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POST-OPTIMALITY ANALYSIS VIA THE REVERSE SIMPLEX METHOD
AND THE TARRY METHOD

by

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C. van de Panne¹

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ABSTRACT

In practical applications of linear programming, not only the optimal solution but also solutions which have a somewhat lower value of the objective function are of interest. It is therefore desirable to generate all extreme-point solutions satisfying the constraints and giving a value of the objective function which differs by at most a given amount from the value for the optimal solution. Two methods are considered for generating these extreme points. The first method is called the reverse Simplex method, since it reverses the Simplex method for linear programming, the second is basic on the Tarry method for traversing a network such that all nodes are visited. The two methods are explained in detail, applied to an example and compared with each other.

1. INTRODUCTION

Linear programming can be used for the solution of a great variety of problems. The solution found by linear programming is a unique one, or, if not, there are only a few extreme-point solutions. However, the optimal solution or solutions are not the only ones of interest, since the decision maker will usually also be interested in near-optimal solutions. This is so, because in many cases it will be impossible to specify exactly the constraints of the problem; furthermore, the objective function does not always reflect precisely the preferences of the decision maker.

The linear programming formulation of a production planning problem can be said to give the most important objectives and constraints of the problem; other factors may and frequently do remain outside the formulation, either because it is too difficult to incorporate them or because the decision maker is only vaguely aware of them. The only way to deal with these factors is to face the decision maker with alternative decisions and to leave it to him to select the alternative he prefers. The difficulty then is that the number of all conceivable alternatives is so large that the decision maker cannot evaluate them.

This suggests the following approach. First the optimal solution for the linear programming formulation is determined. Any additional binding constraints imposed on the solution will always lower the objective function and so will any other objectives which the decision maker has in mind besides the main objective of maximizing profits. However, these factors will only be able to decrease the objective function of the linear programming problem by a limited amount. The optimal solution for the

decision maker is therefore one which satisfies the constraints of the linear programming problem and which has a value of the objective function differing only by a limited amount from the optimal solution of the programming problem. The decision maker may therefore find his optimal solution by considering all solutions which satisfy these requirements. There will of course be an infinity of these solutions, but it suffices to give only the extreme-point solutions.

The extreme-point solution can be determined by linear programming using a method in which the amount by which the value of the objective may differ from that of the optimal value in the programming formulation is varied from 0 to an amount the decision maker does not want to lose in any case. This method can be said to use the Simplex method in the reverse direction and is therefore called the reverse Simplex method.

In the next section this approach will be presented and applied to a simple example.

A different approach, based on the theory of graphs which follows from a proposal of Charnes and Cooper for a problem of a similar kind, is given in Section 3. Both methods are explained in detail, applied to a small example and compared with each other.

2. GENERATION OF NEAR-OPTIMAL SOLUTIONS BY THE REVERSE SIMPLEX METHOD

2.1. Formulation of the Problem

The linear programming problem can be formulated as follows. Maximize

$$(2.1) \quad f(x) = p'x$$

subject to

$$(2.2) \quad Ax = b ,$$

$$(2.3) \quad x \geq 0.$$

A is an $m \times n$ matrix, p and x are n -element column vectors and b is column vector of m elements. Let the value of the objective function for the optimal solution² x^0 be f^0 . It is assumed that the optimal solution is unique. The difference of f^0 and the value of the objective function for some solution is called the loss of that solution; any feasible solution which is not optimal has a positive loss.

Suppose that the maximum loss allowed is k . This means that only solutions are considered which satisfy, in addition to (2.2) and (2.3), the inequality

$$(2.4) \quad f(x) = p'x \geq f^0 - k.$$

The problem is now to generate all extreme-point solutions satisfying (2.2), (2.3) and (2.4). For $k = 0$, we find, of course, the optimal solution x^0 . For increasing values of k an increasing number of extreme-point solutions must be found; for some finite value of k the extreme-point solutions of (2.2), (2.3) and (2.4) are in any case the same as those of (2.2) and (2.3) if these are finite. In practical problems the value of k will be relatively small, so that the number of extreme-point solutions of (2.2) - (2.4) will not be too large.

²Throughout this paper, any superindex is to be interpreted as such and not as an exponent.

To find all these extreme-point solutions, two methods may be used. The first of these methods is what is called here the reverse Simplex method and will be treated in the following subsections; the second method which is based on graph theory is presented in the next section.

2.2. The Reverse Simplex Method

The idea on which the reverse Simplex method is based is quite simple. In the ordinary Simplex method variables are introduced into the basis which increase the value of the objective function until the optimal solution is found. In the reverse Simplex method the optimal solution is taken as a starting point for the generation of the extreme-point solution of (2.2) - (2.4). In this and in subsequent solutions variables are introduced into the basis which decrease the value of the objective function; the maximal loss k sets a lower bound to this decrease.

