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MAXIMAL TWO-WAY FLOWS

by

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The most familiar network flow problem is that of finding the maximal integer flow from a source s to a sink t in a network G. In this paper we discuss the problem of simultaneous flows from s to t and from t to s. The main result of this paper is a max-flow min-cut theorem for this type of problem. The method of proof used indicates a procedure for finding the maximal flows. Finally, the problem of feasibility is discussed.

Introduction.

The most familiar network flow problem is that of finding the maximal integer flow from a source s to a sink t in a network G with integer capacities (see [1]). In this paper we discuss the problem of simultaneous flows from s to t and from t to s. For example, one might wish to know how much traffic a specific system of roads could carry between two points going in either direction. If the network is undirected (e.g., there are no one-way streets), then this problem is identical with the problem of finding a maximal flow from s to t. The answer to this is the well known Max-flow Min-cut theorem (see [1]). When one has some directed arcs, however, the problems are no longer equivalent. Here we obtain a Max-flow Min-cut theorem for two-way flows in certain special networks, called Euler networks, which have some arcs directed and some undirected. They are defined below. The method of proof used here indicates a procedure for actually constructing a maximal flow. It is the analogue to the cross path method used in [2] for the one-way flow.

We consider here *mixed* networks, that is networks with both directed and undirected arcs. For convenience in what follows, we consider all arcs to have capacity one, but we permit multiple arcs between any two nodes. For two nodes a,b in the network G, an <u>a $\rightarrow$ b path</u> is a path consisting possibly of both directed and undirected arcs such that the directed arcs occur in the direction from a to b along the path. We permit nodes to be used more than once in a path, but arcs may only be used once. A <u>flow</u> is a collection of paths no two having any arc in common. An <u>a $\rightarrow$ b flow</u> is a flow consisting of a $\rightarrow$ b paths; a <u>b $\rightarrow$ a flow</u> is a flow consisting of b $\rightarrow$ a paths. A <u>two-way flow</u> or <u>a $\leftrightarrow$ b flow</u> is a flow consisting of a $\rightarrow$ b paths and b $\rightarrow$ a paths. An <u>a $\rightarrow$ b</u> (respectively <u>b $\rightarrow$ a</u>) <u>cut-set</u> is a collection of arcs (again these may be directed and undirected together) such that their removal from G eliminates all a $\rightarrow$ b paths (respectively b $\rightarrow$ a paths). An <u>a $\leftrightarrow$ b cut-set</u> is a collection of arcs whose removal eliminates both a $\rightarrow$ b and b $\rightarrow$ a paths.

A network is said to be <u>Euler</u> if at each node there is an even number of undirected arcs and there is the same number of incoming directed arcs as outgoing directed arcs. A circuit (i.e., closed path) or a collection of paths or a flow is called <u>Euler</u> if the subnetwork consisting the arcs and nodes of the circuit or collection of paths or flow respectively is Euler. A network G is called an Euler directed network if it is Euler and all arcs are directed. Similarly, it is called an Euler undirected network if it is Euler and all its arcs are undirected. Clearly the subnetwork of G consisting of all directed arcs and the nodes to which they are attached is an Euler directed subnetwork D. Similarly, all the undirected arcs of G determine an Euler undirected subnetwork U. G = U + D. We note that neither D nor U need be connected, even if G is connected. By a <u>connected network</u> we mean one such that if all directed arcs are replaced by undirected arcs, the resulting network is connected as an undirected network. A connected component of a network is a maximal connected subnetwork.

Main Result.

Lemma 1 Let G be a connected Euler network. Then if u and v are any two nodes of G, there is an Euler circuit containing them both.

Proof. Let $G = D + U$ as above. Since both D and U are Euler, each of their connected components must be Euler also. Then by [2] 3.1.1 and 3.1.3, each of these components can be considered to be a single circuit. Thus since G is connected, each of the components shares some node with some other component. Hence all of G can be considered to be a single circuit. In particular, it is clearly an Euler circuit containing u and v. We note that there may certainly be Euler circuits containing u and v which do not exhaust G also.

