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Stochastic Duels with Time-of-Flight Included

C. J. Ancker, Jr.

19 May 1966

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Stochastic Duels with Time-Of-Flight Included*

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ABSTRACT

The fundamental duel in the Theory of Stochastic Duels is extended to include projectile time-of-flight. As a preliminary, the marksman firing at a passive target is studied. Two firing procedures are assumed. In one, refiring proceeds as rapidly as possible, whereas in the other the duelist delays until he observes the effect of each round. Both fixed (discrete) and continuous (random) firing-times are considered. Also included is the case where time-of-flight varies uniformly with elapsed time. General solutions and examples are given. The analysis indicates in a quantitative way how time-of-flight may radically influence the outcome.

* Presented at the Twenty-Ninth National Meeting of the Operations Research Society of America, 19 May 1966, Santa Monica, California.

I. INTRODUCTION

In a number of previous papers [1]-[9], the basic one-versus-one stochastic duel has been developed and then extended to include such things as limited ammunition supply, limited time-duration, surprise, and variable kill probabilities. In addition, to some extent, the case of more than two adversaries has also been considered. Our principal concern here is to extend the fundamental duel to those situations in which projectile time-of-flight has a significant effect on the outcome. Previously, time-of-flight has always been assumed to be zero. There are, of course, many situations in which time-of-flight plays an important role in combat. Clearly, one such case occurs where the distance between the combatants is rather large. An example is the artillery counter-battery duel.

In the basic model, two opponents, A and B, start with unloaded weapons and with unlimited supplies of ammunition. They load (starting together at time zero) and fire, repeating this process until one or the other or (under certain conditions) both are killed. There is no time-limit on the duel thus making certain one of the two (or three) outcomes occurs. On each round fired, A has a fixed probability, p_A, of killing B, and similarly, B's kill probability is p_B. The time between rounds fired is different for each contestant and is taken to be either a random variable or a constant. Finally, we assume each contestant's time-of-flight to be either a random variable or a constant.

Time-of-flight is incorporated in two different ways. In the first procedure, as soon as the weapon is fired it is immediately prepared and refired without delay. Thus, as far as the winning combatant is concerned, the succession of events is a series of firings (separated only by the time between rounds); these events continue until the killing round is fired, and then the actual kill occurs at a later time thus including the time-of-flight. Of course, it is possible that more rounds may be fired in the time between the firing of the killing round and the time it lands, but this is of no consequence.

In the second procedure, the first round is prepared and fired, and the contestant waits until it lands. He then prepares and fires the second round, and so on. This might correspond to artillery under observed fire conditions where each round (salvo, volley) is observed and corrections made before the next one is fired. The principal objective of the analysis is to determine the probability of the three outcomes (with time-of-flight included, the probability of a draw is always non-zero).

It is convenient to consider first the simpler problem of the marksman firing at a passive target. This problem is intrinsically interesting, and more importantly, the duel is mathematically equivalent to a situation in which two marksmen are firing at two targets and the first one who scores a hit is declared the winner.

In what follows we shall derive the general solutions to a number of models and then illustrate the results by example. In all cases where continuous density functions are employed we use the negative exponential density function in the example. This is done for simplicity and economy. Although examples using the negative exponential can be handled directly without the use of characteristic functions (which use is fundamental to our method), they do illustrate the method, which is our purpose.

II. THE MARKSMAN FIRING AT A PASSIVE TARGET

The objective is to derive the probability density function of the time to a hit.

A. Random Firing-Times

In this model the time between rounds fired by the marksman is a continuous random variable with probability density function $f(t)$. We shall subdivide this section into two parts, the first describing the procedure in which there is no delay between rounds fired, and the second the procedure with delay.

1. No Delay Between Rounds Fired.

In this case the marksman loads, lays, and fires at his normal rate not waiting to observe the effect, until finally one round hits the target. Note that for long times-of-flight and a fast rate of fire there may be several rounds in the air at once. If the random variable, time between rounds is denoted by T, and if τ is the random variable, time-of-flight, then one complete sequence might appear graphically as in Figure 1.