The main difference between the ordinary Simplex method and the reverse Simplex method is that in the ordinary Simplex method only one variable is introduced into the basis; in the reverse Simplex method all variables which decrease the objective function are sooner or later introduced into the basis. Corresponding with the normal Simplex method is the effect of iterations on the objective function: in the normal Simplex method the objective function increases in each iteration, while in the reverse Simplex method it decreases with each iteration, provided there is no degeneracy.

Let us consider an iteration in the ordinary Simplex method. Tableau I of Table 1 gives a representation of the relevant parts of a Simplex tableau. x_j is the variable entering the basis since $p_j < 0$. x_r leaves the basis since it is connected with

$$(2.5) \quad \min_i \left(\frac{b_i}{a_{ij}} \mid a_{ij} > 0 \right) = \frac{b_r}{a_{rj}} .$$

Tableau I is then transformed into Tableau II with a_{rj} as the pivot element of the transformation. In Tableau II we find that the value of the objective function has increased by $-p_j b_r / a_{rj}$; furthermore, we find that the element in the column of x_r and in the last row, $-p_j / a_{rj}$, is positive.

TABLE 1. SUCCESSIVE TABLEAUX FOR A SIMPLEX ITERATION

Tableau	Value Basic Variable	x_j	x_r
I	b_1	a_{1j}	0
	.	.	.
	b_r	$\underline{a_{rj}}$	1
	.	.	.
	b_m	a_{mj}	0
	b_0	p_j	0
II	$b_1 - \frac{a_{1j} b_r}{a_{rj}}$	0	$-\frac{a_{1j}}{a_{rj}}$
	.	.	.
	$\frac{b_r}{a_{rj}}$	1	$\frac{1}{a_{rj}}$
	.	.	.
	$b_m - \frac{a_{mj} b_r}{a_{rj}}$	0	$-\frac{a_{mj}}{a_{rj}}$
	$b_0 - \frac{p_j b_r}{a_{rj}}$	0	$-\frac{p_j}{a_{rj}}$

Let us now consider introducing x_r again into the basis in Tableau II. Since $-p_j/a_{rj}$ is positive, the objective function decreases while doing so. The variable to leave the basis is found by comparing the ratios

$$(2.6) \quad \frac{b_1 - \frac{a_{1j}b_r}{a_{rj}}}{\frac{-a_{1j}}{a_{rj}}}, \dots, \frac{\frac{b_r}{a_{rj}}}{\frac{1}{a_{rj}}}, \dots, \frac{b_m - \frac{a_{mj}b_r}{a_{rj}}}{\frac{-a_{mj}}{a_{rj}}}$$

for positive denominators; these ratios reduce to

$$(2.7) \quad b_r + \frac{b_1 a_{rj}}{-a_{1j}}, \dots, b_r, \dots, b_r + \frac{b_m a_{rj}}{-a_{mj}}$$

for negative a_{1j}, \dots, a_{mj} . The minimum ratio is obviously b_r , so that x_r is the leaving basic variable. The resulting tableau which is obtained by using $1/a_{rj}$ as a pivot, is then again Tableau I. Hence we have proved that for each iteration of the Simplex method there is an iteration reversing the Simplex iteration by introducing a nonbasic variable with a positive element in the last row into the basis and determining the leaving basic variable in the usual manner.

We now consider the case in which p_j is positive and x_j is introduced into the basis; this means that the value of the objective function will decrease. The variable to leave the basis is determined as usual and turns out to be x_r ; Tableau I is therefore transformed into Tableau II with a_{rj} as the pivot. The objective function has decreased by $-p_j b_r/a_{rj}$ and for the element of Tableau II in the last row and in the column of x_r we find $-p_j/a_{rj}$, which is negative.

The ordinary Simplex method may now be applied to Tableau II by introducing x_r into the basis because $-p_j/a_{rj}$ is negative. Comparing the ratios as in (2.6) and (2.7), we find that x_j leaves the basis and Tableau I is

obtained again. We have therefore proved that to each iteration of what may be called the reverse Simplex method there corresponds an iteration of the ordinary Simplex method.

The reverse Simplex method will now be described in more detail. This method uses Simplex Tableaux which are quite the same as those of the ordinary Simplex method. Table 1 gives a representation of the Simplex tableau at iteration s , in which the tableau is rearranged in such a way that the columns of basic variables are arranged in a unit matrix.

TABLE 2. REPRESENTATION OF SIMPLEX TABLEAU AT THE s -th ITERATION OF THE REVERSE SIMPLEX METHOD

Val. Bas. Var.	Non-Basic Variables				Basic Variables				$f(x)$
b_1^s	a_{11}^s	.	a_{1j}^s	.	1	.	0	.	0
.
b_i^s	a_{i1}^s	.	a_{ij}^s	.	0	.	1	.	0
.
b_m^s	a_{m1}^s	.	a_{mj}^s	.	0	.	0	.	0
b^s	p_1^s	.	r_j^s	.	0	.	0	.	1

Table 2 can also serve as a representation of the starting tableau in which case $s = 1$. The starting tableau is the tableau of the optimal solution of the original problem in which the last element of the first column, the value of the objective function for the optimal solution f^0 , has been replaced by zero so that $b^1 = 0$. This means that in this and the following tableaux the value of the objective function is measured as a deviation from f^0 , so that b^s is equal to minus the loss of solution x^s .