Definition. Let $P_1, \ldots, P_k$ be an $a \leftrightarrow b$ flow in G. Let $C_1, \ldots, C_n$ be a family of Euler circuits of $G - (P_1 + \ldots + P_k)$, where subtraction means just that the arcs of $P_1 + \ldots + P_k$ are deleted from G. Assume that no two of the C_j have any arcs in common. We do not exclude the degenerate case of any of the C_i being simply a single node. Then $C_1, \ldots, C_n$ is called a system of alternating circuits with respect to the flow $P_1, \ldots, P_k$ if:

(1) C_1 contains a and some node v_1 in one of the P_i, call it R_1.

(2) For $j > 1$, C_j contains a node u_{j-1} on the path R_{j-1} which is at least as close to a on R_{j-1} as v_{j-1} is.

(3) C_j contains a node v_j on one of the P_i, say R_j. R_j need not be distinct from previous R's.

See Figure 1. (In Figure 1 we do not indicate which are the directed arcs, nor which direction each path has.)

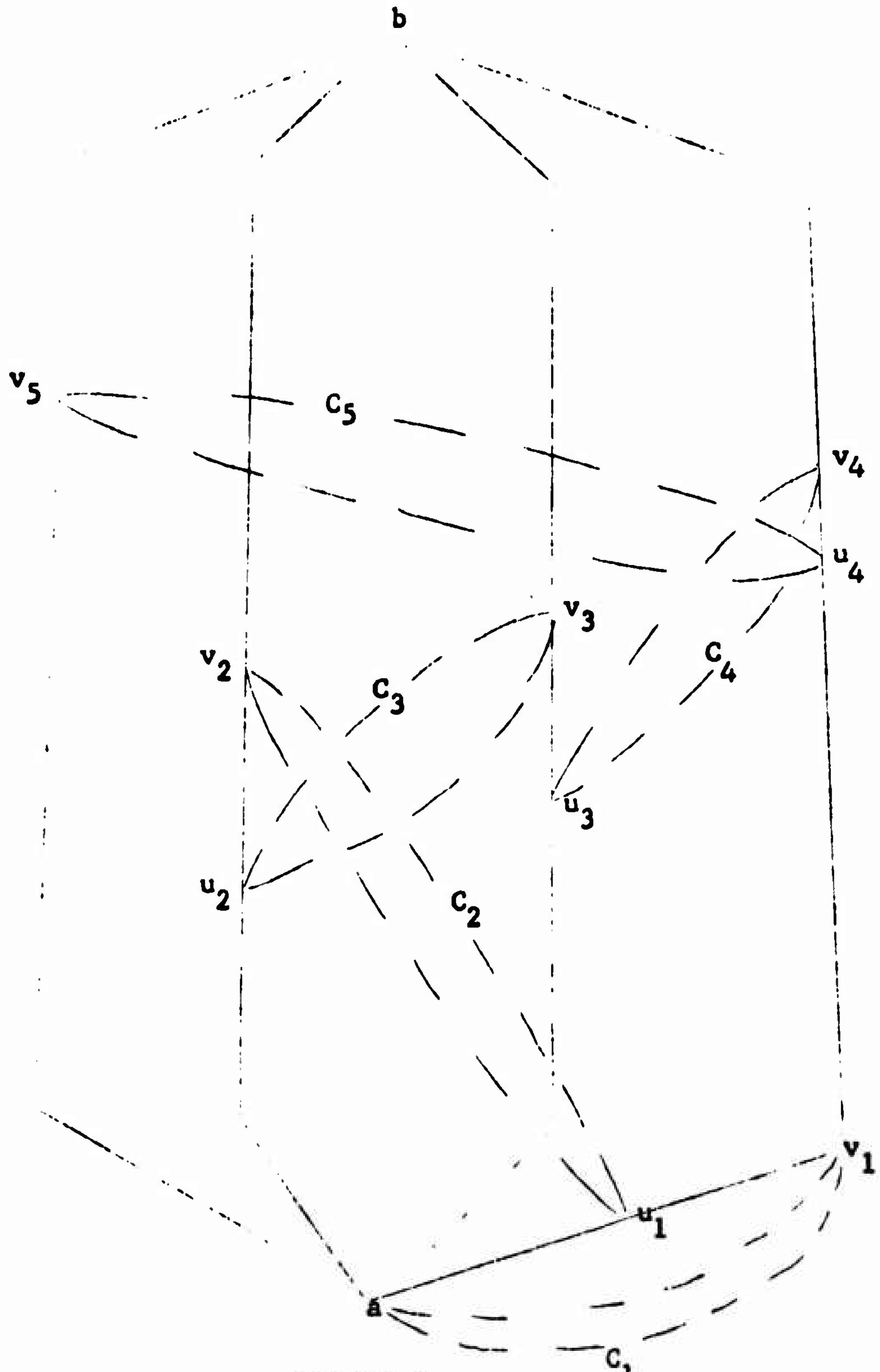

FIGURE 1

Lemma 2 Let $P_1,\ldots,P_k$ be an $a \Leftrightarrow b$ flow containing [illegible] $a \to b$ paths and $k-$ [illegible] $b \to a$ paths. Let $C_1,\ldots,C_n$ be a system of alternating circuits associated with $P_1,\ldots,P_k$, and assume $v_n = b$. Then $P_1+\ldots+P_k + C_1+\ldots+C_n$ forms an $a \Leftrightarrow b$ flow of $k + 2$ consisting of [illegible]$+1$ $a \to b$ and $k-$[illegible]$+1$ $b \to a$ paths.

Proof We use induction on n. If $n = 1$, then we are done because C_1 contains a and b and has no arcs in common with any of the P_i. So assume that the Lemma is true for $n < r$, and let $n = r > 1$.

We may assume that among those U_i having $R_i = R_1$, none lie closer to a on this path R_1 than v_1 does. Otherwise, C_1, C_{i+1}, $C_{i+2},\ldots,C_n$ is a system of alternating circuits, and we are done by induction. Thus we have the situation of Figure 2, where for any i with $R_i = R_1$, u_i must lie in E.