Let T_F be the time at which the killing round is fired and T_K be the time at which the target is hit. Since the probability of a hit on each round fired is not unity, and since the time between rounds fired is a random

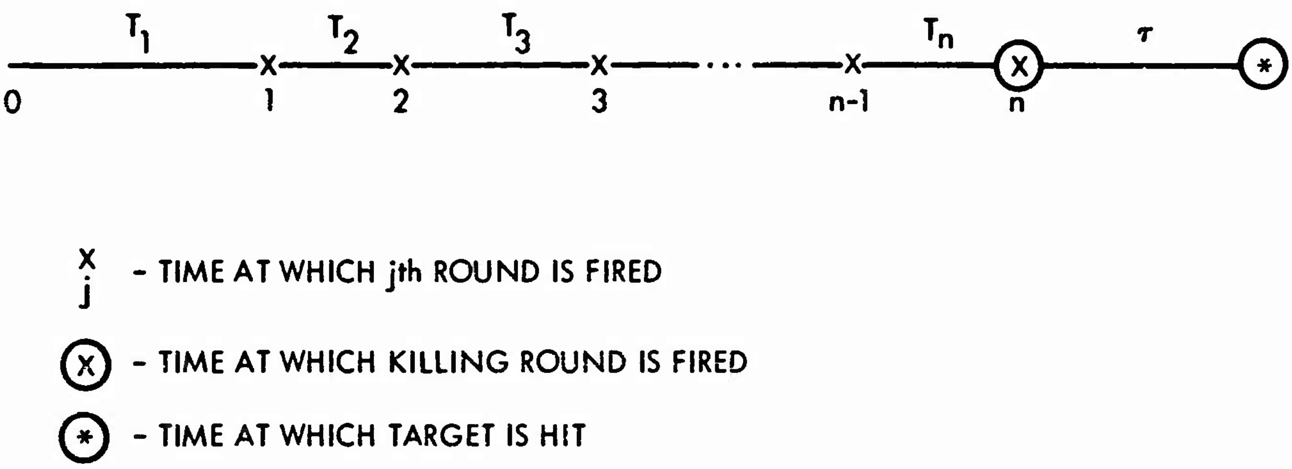


Figure 1.

The Marksman Firing at a Target with Random Firing-Times and with No Delay

variable, T_F and T_K are clearly random variables. We shall proceed as in the former papers. First, let us derive the probability density function of T_F which we shall call h(t). If p is the marksman's hit probability and $q = 1-p$, then

$$h(t) = pf(t) + pq\, f(t)*f(t) + pq^2\, f(t)*f(t)*f(t) + \ldots$$

$$+ pq^{n-1}f^{n*}(t) + \ldots$$

$$= p \sum_{j=1}^{\infty} q^{j-1}f^{j*}(t)$$

or, converting both sides into characteristic functions,

$$\Phi(u) = p \sum_{j=1}^{\infty} q^{j-1}\phi^{j}(u) = \frac{p\phi(u)}{1-q\phi(u)} \tag{1}$$

where

$*$ denotes the convolution integral,

$n*$ denotes n iterated convolutions of a function with itself,

$\phi(u) = \int_0^\infty e^{iut} f(t)dt$ is the characteristic function of f(t),

and

$\Phi(u)$ is the characteristic function of h(t).

Thus, inverting,

$$h(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iut}\phi(u)du}{1-q\phi(u)} \tag{2}$$

and now if the density function of τ is g(t), then the density of T_K is

$$\ell(t) = h(t)*g(t)$$

or in terms of characteristic functions

$$T(u) = \Phi(u)\psi(u) \tag{3}$$

where

$\psi(u)$ is the characteristic function of g(t),

and

$T(u)$ is the characteristic function of $\ell(t)$.