For the solution of the starting tableau we have $b_i^1 > 0$ for each i and $p_j^1 > 0$ for each j ; if we had not assumed that the optimal solution is unique, the weak inequality sign would apply. Introduction of a nonbasic variable x_j into the basis at a level θ leads to a loss of $p_j^1 \theta$ which is positive. The variable to leave the basis is determined, just as in the Simplex method as the one corresponding with

$$(2.8) \quad \theta_j^1 = \min_i \left\{ \frac{b_i^1}{a_{ij}^1} \mid a_{ij}^1 > 0 \right\} = \frac{b_r^1}{a_{rj}^1},$$

where r is the index of the leaving basic variable. The loss corresponding with the new extreme-point solution, which is denoted as k_{1j} , is

$$(2.9) \quad k_{1j} = p_j^1 \frac{b_r^1}{a_{rj}^1}.$$

k_{1j} is determined for each j . Because the extreme-points are ranked in order of increasing losses, the smallest k_{1j} is selected. Since the losses may not exceed the maximum loss allowed k , the loss of the second solution is determined as

$$(2.10) \quad k_1 = \min_i (k_{1j}, k);$$

we assume that this minimum is unique.

The next tableau is then generated by introducing the corresponding nonbasic variable x_j into the basis, the leaving basic variable being determined by (2.8). The case $k_1 = k$ will be dealt with below. If in (2.8) for some j , $a_{ij}^1 \leq 0$ for all i , no corresponding extreme-point solution can be found in the usual manner, because in this case x_j can be increased indefinitely; the loss then also increases indefinitely. However, the maximal loss k provides a bound for the solution, so that this case is similar to the case $k_1 = k$, and can be dealt within the same manner.

We have now found an extreme-point solution with a loss of k_1 . All other extreme-point solutions we shall generate have higher losses. There are no other extreme-point feasible solutions with a loss between 0 and k_1 , since if there was such a solution, the optimal solution, which is our starting solution, could be generated from this solution in one or more steps of the ordinary Simplex method. If it could be generated in one step, then it should be possible to perform the corresponding iteration in the reverse direction, as shown before. If it could be generated in more steps, the last solution before the optimal one has an even smaller loss; from this solution the optimal solution could be generated in one step and it should then be possible to make the corresponding iteration in the reverse direction.

Having found a second extreme-point solution, we are now looking for another one ranking next in order of loss. First, there are the other extreme-point solutions which can be reached from the first tableau. Secondly, there are the extreme-point solutions which can be reached from the second tableau. The latter solutions are determined by introducing into the basis the nonbasic variables x_j having³ $p_j^2 > 0$; a solution obtained by introducing x_j with $p_j^2 < 0$ will have a smaller loss than k_1 and must in this case be the optimal solution. It is not possible that an extreme-point solution with a loss between 0 and k_1 is reached, since x_j with $p_j^2 > 0$ is introduced into the basis, the value of the objective function is decreased, so that the loss must be greater than k_1 . We therefore determine in the second tableau for each j with $p_j > 0$,

$$(2.11) \quad \theta_j^2 = \min_i \left(\begin{array}{c} b_i^2 \\ a_{ij}^2 \\ ij \end{array} \middle| a_{ij}^2 > 0 \right) = \frac{b_r^2}{a_{rj}^2} .$$

³Again, only the strict inequalities are considered.

The increase in loss by introducing x_j into the basis is therefore

$$(2.12) \quad p_j^2 \frac{b_r^2}{a_{rj}^2},$$

which means that the total loss of this solution, defined as k_{2j} , is

$$(2.13) \quad k_{2j} = k_1 + p_j^2 \frac{b_r^2}{a_{rj}^2}.$$

The extreme-point solution ranking next in order of loss is then the one corresponding with

$$(2.14) \quad k_2 = \min_j (k_{1j}, k_{2j} \mid k_{1j} > k_1, p_{2j} > 0);$$

again the minimum is assumed to be unique.

If the minimum corresponds with one of the k_{1j} 's, the next solution is generated from Tableau 1, if it corresponds with one of the k_{2j} , it is generated from Tableau 2; for the case $k_2 = k$, see below.

A general iteration, say iteration s , may now be described as follows. First, determine for each j with $p_j^s > 0$.

$$(2.15) \quad \min_i \left\{ \frac{b_i^s}{a_{ij}^s} \mid a_{ij}^s > 0 \right\} = \frac{b_r^s}{a_{rj}^s}$$

and

$$(2.16) \quad k_{sj} = k_{s-1} + p_j^s \frac{b_r^s}{a_{rj}^s}.$$

After this, determine

$$(2.17) \quad k_s = \min_i (k_{1j}, k_{2j}, \dots, k_{sj}, k \mid k_{1j}, \dots, k_{s-1,j} > k_{s-1})$$

In case the minimum is not determined by the maximum loss k , the next tableau is generated from the one corresponding with the minimum.