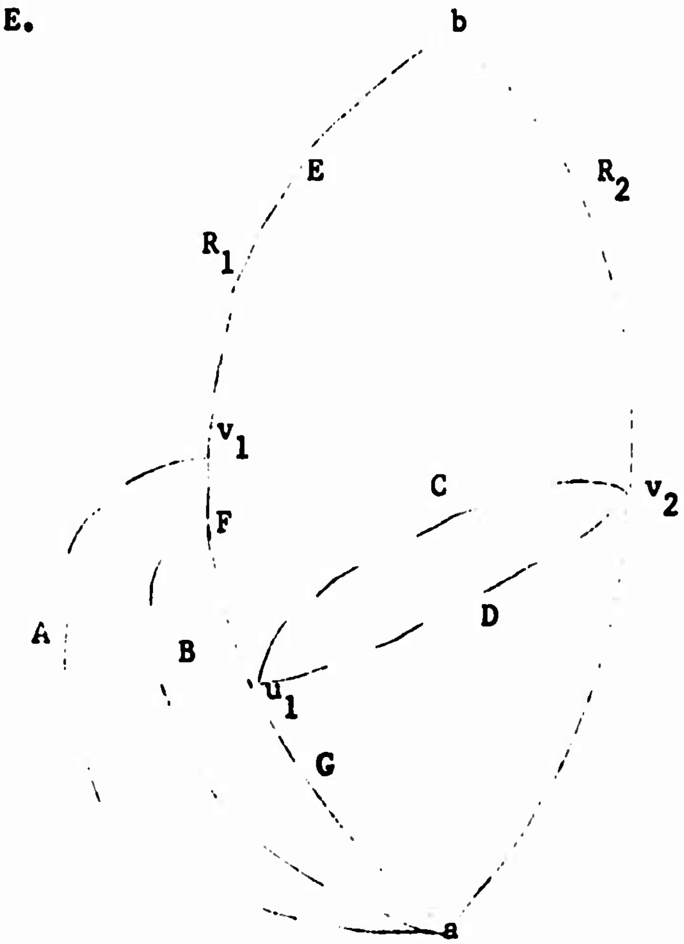

FIGURE 2

Let A be the part of C_1 going from a to v_1 if $G + F + E (= R_1)$ is an $a \to b$ path. If $G + F + E$ is a $b \to a$ path, then choose A to be the part of C_1 which goes from v_1 to a. Then in either case, $A + E$ is a path going the same direction as R_1, while $B + F + D + C + G$ is a circuit. Then replacing R_1 by $A + E$, and C_1 and C_2 by $B + F + D + C + G$, we have a new flow with the same number of paths in each direction as we started with, but with a system of circuits with $n-1$ members. This system we claim is alternating with respect to the new flow. For (1) is satisfied by $B + F + D + C + G$ (with R_2 replacing R_1), and (2) and (3) are satisfied since all the u_i and v_i, $i > 1$ are left unchanged. Also, v_n still is b, so the hypotheses of the lemma are satisfied. Hence we can apply induction and conclude that this new flow and system of circuits form a flow of $\ell+1$ $a \to b$ paths and $k-\ell+1$ $b \to a$ paths. But since the new paths and circuits use exactly the same arcs as the original ones, this $k + 2$ $a \leftrightarrow b$ flow is precisely the desired one. Q.E.D.

Now there may be many different systems of alternating circuits associated with a given set of paths. Consider those systems consisting only of Euler circuits. Let v be any node on, say, P_i. If for some alternating system of Euler circuits $C_1,\ldots,C_n$ we have one of the v_j on P_i and at least as far from a on P_i as v is, then we call the node v <u>accessible</u> (with respect to the given paths $P_1,\ldots,P_k$). (See Figure 3.)