Inverting (3),

$$\ell(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iut}\phi(u)\psi(u)du}{1-q\phi(u)} . \tag{4}$$

In the important special case where τ is a constant, then g(t) is $\delta(t-\tau)$ and $\psi(u) = e^{iu\tau}$ and

$$\ell(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iu(t-\tau)} \phi(u)\,du}{1-q\phi(u)} \quad . \tag{5}$$

Example 1

Let

$$f(t) = r\,e^{-rt} \text{ and } g(t) = \frac{1}{\tau}\; e^{-t/\tau}$$

where r is the rate of fire and τ is the mean time-of-flight. Then

$$\phi(u) = \frac{r}{r-iu} \text{ and } \psi(u) = \frac{1}{1-i\tau u} \quad .$$

Thus, from (4),

$$\ell(t) = \frac{pr}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iut}\,du}{(pr-iu)\,(1-i\tau u)} \quad .$$

The integrand has two poles in the lower half of the complex plane at $u = -ipr$, and $u = -\frac{i}{\tau}$. This expression may be integrated around the contour in Figure 2.

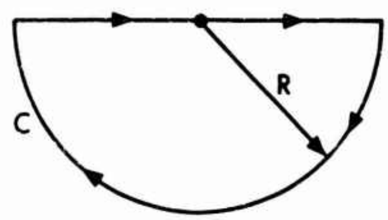


Figure 2.

Path of Integration in Examples 1, 2, and 3

As $R \to \infty$ the integral on C tends to zero and we have by the residue theorem

$$\ell(t) = \frac{pr(e^{-prt} - e^{-t/\tau})}{1-pr\tau} \quad . \tag{6}$$

Example 2

Similarly if g(t) is a constant, τ, and f(t) is as in Example 1, we have from (5),

$$\ell(t) = \frac{pr}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-iu(t-\tau)}du}{pr-iu}$$

$$= pre^{-pr(t-\tau)} \qquad t \geq \tau \qquad (7)$$

$$= 0 \, . \qquad t < \tau$$

2. With Delay Between Rounds Fired.

This case differs from the previous one in that after each firing there is a delay until the round lands. Graphically, this appears as follows (the notation is the same as before):

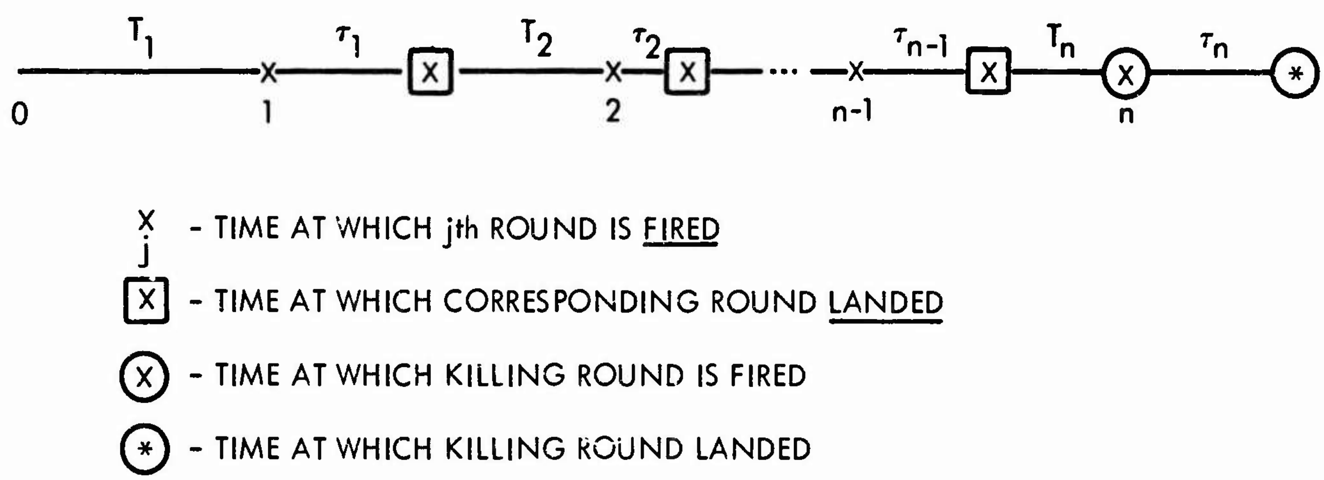


Figure 3.