In the case $k_s = k$, all other extreme-points which can be reached from the tableaux which have been generated so far have a loss exceeding the

maximum loss k . These extreme points may be found by introducing nonbasic variables with positive p_j^s , which have not yet been introduced in the tableaux generated so far; these variables take such values that the corresponding solutions have a loss of exactly k . This means that a nonbasic variable x_j in Tableau n with $p_j^n > 0$ and $k_{nj} > k$ is given a value

$$(2.18) \quad x_j = \frac{k - k_{n-1}}{p_j^n};$$

the corresponding values of the basic variables in the tableau are then

$$(2.19) \quad x_i = b_i^n - a_{ij}^n x_j.$$

In the tableaux these solutions can be obtained by adding to $b_0^n = k_{n-1}$ the maximal loss k and pivoting on the p_j^n 's.

The initial subtraction of f^0 and the present addition of k make it possible to interpret the last row, which was originally a transformation of the equation

$$(2.20) \quad 0 = -p^0x + f(x)$$

as a transformation of

$$(2.21) \quad -f^0 + k = -p^0x + y,$$

which is equivalent to (2.4) with y as slack variable. Pivoting on any p_j can now be interpreted as y leaving the basis.

Because no further iterations take place it is not necessary to generate all elements of the Simplex tableaux of these solutions; only the values of variables and possibly the elements of the last row are needed.

Hence we may state more formally; If for some iterations s , $k_s = k$, add in Tableau n for $n = 1, \dots, s$, to f^n the maximum loss k and generate for each Tableau n and each nonbasic variable j with $p_j^n > 0$ and $k_{nj} > k$ a new solution by pivoting on p_j^n .

That all extreme-point solutions satisfying (2.2) - (2.4) are generated by this procedure can be proved as follows. From any extreme-point solution of (2.2) - (2.4) it is possible to reach the optimal solution in a finite number of steps by the Simplex method. The reverse Simplex method proceeds from the optimal solution in the reverse direction, investigating every extreme-point solution downwards; if there is a path going upwards, we are sure of tracing this when investigating every path going downwards.

2.3. Example

To illustrate the reverse Simplex method, a small example will be used⁴. It involves a production planning problem with 4 products and 2 constraints. The objective function which is to be maximized is

$$(2.22) \quad f(x) = 3x_1 + 4x_2 + 5x_3 + 6x_4$$

and the constraints are

$$(2.23) \quad \begin{cases} x_1 + x_2 + x_3 + x_4 \leq 18, \\ 2x_3 + 3x_4 \leq 6, \end{cases}$$

and

$$(2.24) \quad x_1, x_2, x_3, x_4 \geq 0.$$

⁴ See C. van de Panne [3].

TABLE 3. SIMPLEX TABLEAUX FOR AN APPLICATION OF THE REVERSE METHOD TO AN EXAMPLE

Tableau	Basic Var.	Val. Bas. Var.	x_1	x_2	x_3	x_4	x_5	x_6	f
Init. Tabl. Orig. Prob.	x_5	18	1	1	1	1	1	0	0
	x_6	6	0	0	2	3	0	1	0
	f	0	-3	-4	-5	-6	0	0	1
1	x_2	16	<u>1</u>	1	1/3	0	<u>1</u>	-1/3	0
	x_4	2	0	0	<u>2/3</u>	1	0	<u>1/3</u>	0
	f	76 0	1	0	1/3	0	4	2/3	1
	k_{1j}		16	-	1	-	64	4	20
2	x_2	15	<u>1</u>	1	0	-1/2	<u>1</u>	-1/2	0
	x_3	3	0	0	1	1 1/2	0	<u>1/2</u>	0
	f	-1	1	0	0	-1/2	4	<u>1/2</u>	1
	k_{2j}		16	-	-	-	61	4	20
3	x_2	18	<u>1</u>	1	1	1	<u>1</u>	0	0
	x_6	6	0	0	2	3	0	1	0
	f	-4	1	0	-1	-2	4	0	1
	k_{3j}		22	-	-	-	76	-	20
4	x_1	16	1	1	1/3	0	<u>1</u>	-1/3	0
	x_4	2	0	0	<u>2/3</u>	1	0	<u>1/3</u>	0
	f	-16	0	-1	0	0	3	1	1
	k_{4j}		-	-	16	-	64	22	20

TABLE 3. (Continued)

Tableau	Basic Var.	Val. Bas. Var.	x_1	x_2	x_3	x_4	x_5	x_6	f
5	x_1	15	1	1	0	-1/2	<u>1</u>	-1/2	0
	x_3	3	0	0	1	1 1/2	0	<u>1/2</u>	0
	f	-16	0	-1	0	0	3	1	1
	k_{5j}		-	-	-	16	61	22	20

The initial tableau for an application of the normal Simplex method is then the first tableau of Table 3. The tableau of the optimal solution is reached after a few iterations and is given in Tableau 1; the corresponding value of the objective function is 76 and the maximum loss is assumed to be 20.