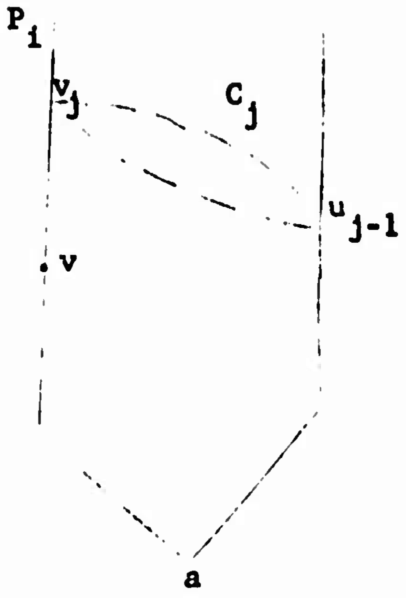

FIGURE 3

Lemma 3 Let G be an Euler network, and $P_1,\ldots,P_k$ an Euler $a \leftrightarrow b$ flow. Suppose x is an accessible node of P_1, and y is a node of $P_i (1 \leq i \leq k)$ such that in $G - (P_1+\ldots+P_k)$ there is an Euler circuit containing x and y. Then y is accessible.

Proof Since x is accessible, we can let $C_1,\ldots,C_n$ be an alternating family of Euler circuits with v_n on P_1 and x on P_1 no farther from a than v_n. Let C be an Euler circuit containing x and y. (See Figure 4.)

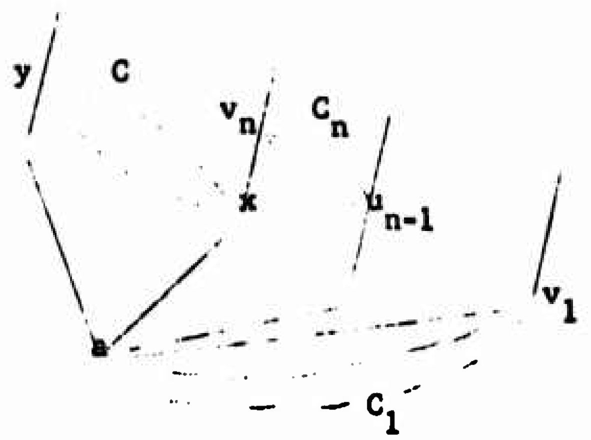

FIGURE 4

Then if C is in a different component of $G - (P_1+\ldots+P_k)$ from all of the C_i, we have $C_1,\ldots,C_n,C$ forming a system of alternating Euler circuits, and y is accessible. If C is in the same component K of $G - (P_1+\ldots+P_k)$ as any of the C_i, let i_o be the minimal i such that C_i is in the component. As G is Euler, and $P_1+\ldots+P_k$ is Euler, $G - (P_1+\ldots+P_k)$ is Euler, and hence K is Euler. By Lemma 1, there is an Euler circuit containing U_{i_o-1} and y, say C_o. Then $C_1,\ldots,C_{i_o-1},C_o$ is a system of alternating circuits, and y is accessible. Q.E.D.