The Marksman Firing at a Target With Random Firing Times and With Delay

Thus, the time to fire the killing round is

$$h(t) = p \sum_{j=1}^{\infty} q^{j-1} f^{j*}(t) * g^{(j-1)*}(t)$$

or

$$\Phi(u) = \frac{p\phi(u)}{1-q\phi(u)\psi(u)} \quad . \tag{8}$$

Inverting,

$$h(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iut}\phi(u)du}{1-q\phi(u)\psi(u)} \quad , \tag{9}$$

and

$$\ell(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iut}\ T(u)du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iut}\Phi(u)\psi(u)du$$

$$= \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iut}\phi(u)\psi(u)du}{1-q\phi(u)\psi(u)} \quad . \tag{10}$$

For $g(t)$ a constant, τ,

$$\ell(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iu(t-\tau)}\phi(u)du}{1-qe^{iu\tau}\phi(u)} \quad . \tag{11}$$

Equation (11) is difficult to deal with because of the essential singularity in the denominator. It is included for completeness.

Example 3

Using the same assumptions as in Example 1,

$$\ell(t) = \frac{pr}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iut}\ du}{(r-iu)(1-i\tau u)-qr} \quad .$$

The poles are at $u = -(i/\tau)\left\{(1+r\tau) \pm \sqrt{(1+r\tau)^2 - 4pr\tau}\right\}$, which gives on integration

$$\ell(t) = \frac{pr\tau\, e^{-(t/\tau)(1+r\tau)} \sinh\left[(t/\tau)\sqrt{(1+r\tau)^2 - 4\,pr\tau}\right]}{\sqrt{(1+r\tau)^2 - 4pr\tau}} \,. \tag{12}$$

It is obvious that the inclusion of time-of-flight can greatly change the outcome. For instance in Example 2, we may write

$$\ell(t) = \ell_0(t)e^{pr\tau} \quad ,$$

where $\ell_0(t)$ is the solution with zero time-of-flight. Clearly the factor $e^{pr\tau}$ can range from 1 to ∞, depending on the value of the parameters p, r, and τ, which is to be expected.

B. <u>Fixed Firing-Times</u>

We shall not consider all the possible cases here, as they are not all physically meaningful. Thus only the no-delay situation with fixed time-of-flight is given (see Figure 4). The time between rounds is a, and τ is the

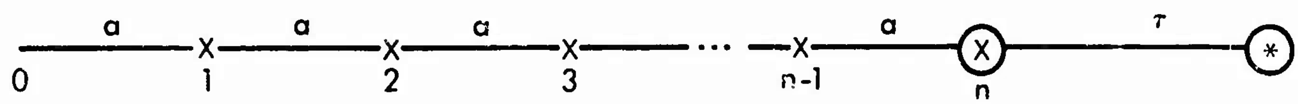


Figure 4.
The Marksman Firing at a Target
With Fixed Firing Times and With No Delay

time of flight. The solution may be written down at once:

$$h(t) = P(t=na) = pq^{n-1} \qquad n=1,2,\ldots. \tag{13}$$

and

$$\ell(t) = P(t=na+\tau) = pq^{n-1}\,. \qquad n=1,2,\ldots \tag{14}$$

III. THE NO-DELAY DUEL

A. Random Firing-Times

If we let T_A and T_B be the random times at which A and B respectively fire killing rounds, then Figure 5 illustrates the ways that various outcomes may occur. Note that τ_A and τ_B are their respective times-of-flight, which we take to be random for the moment.

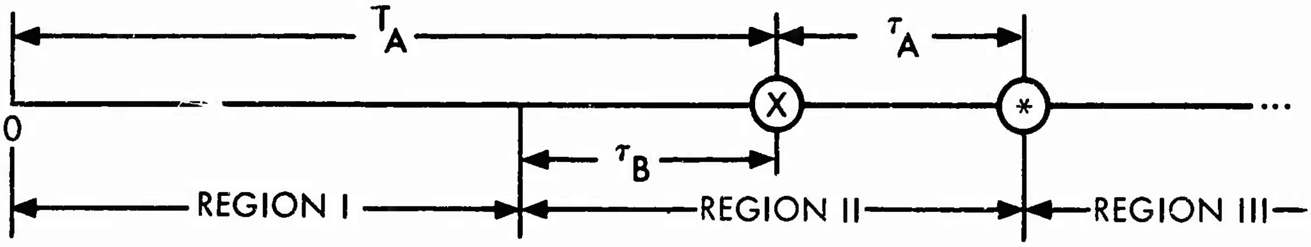


Figure 5.
The No-Delay Duel with Random Firing Times

In Figure 5, if T_B falls in Region I, then B wins; if it falls in Region II, both contestants are killed, and we call the outcome a draw; if it falls in Region III, A wins.