Starting the procedure we replace 76 in Tableau 1 by 0. Considering the introduction of x_1 into the basis, we find

$$\theta_1^1 = 16$$

and

$$k_{11} = 1 \cdot 16 = 16.$$

The eventual pivot element $a_{rj}^1 = 1$ is underlined and the corresponding loss $k_{11} = 16$ is entered in an additional row. For x_3 , x_5 and x_6 we find in a similar manner

$$k_{13} = 1, k_{14} = 64, k_{16} = 4.$$

Below f the value of k, 20 is given. The extreme-point with the lowest loss is found by taking the smallest element of the row k_{1j} ; this 1, the element in the x_3 -column, so that x_3 enters the basis. Tableau 2 is then found by

pivoting on the element $2/3$. For the value of the objective function f we find -1 , which corresponds with k_{13} .

In Tableau 2, the k_{2j} are to be determined; this is only done for the nonbasic variables with a nonnegative entry in the f -row so that x_4 is not considered. For x_1 , for instance, we find

$$\theta_1^2 = 15$$

and

$$k_{21} = k_1 + r_1^2 \theta_1^2 = 1 + 1 \cdot 15 = 16;$$

it is a coincidence that $k_{11} = k_{21}$. k_{25} and k_{26} are formed in the same way.

Determining the minimum of the relevant k_{1j} and k_{2j} , we find

$$k_2 = k_{16} = 4.$$

This minimum is not unique since also $k_{26} = 4$, but this causes no difficulties.

Tableau 3 is generated by introducing x_6 in the basis of Tableau 1 and pivoting in the element $1/3$.

The Tableaux 4 and 5 are generated in the same way. In Tableau 5 the minimum of the relevant losses turns out to be the maximum loss $k = 20$. The nonbasic variables with k_{rj} 's higher than 20 are then introduced in the basis. The relevant solutions are found by adding in each tableau to the value of f the value of k which is 20. The solutions are then found by pivoting on the p_j^n . For instance, introducing x_5 into the basis in Tableau 1 in such a way that the loss is 20, we have to pivot on the element 4. The relevant parts of the two successive tableaux are then as follows:

	V.b.V.	x_5
x_2	16	1
x_4	2	0
f	20	<u>4</u>

	V.b.V.	x_5
x_2	11	0
x_4	2	0
x_5	5	1

All extreme-point solutions which can be found in this manner are given in Table 4, No. 6-13.

TABLE 4. LIST OF EXTREME-POINT NEAR-OPTIMAL SOLUTIONS

Solution No.	Value Obj. F.	x_1	x_2	x_3	x_4	x_5	x_6
1	76	-	16	-	2	-	-
2	75	-	15	3	-	-	-
3	72	-	18	-	-	-	6
4	60	16	-	-	2	-	-
5	60	15	-	3	-	-	-
6	56	-	11	-	2	5	-
7	56	-	10 1/4	3	-	4 3/4	-
8	56	16	2	-	-	-	6
9	56	-	14	-	-	4	6
10	56	14 2/3	-	-	2	1 1/3	-
11	56	17 1/3	-	-	2/3	-	4
12	56	13 2/3	-	3	-	1 1/3	-
13	56	17	-	1	-	-	4

Note that the total number of extreme-point solutions, 13, is fairly high for a problem of such a small size. The maximum loss, 20, is relatively high, but even if it had been chosen much lower, say between 4 and 16, there would have been 9 extreme-point solutions. The decisionmaker's task of selecting the preferred solutions is therefore not likely to be a trivial one.

3. GENERATION OF NEAR-OPTIMUM SOLUTIONS BY THE TARRY METHOD

3.1. The Tarry Method

An important difference between the ordinary Simplex method and the reverse Simplex method is that in the first method only the current tableau, or some parts of it, are necessary to perform an iteration, while in the reverse method as it was presented here, any of the previous tableaux may have to be used. When using computers for larger problems, the memory space required to store all previous tableaux or the corresponding inverses may soon exhaust the available space.

An alternative for the reverse Simplex method is to store only the k_{1j} , k_{2j} , . . . , k_{3j} and to generate each tableau from that of the optimal solution when necessary. Any tableau problem can be generated from the optimal tableau in at most m steps, but usually less.

In view of this, it is useful to consider an alternative method in which iterations use, apart from some other information, only the preceding tableau, so that the product form of the inverse can be used as in the ordinary Simplex method. This means that we want to consider a method in which the successive solutions are adjacent extreme points as in the Simplex method.

We may proceed as follows. The starting point is the tableau of the optimal solution. In this tableau, f^0 is replaced by k , so that $f(x)$ is measured from its lower bound $f^0 - k$. We now want to generate all extreme-point feasible solutions in which $f(x)$, measured as indicated, is nonnegative. This can be done by generating all basic feasible solutions of the equation system given by the tableau in which $f(x)$ is now considered as an ordinary basic variable. Any of the non-basic variables may now be introduced into the basis. The leaving basic variable is determined by comparing positive ratios for all rows, including the f-row. In the resulting tableau we have found a new extreme-point feasible solution. In this tableau we may introduce any of the nonbasic variables into the basis, determining the leaving basic variable

as before. The resulting tableau may be treated in the same manner, and so on.