<u>Lemma 4</u> Let m be the minimum size of an $a \leftrightarrow b$ cut-set in an Euler network G. Let $P_1,\ldots,P_k$ be an Euler collection of paths forming an $a \leftrightarrow b$ flow. Then b is accessible if $k < m$.

<u>Proof</u> Assume b is not accessible. By beginning at a and moving along a path P_i to b, (this may be in the same direction or the opposite direction from the direction of P_i), we must reach a last node on P_i which is accessible, and all nodes beyond this must not be accessible. Let e_i be the arc of P_i connecting the last accessible node of P_i with the first inaccessible one, $i = 1,2,\ldots,k$. Then since $k < m$, $e_1,\ldots,e_k$

does not form an $a \Leftrightarrow b$ cut-set. Hence in $G - (e_1+\ldots+e_k)$ there is an $a \to b$ path or a $b \to a$ path P. Traveling from a to b along this path, there is a first inaccessible node u. Then there is a last accessible node before u, say v. Let R be the part of the path P between v and u. Then R lies in $G - (P_1+\ldots+P_k)$. For suppose it didn't. Then some arc e of R would lie on one of the P_i, and thus connect either two accessible nodes, two inaccessible nodes, or one of each. But it can't be one of each, since then it would have to be e_i, and $R \subseteq G - (e_1+\ldots+e_k)$. It can't be two accessible nodes, by definition of R. Similarly, it can't be two inaccessible nodes by definition of R. This is a contradiction. So $R \subseteq G - (P_1+\ldots+P_k)$. But since G is Euler, and $P_1+\ldots+P_k$ is Euler, $G - (P_1+\ldots+P_k)$ is Euler, and so is the component of $G - (P_1+\ldots+P_k)$ containing R. Thus by Lemma 1, u and v lie on an Euler circuit of $G - (P_1+\ldots+P_k)$. Now Lemma 3 implies that u is accessible. This is a contradiction, so b is accessible. Q.E.D.

__Theorem 1__. If G is an Euler connected network, and if m is the minimal size of an $a \Leftrightarrow b$ cut-set, then m is even, and there is an $a \Leftrightarrow b$ flow of m in G such that its constituent paths are an Euler collection of paths.

__Proof__. Lemma 1 implies that there are at least two paths, one from a to b, and one from b to a which form an Euler collection. Starting with these two, we may apply Lemma 4 if $2 < m$, and then Lemma 2 to obtain 4 paths forming an Euler collection. Continuing in this way, we may keep adding 2 paths at a time until we have an Euler collection of k paths, and $k = m$. m must be even since we began with 2.

Feasibility

We say that a flow of (s,t) is feasible if there is a flow with s $a \to b$ paths and t $b \to a$ paths. Certainly if (s,t) is feasible, then (x,y) is feasible for $x \leq s$, $y \leq t$. So the interesting question is: What (s,t) are feasible for $s + t = m$? We can answer this for the case where the m paths are an Euler flow.

The proof of Theorem 1 quarantees that a flow of $(m/2, m/2)$ is feasible, since we can start with $(1,1)$ and add two at a time, obtaining successively $(2,2)$, $(3,3)$, etc. But, of course, if somehow we obtained a flow of (x,y) where the $x + y$ paths were an Euler collection, then just as in the proof of Theorem 1, repeated applications of Lemmas 2 and 4 imply that $(x + 1, y + 1)$, $(x + 2, y + 2)$, etc. are feasible. Thus we would ultimately reach a flow of (s,t) with $s + t = m$ and $s - t = x - y$. As m is even, $s - t$ must also be even.

Now suppose we could obtain a flow of $2k$ consisting entirely of undirected paths. Then this could be considered either as a $(2k, 0)$ or a $(0, 2k)$ flow or anything in between. It is clearly an Euler collection of paths. Hence we could obtain a flow of (s,t) with $s + t = m$, $|s - t| = 2k$. What we show now is that the converse of this is also true.

<u>Theorem 2</u> Suppose there is an Euler collection of paths forming an (x,y) flow. Then we have seen that $|x - y|$ must be even, so let it be $2k$. Let P be the subnetwork consisting of these paths. Then there is a flow of $2k$ in P between a and b consisting entirely of undirected paths.

