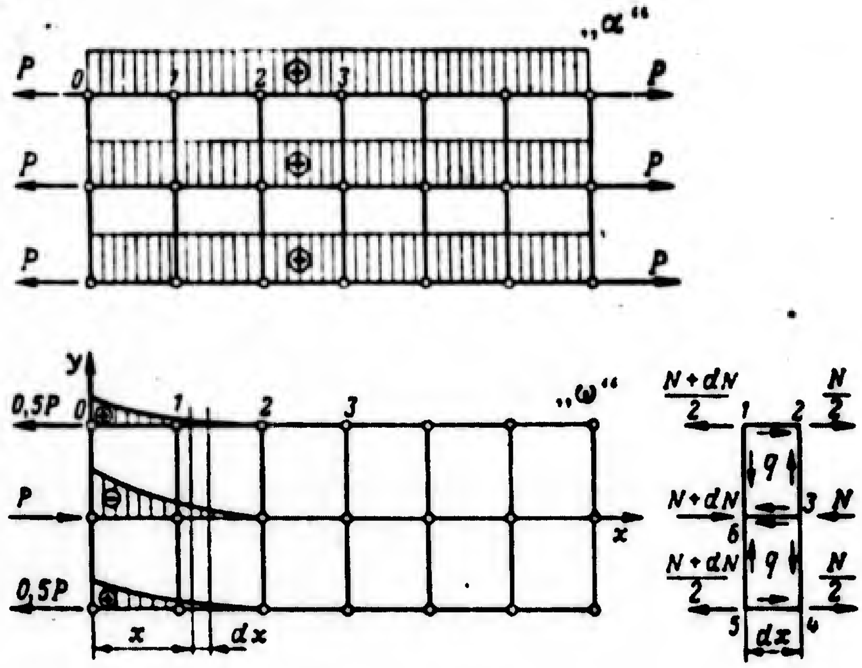


Fig. 4.12

When systems are loaded in this manner, the stressed state may be represented as composed of two states: "α" and "ω" (Fig. 4.12).

In state "α," the forces in the strips are constant over their entire length.

To determine stressed state "ω," it is necessary to ascertain how the self-compensated group of forces applied to the left end of the system decays over the length of the strips (along the x-axis) under the condition that the skin operates only in shear, while the struts are absolutely rigid axially and absolutely compliant in the direction perpendicular to their axis.

We determine the potential energy of deformation

$$\bar{U}=\int_V\left(\frac{\sigma_x^2}{2E}+\frac{\tau_{xy}^2}{2G}\right)dV=\int_0^l \Gamma dx,$$

where Γ is the potential energy of an element of the system having unit length.

To determine one of these unknowns, for example, τ, we apply the equation of statics.

Isolating an element of the panel with length dx (Fig. 4.12), we find from the equilibrium conditions for any strip $q=\frac{dN}{2dx}$.

The potential energy of the unit-length element

$$\Gamma=\frac{\left(\frac{N}{2}\right)^2}{2Ef}+\frac{N^2}{2Ef}+\frac{\left(\frac{N}{2}\right)^2}{2Ef}+2\,\frac{q^2a}{2G\delta}=\frac{3}{4}\,\frac{N^2}{Ef}+\frac{a}{4G\delta}\left(\frac{dN}{dx}\right)^2.$$

The system's deformation potential energy will have a minimum value only if the unknown varied function N = N(x) satisfies the Euler differential equation

$$\frac{\partial\Gamma}{\partial N}-\frac{d}{dx}\,\frac{\partial\Gamma}{\partial N'}=0,$$

where

$$N'=\frac{dN}{dx};\quad \frac{\partial\Gamma}{\partial N}=\frac{3}{2}\,\frac{N}{Ef};\quad \frac{\partial\Gamma}{\partial N'}=\frac{a}{2G\delta}\,\frac{dN}{dx}.$$

In our case, the Euler equation takes the form

$$\frac{3}{2}\,\frac{N}{Ef}-\frac{a}{2G\delta}\,\frac{d^2N}{dx^2}=0. \qquad \text{(a)}$$

Multiplying the right and left members by $\left(-\frac{2G\delta}{a}\right)$, we obtain

$$\frac{d^2N}{dx^2}-\frac{3G\delta}{aEf}N=0. \qquad \text{(b)}$$

The following characteristic equation corresponds to this differential equation:

$$r^2-k^2=0,$$

where

$$k^2=\frac{3G\delta}{aEf}.$$

The roots of this equation are $r_1 = k$ and $r_2 = -k$.

We write the solution of the differential equation in the form

$$N=C_1e^{kx}+C_2e^{-kx}.$$

In the present case, the following are given at the boundaries: 1) at z = 0, the force N = − P; 2) at x = ℓ, the force N = 0 (since, by definition, the strut at x = ℓ is absolutely compliant in the direction of the x-axis). These static boundary conditions are what determines the values of the constants instead of Conditions (4.44), which are identically satisfied, since the value of the unknown function is known at the boundaries and, consequently, its variations in the sections at x = 0 and x = ℓ are equal to zero.

From the condition N = − P at x = 0,

$$-P=C_1+C_2. \tag{c}$$

From the condition N = 0 at x = ℓ,

$$0=C_1e^{kl}+C_2e^{-kl}.$$

we obtain

$$C_1=-C_2\frac{e^{-kl}}{e^{kl}}.$$

Substituting the value of C_1 in Eq. (c), we obtain

$$-P=-C_2\frac{e^{-kl}}{e^{kl}}+C_2=C_2\left(1-\frac{e^{-kl}}{e^{kl}}\right).$$

From this,

$$C_2=-\frac{P}{1-\frac{e^{-kl}}{e^{kl}}}.$$

Consequently,

$$N=-\frac{P}{1-\frac{e^{-kl}}{e^{kl}}}\left(e^{-kx}-\frac{e^{-kl}}{e^{kl}}e^{kx}\right).$$

As $l\to\infty$ we obtain $e^{-kl}\to 0$ and $e^{kl}\to\infty$. Therefore

$$N=-Pe^{-kx}.$$

Taking the numerical data G = E/2.6; δ = 0.1 cm; f = 5 cm^2; a = = 60 cm, we obtain

$$K=\sqrt{\frac{3G\delta}{aEf}}=0{,}0196\ \frac{1}{cm};\quad N=-P2{,}72^{-0{,}0196x}.$$

In sections 1 and 2, the force $N_1=-P\,2{,}72^{-0{,}0196\cdot 60}=-0{,}307P;\ N_2\approx 0.$

Diagrams of the variation of the forces N in the strips are shown in Fig. 4.12, "ω." As we see, the forces produced by the self-compensating force group decay on a length approximately equal to the height of the wall.

The resultant axial forces of states "α" and "ω" are shown in Fig. 4.13.

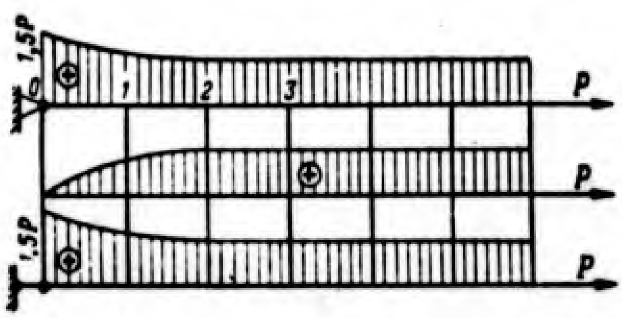


Fig. 4.13

The value of the PKS

$$q=\frac{dN}{2dx}=\frac{0{,}0196}{2}Pe^{-0{,}0196x}.$$

The direction of the PKS is shown in Fig. 4.12.

Since the stressed state has been varied, the differential equation (b) is the continuity equation. To grasp the physical significance of the terms of this equation, we write the strain continuity equation directly for a panel of length dx in section x (Fig. 4.14a). The tangential-force flux q = dN/2dx produces a deformation of the panel under consideration (Fig. 4.14a), with the result that point 6 of the middle strip is displaced along its axis through a distance $\delta_q=\gamma a=\frac{a}{2G\delta}\frac{dN}{dx}$. Point 3, which is situated at a distance dx from point 6, will be displaced through a distance $\delta_q+\frac{\partial\delta_q}{\partial x}dx$, and, consequently, the elementary displacement due to the PKS along the axis of the middle strip will be

$$d\delta_q=\frac{\partial\delta_q}{\partial x}dx=\frac{a}{2G\delta}\frac{d^2N}{dx^2}dx.$$

The elementary displacement $d\delta_c$ (Fig. 4.14b) along the axis of the middle strip due to the deformation of the rods on the segment dx is equal to

$$d\delta_c = -\frac{Ndx}{Ef} - \frac{Ndx}{2Ef} = -\frac{3}{2}\,\frac{N}{Ef}\,dx.$$

From the continuity conditions

$$d\delta_q + d\delta_c = 0$$

or

$$\left(\frac{a}{2G\delta}\,\frac{d^2N}{dx^2} - \frac{3}{2}\,\frac{N}{Ef}\right)dx = 0.$$

Dividing this equality by dx, we obtain the deformation continuity equation, which agrees with the original Euler equation (b).

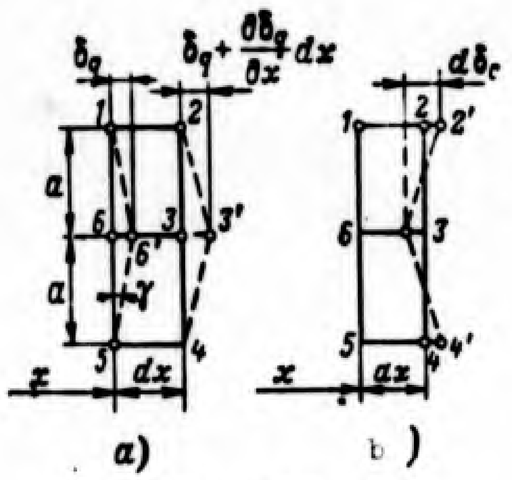


Fig. 4.14

The strain continuity equations can be written in this direct manner only for simple calculating models, while these equations can be obtained easily even for complex systems by minimization of the potential energy.

<u>Example 2.</u>

Figure 4.15 shows a pictureframe monocoque to one end of which is applied a twisting moment M, while the other end is secured to an absolutely rigid support. The cross-sectional area of the framing members is <u>f</u> and the skin thickness is δ. The ribs will be regarded as absolutely rigid in the plane in which they are situated, and absolutely compliant outside of this plane. The skin of the monocoque is assumed to be working only in shear.

Determine the stressed state in the root part of the monocoque.

The unknown stressed state is created by the PKS of free torsion and by secondary stresses.

We know the law of distribution of the secondary-state axial forces in the cross section of the monocoque: it is a self-balancing group of four axial forces N (see, for example, [30]). Let us determine the law of variation of the forces N along the z-axis: N = f(z).

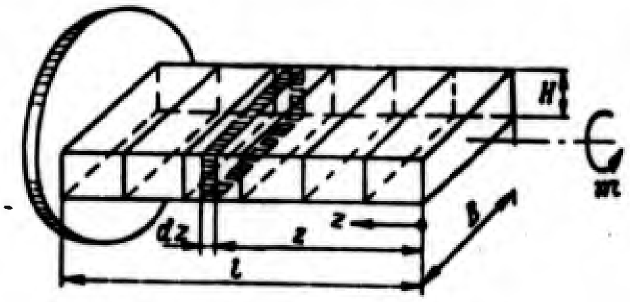


Fig. 4.15

We write the Euler differential equation from the conditions of minimum deformation potential energy:

$$\bar{U}=\int_0^l \Gamma\,dz,$$

where Γ is the potential energy of an elementary unit-length compartment of the monocoque.

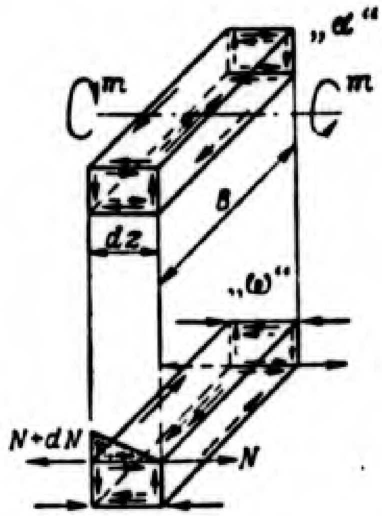


Fig. 4.16

The stressed state of this compartment (Fig. 4.16) is decomposed into two simple stressed states "α" and "ω." In state "α," the PKS along the spar webs and the skin is

$$q_{\mathfrak{M}}=\frac{\mathfrak{M}}{2HB}.$$

From the equilibrium conditions for any framing member, the PKS along the spar webs in state "ω" is

$$q_{\omega}=\frac{dN}{2dz}$$

and on the upper and lower skin panels

$$q_{\omega}=-\frac{dN}{2dz}.$$

Consequently,

$$\Gamma=\frac{4N^2}{2Ef}+\frac{2(q_{\mathfrak{M}}+q_{\kappa})^2H}{2G\delta}+$$
$$+\frac{2(q_{\mathfrak{M}}-q_{\kappa})^2B}{2G\delta}$$

or

$$\Gamma=\frac{2N}{Ef}+\frac{1}{2G}\left[\frac{H+B}{4}\left(\frac{dN}{dz}\right)^2+q_{\mathfrak{M}}(H-B)\frac{dN}{dz}+q_{\mathfrak{M}}^2(H+B)\right].$$

Substituting in the Euler equation (4.43),

$$\frac{\partial\Gamma}{\partial N}=\frac{4N}{Ef},\quad \frac{\partial\Gamma}{\partial\left(\frac{dN}{dz}\right)}=\frac{1}{G\delta}\left[\frac{H+B}{2}\frac{dN}{dz}+q_{\mathfrak{M}}(H-B)\right],$$

$$\frac{d}{dz}\left[\frac{\partial\Gamma}{\partial\left(\frac{dN}{dz}\right)}\right]=\frac{1}{G\delta}\,\frac{H+B}{2}\,\frac{d^2N}{dz^2},$$

we obtain

$$\frac{d^2N}{dz^2}-k^2N=0, \qquad \text{(a)}$$

where

$$k^2=\frac{8G\delta}{(H+B)Ef}.$$

We write the solution of Eq. (a) in the form

$$N=C_1\,\mathrm{sh}\,kz+C_2\,\mathrm{ch}\,kz.$$

From the assigned boundary condition z = 0 and N = 0, we find C_2 = 0; consequently,

$$N=C_1\,\mathrm{sh}\,kz. \qquad \text{(b)}$$

Since the forces are not given for $z = \ell$, their variation here is not zero; hence we shall employ boundary condition (4.44), which assumes the form

$$\left(\frac{\partial\Gamma}{\partial N'}\right)_{z=l}=0$$

or

$$\frac{H+B}{2}\left(\frac{dN}{dz}\right)_{z=l}+q_{\mathfrak{M}}(H-B)=0, \qquad \text{(c)}$$

from which

$$\left(\frac{dN}{dz}\right)_{z=l}=q_{\mathfrak{M}}\frac{2(B+H)}{H+B}=\frac{\mathfrak{M}}{HB}\,\frac{(B-H)}{(B+H)}.$$

From Eq. (b) we have

$$\frac{dN}{dz}=C_1 k\,\mathrm{ch}\,kz$$

or, (for $z = \ell$)

$$\left(\frac{dN}{dz}\right)_{z=l}=C_1 k\,\mathrm{ch}\,kl.$$

Equating the two results for $\left(\frac{dN}{dz}\right)_{z=l}$, we find

$$C_1=\frac{\mathfrak{M}(B-H)}{HB(B+H)k\,\mathrm{ch}\,kl}.$$

Consequently,

$$N=\frac{\mathfrak{M}(B-H)\,\mathrm{sh}\,kz}{HB(B+H)k\,\mathrm{ch}\,kl}. \qquad \text{(d)}$$

Let M = 1000 kgf-m, B = 100 cm, H = 30 cm, δ = 0.1 cm, f = 10 cm^2, ℓ = 400 cm, G = E/2.6. Then

$$k=\sqrt{\frac{8G\delta}{(B+H)fE}}=\sqrt{\frac{8\cdot 0{,}1}{130\cdot 10\cdot 2{,}6}}=0{,}0154.$$

For $z = \ell = 400$, according to Eq. (d),

$$N_{z=400}=\frac{100\,000\cdot 70}{3000\cdot 130\cdot 0{,}0154}=1170\ \kappa\Gamma.$$

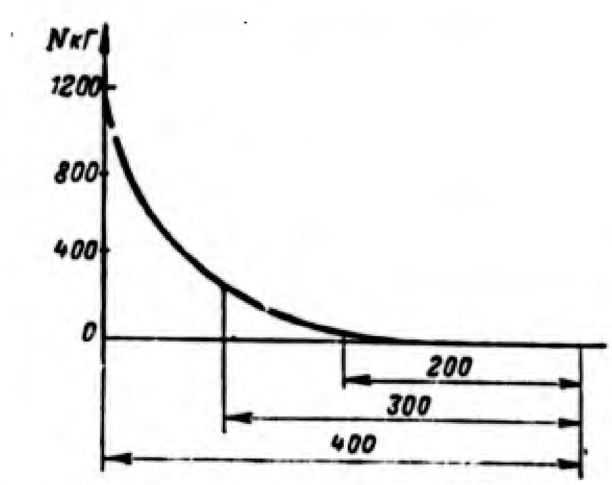


Fig. 4.17. a) N, kgf.

For z = 300,

$$N_{z=300}=1170\,\frac{49\cdot 75}{220}=264\ \kappa\Gamma.$$

For z = 200,

$$N_{z=200}=1170\,\frac{10.6}{220}=56\ \kappa\Gamma.$$

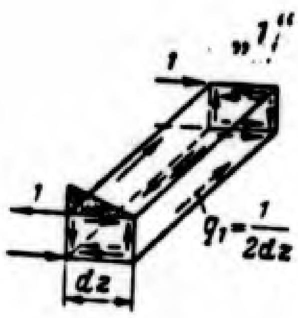


Fig. 4.18

A diagram of the axial forces along one framing member appears in Fig. 4.17.

The physical significance of the boundary condition at z = ℓ consists in the fact that in this section, the cross-sectional distortion (deplanation) is zero because the support is absolutely rigid.

Let us determine the value of the displacement associated with the distortion angle of the monocoque in a section immediately adjacent to the support. For this purpose, it is sufficient to "cross-multiply" states "α" and "ω," indicated in Fig. 4.16, by the unit state shown in Fig. 4.18.

In the unit state, the PKS $q_1=\frac{1}{2dz}$. The direction of this PKS is indicated on Fig. 4.18. The unknown diaplacement

$$\Delta_{1P}=\sum\frac{1}{3}\,\frac{dNdz}{Ef}+\sum\frac{q_1q_\omega F}{G\delta}+\sum\frac{q_1q_{\mathfrak{M}}F}{G\delta}=$$
$$=\frac{4}{3}\,\frac{dNdz}{Ef}+2\,\frac{1}{2dz}\,\frac{dN}{2dz}\,\frac{Hdz}{G\delta}+2\,\frac{1}{2dz}\,\frac{dN}{2dz}\,\frac{Bdz}{G\delta}-$$
$$-2\,\frac{1}{2dz}\,\frac{q_{\mathfrak{M}}Bdz}{G\delta}+2\,\frac{1}{2dz}\,\frac{q_{\mathfrak{M}}Hdz}{G\delta}=\frac{4}{3}\,\frac{dNdz}{Ef}+$$
$$+\frac{H+B}{2G\delta}\,\frac{dN}{dz}+\frac{1}{G\delta}\,q_{\mathfrak{M}}(H-B).$$

Disregarding quantities of the second order of smallness, we obtain

$$\Delta_{1P}=\frac{1}{G\delta}\left[\frac{H+B}{2}\,\frac{dN}{dz}+q_{\mathfrak{M}}(H-B)\right],$$

where the right member is equal to $\frac{\partial\Gamma}{\partial\left(\frac{dN}{dz}\right)}$

Satisfying the condition that the displacement $\Delta_{1P}=0$ at z = ℓ, we obtain

$$\frac{H+B}{2}\left(\frac{dN}{dz}\right)_{z=l}+q_{\text{ж}}(H-B)=0,$$

which is equivalent to the condition $\left(\frac{\partial\Gamma}{\partial\left(\frac{dN}{dz}\right)}\right)_{z=l}=0,$ which was obtained earlier in a very simple manner. The other route has been indicated only to bring out the physical significance of this boundary condition.

Manu-script Page No.	[Footnotes]
72	The surface forces include the pressures of liquids, gases or one body on another. Volume forces are gravitational and inertial.

Manu-script Page No.	[Transliterated Symbols]
105	ПКС = PKS = potok kasatel'nykh sil = tangential-force flux
109	sh = sinh
109	ch = cosh

Chapter 5

CERTAIN APPROXIMATE METHODS OF SOLUTION FOR PROBLEMS IN STRUCTURAL MECHANICS

1. THE RITZ METHOD

Major mathematical difficulties are frequently encountered in determining unknown functions from the condition of zero variation of potential energy; they arise in integrating the differential equations. The idea of the Ritz method consists in representing a function v(x) that realizes an extreme of the functional in the form of a series

$$v(x)=\sum_{i=1}^{n} b_i\psi_i(x)$$

with arbitrary parameters b_i. Then the functional Φ is transformed into a function of unknown parameters b_i. These parameters are so selected that the function

$$\Phi(b_i)$$

reaches an extreme. Consequently, the b_i must be determined from an equation system of the form

$$\frac{\partial \Phi}{\partial b_i}=0, \qquad (i=1,\ 2,\ 3,\ \ldots,\ n)$$

Without loss of generality, we shall set forth the Ritz method in an example in which the potential energy is represented by a functional of the form

$$\Phi=\int_a^b F(x,\ v,\ v',\ v'')\,dx. \tag{5.1}$$

The solution of the function v that appears in this expression is obtained in the form of a series containing arbitrary parameters b_i, where $i = 1, 2, \ldots, n$:

$$v(x)=\sum_{i=1}^{n} b_i \psi_i (x). \tag{5.2}$$

Here $\psi_i(x)$ are functions that satisfy assigned geometrical boundary conditions.

Substituting the adopted function (5.2) and its derivatives into Expression (5.1), we obtain a function

$$\Phi=\int_0^b f(x, b_1, b_2, \ldots b_n)dx.$$

Integrating with respect to x, we obtain a function that depends only on the parameters:

$$\Phi=\Phi(b_1, b_2, \ldots b_n). \tag{5.3}$$

The parameters b_i must be selected such that the function $\Phi(b_i)$ reaches an extreme.

This presupposes that

$$\left.\begin{array}{l} \dfrac{\partial\Phi}{\partial b_1}=0, \\ \dfrac{\partial\Phi}{\partial b_2}=0, \\ \cdot \;\cdot \;\cdot \;\cdot \\ \dfrac{\partial\Phi}{\partial b_n}=0. \end{array}\right\} \tag{5.4}$$

Thus, we obtain n algebraic equations, from the simultaneous solution of which we can determine the n independent parameters.

It can be shown that an infinite series incorporating a complete sequence of functions actually minimizes the functional (5.1).* But we shall take a series with a finite number of terms. Consequently, the solution (5.2) will not contain all of the b_i. Hence the static boundary conditions will be satisfied approximately. Thus it is desirable to select in advance such functions ψ_i as satisfy both geometrical (deflection, angle of turn) and static (transverse force, bending moment) boundary conditions.

Analysis of the physical aspect of a problem frequently enables us to postulate a possible type of variation for the unknown function. It is this assumption that should be taken into account in constructing an approximating series of the form (5.2). Use of the Ritz method usually involves a large volume of computing work, but the calculations are simplified when the series converge rapidly. A practical estimate of the rapidity of series convergence is made during the calculation. Progressively increasing the number of parameters, we calculate the errors between one solution and the next. If these errors diminish rapidly and tend to zero, then the approximating series was well selected. Series with functions ψ_1 that satisfy both geometrical and static boundary conditions usually converge rapidly.

This method and other methods that do not reduce the variational problem to differential equations are known as direct methods.

By way of example, let us determine the maximum deflection w and the maximum bending moment M that arise when a uniformly distributed load q acts on a hinged beam of length ℓ (Fig. 5.1).

We represent the elastic line in the form of a sinusoid with only a single arbitrary parameter b:

$$w = b \sin \frac{\pi x}{l}.$$

Its derivatives are

$$w' = \frac{b\pi}{l} \cos \frac{\pi x}{l}; \quad w'' = -\frac{b\pi^2}{l^2} \sin \frac{\pi x}{l}; \quad w''' = -\frac{b\pi^3}{l^3} \cos \frac{\pi x}{l}.$$

The selected function satisfies not only the geometrical boundary conditions

$$\text{at } x = 0 \text{ and } x = \ell \text{ the deflection } w = 0 \qquad \text{(a)}$$

but also the static conditions

$$\text{at } x = 0 \text{ and } x = \ell \text{ the moment } M = -EJw'' = 0. \qquad \text{(b)}$$

The potential energy of the system is

$$\Pi=\frac{EJ}{2}\int_0^l (w'')^2 dx-\int_0^l qw\,dx=$$

$$=\frac{EJ}{2}\,\frac{b^2\pi^4}{l^4}\int_0^l \sin^2\frac{\pi x}{l}\,dx-qb\int_0^l \sin\frac{\pi x}{l}\,dx=$$

$$=\frac{EJ}{4}\,\frac{b^2\pi^4}{l^3}-\frac{2lqb}{\pi}.$$

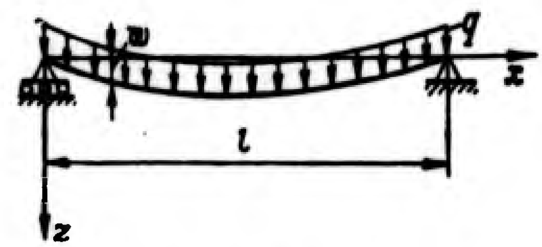


Fig. 5.1

According to (5.4),

$$\frac{\partial\Pi}{\partial b}=\frac{EJb\pi^4}{2l^3}-\frac{2ql}{\pi}=0,$$

from which

$$b=\frac{4ql^4}{EJ\pi^5}.$$

Consequently,

$$w=\frac{4ql^4}{EJ\pi^5}\sin\frac{\pi x}{l}.$$

The maximum deflection (at $l/2$) is

$$w_{max}=\frac{ql^4}{76{,}5\,EJ}.$$

According to the exact solution

$$w'_{max}=\frac{5}{384}\,\frac{ql^4}{EJ}=\frac{ql^4}{76{,}8EJ}.$$

Thus, the error is less than 0.5%. The bending moment

$$M=-EJw''=\frac{EJb\pi^2}{l^2}\sin\frac{\pi x}{l}=\frac{4ql^2}{\pi^3}\sin\frac{\pi x}{l}.$$

The maximum bending moment

$$M_{max}=\frac{4ql^2}{\pi^3}=\frac{ql^2}{7{,}75}.$$

According to the exact solution

$$M_{max}=\frac{ql^2}{8}.$$

The error is approximately 3%.

The intersecting force

$$Q=-EJw'''=EJ\frac{4ql^4\pi^3}{EJ\pi^5l^3}\cos\frac{\pi x}{l}=\frac{4ql}{\pi^2}\cos\frac{\pi x}{l}.$$

at $x = 0$,

$$Q=\frac{4ql}{\pi^2}=0{,}407ql.$$

Here the error is now about 20% as compared with the exact value.

Thus, good accuracy is obtained for the displacements and poorer accuracy for the force factors when the problem is solved in the displacements. However, the calculation can always be refined by increasing the number of terms in the series. In the case under consideration, the error of the calculation can be reduced by increasing the number of parameters, i.e., by taking

$$w=\sum_{i=1}^{n}b_i\sin\frac{i\pi x}{l}. \tag{5.5}$$

If we assign a series that satisfies only the geometrical boundary conditions (a), for example,

$$w=\sum_{i=1}^{n}b_ix^i(l-x)^i, \tag{5.6}$$

the reader can verify independently that it is necessary to keep a considerably larger number of terms to obtain sufficient accuracy in the series (5.6) than in the solution (5.5), which also satisfies the static boundary conditions (b).

2. TORSION OF A PRISMATIC WING WITH A RHOMBOID PROFILE

Let us apply the Ritz method to the problem of twisting of a prismatic wing with a rhomboid profile. In accordance with (5.2), we take a twisting stress function ϕ in the form of a series

$$\phi=b_1\psi_1+b_2\psi_2+b_3\psi_3\ldots, \qquad (1)$$

where ψ_1, ψ_2, ψ_3 are functions that vanish on the contour [24].

We show the solution for only one term of Series (1).

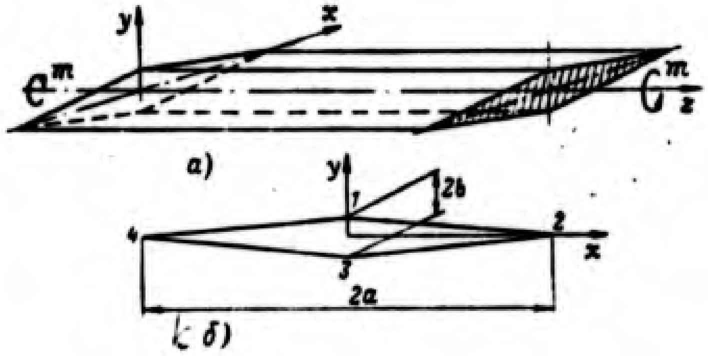


Fig. 5.2

Thus, we assume in first approximation

$$\phi=b_1\psi_1. \qquad (2)$$

To determine the function ψ_1, we write an equation describing the contour of the rhomboid profile (Fig. 5.2).

The equations of the straight line forming the boundaries of the contour are

$$\left.\begin{array}{ll}
\text{for line } 1\text{–}2 & \frac{x}{a}+\frac{y}{b}-1=0,\\
\text{for line } 2\text{–}3 & \frac{x}{a}-\frac{y}{b}-1=0,\\
\text{for line } 3\text{–}4 & -\frac{x}{a}-\frac{y}{b}-1=0,\\
\text{for line } 4\text{–}1 & -\frac{x}{a}+\frac{y}{b}-1=0.
\end{array}\right\} \qquad (3)$$

The function ψ_1, which consists of the product of the left-hand members of Eqs. (3):

$$\psi_1=\left(\frac{x}{a}+\frac{y}{b}-1\right)\left(\frac{x}{a}-\frac{y}{b}-1\right)\left(-\frac{x}{a}-\frac{y}{b}-1\right)\times$$
$$\times\left(-\frac{x}{a}+\frac{y}{b}-1\right)=\frac{x^4}{a^4}+\frac{y^4}{b^4}-\frac{2x^2y^2}{a^2b^2}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1,$$

vanishes on the contour.

Hence the stress function

$$\phi=b_1\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}-\frac{2x^2y^2}{a^2b^2}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1\right) \tag{4}$$

satisfies the boundary condition on the contour.

We find the parameter b_1 from the condition $\frac{\partial \overline{\Pi}}{\partial b_1}=0$, where, according to (4.35),

$$\overline{\Pi}=\frac{1}{2G}\iint\left\{\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]-4G\theta\phi\right\}dx\,dy. \tag{5}$$

Introducing ϕ from (4) and the values of $\left(\frac{\partial\phi}{\partial x}\right)^2, \left(\frac{\partial\phi}{\partial y}\right)^2$, into this expression, we obtain

$$\begin{aligned}\overline{\Pi}=\frac{1}{G}\int_0^a\int_0^b\Big\{8b_1^2\Big[&\frac{x^6}{a^8}+\frac{x^2y^4}{a^4b^4}+\frac{x^2}{a^4}-\frac{2x^4y^2}{a^6b^2}-\frac{2x^4}{a^6}+\frac{2x^2y^2}{a^2b^2}+\\&+\frac{y^6}{b^8}+\frac{x^4y^2}{a^4b^4}+\frac{y^2}{b^4}-\frac{2x^2y^4}{a^2b^6}-\frac{2y^4}{b^6}+\frac{2x^2y^2}{a^2b^4}\Big]-\\&-2G\theta b_1\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}-\frac{2x^2y^2}{a^2b^2}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1\right)\Big\}\,dx\,dy.\end{aligned} \tag{6}$$

On integration of this expression, we obtain

$$\overline{\Pi}=\frac{4}{G}\left[8b_1^2\,\frac{2}{45}\left(\frac{a}{b}+\frac{b}{a}\right)-\frac{20}{45}\,G\theta b_1 ab\right]. \tag{7}$$

From the condition

$$\frac{\partial\overline{\Pi}}{\partial b_1}=\frac{4}{G}\left[\frac{32b_1}{45}\left(\frac{a}{b}+\frac{b}{a}\right)-\frac{20}{45}\,G\theta ab\right]=0 \tag{8}$$

we find

$$b_1=\frac{5}{8}\,\frac{G\theta ab}{\left(\frac{a}{b}+\frac{b}{a}\right)}. \tag{9}$$

Hence the stress function

$$\phi=\frac{5}{8}\,\frac{G\theta ab}{\left(\frac{a}{b}+\frac{b}{a}\right)}\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}-\frac{2x^2y^2}{a^2b^2}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1\right). \tag{10}$$

According to Formulas (4.34), we find for the tangential stresses

$$\tau_{zx}=\frac{\partial\phi}{\partial y}=-\frac{5G\theta ab}{8\left(\frac{a}{b}+\frac{b}{a}\right)}\left(\frac{4y^3}{b^4}-\frac{4x^2y}{a^2b^2}-\frac{4y}{b^2}\right); \tag{11}$$

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} = -\frac{5G\theta ab}{8\left(\frac{a}{b}+\frac{b}{a}\right)}\left(\frac{4x^3}{a^4}-\frac{4xy^2}{a^2b^2}-\frac{4x}{a^2}\right). \tag{12}$$

We determine the angle θ from the expression

$$\mathfrak{M} = 2\iint \phi \, dxdy^*. \tag{13}$$

Substituting the value of ϕ from Formula (10) into (13), we find on integration

$$\mathfrak{M} = \frac{5}{8}\,\frac{G\theta ab}{\left(\frac{a}{b}+\frac{b}{a}\right)}\,\frac{80}{45}\,ab = \frac{10}{9}\,\frac{G\theta a^2b^2}{\left(\frac{a}{b}+\frac{b}{a}\right)}. \tag{14}$$

From this,

$$\theta = \frac{9\left(\frac{a}{b}+\frac{b}{a}\right)\mathfrak{M}}{10Ga^2b^2} = \frac{\mathfrak{M}}{GJ}, \tag{15}$$

where

$$J = \frac{10}{9}\,\frac{a^2b^2}{\left(\frac{a}{b}+\frac{b}{a}\right)}.$$

Consequently,

$$\tau_{zx} = \frac{9}{4}\,\frac{\mathfrak{M}}{ab^3}\left(\frac{y^3}{a^2}-\frac{x^2y}{a^2}-y\right); \tag{16}$$

$$\tau_{zy} = -\frac{9}{4}\,\frac{\mathfrak{M}}{a^3b}\left(\frac{x^3}{a^2}-\frac{xy^2}{b^2}-x\right). \tag{17}$$

On the horizontal axis of symmetry ($y = 0$), we have

$$(\tau_{zx})_{y=0} = 0 \text{ and } (\tau_{zy})_{y=0} = -\frac{9}{4}\,\frac{\mathfrak{M}}{a^3b}\left(\frac{x^3}{a^2}-x\right). \tag{18}$$

For

$$x = 0 \qquad (\tau_{zy})_{y=0} = 0;$$
$$x = \pm a \qquad (\tau_{zy})_{y=0} = 0.$$

We find the extreme value of $(\tau_{zy})_{y=0}$ from the condition

$$\left(\frac{d\tau_{zy}}{dx}\right)_{y=0} = -\frac{9}{4}\,\frac{\mathfrak{M}}{a^3b}\left(\frac{3x^2}{a^2}-1\right) = 0.$$

From this, x at the extreme value is

$$x=\sqrt{\frac{a^2}{3}}=\frac{a}{\sqrt{3}}.$$

Consequently,

$$(\tau_{zy})_{\substack{max\\ y=0}}=-\frac{9}{4}\,\frac{\mathfrak{M}}{a^3b}\left(\frac{a^3}{3\sqrt{3}a^2}-\frac{a}{\sqrt{3}}\right)=\frac{3}{2}\,\frac{\mathfrak{M}}{\sqrt{3}a^2b}. \tag{19}$$

On the vertical axis of symmetry (x = 0),

$$\tau_{zy}=0 \text{ and } (\tau_{zx})_{x=0}=\frac{9}{4}\,\frac{\mathfrak{M}}{ab^3}\left(\frac{y^3}{b^2}-y\right). \tag{20}$$

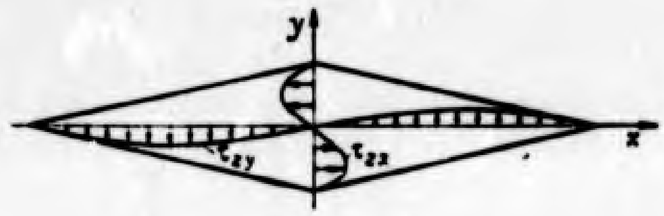


Fig. 5.3

We conclude from comparison of Formulas (18) and (20) that

$$(\tau_{zx})_{\substack{max\\ x=0}}=\frac{3}{2}\,\frac{\mathfrak{M}}{\sqrt{3}ab^2}. \tag{21}$$

Since b << a, $(\tau_{zx})_{max} >> (\tau_{zy})_{max}$.

The variations of $(\tau_{zy})_{y=0}$ and $(\tau_{zx})_{x=0}$ are shown in Fig. 5.3.

For b = a (Fig. 5.4), we obtain from Expression (14)

$$\mathfrak{M}=\frac{5}{9}\,G\theta a^4=0{,}5555G\theta a^4. \tag{22}$$

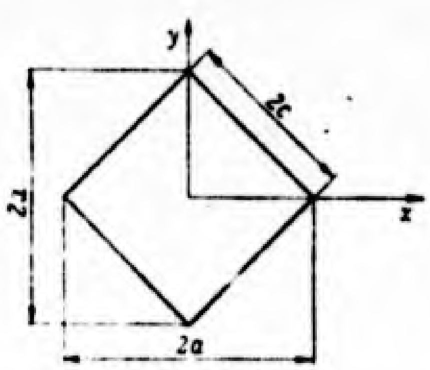


Fig. 5.4

The length of a side of the square is

$$2C=a\sqrt{2}.$$

According to the exact solution for the square [24],

$$\mathfrak{M}=0{,}1406G\theta(2c)^4=0{,}1406G\theta(a\sqrt{2})^4=0{,}5624C\theta a^4. \tag{23}$$

The value of the moment according to Formula (22) is 1.25% smaller than the exact value according to (23).

To obtain a more exact solution, the functions ψ_2, ψ_3, ... of the series (1) may be taken in the form

$$\psi_2=x^2\psi_1;\quad \psi_3=y^2\psi_1;\quad \psi_4=x^2y^2\psi_1\ldots$$

Substituting these values in the series (1), we obtain

$$\phi=\left(\frac{x^4}{a^4}+\frac{y^4}{b^4}-\frac{2x^2y^2}{a^2b^2}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1\right)\times \\ \times(b_1+b_2x^2+b_3y^2+b_4x^2y^2+\ldots). \tag{24}$$

Here we have included only terms with even powers, since the stress function ϕ must be symmetrical about the x- and y-axes for symmetrical sections about these axes.

3. THE BUBNOV-GALERKIN METHOD

The Bubnov-Galerkin method is used in approximate solution of differential equations* obtained in the usual manner or by minimization of potential energy.

Without compromising the generality of the method under consideration, we shall illustrate it, as we did with the Ritz method, on a one-dimensional problem, when its solution reduces to a differential equation of the form

$$L(x, v, v', v'')=0. \tag{5.7}$$

Here, as in the Ritz method, the function v is represented in the form of a series:

$$v(x)=\sum_{i=1}^{n} b_i\psi_i(x), \tag{5.8}$$

where $\psi_i(x)$ are assigned functions that satisfy the boundary conditions.

Substituting the expression for Function (5.8) and its derivatives into Expression (5.7), we obtain

$$L(x, b_1, b_2 \ldots b_n) \neq 0. \tag{5.9}$$

The left member of Eq. (5.9) will be equal to zero only provided that the series (5.8) is an exact expression of the function v; Otherwise, this member will only approach zero as the function v approaches its true value, which is brought about by proper selection of the parameters b_i. For this purpose, we multiply the left member of differential equation (5.7) successively by the functions $\psi_1(x)$, $\psi_2(x)$,...,$\psi_n(x)$ and, integrating with respect to $\underline{x}$, obtain a system of $\underline{n}$ algebraic equations:

$$\left.\begin{array}{ll} \int\limits_a^b L(x, b_1, b_2 \ldots b_n)\psi_1\,dx=0, & (1) \\ \int\limits_a^b L(x, b_1, b_2 \ldots b_n)\psi_2\,dx=0, & (2) \\ \int\limits_a^b L(x, b_1, b_2 \ldots b_n)\psi_n\,dx=0. & (n) \end{array}\right\} \tag{5.10}$$

From the simultaneous solution of these equations, we obtain all $\underline{n}$ parameters, which determine the approximate value of the unknown function.

This simple method of approximate solution for the differential equations can be justified as follows. As was shown in Section 7 of Chapter 4, the minimum condition for the functional of the two dimensional problem reduces to the equation

$$\iint \left[\sum_{i=1}^{n} L_i \delta f_i\right] dx\,dy = 0. \tag{5.11}$$

For simplicity, we have taken a differential equation in the form (5.7). In this case, Eq. (5.11) will assume the form

$$\int\limits_a^b L(x, v, v', v'')\,\delta v\,dx = 0. \tag{5.12}$$

Taking, as before, the expression of the function v in the form of a series (5.8), i.e.,

$$v(x)=b_1\psi_1(x)+b_2\psi_2(x)+\ldots+b_n\psi_n(x), \tag{a}$$

we obtain the variation of the function v(x) in the form

$$\delta v=\frac{\partial v}{\partial b_1}\delta b_1+\frac{\partial v}{\partial b_2}\delta b_2+\ldots+\frac{\partial v}{\partial b_n}\delta b_n$$

or

$$\delta v=\psi_1\delta b_1+\psi_1\delta b_2+\ldots+\psi_n\delta b_n. \tag{b}$$

Substituting the function v(x) of Expression (a) with its derivatives and Expression (b) into Eq. (5.12), we obtain

$$\int_a^b L(x_1,\ b_1,\ b_2\ldots b_n)(\psi_1\delta b_1+\psi_2\delta b_2+\ldots+\psi_n\delta b_n)\,dx=0$$

or

$$\int_a^b L(x,\ b_1,\ b_2\ldots b_n)\psi_1\delta b_1\,dx+\int_a^b L(x,\ b_1,\ b_2\ldots b_n)\psi_2\delta b_2\,dx+$$
$$+\ldots+\int_a^b L(x,b_1,b_2\ldots b_n)\psi_n\delta b_n\,dx=0. \tag{c}$$

Remembering that the variations δb_1, δb_2, ...,δb_n in the interval (a, b) are arbitrary, Eq. (c) is satisfied only provided that

$$\int_a^b L(x,\ b_1,\ b_2\ldots b_n)\psi_1\,dx=0;$$
$$\int_a^b L(x,\ b_1,\ b_2\ldots b_n)\psi_2\,dx=0;$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\int_a^b L(x,\ b_1,\ b_2\ldots b_n)\psi_n\,dx=0.$$

Thus we have arrived at the Bubnov-Galerkin equations (5.10). Let us perform the calculation by the Bubnov-Galerkin method for the hinged beam considered in Section 1 of Chapter 5.

We know that for the system under consideration (see Fig. 5.1), the differential equation of the elastic line takes the form

$$EJ\,\frac{d^4w}{dx}-q=0.$$

We recall once again that this equation is an equation of equilibrium. Taking, as before, the deflection function in the form

$$w=b_i\sin\frac{i\pi x}{l}$$

and applying Expression (c), we obtain

$$\int_0^l\left(EJ\,\frac{d^4w}{dx^4}-q\right)\delta b_i\sin\frac{i\pi x}{l}\,dx=0.$$

The integrand is the work of the external and internal forces (i.e., $EJ\frac{d^4w}{dx^4}-q$)dx in accomplishing the possible displacements $\delta b_i\sin\frac{i\pi x}{l}$. Remembering the arbitrariness of the variation δb_1 and taking one term of the series, we obtain

$$\int_0^l\left(EJ\,\frac{d^4w}{dx^4}-q\right)\sin\frac{\pi x}{l}\,dx=0.$$

Remembering that

$$\frac{d^4w}{dx^4}=b\,\frac{\pi^4}{l^4}\sin\frac{\pi x}{l},$$

we find

$$\frac{EJb\pi^4}{l^4}\int_0^l\sin^2\frac{\pi x}{l}\,dx-q\int_0^l\sin\frac{\pi x}{l}\,dx=\frac{EJb\pi^4}{2l^3}-\frac{2ql}{\pi}=0.$$

From this

$$b=\frac{4ql^4}{EJ\pi^5}\ \text{and}\ w=\frac{4ql^4}{EJ\pi^5}\sin\frac{\pi x}{l}.$$

The same solution was obtained by the Ritz method. This agreement is accounted for by the fact that the approximating function satisfied not only the kinematic boundary conditions w = 0 at x = 0 and x = ℓ, but also the static conditions – the moment M = EJw" = 0 at x = 0 and x = ℓ.

4. THE MIXED METHOD

To avoid having to integrate the partial differential equation.

(4.54), which the unknown functions satisfies in the solution of the two dimensional problem, we may follow the same procedure as in solving the one dimensional problem, representing the approximate value of the unknown function in the form of a series with assigned functions that depend on the coordinates x and y, i.e.,

$$v(x, y)=\sum_{i=1}^{n} b_i\psi_i(x, y). \tag{5.13}$$

In numerous cases, however, it is difficult to assign suitable functions in two directions. Thus it is sometimes more convenient, and the results more accurate, if we use the so-called mixed method, the sense of which consists in assigning the functions in one of the directions, in which the variation of the true function is known quite reliably, while the functions remain unknown in the other direction, which is usually perpendicular to the first, and are determined from solutions of ordinary differential equations. Thus, if the functions are assigned in the direction of the y-axis, an expression for the function $v(x, y)$ may be found in the form of a series

$$v(x, y)=\sum_{i=1}^{n} \psi_i(y)\varphi_i(x), \tag{5.14}$$

where $\psi_i(y)$ is an assigned function, dependent only on the coordinate y and $\varphi_i(x)$ is the unknown function, which depends only on the coordinate x.

Substituting the value of v defined by Series (5.14) into the expression for the functional of the two dimensional problem [for example, (4.52)], we obtain the functional of the one dimensional problem on integration with respect to y.

Minimization of this functional with respect to the unknown functions yields a system of ordinary differential equations that, together with the boundary conditions, fully define the values of the unknown functions $\varphi_i(x)$ and, consequently, the approximate solution to Expression (5.14) will have been found.

The mixed method was proposed by L.V. Kantorovich [10] in 1933. Owing to a series of monumental studies by V.Z. Vlasov, this method came into extensive use in the design of complex engineering structures. Thus the mixed method is frequently known as the Kantorovich-Vlasov method in its technical applications. Further development and generalization of V.Z. Vlasov's theory for the design of airframe structures by the mixed method may be found in I.F. Obraztsov's book [16].

5. FINITE-DIFFERENCE EQUATIONS OF THE MIXED METHOD

To solve problems of structural mechanics by the mixed method, it is necessary to find functions that depend on a single coordinate, for example, x from a solution of differential equations.

If the system is regular along the x-axis (sections perpendicular to the x-axis are similar to one another as regards rigidity and geometrical characteristics) and the external forces vary simply or are constant along this axis, no particular difficulty is encountered in solving the differential equations describing the stressed and strained states of these systems. Otherwise, analytical solution of the differential equations might be found too complicated for practical use. Then equations in finite differences may be employed as an approximate solution to the problem.

The equation in finite differences for some section m (Fig. 5.5) may be represented in the form

$$\delta_{m,m-2}X_{m-2}+\delta_{m,m-1}X_{m-1}+\delta_{m,m}X_m+\delta_{m,m+1}X_{m+1}+ \\ +\delta_{m,m+2}X_{m+2}+\Delta_{m,p}=0. \qquad (5.15)$$

If the rods in the direction perpendicular to the x-axis (struts, ribs, bulkheads) are absolutely rigid, the extreme terms $\delta_{m,m-2}X_{m-2}$ and $\delta_{m,m+2}X_{m+2}$ of Eq. (5.15) vanish. Then we obtain an equation with only the three unknowns X_{m-1}, X_m and X_{m+1}, i.e.,

$$\delta_{m,m-1}X_{m-1}+\delta_{m,m}X_m+\delta_{m,m+1}X_{m+1}+\Delta_{m,p}=0^{*}. \qquad (5.16)$$

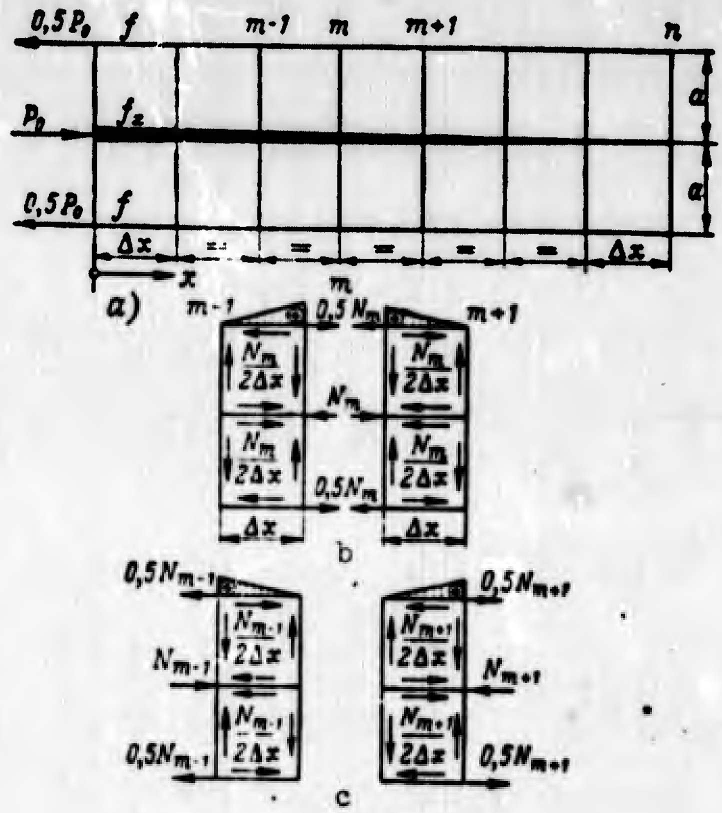


Fig. 5.5

Applying Eq. (5.15) or (5.16) to all sections m = 0, 1, 2, 3, ...,n (Fig. 5.6), we may obtain a system of algebraic equations for determination of the unknowns X_1, X_2, X_3, ..., X_n.

Since the sections m = 0, ., 2, 3, ..., n are separated from one another by a finite distance Δx, the complex law of variation of the rigidity characteristics of both the individual elements and the external forces acting on the system may, in determining the coefficients δ and the absolute terms Δ for equations of the form (5.15) or (5.16), be replaced on a distance Δx by their constant averaged values, as opposed to the variable coefficients of the corresponding differential equations. As a result of the substitution, we obtain, instead of the variable coefficients and absolute terms, constant coefficients and terms for any section m.

The following example will illustrate casting and solution of the equation in finite differences. Figure 5.5 shows the system that was analyzed in the example of Section 7, Chapter 4 in determination of the "ω" state; the only difference consists in the fact that the middle strip has a variable cross-sectional area. In this case, therefore, we are dealing with a nonregular system. To determine the "ω" state, we shall use equations in finite differences.

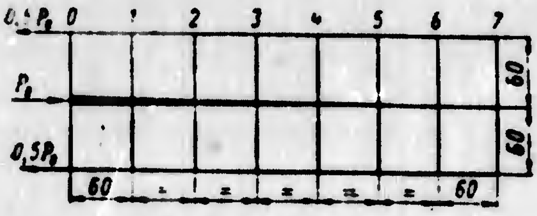


Fig. 5.6

We shall assume that the cross-sectional area of the middle strip f_x is varied nonlinearly along the x-axis.

We mark a series of cross sections separated from one another by finite distances Δx so small that we may assume the cross sections of the middle strip to vary linearly on the segments between sections.

We write the finite-difference equation for some section m (Fig. 5.5a). Figure 5.5b shows the stressed state of the segments m, m − 1 and m, m + 1 with external forces N_m acting in section m and linear variation of these forces on segments of length Δx.

The strain consistency equation of section m may be written on the basis of the condition that the sum of the works done by the forces in accomplishing the displacements in section m is equal to zero.

Let us determine the work of the forces N_m on the displacements in section m: here we shall take averaged values of the cross-sectional areas for the middle strip, i.e.,

$$f_{m,m-1}=\frac{f_m+f_{m-1}}{2};\quad f_{m,m+1}=\frac{f_m+f_{m+1}}{2},$$

where f_{m-1}, f_m, f_{m+1} are the cross-sectional areas of the middle strip in the respective sections m − 1, m, m + 1.

The work of the forces N_m on the displacements in section m is

$$\Delta A_{m,m}=4\frac{0{,}5N_m\Delta x}{2}\,\frac{2\cdot 0{,}5}{3Ef}+\frac{N_m\Delta x}{2}\,\frac{2}{3}\,\frac{1}{Ef_{m,m-1}}+\frac{N_m\Delta x}{2}\,\frac{2}{3}\,\frac{1}{Ef_{m,m+1}}+$$
$$+4\frac{N_m}{2\Delta x}\,\frac{a\Delta x}{2\Delta xG\delta}=\frac{1}{3}\,\frac{N_m\Delta x}{Ef}+\frac{1}{3}\,\frac{N_m\Delta x}{E}\left(\frac{1}{f_{m,m-1}}+\frac{1}{f_{m,m+1}}\right)+$$
$$+\frac{N_m a}{\Delta xG\delta}=N_m\left[\frac{1}{3E}\left(\frac{\Delta x}{f}+\frac{\Delta x}{f_{m,m-1}}+\frac{\Delta x}{f_{m,m+1}}\right)+\frac{a}{\Delta xG\delta}\right].$$

The work of the forces N_{m-1} and N_{m+1} (Fig. 5.5c) on the displacements in section m is accordingly

$$\Delta A_{m,m-1}=N_{m-1}\left[\frac{1}{3E}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{m,m-1}}\right)-\frac{a}{2\Delta xG\delta}\right];$$
$$\Delta A_{m,m+1}=N_{m+1}\left[\frac{1}{3E}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{m,m+1}}\right)-\frac{a}{2\Delta xG\delta}\right].$$

Thus, the equation in finite differences for section m with G = E/ /2.6 takes the form

$$N_{m-1}\left[\frac{1}{3}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{m,m-1}}\right)-\frac{1{,}3a}{\Delta x\delta}\right]+N_m\left[\frac{1}{3}\left(\frac{\Delta x}{f}+\frac{\Delta x}{f_{m,m-1}}+\frac{\Delta x}{f_{m,m+1}}\right)+\right.$$
$$\left.+\frac{2{,}6a}{\Delta x\delta}\right]+N_{m+1}\left[\frac{1}{3}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{m,m+1}}\right)-\frac{1{,}3a}{\Delta x\delta}\right]=0. \qquad (5.17)$$

Here, as in the example of Section 7, Chapter 4, the struts are assumed to be absolutely rigid. If the elasticity of the struts is taken into account – something that represents no difficulty – the left member of Eq. (5.17) will consist of five terms.

For numerical calculation, the data taken earlier (δ = 0.1 cm; f = 5 cm^2; a = 60 cm) is complemented by the following additional data: 1) Δx = 60 cm; 2) cross-sectional areas of center strip in sections 0 and 1 respectively f_0 = 15 cm^2 and f_1 = 10 cm^2, but 5 cm^2 in all subsequent sections (see Fig. 5.6).

First we perform the calculation with the assumption that the forces N_3 = 0 in Section 3. We begin with Section 1, for which Eq. (5.17) acquires the form

$$-P_0\left[\frac{1}{3}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{1,0}}\right)-\frac{1{,}3a}{\Delta x\delta}\right]+N_1\left[\frac{1}{3}\left(\frac{\Delta x}{f}+\frac{\Delta x}{f_{1,0}}+\frac{\Delta x}{f_{1,2}}\right)+\right.$$
$$\left.+\frac{2{,}6a}{\Delta x\delta}\right]+N_2\left[\frac{1}{3}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{1,2}}\right)-\frac{1{,}3a}{\Delta x\delta}\right]=0.$$

Remembering that

$$f_{1,0}=\frac{10+15}{2}=12,5\ \text{cm}^2 \text{ and } f_{1,2}=\frac{10+5}{2}=7,5\ \text{cm}^2$$

we obtain

$$-P_0\left[\frac{1}{3}\left(\frac{60}{4\cdot 5}+\frac{60}{2\cdot 12,5}\right)-\frac{1,3\cdot 60}{60\cdot 0,1}\right]+N_1\left[\frac{1}{3}\left(\frac{60}{5}+\frac{60}{12,5}+\frac{60}{7,5}\right)+\right.$$
$$\left.+\frac{2,6\cdot 60}{60\cdot 0,1}\right]+N_2\left[\frac{1}{3}\left(\frac{60}{4\cdot 5}+\frac{60}{2\cdot 7,5}\right)-\frac{1,3\cdot 60}{60\cdot 0,1}\right]=-P_0\left[\frac{1}{3}(3+2,4)-13\right]+$$
$$+N_1\left[\frac{1}{3}(12+4,8+8)+26\right]+N_2\left[\frac{1}{3}(3+4)-13\right]=$$
$$=P_0(1,8-13)+N_1(8,3+26)+N_2(2,34-13)=0$$

or

$$+11,2P_0+34,3N_1-10,7N_2=0. \qquad \text{(a)}$$

Then we consider Section 2, for which Eq. (5.17) assumes the form

$$N_1\left[\frac{1}{3}\left(\frac{\Delta x}{4f}+\frac{\Delta x}{2f_{2,1}}\right)-13\right]+N_2\left[\frac{1}{3}\left(\frac{\Delta x}{f}+\frac{\Delta x}{f_{2,1}}+\frac{\Delta x}{f_{2,3}}\right)+26\right]=0.$$

Remembering that

$$f_{2,1}=f_{1,2}=7,5\ \text{cm}^2 \text{ and } f_{2,3}=5\text{cm}^2$$

we obtain

$$-10,7N_1+N_2\left[\frac{1}{3}\left(\frac{60}{5}+\frac{60}{7,5}+\frac{60}{5}\right)+26\right]=$$
$$=-10,7N_1+N_2\left[\frac{1}{3}(12+8+12)+26\right]=0$$

or

$$-10,7N_1+36,7N_2=0. \qquad \text{(b)}$$

From Eqs. (a) and (b), we obtain

$$N_2=\frac{10,7}{36,7}N_1=0,292N_1 \text{ and } N_1=-\frac{11,2P_0}{31,2}=-0,358P_0.$$

Thus:

$$N_1=-0,358P_0;\quad N_2=-0,0104P_0;\quad N_3=0.$$

It is clear from the calculation that in the present case, the forces decay more slowly than in the calculation performed previously with constant and uniform areas of all strips. Here again, however, the self-balancing force group decays very quickly and the assumption that we adopted to the effect that the force $N_3 = 0$ in Section 3 was quite justified. If a few more sections are introduced into the calculation, we may once again satisfy ourselves that it does not introduce any appreciable correction into the calculation performed if we assume that the force $N_3 = 0$ in Section 3.