The difficulty is that we may find the same solution again and again without being sure that all extreme-point feasible solutions are generated. Since the only element of choice in each iteration is that of the choice of the nonbasic variable which is to enter the basis, the problem is to find a rule for the new basic variable at each iteration which results in generating in a sequence of iterations all extreme-point feasible solutions.

Charnes and Cooper [1] have proposed to use in problems like these a method devised by Tarry⁵ for finding a path through a network such that each node is visited at least once. This method and its application to our present problem will now be explained.

As an example let us consider an even smaller problem than that used in Section 2. The original problem is to maximize

$$(3.1) \quad f = 5x_1 + 6x_2$$

subject to

$$(3.2) \quad 2x_1 + 3x_2 \leq 6,$$

$$(3.3) \quad x_1, x_2 = 0.$$

The maximum loss allowed is 5.

⁵See König [2], p. 41 seq.

TABLE 5. GENERATION OF FEASIBLE EXTREME POINTS FOR A SMALL EXAMPLE

Tableau	Basic Variable	Val. Basic Variable	x_1	x_2	x_3	f
0	x_3	6	<u>2</u>	3	1	0
	f	0	-5	-6	0	1
1	x_1	3	1	<u>1 1/2</u>	1/2	0
	f	15 5	0	1 1/2	<u>2 1/2</u>	1
2	x_2	2	<u>2/3</u>	1	1/3	0
	f	2	-1	0	<u>2</u>	1
3	x_2	1 2/3	<u>5/6</u>	1	0	-1/6
	x_3	1	-1/2	0	1	<u>1/2</u>
4	x_1	2	1	<u>1 1/5</u>	0	-1/5
	x_3	2	0	3/5	1	2/5

The set-up tableau is then as given in Tableau 0 of Table 5; Tableau 1 gives the optimal solution, in which the value of the objective function, 15, is replaced by 5. Introducing x_2 , the first nonbasic variable into the basis, we find that x_1 leaves the basis. If in Tableau 2 the first nonbasic variable x_1 , is introduced into the basis, we find Solution 1 again. If x_3 is selected as a nonbasic variable, f leaves the basis; the resulting tableau is Tableau 3. Introducing now x_1 , into the basis we find Tableau 4. Introducing then f into the basis, Tableau 1 is found again. There exists no other extreme point feasible solutions apart from those of Tableaux 1-4, as can be ascertained by introducing other nonbasic variables in each of the tableaux.

In larger problems it would not have been so easy to find a sequence of adjacent basic feasible solutions containing every basic feasible solution. To find such a sequence, it must be realized that the various solutions can be arranged in a network representation. From the tableaux of Table 5 we find that from Solution 1 we may reach Solutions 2 and 4, from Solution 2 we may reach Solutions 1 and 3, from Solution 3 we may reach Solutions 2 and 4, and from Solution 4 we may reach Solutions 1 and 3. These relations are given in the network Figure 1, in which the nodes correspond with the solutions.

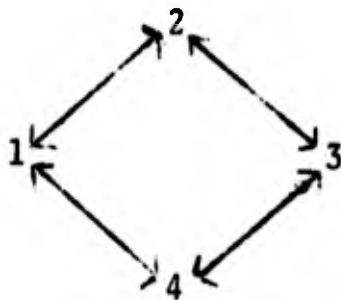


Figure 1.

Even in such a small network it is easily possible to have a path which misses a node: for example the path 1 - 2 - 1 - 4 - 1 does not include node 3. For larger programming problems, the network is much more complicated; for instance, in the problem used in Section 2 each node has 4 edges leading from it, since there are in each tableau 4 nonbasic variables. For such problems it is much less easy to find a path visiting all nodes.

The problem of finding a path through a network such that all nodes are visited is a well-known problem in graph theory and can be solved in various ways. Following Charnes and Cooper, we shall consider the method devised by Tarry.

This method requires that at each node it is known through which edge it was first reached; further it must be known whether an edge has already been traversed in a certain direction. The rule of Tarry's method is then as follows:

At any node, go to another node along an edge which has not yet been traversed in that direction, but do not go along the edge that was used to reach the first-mentioned node for the first time unless there is no other choice.

If this rule is followed, every edge will be traversed twice, once in one direction and once in the opposite direction. Each node will be visited at least once. Applying this rule to the network of Figure 1, we find the following. Solution 1 is used as a starting point; we can go to Solutions 1 and 4, but choose to go to Solution 2. In Solution 2 we can in principle go to Solutions 1 and 3, but we have to go to Solution 3, since Solution 1 was used to reach Solution 2 for the first time. For similar reasons we go to Solution 3, 4, and 1. In Solution 1, we cannot go to Solution 2, since we went already once in that direction. Though Solution 1 was reached first coming from Solution 4, there is no other choice, so that we return to Solution 4. In Solution 4, we go back to Solution 3, then to Solution 2, then to Solution 1, all for the same reasons. Every edge has then been traversed twice and every solution has been visited.