Thus, the probability that A wins is

$$P(A) = P\ (T_A+\tau_A<T_B) \quad .$$

This means that if A's time-of-flight is added to A's time-to-kill we can immediately use the results of the fundamental duel [1] by replacing $\Phi_A(u)$ by $T_A(u) = \Phi_A(u)\psi_A(u)$ where the notation carries on in an obvious way from the marksman results given above. The subscripts refer of course to a particular contestant. Thus,

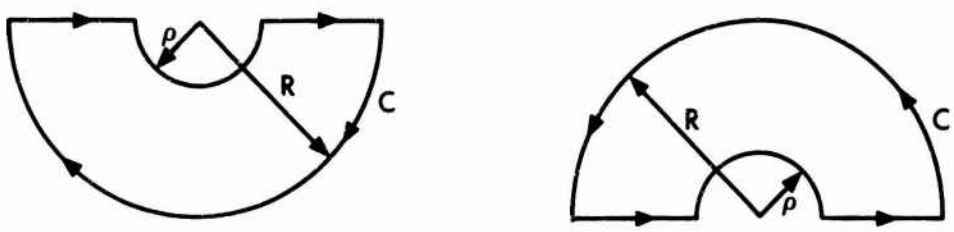
$$P(A) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \psi_A(-u)\Phi_A(-u)\,[\Phi_B(u)-1]\,\frac{du}{u} \tag{15a}$$

$$= 1/2 + \frac{1}{2\pi i}\,(P)\int_{-\infty}^{+\infty} \psi_A(-u)\Phi_A(-u)\Phi_B(u)\,\frac{du}{u} \tag{15b}$$

$$= \frac{1}{2\pi i}\int_L \psi_A(-u)\Phi_A(-u)\Phi_B(u)\,\frac{du}{u} \tag{15c}$$

$$= 1 + \frac{1}{2\pi i}\int_U \psi_A(-u)\Phi_A(-u)\Phi_B(u)\,\frac{du}{u}\ , \tag{15d}$$

where $(P)\int$ means the Cauchy principal value of the integral and $\int_L$, $\int_U$ mean contour integration in the complex plane around the contours shown in Figures 6a and 6b, respectively. The radius ρ is finite but less than the distance to



(a) Figure 6. (b)

The Paths of Integration for Equations (15c), (15d), (16b), (19c), (20b), (30a) and (30b).

the nearest singularity, and of course, the integral must vanish on C as $R\to\infty$. The conditions under which contour integration may be used have been shown [1] to be those in which $f_{A,B}(t)$ are differentiable functions of bounded variation and in which $\lim_{t\to\infty} f(t)=0$. This holds throughout the remainder of the paper.

P(B) is obtained by interchanging A and B in (15) and draws are given by

$$P(AB) = 1 - P(A) - P(B)$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \phi_A(-u)\phi_B(u)[\psi_B(u)-\psi_A(-u)]du/u \qquad (16a)$$

$$= \frac{1}{2\pi i}\int_{L,U} \phi_A(-u)\phi_B(u)[\psi_B(u)-\psi_A(-u)]\, du/u \quad . \qquad (16b)$$

<u>Example 4</u>

Let all the density functions be exponential with r_A and r_B as the rates of fire and τ_A and τ_B as the mean times-of-flight. Then, from (15d),

$$P(A) = \frac{1}{2\pi i}\int_L \left(\frac{1}{1+i\tau_A u}\right)\left(\frac{p_A r_A}{p_A r_A + iu}\right)\left(\frac{p_B r_B}{p_B r_B - iu}\right)\frac{du}{u}$$

$$= \frac{1}{1+\tau_A p_B r_B}\left(\frac{p_A r_A}{p_A r_A + p_B r_B}\right) \quad . \qquad (17)$$

The part in brackets is the solution to the corresponding fundamental duel with zero time-of-flight. Thus, the coefficient in front corrects the fundamental duel to include time-of-flight. Clearly, time-of-flight can be very important since for a given $p_B r_B$ the correction factor may vary from 0 to 1 depending on the value of τ_A.