Manuscript Page No.	[Footnotes]
113	A number of works by Soviet scientists have been devoted to this problem [10].
120	The derivation of this equation may be found in [24].
121	The equilibrium or strain-continuity equations.
127	V.N. Belyayev's equation of three axial forces [3] proceeds from this equation as a particular case.

Chapter 6

APPLICATION OF APPROXIMATE METHODS TO CALCULATIONS FOR RECTANGULAR PLATES

1. POTENTIAL ENERGY OF THE PLATE

A plate is a flat body whose thickness is small as compared with its other linear dimensions. The plane that divides the thickness of a plate equally is known as the median plane. We denote the thickness of the plate by h and its sides by a and b. As a convention, we shall set up a rectangular coordinate system in such a way that the xoy-plane is in the median plane of the undeformed plate, and the z-axis is directed downward (Fig. 6.1).

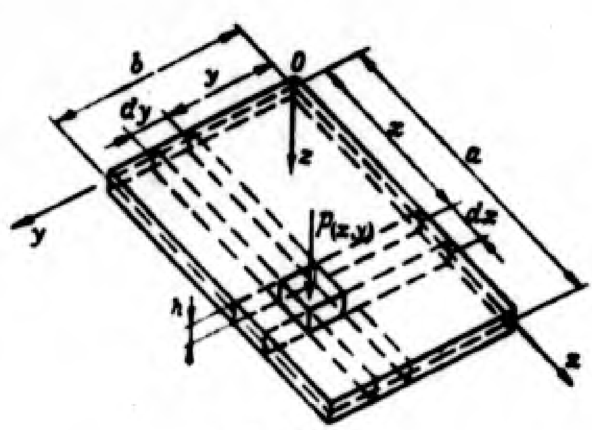


Fig. 6.1

The calculating model for elastic plates is based on the following hypotheses.

1. Kinematic hypothesis. Any element that is rectilinear and normal to the median plane of a plate before deformation remains rectilinear and normal to the deformed median surface of the plate.

Henceforth the deflections of the plates will be assumed to be small.

2. The static hypothesis. As in the elementary theory of bending, we shall disregard the normal stresses σ_z perpendicular to the median plane of the plate. In adopting this hypothesis, we shall regard the plate as under conditions of the plane stressed state.

Consequently, the expression for the deformation potential energy of the plate will take the form

$$U = \frac{1}{2} \int_0^a \int_0^b \int_{-\frac{h}{2}}^{+\frac{h}{2}} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy})\, dx\, dy\, dz. \tag{6.1}$$

In accordance with the second assumption, we obtain the strain components for the plate in bending from (4.6):

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y); \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x);$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)}{E} \tau_{xy}. \tag{6.2}$$

Substituting these expressions for the strain components in (6.1), we obtain

$$U = \frac{1}{2} \int_0^a \int_0^b \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{1}{E} [\sigma_x^2 + \sigma_y^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\tau_{xy}^2]\, dx\, dy\, dz. \tag{6.3}$$

Now let us express the stress components as functions of the plate's deflection w. We obtain from Expressions (6.2)

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\mu^2}(\varepsilon_x + \mu\varepsilon_y), \\ \sigma_y &= \frac{E}{1-\mu^2}(\varepsilon_y + \mu\varepsilon_x), \\ \tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy}. \end{aligned}\right\} \tag{6.4}$$

The relationships between the strain components and the displacement components take the form

$$\varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_y = \frac{\partial v}{\partial y}; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{6.5}$$

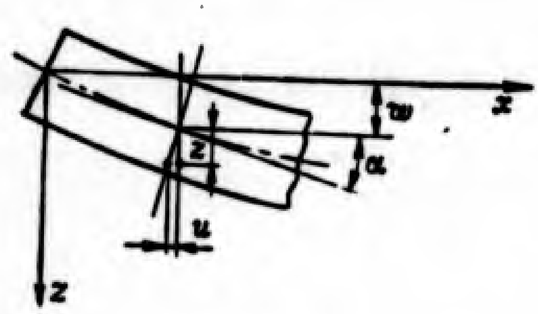


Fig. 6.2

To establish the relation between the displacement along the x-axis and the deflections $\underline{w}$ of the plate's median surface, we refer to Fig. 6.2.

The rotation angle of the tangent to the x-axis

$$\alpha \approx \operatorname{tg} \alpha = \frac{\partial w}{\partial x}.$$

Consequently, the displacement $\underline{u}$ along the x-axis of a point of the plate at a distance $\underline{z}$ from the median surface is

$$u = -\frac{\partial w}{\partial x} z.$$

Similarly,

$$v = -\frac{\partial w}{\partial y} z.$$

Consequently, the strain components

$$\left.\begin{aligned} \varepsilon_x &= -z w_{xx}, \\ \varepsilon_y &= -z w_{yy}, \\ \gamma_{xy} &= -2z w_{xy}, \end{aligned}\right\} \tag{6.6}$$

where $w_{xx} = \frac{\partial^2 w}{\partial x^2}$ etc.

Substituting the values of the strain components (6.6) into Eqs. (6.4), we obtain

$$\left.\begin{aligned} \sigma_x &= -\frac{Ez}{1-\mu^2}(w_{xx} + \mu w_{yy}), \\ \sigma_y &= -\frac{Ez}{1-\mu^2}(w_{yy} + \mu w_{xx}), \\ \tau_{xy} &= -\frac{Ez}{1+\mu} w_{xy}. \end{aligned}\right| \tag{6.7}$$

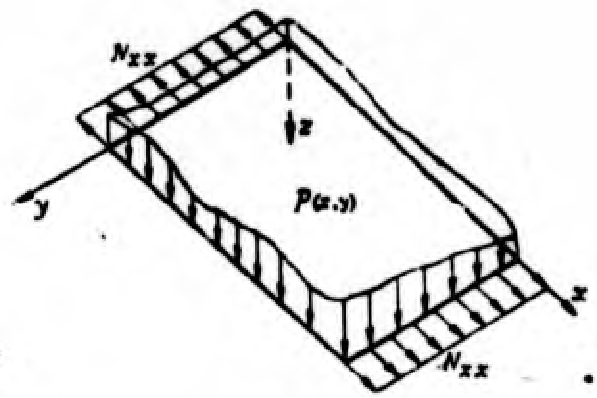


Fig. 6.3

Substituting Expressions (6.7) into Formula (6.3), we obtain

$$U=\int_0^a\int_0^b \frac{D}{2}\,[w_{xx}^2+w_{yy}^2+2\mu w_{xx}w_{yy}+2(1-\mu)w_{xy}^2]\,dy\,dx, \qquad (6.8)$$

where $D=\frac{Eh^3}{12(1-\mu^2)}$ is the cylindrical rigidity.

We determine the total potential energy $\Pi = U - W$ for the plate acted upon by a transverse load p(x, y) and tensile forces per unit length uniformly distributed along sides x = 0 and x = a (Fig. 6.3).

For the initial state we take the plate loaded only by the forces N_{xx}. Then we apply the transverse load p(x, y). Under the influence of this load, the plate will bend and the external forces p(x, y) and N_{xx} will perform a work

$$W=W_p+W_N,$$

where W_p is the work of the external forces p(x, y) and W_N is the work of the external forces N_{xx}.

The work of the external forces p(x, y) is

$$W_p=\int_0^a\int_0^b p(x, y)\,w\,dx\,dy.$$

To determine the work of the external forces N_{xx}, let us examine the strain of element AB on bending of the median plane (Fig. 6.4).

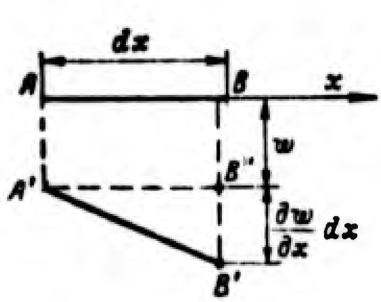


Fig. 6.4

When only the forces N_{xx} were operating, element AB was on the x-axis and had a length dx. After bending of the plate, element AB will have shifted to position A'B'* and point B will have been displaced in the direction of the x-axis by an amount

$$A'B'' - AB = \sqrt{dx^2 - \left(\frac{\partial w}{\partial x} dx\right)^2} - dx \approx$$
$$\approx dx - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx - dx \approx$$
$$\approx -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx.$$

Hence the sides x = 0 and x = a of the plate will have drawn together by $-\frac{1}{2}\int_0^a w_x^2\,dx$ and the work of the external forces applied to these sides is

$$W_x = -\frac{1}{2}\int_0^a\int_0^b N_{xx} w_x^2\,dy\,dx.$$

Thus, the total potential energy in the present case is

$$\Pi = \int_0^a\int_0^b \left\{\frac{D}{2}\left[w_{xx}^2 + w_{yy}^2 + 2\mu w_{xx} w_{yy} + 2(1-\mu) w_{xy}^2\right] + \right.$$
$$\left. + \frac{1}{2} N_{xx} w_x^2 - p(x, y) w \right\} dy\,dx. \qquad (6.9)$$

From the minimum condition of Expression (6.9), we obtain the differential equation equilibrium for the plate element.

Actually, according to Expression (4.54), we get

$$L(w) = -p_{(x,y)} - N_{xx} w_{xx} + \frac{D}{2}\left[\frac{\partial^2}{\partial x^2}(2w_{xx} + 2\mu w_{yy}) + \right.$$
$$\left. + \frac{\partial^2}{\partial x\,\partial y} 4(1-\mu) w_{xy} + \frac{\partial^2}{\partial y^2}(2w_{yy} + 2\mu w_{xx})\right] = 0$$

or

$$D(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) - p_{(x,y)} - N_{xx} w_{xx} = 0. \qquad (6.10)$$

If, in addition to the forces N_{xx}, tensile forces per unit length N_{yy} and tangential forces per unit length N_{yx} act in the median plane of the plate, the differential equation of the bent surface of the plate will assume the form

$$D(w_{xxxx}+2w_{xxyy}+w_{yyyy})-p_{(x,y)}-N_{xx}w_{xx}-$$
$$-2N_{xy}w_{xy}-N_{yy}w_{yy}=0. \qquad (6.11)$$

When only the transverse load $p_{(x,y)}$ is operating, we obtain Eq. (6.11) in the form

$$D(w_{xxxx}+2w_{xxyy}+w_{yyyy})-p_{(x,y)}=0. \qquad (6.12)$$

2. BENDING OF HINGED RECTANGULAR PLATES

1. Let us use the Bubnov-Galerkin method to find an approximate expression for the deflection of a hinged rectangular plate under a uniformly distributed transverse load p_0.

The deflections w must satisfy the equilibrium condition

$$w_{xxxx}+2w_{xxyy}+w_{yyyy}-\frac{p_0}{D}=0 \qquad (a)$$

with the following boundary conditions:

$$w=0, \quad w_{xx}=0 \quad \text{for } x=0 \text{ and } x=a;$$
$$w=0, \quad w_{yy}=0 \quad \text{for } y=0 \text{ and } y=a.$$

These boundary conditions will be satisfied if we express the deflection in the form of a double Fourier series

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \qquad (b)$$

In the approximate calculation, we assign the deflection functions in the form of only a single term of series (b), i.e.,

$$w=A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \qquad (c)$$

Substituting this deflection expression in Eq. (a), we obtain

$$A\left[\left(\frac{\pi}{a}\right)^4+2\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2+\left(\frac{\pi}{b}\right)^4\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}-\frac{p_0}{D}=0$$

or

$$A\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}-\frac{p_0}{D}=0.$$

Then, using the Bubnov-Galerkin method, we get

$$\int_0^a\int_0^b\left[A\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}-\frac{p_0}{D}\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy=0$$

or

$$A\pi^4\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^2\int_0^a\sin^2\frac{\pi x}{a}\,dx\int_0^b\sin^2\frac{\pi y}{b}\,dy-$$

$$-\frac{p_0}{D}\int_0^a\sin\frac{\pi x}{a}\,dx\int_0^b\sin\frac{\pi y}{b}\,dy=0.$$

On integration,

$$A\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\frac{ab}{4}-\frac{4p_0}{D}\,\frac{ab}{\pi^2}=0,$$

from which

$$A=\frac{16\,p_0}{D\pi^6\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}. \tag{6.13}$$

Therefore,

$$w=\frac{16\,p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{D\pi^6\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}. \tag{6.14}$$

The maximum deflection at the center of the plate

$$w_{max}=\frac{16\,p_0}{D\pi^6\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}. \tag{6.15}$$

In the case of a square plate, we have with $\mu = 0.3$

$$w_{max}=0.0454\,\frac{p_0a^4}{Eh^3} \tag{6.16}$$

with an error of about 2.5% of the exact solution obtained by Navier [26].

2. Using the Ritz method, we would substitute $w=A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$ into Eq. (6.9) for $N_{xx} = 0$, to obtain

$$\Pi=\int_0^a\int_0^b\Big\{\frac{D}{2}\Big[A^2\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}-$$

$$-2(1-\mu)A^2\frac{\pi^4}{b^2a^2}\left(\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}-\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}\right)\Big]-$$

$$-A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,p_0\Big\}\,dx\,dy.$$

Remembering that

$$\int_0^a\int_0^b \sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\,dx\,dy=\frac{ab}{4},$$

$$\int_0^a\int_0^b \cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b}\,dx\,dy=\frac{ab}{4}$$

and

$$\int_0^a\int_0^b \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy=\frac{4ab}{\pi^2},$$

we obtain

$$\Pi=DA^2\frac{\pi^4}{8}ab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2-A\frac{4ab}{\pi^2}p_0.$$

The condition $\frac{\partial\Pi}{\partial A}=0$ gives

$$DA\frac{\pi^4}{4}ab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2-\frac{4p_0ab}{\pi^2}=0.$$

Consequently,

$$A=\frac{16p_0}{D\pi^6\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}.$$

Thus,

$$w=\frac{16p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{D\pi^6\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}.$$

Since the approximating function (b) also satisfies static boundary conditions, the Bubnov-Galerkin and Ritz method both yield identical expressions for the deflection of the plate, but the Ritz method involves more labor. Hence if a selected function satisfies both geometrical and static boundary conditions, it is preferable to use the Bubnov-Galerkin method.

3. STABILITY OF PLATES

Let us consider a method of determining the critical load for a rectangular plate with hinge support at all four margins and subject to compressive forces N_{xx} that are uniformly distributed along opposite edges (Fig. 6.5).

We shall assume that there is no load perpendicular to the median plane of the plate. As long as the forces N_{xx} have not reached a certain critical value, the plate will retain its flat equilibrium shape in a stable manner. When, however, the external forces N_{xx} reach a certain critical value, the deflections that arise as a result of very small disturbances produced by the transverse load no longer vanish after the load is removed, and the plate does not return to its initial flat equilibrium shape. The flat shape has become unstable and the distorted shape stable. Let us determine the value that the forces N_{xx} must reach if this shape of the plate is to be retained. It is this value that will be the critical load for the plate.

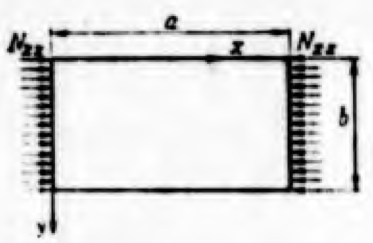


Fig. 6.5

The differential equation of the elastic plane of the plate may be obtained from Eq. (6.10) for $p_{(x,y)} = 0$:

$$w_{xxxx}+2w_{xxyy}+w_{yyyy}+\frac{N_{xx}}{D}w_{xx}=0.$$

Substituting the deflection function

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

which satisfies the boundary conditions, into this equation, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2-N_{xx}\pi^2\frac{m^2}{a^2}\right]A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=0.$$

Rejecting the trivial solution

$$A_{mn}=0 \quad \text{or} \quad w=0,$$

we obtain

$$D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2-N_{xx}\pi^2\frac{m^2}{a^2}=0$$

or

$$N_{xx}=\pi^2 D\frac{a^2}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2=\frac{\pi^2 D}{b^2}\left(\frac{mb}{a}+\frac{n^2 a}{mb}\right)^2. \qquad (6.17)$$

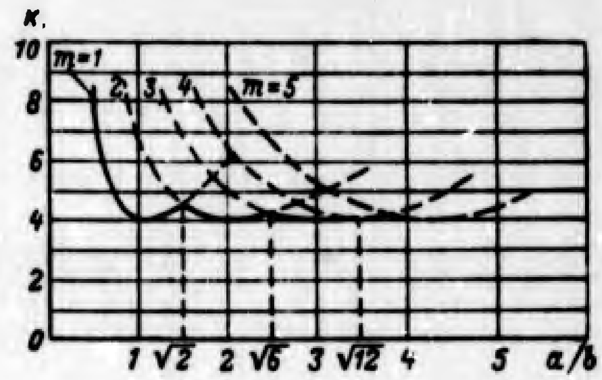


Fig. 6.6

As will be seen from Eq. (6.17), the value of N_{xx} will be smallest for n = 1. Consequently, only a single half-wave is formed in the direction of the y-axis when the plate buckles.

Thus, the critical force per unit length N_{xx} is

$$N_{кр}=\frac{\pi^2 D}{b^2}\left(\frac{mb}{a}+\frac{a}{mb}\right)^2=k\frac{\pi^2 D}{b^2}.$$

From this, the critical stress

$$\sigma_{кр}=\frac{N_{кр}}{h}=\frac{k\pi^2 E}{12(1-\mu^2)}\,\frac{h^2}{b^2}, \qquad (6.18)$$

where $k=\left(\frac{mb}{a}+\frac{a}{mb}\right)^2$ is a numerical coefficient that depends on the ratio a/b of the sides and on the number of half-waves m in the direction of compression.

For m = 1, we obtain

$$k=\left(\frac{b}{a}+\frac{a}{b}\right)^2.$$

On varying the length a of the plate with b = const, we obtain the curve of k for m = 1 (Fig. 6.6) as a function of the ratio a/b. For a = = b, this curve has a minimum value (k = 4). Curves for m = 2, 3, 4 and

5 are also plotted on Fig. 6.6. Given these curves, we can determine the critical stress and number of half-waves for any ratio a/b. Thus, for example, for a/b = 0.8, we find that k = 4.2 and m = 1. Hence

$$\sigma_{\kappa p}=\frac{4,2\pi^2 E}{12(1-\mu^2)}\,\frac{h^2}{b^2}.$$

For the ratio $\frac{a}{b}=\sqrt{2}$, which corresponds to k = 4.49, the number of half-waves m may be equal to 1 or 2. For the ratios a/b $=\sqrt{6}, \sqrt{12}$, the respective numbers of half-waves are 2 or 3, 3 or 4.

With other boundary conditions at the margins of the plate, the structure of the formula for determining the critical stresses remains as before, namely:

$$\sigma_{\kappa p}=\frac{k\pi^2 E}{12(1-\mu^2)}\,\frac{h^2}{b^2},$$

but the coefficient k varies as a function of the boundary conditions.

Thus, for example, if the sides of the plate at x = 0, x = a and y = 0 are hinged and side y = b is free, the coefficient k will take the following values as a function of the side ratio:

$\frac{a}{b}$	0,5	1,0	1,2	1,4	1,6	1,8	2,0	2,5	3,0	4,0	5
k	4,40	1,44	1,135	0,952	0,835	0,755	0,698	0,610	0,564	0,516	0,506

and, for longer plates, can be determined by the formula

$$k=0,456+\frac{b^2}{a^2}.$$

In case the exact solution of Eq. (6.11) is unknown, the energy method will be found most helpful in that it enables us to determine the critical loads with accuracy sufficient for practical application. When this method is used, it is assumed that when the plate is acted upon by critical forces applied in its median plane, it buckles out slightly in a state of indifferent equilibrium. In buckling in this way, the plate accumulates a bending-deformation energy U and the forces in the median plane will do a work W. If the work W is smaller than the energy of the bending deformation, the flat equilibrium shape of the plate will be stable. The critical value of the load, at which the rectilinear equilibrium shape passes from a stable equilibrium state to an unstable one, is determined by the condition

$$U=W$$

or

$$\Pi=U-W=0.$$

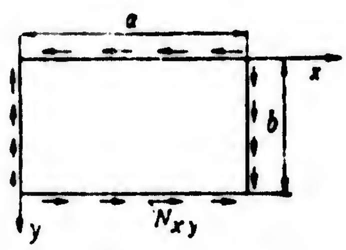


Fig. 6.7

Thus, for example, in investigating the stability of a plate acted upon by shearing forces per unit length N_{xy} (Fig. 6.7), the expression for w is taken as before in the form of the double trigonometric series

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

From the condition $\frac{\partial \Pi}{\partial A_{mn}}=0$ we obtain the system of equations for determination of the unknown parameters. From the simultaneous solution of these equations, the critical force

$$(N_{xy})_{\kappa p}=k\frac{\pi^2 D}{b^2}$$

or

$$\tau_{\kappa p}=\frac{(N_{xy})_{\kappa p}}{h}=k\frac{\pi^2 E}{12(1-\mu^2)}\frac{h^2}{b^2}, \qquad (6.19)$$

where b is the shorter side (i.e., $b \leq a$) and k is a coefficient that depends on the ratio of the sides.

When six terms of the series are considered, the coefficient k takes the following values:

a/b	1,0	1,2	1,4	1,6	1,8	2,0	2,5	3
k	9,4	8,0	7,3	7,0	6,8	6,6	6,3	6,1

Approximate values of k for other values of a/b of the plate can be determined from the formula [25]

$$k=5{,}35+4\left(\frac{b}{a}\right)^2.$$

The experimental values of the critical stresses are in good agreement with the theoretical ones only provided that the compressive or shearing stresses do not exceed the proportionality limit of the material. This is understandable in the light of the fact that it was assumed in the stability analysis of the plates that the material is subject to Hooke's law. At stresses above the proportionality limit, the theoretical formulas give critical-stress values on the high side.

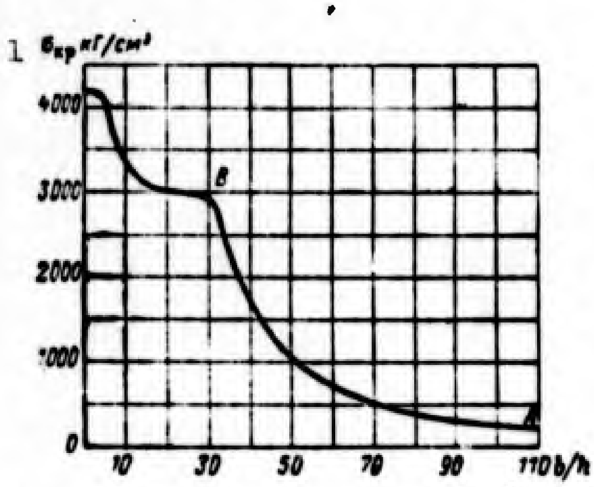


Fig. 6.8. 1) σ_{kr}, kgf/cm^2.

It may be assumed with an accuracy sufficient for practical application that Formula (6.19) is valid to

$$\tau_{кр}\leqslant\tau_{тек}\approx 0{,}5\sigma_{тек}.$$

Let us ascertain the limit of applicability of the formulas defining the critical stresses in compression. For a square plate hinged at the margins

$$\sigma_{кр}=\frac{4\pi^2 D}{b^2 h}=\frac{\pi^2 E}{3(1-\mu^2)}\,\frac{h^2}{b^2}. \qquad (6.20)$$

It is seen from Fig. 6.6 that this formula is also sufficiently accurate for plates with a side ratio $a/b > 1$.

Curve AB of the variation of σ_{kr} as a function of b/h for a plate of D16T duralumin with $E = 7.2\cdot10^5$ kgf/cm^2, $\sigma_{tek} = 3000$ kgf/cm^2 and a short-term ultimate strength $\sigma_v = 4200$ kgf/cm^2, plotted by Formula (6.20), is shown in Fig. 6.8.

The smallest value of b/h, which determines the limit of applicability of Formula (6.20), i.e., point B, is determined from the equation

$$3000 = \frac{\pi^2 7{,}2 \cdot 10^5}{3(1-0{,}3^2)} \frac{h^2}{b^2} = 26 \cdot 10^5 \frac{h^2}{b^2}$$

or

$$\left(\frac{b}{h}\right)_{min} = \sqrt{\frac{26 \cdot 10^5}{3000}} \approx 30.$$

The nature of the curve $\sigma_{kr} = f(b/h)$ on the segment from point B to $b/h = 0$ is shown in Fig. 6.8; for small values of b/h, the critical stresses exceed σ_v, but it is frequently assumed for the sake of a strength margin that $\sigma_{kr} = \sigma_v$.

4. PLATES REINFORCED BY LONGITUDINAL RIBS (STRINGERS). THE CONCEPT OF THE REDUCTION COEFFICIENT

In all cases of plate buckling, the value of the critical stresses is proportional to h^2/b^2. Since the critical stresses increase in proportion to the square of the thickness h, it is more advantageous from a weight standpoint to use light-alloy panels, which have a thicker skin than steel panels at a given weight. For example, a duralumin panel will be about 3 times thicker than a steel panel for a given shape and weight.

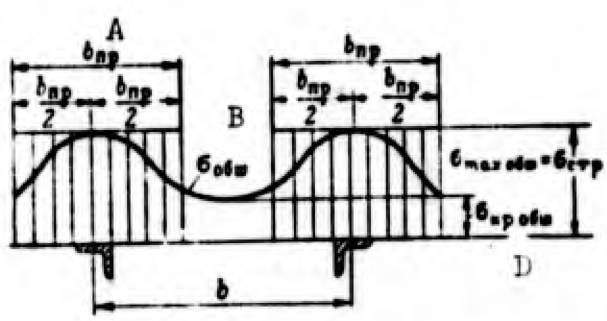


Fig. 6.9. A) b_{pr}; B) σ_{obsh}; C) $\sigma_{max\ obsh} = \sigma_{str}$; D) $\sigma_{kr.obsh}$.

On the other hand, the magnitude of the plate critical stresses may be increased quite effectively by reducing the width b, since it is sufficient to place a single intermediate stringer and thus halve the width b so that the critical stresses are increased by a factor of 4. Thus the panels of thin-walled structures are usually stiffened by stringers.

Since the stringers are used for attachment of the skin (plate), their rigidity should be considerably higher than that of the skin. Then, when a panel consisting of a skin and stringers is compressed, the skin, losing stability, will not fail but, bearing against neighboring stringers, will be capable of carrying its share of the compressive load.

The manner in which the compressive stresses are distributed over the width of a stringer-reinforced plate with stresses σ_{str} in the stringers is shown in Fig. 6.9. As will be seen, the load-carrying ability of the reinforced skin is raised considerably as a result of the increase in the stresses on the zones of the skin adjacent to the stringers.

The total load acting on the entire reinforced panel

$$P=P_{стр}+P_{обш},$$

where P_{str} is the part of the load taken by the stringers and P_{obsh} is the part of the load taken by the skin.

In determining P_{obsh}, it is helpful to replace the true σ_{obsh} diagram by a fictitious one with a constant stress σ_{str} on a certain reduced width $b_{pr} < b$.

Given identical stringers of area <u>f</u> and a constant skin thickness with $\sigma_{str} = \sigma_{kr.str}$, we obtain

$$P=\sigma_{кр.стр}\left(\sum f+\sum \varphi bh\right), \tag{6.21}$$

where $\varphi = b_{pr}/b$ is the ratio of the reduced skin width to the actual width. This ratio is known as the <u>reduction coefficient</u>.

On the basis of the assumption that the compressive load carried by the skin is taken up only on skin zones of width b_{pr} when $\sigma_{kr.str}$ >>

$\gg \sigma_{kr.obsh}$, we obtain the following formula for determination of b_{pr}:

$$b_{пр}=1{,}9h\sqrt{\frac{E}{\sigma_{кр.стр}}}. \quad (6.22)$$

Actually, for a changed plate of width b_{pr}, we have

$$\sigma_{кр.стр}=\frac{4\pi^2 D}{hb_{пр}^2}=\frac{\pi^2 Eh^2}{3(1-\mu^2)b_{пр}^2}=3{,}61E\frac{h^2}{b_{пр}^2}.$$

From this,

$$b_{пр}=1{,}9h\sqrt{\frac{E}{\sigma_{кр.стр}}}.$$

Remembering that

$$\sigma_{кр.общ}=3{,}61E\frac{h^2}{b^2} \quad \text{or} \quad E=\frac{\sigma_{кр.общ}b^2}{3{,}61h^2}$$

the quantity b_{pr} may be written in the form

$$b_{пр}=1{,}9h\sqrt{\frac{\sigma_{кр.общ}b^2}{\sigma_{кр.стр}3{,}61h^2}}$$

or

$$b_{пр}=b\sqrt{\frac{\sigma_{кр.общ}}{\sigma_{кр.стр}}}. \quad (6.23)$$

Consequently,

$$\varphi=\frac{b_{пр}}{b}=\sqrt{\frac{\sigma_{кр.общ}}{\sigma_{кр.стр}}}\leqslant 1. \quad (6.24)$$

If the maximum skin stresses (see Fig. 6.9) are

$$\sigma_{max}=\sigma_{стр}<\sigma_{кр.стр},$$

then

$$\varphi=\sqrt{\frac{\sigma_{кр.общ}}{\sigma_{max}}}. \quad (6.25)$$

In the design of aircraft structures, it is usually possible to specify spar flanges with $\sigma_{kr} \geq \sigma_v$ [9].

In specifying the stringers, it is also desirable to have profile dimensions such that the plates forming its configuration will have critical stresses equal to the yield point of the material [9].

In this case,

$$b_{пр}=1{,}9h\sqrt{\frac{E}{\sigma_{тек}}}.$$

For a panel made from D16T duralumin,

$$b_{пр}=1{,}9h\sqrt{\frac{7{,}2\cdot10^5}{3000}}\approx 30h. \quad (6.26)$$

Since the stringer may also suffer general stability loss, an effort is made in design to space the transverse couplings (ribs, bulkheads) in such a way as to satisfy the condition

$$\sigma_{кр.общ}=\sigma_{кр.мест}.$$

Here

$$\sigma_{кр.общ}=\frac{C\pi^2E}{\left(\frac{l}{i}\right)^2} \text{ (Euler formula)} \quad (6.27)$$

$$\sigma_{кр.мест}=\frac{\sum\sigma_i\Delta f_i}{f}, \quad (6.28)$$

where σ_i and Δf_i are the critical stresses and areas of the elements (plates) of the profile and f is the total area of the profile.

Example.

Determine the breaking force P of a duralumin panel. The panel consists of two longitudinal framing stringers, each with an area f = = 1 cm^2, and a skin with a thickness h = 0.2 cm and a width b = 18 cm; the stringer critical stresses are $\sigma_{kr.obsh} = \sigma_{kr.mest} = 3000$ kgf/cm^2. According to the diagram of Fig. 6.8, we have for the ratio b/h = 18/ /0.2 = 90

$$\sigma_{кр.общ}=320 \text{ кГ/см}^2.$$

Consequently,

$$\varphi=\sqrt{\frac{320}{3000}}=0{,}327.$$

We find from Formula (6.21) that

$$P=3000(2\cdot1+0{,}327\cdot18\cdot0{,}2)=3000(2+1{,}18)=9550\ \kappa\Gamma,$$

of which 37% is taken up by the skin.

In a more general case, the panels may have stringers and skin made from different materials. In this case, Formula (6.25) will obviously take the form

$$\varphi=\frac{E_{\text{обш}}}{E_{\text{стр}}}\sqrt{\frac{\sigma_{\text{кр.обш}}}{\sigma_{\max}}},$$

where E_{obsh} is the Young's modulus of the skin and E_{str} is the Young's modulus of the stringer.

From the condition of equal relative elongations ($\varepsilon_{obsh} = \varepsilon_{str}$) for the skin and stringer near a riveted joint, we have

$$\sigma_{\max}=\sigma_{\text{стр}}\frac{E_{\text{обш}}}{E_{\text{стр}}}.$$

Consequently,

$$\varphi=\frac{E_{\text{обш}}}{E_{\text{стр}}}\sqrt{\frac{\sigma_{\text{кр.обш}}}{\sigma_{\max}}}=\frac{E_{\text{обш}}}{E_{\text{стр}}}\sqrt{\frac{\sigma_{\text{кр. обш}}E_{\text{стр}}}{\sigma_{\text{стр}}E_{\text{обш}}}}$$

or

$$\varphi=\sqrt{\frac{\sigma_{\text{кр.обш}}E_{\text{обш}}}{\sigma_{\text{стр}}E_{\text{стр}}}}, \qquad (6.29)$$

where

$$\sigma_{\text{кр. обш}}=kE_{\text{обш}}\frac{h^2}{b^2};$$

k = 3.61 for hinged margins and k = 6.3 for restrained margins (double-row joint).

Manu-
script
Page
No.

[Footnotes

136 Since the deflections are assumed to be small, it may also be assumed that segment A'B' has a length dx.

Manuscript Page No.	[Transliterated Symbols]
141	кр = kr = kriticheskiy = critical
144	тек = tek = tekuchest' = yield
144	в = v = vremennoy = short-term
145	пр = pr = privedennyy = reduced
145	обш = obsh = obshivka = skin
145	стр = str = stringer = stringer
148	мест = mest = mestnyy = local

Part Three

DESIGNING WING-AND-BODY-TYPE SHELLS FOR AIRCRAFT

Chapter 7

SOME APPLICATIONS OF FINITE-DIFFERENCE EQUATIONS TO THE DETERMINATION OF THE STRESSED STATE IN A CYLINDRICAL SHELL

1. CYLINDRICAL-SHELL CALCULATIONS AT A JOINT

We are to determine the stressed state in bending at a cylindrical-shell joint where the shell is made of a sheet material with constant thickness δ, and is strengthened by longitudinal members for a length L. We shall assume that the law of plane sections is violated for the length L owing to the spot welding of two parts of the shell. We shall direct our attention to the right side of the system (Fig. 7.1a).

The condition requiring symmetry about the longitudinal plane enables us to consider only half of the structure, for example, the side bounded by the angles from $\varphi = 0$ to $\varphi = \pi$ (Fig. 7.1b).

At joints 0 and 1, the concentrated forces

$$P_0 = -P_1 = \frac{M}{4r\cos\varphi_1}, \qquad (7.1)$$

will act on the left side of the system, while on the right side, far from the attachment joints, the bending moment M produces normal stresses that concentrate in accordance with the law of plane sections.

Remembering that $M = P_0 4r\cos\varphi_1$, we obtain the stresses in the section $x = L$:

$$\sigma = \frac{P_0\cos\varphi}{\eta_L}, \qquad (7.2)$$

where

$$\eta_L = \frac{\pi\delta r}{4\cos\varphi_1} + f_L\cos\varphi_1;$$

Here f_L is the area of the longitudinal element $x = L$, which may equal zero.

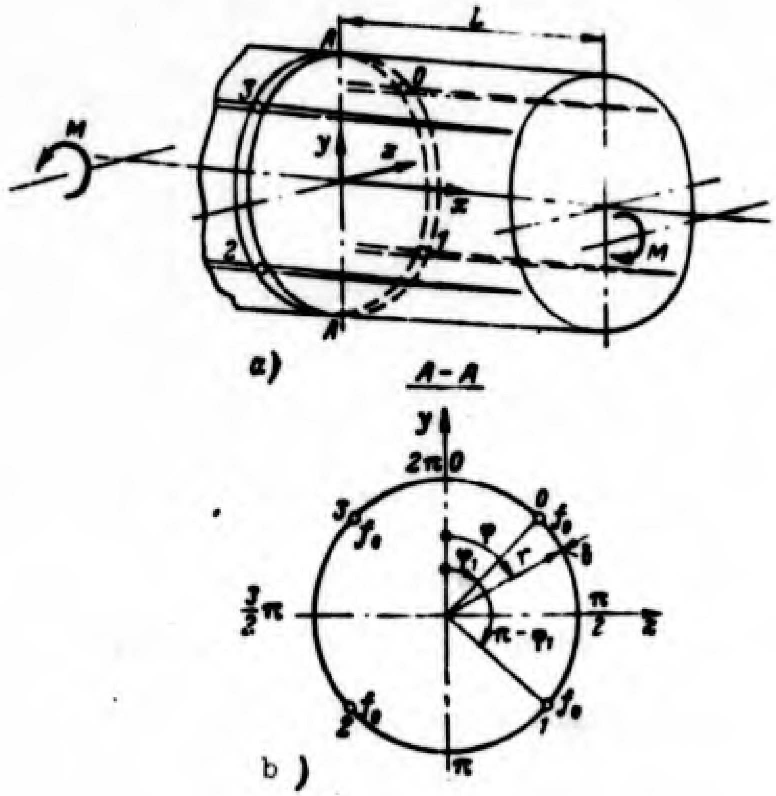


Fig. 7.1

The force factors P and σ, (Fig. 7.2a) are represented in the form of two states "α" and "ω" (Fig. 7.2b and c). In state "α," the effect of the bending moment is represented in the form of stresses distributed over the contour in accordance with the law of plane sections. In state "ω," the right side of the system is free from load, while to the left side there is applied a group of self-balancing forces so selected that when they are added to state "α," only concentrated forces will remain at the left end of the system. As a consequence, the sum "α" + "ω" will be the actual external-load distribution.

We shall not go into detail about the determination of the stressed state "α" which is simple to determine; we shall consider the determination of the stressed state "ω." In this state, a group of self balancing forces is applied at section x = 0; the group consists of the couple $P_0 = - P_0$ and the stresses $\sigma_0^{\omega} = -\frac{P_0 \cos\varphi}{r_0}$.

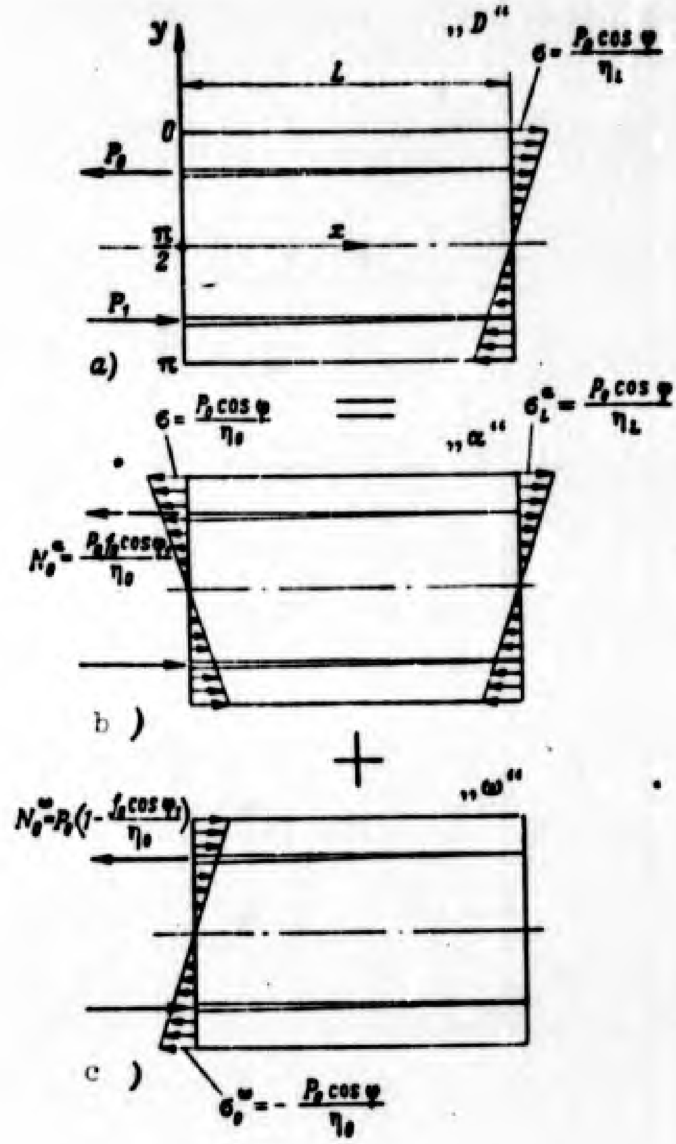


Fig. 7.2

The group of self-balancing forces proves to have a substantial effect on a stress only in the immediate area of the point of application. We shall investigate the attenuation of the forces in this group along the x axis.

We let P_x be the running value of the force P_0 along the x axis. Then the stresses

$$\sigma^{\omega}_{(x,\varphi)} = -\frac{P_x \cos\varphi}{\eta_x}, \tag{7.3}$$

where

$$\eta_x = \frac{\pi \delta r}{4\cos\varphi_1} + f_x \cos\varphi_1.$$

For f = const, the system would be regular and the internal forces could be determined as in the case of Section 7 of Chapter 6.

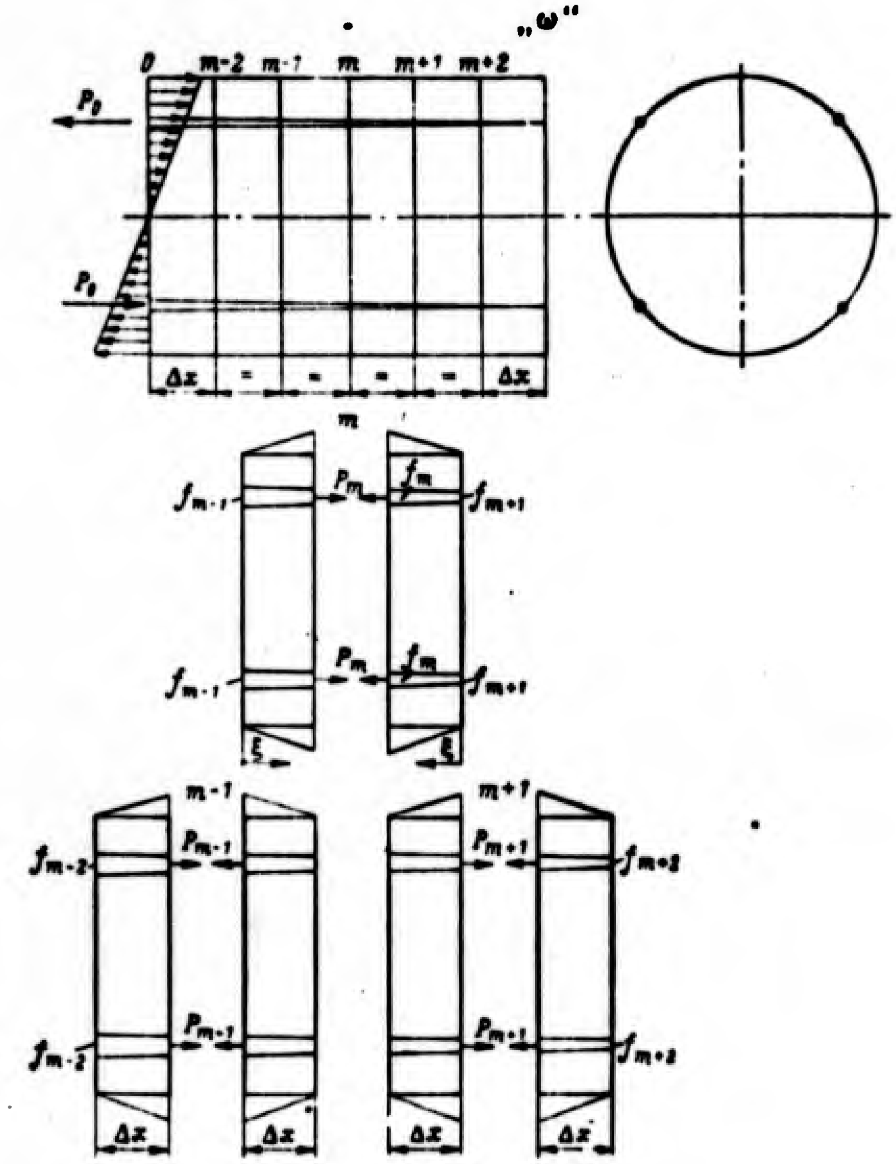


Fig. 7.3

Where the area of the reinforcing element changes along the shell length in accordance with some given law, a differential equation may be obtained with variable coefficiengs described by complex expressions. It therefore may be difficult to solve the equation. In such cases, it is desirable to make use of finite-difference equations.

Let us set up a finite-difference equation for some section m. To do this, we mark off on the length L a number of sections with spans of length Δx (Fig. 7.3).

If we assume that the axial forces P_x on the sections of length Δx vary in accordance with a linear law, then the running value of the force P_m acting in section m will equal

$$P_{m(\xi)}=P_m\frac{\xi}{\Delta x},$$

where ξ is the auxiliary coordinate for the section of length Δx.

The bending moment is

$$M_{m(\xi)}=P_{m(\xi)}4r\cos\varphi_1.$$

As a consequence,

$$\sigma_{(\xi,\varphi)}=-\frac{P_m\xi}{\eta_\xi\Delta x}\cos\varphi, \qquad (7.4)$$

where

$$\eta_\xi=\frac{\pi\delta r}{4\cos\varphi_1}+f_\xi\cos\varphi_1.$$

The force in the longitudinal member is

$$N_{m(\xi)}=P_m\frac{\xi}{\Delta x}-\frac{P_m\xi f_\xi}{\eta_\xi\Delta x}\cos\varphi_1=P_m\frac{\xi}{\Delta x}\left(1-\frac{f_\xi}{\eta_\xi}\cos\varphi_1\right). \qquad (7.5)$$

Let us consider the section from m to m − 1. The shear flow from $\varphi = 0$ to $\varphi = \varphi_1$ will equal

$$q_\varphi=\frac{\int_0^\varphi\sigma_{(m,\varphi)}\delta\,ds}{\Delta x}=-P_m\frac{\delta r}{\eta_m\Delta x}\sin\varphi, \qquad (7.6)$$

where $\sigma_{(m,\varphi)}$ is the running value of the stress acting in section m;

$$\eta_m=\frac{\pi\delta r}{4\cos\varphi_1}+f_m\cos\varphi_1.$$

The shear flow beneath the longitudinal member from $\varphi = \varphi_1$ to $\varphi = \pi/2$ will equal

$$q_\varphi=\frac{\int_0^\varphi\sigma_{(m,\varphi)}\delta\,ds}{\Delta x}+\frac{\sigma_{(m,\varphi_1)}f_m}{\Delta x}+\frac{P_m}{\Delta x}=$$
$$=P_m\frac{\delta r}{\eta_m\Delta x}\left(\frac{\pi}{4\cos\varphi_1}-\sin\varphi\right). \qquad (7.7)$$

The strain continuity equation for section m may consists of conditions requiring that the sum of work done by the forces owing to displacements in section m equals zero. Let us find the work done by the forces in one side half of the considered part; here we use the average values of the quantities

$$\bar{f}=\frac{f_m+f_{m-1}}{2} \text{ и } \bar{\eta}=\frac{\pi\delta r}{4\cos\varphi_1}+\bar{f}\cos\varphi_1.$$

This assumption is quite acceptable, since it has little effect on the accuracy of the calculations, but simplifies them considerably. Keeping in mind Formulas (7.4), (7.5), (7.6), (7.7) we write the expression for the virtual work of the forces P_m acting on the lateral strip m, m − 1:

$$\Delta A_{m,m}=2\int_0^{\Delta x}\int_0^{\frac{\pi}{2}}\frac{P_m\xi^2\cos^2\varphi\delta\, ds\, d\xi}{\bar{\eta}^2\Delta x^2E}+2\int_0^{\Delta x}\frac{P_m\xi^2}{\Delta x^2}\left(1-\frac{\bar{f}}{\bar{\eta}}\cos\varphi_1\right)^2\frac{d\xi}{E\bar{f}}+$$

$$+2\int_0^{\Delta x}\int_0^{\varphi_1}P_m\frac{\delta^2r^2\sin^2\varphi\, ds\, d\xi}{\eta_m^2\Delta x^2G\delta}+$$

$$+2\int_0^{\Delta x}\int_{\varphi_1}^{\frac{\pi}{2}}P_m\frac{\delta^2r^2}{\eta_m^2\Delta x^2}\left(\frac{\pi}{4\cos\varphi_1}-\sin\varphi\right)^2\frac{ds\, d\xi}{G\delta}.$$

After integration and simplification we obtain

$$\Delta A_{m,m}=\frac{2}{3}\frac{P_m}{E}\lambda_{m,m-1}+\frac{2P_m\delta r^3}{\eta_m^2\Delta xG}\gamma,$$

where

$$\gamma=\left[\frac{\pi^2\left(\frac{\pi}{2}-\varphi_1\right)}{16\cos^2\varphi_1}-\frac{\pi}{4}\right];$$

$$\lambda_{m,m-1}=\frac{\Delta x}{\bar{\eta}^2}\left(\frac{\bar{\eta}^2}{\bar{f}}-2\bar{\eta}\cos\varphi_1+\bar{f}\cos^2\varphi_1+\frac{\pi\delta r}{4}\right).$$

The subscripts on λ indicate that $\bar{\eta}$ and $\bar{f}$ refer to the zone m, m − 1.

For zone m, m + 1, we obtain

$$\Delta A_{m,m}=\frac{2}{3}\frac{P_m}{E}\lambda_{m,m+1}+\frac{2P_m\delta r^3}{\eta_m^2\Delta xG}\gamma.$$

The work of the force factors due to the forces P_{m-1} going for displacements in section m will equal

$$\Delta A_{m,m-1}=\frac{P_{m-1}}{3E}\lambda_{m,m-1}-\frac{2P_{m-1}\delta r^3}{\eta_m\eta_{m-1}\Delta xG}\gamma,$$

while that due to the forces P_{m+1} will be

$$\Delta A_{m,m+1}=\frac{P_{m+1}}{3E}\lambda_{m,m+1}-\frac{2P_{m+1}\delta r^3}{\eta_m\eta_{m+1}\Delta xG}\gamma.$$

Thus for section m the finite-difference equation for G = E/2.6 will have the form

$$P_{m-1}\left(\frac{\lambda_{m,m-1}}{3}-\frac{5,2\delta r^3\gamma}{\eta_m\eta_{m-1}\Delta x}\right)+P_m\left[\frac{2}{3}(\lambda_{m,m-1}+\lambda_{m,m+1})+\right.$$
$$\left.+\frac{10,4\delta r^3\gamma}{\eta_m^2\Delta x}\right]+P_{m+1}\left(\frac{\lambda_{m,m+1}}{3}-\frac{5,2\delta r^3}{\eta_m\eta_{m+1}\Delta x}\right)=0. \qquad (7.8)$$

Owing to the change in shear flow q_φ, stresses σ_φ will also appear; we neglected them in the derivation of this equation. When we take the stresses into account, the left side of Eq.(7.8) will consist of five terms.

As a numerical example, let us perform a calculation using the following values: P_0 = 15,000 kgf; δ = 0.2 cm; r = 25 cm; φ_1 = 40°, L = 45 cm; Δx = 15 cm; f_0 = 6 cm^2. The change in the cross-sectional area of the longitudinal member along the x axis is shown in Fig. 7.4b.

We shall consider Section 1 (Fig. 7.4a), for which Eq. (7.8) takes the form:

$$P_0\left(\frac{\lambda_{1,0}}{3}-\frac{5,2\delta r^3\gamma}{\eta_1\eta_0\Delta x}\right)+P_1\left[\frac{2}{3}(\lambda_{1,0}+\lambda_{1,2})+\right.$$
$$\left.+\frac{10,4\delta r^3\gamma}{\eta_1^2\Delta x}\right]+P_2\left(\frac{\lambda_{1,2}}{3}-\frac{5,2\delta r^3\gamma}{\eta_1\eta_2\Delta x}\right)=0.$$

For our initial values, we obtain

$$-1,31P_0+8,71P_1-1,56P_2=0. \qquad (1)$$

We now consider Section 2, for which Eq. (7.8) takes the form:

$$P_1\left(\frac{\lambda_{2,1}}{3}-\frac{5,2\delta r^3\gamma}{\eta_2\eta_1\Delta x}\right)+P_2\left[\frac{2}{3}(\lambda_{2,1}+\lambda_{2,3})+\frac{10,4\delta r^3\gamma}{\eta_2^2\Delta x}\right]=0.$$

Finding the coefficients on the unknowns, we obtain

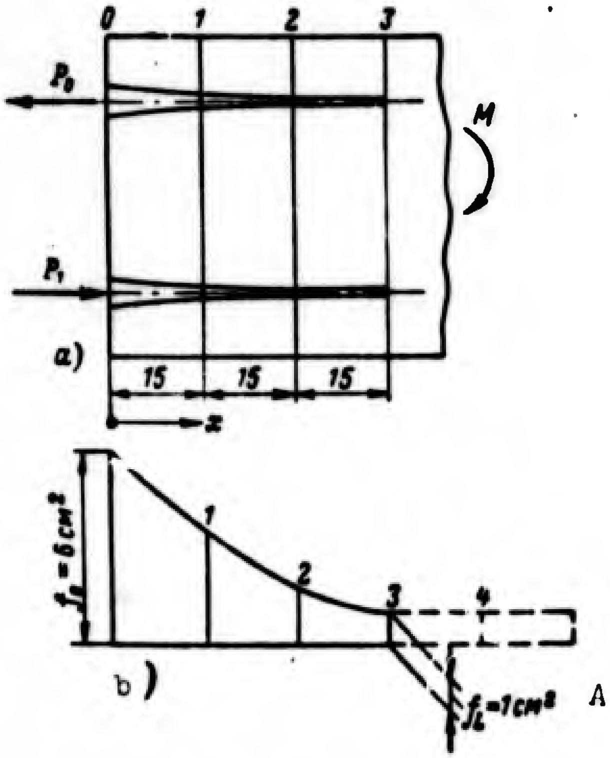


Fig. 7.4. A) $f_L = 1\ cm^2$.

$$-1{,}56P_1+16{,}15P_2=0. \qquad (2)$$

From Eqs. (1) and (2), we obtain

$$P_2=0{,}0965P_1 \text{ and } P_1=0{,}153P_0.$$

In state "ω" the forces N in Sections 0, 1, 2 and 3 (see Fig. 7.5a) will equal:

in Section 0

$$N_0^{\omega}=P_0-\sigma_{(+)}f_0=P_0\left(1-\frac{f_0\cos\varphi_1}{\varphi_0}\right)=$$

$$=15\,000\left(1-\frac{6\cdot 0{,}766}{9{,}72}\right)=7930\ \kappa\Gamma;$$

in Section 1

$$N_1^{\omega}=1480\ \kappa\Gamma;$$

in Section 2

$$N_2^{\omega}=179\ \kappa\Gamma;$$

in Section 3

$$N_3^{\omega}=0.$$

The calculations validate our assumption as to the rapid attenuation of the axial forces; just as if the longitudinal members with their finite area $f_1 = 1\ cm^2$ did not extend far along the x axis, the stress states "ω" remains as before. In order to see this, we add two more spans with constant longitudinal-member area $f_L = 1\ cm^2$ (see Fig. 7.4b). We then obtain the following system of equations:

$$-1{,}31P_0+8{,}71P_1-1{,}56P_2=0; \qquad (1)$$

$$-1{,}31P_1-16{,}15P_2-0{,}850P_3=0; \qquad (2)$$

$$-0{,}850P_2+22{,}9P_3-0{,}020P_4=0; \qquad (3)$$

$$-0{,}020P_3+25{,}2P_4=0. \qquad (4)$$

From the solution to these equations we obtain

$$P_1=0{,}153P_0;$$

$$P_2=0{,}0968P_1;$$

$$P_3=0{,}037P_2;$$

$$P_4=0{,}0008P_3.$$

We thus see that the increase in the previously assumed longituindal-member length essentially has no effect on the stressed state "ω."

Let us now determine the mean shear flows acting on a longitudinal member. The mean PKS for the zone 0-1 from $\varphi = 0$ to $\varphi = \varphi_1$ will equal

$$q_\varphi=\frac{\int\limits_0^\varphi \sigma_{(0,\varphi)}\delta\, ds-\int\limits_0^\varphi \sigma_{(1,\varphi)}\delta\, ds}{\Delta x}=$$

$$=-\frac{\delta r\int\limits_0^\varphi\left(\frac{P_0\cos\varphi}{\eta_0}-\frac{P_1(\cos\varphi)}{\eta_1}\right)d\varphi}{\Delta x},$$

or

$$q_\varphi=-\frac{\delta r}{\Delta x}\left(\frac{P_0}{\eta_0}-\frac{P_1}{\eta_1}\right)\sin\varphi.$$

For $\varphi = \varphi_1$, we obtain

$$q_{\varphi_1}=-\frac{0{,}2\cdot 25}{15}\left(\frac{15\,000}{9{,}72}-\frac{0{,}153\cdot 15\,000}{7{,}87}\right)0{,}643=$$

$$= -\frac{5(1542-291)0,643}{15} = -\frac{1251\cdot 0,643}{3} = -268 \text{ kgf/cm}$$

The PKS beneath the longitudinal member from $\varphi = \varphi_1$ to $\varphi = \pi/2$ will equal

$$q_{\varphi} = \frac{N_0^{\omega} - N_1^{\omega}}{\Delta x} - \frac{1251}{3}\sin\varphi = \frac{7930-1480}{15} -$$

$$-\frac{1251}{3}\sin\varphi = 430 - 417\sin\varphi.$$

for $\varphi = \varphi_1$,

$$q_{\varphi_1} = 430 - 268 = 162 \text{ kgf/cm}$$

On zone 1-2 for $\varphi = \varphi_1$,

$$q_{\varphi_1} = -\frac{0,2\cdot 25}{15}\left(\frac{0,153\cdot 15\,000}{7,87} - \frac{0,0148\cdot 15\,000}{6,35}\right)0,643 =$$

$$= -\frac{1}{3}(291-35)0,643 = -55 \text{ kgf/cm}$$

The PKS beneath the longitudinal member at $\varphi = \varphi_1$ will equal

$$q_{\varphi_1} = \frac{1480-179}{15} - 55 = 87 - 55 = 32 \text{ kgf/cm}$$

On zone 2-3 for $\varphi = \varphi_1$,

$$q_{\varphi_1} = -\frac{35}{3}0,643 = -7,5 \text{ kgf/cm}$$

while beneath the longitudinal member,

$$q_{\varphi_1} = \frac{179}{15} - 7,5 = 4,5 \text{ kgf/cm}$$

The force factors for state "ω" are shown in Fig. 7.5a.