3.2. Application to an Example

The Tarry method will now be applied to the example used in Section 2. The initial solution is the optimal solution of the linear programming problem; it is given in Tableau 0 of Table 6. When the Tarry method allows this, we shall always introduce into the basis the nonbasic variable with the smaller number; for this purpose f is considered to be x_7 . The information required for an application of the Tarry method is given in Table 7. All solutions generated by the method are numbered according to the iterations in which they appear. Since solutions reappear many times, they are indicated in Table 7 by the number of the iteration in which they first occur. Each solution is indicated by strokes in the columns of basic variables; the

spaces in the columns of nonbasic variables are used for the number of the iteration in which the nonbasic variable concerned is introduced into the basis.

In Tableau 0 of Table 6, x_1 is introduced into the basis; we therefore enter 1 as the number of the iteration in the column of x_1 in the row of Solution 0 in Table 7. From Tableau 0 we find that x_2 leaves the basis; the resulting solution which is indicated as Solution 1 is given in Tableau 1. This solution is then also entered in Table 7 with strokes in the column of x_1 , x_4 , and x_5 .

TABLE 6. SIMPLEX TABLEAUX FOR AN APPLICATION OF THE TARRY METHOD

Tabl.	Basic Variable	Value Bas. Var.	x_1	x_2	x_3	x_4	x_5	x_6	f
0	x_2	16	<u>1</u>	1	1/3	0	1	-1/3	0
	x_4	2	0	0	<u>2/3</u>	1	0	<u>1/3</u>	0
	f	20	1	0	1/3	0	<u>4</u>	2/3	1
1	x_1	16	1	<u>1</u>	1/3	0	1	-1/3	0
	x_4	2	0	0	<u>2/3</u>	1	0	1/3	0
	f	4	0	-1	0	0	<u>3</u>	<u>1</u>	1
2	x_1	15	1	<u>1</u>	0	-1/2	1	-1/2	0
	x_3	3	0	0	1	<u>1 1/2</u>	0	1/2	0
	f	4	0	-1	0	0	<u>3</u>	<u>1</u>	1
3	x_2	15	<u>1</u>	1	0	-1/2	1	-1/2	0
	x_3	3	0	0	1	<u>1 1/2</u>	0	<u>1/2</u>	0
	f	19	1	0	0	-1/2	<u>4</u>	1/2	1

TABLE 6. (Continued)

Tabl.	Basic Variable	Value Bas. Var.	x_1	x_2	x_3	x_4	x_5	x_6	f
6	x_2	10 $\frac{1}{4}$	<u>$\frac{3}{4}$</u>	1	0	$-\frac{3}{8}$	0	$-\frac{5}{8}$	$-\frac{1}{4}$
	x_3	3	0	0	1	<u>$1 \frac{1}{2}$</u>	0	<u>$\frac{1}{2}$</u>	0
	x_5	4 $\frac{3}{4}$	$\frac{1}{4}$	0	0	$-\frac{1}{8}$	1	$\frac{1}{8}$	<u>$\frac{1}{4}$</u>
7	x_1	13 $\frac{2}{3}$	1	<u>$1 \frac{1}{3}$</u>	0	$-\frac{1}{2}$	0	$-\frac{5}{6}$	$-\frac{1}{3}$
	x_3	3	0	0	1	<u>$1 \frac{1}{2}$</u>	0	$\frac{1}{2}$	0
	x_5	1 $\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	1	<u>$\frac{1}{3}$</u>	<u>$\frac{1}{3}$</u>
8	x_1	14 $\frac{2}{3}$	1	<u>$1 \frac{1}{3}$</u>	$\frac{1}{3}$	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$
	x_4	2	0	0	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0
	x_5	1 $\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	1	<u>$\frac{1}{3}$</u>	<u>$\frac{1}{3}$</u>
9	x_2	11	<u>$\frac{3}{4}$</u>	1	$\frac{1}{4}$	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$
	x_4	2	0	0	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0
	x_5	5	$\frac{1}{4}$	0	$\frac{1}{12}$	0	1	$\frac{1}{6}$	<u>$\frac{1}{4}$</u>
12	x_2	14	$\frac{3}{4}$	1	$1 \frac{1}{4}$	$1 \frac{1}{2}$	0	0	$-\frac{1}{4}$
	x_5	4	<u>$\frac{1}{4}$</u>	0	$-\frac{1}{4}$	$-\frac{1}{2}$	1	0	<u>$\frac{1}{4}$</u>
	x_6	6	0	0	<u>2</u>	<u>3</u>	0	1	0
13	x_1	16	1	0	-1	-2	<u>4</u>	0	<u>1</u>
	x_2	2	0	1	<u>2</u>	<u>3</u>	-3	0	-1
	x_6	6	0	0	2	3	0	1	0

TABLE 6. (Continued)