<u>Proof.</u> We define a <u>critical</u> $a - b$ cut-set in P, or just critical cut-set, to be a set of arcs such that their removal leaves a and b in

different components, and such that no subset of these arcs has this property. Clearly, every critical cut-set separates the network into exactly two components, and each arc in it has one end in each component. Let E be a critical cut-set in P, and let the two components of $P - E$ be C and D, where $a \in C$ and $b \in D$. Now consider $C + E$. Each node of C here has as many incoming directed arcs as outgoing directed arcs, by the Euler condition on P. Summing over all nodes in C, the total number of incoming arcs is equal to the total number of outgoing arcs. Each directed arc in C contributes one to each of these sums. Each directed arc in E, however, contributes to only one of the sums. Thus, since the sums are equal, there must be exactly as many arcs in E directed from C to D as there are from D to C. (Of course, there may also be other arcs which are undirected.)

Let $P_1,\dots,P_x$ be the $a \to b$ paths, and $P_{x+1},\dots,P_{x+y}$ the $b \to a$ paths constituting P. Traveling along P_i (in the $a \to b$ direction for $i \le x$, and in the $b \to a$ direction for $i \ge x$) we use some of the arcs of E to get from C to D, and some to get from D to C. Let f_i be the number of directed arcs of E used by P_i to get from C to D, and let u_i be the number of undirected arcs of E used by P_i to get from C to D. Similarly, let b_i be the number of directed of arcs of E used by P_i to get from D to C, and v_i the number of undirected arcs of E used by P_i to get from D to C. Then we have:

(1) $\quad f_i + u_i = b_i + v_i + 1 \qquad 1 \le i \le x$

(2) $\quad f_i + u_i + 1 = b_i + v_i \qquad x + 1 \le i \le x + y$

This is because the $a \to b$ paths cross from C to D exactly one more time than from D to C, and similarly for $b \to a$ paths.

Summing (1) from $i = 1$ to x and (2) from $i = x + 1$ to $x + y$, and adding these results, we get:

$$(3) \qquad \sum_{i=1}^{x+y} f_i + \sum_{i=1}^{x+y} u_i + y = \sum_{i=1}^{x+y} b_i + \sum_{i=1}^{x+y} v_i + x$$

Now since all the arcs of E are used by the $x + y$ paths by assumption, we know that $\sum_{i=1}^{x+y} f_i = \sum_{i=1}^{x+1} b_i$ by what we observed above. Thus

$$(4) \qquad |y - x| = |\sum u_i - \sum v_i| \leq \sum u_i + v_i$$

The expression $\sum u_i + v_i$ is just the total number of undirected arcs in E. Thus we have shown that for any critical $a - b$ cut-set E, there are at least $|x - y| = 2k$ undirected arcs in it.

Now we claim that there is a flow of 2k undirected paths between a and b in P. For let $P = D + U$, the directed and undirected parts. Then if there is no flow of 2k in U alone between a and b, by the Max-flow Min-cut Theorem [1], there is a minimal $a - b$ cut-set E_u in U with fewer than 2k arcs. Now adjoin one at a time (in any order) the arcs of D until adjoining any more would result in a and b no longer being in different components. Let E_D be the remaining arcs of D. Then clearly $E = E_U + E_D$ is a critical cut-set for P. But E has only as many undirected arcs as E_U, and this is fewer than 2k, contradicting (4). Thus there must be a flow of 2k undirected paths between a and b in P. Hence the Theorem is proved.

We note that any collection of 2k undirected paths between a and b is an Euler collection. Thus we have a procedure or algorithm for finding all feasible Euler maximal flows. Namely, for $G = D + U$ as usual, find a maximal flow in U between a and b. Then if this is a 2k flow, assign the 2k paths either $a \to b$ or $b \to a$ directions. Then apply the method of Theorem 1 (i.e., repeated application of Lemma 2) to obtain the maximal two-way flow.

Now none of the original undirected paths may be intact when the procedure terminates, but the initial difference between the numbers of $a \to b$ paths and $b \to a$ paths among them is preserved. Figure 5 is an example of an Euler network where the maximal undirected flow is 2, the maximal two-way flow is 6, and no maximal two-way flow contains any undirected path.