In state "α" the system experiences pure bending (Fig. 7.5b). In this case, the normal stresses in any section x are found from the formula

$$\sigma^{\alpha} = \frac{P_0 \cos\varphi}{\eta_x},$$

where

$$\eta_x = \frac{\pi\delta r}{4\cos\varphi_1} + f_x \cos\varphi_1.$$

Thus in state "α" (Fig. 7.5b) we have:

in Section 0

$$N_0^{\alpha} = 15\,000\,\frac{6\cdot 0,766}{9,72} = 7070 \text{ kgf}$$

in Section 1

$$N_1^s = 15\,000\frac{3{,}6\cdot 0{,}766}{7{,}87} = 5270 \text{ kgf}$$

in Section 2

$$N_2^s = 15\,000\,\frac{1{,}6\cdot 0{,}766}{6{,}35} = 2890 \text{ kgf}$$

in Section 3

$$N_3^s = 0.$$

The mean PKS on zone 0-1 from $\varphi = 0$ to $\varphi = \varphi_1$ will equal

$$q_\varphi^s = \frac{\int\limits_0^\varphi \sigma_{(0,\varphi)}^s \delta\, ds - \int\limits_0^\varphi \sigma_{(1,\varphi)}^s \delta\, ds}{\Delta x} = \frac{P_0 \delta r}{\Delta x}\left(\frac{1}{\eta_0} - \frac{1}{\eta_1}\right)\sin\varphi.$$

For $\varphi = \varphi_1$,

$$q_{\varphi_1}^s = \frac{15\,000\cdot 0{,}2\cdot 25}{15}\left(\frac{1}{9{,}72} - \frac{1}{7{,}87}\right)0{,}643 =$$

$$= \left(\frac{5000}{9{,}72} - \frac{5000}{7{,}87}\right)0{,}643 = -121\cdot 0{,}643 = -77{,}5.$$

The PKS beneath the longitudinal member from $\varphi = \varphi_1$ to $\varphi = \pi/2$ is

$$q_\varphi^s = \frac{N_0^s - N_1^s}{\Delta x} - 121\sin\varphi.$$

For $\varphi = \varphi_1$

$$q_{\varphi_1} = \frac{7070 - 5270}{15} - 77{,}5 = 120 - 77{,}5 = 42{,}5.$$

On zone 1-2 for $\varphi = \varphi_1$,

$$q_{\varphi_1}^s = \left(\frac{5000}{7{,}87} - \frac{5000}{6{,}35}\right)0{,}643 =$$

$$= (636 - 788)\; 0{,}643 = -97{,}6.$$

The PKS beneath the longitudinal member (zone 1-2) for $\varphi = \varphi_1$ that will equal

$$q_\varphi^s = \frac{5270 - 2890}{15} - 97{,}6 =$$

$$= 158{,}6 - 97{,}6 = 61.$$

On zone 2-3 for $\varphi = \varphi_1$,

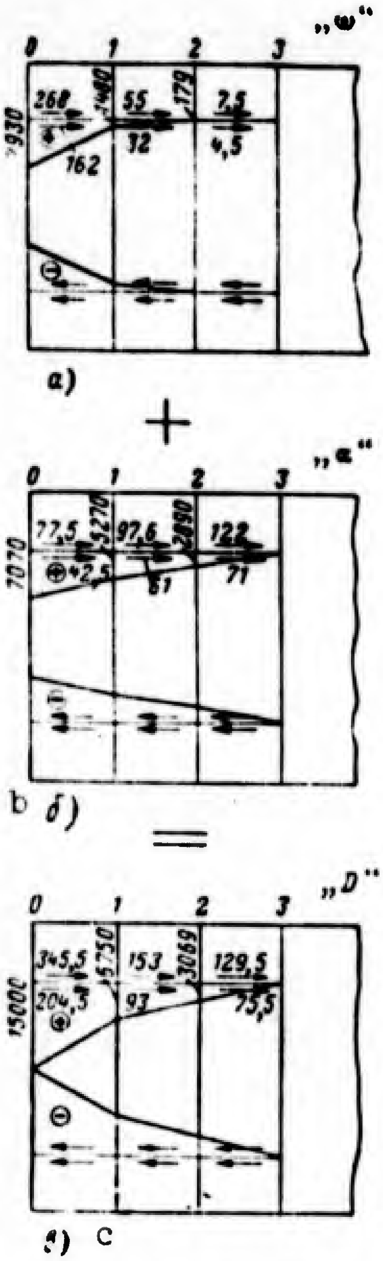


Fig. 7.5

$$q^{*}_{\varphi_1}=\left(\frac{5000}{6,35}-\frac{5000}{5,12}\right)0,643=$$
$$=(788-978)0,643=-122,$$

while beneath the longitudinal member

$$q_{\varphi_1}=\frac{2890}{15}-122=193-122=71.$$

Let us now determine the force factors in state "D" (Fig. 7.5c):

$$N_0=7930+7070=15000 \text{ kgf}$$
$$N_1=1480+5270=6750 \text{ kgf}$$
$$N_2=179+2890=3069$$
$$N_3=0. \text{ kgf}$$

The PKS for $\varphi = \varphi_1$ will be

on zone 0-1:

$$-268-77,5=-345,5 \text{ kgf/cm}$$
$$162+42,5=+204,5 \text{ kgf/cm}$$

on zone 1-2:

$$-55-97,6=-152,6 \text{ kgf/cm},$$
$$+32+61=+93 \text{ kgf/cm};$$

on zone 2-3:

$$-7,5-122=-129,5 \text{ kgf/cm};$$
$$4,5+71=+75,5 \text{ kgf/cm}.$$

Despite the fact that the Δx are long with respect to length L,

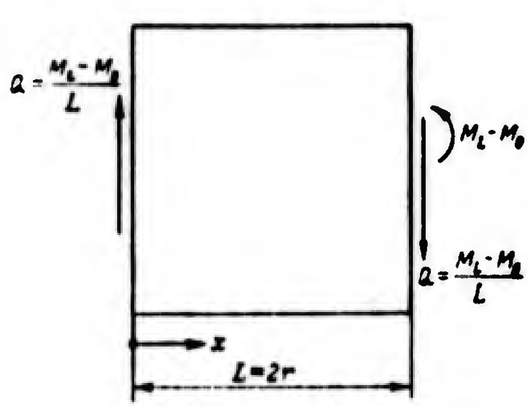


Fig. 7.6

the values of the axial forces in the longitudinal members are quite exact. The average PKS values are far less accurate, especially for zone 0-1, where the most vigorous attenuation of the axial forces occurs.

In order to obtain a more exact value for the PKS, it is necessary to reduce the distance between spans, which leads to more cumbersome calculations. The computational difficulties appearing, however, may easily be eliminated through the use of computers.

It follows from the calculations given that in sections lying more than 2r away from the joint, we can determine the stresses in accordance with the law of plane sections with a satisfactory degree of accuracy. If for $x = 0$ and $x = L$ the bending moments are roughly equal, then in order to increase the safety factor, we can determine the stressed state due to the greater bending moment. In the opposite case, the recommended procedure is as follows: if $M_0 > M_L$, the stressed state should be determined in accordance with the method presented above for M_0; we then add to this state the stresses due to $M_L - M_0$, distributed in accordance with the hypothesis of plane sections and the accompanying PKS stresses; here we assume that $M_L - M_0$ varies linearly, reaching the zero value at $x = 0$ (Fig. 7.6).

2. DETERMINATION OF STRESS STATE IN TORSION FOR CYLINDRICAL SHELL WITH CUT

We shall determine the stressed state in a shell of the type used for a flying-craft body (Fig. 7.7a), consisting of a center bay with a cut and closed bays bounding the central bay on both the right and the left.

Since the bulkheads are usually quite stiff in their own plane, we shall assume that they are perfectly rigid, and that they are perfectly flexible in the direction perpendicular to their plane.

We assume that the framing bar has a constant cross-sectional area $\underline{f}$, while the skin thickness equals δ everywhere. In this case, the system will have a transverse plane of elastic symmetry. The transverse plane of elastic symmetry and the position of the coordinate axes are shown in Fig. 7.7a. Since the torsional moments M are applied to the body, the internal forces must be inversely symmetric. As a consequence, at section x = 0, the stresses must equal zero, since it is only in this case that the internal force factors about the plane of elastic symmetry will be inversely symmetric.

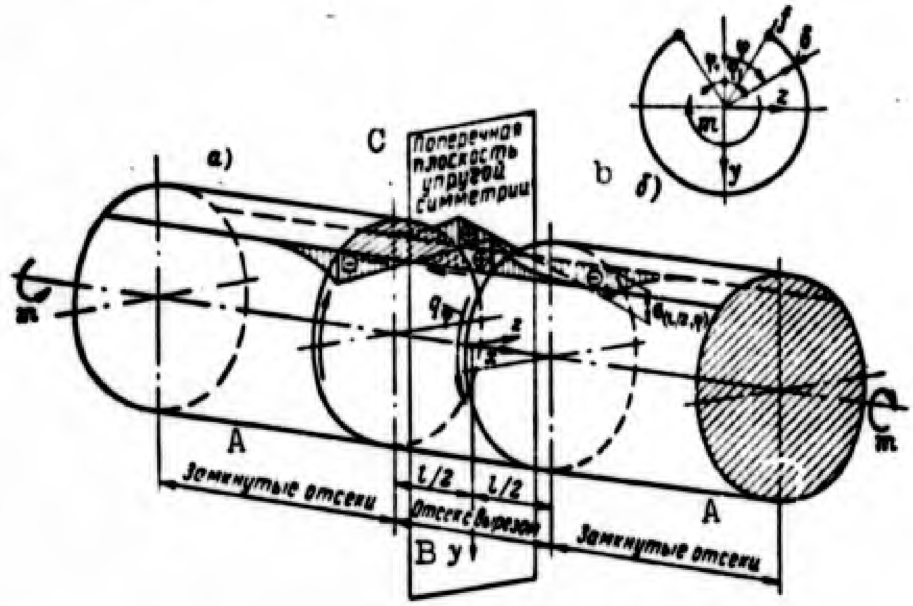


Fig. 7.7. A) Closed bays; B) bay with cutout; C) transverse plane of elastic symmetry.

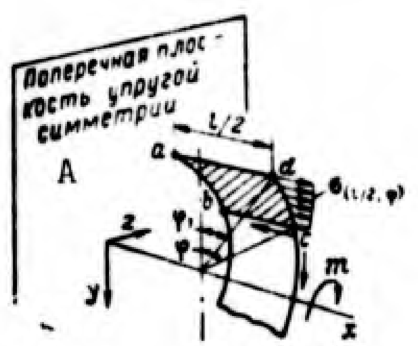


Fig. 7.8. A) Transverse plane of elastic symmetry.

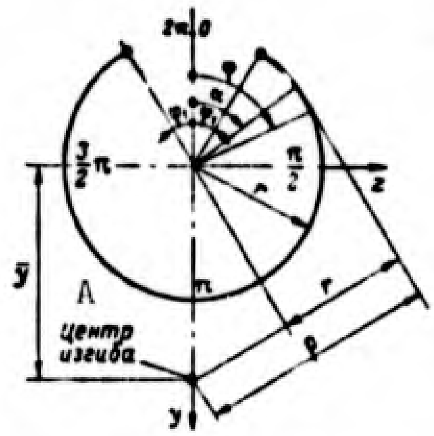


Fig. 7.9. A) Center of bending.

The framework consisting of the framing bar and bulkheads has a very small frame effect, which may be deflected; we assume a hinged attachment of the longitudinal assembly to the bulkheads.

The stresses may be written in the form

$$\sigma_{(x,\varphi)}=\psi(x)\omega(\varphi), \tag{7.9}$$

where the function $\psi(x)$, which depends solely on x, is unknown, while the function $\omega(\varphi)$, which depends solely on the angle φ, is the sector area

$$\omega=\int_0^s \varrho\, ds=\int_0^\alpha \varrho r\, d\alpha, \tag{7.10}$$

where ρ is the length of the perpendicular dropped from the center of bending of the open contour to the tangent at the point of the contour having coordinate s.

The change in stresses along the contour of the body cross section obtained from (7.10) is proportional to the sector area ω, and agrees well with the more exact calculation of [29].

Determination of Stressed State in Bay with Cut

Despite the fact that this is a fairly simple structure, the stressed state is described by very complicated formulas. For simplicity we shall assume that the function $\psi(x)$ varies in accordance with a linear law, i.e., we let

$$\psi(x)=kx, \tag{7.11}$$

where k is a constant. As a consequence, the stresses for a bay with a cut will be

$$\sigma_{(x,\varphi)}=kx\,\omega(\varphi). \tag{7.12}$$

Since the stresses $\sigma_{(x,\varphi)}$ vary along the x axis in accordance with a linear law, then over the span of the bay with cut, the linear PKS q_φ will depend solely on the angle φ.

The value of q_φ is found from the conditions requiring equilibrium of the part of the shell abcd (shown hatched in Fig. 7.8). On the left side (ab) for x = 0 the stresses will be $\sigma_{(0,\varphi)} = 0$, and on the right side (cd) for $x = \ell/2$ we have

$$\omega=\int_0^s \varrho\, ds=\int_0^\alpha (r+\bar{y}\cos\alpha)\, r\, d\alpha+C=r(r\varphi+\bar{y}\sin\varphi)+C.$$

Thus, from the conditions requiring equilibrium of the part of the shell abcd, we obtain

$$q_{\varphi}=+\frac{\sigma_{(l/2,\varphi_1)}f+\int\limits_{\varphi_1}^{\varphi}\sigma_{(l/2,\varphi)}\delta\,ds}{\frac{l}{2}}. \tag{7.13}$$

It is clear from Fig. 7.9 that

$$\sigma_{(l/2,\varphi)}=\frac{kl}{2}\omega.$$

The contour distribution of the stresses σ and, consequently, of ω as well, must be inversely symmetric about the $\underline{y}$ axis, so that for $\varphi = \pi$, the sector area must equal zero, or

$$r(r\pi+\bar{y}\sin\varphi)+C=0.$$

From this we have

$$C=-r^2\pi.$$

As a consequence,

$$\omega=r\left[r(\varphi-\pi)+\bar{y}\sin\varphi\right] \tag{7.14}$$

and

$$\sigma_{(l/2,\varphi)}=\frac{kl}{2}r[r(\varphi-\pi)+\bar{y}\sin\varphi]. \tag{7.15}$$

Substituting (7.15) into Expression (7.13), we obtain

$$q_{\varphi}=+\frac{\frac{kl}{2}r\left[r(\varphi_1-\pi)+\bar{y}\sin\varphi_1\right]f+\int\limits_{\varphi_1}^{\varphi}\frac{kl}{2}r\left[r(\varphi-\pi)+\bar{y}\sin\varphi\right]\delta\,ds}{\frac{l}{2}},$$

or

$$q_{\varphi}=+kr\left\{\left[r(\varphi_1-\pi)+\bar{y}\sin\varphi_1\right]f+\delta r\int\limits_{\varphi_1}^{\varphi}\left[r(\varphi-\pi)+\bar{y}\sin\varphi\right]d\alpha\right\}.$$

Following integration we obtain

$$q_{\varphi}=+kr\left\{Af+\delta r\left\{r\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]-\bar{y}(\cos\varphi-\cos\varphi_1)\right\}\right\}. \tag{7.16}$$

where

$$A=r(\varphi_1-\pi)+\bar{y}\sin\varphi_1.$$

We find the coefficient $\underline{k}$ from the conditions requiring equilibrium, for example, of the entire right side of the body bounding the cut. On this portion there acts the external torsional moment $\dot{M}$, which must be balanced by the moment due to q_φ, acting at section $x = 0$. As a consequence,

$$2\int_{\varphi_1}^{\pi} q_\varphi r\,ds+\mathfrak{M}=0. \tag{7.17}$$

In view of the fact that

$$\int_{\varphi_1}^{\pi} q_\varphi r\,ds=+kr^3\left\{\left\{Af\varphi+\left\{r\left[\frac{\varphi^3}{6}-\frac{\varphi_1^3\varphi}{2}-\frac{\pi\varphi^2}{2}+\pi\varphi_1\varphi\right]-\right.\right.\right.$$

$$\left.\left.\left.-\bar{y}\left[\sin\varphi-\varphi\cos\varphi_1\right]\right\}\delta r\right\}\right\}_{\varphi_1}^{\pi}=$$

$$=+kr^3\left\{Af(\pi-\varphi_1)-r^2\frac{\delta(\pi-\varphi_1)^3}{3}+\bar{y}r\delta[(\pi-\varphi_1)\cos\varphi_1+\sin\varphi_1]\right\}$$

or

$$2\int_{\varphi_1}^{\pi} q_\varphi r\,ds=2kr^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\delta\Phi\right). \tag{7.18}$$

Here $\xi = \pi - \varphi_1$ and $\Phi = \xi\cos\varphi_1 + \sin\varphi_1$.

Substituting (7.18) into Eq. (7.17), we obtain

$$+2kr^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\delta\Phi\right)+\mathfrak{M}=0.$$

As a consequence,

$$k=-\frac{\mathfrak{M}}{2r^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}-\bar{y}r\delta\Phi\right)}\ [\text{кГ/см}^5]. \tag{7.19}$$

We now find the position of the center of bending $\bar{y}$:

$$\bar{y}=\frac{1}{J_y}\int_{\varphi_1}^{2\pi-\varphi_1} S_{y(\varphi)}r\,ds.$$

Here

$$S_{y(\varphi)}=fr\sin\varphi_1+\int_{\varphi_1}^{\varphi}\delta r^2\sin\alpha\,d\alpha=fr\sin\varphi_1+\delta r^2(\cos\varphi_1-\cos\varphi);$$

$$(2\pi-\varphi_1)\geqslant\varphi\geqslant\varphi_1;$$

$$J_y=2fr^2\sin^2\varphi_1+2\int_{\varphi_1}^{\pi}\delta r^3\sin^2\varphi\,d\varphi=2fr^2\sin^2\varphi+2\delta r^3\left(\frac{\pi-\varphi_1}{2}+\frac{\sin 2\varphi_1}{4}\right).$$

Since

$$\int\limits_{\varphi_1}^{2\pi-\varphi_1} S_y r\,ds = \int\limits_{\varphi_1}^{2\pi-\varphi_1} [fr \sin\varphi_1 + \delta r^2(\cos\varphi_1 - \cos\varphi)]\, r^2 d\varphi =$$

$$= r^3 [f\varphi \sin\varphi_1 + \delta r(\varphi\cos\varphi_1 - \sin\varphi)]_{\varphi_1}^{2\pi-\varphi_1} =$$

$$= 2r^3 \{f_1(\pi-\varphi_1)\sin\varphi_1 + \delta r[(\pi-\varphi_1)\cos\varphi_1 + \sin\varphi_1]\},$$

or

$$\int\limits_{\varphi_1}^{2\pi-\varphi_1} S_y r\,ds = 2r^3(\xi f_1 \sin\varphi_1 + \delta r\Phi),$$

then

$$\bar{y} = \frac{r(\xi f_1 \sin\varphi_1 + \delta r\Phi)}{f\sin^2\varphi_1 + \delta r\left(\frac{\xi}{2} + \frac{\sin 2\varphi_1}{4}\right)}. \qquad (7.20)$$

In order to test the formulas obtained, we turn to a numerical example.

Let

$$\mathfrak{M} = 10^5 \text{ kgf}\cdot\text{cm} \quad l = 2r; \quad \text{cm};\ r = 25\ \text{см};$$

$$\delta = 0{,}2\ \text{cm} \quad f = 2\ \text{cm}^2 \quad \varphi_1 = 30^\circ = \frac{\pi}{6}.$$

Then

$$\sin\varphi_1 = 0{,}5; \quad \cos\varphi_1 = 0{,}866; \quad \sin 2\varphi_1 = 0{,}866;$$

$$\pi - \varphi_1 = \xi = \frac{5}{6}\pi = 2{,}618;$$

$$\Phi = 2{,}618\cdot 0{,}866 + 0{,}5 = 2{,}767; \quad \xi^3 = 17{,}944.$$

Let us determine the position of the center of bending, the value of the coefficient k, and the PKS q_φ:

$$\bar{y} = \frac{25(2{,}618\cdot 2\cdot 0{,}5 + 0{,}2\cdot 25\cdot 2{,}767)}{2\cdot 0{,}5^2 + 0{,}2\cdot 25\left(\frac{2{,}618}{2} + \frac{0{,}866}{4}\right)} =$$

$$= \frac{25(2{,}618 + 13{,}8)}{0{,}5 + 7{,}62} = \frac{410}{8{,}12} = 50{,}62\ \text{см};$$

$$k = -\frac{\mathfrak{M}}{2\cdot 25^3\left(A\,2\cdot 2{,}618 - \frac{25^2\cdot 0{,}2\cdot 17{,}944}{3} + 50{,}62\cdot 25\cdot 0{,}2\cdot 2{,}767\right)} =$$

$$= -\frac{\mathfrak{M}}{2\cdot 15\,620(A\cdot 5{,}236 - 750 + 700)}.$$

Here

$$A=25\left(\frac{\pi}{6}-\pi\right)+50{,}62\cdot 0{,}5=-25\,\frac{5}{6}\,3{,}14+25{,}31=-40{,}14.$$

Thus

$$k=-\frac{\mathfrak{M}}{2\cdot 15\,620(-210-750+700)}=\frac{\mathfrak{M}}{2\cdot 15\,620\cdot 260}=\frac{\mathfrak{M}}{8{,}12\cdot 10^{6}},$$

$$q_\varphi=0{,}308\left[-80{,}28+125\left(\frac{\varphi^2}{2}-\pi\varphi\right)+189-253\cos\varphi+218\right],$$

or

$$q_\varphi=0{,}308\left[-80{,}28+125\left(\frac{\varphi^2}{2}-\pi\varphi\right)-253\cos\varphi+407\right].$$

For $\varphi = \varphi_1 = \pi/6$

$$q_\varphi=0{,}308(-80{,}28)=-24{,}7 \text{ kgf/cm}$$

For $\varphi = \pi/2$

$$q_\varphi=0{,}308\left[-80{,}28+125\left(\frac{\pi^2}{8}-\frac{\pi^2}{2}\right)+407\right]=$$
$$=-0{,}308\left(-80{,}28-125\,\frac{3}{8}\,\pi^2+407\right)=$$
$$=0{,}308(-80{,}28-462+407)=-0{,}308\cdot 135=-41{,}6 \text{ kgf/cm}$$

For $\varphi = \pi$

$$q_\varphi=0{,}308\left[-80{,}28+125\left(\frac{\pi^2}{2}-\pi^2\right)+253+407\right]=$$
$$=0{,}308\left(-80{,}28-125\,\frac{\pi^2}{2}+660\right)=$$
$$=-0{,}308\cdot 37{,}78=-11{,}65 \text{ kgf/cm}$$

For $\varphi = 3\pi/2$

$$q_\varphi=0{,}308\left[-80{,}28+125\left(\frac{9\pi^2}{2}-\frac{3}{2}\,\pi^2\right)+407\right]=$$
$$=+0{,}308\left[-80{,}28-125\,\frac{3}{8}\,\pi^2+407\right]=$$
$$=0{,}308[-80{,}28-462+407]=-41{,}6 \text{ kgf/cm}$$

For $\varphi = 2\pi - \varphi_1$

$$q_\varphi=0{,}308\left[-80{,}28+125\left(\frac{4\pi^2-4\pi\varphi_1+\varphi_1}{2}-2\pi^2+\pi\varphi_1\right)-\right.$$
$$\left.-253\cos(2\pi-\varphi_1)+407\right]=$$
$$=0{,}308\left[-80{,}28+125\left(\frac{\varphi_1^2}{2}-\pi\varphi_1\right)-253\cos\varphi_1+407\right]=-24{,}7 \text{ kgf/cm}$$

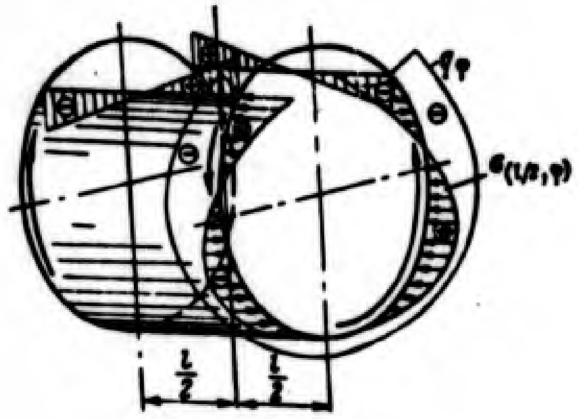


Fig. 7.10

We also determine the stresses at section $x = \ell/2$. In accordance with Formula (7.15)

$$\sigma_{(l/2,\varphi)} = \frac{10^6 \cdot 25^2}{8{,}12 \cdot 10^6}[25(\varphi - \pi) + 50{,}62 \sin\varphi] =$$
$$= 7{,}7(25\varphi - 78{,}5 + 50{,}62 \sin\varphi).$$

For $\varphi = \varphi_1 = \pi/6 = 30°$

$$\sigma_{(l/2,\pi/6)} = 7{,}7(13{,}1 - 78{,}5 + 25{,}3) = -7{,}7 \cdot 40{,}1 = -308 \text{ kgf/cm}^2$$

For $\varphi = \pi/2$

$$\sigma_{(l/2,\pi)} = 7{,}7\left(25\frac{\pi}{2} - 78{,}5 + 50{,}62 \sin\frac{\pi}{2}\right) = 7{,}7 \cdot 11{,}32 = 87.$$

For $\varphi = 3\pi/4 = 135°$

$$\sigma_{\left(l/2\ \frac{3}{4}\pi\right)} = 7{,}7\left(25\frac{3}{4}\pi - 78{,}5 + 50{,}62 \sin\frac{3}{4}\pi\right) =$$
$$= 7{,}7(58{,}8 - 78{,}5 + 50{,}62 \cdot 0{,}707) = 7{,}7 \cdot 16{,}1 = 124.$$

For $\varphi = \pi$

$$\sigma_{(l/2,\pi)} = 7{,}7(25 \cdot 3{,}14 - 78{,}5) = 0.$$

For $\varphi = 3\pi/2$

$$\sigma_{\left(l/2,\ \frac{3}{2}\pi\right)} = 7{,}7\left(25\frac{3}{2}\pi - 78{,}5 + 50{,}62 \sin\frac{3}{2}\pi\right) = -87 \text{ kgf/cm}^2$$

The nature of the variation in q_φ and $\sigma_{(\ell/2,\varphi)}$, acting along the bulkhead contour from the direction of the skin on the bulkhead with the cut is shown in Fig. 7.10.

Determination of Stressed State in Closed Bays

Let us consider the part of the body to the right of the bay with the cut. To this part there is applied an external torsional moment $\mathfrak{M}$ (Fig. 7.11) which is balanced by forces acting on the left bulkhead; these forces are known from the calculations for the bay with cut.

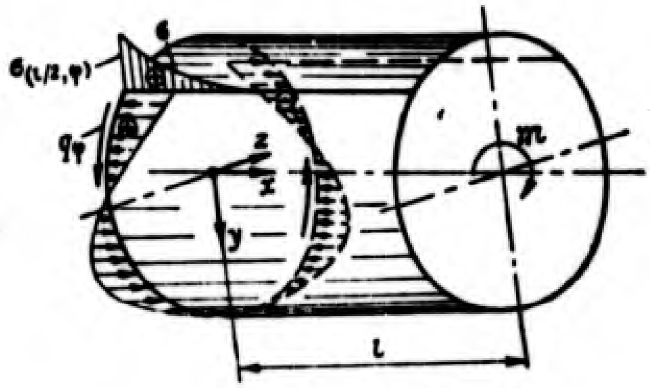


Fig. 7.11

We move the coordinate-system origin from the center of the bay with the cut to the center of the left end of the considered portion of the body (Fig. 7.11). In this section there acts the group of self balancing forces produced by the stresses $\sigma_{(\ell/2,\ \varphi)}$, distributed in proportion to the sector areas ω. Let us see how this group of self balancing forces varies along the x axis. Representing the law governing the variation in this group of forces in the form of the function $\psi(x)$, we obtain

$$\sigma_{(x,\varphi)}=\psi(x)\,\omega(\varphi)=\psi(x)\,r\,[r(\varphi-\pi)+\bar{y}\sin\varphi].$$

The strain potential energy is

$$U=\int_0^l\oint\left(\frac{\sigma^2_{(x,\varphi)}}{2E}+\frac{\tau^2_{(x,\varphi)}}{2G}\right)r\delta\,d\varphi\,dx=$$

$$=\int_0^l\oint^{2\pi-\varphi_1}\left(\frac{\sigma^2_{(x,\varphi)}}{2E}+\frac{q^2_{(x,\varphi)}}{2G\delta^2}\right)r\delta\,d\varphi\,dx=$$

$$=\int_0^l\frac{r\delta}{2E}\oint\left[\sigma^2_{(x,\varphi)}+\frac{2(1+\mu)}{\delta^2}\,q^2_{(x,\varphi)}\right]d\varphi\,dx,$$

or

$$U=\int_0^l \Gamma dx,$$

where

$$\Gamma=\frac{r\delta}{2E}\oint\left[\sigma^2_{(x,\varphi)}+\frac{2(1+\mu)}{\delta^2}q^2_{(x,\varphi)}\right]d\varphi. \tag{7.21}$$

We determine the value of the integral:

$$\oint\left[\sigma^2_{(x,\varphi)}+\frac{2(1+\mu)}{\delta^2}q^2_{(x,\varphi)}\right]d\varphi=$$
$$=\oint\sigma^2_{(x,\varphi)}d\varphi+\frac{2(1+\mu)}{\delta^2}\oint q^2_{(x,\varphi)}d\varphi. \tag{7.22}$$

The first integral equals

$$\oint\sigma^2_{(x,\varphi)}d\varphi=\int_{\varphi_1}^{2\pi-\varphi_1}\psi^2(x)r^2[r(\varphi-\pi)+\bar{y}\sin^2\varphi]^2d\varphi=\psi^2(x)B, \tag{7.23}$$

where

$$B=2r^2\left\{\frac{r^2(\pi-\varphi_1)^3}{3}-2r\bar{y}[(\pi-\varphi_1)\cos\varphi_1+\sin\varphi_1]+\right.$$
$$\left.+\bar{y}^2\left(\frac{\pi-\varphi_1}{2}+\frac{\sin 2\varphi_1}{4}\right)\right\}.$$

Before proceeding to integrate the second term of Expression (7.22), we determine the PKS $q_{(x,\ \varphi)}$. From the conditions requiring equilibrium of the elementary portion of the shell (shown hatched in Fig. 7.12), we obtain

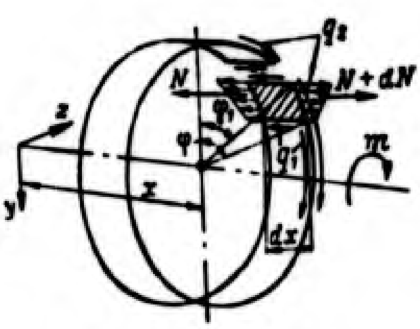


Fig. 7.12

or

$$\left.\begin{aligned}-q_1dx+dN&=0\\ q_1&=\frac{dN}{dx}.\end{aligned}\right\} \tag{7.24}$$

From the conditions requiring equilibrium of the cross section of the shell located a distance x away from the coordinate-system origin we obtain

$$\int_{\varphi_1}^{2\pi-\varphi_1}q_1 r\,ds+q_2\Omega+\mathfrak{M}=0. \tag{7.25}$$

From this we have

$$q_2 = -\frac{\mathfrak{M}}{\Omega} - \frac{\oint q_1 r\, ds}{\Omega}, \qquad (7.26)$$

where Ω is twice the area bounded by the contour of the cross section.

The total shear flow is

$$q_{(x,\varphi)} = q_1 + q_2 = \frac{dN}{dx} - \frac{\mathfrak{M} + \oint \frac{dN}{dx} r\, ds}{\Omega}. \qquad (7.27)$$

The running value of the force is

$$N_{(x,\varphi)} = \sigma_{(x,\varphi_1)} f + \int_{\varphi_1}^{\varphi} \sigma_{(x,\varphi)} \delta\, ds = \psi(x) r [r(\varphi_1 - \pi) + \bar{y} \sin \varphi_1] f +$$

$$+ \psi(x) \int_{\varphi_1}^{\varphi} r [r(\varphi - \pi) + \bar{y} \sin \varphi] \delta r\, d\varphi =$$

$$= \psi(x) r \left\{ [r(\varphi_1 - \pi) + \bar{y} \sin \varphi_1] f + \delta r \int_{\varphi_1}^{\varphi} [r(\varphi - \pi) + \bar{y} \sin \varphi]\, d\varphi \right\} =$$

$$= \psi(x) r \left\{ \left\{ Af + \delta r \left\{ r \left[\frac{\varphi^2 - \varphi_1^2}{2} - \pi(\varphi - \varphi_1) \right] - \bar{y} (\cos \varphi - \cos \varphi_1) \right\} \right\} \right\}.$$

As a consequence,

$$\frac{dN}{dx} = + \psi'(x) r \left\{ \left\{ Af + \delta r \left\{ r \left[\frac{\varphi^2 - \varphi_1^2}{2} - \pi(\varphi - \varphi_1) \right] - \bar{y} (\cos \varphi - \cos \varphi_1) \right\} \right\} \right\} \qquad (7.28)$$

or

$$\frac{dN}{dx} = \frac{\psi'(x)}{k} q_\varphi, \qquad (7.29)$$

where the PKS q_φ is determined from Formula (7.16).

We next determine the value

$$\oint \frac{dN}{dx} r\, ds = \int_{\varphi_1}^{2\pi - \varphi_1} \frac{dN}{dx} r\, ds = \frac{\psi'(x)}{k} 2 \int_{\varphi_1}^{\pi} q_\varphi r\, ds.$$

Keeping in mind Expression (7.18), we obtain

$$\int_{\varphi_1}^{2\pi - \varphi_1} \frac{dN}{dx} r\, ds = + \psi'(x) 2 r^3 \left(Af\xi - \frac{r^2 \delta \xi^3}{3} + \bar{y} r \delta \Phi \right).$$

where, as before,

$$\xi = \pi - \varphi_1 \text{ and } \Phi = \xi \cos \varphi_1 + \sin \varphi_1.$$

Thus,

$$q_{(x,\varphi)} = \psi'(x)\frac{q_\varphi}{k} - \frac{\psi'(x)\,2r^3\left(A f\xi - \frac{r^2\delta\xi^3}{3} + \bar{y}r\,\delta\Phi\right)}{\Omega} - \frac{\mathfrak{M}}{\Omega}. \tag{7.30}$$

Since

$$\frac{q_\varphi}{k} = \delta r^2\left\{r\left[\frac{\varphi^2-\varphi_1^2}{2} - \pi(\varphi-\varphi_1)\right] - \bar{y}(\cos\varphi - \cos\varphi_1)\right\} + \omega(\varphi_1)f,$$

we obtain

$$q_{(x,\varphi)} = \psi'(x)\left\{\left\{\delta r^2\left\{r\left[\frac{\varphi^2-\varphi_1^2}{2} - \pi(\varphi-\varphi_1)\right] - \right.\right.\right.$$
$$\left.\left.\left. - \bar{y}(\cos\varphi - \cos\varphi_1)\right\} + C\right\}\right\} - \frac{\mathfrak{M}}{\Omega}, \tag{7.31}$$

where

$$C = \omega(\varphi_1)f - \frac{2r^3\left(A f\xi - \frac{r^2\delta\xi^3}{3} + \bar{y}r\right)\delta\Phi}{\Omega}.$$

We now determine $q^2_{(x,\varphi)}$:

$$q^2_{(x,\varphi)} = \psi''(x)\left\{\left\{\delta^2 r^6\left[\frac{\varphi^2-\varphi_1^2}{2} - \pi(\varphi-\varphi_1)\right]^2 + \right.\right.$$
$$+ \bar{y}^2\delta^2 r^4(\cos\varphi - \cos\varphi_1)^2 + C^2 -$$
$$- 2\delta^2 r^5\left[\frac{\varphi^2-\varphi_1^2}{2} - \pi(\varphi-\varphi_1)\right]\bar{y}(\cos\varphi - \cos\varphi_1) +$$
$$\left.\left. + 2\delta r^3\left[\frac{\varphi^2-\varphi_1^2}{2} - \pi(\varphi-\varphi_1)\right]C - 2\bar{y}\delta r^2(\cos\varphi - \cos\varphi_1)C\right\}\right\} -$$
$$- 2\psi'(x)\left\{\left\{\delta r^2\left\{r\left[\frac{\varphi^2-\varphi_1^2}{2} - \pi(\varphi-\varphi_1)\right] - \right.\right.\right.$$
$$\left.\left.\left. - \bar{y}(\cos\varphi - \cos\varphi_1)\right\}\frac{\mathfrak{M}}{\Omega} + C\frac{\mathfrak{M}}{\Omega}\right\}\right\} + \frac{\mathfrak{M}^2}{\Omega^2}.$$

Let ng η represent the first sum of the terms in the double races and γ the second sum, we obtain

$$q^2_{(x,\varphi)} = \psi''(x)\,\eta - 2\psi'(x)\,\gamma + \frac{\mathfrak{M}^2}{\Omega^2}.$$

As a consequence,

$$\Gamma=\frac{r\delta}{2E}\oint\left[\sigma^2_{(x,\varphi)}+\frac{2(1+\mu)}{\delta^2}\left(\psi''(x)\eta-2\psi'(x)\gamma+\frac{\mathfrak{M}^2}{\Omega^2}\right)\right]d\varphi$$

or, taking into account Expression (7.23), we obtain

$$\Gamma=\frac{r\delta}{2E}\left\{\psi^2(x)B+\oint\left[\frac{2(1+\mu)}{\delta^2}\left(\psi''(x)\eta-2\psi'(x)\gamma+\frac{\mathfrak{M}^2}{\Omega^2}\right)\right]d\varphi\right\}.$$

After integrating over the contour

$$\Gamma=\frac{r\delta}{2E}\left[\psi^2(x)B+\frac{2(1+\mu)}{\delta^2}(\psi''(x)\bar{\eta}-2\psi'(x)\bar{\gamma}+\frac{\mathfrak{M}^2}{\Omega^2}2\pi\right],$$

where

$$\bar{\eta}=\oint\eta\,d\varphi \text{ and } \bar{\gamma}=\oint\gamma\,d\varphi.$$

Using the equation

$$\frac{\partial\Gamma}{\partial\psi}-\frac{d}{dx}\left[\frac{\partial\Gamma}{\partial(\psi'_{(x)})}\right]=0,$$

we obtain

$$\psi_{(x)}B-\psi''_{(x)}\bar{\eta}\frac{2(1-\mu)}{\delta^2}=0$$

or

$$\psi''_{(x)}-\alpha^2\psi(x)=0, \tag{7.32}$$

where

$$\alpha^2=\frac{\delta^2B}{2(1+\mu)\bar{\eta}}. \tag{7.33}$$

The solution of Eq. (7.32) takes the form

$$\psi_{(x)}=C_1e^{-\alpha x}+C_2e^{\alpha x}. \tag{7.34}$$

Then

$$\frac{\sigma_{(x,\varphi)}}{\omega}=C_1e^{-\alpha x}+C_2e^{\alpha x}.$$

From the boundary condition at x = ∞ it follows that $\frac{\sigma_{(x,\varphi)}}{\omega}=0$ and $C_2 = 0$. Therefore

$$\frac{\sigma_{(x,\varphi)}}{\omega(\varphi)}=C_1e^{-\alpha x}.$$

From the boundary condition at x = 0 we obtain $C_1=\frac{\sigma_{(l/2,\varphi)}}{\omega(\varphi)}$ Thus,

$$\sigma_{(x,\varphi)}=\sigma_{(l/2,\varphi)}e^{-\alpha x} \tag{7.35}$$

and

$$\psi_1(x)=\frac{\Phi(1/2,\varphi)}{\omega(\varphi)}\,e^{-\alpha x}. \qquad (7.36)$$

We determine the value

$$\bar{\eta}=\oint h\,d\varphi,$$

where

$$\begin{aligned}\eta={}&\delta^2 r^6\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]^2+\\&+\bar{y}^2\delta^2 r^4(\cos\varphi-\cos\varphi_1)^2+C^2-\\&-2\delta^2 r^5\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]\bar{y}(\cos\varphi-\cos\varphi_1)+\\&+2\delta r^3\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]C-2\bar{y}\delta r^2(\cos\varphi-\cos\varphi_1)C\end{aligned}$$

and

$$C=\omega(\varphi_1)f-\frac{2r^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\,\delta\Phi\right)}{\Omega}.$$

In view of the fact that

$$\oint\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]^2 d\varphi=2\int\limits_{\varphi_1}^{\pi}\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]^2 d\varphi=\frac{4}{15}\xi^5;$$

$$2\int\limits_{\varphi_1}^{\pi}(\cos\varphi-\cos\varphi_1)^2 d\varphi=\xi+1{,}5\sin 2\varphi_1+2\xi\cos^2\varphi_1;$$

$$\begin{aligned}\oint C^2 d\varphi={}&2\int\limits_{\varphi_1}^{\pi}\omega^2(\varphi_1)f^2 d\varphi-2\int\limits_{\varphi_1}^{\pi}2\omega(\varphi_1)f\,\frac{2r^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\,\delta\Phi\right)}{\Omega}\,d\varphi+\\&+\int\limits_0^{2\pi}\frac{4r^6\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\,\delta\Phi\right)^2}{\Omega^2}\,d\varphi=\\&=2\omega^2(\varphi_1)f^2\xi-4\omega(\varphi_1)f\,\frac{2r^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\,\delta\Phi\right)}{\Omega}\,\xi+\\&+\frac{4r^6\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\,\delta\Phi\right)^2 2\pi}{\Omega^2};\end{aligned}$$

$$2\int\limits_{\varphi_1}^{\pi}\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right](\cos\varphi-\cos\varphi_1)\,d\varphi=2\Phi+\frac{2}{3}\xi^3\cos\varphi_1;$$

$$2\int_{\varphi_1}^{\pi}\left[\frac{\varphi^2-\varphi_1^2}{2}-\pi(\varphi-\varphi_1)\right]d\varphi=-\frac{2}{3}\xi^3;$$

$$2\int_{\varphi_1}^{\pi}(\cos\varphi-\cos\varphi_1)\,d\varphi=-2\Phi,$$

We obtain

$$\bar{\eta}=\delta^2r^6\,\frac{4}{15}\,\xi^5+\bar{y}^2\delta^2r^4(\xi+1{,}5\sin 2\varphi_1+2\xi\cos^2\varphi_1)+$$
$$+D-4\delta^2r^5\bar{y}\left(\Phi+\frac{1}{3}\xi^3\cos\varphi_1\right)-\frac{4}{3}\,\delta r^3\xi^3C+4\bar{y}\delta r^2\Phi C, \qquad (7.37)$$

where D represents the expression for the third integral.

Manipulating Expression (7.33), we obtain

$$\alpha^2=\frac{\dfrac{\xi^3}{3}-2\,\dfrac{\bar{y}}{r}\,\Phi+\left(\dfrac{\bar{y}}{r}\right)^2\left(\dfrac{\xi}{2}+\dfrac{\sin 2\varphi_1}{4}\right)}{2(1+\mu)\left[\dfrac{2r^2}{15}\,\xi^5+\bar{y}^2E+\dfrac{D}{2\delta^2r^4}-2r\,\bar{y}\left(\Phi+\dfrac{1}{3}\,\xi^3\cos\varphi_1\right)-\dfrac{2}{3}\,\dfrac{\xi^3C}{\delta r}+\dfrac{2\,\bar{y}\,\Phi C}{\delta r^2}\right]}, \qquad (7.38)$$

where

$$E=\frac{\xi}{2}+0{,}75\sin 2\varphi_1+\xi\cos^2\varphi_1.$$

We determine the value of α using the geometric data for the body as given in the sample calculation for determination of the stressed state in the bay with the cut.

Letting $\mu = 0.3$, we obtain

$$\alpha^2=\frac{5{,}981-11{,}206+6{,}254+\left(\dfrac{50{,}62}{25}\right)^2\left(\dfrac{2{,}618}{2}+\dfrac{0{,}866}{4}\right)}{2{,}6\,(10248{,}4+10049{,}6+139{,}6-20108{,}7-14{,}7+94{,}4)}=$$
$$=\frac{1{,}029}{1063}=0{,}000968.$$

As a consequence,

$$\alpha=\sqrt{0{,}000968}=0{,}0311.$$

Thus, for the given case,

$$\sigma_{(x,\varphi)}=\sigma_{(l/2,\varphi)}e^{-0{,}0311x},$$

where

$$\sigma_{(l/2,\varphi)}=7{,}7\,(25\varphi-78{,}5+50{,}62\sin\varphi).$$

For x = 0, $\sigma_{(x,\varphi)}=\sigma_{(l/2,\varphi)}$;

for

$$x=r=25 \qquad \sigma_{(x,\varphi)}=0{,}459\sigma_{(l/2,\varphi)};$$

for

$$x=2r=50 \qquad \sigma_{(x,\varphi)}=0{,}211\sigma_{(l/2,\varphi)};$$

for

$$x=3r=75 \qquad \sigma_{(x,\varphi)}=0{,}097\sigma_{(l/2,\varphi)}.$$

The PKS $q_{(x,\ \varphi)}$ is computed from Formula (7.30). In the given case

$$\psi'_{(x)}=-\frac{\sigma_{(l/2,\varphi)}}{\omega}\alpha e^{-\alpha x}=-\frac{7{,}7\cdot 0{,}0311}{25}e^{-0{,}0311x};$$

$$\psi'_{(x)}\frac{q_\varphi}{k}=-0{,}239\,e^{-0{,}0311x}\Big[-80{,}28+125\left(\frac{\varphi^2}{2}-\pi\varphi\right)-$$

$$-253\cos\varphi+467\Big];$$

$$\frac{\psi'_{(x)}2r^3\left(Af\xi-\frac{r^2\delta\xi^3}{3}+\bar{y}r\delta\Phi\right)}{\Omega}=+\frac{7{,}7\cdot 0{,}0311}{25}e^{-0{,}0311x}2049{,}2=$$

$$=+19{,}6e^{-0{,}0311x}$$

and

$$\frac{\mathfrak{M}}{\Omega}=\frac{10^5}{2\pi r^2}=25{,}4\ \text{kgf/cm}.$$

As a consequence, the PKS is

$$q_{(x,\varphi)}=-0{,}239\left[-80{,}28+125\left(\frac{\varphi^2}{2}-\pi\varphi\right)-253\cos\varphi+407\right]e^{-0{,}0311x}-$$

$$-19{,}6\,e^{-0{,}0311x}-25{,}4,$$

where the first term is taken from φ_1 to $2\pi-\varphi_1$.

For x = 0, along the upper arch between stringers

$$q_0=-19{,}6-25{,}4=-45\ \text{kgf/cm}$$

while along the lower arch:

for $\varphi=\varphi_1$ $\quad q=-0{,}239(-80{,}28)-45=19{,}2-45=-25{,}8;$

for $\varphi=\frac{\pi}{2}$ $\quad q=+32{,}2-45=-12{,}8;$

for $\varphi=\pi$ $\quad q=+9-45=-36;$

for $\varphi=\frac{3}{2}\pi$ $\quad q=-12{,}8;$

for $\varphi=2\pi-\varphi_1$ $\quad q=-25{,}8.$

For x = r = 25, along the upper arch between stringers we have

$$q_0 = -0{,}459 \cdot 19{,}6 - 25{,}4 = -9 - 25{,}4 = -34{,}4,$$

while along the lower arch:

for $\varphi = \varphi_1$ $q = 0{,}459 \cdot 19{,}2 - 34{,}4 = 8{,}8 - 34{,}4 = -25{,}6;$

for $\varphi = \frac{\pi}{2}$ $q = 0{,}459 \cdot 32{,}2 - 34{,}4 = 14{,}8 - 34{,}4 = -19{,}6;$

for $\varphi = \pi$ $q = 0{,}459 \cdot 9 - 34{,}4 = 4{,}1 - 34{,}4 = -30{,}3;$

for $\varphi = \frac{3}{2}\pi$ $q = -19{,}6;$

for $\varphi = 2\pi - \varphi_1$ $q = -25{,}6.$

For x = 2r = 50, along the upper arch between stringers we have

$$q_0 = -0{,}211 \cdot 19{,}6 - 25{,}4 = -4{,}1 - 25{,}4 = -29{,}5,$$

while along the lower arch:

for $\varphi = \varphi_1$ $q = 0{,}211 \cdot 19{,}2 - 29{,}5 = -25{,}5;$

for $\varphi = \frac{\pi}{2}$ $q = 0{,}211 \cdot 32{,}2 - 29{,}5 = -22{,}7;$

for $\varphi = \pi$ $q = 0{,}211 \cdot 9 - 29{,}5 = -27{,}6;$

for $\varphi = \frac{3}{2}\pi$ $q = -22{,}7;$

for $\varphi = 2\pi - \varphi_1$ $q = -25{,}5.$

For x = 3r = 75, we have along the upper arch between stringers

$$q_0 = -0{,}097 \cdot 19{,}6 - 25{,}4 = -27{,}3,$$

while along the lower arch:

for $\varphi = \varphi_1$ $q = 0{,}097 \cdot 19{,}2 - 27{,}3 = -25{,}4;$

for $\varphi = \frac{\pi}{2}$ $q = 0{,}097 \cdot 32{,}2 - 27{,}3 = -24{,}2;$

for $\varphi = \pi$ $q = 0{,}097 \cdot 9 - 27{,}3 = -26{,}4;$

for $\varphi = \frac{3}{2}\pi$ $q = -24{,}2;$

for $\varphi = 2\pi - \varphi_1$ $q = -25{,}4.$

The stressed state is shown in Fig. 7.13. As we can see, the normal stresses are attenuated rapidly and are nearly zero at a distance of 1.5D; near the bulkheads bounding the bay with the cut, the PKS is considerably greater than yielded by the Bredt formula:

$$q=\frac{\mathfrak{M}}{\Omega}.$$

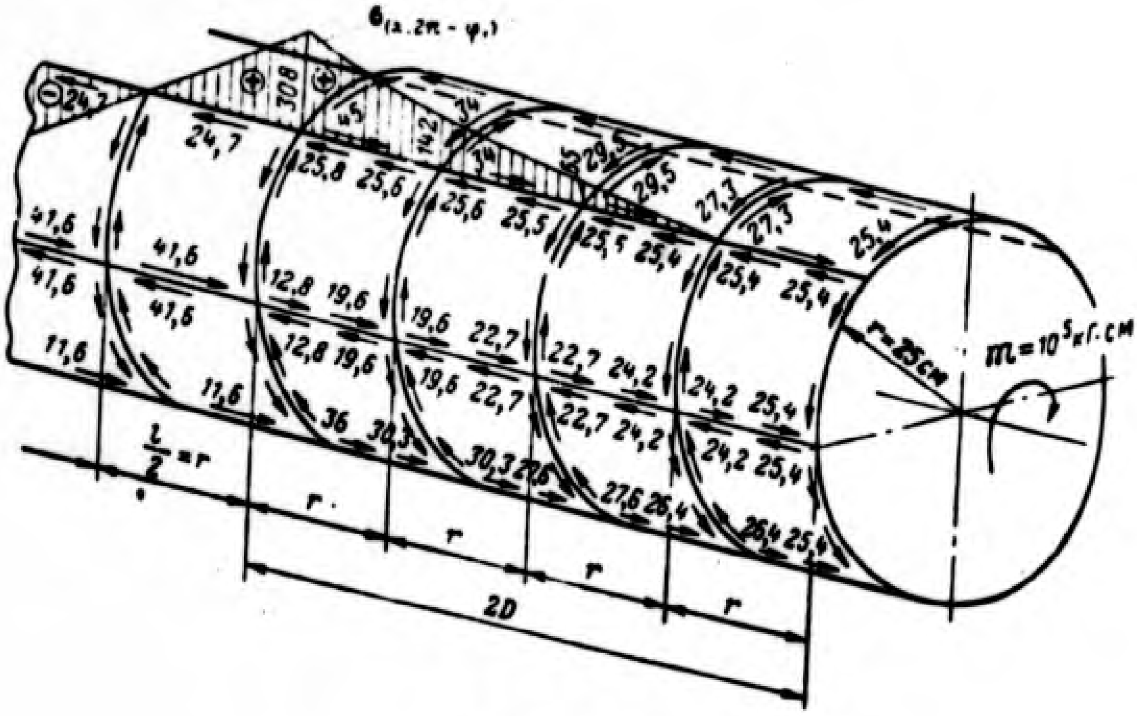


Fig. 7.13

In sitting up the finite-difference equations of Section 1 and determining the potential energy in Section 2, we assumed that the shell was perfectly stiff in the transverse direction and thus the computations given in this chapter should be considered to be first-approximation calculations.

Chapter 8

FREE BENDING AND FREE TORSION IN HIGHLY TAPERED SHELLS*

We shall consider shells of the type used for flying-craft wings working within the elastic-deformation range under the assumption that the ribs are perfectly stiff in their own plane, while they are perfectly flexible in the direction perpendicular to their plane.

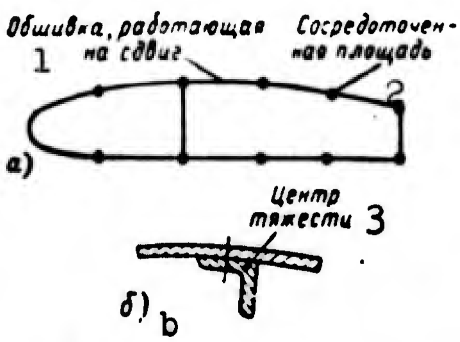


Fig. 8.1. 1) Skin working in shear; 2) concentrated area; 3) center of gravity.

On the assumption that beams can only carry normal stresses and the skin only shear stresses, in calculation schemes we take as the beam area the area of the longitudinal member with the associated area of the skin, multiplied by a reduction factor.

After we have associated with the area of each beam the immediately adjacent skin working effectively under the normal stresses, we obtain a calculation scheme, for example, a wing section, consisting of several separate points – the centers of gravity of the beam areas (Fig. 8.1b) with the skin between them, working only under the shear stresses (Fig. 8.1a).

If the wing has no stringers, the beam area in the calculation scheme will consist solely of the skin area, contracted to the given point.

Such a calculation scheme has found wide application in the design of thin-walled structures, but where the cross-sectional contour is simple (see Fig. 8.1a), another, converse calculation scheme is used on occasion; here the effectively working skin area and the stringer area are not contracted to a point but, quite the opposite, are distributed uniformly over the section contour. As a result, we obtain a shell without stringers (Fig. 8.2a and b) having a thickness δ_σ and working under normal stresses. The thickness of the skin δ_τ that is working under shear stresses remains equal to the true skin thickness.

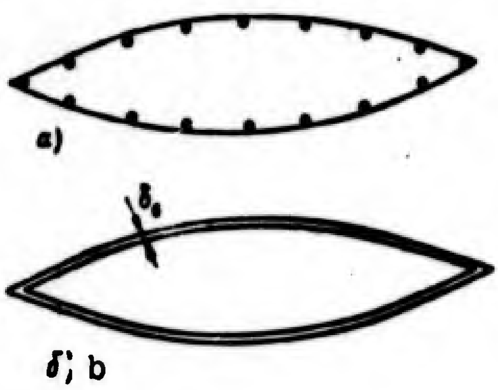


Fig. 8.2

When we are determining stresses and strains, the second calculation scheme enables us to make wide use of integration in closed form, which eliminates the tiresome tabular summation needed when the first, discrete scheme is used.

Henceforth we shall arbitrarily refer to the first calculation scheme as the scheme "with skin working solely in shear," and to the second as the scheme "with skin working under shear and normal stresses."

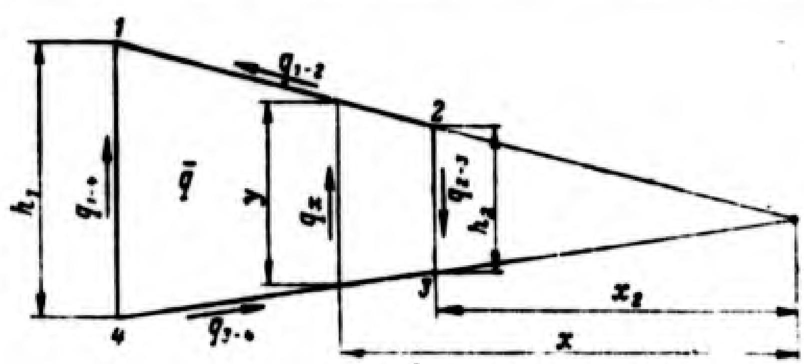


Fig. 8.3

In both the first and second calculation schemes, the thickness of the shell wall is so small as compared with the over-all dimensions of the structure's cross section that both the normal and shear stresses may be assumed to be constant over the wall thickness.

When we consider tapered shells, the calculation schemes will contain trapezoidal skin elements. We recall (see [30]) that for such an element, the average PKS $\bar{q}$ is represented in terms of the PKS acting at the edges (Fig. 8.3), in the following manner:

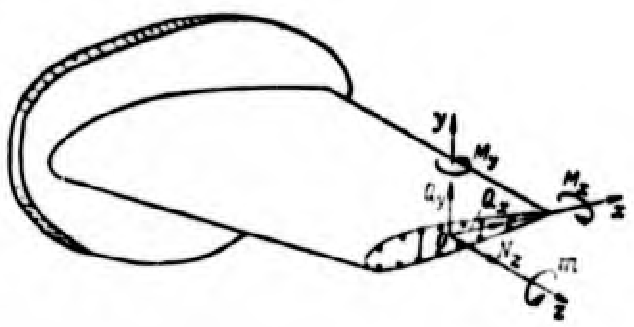
$$\left.\begin{aligned}\bar{q}&=q_{1-2}=q_{3-4},\\ \bar{q}&=q_{2-3}\frac{h_2}{h_1},\\ \bar{q}&=q_{1-4}\frac{h_1}{h_2},\\ q_x&=q_{2-3}\frac{x_2^2}{x^2}.\end{aligned}\right\} \tag{8.1}$$

1. DETERMINATION OF NORMAL STRESSES

Figure 8.4 shows a multistringer shell of the wing type, loaded at the end by the bending moments M_x, M_y, the torsional moment $\dot{M}$ about the $\underline{z}$ axis, the transverse forces Q_y, Q_x, and the longitudinal force N_z. If we had to make use of the familiar methods of static indeterminacy, it would be difficult, since this type of shell has a very high degree of static indeterminacy. In order to eliminate the high degree of static indeterminacy, we introduce the hypothesis of plane distribution of the relative elongations, i.e., we assume that the relative elongation ε along the height of the shell varies in proportion to the distance from the neutral layer of its cross section [9].