In a similar manner

$$P(AB) = \frac{p_A r_A p_B r_B[\tau_B(1+\tau_A p_A r_A) + \tau_A(1+\tau_B p_B r_B)]}{(1+\tau_B p_A r_A)(1+\tau_A p_B r_B)(p_A r_A + p_B r_B)} \quad . \qquad (18)$$

For the case of fixed time-of-flight, which is perhaps of more practical interest than the random case, the general solution is given by replacing $\psi_A(u)$ in (15) and (16) by $\exp[i\tau_A u]$ where appropriate. Thus

$$P(A) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \exp[-i\tau_A u]\Phi_A(-u)\,[\Phi_B(u)-1]\,\frac{du}{u} \tag{19a}$$

$$= \frac{1}{2\pi i}\,(P)\int_{-\infty}^{+\infty} \exp[-i\tau_A u]\Phi_A(-u)\Phi_B(u)\;du/u \tag{19b}$$

$$= \frac{1}{2\pi i}\int_L \exp[-i\tau_A u]\Phi_A(-u)\Phi_B(u)\;du/u\quad, \tag{19c}$$

and

$$P(AB) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \Phi_A(-u)\Phi_B(u)\left\{\exp[i\tau_B u] - \exp[-i\tau_A u]\right\}\frac{du}{u} \tag{20a}$$

$$= \frac{1}{2\pi i}\int_U \exp[i\tau_B u]\Phi_A(-u)\Phi_B(u)\frac{du}{u} - \frac{1}{2\pi i}\int_L \exp[-i\tau_A u]\Phi_A(-u)\Phi_B(u)\,\frac{du}{u}\;. \tag{20b}$$

<u>Example 5</u>

The exponential assumptions in Equation (19c) give

$$P(A) = \frac{1}{2\pi i}\int_L \exp[-i\tau_A u]\left(\frac{p_A r_A}{p_A r_A+iu}\right)\left(\frac{p_B r_B}{p_B r_B-iu}\right)\frac{du}{u}$$

$$= \frac{p_A r_A \exp[-p_B r_B \tau_A]}{p_A r_A+p_B r_B}\quad, \tag{21}$$

and

$$P(AB) = \frac{1}{2\pi i}\int_U \exp[i\tau_B u]\left(\frac{p_A r_A}{p_A r_A+iu}\right)\left(\frac{p_B r_B}{p_B r_B-iu}\right)\frac{du}{u} - \frac{1}{2\pi i}\int_L \exp[-i\tau_A u]\left(\frac{p_A r_A}{p_A r_A+iu}\right)\left(\frac{p_B r_B}{p_B r_B-iu}\right)\frac{du}{u}$$

$$= \frac{p_A r_A(1-\exp[-p_B r_B \tau_A]) + p_B r_B(1-\exp[-p_A r_A \tau_B])}{p_A r_A + p_B r_B} \tag{22}$$

In this case the correction factor in P(A) is $\exp[-p_B r_B \tau_A]$ which again may vary from 0 to 1 depending on τ_A.