Tabl.	Basic Variable	Value Bas. Var.	x_1	x_2	x_3	x_4	x_5	x_6	f
14	x_1	17	1	1/2	0	-1/2	2 1/2	0	1/2
	x_3	1	0	<u>1/2</u>	1	1 1/2	-1 1/2	0	-1/2
	x_6	4	0	-1	0	0	<u>3</u>	1	<u>1</u>
15	x_1	17 1/3	1	2/3	1/3	0	2	0	1/3
	x_4	2/3	0	<u>1/3</u>	<u>2/3</u>	1	-1	0	-1/3
	x_6	4	0	-1	0	0	<u>3</u>	1	<u>1</u>
32	x_2	18	1	1	1	1	1	0	0
	x_6	6	0	0	<u>2</u>	<u>3</u>	0	1	0
	f	16	<u>1</u>	0	-1	-2	<u>4</u>	0	1

TABLE 7. INFORMATION FOR AN APPLICATION OF THE TARRY METHOD

Sol No.	x_1	x_2	x_3	x_4	x_5	x_6	f
0	1	-	5	-	36	38	-
1	-	52*	2	-	21	23	-
2	-	3	-	51*	28	30	-
3	50*	-	-	4	6	34	-
6	7	-	-	11	-	41	49*
7	-	48*	-	8	-	26	29
8	-	9	47*	-	-	19	22
9	46*	-	10	-	-	12	37
12	13	-	40	45*	-	-	42
13	-	-	14	17	44*	-	32
14	-	31*	-	15	25	-	27
15	-	16	24*	-	18	-	20
32	43*	-	33	35	39	-	-

To indicate the solution from which Solution 1 was reached for the first time, a star is placed in the column of x_2 , since by introducing this variable into the basis Solution 0 is generated. In Solution 1, we cannot introduce x_2 into the basis unless there is no other possibility, so that x_2 is introduced into the basis. Since this is the second iteration, a 2 is put in the column of x_3 in the row of Solution 1 in Table 7. From Tableau 1 it is found that x_4 leaves the basis. After the transformation, Tableau 2 containing Solution 2 is obtained, which is entered in Table 7 with strokes in the columns of x_1 , x_3 , and f and a star in the column of x_4 . Solution 3 is generated in the same manner.

In Solution 3, x_4 is introduced into the basis, yielding Solution 0 again. Hence Solution 4 is equivalent with Solution 0. In order to save space, we do not reproduce the tableau of this solution again, but if a computer was used, this tableau would be generated from Tableau 3. In Table 7 the nonbasic variable with the next blank space in the row of Solution 0 is selected as the new basic variable; this happens to be x_3 , so that the number of the next iteration, 5, is entered in the space. The resulting solution happens to be the same as Solution 3. In this solution x_4 is introduced into the basis. Solution 6 turns out to be a solution which has not been generated before, so that it is entered in Tableau 7.

All other iterations are performed in the same manner. If in some row of Table 7 no blank space is left except the starred one, then the corresponding variable of this starred space enters the basis; the solution concerned will then not be generated again.

After 52 iterations we return to Solution 0; every solution has then been generated four times which is the number of nonbasic variables in each solution. Note that the last new solution was generated in the 32nd iteration, so that the last 20 iterations only served to prove that there is no other solution. In fact, the first 15 iterations, which is 30 percent of

all iterations gave all solutions except one. In other problems, however, things may turn out differently.

3.3. Comparison and Concluding Remarks

The number of iterations in the Tarry method is equal to the number of extreme-point feasible solutions times the number of nonbasic variables in a tableau. In linear programming with inequalities only, the number of nonbasic variables is equal to the number of variables (including slack variables.) Thus, if there are n variables in a problem with inequalities only, every extreme-point feasible solution has to be generated n times in order to be able to use an adjacent extreme-point method in which the product form of the inverse can be applied. Since the number of variables in most cases exceeds the number of constraints, the reverse Simplex method in the version in which each previous tableau is regenerated from the optimal tableau will usually require fewer computations.

Another advantage which the reverse Simplex method has over the Tarry method is that it is easier in the first method to vary the value of k , since all solutions are ranked according to increasing loss. An increase in k will leave all solutions having a smaller loss than k the same, while those having a loss of exactly k can be easily adapted. In the Tarry method the entire procedure would have to be started all over again.

Finally it must be observed that if the post-optimality analysis results in a large number of extreme-point feasible solutions, it will have missed its main purpose anyway, which was to enable the decision maker to select his decisions from a limited number of alternatives, since it will usually not be possible for him to go through a large number of alternatives and their linear combinations.

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13 ABSTRACT In practical applications of linear programming, not only the optimal solution but also solutions which have a somewhat lower value of the objective function are of interest. It is therefore desirable to generate all extreme-point solutions satisfying the constraints and giving a value of the objective function which differs by at most a given amount from the value for the optimal solution. Two methods are considered for generating these extreme points. The first method is called the reverse Simplex method, since it reverses the Simplex method for linear programming, the second is basic on the Tarry method for traversing a network such that all nodes are visited. The two methods are explained in detail, applied to an example and compared with each other.			

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