The possibility of extending this hypothesis to shells of the wing type has been verified by numerous static tests of structures under actual conditions. The tests showed that the normal stresses obtained by calculations based on the hypothesis of plane distribution of relative elongations agree well with the experimental values for sections far from the attachment.

Since the normal stresses due to bending, distributed in accordance with our deformation hypothesis, will not depend on the constraint conditions, this stressed state is called <u>free bending</u>, while torsion that does not take into account the effect of the constraint is called <u>free torsion</u>.



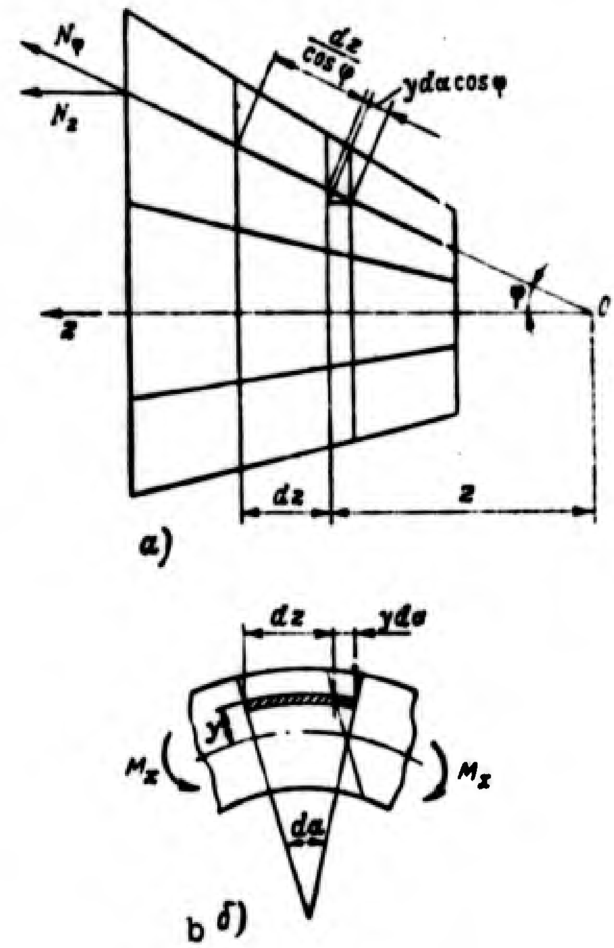


Fig. 8.4

Fig. 8.5

Let us consider the method of normal-stress calculations developed by I.A. Sverdlov [9].

In determining the normal stresses we shall assume that the shell taper is great only in the Oxz plane (see Fig. 8.4). From the shell, we take a bay of length dz formed by two planes with coordinates z = const and z + dz = const (Fig. 8.5a); after the shell has been loaded, for example, by a bending moment M_x, these planes form an angle $d\alpha$ (Fig. 8.5b). As a result, any longitudinal element $\underline{i}$ a distance y_i away from the neutral layer will be displaced an amount $y_i\, d\alpha$ along the $\underline{z}$ axis. Corresponding to this displacement there will be an elongation or contraction of

$$y_i\, d\alpha \cos\varphi.$$

in the longitudinal element forming the angle φ with the z axis.

In view of the fact that the element length equals dz/cos φ, we obtain the relative elongation of the longitudinal member in the form:

$$\varepsilon_\varphi = \frac{y_i d\alpha}{dz} \cos^2 \varphi.$$

As a consequence, the stress is

$$\sigma_\varphi = E \frac{y_i d\alpha}{dz} \cos^2 \varphi. \tag{a}$$

The axial force is

$$N_\varphi = f_i E \frac{y_i d\alpha}{dz} \cos^2 \varphi. \tag{b}$$

Here f_i is the area of the longitudinal member, cut perpendicular to its axis.

The axial force along the z axis equals

$$N_z = N_\varphi \cos \varphi = f_i E \frac{y_i d\alpha}{dz} \cos^3 \varphi, \tag{c}$$

while the stress along the axis is

$$\sigma_z = \frac{N_z}{f_i} \cos \varphi = E \frac{y_i d\alpha}{dz} \cos^4 \varphi. \tag{d}$$

We find the constant value dα/dz for section z from the equilibrium equation

$$M_x = \sum \sigma_z y_i f_z = \sum \sigma_z y_i \frac{f_i}{\cos \varphi} = \sum \frac{d\alpha}{dz} E y_i^2 \cos^3 \varphi f_i.$$

From which we have

$$\frac{d\alpha}{dz} = \frac{M_x}{EJ_x},$$

where

$$J_x = \sum y_i^2 \cos^3 \varphi f_i.$$

Substituting the value of dα/dz into Formulas (d), (a), (c) and (b), we obtain the desired stresses

$$\sigma_z = \frac{M_x y_i}{J_x} \cos^4 \varphi;$$

$$\sigma_\varphi = \frac{M_x y_i}{J_x} \cos^2 \varphi.$$

The axial forces are

$$N_z=\frac{M_x S_{x_i}}{J_x},$$
$$N_\varphi=\frac{M_x S_{x_i}}{J_x \cos\varphi}.$$

Here

$$S_{x_i}=y_i \cos^3\varphi f_i.$$

Taking into account M_y (see Fig. 8.4), we obtain

$$\sigma_z=\left(\frac{M_x}{J_x}y_i-\frac{M_y}{J_y}x_i\right)\cos^4\varphi; \tag{8.2}$$

$$\sigma_\varphi=\frac{\sigma_z}{\cos^2\varphi}; \tag{8.3}$$

$$N_z=\frac{M_x S_{x_i}}{J_x}-\frac{M_y S_{y_i}}{J_y}; \tag{8.4}$$

$$N_\varphi=\frac{N_z}{\cos\varphi}. \tag{8.5}$$

Here $\underline{x}$, $\underline{y}$ are the running coordinates in the principle central axes of inertia of the shell cross section;

$$\left.\begin{aligned}J_x&=\sum y_i^2 f_i\cos^3\varphi\\ J_y&=\sum x_i^2 f_i\cos^3\varphi\end{aligned}\right\}\ \text{principle central moments of inertia}$$

$$\left.\begin{aligned}S_{x_i}&=y_i f_i\cos^3\varphi\\ S_{y_i}&=x_i f_i\cos^3\varphi\end{aligned}\right\}\ \text{static moments of } i\underline{\text{th}} \text{ element;}$$

N_z is the force for the i<u>th</u> element, acting along the $\underline{z}$ axis; N_φ is the force acting along the axis of the i<u>th</u> element.

2. DETERMINATION OF SHEAR STRESSES IN BENDING OF A SHELL WITH OPEN CROSS-SECTION CONTOUR

We have a bay of an open shell with length $\Delta\ell$ equal to the distance between ribs (Fig. 8.6) with an arbitrary number of longitudinal members each having a reduced area f_i.

For simplicity we limit ourselves to the case in which the effect of the remote portion on the right-hand section of the bay being considered reduces solely to the bending moment M_x and the transverse force Q_y.

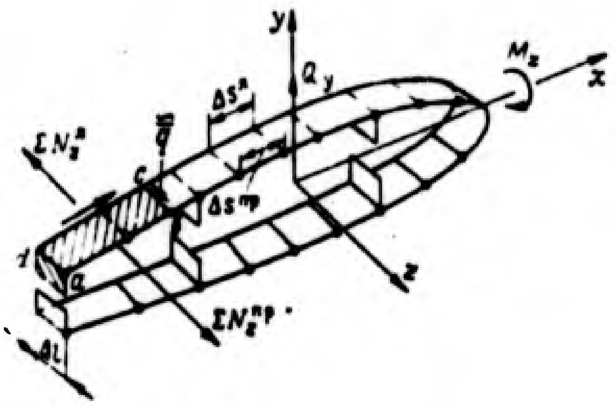


Fig. 8.6

In order to find the mean PKS $\tilde{\bar{q}}$ in any section c-b, we consider the equilibrium of the part of the bay shell shown hatched in Fig. 8.6 whose length is equal to the distance from the free edge d-a to section c-b. The symbol "~" above $\bar{q}$ indicates that the PKS refers to an open contour.

The effect of the remote portions of the shell on the hatched element abcd reduces to the unknown PKS $\tilde{\bar{q}}$ and the resultant forces from the right side ΣN_z^{pr} and from the left side ΣN_z^{ℓ}. Thus the equilibrium condition for the element abcd about the z axis may be written as

$$-\sum N_z^{\text{л}} + \sum N_z^{\text{пр}} - \tilde{\bar{q}} \Delta l = 0.$$

from which we have

$$\tilde{\bar{q}} = \frac{\sum N_z^{\text{пр}} - \sum N_z^{\text{л}}}{\Delta l}. \tag{8.6}$$

The equivalent force from the right side of the bay is

$$\sum N_z^{\text{пр}} = \sum \frac{M_x S_{x_l}^{\text{пр}}}{J_x^{\text{пр}}}, \tag{8.7}$$

while from the left side it is

$$\sum \quad = \sum \frac{M_x S_{x_l}^{\text{л}}}{J_x^{\text{л}}} - \sum \frac{Q_x \Delta l S_{x_l}^{\text{л}}}{J_x^{\text{л}}},$$

or

$$\sum \tilde{N}_z^{л} = \sum \frac{(M_y - Q_y \Delta l)}{J_x^{л}} S_{x_l}^{л}, \tag{8.8}$$

where $S_{x_1}^{pr}$ and J_x^{pr} refer to the right-hand section of the bay and $S_{x_1}^{\ell}$ and J_x^{ℓ} refer to the left-hand section.

According to Formulas (8.1), the PKS due to the right-hand and left-hand sides of the bay will equal

$$\tilde{q}^{np} = \overset{*}{q} \frac{\Delta S^{л}}{\Delta S^{np}} \text{ (from the right side)}; \tag{8.9}$$

$$\tilde{q}^{л} = \overset{*}{q} \frac{\Delta S^{np}}{\Delta S^{л}} \text{ (from the left side)}. \tag{8.10}$$

Here ΔS^{pr} and ΔS^{ℓ} are the distances between stringers from the left and right sides, respectively (see Fig. 8.6).

For the second calculation scheme of Fig. 8.6, in accordance with Formulas (8.2), (8.3), (8.6), (8.7) and (8.8):

$$\sigma_z = \left(\frac{M_x}{J_x} y_l - \frac{M_y}{J_y} x_l\right) \cos^4 \varphi; \tag{8.11}$$

$$\sigma_\gamma = \frac{\sigma_z}{\cos^2 \varphi}; \tag{8.12}$$

$$\overset{*}{q} = \frac{\sum N_z^{np} - \sum N_z^{л}}{\Delta l}; \tag{8.13}$$

$$\sum N_z^{np} = \frac{M_x}{J_x^{np}} S_x^{np}; \tag{8.14}$$

$$\sum N_z^{л} = \frac{M_x - Q_y \Delta l}{J_x^{л}} S_x^{л}. \tag{8.15}$$

Here

$$J_x^{np} = \oint y^{np^2} \cos^4 \varphi \, dF = \oint y^{np^2} \cos^4 \varphi \frac{z^{np} \delta_э d\varphi}{\cos^2 \varphi}$$

or

$$\left.\begin{aligned} J_x^{np} &= \oint y^{np^2} \cos^2 \varphi z^{np} \delta_э d\varphi, \\ J_x^{л} &= \oint y^{л^2} \cos^2 \varphi z^{л} \delta_э d\varphi; \end{aligned}\right\} \tag{8.16}$$

$$\left.\begin{aligned} S_x^{np} &= \int y^{np} \cos^2 \varphi z^{np} \delta_э d\varphi, \\ S_x^{л} &= \int y^{л} \cos^2 \varphi z^{л} \delta_э d\varphi. \end{aligned}\right\} \tag{8.17}$$

The origin for the z coordinate is shown in Fig. 8.5a.

In determining the moments of inertia, integration is taken around the entire contour, while in determining the static moments it extends from the free edge to the section under consideration.

3. BENDING AND TORSION IN CLOSED SHELLS

Assume we have a calculation scheme of the first (Fig.8.7a), loaded by the transverse force Q_y, the bending moment M_x, and the torsional moment M acting about the z axis.

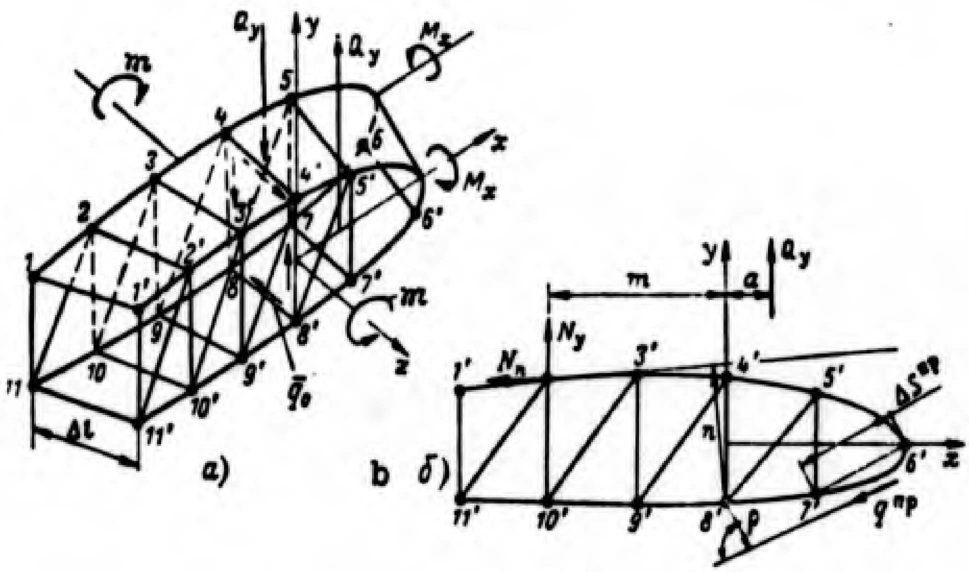


Fig. 8.7

If the system has one wall 4-4'-8'-8 (contour with simple closure) then traversing the contour in the clockwise direction, we can find the mean PKS for panels 1-2, 2-3, 3-4 from Formula (8.6):

$$\bar{q}=\frac{\sum N_z^{np}-\sum N_z^{\text{л}}}{\Delta l}.$$

It is not possible to extend this formula to panels 4-5, 5-6, 6-7 and 7-8, since we do not know the PKS for the wall 4-4'-8'-8. To avoid this difficulty, we let $\bar{q}_0$ be the unknown mean PKS for the wall. Then the mean PKS for the closed-contour panels will be

$$\bar{q}=\frac{\sum N_z^{np}-\sum N_z^{\text{л}}}{\Delta l}+\bar{q}_0$$

or

$$\bar{q}=\dot{q}+\bar{q}_0. \qquad (8.18)$$

We find the unknown value $\bar{q}_0$ from the as-yet unused equilibrium condition of statics, which states that the sum of the moments due to all forces (external and internal) acting on the system as a whole or on part of it about any axis parallel to the z axis must equal zero.

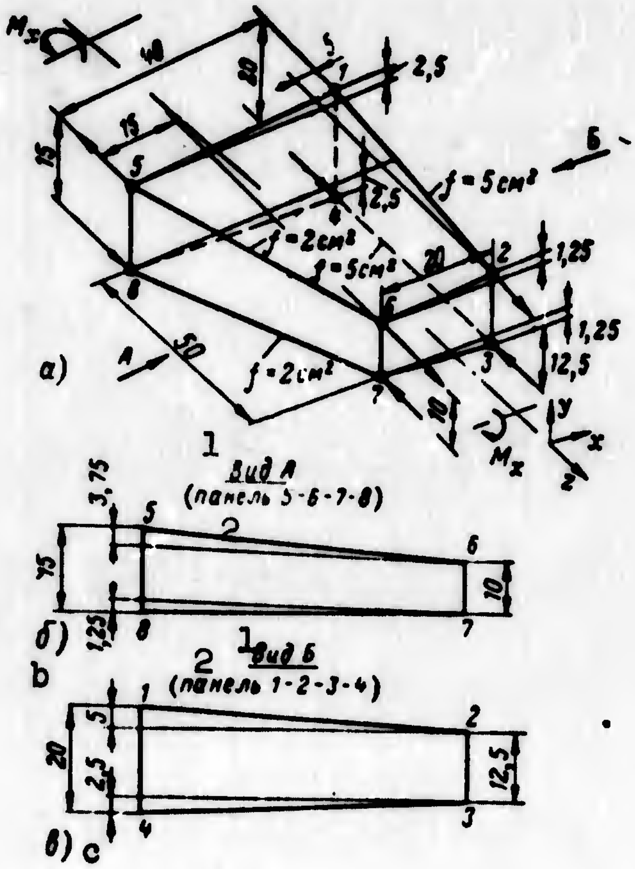


Fig. 8.8. 1) View; 2) panel.

Let us write the equilibrium condition for the right rib 1'-2'-3'-4'-5'-6'-7'-8'-9'-10'-11' (Fig. 8.7b).

If we take as the center of moments, for example, the point (joint) 8', we obtain

$$\sum q^{пр}\rho\Delta S^{пр}-\sum N_x n+\sum N_y m-\mathfrak{M}-Q_y a=0, \qquad (8.19)$$

where q^{pr} is the PKS acting on the rib flange; it clearly equals, in accordance with Formula (8.9)

$$q^{\text{пр}}=\bar{q}\,\frac{\Delta S^{\text{л}}}{\Delta S^{\text{пр}}},$$

or

$$q^{\text{пр}}=(\bar{q}+\bar{q}_0)\frac{\Delta S^{\text{л}}}{\Delta S^{\text{пр}}}; \qquad (8.20)$$

Here ρ is the length of the perpendicular drop from the center of moments to the line of action of the PKS q^{pr}; N_n is the component of N_z that acts along the flange or in the direction of the <u>x</u> axis; N_y is the component of N_z that acts in the direction of the <u>y</u> axis; n, m are the arms of forces N_n and N_y, respectively (see Fig. 8.7b).

Since in the solution for the simply closed shell all the force factors are determined from the conditions of statics, we can conclude that simply closed shells are statically determinate.

<u>Example</u>.

We are to determine the stresses state in a four-flange bay acted on by the bending moment M_x = 1000 kgf·m. The geometric characteristics are shown in Fig. 8.8.

We first determine the forces acting on the joints at the right- and left-sides of the bay. We use the formula

$$N_z=\frac{M_x S_{x_i}}{J_x},$$

where

$$S_{x_i}=y_i f_i \cos^3\varphi \quad \text{и} \quad J_x=\sum y_i^2 f_i \cos^3\varphi.$$

In the given case

$$\cos(1-2,\ z)=0{,}99; \qquad f_{1-2}=5\ \text{см}^2;$$
$$\cos(6-5,\ z)=0{,}955; \qquad f_{6-5}=2\ \text{см}^2.$$

For the left side of the bay

$$J_x^{\text{л}}=2(5\cdot10^2\cdot0{,}99^3+2\cdot7{,}5^2\cdot0{,}955^3)=1166\ \text{см}^4;$$
$$S_{x_1}^{\text{л}}=5\cdot10\cdot0{,}99^3=48{,}5; \qquad S_{x_4}^{\text{л}}=-48{,}5;$$
$$S_{x_6}^{\text{л}}=2\cdot7{,}5\cdot0{,}955^3=13{,}05; \qquad S_{x_8}^{\text{л}}=-13{,}05.$$

As a consequence,

$$N^{л}_{z_1}=\frac{100\,000\cdot 48,5}{1166}=4170; \qquad N^{л}_{z_3}=-N^{л}_{z_1};$$

$$N^{л}_{z_6}=\frac{100\,000\cdot 13,05}{1166}=1118; \qquad N^{л}_{z_8}=-N^{л}_{z_6}.$$

For the right side of the bay

$$J^{пр}_{x}=2\left[5\left(\frac{12,5}{2}\right)^{2}\cdot 0,99^{3}+2\cdot 5^{2}\cdot 0,955^{3}\right]=466\ \text{cm}^{4};$$

$$S^{пр}_{x_1}=5\,\frac{12,5}{5}\,0,99^{3}=30,3; \qquad S^{пр}_{x_3}=-30,3;$$

$$S^{пр}_{x_6}=2,5\cdot 0,955^{3}=8,7; \qquad S^{пр}_{x_8}=-8,7.$$

As a consequence,

$$N^{пр}_{z_1}=\frac{100\,000\cdot 30,3}{466}=6500; \qquad N^{пр}_{z_3}=-N^{пр}_{z_1};$$

$$N^{пр}_{z_6}=\frac{100\,000\cdot 8,7}{466}=1870; \qquad N^{пр}_{z_8}=-N^{пр}_{z_6}.$$

TABLE 8.1

1 № стержней	2 Координаты			3 Квадраты координат			l^2	l	4 Косинусы углов		
	x	y	z	x^2	y^2	z^2			x/l	y/l	z/l
1—2	—5	—5	50	25	25	2500	2550	50,5	—0,099	—0,099	0,99
1—4	0	—20	0	0	400	0	400	20,0	0	—1	0
1—5	—40	—2,5	0	1600	6,25	0	1606,25	~40,0	—1	—0,0625	0
3—2	0	12,5	0	0	156	0	156	12,5	0	1	0
3—4	5	—2,5	—50	25	6,25	2500	2531,25	50,3	0,0992	—0,0497	—0,993
3—7	—20	1,25	0	400	1,56	0	401,56	20	—1	0,0625	0
6—2	20	1,25	0	400	1,56	0	401,56	20	1	0,0625	0
6—5	—15	3,75	—50	225	14,1	2500	2739,1	52,3	—0,288	0,0717	—0,955
6—7	0	—10	0	0	100	0	100	10,0	0	—1	0
8—5	0	15	0	0	225	0	225	15,0	0	1	0
8—4	40	—2,5	0	1600	6,25	0	1606,25	40	1	—0,0625	0
8—7	15	1,25	50	225	1,56	2500	2726,56	52,25	0,288	0,0249	0,956

1) Beam number; 2) coordinates; 3) squares of coordinates; 4) cosines of angles.

We turn to the determination of the force components N_z in the direction of the longitudinal members (flanges, strut ribs) for which we set up the auxiliary table of geometric relationships (Table 8.1).

Joint 1

$$\sum Z = -4170 + N_{1-2}\,0{,}99 = 0; \qquad (1)$$

$$N_{1-2} = 4210$$

$$\sum Y = -4210 \cdot 0{,}099 + N_{1-4}(-1) + N_{1-5}(-0{,}0625) = 0; \qquad (2)$$

$$\sum X = -0{,}099 \cdot 4210 + N_{1-5}(-1) = 0. \qquad (3)$$

From the solution to Eqs. (3) and (2) we obtain

$$N_{1-4} = -391 \text{ kgf}$$
$$N_{1-5} = -417 \text{ kgf}$$

Joint 2

$$\sum Z = 0 \qquad 6500 - N_{2-3}0{,}99 = 0, \qquad N_{2-1} = 6570 \text{ kgf}$$
$$\sum Y = 0 \qquad 6570 \cdot 0{,}099 - N_{2-6}0{,}0625 - N_{2-3} = 0, \qquad N_{2-3} = 610 \text{ kgf}$$
$$\sum X = 0 \qquad 6570 \cdot 0{,}099 - N_{2-6} = 0, \qquad N_{2-6} = 650 \text{ kgf}$$

Joint 3

$$\sum Z = 0 \qquad -6500 - N_{3-4} \cdot 0{,}993 = 0, \qquad N_{3-4} = 6550 \text{ kgf}$$
$$\sum X = 0 \qquad -6550 \cdot 0{,}0992 - N_{3-7} = 0, \qquad N_{3-7} = -650 \text{ kgf}$$
$$\sum Y = 0 \qquad 6550 \cdot 0{,}0497 - 650 \cdot 0{,}0625 + N_{3-2} = 0, \qquad N_{3-2} = -325 + {} + 41 = -284 \text{ kgf}$$

Joint 4

$$\sum Z = 0 \qquad 4170 + N_{4-3}0{,}993 = 0, \qquad N_{4-3} = -4210 \text{ kgf}$$
$$\sum X = 0 \qquad 4210 \cdot 0{,}0992 - N_{4-8} = 0. \qquad N_{4-8} = 418 \text{ kgf}$$
$$\sum Y = 0 - 4210 \cdot 0{,}0497 + 418 \cdot 0{,}0625 + N_{4-1} = 0, \qquad N_{4-1} = 209 - 26 = {} = 183 \text{ kgf}$$

Joint 5

$$\sum Z = 0 \qquad -1118 + N_{5-6}0{,}995 = 0, \qquad N_{5-6} = 1170 \text{ kgf}$$
$$\sum X = 0 \qquad 1170 \cdot 0{,}288 + N_{5-1} = 0, \qquad N_{5-1} = -337 \text{ kgf}$$
$$\sum Y = 0 \qquad -1170 \cdot 0{,}0717 - 337 \cdot 0{,}0625 - N_{5-8} = 0, \qquad N_{5-8} = -84 - {} - 12 = -105 \text{ kgf}$$

Joint 6

$$\sum Z=0 \qquad 1870-N_{6-5}0{,}955=0, \quad N_{6-5}=1960 \text{ kgf}$$

$$\sum X=0 \qquad -1960\cdot 0{,}288+N_{6-2}=0, \quad N_{6-2}=565 \text{ kgf}$$

$$\sum Y=0 \qquad 1960\cdot 0{,}0717+565\cdot 0{,}625-N_{6-7}=0, \quad N_{6-7}=141+35= =176 \text{ kgf}$$

Joint 7

$$\sum Z=0 \qquad -1870-N_{7-8}0{,}955=0, \quad N_{7-8}=-1960 \text{ kgf}$$

$$\sum X=0 \qquad 1960\cdot 0{,}288+N_{7-3}=0, \quad N_{7-3}=-565 \text{ kgf}$$

$$\sum Y=0 \qquad 1960\cdot 0{,}0249+565\cdot 0{,}0625+N_{7-6}=0, \quad N_{7-6}=-48{,}8-35{,}1= =-83{,}9 \text{ kgf}$$

Joint 8

$$\sum Z=0 \qquad 1118+N_{8-7}0{,}955=0, \quad N_{8-7}= =-1170 \text{ kgf}$$

$$\sum X=0 \qquad -1170\cdot 0{,}288+N_{8-4}=0, \quad N_{8-4}=337 \text{ kgf}$$

$$\sum Y=0 \qquad -1170\cdot 0{,}0249-337\cdot 0{,}0625+N_{8-5}=0, \quad N_{8-5}=29{,}2+ +21{,}1=50{,}3 \text{ kgf}$$

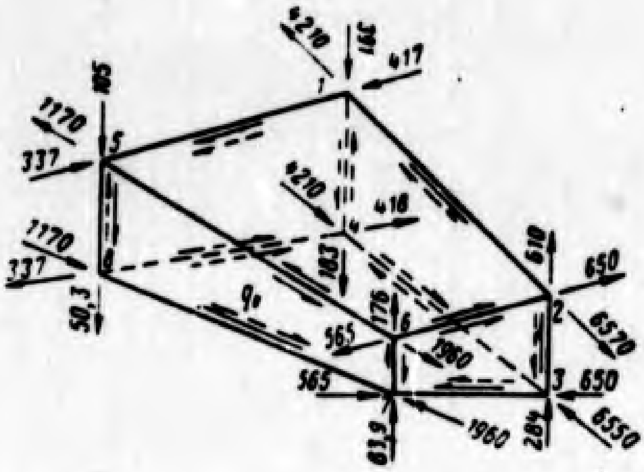


Fig. 8.9

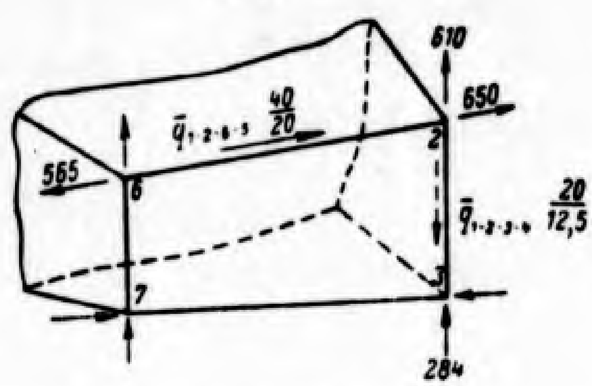


Fig. 8.10

The forces N and the positive direction taken for the PKS are shown in Fig. 8.9.

Let us determine the PKS:

$$-\bar{q}_{1-2-3-4}50+\bar{q}_0 50+6500+1870-4170-1118=0,$$

or

$$\bar{q}_{1-2-3-4}=61{,}6+\bar{q}_0.$$

From which we have

$$\bar{q}_{4-3-7-8}=\bar{q}_{1-2-6-5}=\bar{q}_0+15.$$

or

$$-\bar{q}_{1-2-6-5}50+\bar{q}_0 50+N_{z_6}-N_{z_6}=0$$

or

$$-\bar{q}_{1-2-6-5}50+\bar{q}_0 50+1870-1118=0.$$

From symmetry

$$\bar{q}_{1-2-6-5}=\bar{q}_0+15;$$

$$-\bar{q}_{1-2-3-4}50+\bar{q}_0 50+N_{z_6}+N_{z_7}-N_{z_6}-N_{z_7}=0,$$

From the equilibrium conditions for rib 2-3-7-6 it follows that the sum of the moments of all forces about joint 7 will equal zero (Fig. 8.10), i.e.,

$$\sum M_7=\bar{q}_{1-2-6-5}\,\frac{40}{20}\cdot 20\cdot 10+\bar{q}_{1-2-3-4}\,\frac{20}{12{,}5}\,12{,}5\cdot 20+$$
$$+(650-565)\cdot 10-(610+284)\cdot 20=0$$

or

$$\bar{q}_{1-2-6-5}+\bar{q}_{1-2-3-4}=\frac{-850+17880}{400}=42{,}6,$$

so that

$$\bar{q}_0+15{,}0+\bar{q}_0+61{,}6=42{,}6.$$

From this we have

$$\bar{q}_0=17{,}1 \text{ kgf/cm},$$

and then

$$\bar{q}_{1-2-6-5}=-1{,}9 \text{ kgf/cm}$$
$$\bar{q}_{1-2-3-4}=44{,}6 \text{ kgf/cm}$$
$$\bar{q}_{4-3-7-8}=-1{,}9 \text{ kgf/cm}$$

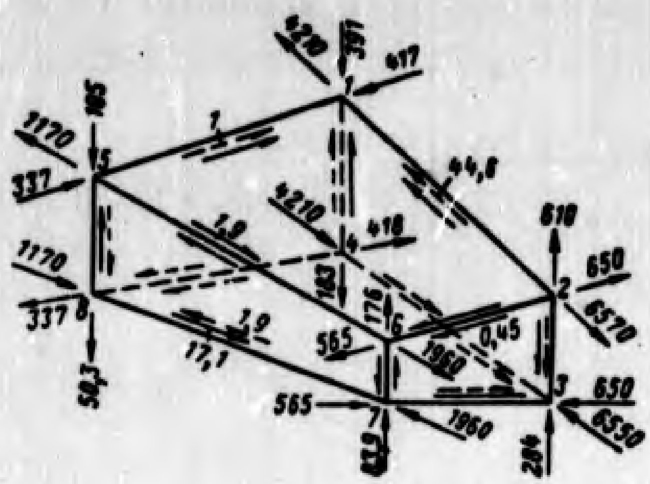


Fig. 8.11

It follows from the equilibrium conditions for beam 6-2 that

$$\bar{q}_{2-3-7-6}20-1{,}9\ \frac{40}{20}\ 20+(650-565)=0;$$

$$\bar{q}_{2-3-7-6}=-0{,}45\ \text{kgf},$$

while from the equilibrium conditions for beam 1-5 (see Fig. 8.9) we have

$$\bar{q}_{1-4-8-5}40+\bar{q}_{1-2-6-5}\ \frac{20}{40}\ 40-337-417=0;$$

$$\bar{q}_{1-4-8-5}=-1\ \text{kgf}.$$

The PKS found and the axial forces are shown in Fig. 8.11.

4. DETERMINATION OF SHELL DISPLACEMENTS

When external forces act on a shell, any longitudinal element of it set off by sections z = const, z + dz = const and s = const, s + ds = = const will be affected by stresses σ_P, σ_x, and τ_P; here σ_P acts along the x axis.

Since the stresses $\sigma_x = 0.1\sigma_P$ [30] have little effect on the stressed state, they may be neglected. Then the element will be affected only by the stresses shown in Fig. 8.12.

We shall call this set of external and internal forces the state "P."

In order to obtain an expression to determine the deflection f of point K in section I-I, located far from the constraint, we apply a force $Q_y = 1$ kgf in accordance with the unknown displacement (Fig. 8.13).

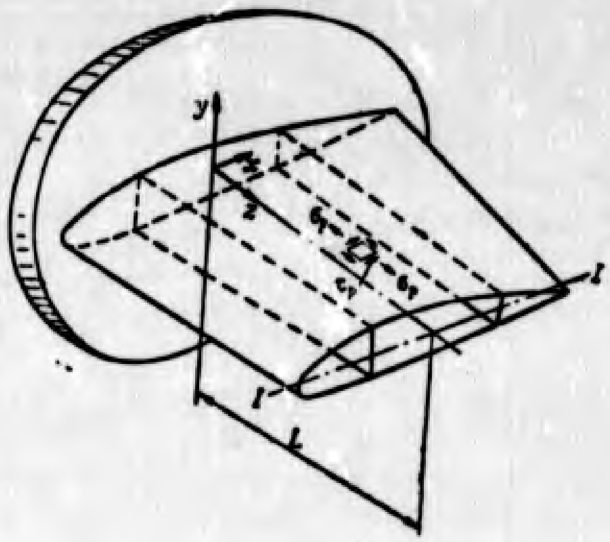


Fig. 8.12

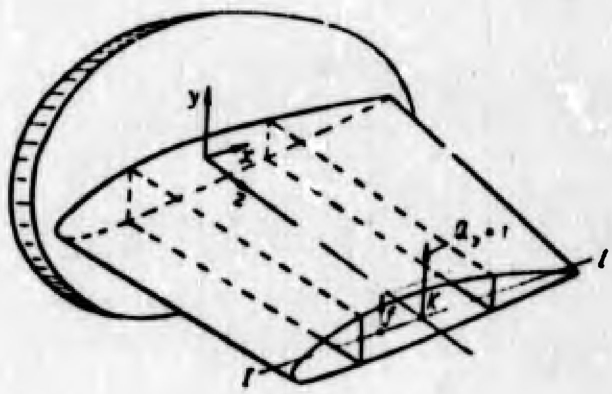


Fig. 8.13

The force $Q_y = 1$ kgf produces stresses σ_1 and τ_1 in the shell elements; σ_1 acts along the $\underline{z}$ axis.

The work expended by the internal forces of this state on displacements caused by the forces of state "P" for an element of length dz and width ds equals roughly

$$-\sigma_1 \delta ds \frac{\sigma_p dz}{E} - \tau_1 \delta ds \frac{\tau_p dz}{G},$$

while the total internal-force work is

$$-\oint\int_0^l \frac{\sigma_1 \sigma_p \delta dz\, ds}{E} - \oint\int_0^l \frac{\tau_1 \tau_p \delta dz\, ds}{G}.$$

The exact value of the internal-force work also contains additional terms (see [30], page 257).

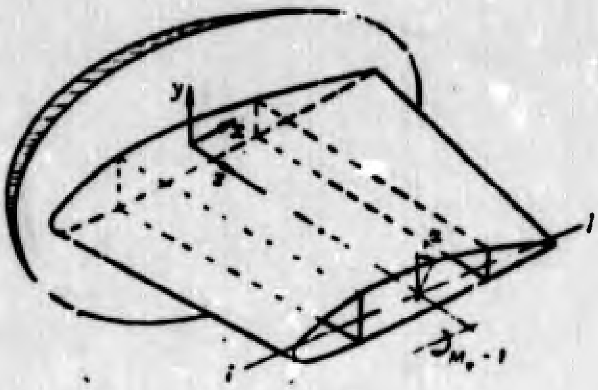


Fig. 8.14

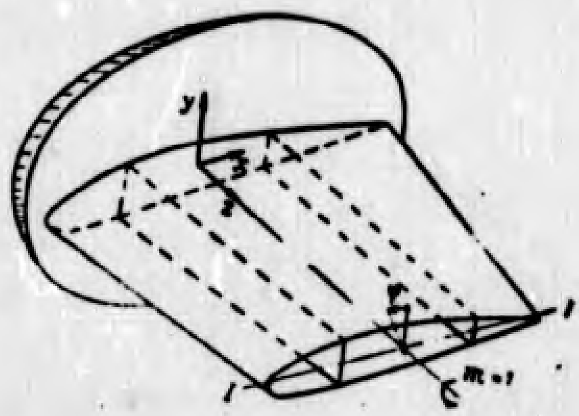


Fig. 8.15

The work done by the external forces is $\Delta_{1P} = f \cdot 1$, where $\underline{f}$ is the unknown deflection. On the basis of the principle of possible displacements

$$f \cdot 1 - \oint\int_0^l \frac{\sigma_1\sigma_p\delta dz\, ds}{E} - \oint\int_0^l \frac{\tau_1\tau_p\delta dz\, ds}{G} = 0.$$

From this we obtain

$$f = \oint\int_0^l \frac{\sigma_1\sigma_p\delta dz\, ds}{E} + \oint\int_0^l \frac{\tau_1\tau_p\delta dz\, ds}{G},$$

or

$$f = \oint\int_0^l \frac{\sigma_1\sigma_p\delta dz\, ds}{E} + \oint\int_0^l \frac{q_1 q_p dz\, ds}{G\delta}. \qquad (8.21)$$

Since for sections far from the constraint, the stress distribution is assumed to be governed by the law of plane distribution of relative elongations, then σ_p and σ_1 will be statically determinate, while τ_p or q_p will be statically indeterminate for a multiply closed contour; when we determine the displacements in a single state we employ the fundamental statically determinate force (singly closed contour) so that the stresses τ_1 and the PKS q_1 are statically determinate.

In order to obtain the angle of rotation α for section I-I about the x axis, we apply to this section the moment $M_x = 1$ (Fig. 8.14).

Then

$$\alpha = \oint \int_0^l \frac{\sigma_1 \sigma_p \delta \, dz \, ds}{E} + \oint \int_0^l \frac{\tau_1 \tau_p \delta \, dz \, ds}{G} \Bigg\} \tag{8.22}$$

or

$$\alpha = \oint \int_0^l \frac{\sigma_1 \sigma_p \delta \, dz \, ds}{E} + \oint \int_0^l \frac{q_1 q_p \delta \, dz \, ds}{G\delta},$$

where σ_1 and q_1 are produced by $M_x = 1$.

Finally, in order to obtain the torsion angle φ for section I-I about the z axis, we apply to this section the moment $\dot{M} = 1$ (Fig. 8.15).

When $\dot{M} = 1$ acts, the normal stresses $\sigma_1 = 0$, so that the unknown torsion angle will be

$$\varphi = \oint \int_0^l \frac{\tau_1 \tau_p \delta \, dz \, ds}{G},$$

or

$$\varphi = \oint \int_0^l \frac{q_1 q_p \, dz \, ds}{G\delta}. \tag{8.23}$$

5. BENDING AND TORSION IN A SYSTEM WITH MULTIPLY CLOSED SECTIONS

The determination of normal stresses far from the constraint does not depend on the number of closed content or the number of longitudinal bars, i.e., in other words it does not depend on the degree of static indeterminacy. Thus, here also the normal stresses are determined, as before, from Formulas (8.2) and (8.3):

$$\sigma_z=\left(\frac{M_x}{J_x}y_i-\frac{M_y}{J_y}x_i\right)\cos^4\varphi;$$

$$\sigma_\varphi=\frac{\sigma_z}{\cos^2\varphi}.$$

We recall that static indeterminacy was eliminated from the calculations when we determined the normal stresses owing to the earlier assumption as to the distribution of these stresses in accordance with the law of plane distribution of relative deformations.

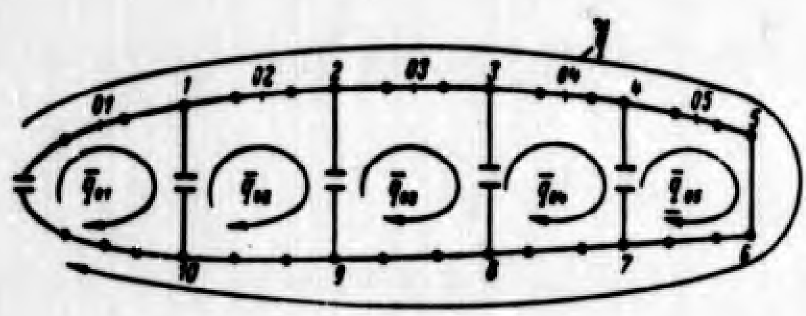


Fig. 8.16

Things are different when we come to determine shear stresses. Here we have advance knowledge of no law that will describe the variation in PKS over the entire multiply closed contour with sufficient accuracy. As a consequence, in order to determine the PKS it is necessary to solve a statically indeterminate problem.

Actually, in the solution for a singly closed contour, all equations of statics were used and, as a consequence, if the shell has n closed contours, to determine the PKS it would be necessary to use n − 1 deformation equations. If we also wish to determine the relative angle of rotation θ, it would be necessary to increase the number of equations to n.

The deformation equations can easily be set up when we consider the fact that the ribs in shells of, for example, the wing type are so rigid in their own plane that the entire multiply closed contour essentially does not change configuration during loading of the shell. As a consequence, if the entire multiply closed contour (Fig. 8.16) is turned through an angle θ about the z axis, then each component of the contour will be turned to the same angle θ, i.e.,

$$\theta_1=\theta_2=\theta_3\ldots=\theta_n=\theta. \tag{8.24}$$

In accordance with Formula (8.23), the angle through which a bay of length $\Delta\ell$ twists will equal

$$\varphi=\oint\int_0^{\Delta l}\frac{q_1 q_p dz\, ds}{G\delta}.$$

Here over the length of the bay $\Delta\ell$, we replace the flow variables q_1 and q_p by their mean values $\bar{q}_1$ and $\bar{q}_p$. Then

$$\varphi=\Delta l\oint\frac{\bar{q}_1\bar{q}_p ds}{G\delta}.$$

As a consequence, the relative angle of rotation θ is

$$\theta=\frac{\varphi}{\Delta l}=\oint\frac{\bar{q}_1\bar{q}_p ds}{G\delta}.$$

Since $\bar{q}_1$ refers to a statically determinate system (see Part 5, Chapter 3) for $M = 1$, then regardless of the number of closed shell-section contours, for the ith contour, it will always be the case that

$$\bar{q}_i=\frac{1}{\bar{\Omega}_i},$$

where $\bar{\Omega}_1$ is twice the area bounded by the contour of the average section of the ith singly closed cavity of the bay. As a consequence,

$$\theta_i=\frac{1}{\bar{\Omega}_i}\oint\frac{\bar{q}_{pi}ds}{G\delta}.$$

Here we integrate over the average section; $\bar{q}_{pi}$ is the total PKS acting along the ith contour.

The total PKS $\bar{q}_p$ of the multiply closed shell may be represented as consisting of the PKS $\tilde{\bar{q}}$ of the open contour and the additional PKS $\bar{q}_{01}, \bar{q}_{02}, \bar{q}_{03}\ldots\bar{q}_{0n}$, that compensate the PKS in the cuts, set equal to zero in determination of $\tilde{\bar{q}}$.

Thus if in the determination of $\tilde{\bar{q}}$ we assume that the contour is opened by making the cuts shown in Fig. 8.15, the total PKS will equal, if we drop the subscript "P":

for the first contour:

at the tip

$$\bar{q}_1 = \dot{\bar{q}} + \bar{q}_{01},$$

at wall 1-10

$$\bar{q}_1 = \bar{q}_{01} - \bar{q}_{02};$$

For the second contour:

at wall 1-10

$$\bar{q}_2 = \bar{q}_{01} - \bar{q}_{02},$$

along skin 1-2 and 9-10

$$\bar{q}_2 = \dot{\bar{q}} + \bar{q}_{02},$$

at wall 2-9

$$\bar{q}_3 = \bar{q}_{02} - \bar{q}_{03} \quad \text{etc.}$$

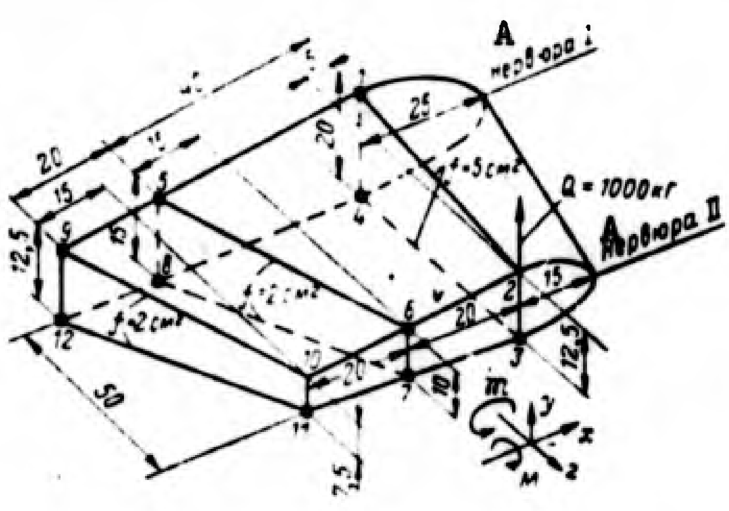


Fig. 8.17. A) Rib.

The relative angles of rotation of each contour will equal

$$\theta_1 = \frac{1}{\overline{\Omega}_1} \oint \frac{\overline{q}_1 ds}{G\delta}; \quad (1)$$

$$\theta_2 = \frac{1}{\overline{\Omega}_2} \oint \frac{\overline{q}_2 ds}{G\delta}; \quad (2)$$

$$\theta_3 = \frac{1}{\overline{\Omega}_3} \oint \frac{\overline{q}_3 ds}{G\delta}; \quad (3)$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\theta_n = \frac{1}{\overline{\Omega}_n} \oint \frac{\overline{q}_n ds}{G\delta}, \quad (4)$$

where we integrate only over the corresponding contour. The n equations contain n unknown PKS $\overline{q}_{01}, \overline{q}_{02}, \ldots, \overline{q}_{0n}$ and the unknown angle θ.

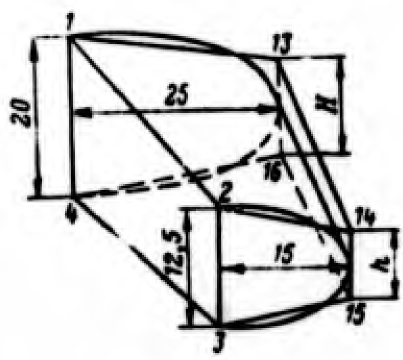


Fig. 8.18

We obtain the one equation still needed from the condition requiring that all forces acting on a rib equal zero, just as in the case of calculations for a singly closed shell.

It should be noted that the unknowns $\overline{q}_{01}$, $\overline{q}_{02}$, $\overline{q}_{03}$, etc., can be determined with a high degree of accuracy by slide rule if the cuts made in determining $\tilde{\overline{q}}$ are taken not at the points indicated in Fig. 8.16 but at points 01, 02, 03, 04, 05, as shown in the same figure.

Example.

We are to determine the stressed state and relative angle of rotation θ of the triply closed shell of Fig. 8.17 acted on by a force $Q = 1000$ kgf, a bending moment $M = 1000$ kgf·m, and a torsional moment $\dot{M} = 1000$ kgf·m. The wing-tip contour is described by a semiellipse. The ribs are assumed to be perfectly rigid. The skin and spar walls have a thickness $\delta = 1$ mm $= 0.1$ cm everywhere.

We first reduce the tip to a trapezoidal bay for which the cross-sectional areas along ribs I and II equal, respectively, the areas of the initial tip bounded by the semiellipses (Fig. 8.18):

$$\frac{(20+H)}{2}25=\frac{1}{2}\,\frac{\pi}{4}\,2\cdot2\cdot5\cdot20,\qquad H=11{,}4\ \text{cm},$$

$$\frac{(12{,}5+h)}{2}15=\frac{1}{2}\,\frac{\pi}{4}\,2\cdot15\cdot12{,}5,\quad h=7{,}15\ \text{cm}.$$

We now proceed to determine the axial forces in the strips. The bending moment acting on the left side is M = 100,000 − 50•1000 = = 50,000 kgf•cm, while on the right side it is M = 100,000 kgf•cm. We make the x axis coincide with the axis of symmetry of the cross section. The calculations are shown in Table 8.2.

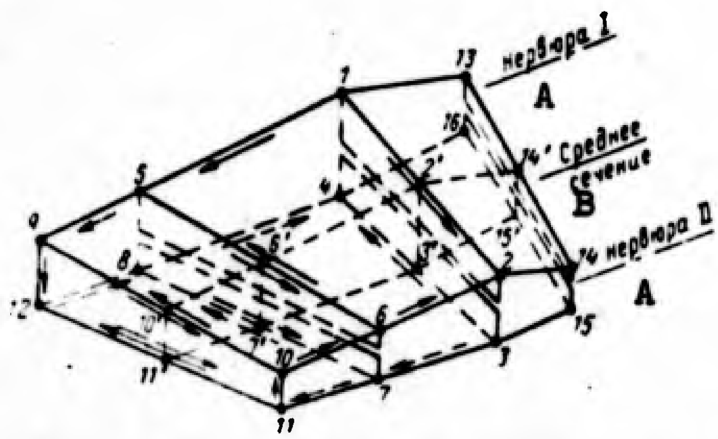


Fig. 8.19. A) Rib; B) mean section.

On the basis of the axial forces obtained, we can determine the PKS $\tilde{\tilde{q}}$ for the open profile. The positive direction taken for $\tilde{\tilde{q}}$ when the equations were set up is indicated in Fig. 8.19:

$$\tilde{\tilde{q}}_{1-2-6-5}50+N_{z_2}-N_{z_1}=0,\quad \tilde{\tilde{q}}_{1-2-6-5}=\frac{(1850-5860)}{50}=-80{,}2\,\text{kgf/cm}$$

$$\tilde{\tilde{q}}_{5-6-10-9}=\frac{(1850+500-5860-1680)}{50}=-104\ \text{kgf/cm}$$

$$\tilde{\tilde{q}}_{9-10-11-12}=\frac{(1850+500+415-5860-1680-1260)}{50}=-120{,}5\,\text{kgf/cm}$$

$$\tilde{\tilde{q}}_{3-4-7-8}=\frac{(-5860+1850)}{50}=-80{,}2\ \text{kgf/cm}$$

$$\tilde{\tilde{q}}_{7-8-12-11}=\frac{(-5860-1680+1850+500)}{50}=-104\ \text{kgf/cm}$$

For convenience in the remaining calculations, we decompose the axial forces into their components (see Table 8.2).

We check the equilibrium of the right-hand section in the y-axis

TABLE 8.2

Стер. 1	Δx	Δy	Δz	$(\Delta x)^2$	$(\Delta y)^2$	$(\Delta z)^2$	l^2	l	$\Delta x/l$	$\Delta y/l$	$\Delta z/l$	$(\Delta z/l)^3$	f	$f' = f\left(\frac{\Delta z}{l}\right)^3$	y_i	y_i^2	Sx_i
1–2	–5	–5	50	25	25	2500	2550	50,5	–0,099	–0,099	0,99	0,97	5	4,85	10	100	48,5
4–3	–5	2,5	50	25	6,25	2500	2531	50,3	–0,0992	0,0497	0,993	0,98	5	4,90	–10	100	–49,0
5–6	15	–3,75	50	225	14,1	2500	2739	52,3	0,288	–0,0717	0,955	0,871	2	1,74	7,5	56,25	13,05
8–7	15	1,25	50	225	1,56	2500	2727	52,25	0,288	0,0249	0,955	0,871	2	1,74	–7,5	56,25	–13,05
9–10	15	–3,75	50	225	14,1	2500	2739	52,3	0,288	–0,0717	0,955	0,871	2	1,74	6,25	39,1	10,88
12–11	15	1,25	50	225	1,56	2500	2727	52,25	0,288	0,0249	0,955	0,871	2	1,74	–6,25	39,1	–10,88
2–1														4,85	6,25	39,1	30,3
3–4														4,90	–6,25	39,1	–30,6
6–5														1,74	5	25	8,7
7–8														1,74	–5	25	–8,7
10–9														1,74	3,75	14,08	6,53
11–12														1,74	–3,75	14,08	–6,53
1–5	–40	–2,5	0	1600	6,25	0	1606	40,08	–1	–0,0625	0						
1–13	25	–4,3	0	625	18,49	0	643,5	25,37	0,985	–0,1695	0						
2–6	–20	–1,25	0	400	1,56	0	401,6	20	–1	–0,062	0						
2–14	15	–2,075	0	225	7,156	0	232,16	15,27	0,982	–0,175	0						

Стержни 1	J_{x_i}	S_{x_i}/J_x	N_{z_i}	ΔM	N_{φ_i}	N_{x_i}	N_{y_i}	Δy_3	Δx_3	$N_{x_i}\cdot\Delta y_3$	$N_{y_i}\Delta x_3$
1–2	485	0,0371	1850	18 500	1870	–185,0	–185,0				
4–3	490	0,0371	–1850	18 500	–1863	+184,8	92,59				
5–6	98,0	0,0100	500	3 750	523,6	+150,8	37,54				
8–7	98,0	–0,0100	–500	3 750	–523,6	–150,8	13,04				
9–10	68,0	0,0083	415	2 594	434,6	+125,7	31,16				
12–11	68,0 $J_x=1307$	–0,0083	–415	2 594 $\approx 50\,000$	–434,6	–125,7 $\Sigma \approx 0$	10,82 $\Sigma = -370,15$				
2–1	189,5	+0,586	5860	36 625	5919	586	586	12,5	0	7325	0
3–4	191,5	–0,586	–5860	36 625	–5901	–586	292	0	0	0	0
6–5	43,5	+0,168	1680	8 400	1759	–507	126	11,25	–20	–5703	2520
7–8	43,5	–0,168	–1680	8 400	–1759	507	43,8	1,25	–20	633,75	876
10–9	24,5	+0,126	1260	4 725	1319	–380	94,5	10	–40	–3800	3780
11–12	24,5 $J_x=517,0$	–0,126	–1260	4 725 $\approx 100\,000$	1319	380 0	32,8 1175,1	2,5	–40	950 –594,25	1312 8488 кГ/см
1–5											
1–13											
2–6											
2–14											

2

direction.

Keeping in mind that the resultant shear stress on one of the sides will equal the mean PKS multiplied by the length of the opposite side, we obtain

$$(\tilde{q}_{1-2-6-5}40+\tilde{q}_{5-6-10-9}20+\tilde{q}_{3-4-7-8}40+\tilde{q}_{7-8-12-11}20)\cos(6-2,\,y)+$$
$$+\tilde{q}_{9-10-11-12}12{,}5+\sum N_{y_i}+Q=0,\quad i=2,\,3,\,6,\,7,\,10,\,11.$$
$$(80{,}2\cdot 40-104\cdot 20-80{,}2\cdot 40-104\cdot 20)\,0{,}0625-120{,}5\cdot 12{,}5+$$
$$+1175+1000=-661-1506{,}25+2175{,}1=-2167{,}25+2175{,}1\approx 0.$$

We now proceed to determine the mean PKS $\bar{q}$ (Fig. 8.20):

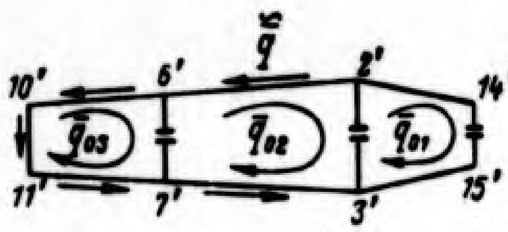


Fig. 8.20

$$\bar{q}_{\text{носка}}=\bar{q}_{01};$$
$$\bar{q}_{2-3-4-1}=\bar{q}_{01}-\bar{q}_{02};$$
$$\bar{q}_{2-6-5-1}=-80{,}2+\bar{q}_{02};$$
$$\bar{q}_{6-7-8-5}=\bar{q}_{02}-\bar{q}_{03};$$
$$\bar{q}_{6-10-9-5}=-104+\bar{q}_{03};$$
$$\bar{q}_{10-11-12-9}=-120{,}5+\bar{q}_{03};$$
$$\bar{q}_{11-7-8-12}=-104+\bar{q}_{03};$$
$$\bar{q}_{7-3-4-8}=-80{,}2+\bar{q}_{02}.$$

In order to determine $\bar{q}_{01}$, q_{02}, q_{03} and the relative angle of rotation, we set up the single equation of statics $\Sigma M = 0$ and three deformation equations.

The sum of the moments for all forces acting on the right rib is set up, for example, about joint 3 (see Fig. 8.19):

$$\sum \bar{q}^{\text{пр}}\Delta s\cdot\varrho+\sum(N_{x_i}\cdot\Delta y_3+N_{y_i}\Delta x_3)-\mathfrak{M}=0.$$

We determine the value of the first term in this equation with no difficulty, and the value of the second term is indicated in Table 8.2.