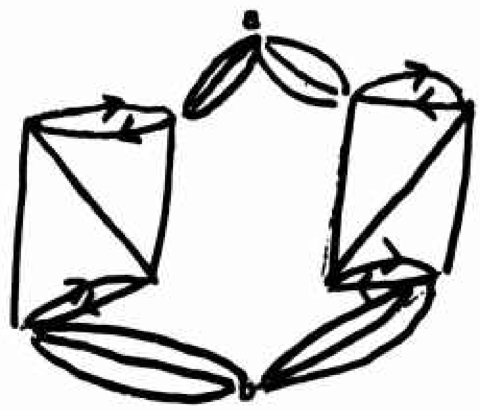

FIGURE 5

Remarks

Now although we know how to find all Euler flows in an Euler network, we know nothing about non-Euler flows in an Euler network. Figure 6 is an example of an Euler network with no undirected paths between a and b, and a maximal double flow of 6. Thus the only Euler two-way flow is (3,3). But also a (4,2) flow is feasible (not an Euler one, of course).

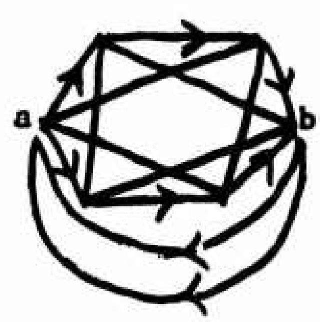

FIGURE 6

The Max-flow Min-cut Theorem [1] holds for any undirected network. It does not need to be Euler. By Theorem 1 above, for Euler directed networks, the maximum two-way flow equals the minimum two-way cut-set. We might guess that if $G = D + U$, as usual, and D is an Euler directed subnetwork, while U is not necessarily Euler, then the maximum two-way flow equals the minimum two-way cut-set. Figure 7 gives a counter-example for this, where the maximum flow is one, but the minimum cut-set is two.

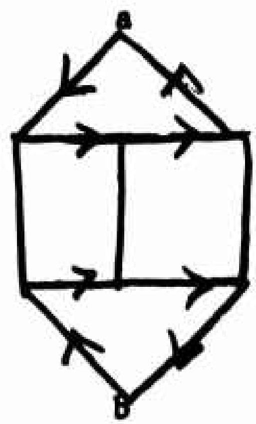

FIGURE 7

Finally, we observe that we can generalize the above results slightly as follows. Consider a network G with A and B two disjoint sets of nodes of G. Call G <u>almost Euler</u> if for all nodes except those of A and B we have the number of undirected arcs there even, and the number of incoming directed arcs equal to the number of outgoing directed arcs there. Similarly, a flow is almost Euler if the arcs used by it determine an almost Euler subnetwork. By using the standard method of introducing two super-nodes a and b, as indicated in Figure 8,

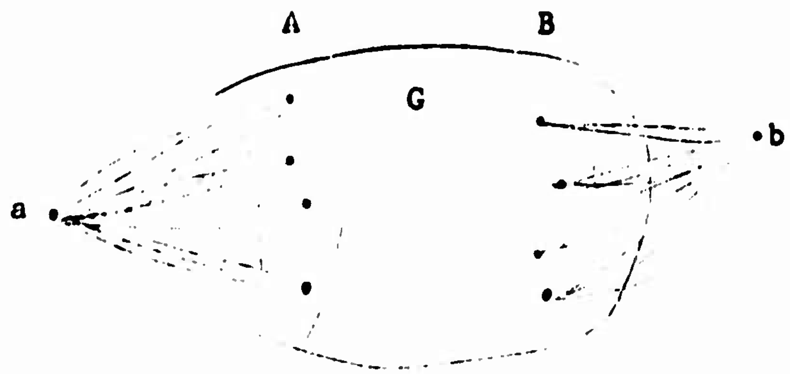

FIGURE 8

we can obtain from Theorems 1 and 2 their analogues, replacing $a \leftrightarrow b$ flows and cut-sets by $A \leftrightarrow B$ flows and cut-sets, and the Euler conditions by almost Euler conditions. (The conditions of evenness are not true for these analogues.)

References

[1] L. R. Ford, Jr. and D. R. Fulkerson, Flows in Networks, Princeton University Press, 1962.

[2] O. Ore, Theory of Graphs, American Mathematical Society Colloquium Publications, Vol. XXXVIII (1962).

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13 ABSTRACT

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