We thus have

$$\bar{q}_{01}(11{,}4\cdot 15+25\cdot 12{,}5)+40\cdot 12{,}5(-80{,}2+\bar{q}_{02})+15\cdot 20(\bar{q}_{02}-\bar{q}_{03})+$$
$$+20\cdot 12{,}5(-104+\bar{q}_{03})+12{,}5\cdot 40(-120{,}5+\bar{q}_{03})-594{,}25+$$
$$+8488-100\,000=0$$

or

$$4{,}835\bar{q}_{01}+8\bar{q}_{02}+4{,}5\bar{q}_{03}=2185. \tag{1}$$

The relative angle of rotation in the mean section is, for the first contour:

$$\theta_1=\frac{1}{\Omega_1}\sum\frac{\bar{q}_1\Delta s}{G\delta},$$

$$\bar{\Omega}_1=2\,\frac{25+15}{2}\,\frac{\frac{20+12{,}5}{2}+\frac{11{,}4+7{,}15}{2}}{2}=$$

$$=2\,\frac{40}{2}\,\frac{(16{,}25+9{,}28)}{2}=510{,}6\ \text{см}^2,$$

$$G\delta\theta_1=\frac{1}{510{,}6}\,[20{,}2\bar{q}_{01}+9{,}28\bar{q}_{01}+20{,}2\bar{q}_{01}+16{,}25\,(\bar{q}_{01}-\bar{q}_{02})]=$$

$$=0{,}129\bar{q}_{01}-0{,}0318\bar{q}_{02}; \tag{2}$$

for the second contour it is:

$$\bar{\Omega}_2=2\,\frac{40+20}{2}\,\frac{\frac{15+10}{2}+\frac{20+12{,}5}{2}}{2}=30(12{,}5+16{,}25)=862{,}5\ \text{см}^2,$$

$$G\delta\theta_2=\frac{1}{862{,}5}[16{,}25\,(\bar{q}_{02}-\bar{q}_{01})+30\,(-80{,}2+\bar{q}_{02})+12{,}5\,(\bar{q}_{02}-\bar{q}_{03})+$$

$$+30\,(-80{,}2+\bar{q}_{02})]=-0{,}0188\bar{q}_{01}+0{,}103\bar{q}_{02}-$$

$$-0{,}0145\bar{q}_{03}-5{,}58; \tag{3}$$

for the third contour it is:

$$\bar{\Omega}_3=2\,\frac{20+20}{2}\,\frac{\frac{12{,}5+7{,}5}{2}+\frac{15+10}{2}}{2}=450\ \text{см}^2,$$

$$G\delta\theta_3=\frac{1}{450}\,[2\cdot 20(-104+\bar{q}_{03})+10(-120{,}5+\bar{q}_{03})+12{,}5(\bar{q}_{03}-\bar{q}_{02})]=$$

$$=-0{,}0278\bar{q}_{02}+0{,}139\bar{q}_{03}-11{,}9. \tag{4}$$

Setting $\theta_1 = \theta_2 = \theta_3$ and using Eq. (1), we obtain

$$\bar{q}_{01}=77{,}18\ \text{kgf/cm};$$

$$\bar{q}_{02}=141\ \text{kgf/cm}$$

$$\bar{q}_{03}=153\ \text{kgf/cm}$$

Substituting these values into (2), (3) and (4), we obtain

$$G\delta\theta_1=0{,}129\cdot 77{,}18-0{,}0318\cdot 141=9{,}956-4{,}484=5{,}47\ \text{kgf/cm}^2$$

$$G\delta\theta_2 = -0{,}0188\cdot 77{,}18 + 0{,}103\cdot 141 - 0{,}0145\cdot 153{,}5 - 5{,}58 =$$
$$= -1{,}45 + 14{,}52 - 2{,}22 - 5{,}58 = 5{,}27 \text{ kgf/cm}^2$$
$$G\delta\theta_3 = -0{,}0278\cdot 141 + 0{,}139\cdot 153{,}5 - 11{,}9 =$$
$$= -3{,}92 + 21{,}34 - 11{,}9 = 5{,}52 \text{ kgf/cm}^2$$
$$G\delta\theta_{\text{сред}} = 5{,}42.$$

For G = 270,000 kgf/cm^2, the relative torsion angle is

$$\theta = \frac{5{,}42}{0{,}1\cdot 270\,000} = 0{,}00207 \text{ rad}/\text{см} = 0{,}0119^\circ/\text{см}.$$

The torsion angle for the right section (rib II) about the left is $\varphi = \theta l = 0.595^\circ$. We take angle φ to be positive if the right-hand section twists counterclockwise with respect to the left.

Finally, the mean PKS equal:

$$\bar{q}_{\text{носка}} = 77{,}2 \text{ kgf/cm}^2$$
$$\bar{q}_{2-3-4-1} = -63{,}8 \text{ kgf/cm}$$
$$\bar{q}_{2-6-5-1} = 60{,}8 \text{ kgf/cm}$$
$$\bar{q}_{6-7-8-5} = -12{,}5 \text{ kgf/cm}$$
$$\bar{q}_{6-10-9-5} = 49{,}5 \text{ kgf/cm}$$
$$\bar{q}_{10-11-12-9} = 33 \text{ kgf/cm}$$
$$\bar{q}_{11-7-8-12} = 49{,}5 \text{ kgf/cm}$$
$$\bar{q}_{7-3-4-8} = 60{,}8 \text{ kgf/cm}$$

The magnitudes and directions of these PKS are indicated in Fig. 8.21.

Knowing the values of the mean PKS, we have no difficulty in using Formulas (8.9) and (8.10) to determine the PKS acting in the sections near ribs I and II as well.

When we make allowance for the taper, the components of the axial forces usually unload the spar walls, but in the given case they load them, since here the direction of the bending moment is taken opposite to the direction of the bending moment due to the intersecting force.

I.A. Sverdlov has shown that the calculation method constructed on the hypothesis of plane distribution of relative elongations is a fairly good approximation to the actual stressed state even in the section directly adjacent to the wing attachment.

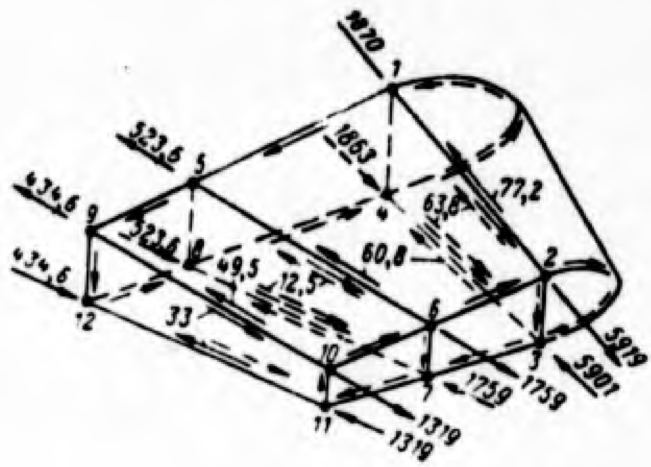


Fig. 8.21

A more exact determination of the stressed state involves determination of "secondary" stresses. This problem may be solved "in stresses," as, for example, in the studies of V.F. Kiselev [11] or "in displacements" [33], [34].

In the next chapter, we shall consider the solution of this problem in displacements, using the example of a wing with skin working only in shear.

Manuscript Page No.	[Footnotes]
	Free bending and free torsion of cylindrical shells have been considered in detail in [30].

Chapter 9

APPLICATION OF THE METHOD OF DISPLACEMENTS TO CALCULATIONS FOR CONSTRAINED BENDING AND TORSION IN WING-TYPE SHELLS

1. METHOD OF DISPLACEMENTS

In calculations for systems with a high degree of static indeterminacy, it is frequently preferable to take as the unknowns not the force factors as we do in the method of forces, but the displacements.

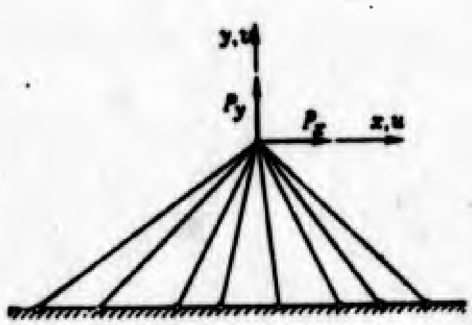


Fig. 9.1

Thus, for example, if we are to determine the stresses in the bars of a plane truss (Fig. 9.1) consisting of a single joint connected by a large number of bars to a rigid support and loaded by forces P_x and P_y, the solution will be far simpler if we take as the unknowns not the stresses in the "redundant" bars, but the displacements u and v. Actually, regardless of the number of redundant constraints, the longitudinal deformation of each bar and, then the stresses in the constraints, may be expressed in terms of a function of two unknown component displacements u and v.

As soon as we have represented all stresses in the bars as functions of u and v, we can set up two equations for equilibrium of the joint: $\Sigma X = 0$ and $\Sigma Y = 0$. From the solution of these equations we determine the values of the unknowns u and v, and then the stresses in all bars.

We can give many more examples of systems for which the application of the method of displacements offers an advantage over the method of forces, but here we shall only consider the application of this method to wing design.

Wing design by the method of displacements involves solution of flexibility matrices and stiffness matrices.

2. FLEXIBILITY MATRIX AND STIFFNESS MATRIX

Let us determine the vertical displacements at joints of the structure when certain generalized forces $P_1 \ldots P_n$ act (Fig. 9.2). We shall assume that the material of the structure is Hookian and, as a consequence, the displacements will be proportional to the forces.

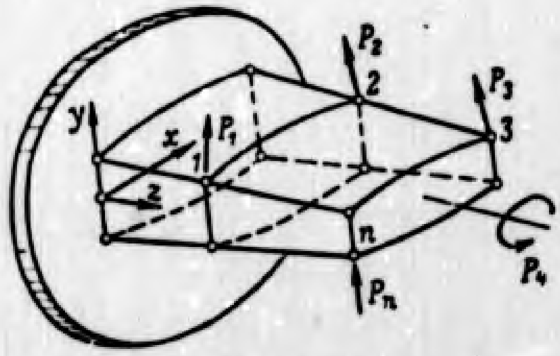


Fig. 9.2

We first consider a case in which the system is acted on by just one force P_1 applied at joint 1. Under the action of this force, another joint, for example, joint 2, will move by a certain amount. Since the total displacement of joint 2 is proportional to P_1, then each of the components Δu_2, Δv_2, Δw_2 of this displacement will also be proportional to the force P_1. Thus, for the vertical component

$$\Delta v_2 = a_{21} P_1,$$

where a_{21} is the influence coefficient of force P_1 on the vertical displacement at joint 2.

When the generalized forces P_1, P_2, ...,P_n act simultaneously, on the basis of the law of independent action of forces, the vertical displacement of joint 2 will equal

$$v_2 = a_{21}P_1 + a_{22}P_2 + \ldots + a_{2n}P_n.$$

Using similar arguments for the other system joints, we obtain the vertical displacements for all n joints that can be displaced in the form of the following linear equations:

$$\left.\begin{array}{l} v_1 = a_{11}P_1 + a_{12}P_2 + \ldots + n_{1n}P_n, \\ v_2 = a_{21}P_1 + a_{22}P_2 + \ldots + a_{2n}P_n, \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ v_n = a_{n1}P_1 + a_{n2}P_2 + \ldots + a_{nn}P_n \end{array}\right\} \qquad (9.1)$$

or

$$v_i = \sum_{k=1}^{n} a_{ik}P_k \quad (i = 1, 2, \ldots n). \qquad (9.1')$$

The constants a_{ik} for the considered structure are called the flexibility influence coefficients.

Solving (9.1) for P_1, P_2, ...P_n as linear functions of v_i, we obtain the following system of linear equations:

$$\left.\begin{array}{l} P_1 = b_{11}v_1 + b_{12}v_2 + \ldots + b_{1n}v_n, \\ P_2 = b_{21}v_1 + b_{22}v_2 + \ldots + b_{2n}v_n, \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ P_n = b_{n1}v_1 + b_{n2}v_2 + \ldots + b_{nn}v_n \end{array}\right\} \qquad (9.2)$$

or

$$P_i = \sum_{k=1}^{n} b_{ik}v_k \quad (i = 1, 2, \ldots n). \qquad (9.2')$$

The constants b_{ik} are called the rigidity influence coefficients.

In connection with the theorem of displacement reciprocity, the influence coefficients will possess symmetry properties:

$$a_{ik} = a_{ki}, \; b_{ik} = b_{ki} \qquad (i, k = 1, 2, \ldots n). \qquad (9.3)$$

In static and dynamic investigations of modern flying-craft structures, the number of terms in Eqs. (9.1) and (9.2) is very large. The solution of such equations using, for example, the Gauss algorithm, involves enormous computational effort. Thus, automatic computers are

The writing of computer programs will be simplified considerably if the equations are written in matrix notation.

Equations (9.1) and (9.2) are written as follows in matrix form:

$$\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}a_{n2}\ldots a_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ P_n \end{bmatrix}; \tag{9.4}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ P_n \end{bmatrix} = \begin{bmatrix} b_{11}b_{12}\ldots b_{1n} \\ b_{21}b_{22}\ldots b_{2n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n1}b_{n2}\ldots b_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \tag{9.5}$$

or in condensed form:

$$[v]=[a]\,[P]; \tag{9.4'}$$

$$[P]=[b]\,[v]. \tag{9.5'}$$

The square matrices [a] and [b] are called the <u>flexibility matrix</u> and the stiffness matrix, respectively.

The solution of the linear equations (9.4) for P_1, P_2, ...,P_n involves <u>inversion of the matrix</u> [a], and this is written in the following manner:

$$[P]=[a]^{-1}\,[v]. \tag{9.6}$$

where $[a]^{-1} = [b]$ is the matrix that is the inverse of matrix [a].

As an illustration, let us determine the flexibility influence coefficients a_{ik} for a homogeneous thin-web beam of the type used for a wing spar (Fig. 9.3). Let the beam have an axis of symmetry <u>x</u>, a flange cross-sectional area <u>f</u>, and a web thickness δ.

For a thin-web structure, the flexibility influence coefficient for any point <u>i</u> is found from Formula (3.6)

$$a_{ik}=\sum\frac{q_iq_kF}{G\delta}+\sum\int\frac{N_iN_k\,dl}{Ef},$$

where the subscripts <u>i</u> and <u>k</u> indicate that the displacement is found from force $P_k = 1$ at the point of application of force $P_i = 1$ in the direction of this same force.

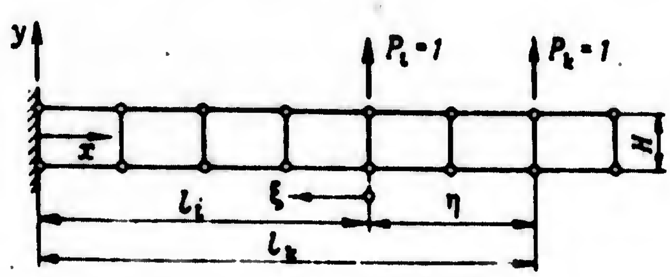


Fig. 9.3

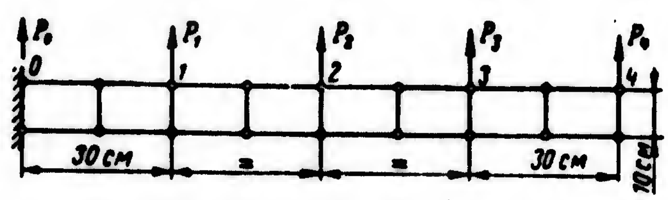


Fig. 9.4

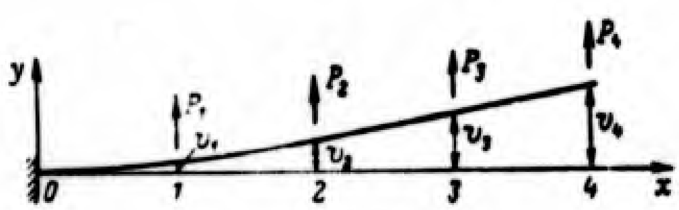


Fig. 9.5

Summation is taken over all elements of the panels for which the force factors due to $P_k = 1$ and $P_i = 1$ overlap.

In the case considered, the PKS will equal:

$$q_i = \frac{1}{H}; \quad q_k = \frac{1}{H},$$

while the stresses in the flanges are determined from the expressions:

$$N_i = \frac{\xi}{H}; \quad N_k = \frac{\xi + \eta}{H},$$

where $\eta = l_k - l_i$; ξ is an auxiliary coordinate (see Fig. 9.3). Thus, neglecting the effect of axial forces in the braces since they are so small, we obtain

$$a_{ik} = \frac{H l_i}{H^2 G \delta} + 2 \int_0^{l_i} \frac{\xi(\xi + \eta)\, d\xi}{H^2 E f}.$$

After integration at G = E/2.6, we obtain

$$a_{ik}=\frac{l_i}{E}\left[\frac{2{,}6}{H\delta}+\frac{l_i}{3H^2f}(3\eta+2l_i)\right]. \qquad (9.7)$$

Let us evaluate the flexibility matrix for the beam shown in Fig. 9.4. We let $f = 3\ \text{cm}^2$ and $\delta = 0.1$ cm. From Formula (9.7)

$$a_{11}=\frac{30}{E}\left[\frac{2{,}6}{10\cdot 0{,}1}+\frac{30}{3\cdot 10^2\cdot 3}(2\cdot 30)\right]=\frac{30}{E}(2{,}6+2)=\frac{138}{E};$$
$$a_{12}=\frac{30}{E}\left[2{,}6+\frac{1}{30}(3\cdot 30+2\cdot 30)\right]=\frac{30}{E}(2{,}6+5)=\frac{228}{E};$$

$$a_{13}=\frac{318}{E};\quad a_{14}=\frac{408}{E};\quad a_{22}=\frac{636}{E};\quad a_{23}=\frac{996}{E};$$
$$a_{24}=\frac{1356}{E};\quad a_{33}=\frac{1854}{E};\quad a_{34}=\frac{2664}{E};\quad a_{44}=\frac{4152}{E}.$$

Using the symmetry property (9.3), we obtain

$$[a]=\frac{1}{E}\begin{bmatrix}138 & 228 & 318 & 408\\ 228 & 636 & 996 & 1356\\ 318 & 996 & 1854 & 2664\\ 408 & 1356 & 2664 & 4152\end{bmatrix}.$$

The beam deflections (Fig. 9.5) will equal:

$$\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}=\frac{1}{E}\begin{bmatrix}138 & 228 & 318 & 408\\ 228 & 636 & 996 & 1356\\ 318 & 996 & 1854 & 2664\\ 408 & 1356 & 2664 & 4152\end{bmatrix}\begin{bmatrix}P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}.$$

If H, f, and δ are variables, in calculating their influence coefficients we may substitute the average values for the portions between ribs. Then for the panel located between ribs r and r + 1 we obtain the constant values H_r, f_r and δ_r (Fig. 9.6).

The displacement from this panel is

$$\Delta a_{ik}=\frac{q_{ir}q_{kr}F_r}{G\delta_r}+2\int_{l_{r+1}}^{l_r}\frac{N_{ir}N_{kr}d\xi}{Ef_r}.$$

Since

$$q_{ir}=\frac{1}{H_r},\quad q_{kr}=\frac{1}{H_r},\quad N_{ir}=\frac{\xi}{H_r},\quad N_{kr}=\frac{\xi+\eta}{H_r},\quad F_r=H_r(l_r-l_{r+1}).$$

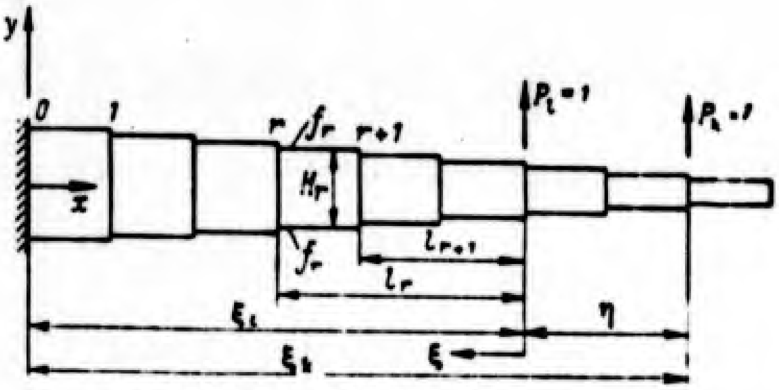


Fig. 9.6

then

$$\Delta a_{ik} = \frac{H_r(l_r + l_{r+1})}{H_r^2 G\delta_r} + 2\int_{l_{r+1}}^{l_r} \frac{\xi(\xi+\eta)\,d\xi}{H_r^2 E f_r}.$$

For $G = E/2.6$ we obtain

$$\Delta a_{ik} = \frac{l_r - l_{r+1}}{E}\left\{\frac{2.6}{H_r\delta_r} + \frac{l_r+l_{r+1}}{3H_r^2 f_r}\left[3\eta + 2\left(l_r + \frac{l_{r+1}^2}{l_r+l_{r+1}}\right)\right]\right\}.$$

The total value of the flexibility coefficient is

$$a_{ik} = \sum \frac{l_r - l_{r+1}}{E}\left\{\frac{2.6}{H_r\delta_r} + \frac{l_r+l_{r+1}}{3H_r^2 f_r}\left[3\eta + 2\left(l_r + \frac{l_{r+1}^2}{l_r+l_{r+1}}\right)\right]\right\}. \qquad (9.8)$$

The sign on the sum extends to all panels making up the length ξ_i.

3. WING CALCULATION SCHEME

For calculations making use of the method of displacements, the wing surface load is transferred to the joints. As a result, a concentrated force P_r will act at each joint, where the subscript r indicates the number of the joint. If the wing has a tail section not connected to the body, then in an approximate calculation this section may be removed and replaced by the effect of forces and moments M_1, M_2, ...,M_n on the wing (Fig. 9.7).

The spars and ribs bend under the action of the forces P_r and moments M_1, ...M_n, while the bays bounded by the spars and ribs will twist.

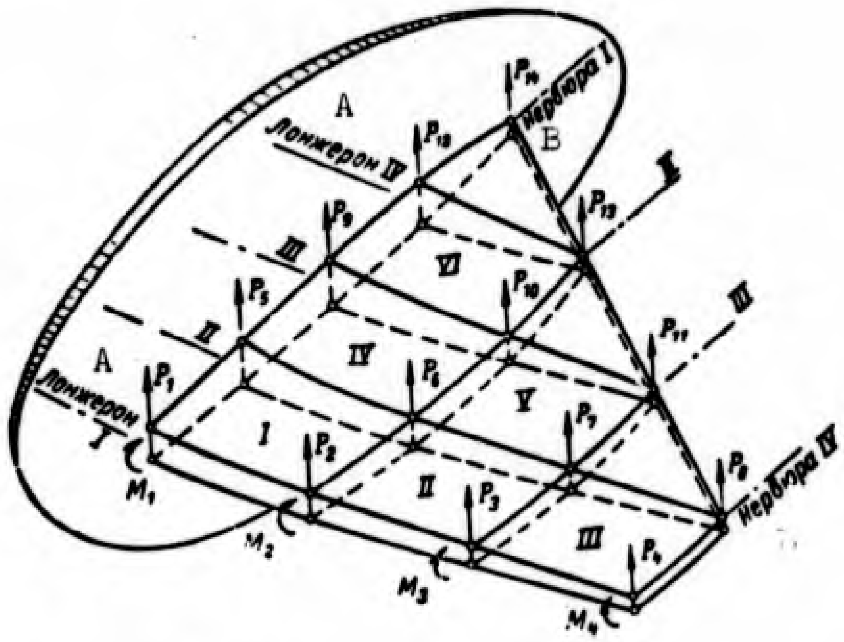


Fig. 9.7. A) Spar; B) rib.

The portion of the joint loads P that causes spar bending is represented by F, the portion of the joint loads P that causes rib bending is represented by $\overline{F}$ and, finally, the portion of the joint loads P responsible for bay twisting is represented by $\hat{F}$.

Thus, if we concentrate our attention, for example, on joint 6 which is touched at angles by four bays and which intersects spar III and rib II, then in the direction of the forces indicated in Fig. 9.8, the joint load for joint 6 will be

$$P_6 = F_6 + \overline{F}_6 + \hat{F}_6^{\mathrm{I}} + \hat{F}_6^{\mathrm{V}} - \hat{F}_6^{\mathrm{II}} - \hat{F}^{\mathrm{IV}},$$

or

$$P_6 = F_6 + \overline{F}_6 + \hat{F}_6,$$

where

$$\hat{F}_6 = \hat{F}_6^{\mathrm{I}} + \hat{F}_6^{\mathrm{V}} - \hat{F}_6^{\mathrm{II}} - \hat{F}_6^{\mathrm{IV}}.$$

For n elastically seated joints we obtain the n equation

$$\left.\begin{array}{l} P_1=F_1+\overline{F}_1+\hat{F}_1, \\ P_2=F_2+\overline{F}_2+\hat{F}_2, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ P_n=F_n+\overline{F}_n+\hat{F}_n. \end{array}\right| \quad (9.9)$$

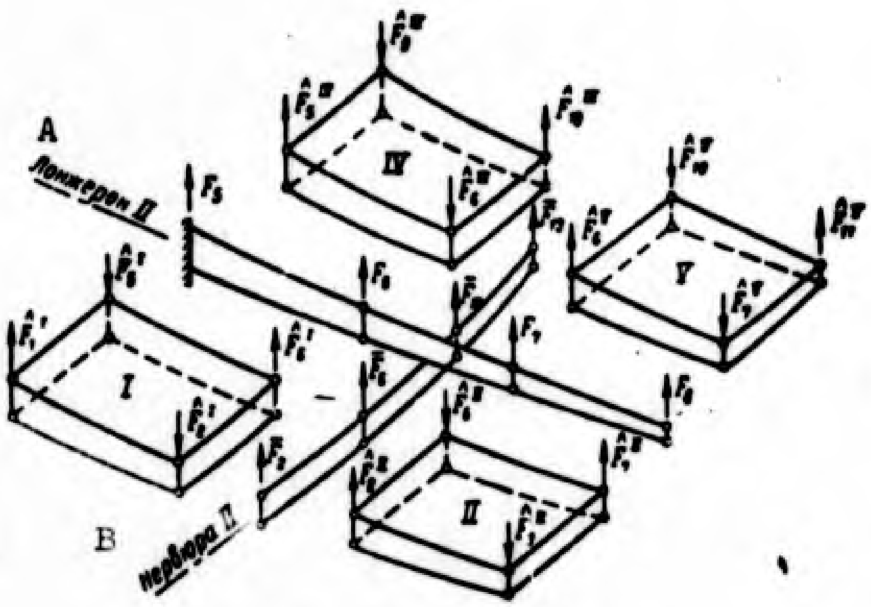


Fig. 9.8. A) Spar; B) rib.

If we represent the right sides of these equations as functions of the joint deflections, then from the solution to the system of equations (9.9) we obtain the value of deflection for each joint. After this it is not difficult to determine all the values for the forces F, $\overline{F}$, and $\hat{F}$; knowing these, we can also find the stressed state of the spars, ribs, and bays.

Thus the fundamental problem in calculations using the method of displacements consists in determining the forces F, $\overline{F}$, and $\hat{F}$ as a function of joint deflections.

In the following sections we give a method for solving such a problem using the wing calculation scheme shown in Fig. 9.7.

4. DETERMINATION OF SPAR JOINT FORCES

We shall consider one wing spar, for example, spar II (Fig. 9.9).

A

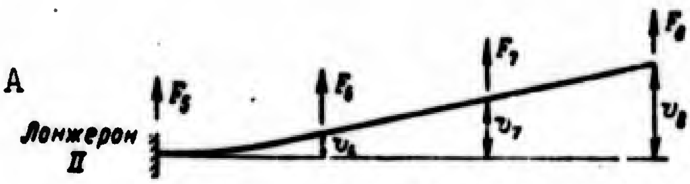


Fig. 9.9. A) Spar.

The deflections of joints 6, 7, 8 will equal:

$$\left.\begin{aligned} v_6&=a_{66}F_6+a_{67}F_7+a_{68}F_8,\\ v_7&=a_{76}F_5+a_{77}F_7=a_{78}F_8,\\ v_8&=a_{86}F_6+a_{87}F_7+a_{86}F_8 \end{aligned}\right\} \qquad (9.10)$$

or, in matrix notation,

$$\begin{bmatrix} v_6\\ v_7\\ v_8 \end{bmatrix}=\begin{bmatrix} a_{66}a_{67}a_{68}\\ a_{76}a_{77}a_{78}\\ a_{86}a_{87}a_{88} \end{bmatrix}\begin{bmatrix} F_6\\ F_7\\ F_8 \end{bmatrix}. \qquad (9.11)$$

Where the relative thickness of the wing is small, in approximate calculations the flexibility influence coefficients may be determined from Formula (9.8) with no allowance for deformation of the wall owing to shear, i.e.,

$$a_{ik}=\sum\frac{l_r^2-l_{r+1}^2}{3H_r^2Ef_r}\left[3\eta+2\left(l_r+\frac{l_{r+1}^2}{l_r+l_{r+1}}\right)\right]. \qquad (9.12)$$

Solving the system of linear equations (9.10) for the forces F by the Gauss method or by inversion of the flexibility matrix [a] for System (9.11), we obtain the forces F_6, F_7, F_8 expressed as functions of deflections:

$$\left.\begin{aligned} F_6&=b_{66}v_6+b_{67}v_7+b_{68}v_8,\\ F_7&=b_{76}v_6+b_{77}v_7+b_{78}v_8,\\ F_8&=b_{86}v_6+b_{87}v_7+b_{88}v_8. \end{aligned}\right\} \qquad (9.13)$$

Applying this same method to the remaining spars, we obtain all joint forces F as a function of joint deflections.

5. DETERMINATION OF RIB JOINT FORCES

We consider one of the wing ribs, for example, rib II (Fig. 9.10). The deflections of joints 6, 10, 13 may be represented as a sum consisting of the deflection v_2 of joint 2 (translational displacement of the rib as a rigid body), and of the displacement produced by rotation of the rib as a rigid body about joint 2 through angle ψ, and by its elastic deformations $\bar{v}$, i.e.,

$$\left.\begin{aligned} v_6&=v_2+\psi l_6+\bar{v}_6,\\ v_{10}&=v_2+\psi l_{10}+\bar{v}_{10},\\ v_{13}&=v_7+\psi l_{13}+\bar{v}_{13}.\end{aligned}\right\} \tag{9.14}$$

Here

$$\left.\begin{aligned} \bar{v}_6&=\bar{a}_{66}\bar{F}_6+\bar{a}_{610}\bar{F}_{10}+a_{613}\bar{F}_{13},\\ \bar{v}_{10}&=\bar{a}_{106}\bar{F}_6+\bar{a}_{1010}\bar{F}_{10}+\bar{a}_{1013}\bar{F}_{13},\\ \bar{v}_{13}&=\bar{a}_{136}\bar{F}_6+\bar{a}_{1310}\bar{F}_{10}+\bar{a}_{1313}\bar{F}_{13},\end{aligned}\right\} \tag{9.15}$$

where $\bar{a}_{ik}$ is the rib flexibility coefficient.

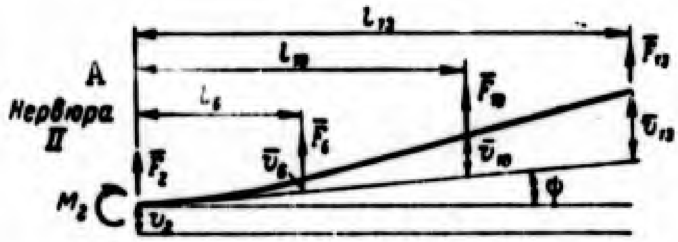


Fig. 9.10. A) Rib.

System (9.14) has five unknowns: v_2, v_6, v_{10}, v_{13}, and the angle ψ, while there are only three equations in this system.

In this case, however (in contrast to the case in which the spar is constrained at one end and all the unknowns in the equation of statics must be used to determine the support reactions), we can write two more equations of statics:

$$\left.\begin{aligned} &\bar{F}_2+\bar{F}_6+\bar{F}_{10}+\bar{F}_{13}=0,\\ &\bar{F}_6 l_6+\bar{F}_{10} l_{10}+\bar{F}_{13} l_{13}-M_2=0.\end{aligned}\right\} \tag{9.16}$$

Solving the equation systems (9.14) and (9.16) simultaneously for the joint forces of the rib $\bar{F}$, we obtain

$$\left.\begin{aligned}\overline{F}_2&=\overline{b}_{22}v_2+\overline{b}_{26}v_6+\overline{b}_{210}v_{10}+\overline{b}_{213}v_{13}+C_2,\\ \overline{F}_6&=\overline{b}_{62}v_2+\overline{b}_{66}v_6+\overline{b}_{610}v_{10}+\overline{b}_{613}v_{13}+C_6,\\ \overline{F}_{10}&=\overline{b}_{102}v_2+\overline{b}_{106}v_6+\overline{b}_{1010}v_{10}+\overline{b}_{1013}v_{13}+C_{10},\\ \overline{F}_{13}&=\overline{b}_{132}v_2+\overline{b}_{136}v_6+\overline{b}_{1310}v_{10}+\overline{b}_{1313}v_{13}+C_{13},\end{aligned}\right\} \qquad (9.17)$$

where $\overline{b}_{ik}$ are the rib stiffness influence coefficients, and the C_i are free terms.

In this manner, we can represent the joint loads as depending on the deflections and the remaining ribs.

6. DETERMINATION OF BAY JOINT FORCES

We shall first consider a single rectangular wing bay, for example, bay V (see Fig. 9.8). We let a^V and b^V represent the sides of bay V, and $\overline{H}^V$ its mean height (Fig. 9.11).

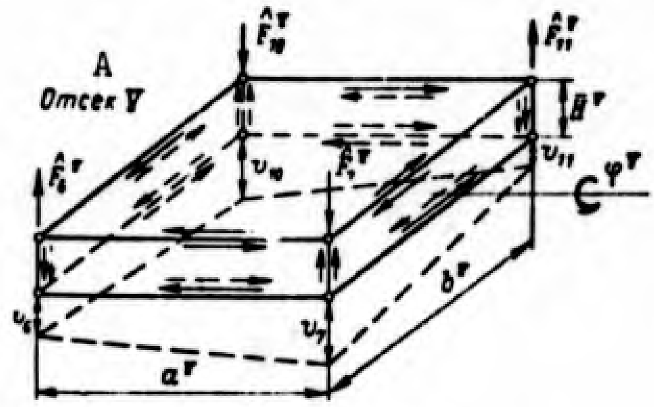


Fig. 9.11. A) Bay.

As a result of a general deformation of the wing, let the corner braces of the bay obtain displacements v_6, v_7, v_{11}, and v_{10}.

We let φ^V represent the angle through which bay V twists owing to elastic deformations. It is clear from Fig. 9.11 that

$$\varphi^V=\frac{v_7-v_{11}+v_{10}-v_6}{b^V}. \qquad (9.18)$$

This is the torsion angle produced by the forces

$$\hat{F}_6^V=-\hat{F}_7^V=\hat{F}_{11}^V=-\hat{F}_{10}^V,$$

which create torsional moments equal in magnitude to $\hat{F}_6^V b^V$. This means that the PKS in all bay walls will be

$$q^{\mathrm{V}}=\frac{\hat{F}_6^{\mathrm{V}}b^{\mathrm{V}}}{2b^{\mathrm{V}}\bar{H}^{\mathrm{V}}}=\frac{\hat{F}_6^{\mathrm{V}}}{2\bar{H}^{\mathrm{V}}}. \qquad (9.19)$$

Since for a wing of low aspect ratio, the height of the bays will be small in comparison with the dimensions of the sides a and b, in approximate calculations in order to determine the torsion angle φ we can consider only the upper and lower skin. Under this assumption, we obtain from Formula (8.23)

$$\varphi^{\mathrm{V}}=2\frac{1}{2b^{\mathrm{V}}\bar{H}^{\mathrm{V}}}\frac{\hat{F}_6^{\mathrm{V}}}{2\bar{H}^{\mathrm{V}}}\frac{b^{\mathrm{V}}a^{\mathrm{V}}}{G\delta^{\mathrm{V}}}=\frac{\hat{F}_6^{\mathrm{V}}a^{\mathrm{V}}}{2(\bar{H}^{\mathrm{V}})^2G\delta^{\mathrm{V}}}. \qquad (9.20)$$

As a consequence,

$$\frac{\hat{F}_6^{\mathrm{V}}a^{\mathrm{V}}}{2(\bar{H}^{\mathrm{V}})^2G\delta^{\mathrm{V}}}=\frac{v_7-v_{11}+v_{10}-v_6}{b^{\mathrm{V}}}$$

or

$$\hat{F}_6^{\mathrm{V}}=k^{\mathrm{V}}(v_7-v_{11}+v_{10}-v_6), \qquad (9.21)$$

where

$$k^{\mathrm{V}}=\frac{2(\bar{H}^{\mathrm{V}})^2G\delta^{\mathrm{V}}}{a^{\mathrm{V}}b^{\mathrm{V}}}.$$

In analogy with Formula (9.21), taking into account Fig. 9.8, we can, by traversing the joints in the counterclockwise direction, at once write

$$\hat{F}_6^{\mathrm{I}}=k^{\mathrm{I}}(v_2-v_6+v_5-v_1);$$
$$\hat{F}_6^{\mathrm{II}}=-k^{\mathrm{II}}(v_3-v_7+v_6-v_2);$$
$$\hat{F}_6^{\mathrm{IV}}=-k^{\mathrm{IV}}(v_6-v_{10}+v_9-v_5).$$

This means that the force due to the four bays that acts at joint 6 will equal

$$\hat{F}_6=k^{\mathrm{I}}(v_2-v_6+v_5-v_1)-k^{\mathrm{II}}(v_3-v_7+v_6-v_2)-$$
$$-k^{\mathrm{IV}}(v_6-v_{10}+v_9-v_5)+k^{\mathrm{V}}(v_7-v_{11}+v_{10}-v_6). \qquad (9.22)$$

In this fashion we can obtain the joint loads due to the bays for the remaining wing joints as well.

7. GENERAL REMARKS ON THE DETERMINATION OF WING JOINT DEFLECTIONS AND THE WING STRESSED STATE

If we express all joint forces of spars, ribs, and bays as functions of deflections and substitute them into Eqs. (9.9), we obtain a system of linear equations containing a large number of unknown joint deflections. Thus, for example, for the wing whose planform is shown in Fig. 9.12, the number of unknowns will equal 63.

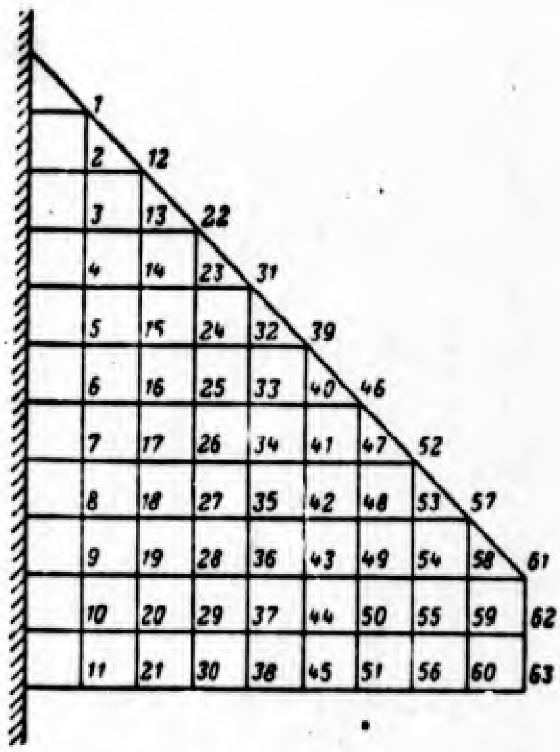


Fig. 9.12

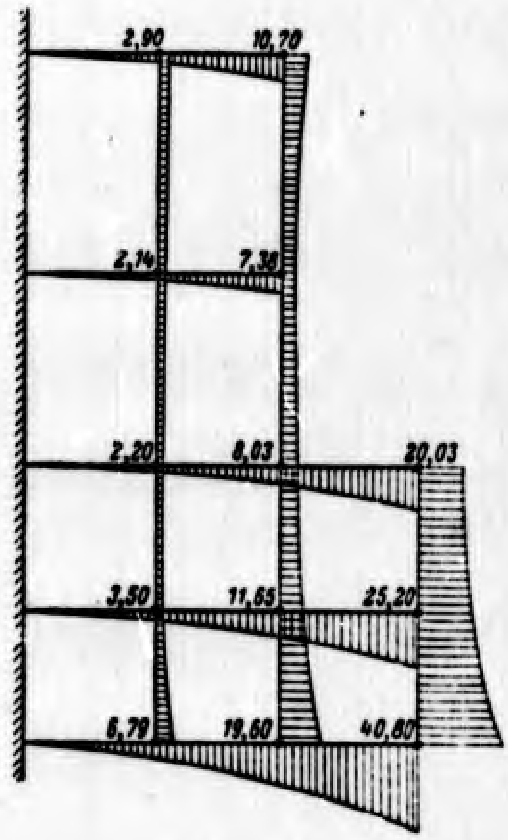


Fig. 9.13

After we have obtained the wing-joint deflections by solving the system of equations (9.9), there will be no difficulty in finding the stressed state for the spars, ribs, and bays, since the forces acting on the spars, ribs, and bays may be found from available formulas expressing the forces F, $\bar{F}_x$, and $\hat{F}$ as functions of joint deflections.

Figure 9.13 shows a wing-deflection pattern obtained by A.N. Bedenko from calculations using a system with 13 unknowns.

Chapter 10

APPLICATION OF THE KANTOROVICH-VLASOV METHOD TO THE DESIGN OF SOLID WINGS

INTRODUCTION

The configuration of solid wings in plan can be described quite exactly by a set of straight lines (Fig. 10.1a) with the form of a trapezoid, rectangle, or triangle. The cross sections of such wings are usually represented as symmetric profiles taking the form of a polygon, rhombus, or lens (Fig. 10.1b).

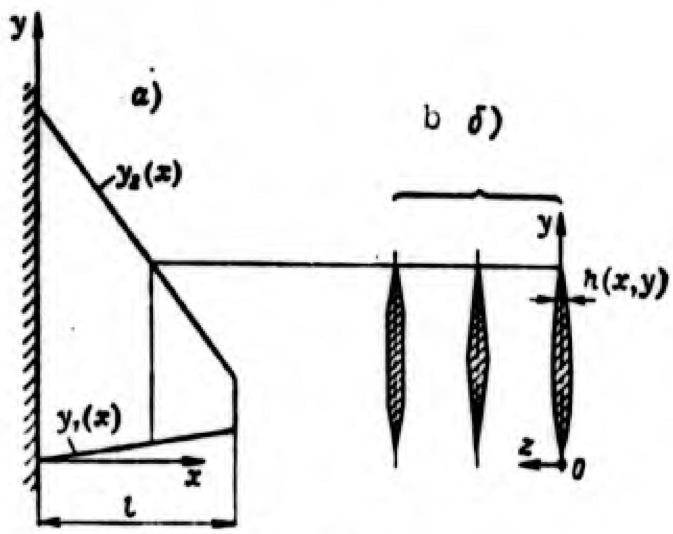


Fig. 10.1

The thickness of solid wings is small in comparison with the overall dimensions. Thus in calculations for such wings we make use of the fundamental equations of the theory of plates with allowance for the fact that in integrating, we must deal with the geometric variables $D(x, y)$, $h(x, y)$.

The coordinate system used in the design of plates (see Fig. 6.1) is rotated through 180° about the x axis; the origin of the system is placed at the wing attachment (see Fig. 10.1a).

The deflection of any point of the wing is represented in accordance with the Kantorovich-Vlasov method in the form of an expansion:

$$w(x, y)=\sum_{i=1}^{n}\varphi_i(x)\psi_i(y), \tag{10.1}$$

where $\varphi_1(x)$ is the unknown function of x and $\psi_1(y)$ is the given function of y.

In the design of solid wings, Expansion (10.1) is usually represented in the form of a polynomial in powers of y:

$$w(x, y)=\varphi_0(x)+\varphi_1(x)y+\varphi_2(x)y^2+\ldots+\varphi_n(x)y^n, \tag{10.2}$$

where $\varphi_0(x)$ is a function of bending, $\varphi_1(x)$ is a function of torsion, and $\varphi_2(x)$ is a function of wing curvature in the direction of the chord for a parabola.

After the curved surface of the wing has been determined, we can also determine its stressed state with the aid of Eq. (6.7). We obtain

$$\begin{aligned} \sigma_x &= -\frac{Ez}{1-\mu^2}(w_{xx}+\mu w_{yy}); \\ \sigma_y &= -\frac{Ez}{1-\mu^2}(w_{yy}+\mu w_{xx}); \\ \tau_{xy} &= -\frac{Ez}{1+\mu}w_{xy}. \end{aligned} \tag{6.7'}$$

As experimental data indicate [35] in approximate calculations we need only keep two terms of this expansion.

Thus, the deflection is determined in the form

$$w(x, y)=\varphi_0(x)+\varphi(x)y. \tag{10.3}$$

1. DIFFERENTIAL EQUATIONS OF THE PROBLEM AND FORMATION OF BOUNDARY CONDITIONS

The total potential energy (6.9) is determined for the considered system from the expression:

$$\Pi=\int_0^l\int_{y_1(x)}^{y_2(x)}\left\{\frac{D(x,y)}{2}\left[w_{xx}^2+w_{yy}^2+2\mu w_{xx}w_{yy}+2(1-\mu)w_{xy}^2\right]-\right.$$
$$\left.-p(x,y)w(x,y)\right\}dy\,dx, \qquad (10.4)$$

where $y_1(x)$, $y_2(x)$ are linear functions describing, respectively, the positions of the wing leading and trailing edges; $D(x,y)=\frac{E[h(x,y)]^3}{12(1-\mu^2)}$ is the cylindrical stiffness; p(x, y) is the transverse surface load, which is positive upward.

Substituting the expression for the deflection (10.3) into Eq. (10.4), we obtain

$$\Pi=\int_0^l\int_{y_1(x)}^{y_2(x)}\left\{\frac{D(x,y)}{2}\left[(\varphi_0''+\varphi_1''y)^2+2(1-\mu)\varphi_1'^2\right]-p(\varphi_0+\varphi_1 y)\right\}dy\,dx. \qquad (a)$$

Here primes indicate derivatives with respect to x.

Integrating Expression (a) over y, we obtain

$$\Pi=\frac{1}{2}\int_0^l\Gamma\,dy,$$

where

$$\Gamma=d_0\varphi_0''^2+2d_1\varphi_0''\varphi_1''+d_2\varphi_1''^2+2(1-\mu)d_0\varphi_1'^2-2q\varphi_0-2m\varphi_1; \qquad (b)$$

$$d_0=\int_{y_1(x)}^{y_2(x)}D(x,y)\,dy;\quad d_1=\int_{y_1(x)}^{y_2(x)}D(x,y)\,y\,dy;$$

$$d_2=\int_{y_1(x)}^{y_2(x)}D(x,y)\,y^2\,dy;$$

$q=\int_{y_1(x)}^{y_2(x)}p(x,y)dy$ is the load per unit length; $m=\int_{y_1(x)}^{y_2(x)}p(x,y)y\,dy$ is the torsional moment about the x axis, per unit length.

From the conditions requiring minimization of potential energy we obtain

$$\Gamma_{\varphi_0}-\frac{d}{dy}\Gamma_{\varphi_0'}+\frac{d^2}{dy^2}\Gamma_{\varphi_0''}=0; \qquad (c)$$

$$\Gamma_{\varphi_1}-\frac{d}{dy}\Gamma_{\varphi_1'}+\frac{d^2}{dy^2}\Gamma_{\varphi_1''}=0. \qquad (d)$$

Since

$$\Gamma_{\varphi_0}=-2q;\quad \Gamma_{\varphi_0'}=0;\quad \Gamma_{\varphi_0''}=2d_0\varphi_0''+2d_1\varphi_1'';$$

$$\Gamma_{\varphi_1} = -2m; \quad \Gamma_{\varphi_1'} = 4(1-\mu)d_0\varphi_1'; \quad \Gamma_{\varphi_1''} = 2d_1\varphi_0'' + 2d_2\varphi_1''.$$

Then Eqs. (c), (d) take the form:

$$(d_0\varphi_0'')'' + (d_1\varphi_1'')'' = q; \tag{10.5}$$

$$(d_1\varphi_0'') + (d_2\varphi_1'')'' - 2(1-\mu)(d_0\varphi_1')' = m. \tag{10.6}$$

Thus the functions of wing bending and torsion are described by two fourth-order differential equations with variable coefficients and free terms.

Eight of the arbitrary constants in these equations are determined from the following boundary conditions.

At x = 0 we let the wing be constrained along the entire chord; we then obtain four given geometric boundary conditions:

$$\varphi_0(0) = 0; \tag{1}$$

$$\varphi_0'(0) = 0; \tag{2}$$

$$\varphi_1(0) = 0; \tag{3}$$

$$\varphi_1'(0) = 0. \tag{4}$$

At the free end of the wing at $x = \ell$, we have four natural boundary conditions:

$$[(d_0\varphi_0'')' + (d_1\varphi_1'')']_{x=l} = 0; \tag{5}$$

$$[d_0\varphi_0'' + d_1\varphi_1'']_{x=l} = 0; \tag{6}$$

$$[(d_1\varphi_0'')' + (d_2\varphi_1'')' - 2(1-\mu)d_0\varphi_1']_{x=l} = 0; \tag{7}$$

$$[d_1\varphi_0'' + d_2\varphi_1'']_{x=l} = 0. \tag{8}$$

It is convenient to combine Eqs. (10.5) and (10.6) into a single differential equation for the torsion function $\varphi_1(x)$.

Integrating Eq. (10.5) and using boundary condition (5), we obtain

$$(d_0\varphi_0'')' + (d_1\varphi_1'')' = -\int_x^l q\,dx = Q(x). \tag{10.7}$$

Integrating this equation and using boundary condition (6), we find

$$d_0\varphi_0'' + d_1\varphi_1'' = -\int_x^l Q(x)\,dx = M(x)$$

or

$$\varphi_0' = -\frac{d_1}{d_0}\varphi_1' + \frac{1}{d_0}M(x). \qquad (10.8)$$

Integrating Eq. (10.6) and using boundary condition (7), we obtain

$$(d_1\varphi_0')' + (d_2\varphi_1')' - 2(1-\mu)d_0\varphi_1' = -\int_x^l m\,dx = \mathfrak{M}(x), \qquad (10.9)$$

where $\dot{M}(x)$ is the torsional moment at section $\underline{x}$.

We now substitute the value φ_0'' from (10.8). We then obtain

$$\left(-\frac{d_1^2}{d_0}\varphi_1'\right)' + \left(\frac{d_1}{d_0}M(x)\right)' + (d_2\varphi_1')' - 2(1-\mu)d_0\varphi_1' = \mathfrak{M}(x)$$

or

$$\left[\left(d_2 - \frac{d_1^2}{d_0}\right)\varphi_1'\right]' - 2(1-\mu)d_0\varphi_1' = \mathfrak{M}(x) - \left[\frac{d_1}{d_0}M(x)\right]'. \qquad (10.10)$$

We have thus obtained a third-order equation which, together with boundary conditions (3), (4) and (8), completely determines the function $\varphi_1(x)$.

We note that boundary condition (8) together with Eq. (10.8) may be represented in the form:

$$\left[\left(d_2 - \frac{d_1^2}{d_0}\right)\varphi_1'\right]_{x=l} = 0. \qquad (8')$$

After we have determined the torsion function φ_1, we can also compute the wing bending function φ_0 by direct integration of Eq. (10.8), using boundary conditions (1) and (2).

2. DESIGN OF RECTANGULAR WING WITH DIAMOND PROFILE

Let us determine the stressed and strained states of a wing acted on by a unit load

$$p(y) = p_0 + (p_b - p_0)\frac{y}{b}$$

and having a diamond-shaped profile (Fig. 10.2).

We shall determine the geometric characteristics of such a profile. The wing thickness is

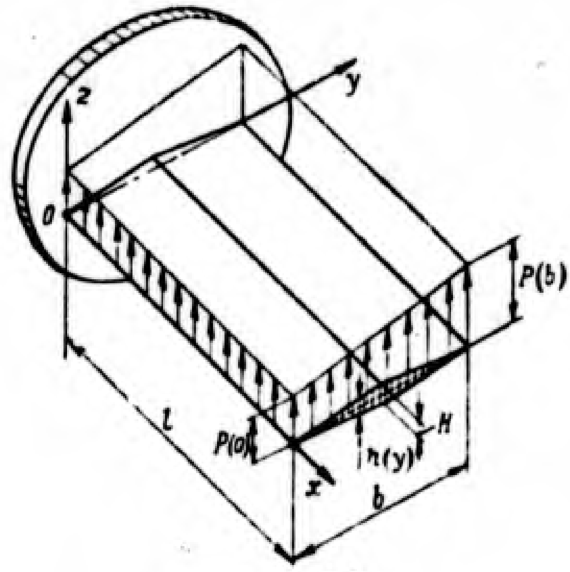


Fig. 10.2

$$h(y)=\frac{2Hy}{b} \qquad \left(0\leqslant y\leqslant\frac{b}{2}\right)$$

and

$$h(y)=2H\left(1-\frac{y}{b}\right) \qquad \left(\frac{b}{2}\leqslant y\leqslant b\right).$$

The cylindrical stiffness is

$$D(y)=\frac{Eh^3(y)}{12(1-\mu^2)}=\frac{E8H^3}{12(1-\mu^2)}\,\frac{y^3}{b^3} \qquad \left(0<y<\frac{b}{2}\right)$$

and

$$D(y)=\frac{E8H^3}{12(1-\mu^2)}\left(1-\frac{y}{b}\right)^3 \qquad \left(\frac{b}{2}<y<b\right)$$

or

$$D(y)=D\frac{8y^3}{b^3} \quad \left(0<y\leqslant\frac{b}{2}\right)$$

and

$$D(y)=D8\left(1-\frac{y}{b}\right)^3 \quad \left(\frac{b}{2}\leqslant y\leqslant b\right),$$

where

$$D=\frac{EH^3}{12(1-\mu^2)}.$$

The coefficients

$$d_0=D\int_0^{b/2}\frac{8y^3}{b^3}dy+D\int_{b/2}^{b}8\left(1-\frac{y}{b}\right)^3dy=\frac{Db}{4};$$

$$d_1=D\int_0^{b/2}\frac{8y^4}{b^3}dy+D\int_{b/2}^{b}8y\left(1-\frac{y}{b}\right)^3dy=\frac{Db^2}{8};$$

$$d_2=D\int_0^{b/2}\frac{8y^5}{b}dy+D\int_{b/2}^{b}8y^2\left(1-\frac{y}{b}\right)^3dy=\frac{Db^3}{15};$$

$$d_2-\frac{d_1^2}{d_0}=\frac{Db^3}{240};\quad \frac{d_1}{d_0}=\frac{b}{2}.$$

We now determine the force factor

$$p(y)=p_0+(p_b-p_0)\frac{y}{b};$$

$$q=\int_0^b p(y)\,dy=\frac{(p_0+p_b)b}{2};$$

$$Q(x)=-\int_x^l q\,dx=-\frac{(p_0+p_b)b(l-x)}{2};$$

$$M(x)=-\int_x^l Q_x\,dx=\frac{(p_0+p_b)b}{4}(l-x)^2;$$

$$m=\int_0^b p(y)y\,dy=\frac{(p_0+2p_b)b^2}{6};$$

$$\mathfrak{M}(x)=-\int_x^l m\,dx=-\frac{(p_0+2p_b)b^2}{6}(l-x).$$

Thus differential equation (10.10) will take the following form for the case under consideration:

$$\frac{Db^3}{240}\varphi_1''-2(1-\mu)\frac{Db}{4}\varphi_1'=-\frac{(p_0+2p_b)b^2}{6}(l-x)+$$

$$+\frac{(p_0+p_b)b^2}{4}(l-x)=-\frac{(p_0-p_b)}{12}b^2(l-x)$$

or

$$\varphi_1''-120(1-\mu)\frac{1}{b^2}\varphi_1'=\frac{20(p_0-p_b)}{Db}(l-x).$$

The substitution $\xi=[1-(x/\ell)]$ yields the following transformation:

$$\varphi_1''(\xi)-120(1-\mu)\frac{l^2}{b^2}\varphi_1'(\xi)=-\frac{20(p_0-p_b)l^4\xi}{Db}.$$

or

$$\varphi_1''(\xi)-5r^2\varphi_1'(\xi)=-\frac{20(p_0-p_b)l^4\xi}{Db}, \tag{I}$$

where the primes now indicate differentiation with respect to ξ, and

$$r^2=24(1-\mu)\frac{l^2}{b^2}.$$

The general integral of Eq. (I) with respect to $\varphi_1'(\xi)$ is:

$$\varphi_1'(\xi)=C_1\,\mathrm{sh}\,k\xi+C_2\,\mathrm{ch}\,k\xi+\eta\xi, \tag{II}$$

where

$$k=r\sqrt{5};$$

$$\eta=\frac{4(p_0-p_b)l^4}{r^2Db}.$$

Boundary conditions (3), (4) and (8') for the function $\varphi_1(\xi)$ have the form

$$\varphi_1(1)=0; \tag{3'}$$

$$\varphi_1'(1)=0; \tag{4'}$$

$$\varphi_1''(0)=0, \tag{8''}$$

Employing boundary condition (4') for Eq. (II), we obtain

$$C_1\,\mathrm{sh}\,k+C_2\,\mathrm{ch}\,k+\eta=0. \tag{A}$$

Differentiating Eq. (II), we obtain

$$\varphi_1''(\xi)=C_1k\,\mathrm{ch}\,k\xi+C_2k\,\mathrm{sh}\,k\xi+\eta. \tag{III}$$

In virtue of boundary condition (8")

$$C_1k+\eta=0,$$

or

$$C_1=-\frac{\eta}{k}.$$

From Eq. (A), we have

$$C_2=-\eta\left(1-\frac{\mathrm{sh}\,k}{k}\right)\frac{1}{\mathrm{ch}\,k}.$$

As a consequence,

$$\varphi_1''(\xi)=\eta\left[-\mathrm{ch}\,k\xi-\left(1-\frac{\mathrm{sh}\,k}{k}\right)\frac{\mathrm{sh}\,k\xi}{\mathrm{ch}\,k}+1\right] \tag{IV}$$

and

$$\varphi_1'(\xi)=\eta\left[-\frac{\operatorname{sh} k\xi}{k}-\left(1-\frac{\operatorname{sh} k}{k}\right)\frac{\operatorname{ch} k\xi}{\operatorname{ch} k}+\xi\right]. \tag{V}$$

Integrating Eq. (V), we obtain

$$\varphi_1(\xi)=\eta\left[-\frac{\operatorname{ch} k\xi}{k^2}-\left(1-\frac{\operatorname{sh} k}{k}\right)\frac{\operatorname{sh} k\xi}{k\operatorname{ch} k}+\frac{\xi^2}{2}\right]+C_3.$$

Boundary condition (3') yields

$$C_3=+\eta\left[\frac{\operatorname{ch} k}{k^2}+\left(1-\frac{\operatorname{sh} k}{k}\right)\frac{\operatorname{sh} k}{k\operatorname{ch} k}-\frac{1}{2}\right].$$

As a consequence,

$$\varphi_1(\xi)=\eta\left[-\frac{\operatorname{ch} k\xi-\operatorname{ch} k}{k^2}-\left(1-\frac{\operatorname{sh} k}{k}\right)\left(\frac{\operatorname{sh} k\xi}{k\operatorname{ch} k}-\frac{\operatorname{sh} k}{k\operatorname{ch} k}\right)-\frac{1-\xi^2}{2}\right],$$

or

$$\varphi_1(\xi)=-\eta\left[\frac{1-\xi^2}{2}-\frac{\operatorname{ch} k-\operatorname{ch} k\xi}{k^2}-\frac{k-\operatorname{sh} k}{\operatorname{ch} k}\left(\frac{\operatorname{sh} k-\operatorname{sh} k\xi}{k^2}\right)\right]. \tag{VI}$$

We now turn to the determination of the value of the function φ_0.

Substituting $\xi = [1-(x/\ell)]$ into Eq. (10.8), we arrive at the following equation:

$$\varphi_0''(\xi)=-\frac{d_1}{d_0}\varphi_1''(\xi)+\frac{l^2}{d_0}M(x).$$

To integrate this equation, we make use of the remaining boundary conditions:

$$\varphi_0(1)=0; \tag{1'}$$

$$\varphi_0'(1)=0. \tag{2'}$$

Taking into account Expression (IV), we obtain

$$\varphi_0''(\xi)=+\frac{b}{2}\eta\left[\operatorname{ch} k\xi-1+\frac{k-\operatorname{sh} k}{\operatorname{ch} k}\operatorname{sh} k\xi\right]+ \\ +\frac{(p_0+p_b)l^4\xi^2}{D}. \tag{VII}$$

Integrating this equation once and using boundary condition (2'), we obtain

$$\varphi_0'(\xi) = +\frac{b}{2}\eta\left[\frac{\operatorname{sh} k\xi}{k} - \xi + \frac{k - \operatorname{sh} k}{\operatorname{ch} k}\,\frac{\operatorname{ch} k\xi}{k}\right] + $$
$$+\frac{(p_0+p_b)l^4}{3D}(\xi^3 - 1). \qquad \text{(VIII)}$$

Integrating Eqs. (VIII) and using boundary condition (1'), we obtain the following value for the wing bending function:

$$\varphi_0(\xi) = \frac{(p_0+p_b)l^4}{3D}\left(1 - \xi - \frac{1-\xi^4}{4}\right) + $$
$$+\frac{b}{2}\eta\left[\frac{1-\xi^2}{2} - \frac{\operatorname{ch} k - \operatorname{ch} k\xi}{k^2} - \frac{k - \operatorname{sh} k}{\operatorname{ch} k}\left(\frac{\operatorname{sh} k - \operatorname{sh} k\xi}{k^2}\right)\right]. \qquad \text{(IX)}$$

Thus, the curved surface of the wing considered may be represented by the expression

$$w(\xi, y) = \varphi_0(\xi) + \varphi_1(\xi)y,$$

or

$$w(\xi, y) = \frac{(p_0+p_b)l^4}{3D}\left(1 - \xi - \frac{1-\xi^4}{4}\right) - \eta\Phi\left(y - \frac{b}{2}\right), \qquad (10.11)$$

where

$$D = \frac{EH^3}{12(1-\mu^2)};$$
$$\xi = \left(1 - \frac{x}{l}\right);$$
$$\eta = \frac{4(p_0 - p_b)l^4}{r^2 Db};$$

$r = \frac{l}{b}\sqrt{24(1-\mu)}$ is an aspect-ratio parameter;

$$\Phi = \frac{1-\xi^2}{2} - \frac{\operatorname{ch} k - \operatorname{ch} k\xi}{k^2} - \frac{k - \operatorname{sh} k}{\operatorname{ch} k}\left(\frac{\operatorname{sh} k - \operatorname{sh} k\xi}{k^2}\right);$$
$$k = r\sqrt{5}.$$

Example.

As an example, we shall determine the deflection, angle of rotation of the free wing end ($\xi = 0$), and maximum stresses σ_x in the section at the constrained end of the wing ($\xi = 1$) for the following hypothetical values: $p_0 = 0.1$ kgf/cm; $p_b = 0.2$ kgf/cm^2; $l = b = 100$ cm; $H = 3$ cm; $\mu = 0.3$; $E = 7 \cdot 10^5$ kgf/cm^2.

Determination of Deflection and Angle of Rotation of Free Wing End

We shall determine the values of the constants D, ξ, r^2, k and

η, as well as the value of Φ, contained in Expression (10.11):

$$D=\frac{7\cdot10^5\cdot3^3}{12(1-0,3^2)}=\frac{7\cdot10^5\cdot27}{12\cdot0,91}=17,3\cdot10^5;$$

$$\xi=0;$$

$$r^2=24(1-0,3)=16,8;$$

$$k=\sqrt{16,8\cdot5}=9,18;$$

$$\eta=\frac{4(0,1-0,2)100^4}{16,8\cdot17,3\cdot10^5\cdot100}=-\frac{0,4\cdot10}{291}=-\frac{4}{291}.$$

Since cosh k ≃ sinh k, then

$$\Phi=\frac{1}{2}+\frac{1}{k^2}-\frac{1}{k}=\frac{1}{2}+\frac{1}{9,18^2}-\frac{1}{9,18}=0,403.$$

From Formula (10.11), the unknown deflection of the wing longitudinal axis will be

$$w\left(0,\frac{b}{2}\right)=\frac{(0,1+0,2)100^4}{3\cdot17,3\cdot10^5}\left(1-\frac{1}{4}\right)=4,33\text{ cm}.$$

The deflection of the leading-edge point at the wing tip will be

$$w(0,b)=4,33-\eta\Phi\left(b-\frac{b}{2}\right)=4,33+\frac{4}{2,91}\,0,403\cdot50=$$
$$=4,33+0,277=4,607\text{ cm}.$$

The deflection of the trailing edge at the wing tip will be

$$w(0,0)=4,33-0,277=4,053\text{ см}.$$

The torsion angle of the free wing end is

$$\varphi=\frac{4,607-4,053}{100}=\frac{0,554}{100}=0,00554\text{ rad}$$

<u>Determination of Maximum Stresses σ_x in Section at Constrained End of Wing (ξ = 1)</u>

The unknown stresses are

$$\sigma_x=-\frac{Ez}{1-\mu^2}(w_{xx}+\mu w_{yy}).$$

Since

$$Z=\frac{h(y)}{2}=\frac{Hy}{b}\qquad\left(0\leqslant y\leqslant\frac{b}{2}\right);$$

$$Z=H\left(1-\frac{y}{b}\right)\qquad\left(\frac{b}{2}\leqslant y\leqslant b\right),$$

$$w_{xx}=\frac{w_{\xi\xi}}{l^2},\quad w_{yy}=0,$$

then

$$\sigma_x = -\frac{EHyw_{\xi\xi}}{(1-\mu^2)bl^2} \qquad \left(0 \leqslant y \leqslant \frac{b}{2}\right);$$

$$\sigma_x = -\frac{EH\left(1-\frac{y}{b}\right)w_{\xi\xi}}{(1-\mu^2)l^2} \qquad \left(\frac{b}{2} \leqslant y \leqslant b\right).$$

Differentiating Expression (10.11) twice with respect to ξ, we obtain

$$w_{\xi\xi} = \frac{(p_0-p_b)l^4\xi^2}{D} - \eta\Phi_{\xi\xi}\left(y-\frac{b}{2}\right),$$

where

$$\Phi_{\xi\xi} = \operatorname{ch} k\xi + \frac{k-\operatorname{sh} k}{\operatorname{ch} k}\operatorname{sh} k\xi - 1.$$

For $\xi = 1$, we obtain $\Phi_{\xi\xi} = k - 1 = 8.18$. As a consequence,

$$w_{\xi\xi} = \frac{(0,1+0,2)\,100^4}{17,3\cdot 10^6} + \frac{4}{291}\cdot 8,18\left(y-\frac{b}{2}\right) =$$

$$= \frac{309}{17,3} + \frac{32,7}{291}\left(y-\frac{b}{2}\right) = 17,30 + 0,1125\left(y-\frac{b}{2}\right).$$

Thus,

$$\sigma_x = -\frac{7,10^5\cdot 3y}{(1-\mu^2)\,100^3}\left[17,3+0,1125\left(y-\frac{b}{2}\right)\right] =$$

$$= -2,3y\left[17,3+0,1125\left(y-\frac{b}{2}\right)\right] \quad \left(0 \leqslant y \leqslant \frac{b}{2}\right);$$

$$\sigma_x = -230\left(1-\frac{y}{b}\right)\left[17,3+0,1125\left(y-\frac{b}{2}\right)\right] \quad \left(\frac{b}{2} \leqslant y \leqslant b\right).$$

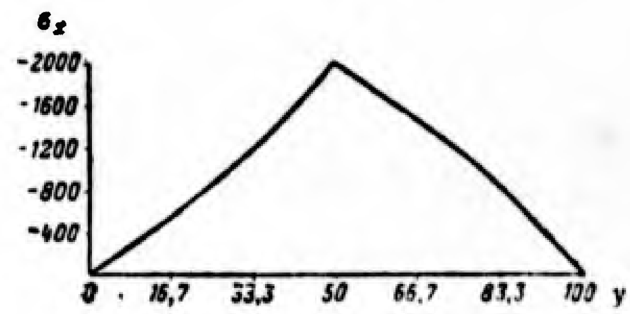


Fig. 10.3

Making use of these formulas, we obtain the following values for the stresses:

for	$y=0$	$\sigma_x=0$; kgf/cm^2
for	$y=\frac{b}{6}=16{,}7$	$\sigma_x=-520$ kgf/cm^2
for	$y=\frac{b}{3}=33{,}3$	$\sigma_x=-1190$ kgf/cm^2
for	$y=\frac{b}{2}=50$	$\sigma_x=-2000$ kgf/cm^2
for	$y=\frac{2}{3}b=66{,}7$	$\sigma_x=-1480$ kgf/cm^2
for	$y=\frac{5}{6}b=83{,}3$	$\sigma_x=-808$ kgf/cm^2
for	$y=b=100$	$\sigma_x=0.$

Figure 10.3 shows a graph of the variation in stresses σ_x in the section at the constrained end of the wing.

3. CALCULATIONS FOR UNIFORMLY LOADED DELTA WING

Let a delta wing be constrained along the root chord of length b_0 (Fig. 10.4) and loaded by a constant transverse load p_0.

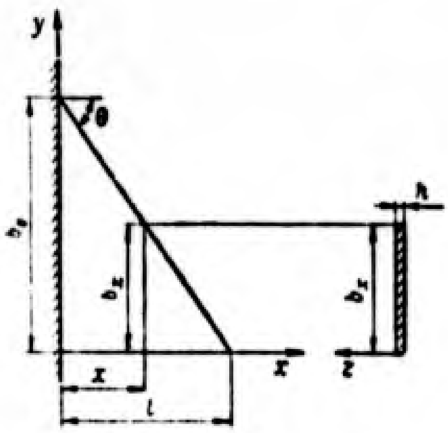


Fig. 10.4

In the section x = const, the chord length is

$$b_x=b_0\xi,$$

where

$$\xi=1-\frac{x}{l};$$

l is the length of the wing along the x axis.

In first-approximation calculations, to determine the deformed state of the wing, the profiles shown in Fig. 10.1b may be reduced to rectangular profiles with an average thickness of

$$\bar{h}=\frac{F_x}{b_x},$$

where F_x is the cross-sectional area of the wing at the section x = = const.

Letting $\bar{h}$ = h = const, we obtain the following geometric characteristics for the delta wing:

$$D=\frac{Eh^3}{12(1-\mu^2)};$$

$$d_0=D\int_0^{b_x} dy=Db_0\xi;\quad d_1=D\int_0^{b_x} y\,dy=\frac{Db_0^2\xi^2}{2};$$

$$d_2=D\int_0^{b_x} y^2\,dy=D\frac{b_0^3\xi^3}{3};\quad d_2-\frac{d_1}{d_0}=\frac{Db_0^3\xi^3}{12}.$$

We now determine the fundamental force factors for the delta wing:

$$q=\int_0^{b_x} p_0\,dy=p_0b_0\xi;$$

$$m=\int_0^{b_x} p_0 y\,dy=p_0\frac{b_0^2\xi^2}{2};$$

$$Q(x)=-\int_x^l q\,dx=-p_0b_0\int_x^l \xi\,dx=-p_0b_0\int_\xi^0 \xi(-l\,d\xi)=-\frac{p_0b_0l\xi^2}{2};$$

$$M(x)=+\int_\xi^0 \frac{p_0b_0l\xi^2}{2}(-l\,d\xi)=\frac{p_0b_0l^2\xi^3}{6};$$

21

$$\mathfrak{M}=-\int_\xi^0 \frac{p_0b_0^2\xi^2}{2}(-l\,d\xi)=-\frac{p_0b_0^2l\xi^3}{6}.$$

Substituting the expressions for the geometric characteristics and the force factors into Eq. (10.10), we obtain

$$\left[\frac{Db_0^3}{12}\xi^3\varphi_1'\right]'-2(1-\mu)Db_0\xi\varphi_1'=$$
$$=-\frac{p_0b_0^2l\xi^3}{6}-\left[\frac{b_0\xi}{2}\,\frac{p_0b_0l^2\xi^3}{6}\right]'.$$

Expressing φ_1' and φ_1'' in terms of the variable $\xi=[1-(x/\ell)]$, we obtain

$$\left[\frac{Db_0^3}{12l^2}\xi^3\varphi_1'(\xi)\right]'+2(1-\mu)\frac{Db_0\xi}{l}\varphi_1'(\xi)=-\frac{p_0b_0^2l\xi^3}{6}-\left[\frac{p_0b_0^2l^2\xi^4}{12}\right]'. \qquad \text{(a)}$$

Since

$$[\xi^3\varphi_1'(\xi)]'=-\frac{\xi^3}{l^3}\varphi_1''(\xi)-\frac{3\xi^2}{l^3}\varphi_1'(\xi)$$

and

$$\left[\frac{p_0 b_0^2 l^2 \xi^4}{12}\right]' = -\frac{p_0 b_0^2 l \xi^3}{3},$$

then Expression (a) will take the form of the Euler differential equation after certain manipulations:

$$\xi^2 \varphi_1'''(\xi) + 3\xi\varphi_1''(\xi) - r^2\varphi_1'(\xi) = -\frac{2p_0 l^4 \xi^2}{Db_0}, \qquad \text{(b)}$$

where

$$r^2 = 24(1-\mu)\frac{l^2}{b_0^2}.$$

We shall seek a solution fo Eq. (b) in the form ξ^1. We assume that $\varphi_1'(\xi) = y = \xi^k$. Then

$$y' = k\cdot\xi^{k-1}; \quad y'' = k(k-1)\xi^{k-2}.$$

Substituting these values into Eq. (b), we obtain the characteristic equation

$$k^2 + 2k - r^2 = 0.$$

The roots of this equation are

$$k_1 = a - 1 \text{ and } k_2 = -a - 1,$$

where

$$a = \sqrt{1 + r^2}.$$

Thus, the general integral for the homogeneous equation

$$\xi^2\varphi_1'''(\xi) + 3\xi\varphi_1''(\xi) - r^2\varphi_1'(\xi) = 0$$

has the form

$$\varphi_1'(\xi) = C_1\xi^{a-1} + C_2\xi^{-a-1}.$$

We seek a particular solution to inhomogeneous equation (b) in the form

$$y_1 = A\cdot\xi^2.$$

We obtain

$$y_1' = 2A\xi; \quad y_1'' = 2A.$$

Substituting these values into Eq. (b), we find

$$2A\xi^2+6A\xi^2-Ar^2\xi^2=-\frac{2p_0l^4\xi^2}{Dh_0}.$$

From which we have

$$A=-\frac{2p_0l^4}{Dh_0(8-r^2)}$$

and

$$y_1=-\frac{2p_0l^4\xi^2}{Db_0(8-r^2)}.$$

As a consequence, the general integral for the inhomogeneous equation will be

$$\varphi_1'(\xi)=C_1\xi^{\alpha-1}+\frac{C_2}{\xi^{\alpha+1}}-\frac{2p_0l^4\xi^2}{Db_0(8-r^2)}.$$

Now boundary conditions (3), (4) and (8') for function $\varphi_1(\xi)$ will have the form

$$\varphi_1(1)=0; \qquad (3')$$
$$\varphi_1'(1)=0; \qquad (4')$$
$$[\xi^3\varphi_1'(\xi)]_{\xi\to 0}=0. \qquad (8'')$$

Employing boundary condition (8"), we can see that $C_2 = 0$. As a consequence,

$$\varphi_1'(\xi)=C_1\xi^{\alpha-1}-\frac{2p_0l^4\xi^2}{Dh_0(8-r^2)}.$$

Employing boundary condition (4'), we obtain

$$C_1=\frac{2p_0l^4}{Db_0(8-r^2)}.$$

This means that

$$\varphi_1'(\xi)=\frac{2p_0l^4}{Db_0(8-r^2)}(\xi^{\alpha-1}-\xi^2). \qquad (I)$$

Differentiating Eq. (I), we find that

$$\varphi_1''(\xi)=\frac{2p_0l^4}{Db_0(8-r^2)}[(\alpha-1)\xi^{\alpha-2}-2\xi]. \qquad (II)$$

Integrating (I), we obtain

$$\varphi_1(\xi)=\frac{2p_0l^4}{Db_0(8-r^2)}\left(\frac{\xi^\alpha}{\alpha}-\frac{\xi^3}{3}\right)+D.$$

According to boundary condition (3')

$$\frac{2p_0l^4}{Db_0(8-r^2)}\left(\frac{1}{\alpha}-\frac{1}{3}\right)+D=0.$$

From which we have

$$D=\frac{2p_0l^4}{Db_0(8-r^2)}\left(\frac{1}{3}-\frac{1}{\alpha}\right).$$

Thus the expression for $\varphi_1(\xi)$ will have the form

$$\varphi_1(\xi)=\frac{2p_0l^4}{Db_0(8-r^2)}\left(\frac{\xi^\alpha-1}{\alpha}-\frac{\xi^3-1}{3}\right). \qquad \text{(III)}$$

We now have no difficulty in determining the value of the bending function φ_0 as well. Equation (10.8) is rewritten as

$$\frac{1}{l^2}\varphi_0''(\xi)=-\frac{d_1}{d_0}\,\frac{1}{l^2}\,\varphi_1''(\xi)+\frac{1}{d_0}M(x)$$

or

$$\varphi_0''(\xi)=-\frac{b_0}{2}\xi\varphi_1''(\xi)+\frac{p_0l^4}{6D}\xi^2.$$

Introducing into this equation the value

$$\varphi_1''(\xi)=\frac{2p_0l^4}{Db_0(8-r^2)}\left[(\alpha-1)\xi^{\alpha-2}-2\xi\right],$$

we obtain

$$\varphi_0''(\xi)=+\frac{p_0l^4}{D(8-r^2)}\left[2\xi^2+\frac{8-r^2}{6}\xi^2-(\alpha-1)\xi^{\alpha-1}\right],$$

or

$$\varphi_0''(\xi)=\frac{p_0l^4}{D(8-r^2)}\left[(20-r^2)\frac{\xi^2}{6}-(\alpha-1)\xi^{\alpha-1}\right]. \qquad \text{(IV)}$$

Integrating Eq. (IV) and using the boundary condition $\varphi_0'(1) = 0$, we have

$$\varphi_0'(\xi)=\frac{p_0l^4}{D(8-r^2)}\left[\frac{20-r^2}{18}(\xi^3-1)-\frac{\alpha-1}{\alpha}(\xi^\alpha-1)\right]. \qquad \text{(V)}$$

Integrating Eq. (V) and making use of the boundary condition $\varphi_0(1) = 0$, we determine the value of the function $\varphi_0(\xi)$:

$$\varphi_0(\xi)=\frac{p_0l^4}{D(8-r^2)}\left[\frac{20-r^2}{18}\left(1-\xi-\frac{1-\xi^4}{4}\right)-\right.$$
$$\left.-\frac{\alpha-1}{\alpha}\left(1-\xi-\frac{1-\xi^{\alpha+1}}{\alpha+1}\right)\right]. \qquad \text{(VI)}$$

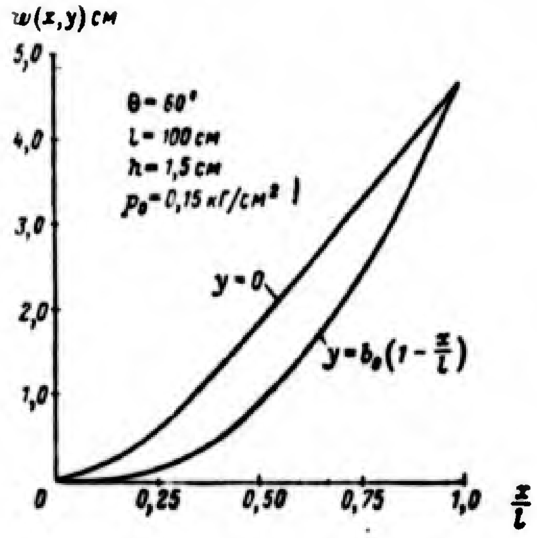


Fig. 10.5. 1) kgf/cm^2.

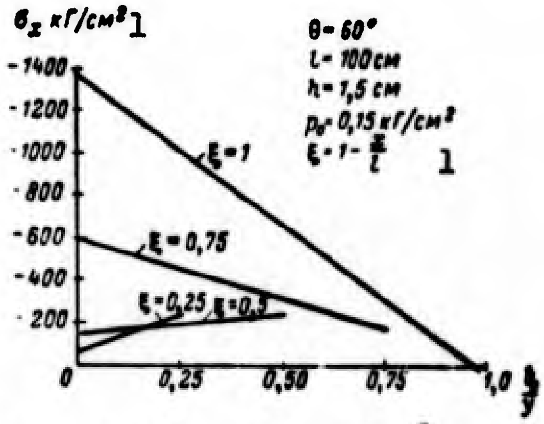


Fig. 10.6. 1) kgf/cm^2.

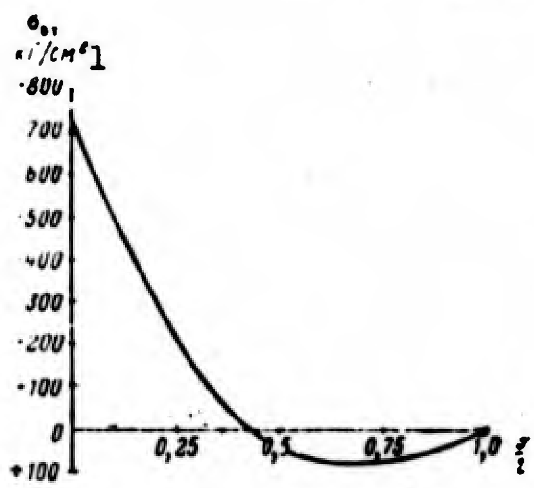


Fig. 10.7 . 1) kgf/cm^2.

Thus, the curved surface of the delta wing may be represented by

$$w(\xi, y)=\frac{p_0 l^4}{D(8-r^2)}\left[\frac{20-r^2}{18}\left(1-\xi-\frac{1-\xi^4}{4}\right)-\right.$$
$$\left.-\frac{a-1}{a}\left(1-\xi-\frac{1-\xi^{a+1}}{a+1}\right)+\frac{2y}{b_0}\left(\frac{\xi^a-1}{a}-\frac{\xi^3-1}{3}\right)\right], \qquad (10.12)$$

where

$$D=\frac{Eh^3}{12(1-\mu^2)};$$
$$\xi=1-\frac{x}{l};$$
$$a=\sqrt{1+r^2};$$
$$r^2=24(1-\mu)\frac{l^2}{b_0^2}.$$

In Fig. 10.5, we have plotted the leading-edge and trailing-edge elastic lines for a hypothetical wing loaded by a transverse unit load; the sweep angle $\Theta = 60°$; $b_0 = 173$ cm; $\mu = 0.3$; $E = 7 \cdot 10^5$ kgf/cm^2 (the remaining values are shown in Fig. 10.5).

For such a wing we use the formula

$$\sigma_x=-\frac{6p_0 l^2}{h^2(8-r^2)}\left\{\frac{20-r^2}{6}\xi^2-(a-1)\xi^{a-1}+\frac{2y}{b_0}[(a-1)\xi^{a-2}-2\xi]\right\} \qquad (10.13)$$

to determine the stresses for four cross sections (Fig. 10.6).

The difference between the stresses computed in accordance with the law of plane sections and in accordance with Formula (10.13) represents the secondary stresses. Figure 10.7 gives a curve for the secondary stresses σ_{vt} in the trailing edge.

Part Four

THERMAL STRESSES

Chapter 11

APPLICATION OF THE METHOD OF FORCES AND FINITE-DIFFERENCE EQUATIONS TO CALCULATIONS FOR THERMAL STRESSES IN THIN-SKIN SYSTEMS

1. DETERMINATION OF THERMAL STRESSES BY THE METHOD OF FORCES

As temperature increases, the elements of an elastic body expand, as we know. In contrast to S.O. systems, in S.N. systems the elements cannot expand freely. As a consequence, heating or cooling of such systems will usually produce thermal stresses.

The stressed state of a structure due to the effect of temperature is selfbalanced. This state consists of the sum of the single states multiplied, respectively, by the unknowns $X_1, X_2, \ldots, X_n$. These unknowns may be found from the solution of the canonical equations:

$$\left.\begin{array}{l} X_1\delta_{11}+X_2\delta_{12}+\ldots+X_n\delta_{1n}+\Delta_{1T}=0, \\ X_1\delta_{21}+X_2\delta_{22}+\ldots+X_n\delta_{2n}+\Delta_{2T}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_1\delta_{n1}+X_2\delta_{n2}+\ldots+X_n\delta_{nn}+\Delta_{nT}=0, \end{array}\right\} \quad (11.1)$$

where the subscript T on the free terms indicates that the displacement is caused by temperature.

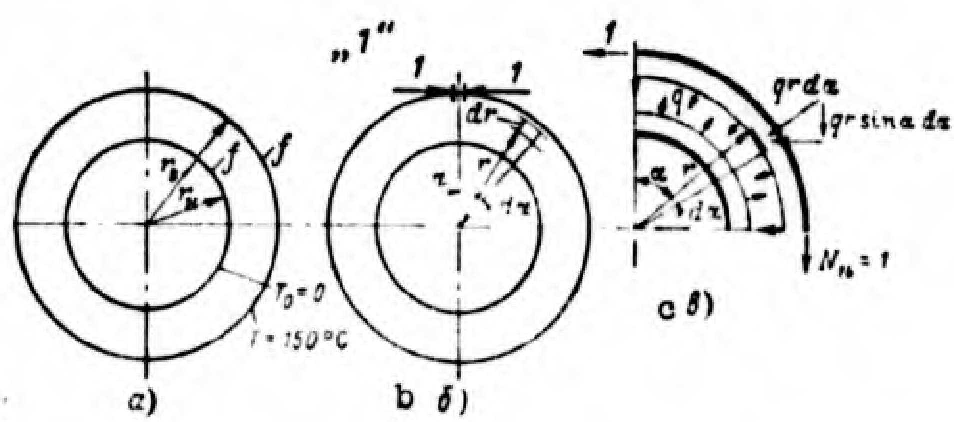


Fig. 11.1

For a bar of length ℓ, the displacement produced by a temperature change $T - T_0 = \Delta T$ will equal its elongation:

$$\Delta l=\alpha(T-T_0)l=\alpha\Delta Tl, \tag{11.2}$$

where α is the linear coefficient of thermal expansion; T_0 is the initial temperature; T is the final temperature.

We shall first consider the determination of thermal stresses in plane systems. We shall find, for example, the stressed state in an annular bulkhead, as shown in Fig. 11.1a, under the condition that the skin and internal flange have zero initial and final temperature while the outer flange is heated from $T_0 = 0$ to $T = 150°$. The material is duralumin, the cross-sectional area of each flange is $\underline{f}$, the skin thickness is δ.

We use a fundamental system with a cut in the outer flange (Fig. 11.1b). On the assumption that the skin works in tension – compression only in the radial direction, we obtain stresses in the flanges $N_{1v} = - N_{1n} = 1$ for the unit state. The per-unit length tensile forces $\underline{q}$ in the skin are shown in Fig. 11.1c for the skin section in the first quadrant. From the equilibrium conditions on the vertical axis we obtain

$$1+\int_0^{\pi} qr\sin\alpha\, d\alpha=0,$$

from which we have

$$1+qr=0.$$

This means that in the unit state the skin will be compressed by forces

$$q=-\frac{1}{r}.$$

The unit displacement is

$$\delta_{11}=\int_0^{2\pi}\frac{N_{1v}^2 r_v\, d\alpha}{Ef}+\int_0^{2\pi}\frac{N_{1n}^2 r_n\, d\alpha}{Ef}+\int_{r_n}^{r_v}\int_0^{2\pi}\frac{\sigma_{\text{общ}}^2\,\delta\, dF}{E},$$

where

$$\sigma_{общ} = -\frac{q}{\delta} = -\frac{1}{\delta r} \quad \text{and} \quad dF = r\,d\alpha\,dr.$$

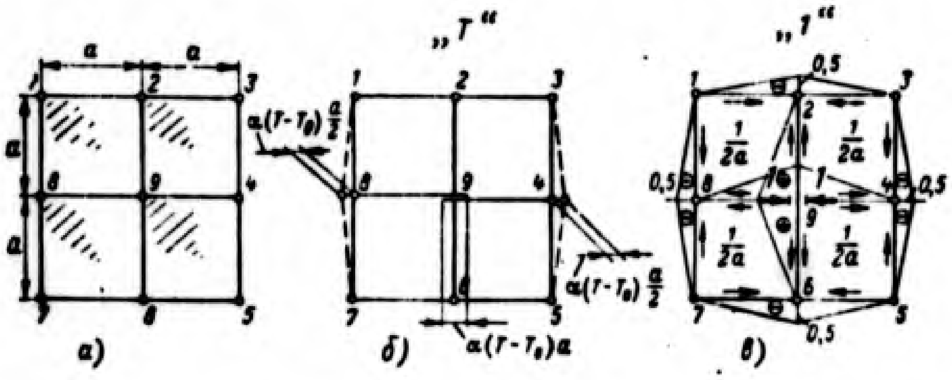


Fig. 11.2

After integration we obtain

$$\delta_{11} = \frac{2\pi r_в}{Ef} + \frac{2\pi r_н}{Ef} + \frac{2\pi}{E\delta} \ln \frac{r_в}{r_н},$$

or

$$\delta_{11} = \frac{2\pi}{Ef}\left(r_в + r_н + \frac{f}{\delta} \ln \frac{r_в}{r_н}\right).$$

In view of the fact that in the cut the displacement due to the temperature is

$$\Delta_{1T} = \alpha \Delta T 2\pi r_в,$$

we obtain

$$X \frac{2\pi}{Ef}\left(r_в + r_н + \frac{f}{\delta} \ln \frac{r_в}{r_н}\right) + \alpha\, 2\pi r_в \Delta T = 0.$$

From which we have

$$X = -\frac{\alpha \Delta T r_в E f}{r_в + r_н + \frac{f}{\delta} \ln \frac{r_в}{r_н}}.$$

If $r_v = 50$ cm, $r_n = 30$ cm, $f = 3.0\ \text{cm}^2$, $\delta = 0.1$ cm, then for $E = 7 \cdot 10^5\ \text{kgf/cm}^2$, $\alpha = 22 \cdot 10^{-6}$ and $\Delta T = 150°C$ we obtain

$$X = -\frac{22 \cdot 10^{-6} \cdot 150 \cdot 50 \cdot 7 \cdot 10^5 \cdot 3}{50 + 30 + \frac{3}{0.1} \ln \frac{50}{30}} = -3640 \text{ kgf}.$$

As a consequence,

$$N_в = -3640\ кГ,\quad \sigma_в = -1215\ \mathrm{kgf/cm^2}$$
$$N_н = 3640\ кГ,\quad \sigma_н = 1215\ \mathrm{kgf/cm^2}$$
$$\sigma^{н}_{обш} = \frac{3640}{\delta \cdot r_н} = \frac{3640}{0{,}1 \cdot 30} = 1215\ \mathrm{kgf/cm^2}$$

The increase in temperature affects the mechanical properties of the material and, in particular, the modulus of elasticity, which for duralumin drops from $E = 7 \cdot 10^5$ kgf/cm^2 at room temperature to $E_{150} =$ $= 5.7 \cdot 10^5$ kgf/cm^2 at T = 150°C.

Using this value of the modulus for α = const = $22 \cdot 10^{-6}$, we obtain

$$X = -\frac{22 \cdot 10^{-6} \cdot 150 \cdot 50 \cdot 7 \cdot 10^5 \cdot 3}{\frac{7}{5{,}7} 50 + 30 + \frac{3}{0{,}1} \ln \frac{50}{30}} = -3270\ \mathrm{kgf}$$

In order to gain a better understanding of the physical meaning of problems involving the determination of thermal stresses, let us consider another such example. As in the preceding example, let the initial and final temperatures of all members of the system shown in Fig. 11.2a equal zero, except for the center flange 8-9-4, which is heated from T_0 = 0°C to T = 150°C. The material is duralumin, the cross-sectional area of each strip is f = 5 cm^2, the skin thickness is everywhere δ = = 0.1 cm, and a = 60 cm. We also assume in this case that the skin works solely in shear.

The system is statically indeterminate with one redundant constraint. We obtain the fundamental system by making a cut at joint 9 in strip 8-9-4 (Fig. 11.2c). The displacements for thermal state "T" is shown in Fig. 11.2b.

The displacement of the unit state (see Fig. 11.2c) will be:

$$\delta_{11} = 4\frac{1}{4a^2}\frac{a^2}{G\delta} + 8\frac{a}{4}\frac{2}{3}\frac{1}{2}\frac{1}{Ef} + 2\frac{a}{2}\frac{2}{3}\frac{1}{Ef} +$$
$$+ 2\frac{a}{2}\frac{2}{3}\frac{1}{E_{150}f} = \frac{1}{G\delta} + \frac{2}{3}\frac{a}{Ef} + \frac{2}{3}\frac{a}{Ef} + \frac{2}{3}\frac{a}{E_{150}f} =$$
$$= \frac{1}{G\delta} + \frac{4}{3}\frac{a}{Ef} + \frac{2}{3}\frac{a}{E_{150}f}.$$

For

$$G=\frac{E}{2,6}=\frac{7\cdot10^5}{2,6}=2,7\cdot10^5\ \text{kgf/cm}^2$$

we obtain

$$\delta_{11}=\frac{1}{10^5}\left(\frac{1}{2,7\cdot0,1}+\frac{4\cdot60}{3\cdot7\cdot5}+\frac{2\cdot60}{3\cdot5,7\cdot5}\right)=\frac{7,4}{10^5}.$$

The stress in the center strip may be obtained from the equation

$$X\delta_{11}+\Delta_{1T}=0$$

or

$$X=-\frac{\alpha(T-T_0)a}{\delta_{11}}=\frac{22\cdot10^{-6}\cdot150\cdot60\cdot10^5}{7,4}=-2680\ \text{kgf}.$$

Multiplying state "1" by X, we obtain the stressed state for the entire system.

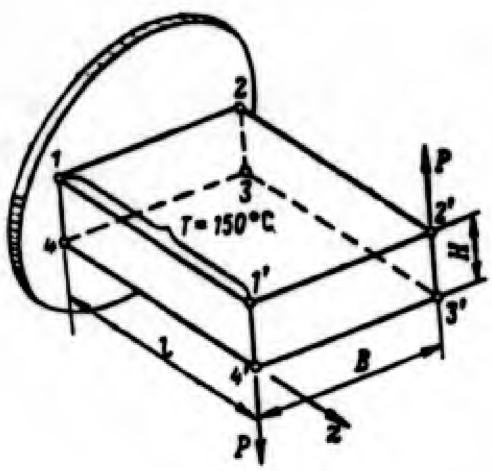


Fig. 11.3

When thermal and external forces act simultaneously on the structure, the free terms in equation system (11.1) will consist of the following sums: $\Delta_{1P}+\Delta_{1T}$, $\Delta_{2P}+\Delta_{2T}$, ... $\Delta_{nP}+\Delta_{2T}$, where Δ_{1P}, Δ_{2P},..., Δ_{nP} are the displacements due to the force factors produced by the external forces.

As an example of the determination of the stressed and strained states under simultaneous action of thermal and external forces on a structure, we shall consider the tip portion of a rigidly clamped four-flange shell, loaded by a torsional moment in the form of a couple (Fig. 11.3) under the condition that the initial and final temperatures of all shell members equal T_0, except for flange 1-1', which is heated from T_0 to T; the shell has an identical thickness δ for the spar webs and the skin, and identical flange areas f.

We shall perform an approximate calculation on the assumption that

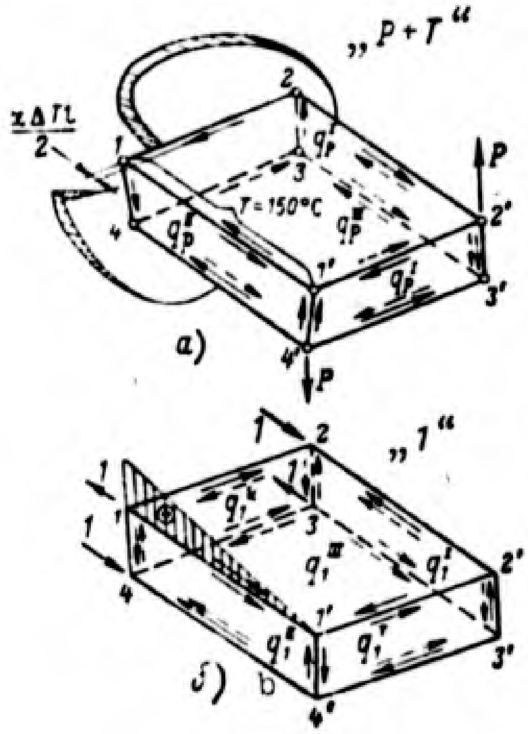


Fig. 11.4

the skin works solely in shear, and that the ribs are perfectly rigid in their own plane.

In the calculations we assume that the axial forces vary linearly along the z axis from the maximum value at section 1-2-3-4 to zero in section 1'-2'-3'-4'. We recall that with this type of axial-force variation, the PKS remains constant. We thus reduce the given structure to a statically indeterminate system with one redundant constraint. Disconnecting joint 1 from the support, we obtain the fundamental S.O. system.

In state "P + T" (Fig. 11.4a), the PKS will be

$$q_P^{\mathrm{I}} = q_P^{\mathrm{II}} = q_P^{\mathrm{III}} = q_P^{\mathrm{IV}} = q_P^{T} = \frac{\mathfrak{M}}{\Omega} = \frac{PB}{2HB} = \frac{P}{2H},$$

where q_P^{IV} is the PKS for the lower skin.

In state "1" (Fig. 11.4b), from the equilibrium conditions for spar 1-1'-4'-4 the PKS will be

$$q_1^{\mathrm{I}} = q_1^{\mathrm{III}} = q_1^{\mathrm{IV}} = q_1^{T} = -\frac{1H}{2lH} = -\frac{1}{2l}.$$

From the equilibrium conditions for braces 1'-4' and 2'-3'

$$q_1^{11}=\frac{1}{2l} \text{ and } q_1^{1}=\frac{1}{2l}.$$

The unknown value of X is found from the equation

$$X\delta_{11}+\Delta_{1P}+\Delta_{1T}=0.$$

Since

$$\delta_{11}=\sum\frac{q_1^2F}{G\delta}+\sum\int\frac{N_1^2\,ds}{Ef}=\frac{B+H}{2l\delta G}+\frac{l}{E_0f}+\frac{l}{3E_Tf}; \quad (1)$$

$$\Delta_{1P}=\sum\frac{q_1q_PF}{G\delta}=-\frac{P(B-H)}{2HG\delta}; \quad (2)$$

$$\Delta_{1T}=\frac{\alpha\Delta Tl}{2}, \quad (3)$$

we then obtain

$$X=X_P+X_T.$$

Here

$$X_P=-\frac{\Delta_{1P}}{\delta_{11}}; \quad (4)$$

$$X_T=-\frac{\Delta_{1T}}{\delta_{11}}. \quad (5)$$

Thus, in the actual state, the axial forces will be

$$N=X_P+X_T \quad (6)$$

and the PKS will be

$$q^{1}=q^{11}=\frac{P}{2H}+\frac{N}{2l}; \quad (7)$$

$$q^{111}=q^{1V}=\frac{P}{2H}-\frac{N}{2l}. \quad (8)$$

We now determine the torsion angle for the end section of the shell:

$$\varphi=\varphi_1+\varphi_2+\varphi_3,$$

where φ_1 is the torsion angle in free torsion; φ_2 is the torsion angle due to the effect of the attachment when only external forces are acting; φ_3 is the torsion angle due to the effect of the attachment when temperature alone is acting.

From the formula

$$\varphi=\frac{l}{\Omega}\sum\frac{q\Delta s}{G\delta}$$

we obtain

$$\varphi_1=\frac{l}{\Omega}\,\frac{P2(B+H)}{2HG\delta};$$

$$\varphi_2=\frac{l}{\Omega}\,\frac{X_P 2(H-B)}{2lG\delta};\quad \varphi_3=\frac{l}{\Omega}\,\frac{X_T 2(H-B)}{2lG\delta};$$

or

$$\varphi_1=\frac{Pl(B+H)}{H\Omega G\delta};$$

$$\varphi_2=-\frac{X_P(B-H)}{\Omega G\delta};\quad \varphi_3=-\frac{X_T(B-H)}{\Omega G\delta},$$

where

$$\Omega=2HB.$$

As a consequence,

$$\varphi=\frac{Pl(B+H)}{H\Omega G\delta}-\frac{X_P(B-H)}{\Omega G\delta}-\frac{X_T(B-H)}{\Omega G\delta}. \tag{9}$$

In the given case for $B > H$, the value of X_P proves to be positive, while X_T is negative and, as a consequence, the second term in Eq. (9) will reduce the angle of twist in free torsion, while the temperature-dependent term will increase it.

In order to determine the effective temperature, we determine the force factors and torsion angle φ for the shell under the following design conditions: $P = 1000$ kgf, $B = 100$ cm, $H = 20$ cm, $l = 100$ cm, $f = 5$ cm^2, $\delta = 0.1$ cm, $T_0 = 0°C$, $T = 150°C$, $E_0 = 7\cdot10^5$, $E_{150} = 5.7\cdot10^5$, $\alpha = 22\cdot10^{-6}$, and $E_0/G_0 = 2.6$.

From Formulas (1), (2), (3):

$$\delta_{11}=\frac{(100+20)\,2{,}6}{2\cdot100\cdot0{,}1\cdot7\cdot10^5}+\frac{100}{7\cdot10^5\cdot5}+\frac{100}{3\cdot5{,}7\cdot10^5\cdot5}=\frac{6{,}26}{10^5};$$

$$\Delta_{1P}=-\frac{1000\,(100-20)\,2{,}6}{2\cdot20\cdot7\cdot10^5\cdot0{,}1}=-\frac{7440}{10^5};$$

$$\Delta_{1T}=\frac{22\cdot10^{-6}\cdot150\cdot100}{2}=\frac{16\,500}{10^5}.$$

As a consequence,

$$X_p = \frac{7440}{6{,}26} = 1190 \text{ kgf};$$

$$X_T = -\frac{16\,500}{6{,}26} = -2640 \text{ kgf};$$

$$N = 1190 - 2640 = -1450 \text{ kgf};$$

$$q^{I} = q^{II} = \frac{1000}{2\cdot 20} + \frac{1450}{2\cdot 100} = 32{,}25 \text{ kgf/cm}$$

$$q^{III} = q^{IV} = \frac{1000}{2\cdot 20} - \frac{1450}{2\cdot 100} = 17{,}75 \text{ kgf/cm}$$

$$\varphi_1 = \frac{1000\cdot 100\cdot 120\cdot 2{,}6}{2\cdot 100\cdot 20\cdot 20\cdot 7\cdot 10^5\cdot 0{,}1} = \frac{557{,}5}{10^5};$$

$$\varphi_2 = -\frac{1190\cdot 80\cdot 2{,}6}{2\cdot 100\cdot 20\cdot 7\cdot 10^5\cdot 0{,}1} = -\frac{88{,}5}{10^5};$$

$$\varphi_3 = \frac{2640\cdot 80\cdot 2{,}6}{2\cdot 100\cdot 20\cdot 7\cdot 10^5\cdot 0{,}1} = \frac{197}{10^5}.$$

Thus,

$$\varphi = \frac{1}{10^5}(557{,}5 - 88{,}5 + 197) = \frac{1}{10^5}(469 + 197).$$

From this it follows that in our case the torsion angle will be increased by about 40% by the temperature.

As we know, not only the quantity E but also the coefficient α depend on temperature. In order to avoid complicating the basic calculations involved in determining the thermal stresses, we shall henceforth assume that E and α are constant, since within the range over which it is desirable to employ a material at elevated temperatures the modulus E decreases slowly as the temperature increases, while the coefficient α slowly rises. Thus the accuracy of the calculation for E = const α = = const will not decrease sharply, since the values of the unknowns X_1, X_2, ...,X_n depend on the product of these quantities.

2. APPLICATION OF THE METHOD OF FINITE DIFFERENCES TO THE CALCULATION OF THERMAL STRESSES IN A WING WITH A THIN SKIN

Using equations in finite differences, we shall determine the stressed state of a four-flange wing with diamond profile (Fig. 11.5). The wing has a skin thickness δ and identical flange areas $\underline{f}$. The temperature of the leading and trailing flanges is greater than the temperature of the remaining wing members by an amount ΔT°C. Since the skin is thin, we shall assume that it works solely in shear.

On the assumption that the ribs are perfectly rigid, we obtain the following finite-difference equation for any section m:

$$X_{m-1}\delta_{m,\,m-1}+X_m\delta_{m,\,m}+X_{m+1}\delta_{m,\,m+1}+\Delta_{m,\,T}=0. \qquad (11.3)$$

It is clear from Fig. 11.6 that

$$\delta_{m,\,m}=\frac{8}{3}\,\frac{\Delta z}{Ef}+2\,\frac{b}{\Delta z G\delta};$$

$$\delta_{m,\,m-1}=\delta_{m,\,m+1}=\frac{2}{3}\,\frac{\Delta z}{Ef}-\frac{b}{\Delta z G\delta},$$

where Δz is the length of the span between ribs.

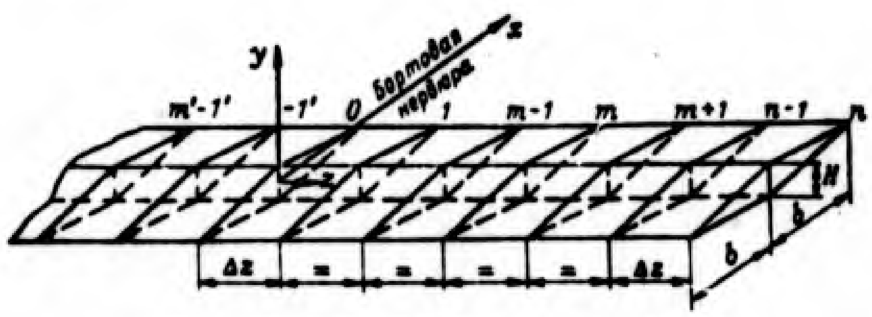


Fig. 11.5. A) Inboard rib.

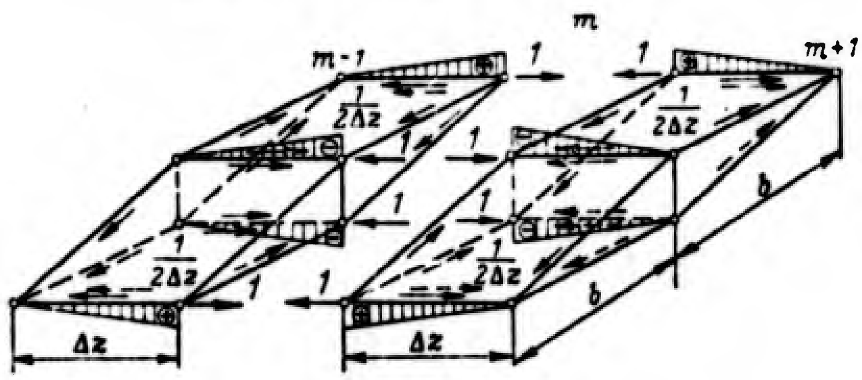


Fig. 11.6

In view of the fact that

$$\Delta\delta_{mT}=\alpha\Delta T\Delta z,$$

for E/G = 2.6 we obtain

$$E\delta_{m,\,m-1}X_{m-1}+E\delta_{m,\,m}X_m+E\delta_{m,\,m+1}X_{m+1}+E\Delta_{mT}=0. \qquad (11.4)$$

where

$$E\delta_{m,m}=\frac{8}{3}\,\frac{\Delta z}{f}+\frac{5{,}2b}{\Delta z\delta};$$

$$E\delta_{m,m-1}=E\delta_{m,m+1}=\frac{2}{3}\,\frac{\Delta z}{f}-\frac{2{,}6b}{\Delta z\delta};$$

$$E\Delta_{m,T}=\Delta zk;$$

$$k=\alpha\Delta TE.$$

Using Eq. (11.4) for section 0, 1, ..., n − 1 (see Fig. 11.5), we obtain n equations to determine the unknowns.

While the system obtained in this manner will contain a large number of equations, there is no fundamental difficulty in its application. We shall illustrate the use of Eq. (11.4) for a limited number of unknowns so that hand computation will be possible.

Variation 1. Let n = 5 and bays 0, −1 and n, n − 1 (see Fig. 11.5) be perfectly rigid. We let δ = 0.1 cm, f = 5 cm^2, b = 50 cm, Δz = 30 cm. Using Eq. (11.4) for section 0, we obtain

$$E\delta_{0,-1'}=0;$$

$$E\delta_{0,0}=\frac{4}{3}\,\frac{30}{5}+\frac{2{,}6\cdot 50}{30\cdot 0{,}1}=8+43{,}4=51{,}4;$$

$$E\delta_{0,1}=\frac{2}{3}\,\frac{30}{5}-\frac{2{,}6\cdot 50}{30\cdot 0{,}1}=4-43{,}4=-39{,}4;$$

$$E\Delta_{0,T}=15k.$$

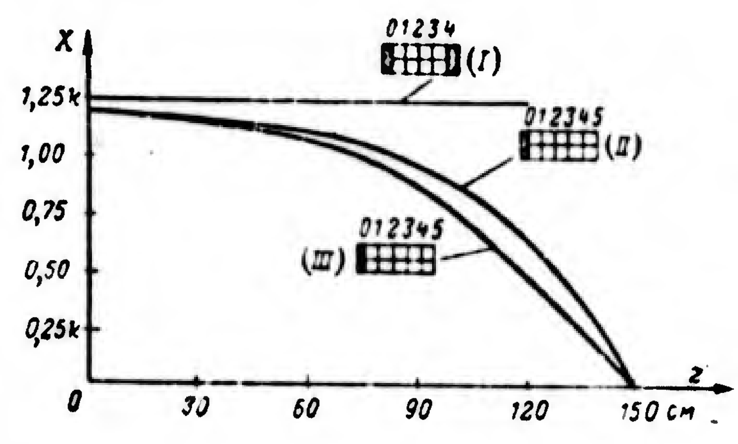


Fig. 11.7

As a consequence, for section 0 Eq. (11.4) will take the form

$$51{,}4X_0-39{,}4X_1+15k=0. \qquad (a)$$

Applying Eq. (11.4) to sections 1, 2, 3, 4 we obtain

$$\left.\begin{array}{lr} -39{,}4X_0-102{,}8X_1-39{,}4X_2+\ .\ .\ .\ +30k=0, & (1)\\ \quad -39{,}4X_1+102{,}8X_2-39{,}4X_3+\ .\ .\ .\ +30k=0, & (2)\\ \quad -39{,}4X_2+102{,}8X_3-39{,}4X_4+30k=0, & (3)\\ \quad -39{,}4X_3+51{,}4X_4+15k=0. & (4)\end{array}\right\} \quad (b)$$

Solving Eq. (a) as well as (1), (2), (3) and (4) of System (b) simultaneously, we obtain

$$X_0=X_1=X_2=X_3=X_4=-1{,}25k.$$

In accordance with our boundary conditions, these unknowns correspond to complete limitation of thermal expansion and, as a consequence, are the maximum values.

Variation 2. If we assume that only bay 0, −1' is perfectly rigid, we then obtain the following system of equations:

$$\left.\begin{array}{lr} +51{,}4X_0-39{,}4X_1+\ .\ .\ .\ +15k=0, & (0)\\ -39{,}4X_0+102{,}8X_1-39{,}4X_2+\ .\ .\ .\ +30k=0, & (1)\\ \quad -39{,}4X_1+102{,}8X_2-39{,}4X_3+\ .\ .\ .\ +30k=0, & (2)\\ \qquad -39{,}4X_2+102{,}8X_3-39{,}4X_4+30k=0, & (3)\\ \qquad\quad -39{,}4X_1+102{,}8X_4+30k=0. & (4)\end{array}\right\} \quad (c)$$

Solving these equations simultaneously we have

$$X_0=-1{,}195k;\quad X_1=-1{,}178k;\quad X_2=-1{,}117k;$$
$$X_3=-0{,}9752k;\quad X_4=-0{,}6656k.$$

Variation 3. Let bay n, n − 1 be elastic, but loaded by a fuel tank; as a consequence, we may assume that it is completely "cold." In this case, equation system (c) remains as before, except for Eq. (4) which now takes the form

$$-39{,}4X_3+125{,}8X_4+15k=0. \qquad (4')$$

Solving Eqs. (0), (1), (2), (3) of System (c) and (4') simultaneously, we have

$$X_0=-1{,}178k;\ X_1=-1{,}1559k;\ X_2=-1{,}0765k;$$
$$X_3=-0{,}8915k;\ X_4=-0{,}4876k.$$

Comparing the curves of Fig. 11.7, we can see how the boundary conditions affect the values of the unknowns.

Chapter 12

APPLICATION OF THE KANTOROVICH-VLASOV METHOD TO CALCULATION OF THERMAL STRESSES

We shall consider a system of the wing type having a width 2b and length 2ℓ. We assume that the temperature distribution over the wing is in accordance with the three-dimensional graph of Fig. 12.1a.

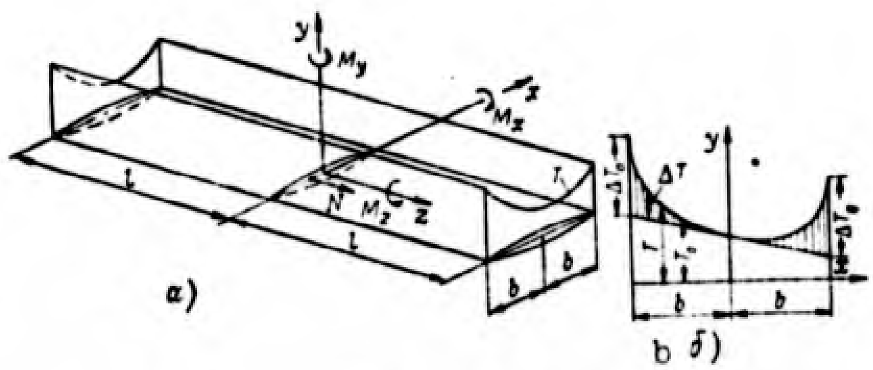


Fig. 12.1

We further assume that the wing may expand freely in the directions of the x and y axes, while thermal expansion is completely restricted in the z-axis direction. Thus the system cross sections remain plane during heating. As a consequence, deformations at any point of a cross section will be determined by the equation of a plane:

$$\varepsilon + \alpha T = a + bx + cy,$$

where the ε are the elastic deformations; the αT are the thermal deformations under free thermal expansion; a, b, c are coefficients determining the position of the cross-section plane after heating of the wing.

In accordance with Hooke's law, the thermal stresses are

$$\sigma_z^T = E\varepsilon = E(a + bx + cy - \alpha T).$$

Since the wing is not loaded by external forces, only selfbalancing force factors can appear in any cross section, so that

$$\sum Z=\int_F \sigma_z^T dF=E\int_F (a+bx+cy-\alpha T)\,dF=0; \qquad (1)$$

$$\sum M_x=\int_F \sigma_z^T y\,dF=E\int_F (a+bx+cy-\alpha T)\,y\,dF=0; \qquad (2)$$

$$\sum M_y=\int_F \sigma_z^T x\,dF=E\int_F (a+bx+cy-\alpha T)\,x\,dF=0. \qquad (3)$$

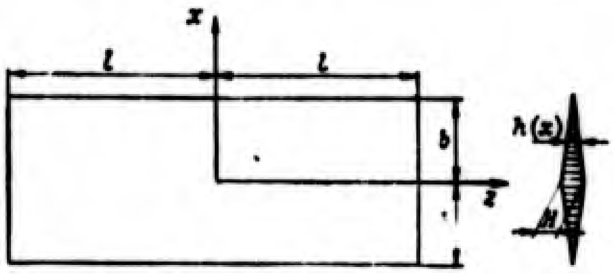


Fig. 12.2

After integration and manipulation, we obtain from these three equations the values of the coefficients

$$a=\frac{N}{F};\quad b=-\frac{M_y}{J_y};\quad c=\frac{M_x}{J_x}$$

and the stresses

$$\sigma_z^T=-\alpha TE+\frac{N}{F}+\frac{M_x}{J_x}y-\frac{M_y}{J_y}x, \qquad (12.1)$$

where

$$N=\alpha E\int T\,dF;\quad M_x=\alpha E\int Ty\,dF;\quad M_y=\alpha E\int Tx\,dF;$$
$$J_x=\int y^2 dF;\quad J_y=\int x^2 dF;$$

here F is the area of the entire wing cross section.

The first term ($-\alpha TE$) in Eq. (12.1) consists of stresses that ensure complete restriction of thermal expansion. The results in force due to the stresses ($-N$) is balanced by the stresses N/F, distributed uniformly over the section. In the general case, the force N acts with a certain eccentricity with respect to the principle central axes x and y, so that bending moments M_x and M_y appear; in accordance with the last two terms of Eq. (12.1), these moments are balanced by forces produced by the stresses distributed in accordance with the law of plane sections.

Since we are considering a free wing, for convenience in the calculations, it is desirable to separate the over-all temperature T into two parts: T_0 and ΔT (Fig. 12.1b). The temperature T_0, which varies along the chord in accordance with a linear law, is noteworthy in that it causes fundamental deformations of the wing without stresses.

In fact, let the wing have a diamond-shaped profile (Fig. 12.2) and be acted on by a temperature

$$T_0 = t - \Delta t \frac{x}{b}.$$

The components of T_0 are shown graphically in Fig. 12.3.

The stress σ_z^T is found from the formula

$$\sigma_z^T = -\alpha T_0 E + \frac{N}{F} - \frac{M_y}{J_y} x.$$

In view of the fact that the thickness of the diamond profile is

$$\left.\begin{aligned} h(x) &= H\left(1 - \frac{x}{b}\right) \quad (0 \leqslant x \leqslant b) \\ \text{and} \qquad h(x) &= H\left(1 + \frac{x}{b}\right) \quad (0 \geqslant x \geqslant -b), \end{aligned}\right\} \tag{12.2}$$

where H is the maximum profile thickness, we obtain

$$N = \alpha E \int_{-b}^{+b} T_0 h(x)\, dx = \alpha E \left[\int_{-b}^{0} \left(t - \Delta t \frac{x}{b}\right) H\left(1 + \frac{x}{b}\right) dx + \right.$$

$$\left. + \int_{0}^{+b} \left(t - \Delta t \frac{x}{b}\right) H\left(1 - \frac{x}{b}\right) dx \right] = \alpha t E H b.$$

Thus

$$\frac{N}{F} = \frac{\alpha t E H b}{H b} = \alpha t E;$$

$$M_y^T = -\alpha E \int_{-b}^{+b} T_0 x h(x)\, dx = \alpha \Delta t E \frac{H b^2}{6};$$

$$J_y = \int_{-b}^{+b} x^2 h(x)\,dx = \frac{Hb^3}{6}.$$

As a consequence,

$$\frac{M_y}{J_y} = \frac{\alpha \Delta t E}{b}.$$

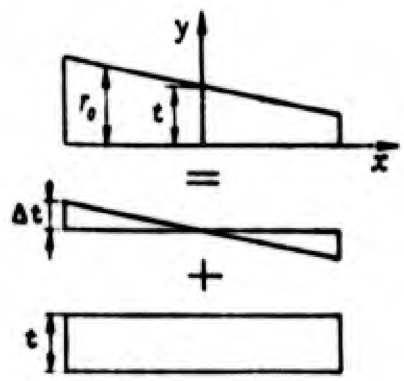


Fig. 12.3

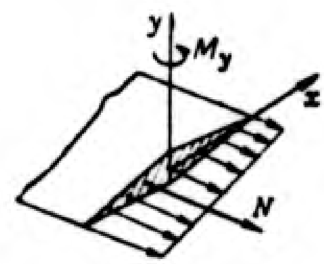


Fig. 12.4

Thus

$$\sigma_z^T = -\alpha\left(t - \Delta t \frac{x}{b}\right)E + \alpha t E - \frac{\alpha \Delta t E x}{b} = 0.$$

For $\sigma_z^T = 0$ the remaining stress components will also equal zero, so that the relative deformations ε_x and ε_z will equal only the relative thermal expansion αT_0.

Taking into account Formula (4.4) and $\gamma_{zx} = \tau_{zx}/G = 0$, we obtain

$$\frac{\partial u}{\partial x} = \alpha T_0; \quad \frac{\partial w}{\partial z} = \alpha T_0; \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0. \qquad \text{(a)}$$

Integrating the first two equations, we obtain

$$u = \alpha \int T_0 dx + f(z) = \alpha\left(tx - \Delta t \frac{x^2}{2b}\right) + f(z);$$
$$w = \alpha T_0 z + f(x).$$

In order to use the third equation of System (a) to determine the unknown functions f(z) and f(x), we must first find

$$\frac{\partial w}{\partial x} = -\frac{\alpha \Delta t z}{b} + \frac{\partial f(x)}{\partial x};$$
$$\frac{\partial u}{\partial z} = \frac{\partial f(z)}{\partial z}.$$

As a consequence,

$$\left[\frac{\partial f(z)}{\partial z}-\frac{\alpha\Delta tz}{b}\right]+\frac{\partial f(x)}{\partial x}=0.$$

The function in brackets depends only on z, the function $\partial f(x)/\partial x$ depends only on x, and the sum of these functions equals zero. This means that they are equal in magnitude to the constant C, but are of unlike sign.

Thus if

$$\frac{\partial f(z)}{\partial z}-\frac{\alpha\Delta tz}{b}=C_1, \qquad \text{(b)}$$

then

$$\frac{\partial f(x)}{\partial x}=-C_1. \qquad \text{(c)}$$

Integrating these equations, we obtain

$$f(z)=\alpha\Delta t\frac{z^2}{2b}+C_1z+C_2;$$

$$f(x)=-C_1x+C_3.$$

The linear two-term expressions involved here represent rigid displacement of the wing. Discarding them we obtain

$$u=\alpha\left(tx-\Delta t\,\frac{x^2}{2b}\right)+\alpha\Delta t\,\frac{z^2}{2b}=\alpha\,\frac{\Delta t}{2b}\,(z^2-x^2)+\alpha tx$$

and

$$w=\alpha\left(t-\Delta t\,\frac{x}{b}\right)z.$$

It is clear from these expressions that the equation for the bent axis of the wing is a parabola, while the cross sections, rotating with respect to one another, remain planes (Fig. 12.5).

Thus we have seen that for a linear temperature variation, a free wing will be deformed without stresses.

Let us now look at the second part of the temperature T – at ΔT, which varies along the curve. As a consequence, any cross section m – n (see Fig. 12.5) will tend to leave its plane, but part A of the wing restrains this warping, and this is the reason for the appearance of thermal stresses. We thus obtain a correct expression for σ_z even where Eq. (12.1) contains ΔT in place of the function T.

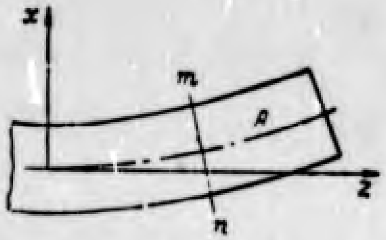


Fig. 12.5

The curve for the temperature T along the wing chord is asymmetric about the y axis, while ΔT is also asymmetric to a certain degree. As a consequence, the component terms of ΔT will be both odd and even. It is possible, however, in many cases to approximate ΔT fairly accurately by just a single term of a power function with an even power n, or by a single term of some other function, for example, a trigonometric function.

Using ρ to represent the approximating function for ΔT, we obtain

$$\Delta T = \Delta T_0 \rho.$$

After substituting $\Delta T_0 \rho$ into Eq. (12.1) in place of T, we find that

$$\sigma_z^T = k\psi(x), \qquad (12.3)$$

where $k = \alpha \Delta T_0 E$ are the stresses ensuring total restraint of thermal expansion;

$$\psi = -\rho + \frac{\int \rho\, dF}{F} + \frac{\int \rho y\, dF}{J_x} y - \frac{\int \rho x\, dF}{J_y} x. \qquad (12.4)$$

Here ψ is a known dimensionless function ensuring that the forces in the wing cross sections will be selfbalancing.

We take as the fundamental stresses those given by Formula (12.3), on the assumption that they occur in any wing cross section. Such a stress state, however, will occur only for a wing of infinite span. Such a state will be called state T.

The solution of (12.3) cannot be used for a wing of small aspect ratio, since at the free ends for $z = \ell$ and $z = -\ell$, the thermal stresses will equal zero. In order to satisfy these boundary conditions, we apply to the ends of the wing the stresses

$$\sigma^{*}_{z=\pm l} = -\sigma^{T}_{z}$$

The running valu of these stresses is represented in the following form:

$$\sigma^{*}_{z} = -\sigma^{T}_{z}\varphi(z)$$

or

$$\sigma^{*}_{z} = -k\varphi(z)\psi(x), \qquad (12.5)$$

where φ(z) is the unknown function, and depends solely on the coordinate z.

The actual stresses are

$$\sigma_{z} = \sigma^{T}_{z} + \sigma^{*}_{z} \qquad (12.6)$$

or

$$\sigma_{z} = k(1-\varphi(z))\psi(x). \qquad (12.7)$$

In the next two sections, we shall show how to determine the function φ(z) with the aid of the Kantorovich-Vlasov method in calculating the thermal stresses for a rectangular plate and a solid wing with diamond-shaped profile.

1. THERMAL STRESSES IN A RECTANGULAR PLATE

Let us determine the thermal stresses σ_z in a rectangular plate of length 2ℓ, width 2b, and thickness h (Fig. 12.6).

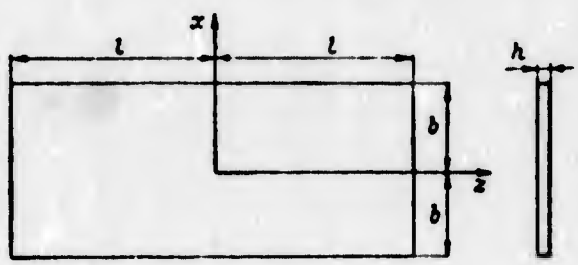


Fig. 12.6

According to (12.6)

$$\sigma_{z} = \sigma^{T}_{z} + \sigma^{*}_{z},$$

where

$$\sigma_z^T = k\psi(x),$$

while

$$\sigma_z^\alpha = -k\varphi(z)\psi(x).$$

Assuming that we know the values $k = \alpha\Delta T_0 E$ and $\psi(x)$, we find that the stresses σ_z^α.

The stressed state due to σ_z^α in an element dx dz taken from an elementary strip of the plate is shown in Fig. 12.7.

Of the three unknowns σ_z^α, σ_x, and τ_{xz}, only the stress σ_z^α is statically indeterminate, since σ_x and τ_{xz} may be represented as functions of σ_z^α using two equations of statics (4.1), which, neglecting body forces, have the form:

$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} = 0; \qquad (1)$$

$$\frac{\partial\tau_{xz}}{\partial z} + \frac{\partial\sigma_x}{\partial x} = 0. \qquad (2)$$

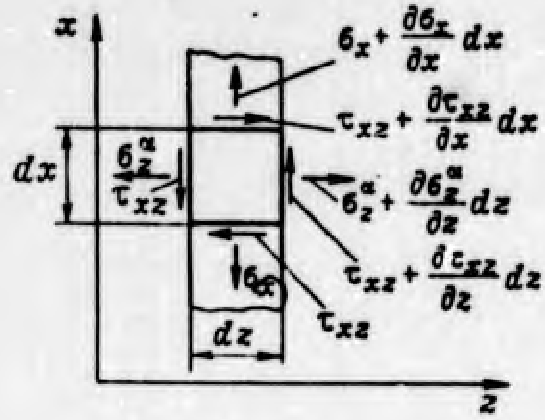


Fig. 12.7

From Eq. (1)

$$\tau_{xz} = -\int_b^x \frac{d\sigma_z^\alpha}{dz}\,dx$$

or

$$\tau_{xz} = k\varphi' \int_b^x \psi\,dx,$$

where

$$\varphi' = \frac{d\varphi(z)}{dz}.$$

From Eq. (2)

$$\sigma_x = -\int_b^x \frac{d\tau_{xz}}{dz}\, dx = -\varphi'' \int_b^x \int_b^x \psi\, dx\, dx.$$

The strain potential energy is

$$\bar{U} = 2\int_0^l \Gamma\, dz,$$

where

$$\Gamma = h\int_{-b}^{+b} \left[\frac{1}{2E}(\sigma_z^2 + \sigma_x^2) - \frac{\mu\sigma_x\sigma_z}{E} + \frac{\tau_{xz}^2}{2G}\right] dx.$$

It can be shown that for the assumed form of the function $\psi(x)$, the integral

$$\int_{-b}^{+b} \mu \frac{\sigma_x\sigma_z}{E}\, dx = 0.$$

Thus the last expression will take the form:

$$\Gamma = h\int_{-b}^{+b} \left[\frac{1}{2E}(\sigma_z^2 + \sigma_x^2) + \frac{\tau_{xz}^2}{2G}\right] dx.$$

Substituting the values of σ_z, σ_x, and τ_{xz} into this expression, we obtain

$$\Gamma = \frac{\delta k^2\varphi^2}{2E}\int_{-b}^{+b} \psi^2\, dx + \frac{\delta k^2\varphi''^2}{2E}\int_{-b}^{+b} \left(\int_b^x \int_b^x \psi\, dx\, dx\right)^2 dx +$$

$$+ \frac{\delta k^2\varphi'^2}{2G}\int_{-b}^{+b} \left(\int_b^x \psi\, dx\right)^2 dx. \qquad (12.8)$$

After integration, we obtain the following constant quantities:

$$\beta = \int_{-b}^{+b} \psi^2\, dx; \quad \gamma = \int_{-b}^{+b} \left(\int_b^x \int_b^x \psi\, dx\, dx\right)^2 dx; \quad \lambda = \int_{-b}^{+b} \left(\int_b^x \psi\, dx\right)^2 dx.$$

Substituting these values into Expression (12.8), we obtain

$$\Gamma = \frac{hk^2}{2E}(\beta\varphi^2 + \gamma\varphi''^2 + 2\lambda\varphi'^2).$$

Minimizing the system potential-energy functional, we obtain the Euler equation in the form:

$$\frac{\partial\Gamma}{\partial\varphi}-\frac{d}{dz}\frac{\partial\Gamma}{\partial\varphi'}+\frac{d^2}{dz^2}\frac{\partial\Gamma}{\partial\varphi''}=0,$$

where

$$\frac{\partial\Gamma}{\partial\varphi}=\frac{hk^2}{2E}2\beta\varphi;\quad \frac{\partial\Gamma}{\partial\varphi'}=\frac{hk^2}{2E}4\lambda\varphi';\quad \frac{\partial\Gamma}{\partial\varphi''}=\frac{hk^2}{2E}2\gamma\varphi''.$$

As a consequence,

$$\gamma\varphi^{IV}-2\lambda\varphi''+\beta\varphi=0. \qquad (12.9)$$

As an example of the solution of differential equation (12.9), we shall determine the stresses in a plate of length $2\ell = 30$ cm and width $2b = 20$ cm (Fig. 12.8), in which there is the following temperature-difference distribution:

$$\Delta T=\Delta T_0\frac{x^2}{b^2}.$$

In this case

$$\psi=-\frac{x^2}{b^2}+\frac{\int\limits_{-b}^{+b}\frac{x^2}{b^2}h\,dx}{F}=\frac{1}{3}-\frac{x^2}{b^2}.$$

As a consequence,

$$\sigma_z^T=k\left(\frac{1}{3}-\frac{x^2}{b^2}\right).$$

State T is shown at the top of Fig. 12.8.

In state "α" the stresses are $\sigma_z^{\alpha}=-k\varphi\psi=-k\varphi\left(\frac{1}{3}-\frac{x^2}{b^2}\right)$ (see Fig. 12.8, center);

$$\tau_{xz}=k\varphi'\int\limits_b^x\psi\,dx=k\varphi'\int\limits_b^x\left(\frac{1}{3}-\frac{x^2}{b^2}\right)dx.$$

From this we have

$$\tau_{xz}=k\varphi'\left(\frac{x}{3}-\frac{x^3}{3b^2}\right). \qquad (12.10)$$

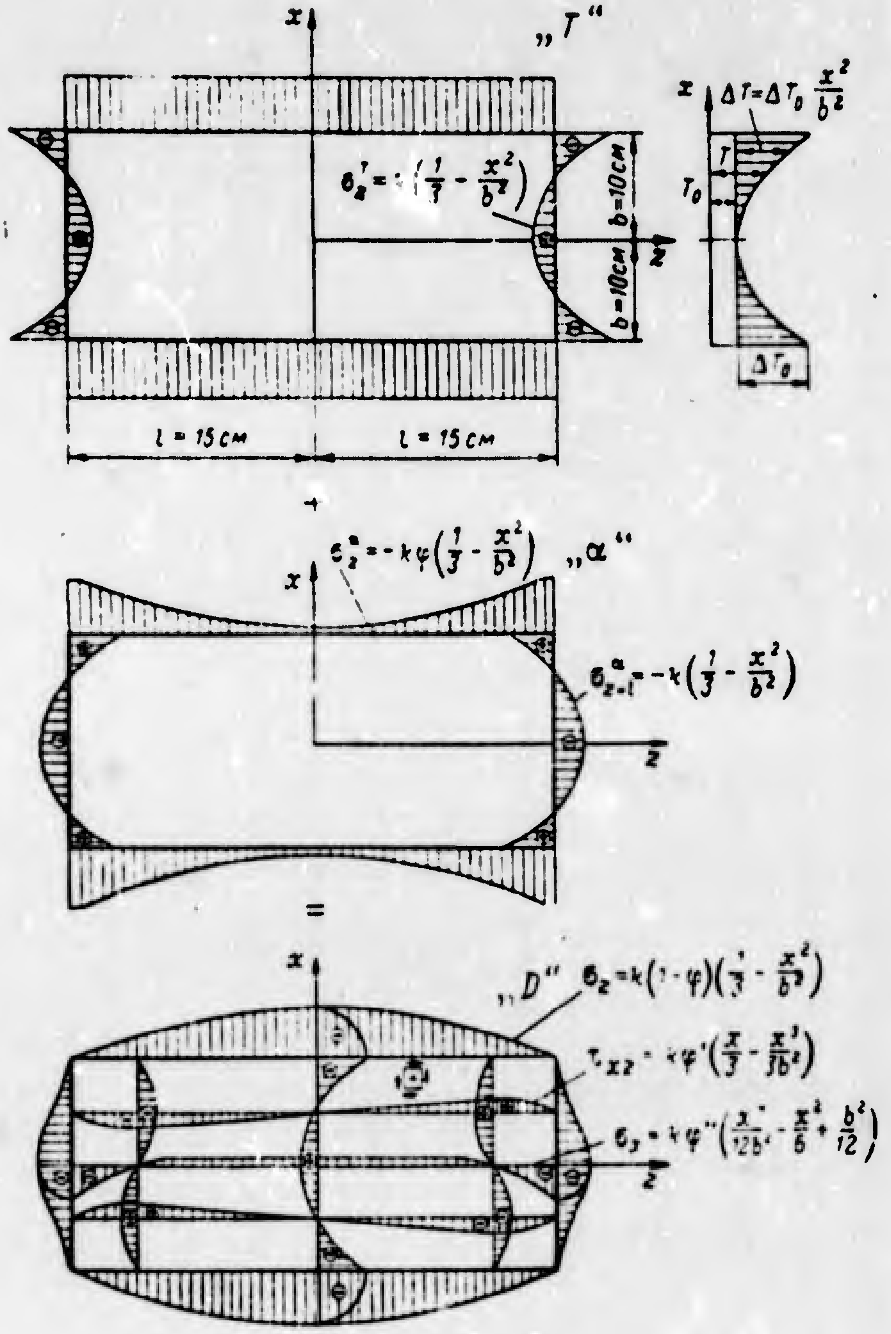


Fig. 12.8

Moreover

$$\sigma_x = -k\varphi'' \int_b^x \int_b^x \varphi \, dx \, dx = -k\varphi'' \int_b^x \left(\frac{x}{3} - \frac{x^3}{3b^2}\right) dx,$$

i.e.,

$$\sigma_x = -k\varphi'' \left(\frac{x^2}{6} - \frac{x^4}{12b^2} - \frac{b^2}{12}\right). \qquad (12.11)$$

We now proceed to determine the constants β, γ, and λ.

$$\beta = \int_{-b}^{+b} \varphi^2 \, dx = \int_{-b}^{+b} \left(\frac{1}{3} - \frac{x^2}{b^2}\right)^2 dx = \frac{8b}{45} = 0{,}178\, b;$$

$$\gamma = \int_{-b}^{+b} \left(\int_b^x \int_b^x \varphi \, dx \, dx \right)^2 dx = \int_{-b}^{+b} \left(\frac{x^2}{6} - \frac{x^4}{12b^2} - \frac{b^2}{12} \right)^2 dx =$$

$$= \frac{16b^5}{2835} = 0{,}00564\, b^5;$$

$$\lambda = \int_{-b}^{+b} \left(\int_b^x \varphi \, dx \right)^2 dx = \int_{-b}^{+b} \left(\frac{x}{3} - \frac{x^3}{3b^2} \right)^2 dx = \frac{16b^3}{945} = 0{,}0169\, b^3.$$

Substituting the values of the constants β, γ and λ into Eq. (12.9), we obtain the following differential equation:

$$5{,}64\, b^4 \varphi^{IV} - 33{,}8\, b^2 \varphi'' + 178 \varphi = 0. \qquad (12.12)$$

From the characteristic equation

$$5{,}64 b^4 k^4 - 33{,}8\, b^2 k^2 + 178 = 0$$

we obtain

$$k_1 = 0{,}207 + 0{,}114\, i;$$
$$k_2 = 0{,}207 - 0{,}114\, i;$$
$$k_3 = -0{,}207 - 0{,}114\, i;$$
$$k_4 = -0{,}207 + 0{,}114\, i.$$

Thus the general solution to differential equation (12.12) may be written as follows:

$$\varphi = e^{\alpha z} (A_1 \cos \beta z + A_2 \sin \beta z) + e^{-\alpha z} (A_3 \cos \beta z - A_4 \sin \beta z), \qquad (12.13)$$

where

$$\alpha = 0{,}207 \text{ and } \beta = 0{,}114.$$

From the conditions requiring symmetry of the system about the x axis, the values of the function φ must be the same for positive and negative values of its argument z, i.e.,

$$e^{\alpha z} (A_1 \cos \beta z + A_2 \sin \beta z) + e^{-\alpha z} (A_3 \cos \beta z - A_4 \sin \beta z) =$$
$$= e^{-\alpha z} (A_1 \cos \beta z - A_2 \sin \beta z) + e^{\alpha z} (A_3 \cos \beta z + A_4 \sin \beta z).$$

From this we have

$$A_1 = A_3 = C_1 \text{ and } A_2 = A_4 = C_2,$$

which enables us to transform Eq. (12.13) to the form

$$\varphi = D_1 \operatorname{ch} \alpha z \cos \beta z + D_2 \operatorname{sh} \alpha z \sin \beta z.$$

In our case

$$\varphi = D_1 \operatorname{ch} 0{,}207\,z \cos 0{,}114\,z + D_2 \operatorname{sh} 0{,}207\,z \sin 0{,}114\,z.$$

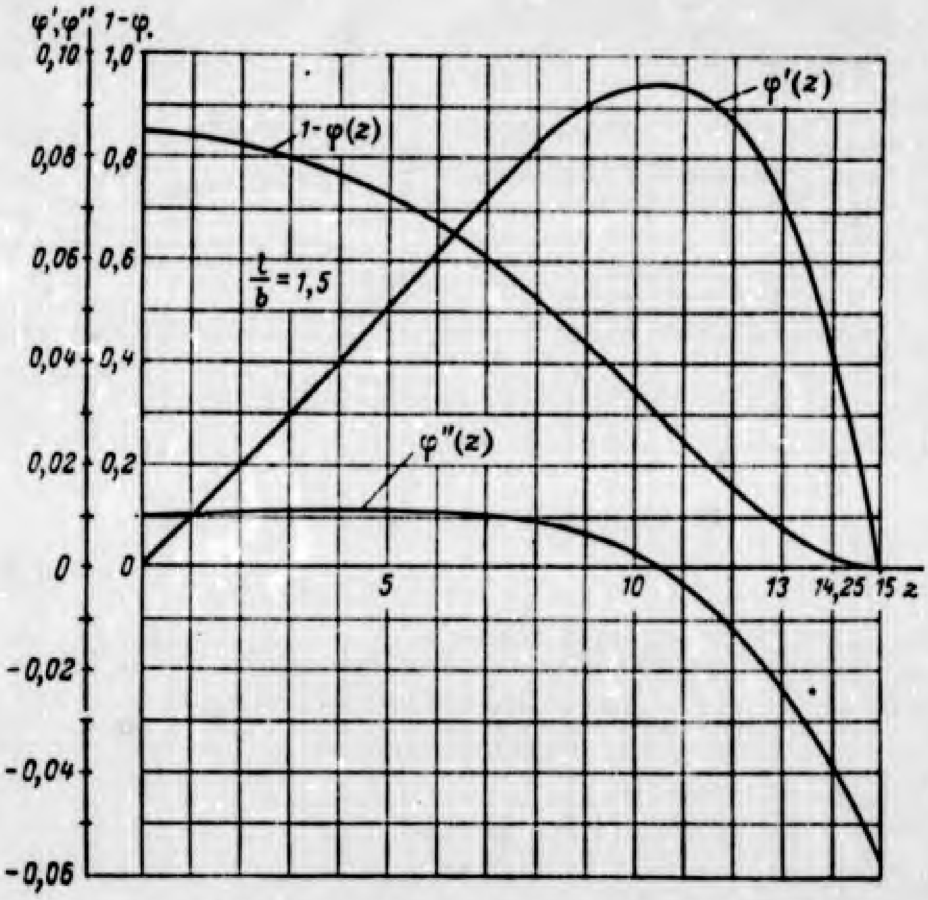


Fig. 12.9

The arbitrary constants D_1 and D_2 are found from the boundary conditions:

for $z = l$, $\varphi = 1$;

for $z = l$, $\tau_{xz} = 0$ or $\varphi' = 0$.

Using the first boundary condition for $z = l = 15$ cm, we obtain

$$1 = D_1 \operatorname{ch} 0{,}207 \cdot 15 \cos 0{,}114 \cdot 15 + D_2 \operatorname{sh} 0{,}207 \cdot 15 \sin 0{,}114 \cdot 15 = -1{,}55 D_1 + 10{,}97 D_2.$$

Using the second boundary condition, we obtain

$$\varphi' = D_1(-\beta \operatorname{ch} \alpha z \sin \beta z + \alpha \operatorname{sh} \alpha z \cos \beta z) + D_2(\beta \operatorname{sh} \alpha z \cos \beta z + \alpha \operatorname{ch} \alpha z \sin \beta z) = -23{,}50 D_1 + 31{,}46 D_2 = 0.$$

We thus have

$$-1{,}55 D_1 + 10{,}97 D_2 - 1 = 0; \qquad (1)$$

$$-23{,}50D_1+31{,}46D_3=0. \quad (2)$$

Solving these equations simultaneously, we obtain

$$D_1=0{,}1510;$$
$$D_3=0{,}1125.$$

As a consequence,

$$\varphi_{(z)}=0{,}151\,\mathrm{ch}\,0{,}207z\cos 0{,}114z+0{,}1125\,\mathrm{sh}\,0{,}207z\sin 0{,}114z \quad (12.14)$$

and

$$\varphi'_{(z)}=0{,}151(-0{,}114\,\mathrm{ch}\,0{,}207z\sin 0{,}114z+0{,}207\,\mathrm{sh}\,0{,}207z\cos 0{,}114z)+$$
$$+0{,}1125(0{,}114\,\mathrm{sh}\,0{,}207z\cos 0{,}114z+0{,}207\,\mathrm{ch}\,0{,}207z\sin 0{,}114z). \quad (12.15)$$

Further,

$$\ddot{\varphi}_{(z)}=D_1[(\alpha^2-\beta^2)\,\mathrm{ch}\,\alpha z\cos\beta z-2\alpha\beta\,\mathrm{sh}\,\alpha z\sin\beta z]+$$
$$+D_3[2\alpha\beta\,\mathrm{ch}\,\alpha z\cos\beta z+(\alpha^2-\beta^2)\,\mathrm{sh}\,\alpha z\sin\beta z]$$

or

$$\ddot{\varphi}_{(z)}=0{,}151(0{,}02985\,\mathrm{ch}\,0{,}207z\cos 0{,}114z-0{,}0472\,\mathrm{sh}\,0{,}207z\sin 0{,}114z)+$$
$$+0{,}1125(0{,}0472\,\mathrm{ch}\,0{,}207z\cos 0{,}114z+$$
$$+0{,}02985\,\mathrm{sh}\,0{,}207z\sin 0{,}114z). \quad (12.16)$$

In Fig. 12.9, we have shown curves for the variation in 1 − φ(z), φ'(z), and φ"(z) in the right half of the plate.

The actual thermal stresses, in accordance with Formulas (12.7), (12.10) and (12.11) will be

$$\sigma_z=k(1-\varphi_{(z)})\left(\frac{1}{3}-\frac{x^2}{b^2}\right);$$
$$\tau_{xz}=k\varphi'_{(z)}\left(\frac{x}{3}-\frac{x^3}{3b^2}\right);$$
$$\sigma_x=k\ddot{\varphi}_{(z)}\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right).$$

The curves for the variation in these stresses are shown in Fig. 12.8, at the bottom.

2. THERMAL STRESSES IN A SOLID WING WITH DIAMOND PROFILE

A solid wing with a diamond-shaped profile (see Fig. 12.2) differs geometrically from the plate with rectangular cross section considered in the preceding section only in that for this case the thickness h(x)

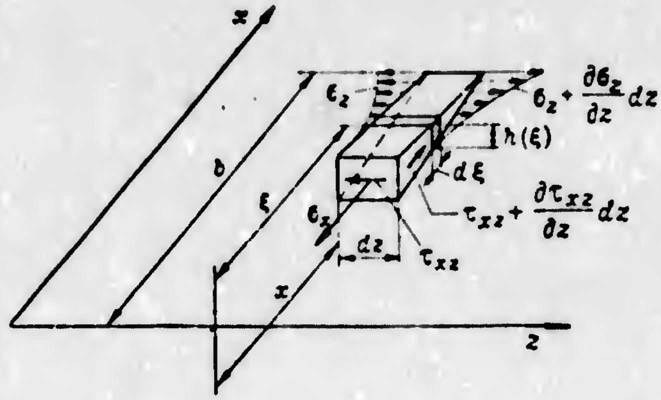


Fig. 12.10

is a variable:

and

$$\left.\begin{aligned} h(x) &= H\left(1-\frac{x}{b}\right) \quad (0 \leqslant x \leqslant b) \\ h(x) &= H\left(1+\frac{x}{b}\right) \quad (0 \geqslant x \geqslant -b), \end{aligned}\right\} \qquad (12.2')$$

where H is the maximum profile thickness.

Thus if we assume

$$\Delta T = \Delta T_0 \frac{x^2}{b^2},$$

then

$$\psi = -\frac{x^2}{b^2} + \frac{2\int\limits_0^b \frac{x^2}{b^2} H\left(1-\frac{x}{b}\right) dx}{F}.$$

In view of the fact that

$$\int\limits_0^b x^2\left(1-\frac{x}{b}\right) dx = \frac{b^3}{12} \text{ and } F = Hb,$$

we obtain

$$\psi = \frac{1}{6} - \frac{x^2}{b^2}. \qquad (12.17)$$

As a consequence,

$$\sigma_z^T = k\left(\frac{1}{6} - \frac{x^2}{b^2}\right); \qquad (12.18)$$

$$\sigma_z^e = -k\varphi\left(\frac{1}{6} - \frac{x^2}{b^2}\right) \qquad (12.19)$$

and

$$\sigma_z=\sigma_z^T+\sigma_z^*=k(1-\varphi)\left(\frac{1}{6}-\frac{x^2}{b^2}\right). \qquad (12.20)$$

In order to find an expression for the τ_{xz} and σ_x accompanying σ_z, we cut an element from the wing at sections z = const and z + dz = const and set up for the portion of length b − x (Fig. 12.10) the equilibrium equations

$$\sum X=0; \qquad (1)$$

$$\sum Z=0. \qquad (2)$$

Expanding equilibrium equation (2), we obtain

$$-\tau_{xz}h(x)dz-\int_b^x\frac{\partial\sigma_z}{\partial z}dzh_{(\xi)}d\xi=0.$$

Since with the introduction of the auxiliary coordinate ξ

$$\frac{\partial\sigma_z}{\partial z}=-k\varphi'\left(\frac{1}{6}-\frac{\xi^2}{b^2}\right)$$

and

$$h(\xi)=H\left(1-\frac{\xi}{b}\right),$$

we obtain

$$-\tau_{xz}h(x)+k\varphi'H\int_b^x\left(\frac{1}{6}-\frac{\xi^2}{b^2}\right)\left(1-\frac{\xi}{b}\right)d\xi=0,$$

or

$$\tau_{xz}h(x)=k\varphi'H\int_b^x\left(\frac{1}{6}-\frac{\xi^2}{b^2}\right)\left(1-\frac{\xi}{b}\right)d\xi.$$

After integration we obtain

$$\tau_{xz}h(x)=\frac{k\varphi'Hb}{12}\left(\frac{2x}{b}-\frac{x^2}{b^2}-\frac{4x^3}{b^3}+\frac{3x^4}{b^4}\right).$$

from which we have

$$\tau_{xz}=\frac{k\varphi'b}{12}\eta_1, \qquad (3)$$

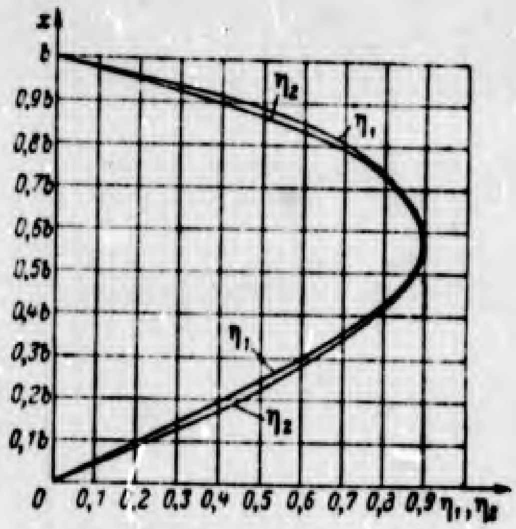


Fig. 12.11

where

$$\eta_1 = \frac{\frac{2x}{b} - \frac{x^2}{b^2} - \frac{4x^3}{b^3} + \frac{3x^4}{b^4}}{1 - \frac{x}{b}}.$$

For the rest of the computation, the function η_1 becomes too complicated, so that we shall approximate it by means of a simpler function.

From the curve of the function η_1 (Fig. 12.11), it is clear that we could, for example, use the approximating function

$$\eta = \left(\gamma \sin \frac{\pi x}{b} - \lambda \sin \frac{2\pi x}{b}\right),$$

where γ and λ are constant coefficients.

But as we must consider a rectangular plate, let us try to compare the expression for τ_{xz} (3) with Expression (12.10) of the preceding section.

To simplify the comparison, we transform Expression (12.10) to the form

$$\tau_{xz} = \frac{k\varphi' b}{12}\left(\frac{4x}{b} - \frac{4x^3}{b^3}\right).$$

The curve $\frac{4x}{b} - \frac{4x^3}{b^3}$ is similar in nature to the curve for η_1, but the abscissas of the points are roughly 40% greater. We thus let

$$\eta_2 = 0{,}58\left(\frac{4x}{b} - \frac{4x^3}{b^3}\right).$$

The curve for the function η_2 is also given in Fig. 12.11.

It is clear from Fig. 12.11 that with an accuracy sufficient for practical purposes, the function η_1 may be replaced by function η_2. Then for a diamond profile, we have in place of Expression (3)

$$\tau_{xz} = \frac{k\varphi' b}{12} 0{,}58\left(\frac{4x}{b} - \frac{4x^3}{b^3}\right),$$

or

$$\tau_{xz} = k\varphi'\psi_1, \tag{12.21}$$

where

$$\psi_1 = 0{,}193b\left(\frac{x}{b} - \frac{x^3}{b^3}\right).$$

Expanding the equilibrium equation (1), we obtain (see Fig. 12.10)

$$-\sigma_x h(x)\,dz - \int_b^x \frac{\partial \tau_{zx}}{\partial z}\,dz\,\eta(\xi)\,d\xi = 0$$

or

$$\sigma_x h(x) = -\frac{0{,}58}{3} k\varphi'' H \int_b^x \left(\frac{\xi}{b} - \frac{\xi^3}{b^3}\right)\left(1 - \frac{\xi}{b}\right) d\xi.$$

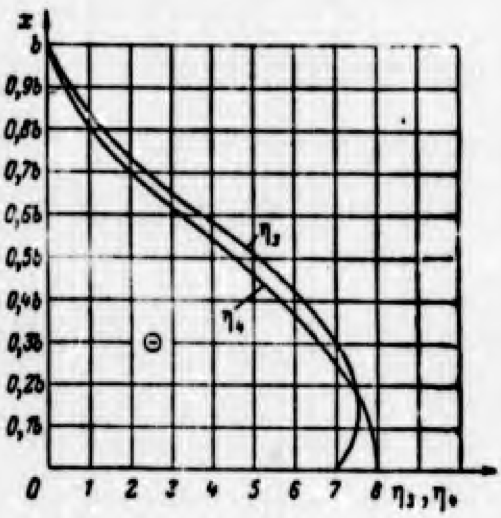


Fig. 12.12

After integration and rearrangement we find that

$$\sigma_x = -\frac{0{,}58}{180} k\varphi'' b^2 \eta_3, \tag{4}$$

where

$$\eta_3 = \frac{\dfrac{30x^2}{b^2} - \dfrac{20x^3}{b^3} - \dfrac{15x^4}{b^4} - \dfrac{12x^5}{b^5} - 7}{1 - \dfrac{x}{b}}.$$

The curve for the function η_3 is plotted in Fig. 12.12.

To approximate σ_x (4) we use Expression (12.11) giving σ_x for a rectangular plate; we transform it to the form:

$$\sigma_x = -\frac{k\varphi'' b^2}{180}\left(\frac{30x^2}{b^2} - \frac{15x^4}{b^4} - 15\right).$$

The curve for the function

$$\eta_4 = 0{,}533\left(\frac{30x^2}{b^2} - \frac{15x^4}{b^4} - 15\right)$$

is shown in Fig. 12.12.

It is clear from Fig. 12.12 that function η_4 is a satisfactory approximation for the function η_3. We stop with this approximation, i.e., for a diamond profile we use

$$\sigma_x = -\frac{0{,}58}{180} k\varphi'' b^2 0{,}533\left(\frac{30x^2}{b^2} - \frac{15x^4}{b^4} - 15\right),$$

or

$$\sigma_x = -k\varphi''\psi_2, \tag{12.22}$$

where

$$\psi_2 = 0{,}0256 b^2\left(\frac{2x^2}{b^2} - \frac{x^4}{b^4} - 1\right).$$

The strain potential energy for one fourth of the wing in plan is

$$\bar{U} = \int_0^l \Gamma\, dz,$$

where

$$\Gamma = \int_0^b h(x)\left[\frac{1}{2E}(\sigma_z^2 + \sigma_x^2) - \frac{\mu\sigma_z\sigma_x}{E} + \frac{\tau_{xz}^2}{2G}\right] dx.$$

Substituting into this expression the values of σ_z, σ_x, and τ_{xz} for $G = \frac{E}{2(1+\mu)}$, we obtain

$$\Gamma = \frac{k^2}{2E}\left[\beta\varphi^2 + \gamma\varphi''^2 - 2\mu\eta\varphi\varphi'' + 2(1+\mu)\lambda\varphi'^2\right],$$

where

$$\beta = \int_0^b h(x)\psi^2\, dx; \quad \gamma = \int_0^b h(x)\psi_2^2\, dx;$$

$$\eta = \int_0^b h(x)\psi\psi_2\, dx; \quad \gamma = \int_0^b h(x)\psi_1^2\, dx.$$

From the conditions requiring minimization of the potential energy, we obtain the following differential equation:

$$\gamma\varphi^{IV}-2[\mu\eta+(1+\mu)\lambda]\varphi''+\beta\varphi=0. \qquad (12.23)$$

We determine the values of the constant coefficients in this equation:

$$\beta=\int_0^b H\left(1-\frac{x}{b}\right)\left(\frac{1}{6}-\frac{x^2}{b^2}\right)^2 dx=0{,}0194Hb;$$

$$\gamma=\int_0^b H\left(1-\frac{x}{b}\right)\left[0{,}0256b^2\left(\frac{2x^2}{b^2}-\frac{x^4}{b^4}-1\right)\right]^2 dx=0{,}000198Hb^5;$$

$$\eta=\int_0^b H\left(1-\frac{x}{b}\right)\left(\frac{1}{6}-\frac{x^2}{b^2}\right)0{,}0256b^2\left(\frac{2x^2}{b^2}-\frac{x^4}{b^4}-1\right)dx=$$

$$=-0{,}000683Hb^3;$$

$$\lambda=\int_0^b H\left(1-\frac{x}{b}\right)\left[0{,}193b\left(\frac{x}{b}-\frac{x^3}{b^3}\right)\right]^2 dx=0{,}00128Hb^3.$$

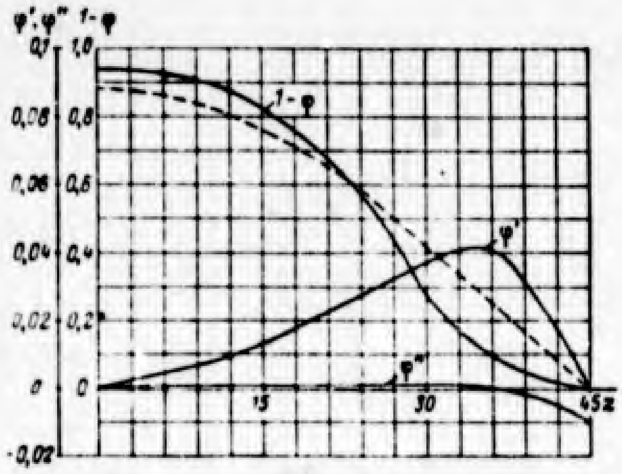


Fig. 12.13

We shall now find the function φ for b = 30 cm and ℓ = 45 cm, i.e., for a wing having an aspect ratio 45/30 = 1.5.

For b = 30 cm, we find

$$\beta=0{,}5833H; \qquad \gamma=4845H;$$
$$\eta=-18{,}44H; \qquad \lambda=34{,}60H.$$

Substituting these values into Eq. (12.23), we obtain

$$4845\varphi^{IV}-2[0{,}3(-18{,}44)+1{,}3\cdot 34{,}60]\varphi''+0{,}5833\varphi=0$$

or

$$4845\varphi^{IV}-78{,}94\varphi''+0{,}5833\varphi=0.$$

From the solution to the characteristic equation

$$4845k^4-78{,}94k^2+0{,}5833=0$$

we obtain

$$k_1=0{,}098+0{,}0378i;$$
$$k_2=0{,}098-0{,}0378i;$$
$$k_3=-0{,}098-0{,}0378i;$$
$$k_4=-0{,}098+0{,}0378i.$$

This means (see Section 1 of Chapter 12) that

$$\varphi=D_1\operatorname{ch}\alpha z\cos\beta z+D_2\operatorname{sh}\alpha z\sin\beta z,$$

where

$\alpha=0{,}098$ and $\beta=0{,}0378.$

In accordance with the boundary conditions:

$z = \ell = 45$ cm and $\varphi = 1$, and also $x = \ell = 45$ cm, and the stresses

$\tau_{zx}=0$ or $\varphi'=0;$

we further obtain

$D_1=0{,}0600$ and $D_2=0{,}0326.$

Thus

$$\varphi=0{,}0600\operatorname{ch}0{,}098z\cos 0{,}0378z+ +0{,}0326\operatorname{sh}0{,}098z\sin 0{,}0378z. \tag{12.24}$$

The curves for $1-\varphi$, φ' and φ'' are plotted in Fig. 12.13. The thermal stresses are found from the formulas

$$\left.\begin{aligned}\sigma_z&=k(1-\varphi)\left(\frac{1}{6}-\frac{x^2}{b^2}\right),\\ \tau_{xz}&=k\varphi'0{,}193b\left(\frac{x}{b}-\frac{x^3}{b^3}\right),\\ \sigma_x&=k\varphi''0{,}0256b^2\left(\frac{2x^2}{b^2}-\frac{x^4}{b^4}-1\right).\end{aligned}\right\} \tag{12.25}$$

Manu-
script
Page
No.

[Footnotes]

The force N sets up a positive moment about the center of gravity of the section (Fig. 12.4), but since the integrand contains $\underline{x}$, the moment will be positive only where there is a "–" sign before the right side of this equation.

Chapter 13

APPLICATION OF THE METHOD OF FINITE DIFFERENCES TO CALCULATIONS FOR WING THERMAL STRESSES. INFLUENCE OF THERMAL STRESSES ON THE REDUCTION IN WING STIFFNESS TORSION

1. THERMAL STRESSES IN A THIN-SKIN WING

As we have already indicated (Section 5, Chapter 5), it is desirable to use the method of finite differences for irregular systems. Thus, the temperature field in a wing or body of a flying craft may vary in accordance with a complicated law not only laterally, but also longitudinally. Moreover, sharp temperature differentials may appear, for example, between the wing center panel and the cantilevers.

Let us consider a cylindrical wing (Fig. 13.1) with symmetric profile (Fig. 13.2) used for a high-speed flying craft which has, as we know, a wing with relative thickness <6%. The temperature distribution over the wingspan and over the chord are shown in Fig. 13.1. Here the temperature field shows sharp differentials along the inboard wing ribs.

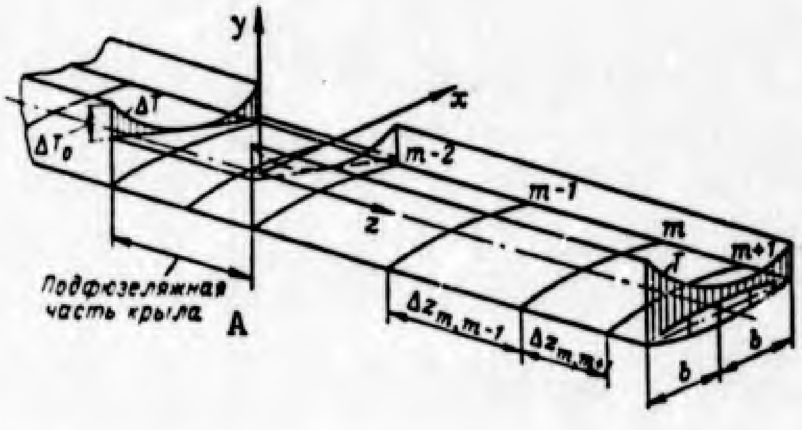


Fig. 13.1. A) Fuselage section of wing.

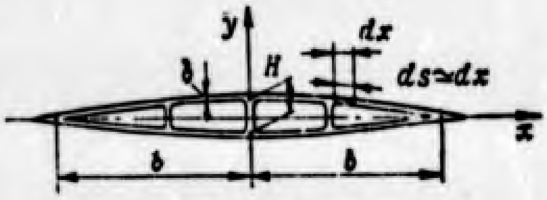


Fig. 13.2

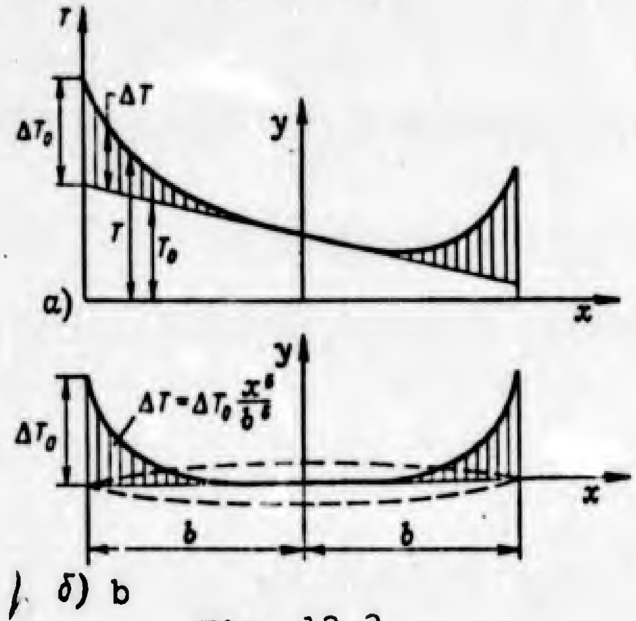


Fig. 13.3

The chord temperature distribution which we assume (Fig. 13.3a) is very asymmetric about the y axis. Isolating ΔT from the temperature we see that this part can be approximated well by a function of the form (Fig. 13.3b)

$$\Delta T = \Delta T_0 \frac{x^6}{b^6}.$$

In virtue of the symmetry of the profile and the function ΔT about the x and y axes, the two last terms in Eq. (12.4) will equal zero and, as a consequence,

$$\psi = -\rho + \frac{\int \rho dF}{F}.$$

In our case $\rho = x^6/b^6$, so that

$$\psi = -\frac{x^6}{b^6} + \frac{\int \frac{x^6}{b^6}\, dF}{F}.$$

In order to avoid dealing in subsequent calculations with integration of piecewise-continuous functions, we shall uniformly distribute the areas of the wall cross sections over the upper and lower wing skins. Then the skin thickness is

$$\delta_{e}=\delta_{\tau}+\frac{\sum f}{4b},$$

where δ_{τ} is the actual skin thickness; Σf is the wall cross-sectional area.

Such a calculation scheme has little effect on the computational accuracy, since the part played by the walls in carrying longitudinal wing stresses is slight.

Also taking into account the fact that for a small relative profile thickness we can assume ds = dx, we obtain the value of ψ from the following expression:

$$\psi=-\frac{x^{6}}{b^{6}}+\frac{2\int\limits_{-b}^{+b}\frac{x^{6}}{b^{6}}\delta_{e}dx}{2\int\limits_{-b}^{+b}\delta_{e}dx}.$$

After integration we obtain

$$\psi=\frac{1}{7}-\frac{x^{6}}{b^{6}}. \tag{13.1}$$

In order to set up a finite-difference equation for each section m, we mark off on the length l of the wing a series of cross sections with spans equal to Δz (Fig. 13.4, top).

In section m and in the adjacent sections m − 2, m − 1, m + 1, m + 2, the stresses σ_z may be represented in the form:

$$\left.\begin{aligned}\sigma_{m-2}&=-X_{m-2}\psi,\\ \sigma_{m-1}&=-X_{m-1}\psi,\\ \sigma_{m}&=-X_{m}\psi,\\ \sigma_{m+1}&=-X_{m+1}\psi,\\ \sigma_{m+2}&=-X_{m+2}\psi,\end{aligned}\right\} \tag{13.2}$$

where X_{m-2}, X_{m-1}, X_m, X_{m+1}, X_{m+2} are the unknown values of the stresses in the sections corresponding to the subscripts; ψ is a known function determined from Expression (12.4).

From the conditions requiring symmetry of the structure cross section and the temperature distribution, in finding the unknowns we need only consider the upper or lower wing skin.

In Fig. 13.4, top, we have shown the stressed states in the upper

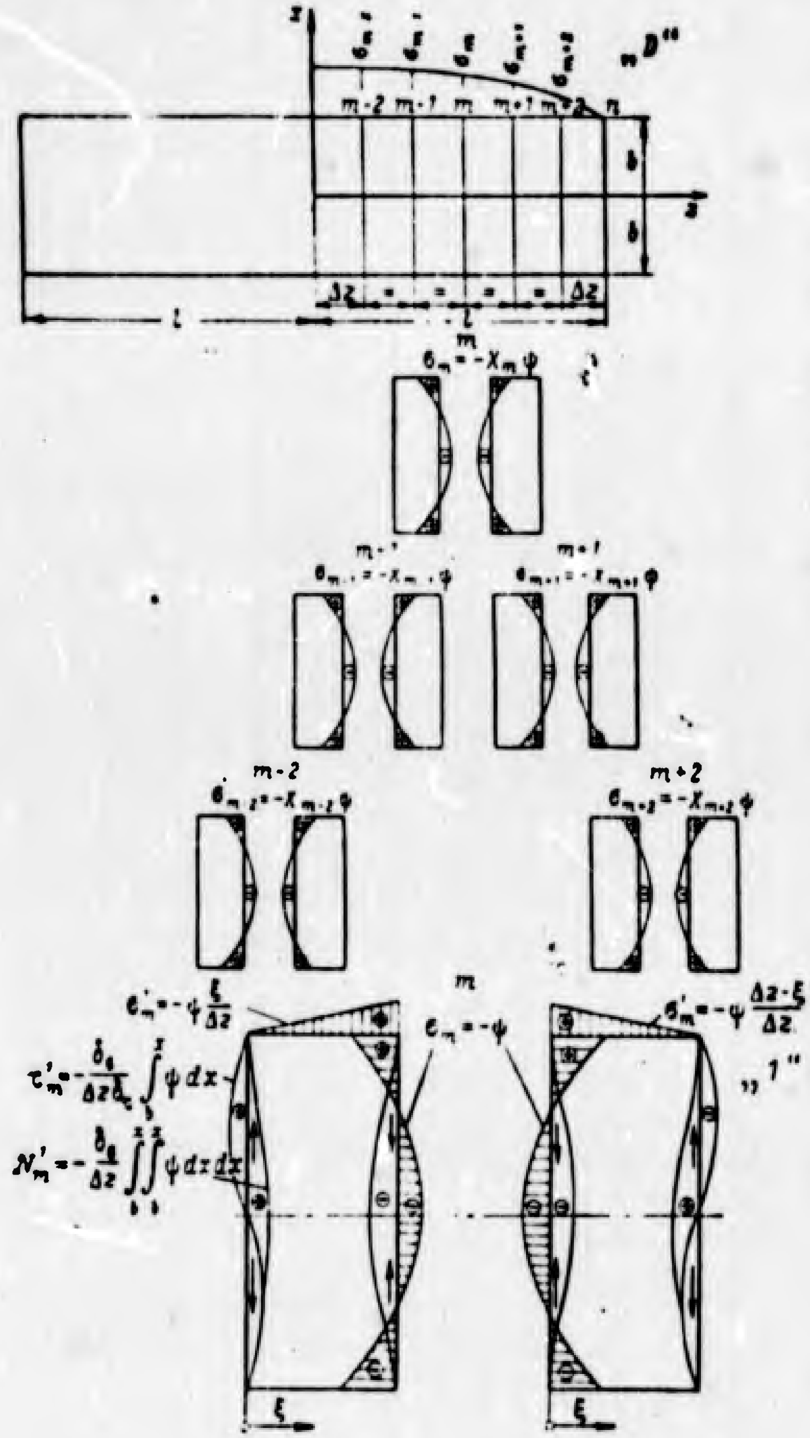


Fig. 13.4

skin produced by the stresses of System (13.2). Here the calculation model for each transverse strip cut off by the two sections z = const and $z + \Delta z$ = const consists of two braces with a cross-sectional area $f_{sh} = \Delta z \delta$ and the strip of skin with thickness δ, which is located between them and which works under normal stresses along the z axis and under shear. In Fig. 13.4, bottom, we have shown in a larger scale the unit state due to the stresses $\sigma'_m = -\psi$ applied in Section m. Let us consider the force factors for this state.

We may assume that σ_z varies linearly along the length Δz, so that the running value of the stresses will be

$$\sigma'_m = -\psi \frac{\xi}{\Delta z} \text{ (for the left-hand strip);} \qquad (13.3)$$

$$\sigma'_m = -\psi \frac{\Delta z - \xi}{\Delta z} \text{ (for the right-hand strip);}$$

Here ξ is the auxiliary coordinate for the zone of length Δz.

Where the stresses ξ_z vary linearly, the accompanying shear stresses on the length Δz will be constant. Let us determine these stresses from the equilibrium equation, for example, for the left-hand strip:

$$\tau'_m \delta_\tau \Delta z = -\int_0^x \sigma'_{m(\xi=\Delta z)} \delta_\sigma \, dx$$

or

$$\tau'_m = \frac{\delta_\sigma}{\Delta z \delta_\tau} \int_0^x \psi \, dx. \qquad (13.4)$$

In Fig. 13.4, bottom, the arrows indicate the action of the forces τ'_m on the braces, and also indicate the nature of the variation in the force factors for the unit state.

We should now have no difficulty in also determining the running value of the forces in the braces. Thus, for example, for the far left-hand brace the force is

$$N'_m = -\int_b^x \tau'_m \delta_1 dx = -\frac{\delta_0}{\Delta z}\int_b^x\int_b^x \psi\, dx\, dx, \tag{13.5}$$

and the stresses are

$$\sigma'_{m,m} = -\frac{\delta_0}{\Delta z^2 \delta_1}\int_b^x\int_b^x \psi\, dx\, dx. \tag{13.6}$$

In order to obtain the true force factors for the transverse strips located in the zone of influence of the stresses σ_{m-2}, σ_{m-1}, σ_m, σ_{m+1}, σ_{m+2}, we need only multiply the appropriate unit state by these stresses.

We find the finite-difference equation for any section m from the conditions requiring that the sum of the works done by all forces of the unit state for the actual displacements equal zero.

The work done by the force factors of the unit state for displacements produced by forces applied in Section m will be, for the identical transverse-strip length Δz

$$A_{m,m} = 2\int_x^{\Delta z}\int_{-b}^{+b}\frac{\sigma'_m \sigma_m \delta_0\, d\xi\, dx}{E} + 6\int_{-b}^{+b}\frac{N'_m N_m\, dx}{E f_{ш}} +$$

$$+2\int_x^{\Delta z}\int_{-b}^{+b}\frac{\tau'_m \tau_m \delta_1\, d\xi\, dx}{G}.$$

Substituting into this expression the values of the force factors from Eqs. (13.3), (13.4), and (13.5), we obtain

$$A_{m,m} = X_m\frac{2\delta_0}{\Delta z^2 E}\int_0^{\Delta z}\xi^2 d\xi\int_{-b}^{+b}\varphi^2\, dx + X_m\frac{6\delta_0^2}{\Delta z^2 E f_{ш}}\int_{-b}^{+b}\left(\int_b^x\int_b^x \varphi\, dx\, dx\right)^2 dx +$$

$$+X_m\frac{2\delta_0^2}{\Delta z^2 G\delta_1}\int_0^{\Delta z} d\xi\int_{-b}^{+b}\left(\int_b^x \psi\, dx\right)^2 dx.$$

After integration from 0 to Δz we have

$$EA_{m,m} = X_m\delta_0\left(2\frac{\Delta z}{3}\beta + 6\frac{\delta_0}{\Delta z^2 f_{ш}}\gamma + 2\frac{E}{\Delta z G}\frac{\delta_0}{\delta_1}\lambda\right). \tag{13.7}$$

Here

$$f_{ш}=\delta_{\tau}\Delta z; \qquad (13.8)$$

$$\beta=\int_{-b}^{+b}\psi^2 dx; \quad \gamma=\int_{-b}^{+b}\left(\int_b^y\int_b^y\psi\, dx\, dx\right)^2 dx; \quad \lambda=\int_{-b}^{+b}\left(\int_b^y \psi\, dx\right)^2 dx. \qquad (13.9)$$

The work done by the force factors of the unit state for displacements due to forces applied in section m − 1 will be:

$$A_{m,\,m-1}=\int_0^{\Delta z}\int_{-b}^{+b}\frac{\sigma'_m\sigma_{m-1}\delta_s d\xi\, dx}{E}+4\int_{-b}^{+b}\frac{N'_m N_{m-1}dx}{Ef_{ш}}+$$
$$+\int_0^{\Delta z}\int_{-b}^{+b}\frac{\tau'_m\tau_{m-1}\delta_\tau d\xi dx}{G}$$

or

$$A_{m,\,m-1}=X_{m-1}\frac{\delta_e}{\Delta z^2 E}\int_0^{\Delta z}\xi(\Delta x-\xi)d\xi\int_{-b}^{+b}\psi^2 dx-$$
$$-X_{m-1}\frac{4\delta_e^2}{\Delta z^2 E f_{ш}}\int_{-b}^{+b}\left(\int_b^x\int_b^x\psi\, dx\, dx\right)^2 dx-$$
$$-X_{m-1}\frac{\delta_e^2}{\Delta z^2 G\delta_\tau}\int_0^{\Delta z}d\xi\int_{-b}^{+b}\left(\int_b^x\psi\, dx\right)^2 dx.$$

from which we have

$$EA_{m,\,m-1}=X_{m-1}\delta_e\left(\frac{1}{2}\,\frac{\Delta z}{3}\,\beta-4\,\frac{\delta_e}{\Delta z^2 f_{ш}}\,\gamma-\frac{E\delta_e}{\Delta z G\delta_\tau}\,\lambda\right). \qquad (13.10)$$

The work done by the unit-state force factors for displacements due to forces applied in section m − 2 extends only to a single brace, so that

$$EA_{m,\,m-2}=X_{m-2}\,\frac{\delta_e^2}{\Delta z^2 f_{ш}}\,\gamma. \qquad (13.11)$$

Using similar reasoning for the virtual work of the forces applied to sections m + 1 and m + 2, we obtain

$$A_{m,\,m+1}=A_{m,\,m-1}$$

and

$$A_{m,\,m+2}=A_{m,\,m-2}.$$

The work in section m of the force factors of the unit state for displacements due to the temperature ΔT will equal:

$$A_{m,T} = \int_{-b}^{+b} \alpha\,\Delta T\,\Delta z\,dP,$$

where

$$dP = -\psi\delta_0 dx.$$

As a consequence,

$$EA_{m,T} = -\int_{-b}^{+b} k\,\Delta z\,\psi\delta_0\,dx. \tag{13.12}$$

In view of the fact that

$$EA_{m,m-2} + EA_{m,m-1} + EA_{m,m} + EA_{m,m+1} + A_{m,m+2} + A_{m,T} = 0,$$

the finite-difference equation for section m is obtained in the form:

$$\begin{gathered} X_{m-2}\delta_{m,m-2} + X_{m-1}\delta_{m,m-1} + X_m\delta_{m,m} + X_{m+1}\delta_{m,m+1} + \\ + X_{m+2}\delta_{m,m+2} + \Delta_{m,T} = 0. \end{gathered} \tag{13.13}$$

Here

$$\left.\begin{aligned} \delta_{m,m-2} &= \delta_{m,m+1} = \frac{\delta_0}{\Delta z^2 f_m}\gamma, \\ \delta_{m,m-1} &= \delta_{m,m-1} = \frac{1}{2}\frac{\Delta z}{3}\beta - 4\frac{\delta_0}{\Delta z^2 f_m}\gamma - \frac{E}{\Delta z G}\frac{\delta_0}{\delta_\tau}\lambda, \\ \delta_{m,m} &= 2\frac{\Delta x}{3}\beta + 6\frac{\delta_0}{\Delta z^2 f_m}\gamma + 2\frac{E}{\Delta z^2 G}\frac{\delta_0}{\delta_\tau}\lambda, \end{aligned}\right\} \tag{13.14}$$

$$\Delta_{m,T} = -\Delta z\int_{-b}^{+b} k\psi\,dx. \tag{13.15}$$

If we are to obtain high computational accuracy, the length Δz of the transverse strip should be small in comparison with its height and, as a consequence, System (13.13) will contain a greater number of equations. Of course, given the present-day level of computer techniques, there should be no difficulty in solving such a system. Even with the use of computers, however, it is desirable to reduce the number of equations insofar as possible. This may be done while retaining the same computational accuracy if we do not try to make the transverse strips all the same length. Then the coefficients and the free terms of System (13.13) will equal

$$\left.\begin{aligned}
&\delta_{m,m-2}=\frac{\delta_0\gamma}{\Delta z_{m,m-1}\cdot\Delta z_{m-1,m-2}f_{m-1}},\\
&\delta_{m,m-1}=\frac{1}{2}\frac{\Delta z_{m,m-1}}{3}\beta-\left[\frac{1}{\Delta z_{m,m-1}\cdot\Delta z_{m,m+1}}+\right.\\
&\left.+\frac{1}{\Delta z_{m,m-1}\cdot\Delta z_{m-1,m-2}}\frac{f_m}{f_{m-1}}+\frac{1}{\Delta z^2_{m,m-1}}\left(1+\frac{f_m}{f_{m-1}}\right)\right]\times\\
&\times\frac{\delta_0\gamma}{f_m}-\frac{E}{\Delta z_{m,m-1}G}\frac{\delta_0}{\delta_\tau}\lambda,\\
&\delta_{m,m}=\left(\frac{\Delta z_{m,m-1}+\Delta z_{m,m+1}}{3}\right)\beta+\left[\frac{1}{\Delta z^2_{m,m-1}}\left(1+\frac{f_m}{f_{m-1}}\right)+\right.\\
&\left.+\frac{2}{\Delta z_{m,m-1}\cdot\Delta z_{m,m+1}}+\frac{1}{\Delta z^2_{m,m+1}}\left(1+\frac{f}{f_{m+1}}\right)\right]\frac{\delta_0\gamma}{f_m}+\\
&+\left(\frac{1}{\Delta z_{m,m-1}}+\frac{1}{\Delta z_{m,m+1}}\right)\frac{E}{G}\frac{\delta_0}{\delta_\tau}\lambda;\\
&\delta_{m,m+1}=\frac{1}{2}\frac{\Delta z_{m,m+1}}{3}\beta-\left[\frac{1}{\Delta z_{m,m-1}\cdot\Delta z_{m,m+1}}+\right.\\
&\left.+\frac{1}{\Delta z_{m,m+1}\cdot\Delta z_{m+1,m+2}}\frac{f_m}{f_{m+1}}+\frac{1}{\Delta z^2_{m,m+1}}\left(1+\frac{f_m}{f_{m+1}}\right)\right]\times\\
&\times\frac{\delta_0\gamma}{f_m}-\frac{E}{\Delta E_{m,m+1}G}\frac{\delta_0}{\delta_\tau}\lambda;\\
&\delta_{m,m+2}=\frac{\delta_0\gamma}{\Delta z_{m,m+1}\cdot\Delta z_{m+1,m+2}\cdot f_{m+1}};\\
&f_{m-1}=\delta_\tau\frac{\Delta z_{m-1,m-2}+\Delta z_{m,m-1}}{2};\\
&f_m=\delta_\tau\frac{\Delta z_{m,m-1}+\Delta z_{m,m+1}}{2};\\
&f_{m+1}=\delta_\tau\frac{\Delta z_{m,m+1}+\Delta z_{m+1,m+2}}{2};
\end{aligned}\right\} \tag{13.16}$$

$$\Delta_{m,T}=-\frac{\alpha E}{2}\left(\Delta z_{m,m-1}\int_{-b}^{+b}\Delta T_{m,m-1}\varphi\,dx+\right.$$
$$\left.+\Delta z_{m,m+1}\int_{-b}^{+b}\Delta T_{m,m+1}\varphi\,dx\right). \tag{13.17}$$

For the sake of a numerical example, we assume that

$$\frac{\delta_0}{\delta_\tau}=1{,}143;\quad b=50\ cm;\quad \Delta z_{0,1}=30\ cm;\quad \Delta z_{1,2}=50\ cm;$$
$$\Delta z_{2,3}=50\ cm;\quad \Delta z_{3,4}=30\ cm;\quad \Delta z_{4,5}=12{,}5\ cm;$$
$$\Delta z_{5,6}=7{,}5\ cm;\quad l=180\ \text{cm (Fig. 13.5)}.$$

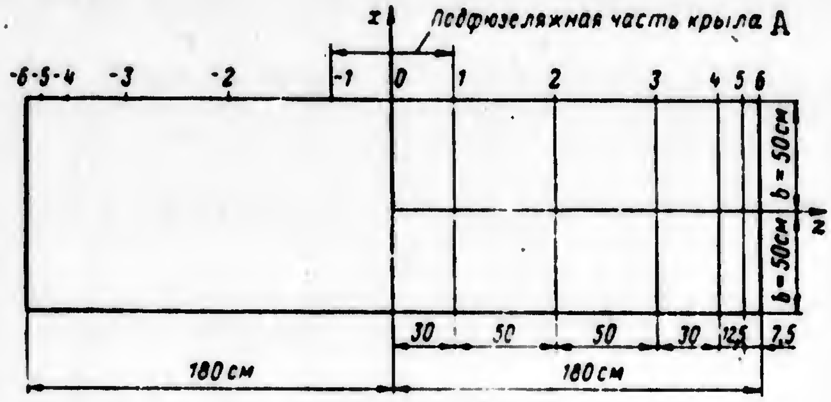


Fig. 13.5. A) Fuselage portion of wing.

We now determine the constant values in Expression (13.9):

$$\beta=\int_{-b}^{+b}\varphi^2 dx=\int_{-b}^{+b}\left(\frac{1}{7}-\frac{x^6}{b^6}\right)^2 dx=\int_{-b}^{+b}\left(\frac{1}{49}-\frac{2x^6}{7b^6}+\frac{x^{12}}{b^{12}}\right)dx=$$

$$=2\left(\frac{b}{49}-\frac{2b}{49}+\frac{b}{13}\right)=2\left(-\frac{13b}{13\cdot 49}+\frac{49b}{13\cdot 49}\right)=\frac{72}{13\cdot 49}\,b=0{,}1130b;$$

$$\gamma=\int_{-b}^{+b}\left[\int_b^x\int_b^x\left(\frac{1}{7}-\frac{x^6}{b^6}\right)dx\,dx\right]^2 dx=\int_{-b}^{+b}\left[\int_b^x\left(\frac{x}{7}-\frac{x^7}{7b^6}\right)dx\right]^2 dx=$$

$$=\int_{-b}^{+b}\left(\frac{x^2}{14}-\frac{x^8}{56b^6}-\frac{b^2}{14}+\frac{b^8}{56b^6}\right)^2 dx=\int_{-b}^{+b}\left(\frac{x^2}{14}-\frac{x^8}{56b^6}-\frac{3b^2}{56}\right)^2 dx=$$

$$=\int_{-b}^{+b}\left(\frac{x^4}{14^2}+\frac{x^{16}}{56^2b^{12}}+\frac{9b^4}{56^2}-\frac{2x^{10}}{14\cdot 56b^6}-\frac{2\cdot 3x^2b^2}{14\cdot 56}+\frac{2\cdot 3x^8b^2}{56^2b^6}\right)dx=$$

$$=2\left(\frac{b^5}{5\cdot 14^2}+\frac{b^5}{17\cdot 56^2}+\frac{9b^5}{56^2}-\frac{2b^5}{11\cdot 14\cdot 56}-\frac{6b^5}{3\cdot 14\cdot 56}+\frac{6b^5}{9\cdot 56^2}\right)=$$

$$=\frac{2b^5}{56^2}\left(\frac{56^2}{5\cdot 14^2}+\frac{1}{17}+9-\frac{2\cdot 56}{11\cdot 14}-\frac{6{,}56}{3\cdot 14}+\frac{3}{9}\right)=$$

$$=\frac{2b^5}{56^2}\left(\frac{3136}{980}+\frac{1}{17}+9-\frac{112}{154}-\frac{336}{42}+\frac{2}{3}\right)=$$

$$=\frac{2b^5}{56^2}(3{,}208+0{,}0588+9-0{,}7272-8+0{,}6667)=0{,}002682b^5;$$

$$\lambda=\int_{-b}^{+b}\left[\int_b^x\left(\frac{1}{7}-\frac{x^6}{b^6}\right)dx\right]^2 dx=\int_{-b}^{+b}\left(\frac{x}{7}-\frac{x^7}{7b^6}\right)^2 dx=$$

$$=\int_{-b}^{+b}\left(\frac{x^2}{49}-\frac{2x^8}{49b^6}+\frac{x^{14}}{49b^{12}}\right)dx=\frac{2b^3}{49}\left(\frac{1}{3}-\frac{2}{9}+\frac{1}{15}\right)=$$

$$=\frac{2b^3}{49}\left(\frac{15}{45}-\frac{10}{45}+\frac{3}{45}\right)=\frac{2b^3}{49}\cdot\frac{8}{45}=\frac{16}{2205}\,b^3=0{,}007256b^3.$$

Substituting the value b = 50 cm into the expressions for β, γ, and λ, we obtain

$$\beta=0{,}113\cdot 50=5{,}650;$$
$$\gamma=0{,}002682\cdot 5^2\cdot 10^5=0{,}002682\cdot 3125\cdot 10^5=8{,}380\cdot 10^5;$$
$$\lambda=0{,}007256\cdot 125\cdot 10^3=0{,}9070\cdot 10^3.$$

Applying Eq. (13.13) to section 0, we obtain

$$X_0\delta_{00}+2X_1\delta_{01}+2X_2\delta_{02}+\Delta_{0r}=0.$$

According to Eqs. (13.16) and (13.17), for E/G = 2.6,

$$\delta_{00}=\frac{\Delta z_{0,-1}+\Delta z_{0,1}}{3}\beta+\left[\frac{1}{\Delta z_{0,-1}^2}\left(1+\frac{\Delta z_{0,-1}+\Delta z_{0,1}}{\Delta z_{-1,-2}+\Delta z_{0,-1}}\right)+\right.$$
$$\left.+\frac{2}{\Delta z_{0,-1}\cdot\Delta z_{0,1}}+\frac{1}{\Delta z_{0,1}^2}\left(1+\frac{\Delta z_{0,-1}+\Delta z_{0,1}}{\Delta z_{0,1}+\Delta z_{1,2}}\right)\right]\frac{2\gamma}{\Delta z_{0,-1}+\Delta z_{0,1}}\frac{\delta_0}{\delta_1}+$$
$$+\left(\frac{1}{\Delta z_{0,-1}}+\frac{1}{\Delta z_{0,1}}\right)\frac{E}{G}\frac{\delta_0}{\delta_1}\lambda=\frac{30+30}{3}\,5{,}65+$$
$$+\left[\frac{1}{30^2}\left(1+\frac{30+30}{50+30}\right)+\frac{2}{30\cdot 30}+\frac{1}{30^2}\left(1+\frac{30+30}{30+50}\right)\right]\times$$
$$\times\frac{2\cdot 8{,}38\cdot 10^5\cdot 1{,}143}{30+30}+\left(\frac{1}{30}+\frac{1}{30}\right)2{,}6\cdot 1{,}143\cdot 0{,}907\cdot 10^3=20\cdot 5{,}65+$$
$$+\frac{1}{30^2}\left(1+\frac{60}{80}+2+1+\frac{60}{80}\right)\frac{9{,}578\cdot 10^5}{30}+\frac{2{,}695\cdot 10^3}{30}=$$
$$=113+\frac{5{,}50\cdot 9{,}578\cdot 10^5}{30^3}+\frac{5{,}39\cdot 10^3}{30}=113+195{,}1+179{,}7=487{,}8:$$
$$\delta_{01}=\frac{30}{2\cdot 3}\,5{,}65-\left[\frac{1}{30\cdot 30}+\frac{1}{30\cdot 50}\,\frac{30+30}{30+50}+\frac{1}{30^2}\left(1+\frac{30+30}{30+50}\right)\times\right.$$
$$\times\frac{2\cdot 9{,}578\cdot 10^5}{30+30}-\frac{1}{30}\,2{,}695\cdot 10^3=\frac{113}{4}-$$
$$-\frac{1}{30^3}\left(1+\frac{30}{50}\,0{,}75+1+0{,}75\right)9{,}578\cdot 10^5-\frac{179{,}7}{2}=28{,}25-$$
$$-\frac{3{,}2\cdot 9{,}578\cdot 10^5}{30^3}-89{,}85=28{,}25-113{,}52-89{,}85=-175{,}1;$$
$$2\delta_{01}=-350{,}2;$$
$$\delta_{02}=\frac{8{,}38\cdot 10^5\cdot 1{,}143}{30\cdot 50(30+50)}=16{,}0;$$
$$2\delta_{02}=32;$$
$$\Delta_{0r}=0.$$

As a consequence,

$$487{,}8X_0-350{,}2X_1+32X_2=0. \qquad (0)$$

Applying Eq. (13.13) to section 1, we obtain

$$X_0\delta_{10}+X_1(\delta_{1,-1}+\delta_{11})+X_2\delta_{12}+X_3\delta_{13}+\Delta_{1T}=0,$$

where

$$\delta_{10}=\delta_{01}=-175{,}1;$$

$$\delta_{1,-1}=\frac{2\cdot 9{,}578\cdot 10^5}{30\cdot 30(30+30)}=35{,}5;$$

$$\delta_{11}=\frac{30+50}{3}\ 5{,}65+\left[\frac{1}{30^2}\left(1+\frac{30+50}{30+30}\right)+\frac{2}{30\cdot 50}+\right.$$

$$\left.+\frac{1}{50^2}\left(1+\frac{30+50}{50+50}\right)\right]\frac{2\cdot 9{,}578\cdot 10^5}{30+50}+\left(\frac{1}{30}+\frac{1}{50}\right)2{,}695\cdot 10^3=$$

$$=150{,}7+111{,}2+143{,}7=405{,}6;$$

$$\delta_{1,-1}+\delta_{11}=35{,}5+405{,}6=441{,}1;$$

$$\delta_{12}=\frac{50}{2\cdot 3}\ 5{,}65-\left[\frac{1}{30\cdot 50}+\frac{1}{50\cdot 50}\ \frac{30+50}{50+50}+\frac{1}{50^2}\left(1+\frac{80}{100}\right)\right]\frac{2\cdot 9{,}578\cdot 10^5}{80}-$$

$$-\frac{1}{50}\ 2{,}695\cdot 10^3=47{,}1-40{,}9-53{,}9=-47{,}7;$$

$$\delta_{13}=\frac{2\cdot 9{,}578\cdot 10^5}{50\cdot 50(50+50)}=7{,}66;$$

$$\Delta_{1T}=-\frac{\alpha E}{2}\ 50\int_{-b}^{+b}\Delta T_0\ \frac{x^6}{b^6}\left(\frac{1}{7}-\frac{x^6}{b^6}\right)dx=$$

$$=-\frac{\alpha\Delta T_0 E 50}{2}\int_{-b}^{+b}\left(\frac{x^6}{7b^6}-\frac{x^{12}}{b^{12}}\right)dx=$$

$$=-50k\left(\frac{b^7}{7\cdot 7b^6}-\frac{b^{13}}{13\cdot b^{12}}\right)=$$

$$=-50kb\left(\frac{1}{49}-\frac{1}{13}\right)=-50kb\left(\frac{13-49}{49\cdot 13}\right)=$$

$$=+50kb\ \frac{36}{673}=\frac{50\cdot 50\cdot 36k}{637}=141{,}3k.$$

As a consequence, the equation acquires the following form: for section 1,

$$-175{,}1X_0+441{,}1X_1-47{,}7X_2+7{,}66X_3+141{,}3k=0; \qquad (1)$$

for section 2,

$$X_0\delta_{20}+X_1\delta_{21}+X_2\delta_{22}+X_3\delta_{23}+X_4\delta_{24}+\Delta_{2T}=0;$$

$$\delta_{20}=\delta_{02}=16{,}0;$$

$$\delta_{21}=\delta_{12}=-47{,}7;$$

$$\delta_{22}=\frac{50+50}{3}\,5{,}65+\left[\frac{1}{50^2}\left(1+\frac{100}{80}\right)+\frac{2}{50^2}+\frac{1}{50^2}\left(1+\frac{100}{80}\right)\right]\times$$
$$\times\frac{2\cdot 9{,}578\cdot 10^5}{50+50}+\left(\frac{1}{50}+\frac{1}{50}\right)2{,}695\cdot 10^3=$$
$$=188{,}3+49{,}8+107{,}8=345{,}9;$$
$$\delta_{23}=\frac{50}{6}\,5{,}65-\left[\frac{1}{50^2}+\frac{1}{50\cdot 30}\,\frac{100}{80}+\frac{1}{50^2}\left(1+\frac{100}{80}\right)\right]\frac{2\cdot 9{,}578\cdot 10^5}{2\cdot 50}-$$
$$-\frac{1}{50}\,2{,}695\cdot 10^3=47{,}1-40{,}9-53{,}9=-47{,}7;$$
$$\delta_{24}=\frac{2\cdot 9{,}578\cdot 10^5}{50\cdot 30\,(50+30)}=16{,}0;$$
$$\Delta_{2T}=141{,}3k\cdot 2=282{,}6k.$$

So that

$$16{,}0X_0-47{,}7X_1+345{,}9X_2-47{,}7X_3+16X_4+282{,}6k=0. \tag{2}$$

for section 3

$$X_1\delta_{31}+X_2\delta_{32}+X_3\delta_{33}+X_4\delta_{34}+X_5\delta_{35}+\Delta_{3T}=0.$$

Here

$$\delta_{31}=\delta_{13}=7{,}66; \qquad \delta_{32}=\delta_{23}=-47{,}7;$$
$$\delta_{33}=420{,}2; \quad \delta_{34}=-274{,}2; \quad \delta_{35}=120{,}2;$$
$$\Delta_{3T}=141{,}3k+141{,}3k\,\frac{30}{50}=(141{,}3+84{,}8)\,k=226{,}1k.$$

As a consequence,

$$7{,}66X_1-47{,}7X_2+420{,}2X_3-274{,}2X_4+120{,}2X_5+226{,}1k=0. \tag{3}$$

for section 4

$$X_2\delta_{42}+X_3\delta_{43}+X_4\delta_{44}+X_5\delta_{45}+\Delta_{4T}=0.$$

Here

$$\delta_{42}=\delta_{24}=16{,}0; \qquad \delta_{43}=\delta_{34}=-274{,}2;$$
$$\delta_{44}=1603{,}9; \qquad \delta_{45}=-2247{,}1;$$
$$\Delta_{4T}=\left(141{,}3\,\frac{30}{50}+141{,}3\,\frac{12{,}5}{50}\right)k=(84{,}8+35{,}3)\,k=120{,}1k.$$

As a consequence,

$$16{,}0X_2-274{,}2X_3+1603{,}9X_4-2247{,}1X_5+120{,}1k=0. \tag{4}$$

For section 5

$$X_3\delta_{53}+X_4\delta_{54}+X_5\delta_{55}+\Delta_{5T}=0.$$

Here

$$\delta_{53}=\delta_{35}=120,2;\quad \delta_{54}=\delta_{45}=-2247,1;$$

$$\delta_{55}=9815,8;$$

$$\Delta_{5T}=\left(141,3\,\frac{12,5}{50}+\frac{141,3\cdot 7,5}{50}\right)k=(35,3+21,2)\,k=56,5k.$$

As a consequence,

$$120,2X_3-2247,1X_4+9815,8X_5+56,5k=0. \tag{5}$$

We thus arrive at the following system of equations:

$$\left.\begin{array}{lr}
487,8X_0-350,2X_1+32X_2+\ldots=0, & (0)\\
-175,1X_0+441,1X_1-47,7X_2+7,66X_3+ & \\
+\ldots+141,3k=0, & (1)\\
16,0X_0-47,7X_1+345,9X_2-47,7X_3+16X_4+ & \\
+\ldots+282,6k=0, & (2)\\
7,66X_1-47,7X_2+420,2X_3-274,2X_4+120,2X_5+ & \\
+226,1k=0, & (3)\\
16X_2-274,2X_3+1603,9X_4-2247X_5+120,1k=0, & (4)\\
+120,2X_3-2247X_4+9815,8X_5+56,5k=0. & (5)
\end{array}\right\} \tag{13.18}$$

From the solution of this system we find:

$$X_0=-0,324k;\quad X_1=-0,540k;\quad X_2=-0,973k;$$
$$X_3=-0,811k;\quad X_4=-0,2905k;\quad X_5=-0,0623k.$$

The actual stresses, according to Formulas (13.2) may be written in the form:

$$\sigma_0=0,324k\varphi=0,324k\left(\frac{1}{7}-\frac{x^6}{b^6}\right);$$

$$\sigma_1=0,540k\left(\frac{1}{7}-\frac{x^6}{b^6}\right);$$

$$\sigma_2=0,973k\left(\frac{1}{7}-\frac{x^6}{b^6}\right);$$

$$\sigma_3=0,811k\left(\frac{1}{7}-\frac{x^6}{b^6}\right);$$

$$\sigma_4=0,2905k\left(\frac{1}{7}-\frac{x^6}{b^6}\right);$$

$$\sigma_5=0,0623k\left(\frac{1}{7}-\frac{x^6}{b^6}\right).$$

In order to understand the effect of the temperature differential along the inboard ribs, we also determine the unknowns for a uniform temperature distribution over the entire wingspan. In this case, all the terms of the system of equations (13.18) remain valid except for the two free terms of Eqs. (0) and (1), which will now equal:

$$\Delta_{0T} = 2\cdot 141{,}3\,\frac{30}{50}\,k = 169k;$$

$$\Delta_{1T} = \left(141{,}3\,\frac{30}{50} + 141{,}3\right)k = 226{,}1k.$$

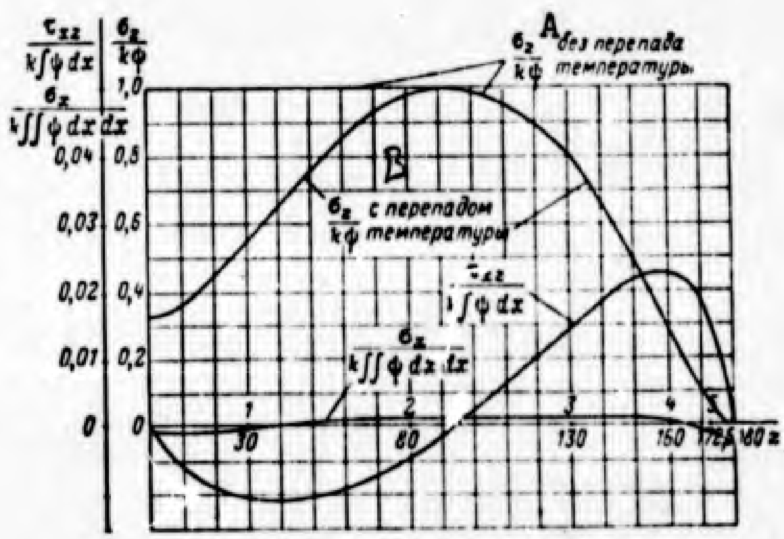


Fig. 13.6. A) Without temperature differential; B) with temperature differential.

Substituting these values into equation system (13.18) and solving, we obtain

$$X_0 = -1{,}00k;\quad X_1 = -1{,}00k;\quad X_2 = -1{,}00k;$$

$$X_3 = -0{,}805k;\quad X_4 = -0{,}290k;\quad X_5 = 0{,}0622k$$

while the stresses will be

$$\sigma_0 = 1{,}00k\left(\frac{1}{7} - \frac{x^6}{b^6}\right);\quad \sigma_1 = 1{,}00k\left(\frac{1}{7} - \frac{x^6}{b^6}\right);$$

$$\sigma_2 = 1{,}00k\left(\frac{1}{7} - \frac{x^6}{b^6}\right);\quad \sigma_3 = 0{,}805k\left(\frac{1}{7} - \frac{x^6}{b^6}\right);$$

$$\sigma_4 = 0{,}290k\left(\frac{1}{7} - \frac{x^6}{b^6}\right);\quad \sigma_5 = 0{,}0622k\left(\frac{1}{7} - \frac{x^6}{b^6}\right).$$

It is clear from the curves given for $\sigma_z/k\psi$ in Fig. 13.6 that for the right-hand half of the curve the thermal stresses at the free end of the wing increase rapidly from the zero value, while for the portion equal to roughly the length of the chord 2b, they reach a maximum value equal to the stresses in a wing of infinite span.

When we take into account the temperature differential along the inboard rib of the wing, the stress pattern for the zone a length 2b away from the wing tip will remain the same as when the differential is neglected, but then the thermal stresses fall rapidly and reach roughly 1/3 of the maximum value on the wing's axis of symmetry.

Let us now determine the stresses $\bar{\tau}_{xz}$ and $\bar{\sigma}_x$ accompanying the longitudinal stresses:

$$\bar{\tau}_0 = 0;$$

$$\bar{\tau}_{0-30} = -\frac{0{,}540-0{,}324}{30}\,\frac{\delta_\sigma}{\delta_\tau}\,k\int_b^x \psi\, dx =$$

$$= -\frac{0{,}216}{30}\,\frac{\delta_\sigma}{\delta_\tau}\,k\int_b^x \left(\frac{1}{7}-\frac{x^6}{7b^6}\right)dx = -0{,}0072\cdot 1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^5}\right) =$$

$$= -0{,}0082k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right);$$

$$\bar{\tau}_{30-80} = -\frac{0{,}973-0{,}540}{50}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) =$$

$$= -\frac{0{,}433}{50}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) = -0{,}0099k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right);$$

$$\bar{\tau}_{80-130} = -\frac{0{,}811-0{,}973}{50}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) =$$

$$= +\frac{0{,}162}{50}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) = 0{,}0037k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right);$$

$$\bar{\tau}_{130-160} = -\frac{0{,}2905-0{,}811}{30}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) =$$

$$= +\frac{0{,}5205}{30}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) = 0{,}0199k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right);$$

$$\bar{\tau}_{160-172{,}5} = -\frac{0{,}0623-0{,}2905}{12{,}5}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) =$$

$$= +\frac{0{,}2282}{12{,}5}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) = 0{,}0209k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right);$$

$$\bar{\tau}_{172{,}5-180} = -\frac{0-0{,}0623}{7{,}5}\,1{,}143k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right) =$$

$$= 0{,}0095k\left(\frac{x}{7}-\frac{x^7}{7b^6}\right);$$

$$\tau_{180} = 0.$$

Further

$$\bar{\sigma}_{x(0-15)} = -\frac{-0,0082}{15}\,\frac{\delta_2}{\delta_1}\,k\int\limits_b^x\int\limits_b^x \varphi\,dx\,dx =$$

$$= -\frac{0,0082\cdot 1,143}{15}\,k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$= -0,00094k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right);$$

$$\bar{\sigma}_{x\,(15-55)} = -\frac{(-0,0099+0,0082)}{40}\,1,143k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$= -\frac{0,0017}{40}\,1,143k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$= -0,000049k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right);$$

$$\bar{\sigma}_{x(55-105)} = +\frac{0,0037+0,0099}{50}\,1,143k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$= +0,00031k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right);$$

$$\bar{\sigma}_{x\,(105-145)} = +\frac{0,0199-0,0037}{40}\,1,143k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$=0,00047k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right);$$

$$\bar{\sigma}_{x\,(145-166,25)} = +\frac{0,0209-0,0199}{21,25}\,1,143k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$=0,000054k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right);$$

$$\bar{\sigma}_{x\,(166,25-176,25)} = -\frac{0,0095-0,0209}{10}\,1,143k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$= -0,00130k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right);$$

$$\bar{\sigma}_{x\,(176,25-180)} = -\frac{0,0095\cdot 1,143k}{3,75}\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right)=$$

$$= -0,00290k\left(\frac{x^4}{12b^2}-\frac{x^2}{6}+\frac{b^2}{12}\right).$$

Figure 13.6 gives the curves for the dimensionless values of the stress τ_{zx} and σ_x.

If we neglect the temperature differential, the values of the stresses τ_{zx} and σ_x at the zone from the wing tip to the maximum values will coincide with the stress curves obtained when we allow for the temperature differential, but from there to the wing axis of symmetry they will equal zero.

The distribution pattern for stresses computed with allowance for

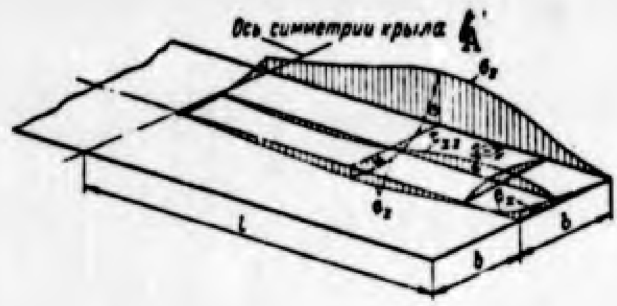


Fig. 13.7. A) Wing axis of symmetry.

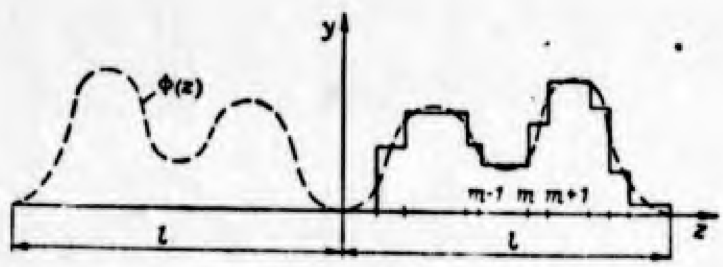


Fig. 13.8

the temperature differential for one-fourth of the wing is shown in Fig. 13.7.

To conclude, in connection with the use of finite-difference equations to determine thermal stresses, we must note that in deriving the expressions for the coefficients on the unknowns in the finite-difference equations, as well as for the free temperature terms entering into these equations, we assumed that the temperature was constant in the longitudinal direction along the wing cantilevers, while that it is represented by the function $\rho(x)$ for the transverse direction.

Let us consider a still more general case in which the temperature also varies along the span in accordance with the function $\Phi(z)$ (Fig. 13.8). In this case, the coefficients on the unknowns in Eqs. (13.13) are determined from the formulas (13.14) or (13.16) used before, since they do not depend on the temperature.

In order to determine the free term Δ_{mT}, it is necessary to replace the actual function $\Phi(z)$ by a step function as, for example, is shown by the solid line in Fig. 13.8. After this substitution, the values of Δ_{mT} may be found from Formula (13.17).

From this it is clear that by using finite-difference equations we can determine, without especial difficulty, the thermal stresses even for the more general case in which the temperature field along the wing-span is described by a function $\Phi(z)$, and in the transverse wing sections by the function $\rho(x)$.

2. EFFECT OF THERMAL STRESSES ON THE REDUCTION IN STIFFNESS IN TORSION

The wing thermal stresses associated with aerodynamic heating reach a maximum value at the end of the aircraft takeoff run. After the aircraft has reached its maximum speed, the temperature along the chord will gradually equalize itself and, as a consequence, the thermal stresses will drop from the maximum value to zero [36], [37].

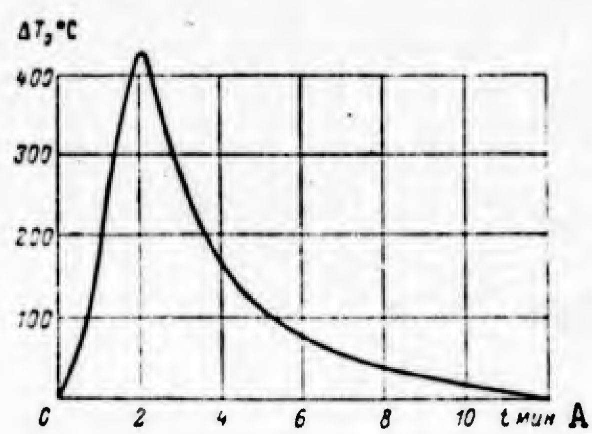


Fig. 13.9. A) Min.

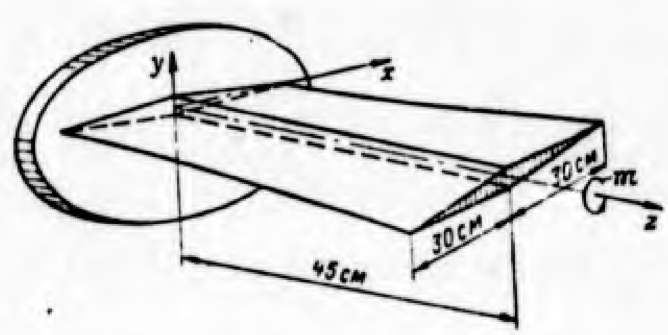


Fig. 13.10

Thus, for example, as we can see from [37], if an aircraft with a steel wing having a thin diamond profile reaches a speed M = 4 at an altitude of 15,000 m in the course of a two-minute takeoff, at the end of the takeoff, the difference between the temperature of the wing center panel and the temperature of the edges will be roughly 425°C, i.e., $\Delta T_0 \simeq 425$°C. During the next 2-3 min of flight at M = 4, at the same altitude, the value ΔT_0 will decrease sharply, and after 10-12 min have elapsed from the beginning of takeoff, we can assume $\Delta T_0 = 0$.

The nature of the time variation in ΔT_0 was shown in Fig. 13.9 [37]. The time variation of the thermal stresses is of the same nature.

As we can see from the thermal-stress calculation, the axial stresses σ_z are compression stresses near the leading and trailing edges of the wing, and tensile stresses in the wing center panel. This load distribution along the chord reduces the wing stiffness in torsion. In order to see this, we consider a wing that carries a torsion moment M under conditions of aerodynamic heating (Fig. 13.10).

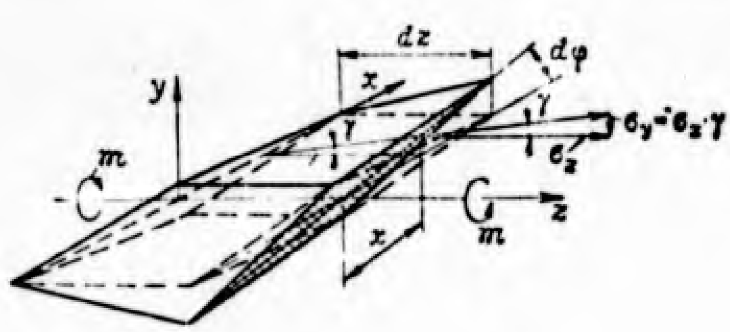


Fig. 13.11

We isolate an elementary strip from the wing by means of the two sections z = const and z + dz = const. It is clear from Fig. 13.11 that under the action of the torsion moment M, the fiber of length dz located a distance x away from the axis of rigidity z will turn about the xz plane by the angle

$$\gamma = \frac{x d\varphi}{dz}.$$

Decomposing the stress vector σ_z along the direction of the fiber and along the y-axis direction, and assuming that the angle γ is small, we obtain

$$\sigma_\gamma = \frac{\sigma_z}{\cos\gamma} \approx \sigma_z \text{ and } \sigma_y = \sigma_z \operatorname{tg}\gamma \approx \sigma_z \gamma.$$

The component σ_y creates the elementary torsion moment

$$\Delta M_T = -\sigma_z x^2 \frac{d\varphi}{dz}\, dF. \tag{13.19}$$

about the z axis.

The minus sign appears here owing to the fact that when σ_z is positive the moment ΔM_T is directed opposite to the torsional moment $\mathfrak{M}$.

By integrating Expression (13.19) over the wing cross section, we obtain the resultant torsional moment due to the thermal stresses σ_z:

$$M_T = -\int_F \sigma_z x^2 \frac{d\varphi}{dz}\, dF. \tag{13.20}$$

Thus, the total torsional moment M_z acting in section z will equal

$$M_z = \mathfrak{M} - \int_F \sigma_z x^2 \frac{d\varphi}{dz}\, dF. \tag{13.21}$$

If we neglect the effect of the attachment, then

$$M_z = (GJ)_T \frac{d\varphi}{dz}, \tag{13.22}$$

where $(GJ)_T$ is the stiffness of the heated wing in torsion.

As a consequence,

$$(GJ)_T \frac{d\varphi}{dz} = \mathfrak{M} - \int_F \sigma_z x^2 \frac{d\varphi}{dz}\, dF$$

or

$$\mathfrak{M} = \left[(GJ)_T + \int_F \sigma_z x^2 dF\right] \frac{d\varphi}{dz}. \tag{13.23}$$

From this we have the torsion angle

$$\varphi_{\text{кр}} = \int_0^z \frac{\mathfrak{M}\, dz}{\left[(GJ)_T + \int_F \sigma_z x^2 dF\right]}. \tag{13.24}$$

The expression in brackets is the effective stiffness of the heated wing in torsion:

$$(GJ)_{эф}=(GJ)_T+\int_F \sigma_z x^2\,dF. \qquad (13.25)$$

Since the value of the integral will be negative when the aircraft is accelerating, the axial thermal stresses will reduce the wing torsional stiffness.

If we consider only the time variation of the second term in Expression (13.25), we can state (see Fig. 13.9) that the effective stiffness will be at a minimum at the end of aircraft acceleration and will then increase gradually, reaching its initial value at $\sigma_z = 0$. As for the first term in this expression $(GJ)_T$, its effect on the reduction in effective stiffness reaches its maximum at $\sigma_z = 0$, i.e., when the wing section has heated up to the maximum temperature over the entire length of the chord.

Comparing Expression (13.25) with the initial wing torsional stiffness GJ, we obtain

$$\nu=\frac{(GJ)_{эф}}{GJ}=\frac{(GJ)_T}{GJ}+\frac{\int_F \sigma_z x^2 dF}{GJ}. \qquad (13.26)$$

As an example, let us find the value of ν and the torsion angle for a wing having a 4% relative thickness (see Fig. 13.10) for the following conditions: the material is nickel-base steel, the maximum temperature does not exceed 600°C, $\Delta T_0(t)$ varies over the time t in accordance with the curve of Fig. 13.9, and $\rho = x^2/b^2$.

We shall assume that at temperatures up to 600°C, nickel-base steel does not change in its properties, so that in our example, which is only designed to illustrate the essential features of the problem, we let $(GJ)_T = GJ$.

With these assumptions, and also assuming that σ_z depends on time, we can write Expression (13.26) in the form:

$$\nu = 1 + \frac{\int_F \sigma_z(t)\, x^2 dF}{GJ}. \qquad (13.27)$$

Keeping in mind the fact that

$$\sigma_z(t) = k(1-\varphi)\psi,$$

where

$$k = \alpha \Delta T_0(t) E,$$

$$\psi = \frac{1}{6} - \frac{x^2}{b^2},$$

we obtain

$$\int_F \sigma_z(t)\, x^2 dF = 2\int_0^b \alpha \Delta T_0(t) E (1-\varphi)\left(\frac{1}{6} - \frac{x^2}{b^2}\right) x^2 H \left(1 - \frac{x}{b}\right) dx$$

or

$$\int_F \sigma_z(t)\, x^2 dF = -\frac{7}{180}\, \alpha \Delta T_0(t) E (1-\varphi) H b^3.$$

In the notation used here for the diamond-profile geometry, the expression for GJ will be

$$GJ = \frac{G}{7{,}2}\, \frac{bH^3}{\left(1 + \frac{H^2}{4b^2}\right)}.$$

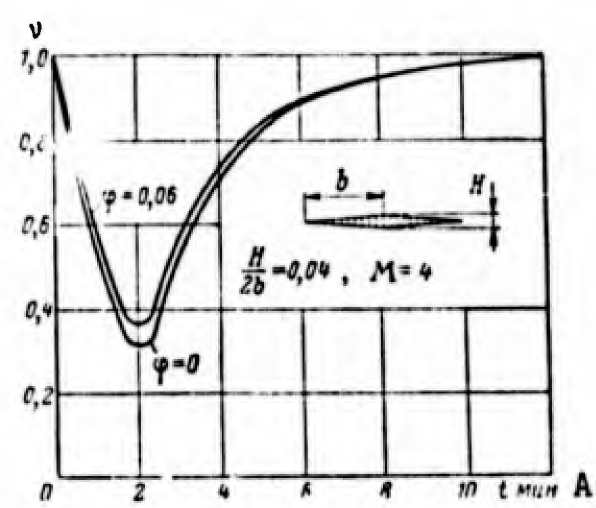


Fig. 13.12. A) Min.

Thus:

$$\nu=1-\frac{7\cdot 7,2}{180}\frac{b^2}{H^2}\left(1+\frac{H^2}{4b^2}\right)\alpha\Delta T_0(t)(1-\varphi)\frac{E}{G}$$

or, with $E/G = 2.6$ and $\alpha = 14\cdot 10^{-6}$

$$\nu=1-10,2\cdot 10^{-6}\frac{b^2}{H^2}\Delta T_0(t)(1-\varphi). \qquad (13.28)$$

For a wing with a 4% relative thickness, $H = 0.04\cdot 2b = 0.08b$, and

$$\nu=1-0,0016\Delta T_0(t)(1-\varphi).$$

For a wing of infinite span,

$$\varphi=0 \text{ and } \nu=1-0,0016\Delta T_0(t).$$

The design curve for the variation in ν with time for $\varphi = 0$, corresponding to our conditions, is shown in Fig. 13.12. Here at the end of acceleration the maximum stresses in the wing edges for $x = \pm b$ will equal

$$\sigma_{z\,max}=14\cdot 10^{-6}\cdot 425\cdot 2,2\cdot 10^{6}\left(\frac{1}{6}-\frac{b^2}{b^2}\right)=-10\,900 \text{ kgf/cm}^2, \qquad (a)$$

while the maximum stresses in the wing center panel ($x = 0$) will be

$$\sigma_{z\,max}=+2180 \text{ kgf/cm}^2. \qquad (b)$$

The ultimate strength is

$$\sigma_b=11\,000 \text{ kgf/cm}^2.$$

The wing considered (see Fig. 13.10) has an aspect ratio of 45/ /30 = 1.5. Thus in accordance with Fig. 12.13, the value of $1 - \varphi$ at section $z = 0$ will equal 0.95. As a consequence, the value of ν at section $z = 0$ for a wing of low aspect ratio will equal

$$\nu=1-0,0016\Delta T_0(t)0,94.$$

Figure 13.12 gives the curve for the variation of ν in time for $1 - \varphi = 0.95$ or $\varphi = 0.06$.

Thus at the end of the acceleration period, the stiffness of a wing with a 4% relative thickness at section $z = 0$ will be reduced by 68% for a wing of infinite span, and by 64% for a wing with aspect ratio of 1.5.

At the end of acceleration, the maximum stresses for $1 - \varphi = 0.94$ and $z = 0$ will be:

at the wing edges ($x = \pm b$)

$$\sigma_{z\,max} = -10\,900 \cdot 0{,}94 = -10\,200 \text{ kgf/cm}^2 \qquad (c)$$

at the wing center panel ($x = 0$)

$$\sigma_{z\,max} = 2050 \text{ kgf/cm}^2$$

In other sections of a wing of low aspect ratio, the stresses will be rapidly attenuated, while for a wing of infinite span, the maximum stresses for Expressions (a) and (b) will be the same in all sections.

Let us now look at the determination of the torsion angle (13.24), which in our case will equal

$$\varphi_{\text{кр}} = \frac{\mathfrak{M}}{GJ} \int_0^z \frac{dz}{1 + \frac{\int \sigma_z x^2 dF}{GJ}}.$$

For a wing with a 4% relative thickness,

$$\varphi_{\text{кр}} = \frac{\mathfrak{M}}{GJ} \int_0^z \frac{dz}{1 - 0{,}0016\Delta T_0(t)(1-\varphi)}, \qquad (13.29)$$

or, according to (12.24)

$$\varphi = 0{,}0600 \operatorname{ch} 0{,}098z \cos 0{,}0378z + 0{,}0326 \operatorname{sh} 0{,}098z \sin 0{,}0378z.$$

The curve for $(1 - \varphi)$ is given in Fig. 12.13. Here it is quite suitable to approximate the complicated function for $(1 - \varphi)$ by a simpler function, namely, the function $0.88 \cos u$, where $u = \pi z/2\ell$. The curve for the function $0.88 \cos u$ is shown in Fig. 12.13 by the dashed line.

In Expression (13.29), we substitute the value $0.88 \cos u$ for $(1 - \varphi)$, and obtain

$$\varphi_{\text{кр}} = \frac{2}{\pi} \frac{\mathfrak{M}l}{GJ} \int_0^u \frac{du}{1 - 0{,}0016\Delta T_0(t)\, 0{,}88 \cos u}. \qquad (13.30)$$

The wing torsion angle at the end of aircraft acceleration, when $\Delta T_0 = 425°C$, will equal

$$\varphi_{\text{кр}} = \frac{2}{\pi} \frac{\mathfrak{M}l}{GJ} \int_0^u \frac{du}{1 - 0{,}6\cos u}.$$

From which we have

$$\varphi_{\text{кр}} = \frac{2}{\pi} \frac{\mathfrak{M}l}{GJ} \frac{2}{\sqrt{1 - 0{,}6^2}} \operatorname{arc\,tg}\left(\sqrt{\frac{1 + 0{,}6}{1 - 0{,}6}} \operatorname{tg}\frac{u}{2}\right),$$

or

$$\varphi_{\text{кр}} = \frac{5}{\pi} \frac{\mathfrak{M}l}{GJ} \operatorname{arc\,tg}\left(2 \operatorname{tg}\frac{u}{2}\right).$$

For $z = \ell$ and the parameter $u = \pi/2$,

$$\varphi_{\text{кр}} = 1{,}77 \frac{\mathfrak{M}l}{GJ}.$$

For a wing of infinite span, $\varphi = 0$. In this case, according to Expression (13.29) at the end of takeoff, the angle

$$\varphi_{\text{кр}} = \frac{\mathfrak{M}z}{GJ} \frac{1}{1 - 0{,}0016 \cdot 425} = 3{,}14 \frac{\mathfrak{M}z}{GJ}.$$

Thus, at the end of acceleration, the torsion angle for a wing of low aspect ratio with a 4% relative thickness will be increased by 77% owing to the thermal stresses, while for a wing of infinite span the increase will be 214%.

Thus at the end of acceleration of a high-speed flying craft, the thermal stresses in solid wings with diamond profile will be great, and will considerably reduce the wing torsional stiffness.

Величины 1	Единица измерения 2	3 Система СИ		6 Система МКГСС (метр, килограмм-сила, секунда)		7 Соотношения между единицами измерения системы МКГСС и системы СИ
		Сокращенное обозначение 4 единицы измерения	5 Размер единицы	Сокращенное обозначение единицы 4 измерения	5 Размер единицы	
Длина 8	метр 19	м		м		
Масса 9	килограмм 20	кг 30		39 кГ·сек²/м		
Время 10	секунда 21	сек 31		сек 31		

Термодинамическая температура 11	градус Кельвина 22	°K		°C		T° K=273°+T° C.
Плоский угол 12	радиан 23	рад 32		рад 32		
Площадь 13	квадратный метр 24	$м^2$		$м^2$		
Объем 14	кубический метр 25	$м^3$		$м^3$		
Скорость 15	метр в секунду 26	м/сек 33		м/сек 33		
Сила 16	ньютон 27	н 34	кг·м/$сек^2$ 37	кГ 40		1 кГ=9,81 н 40 34
Давление (механическое напряжение) 17	ньютон на квадратный метр 28	н/$м^2$ 35		кГ/$м^2$ 41		1 кГ/$м^2$=9,81 н/$м^2$ 41 35
Работа, энергия 18	джоуль 29	дж 36	н·м 38	кГ·м 42		1 кГ·м=9,81 дж 42 36

1) Quantities; 2) unit of measurement; 3) International System of Units; 4) abbreviation for unit of measurement; 5) dimensions of unit; 6) MKGSS (meter, kilogram-force, second) system; 7) relationships between units of MKGSS system and International System; 8) length; 9) mass; 10) time; 11) thermodynamic temperature; 12) plane angle; 13) area; 14) volume; 15) velocity; 16) force; 17) pressure (mechanical stress); 18) work, energy; 19) meter; 20) kilogram; 21) second; 22) degree Kelvin; 23) radian; 24) square meter; 25) cubic meter; 26) meters per second; 27) Newton; 28) Newton's per square meter; 29) Joule; 30) kg; 31) sec; 32) rad; 33) m/sec; 34) n; 35) n/m^2; 36) dzh; 37) kg·m/sec^2; 38) n·m; 39) kgf·sec^2/m; 40) kgf; 41) kgf/m^2; 42) kgf·m.

REFERENCES

1. Adadurov, R.A., Opredeleniye kasatel'nykh napryazheniya vblizi zadelki [Determination of Shear Stresses Near a Constraint], Trudy TsAGI [Transactions of the N.Ye. Zhukovskiy Central Aerohydrodynamics Institute], No. 164, 1947.
2. Balabukh, L.I., Raschet na prochnost' konicheskikh kessonov [Strength Calculations for Conical Shells], Trudy TsAGI, No. 640, 1947.
3. Belyayev, V.N., Raschet svobodnonesushchikh kryl'yev [Design of Freely Supporting Wings], TVF [Air Fleet Engineering], No. 7 and No. 8/9, 1932.
4. Vlasov, V.Z., Tonkostennyye prostranstvennyye sistemy [Thin-Walled Space Systems], Gosstroyizdat [State Publishing House for Literature on Construction, Architecture, and Structural Materials], 1958.

5. Vol'mir, A.S., Gibkiye plastinki i obolochki [Flexible Plates and Shells], GTTI [State Publishing House for Technical and Theoretical Literature], 1956.

6. Geytvud, V.Ye., Temperaturnyye napryazheniya [Thermal Stresses], IL [Foreign Literature Press], 1959.

7. Yelenevskiy, G.S., Strontel'naya mekhanika kryla peremennogo secheniya [Structural Mechanics of Variable-Section Wing], Oborongiz [State Scientific and Technical Publishing House for Defense Literature], 1954.

8. Yelpat'yevskiy, A.N., Konovalov, B.A., Primeneniye odnogo variatsionnogo metoda k raschetu konicheskikh obolochek [Application of One Variational Method to the Design of Conical Shells], Izv. AN SSSR [Bulletin of the Academy of Sciences USSR], No. 8, 1958.

9. Kan S.H., Sverdlov, I.A., Raschet samoleta na prochnost' [Aircraft Strength Calculations], Oborongiz, 1958.

10. Kantorovich, L.V., Krylov, K.I., Priblizhennyye metody vysshego analiza [Approximate Methods of Higher Analysis], Fizmatgiz [State Publishing House for Physical-Mathematical Literature], 1962.

11. Kiselev, V.F., Metod rascheta treugol'nogo kryla na prochnost' s uchetom uprugoy zadelki [A Method for Delta-Wing Strength Calculations with Allowance for Elastic Constraint], Trudy TsAGI, No. 703, Oborongiz, 1957.

12. Klimov, V.I., Raschet otkrytykh obolochek typa aviakonstruktsiy [Designing Open Shells of Aircraft Type], Oborongiz, 1957.

13. Leybenzon, L.S., Variatsionnyye metody resheniya zadach teorii uprugosti [Variational Methods for Solving Problems in Elasticity Theory], Gostekhizdat [State Publishing House for Techni-

cal and Theoretical Literature], 1943.

14. Melan, E., Parkis, G., Temperaturnyye napryazheniya, vyzvannyye statsionarnymi polyami, [Thermal Stresses Caused by Stationary Fields], Fizmatgiz, 1958.

15. Mar'in, V.A., Ustoychivost' tsilindricheskoy paneli pri sdvige. Raschet prostranstvennykh konstruktsiy [Stability of a Cylindrical Panel in Shear. Design of Three-Dimensional Structures], Vol. 5, Gosstroyizdat, 1959.

16. Obraztsov, I.F., Metody rascheta na prochnost' kessonnykh konstruktsiy typa kryla [Strength Calculation Methods for Wing-Type Shell Structures], Oborongiz, 1960.

17. Odinokov, Yu.G., Raschet tonkostennykh konstruktsiy typa kryla fyuzelyazha i opereniya samoleta [Designing Thin-Wall Structures of the Aircraft Wing, Fuselage, and Empennage Type], Trudy Kazanskogo aviatsionnogo instituta [Transactions of the Kazan' Aviation Institute], Vol. XVIII, Kazan', 1946.

18. Papkovich, V.I., Trudy po strontel'noy mekhanike korablya [Reports on the Structural Mechanics of a Ship], Vols. I, II, III and IV, Sudpromgiz [State All-Union Publishing House for the Shipbuilding Industry], 1962.

19. Rabinovich, I.M., Strontel'naya mekhanika sterzhnevykh sistem [Structural Mechanics of Bar Systems], Gostroyizdat, 1946.

20. Romashevskiy, A.Yu., Klimov, V.I., Stroitel'naya mekhanika samoleta [Structural Mechanics of the Airplane], Trudy MAI [Transactions of the Moscow Aviation Institute], 1964.

21. Rostovtsev, G.G., Panovko, Ya.G., Strontel'naya mekhanika samoleta [Structural Mechanics of the Airplane], Vols. II, III, LVVIA [Leningrad Air Force Engineering Academy], 1954.

22. Sibiryakov, V.A., Raschet konicheskoy obolochki na temperatur-

noye vozdeystviye i proizvol'nuyu vneshnyuyu nagruzku [Conical-Shell Calculations for Thermal Effects and Longitudinal External Load], Izv. VUZov, seriya "Aviatsionnaya tekhnika [Bulletin of the Higher Educational Institutions, "Aviation Engineering" Series], No. 2, 1959 and No. 1, 1960.

23. Strugunov, V.M., Teoreticheskoye i eksperimental'noye issledovaniye tonkostennykh balov [Theoretical and Experimental Investigation of Thin-Web Beams], Trudy TsAGI No. 349, 1938.

24. Timoshenko, S.P., Teoriya uprugosti [Elasticity Theory], ONTI [United Scientific Technical Publishing House, 1937.

25. Timoshenko, S.P., Ustoychivost' uprugikh sistem [Stability of Elastic Systems], OGIZ [United State Publishing Houses], 1946.

26. Timoshenko, S.P., Plastinki, i oblochki [Plates and Shells] OGIZ, 1948.

27. Umanskiy, A.A., Stroitel'naya mekhanika samoleta [Structural Mechanics of the Airplane], Oborongiz, 1961.

28. Feppl', A., Feppl', L., Sila i deformatsiya [Force and Deformation], OGIZ, 1933.

29. Feofanov, A.F., Raschety tonkostennykh konstruktsiy [Calculations for Thin-Walled Structures], Oborongiz, 1953.

30. Feofanov, A.F., Stroitel'naya mekhanika tonkostennykh konstruktsiy [Structural Mechanics of Thin-Walled Structures], Oborongiz, 1958.

31. Frolov, V.M., Primeneniye metoda korrektiruyushchikh funktsiy v raschetakh deformatsii konsol'nykh plastin [Application of the Method of Correcting Functions to Calculations for Cantilever Plate Deformation], Trudy TsAGI, 705, Oborongiz, 1957.

32. Wagner, H. and Kimm, G., Bauelemente der Flugzeugen [Structural Elements of Aircraft], Berlin, 1940.

33. Levy, S., Influence Coefficients of Tapered Cantilever Beams Computed on S.E.A.C., Journal of Appl. Mech., March, 1953.

34. Williams, D., Recent Development in the Structural Approach to Aerolasticity Problems, Journ. of Roy. Aeronaut. Soc., Vol. 58, June, 1954.

35. Stein, M., Anderson, E., Hedgepeth, M., Report NACA No. 1131, 1953.

36. Budiansky, B., and Mavere, J., Influence of Aerodynamic Heating on the Effective Torsional Stiffness of Thin Wings, IAS No. 579, 1956.

37. Kochanski, S.H. and Argyris, L.H., Some Effects of Kinetic Heating on the Stiffness of Thin Wings, Aircraft Engineering, No. 344, 1957; No. 348, 1958.