B. Fixed Firing-Times

Let a_1 and b_1 be fixed times between rounds fired. The ratio a_1/b_1 is assumed to be rational and a/b is the reduced ratio if a_1 and b_1 contain a common factor. That is, a/b is relatively prime. We shall only consider the important case where the times-of-flight are fixed at τ_A and τ_B respectively. We have previously [4] shown that if A has fired j rounds then B has fired $[j\frac{a}{b}]$ rounds where the notation [x] means the largest integer less than or equal to x. Let us first consider how A may win. In Figure 7, if A fires the

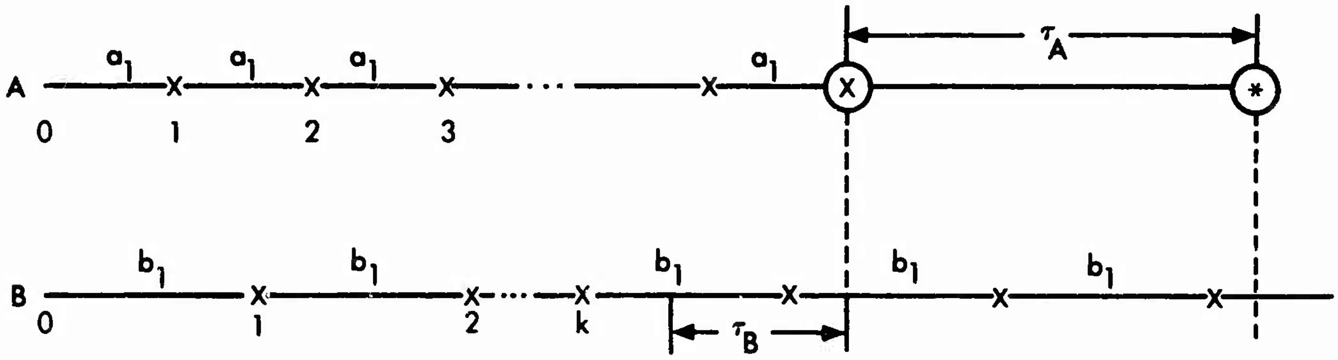


Figure 7.

The No-Delay Duel with Constant Firing-Times

killing round on the j^{th} trial, then it hits at time $ja_1+\tau_A$ and clearly B must fail every firing time in the interval $(0,ja_1+\tau_A)$. The number of rounds fired by B in this interval is $[\frac{ja_1+\tau_A}{b_1}] = [ja/b + \tau_A/b_1]$. Of course, we must account for the possibility of A winning on any round. Thus

$$P(A) = p_A \sum_{j=1}^{\infty} q_A^{\,j-1} q_B^{\,[ja/b+\tau_A/b_1]} \tag{23}$$

and similarly for B,

$$P(B) = p_B \sum_{j=1}^{\infty} q_B^{\,j-1} q_A^{\,[jb/a+\tau_B/a_1]} \, . \tag{24}$$

In the case of draws, we refer again to Figure 7. If A fires a killing round on the $j^{\underline{th}}$ trial, then, for a double kill, B may fire a killing round anywhere in the interval $(ja_1+\tau_A,\ ja_1-\tau_B)$. B's $k^{\underline{th}}$ round (fired just prior to this interval) is his $\left\langle \frac{ja_1-\tau_B}{b_1} \right\rangle^{\underline{th}}$ round. The notation $\langle x \rangle$ means the <u>maximum</u> of the largest integer <u>less than</u> x or zero. Clearly the number of rounds included in the interval is $\left[\frac{ja_1+\tau_A - \langle j\frac{a}{b} - \tau_B/b_1 \rangle b_1}{b_1} \right]$. For the moment let $\langle j\frac{a}{b} - \tau_B/b_1 \rangle = \langle x \rangle$. Thus

$$P(AB) = p_A p_B \sum_{j=1}^{\infty} q_A^{\,j-1} \sum_{k=0}^{[ja/b+\tau_A/b_1 - \langle x \rangle]-1} q_b^{\langle x \rangle + k}$$

$$= p_A \sum_{j=1}^{\infty} q_A^{\,j-1} \left(q_B^{\langle ja/b-\tau_B/b_1 \rangle} - q_B^{[ja/b+\tau_A/b_1]} \right) . \tag{25}$$

<u>Example 6</u>

Let $\frac{a_1}{b_1} = c$, i.e., $a = c$, $b = 1$, where c is a positive integer. Then from (23)

$$P(A) = p_A \sum_{j=1}^{\infty} q_A^{\,j-1} q_B^{[jc+\tau_A/b_1]}$$

$$= p_A q_B^{[\tau_A/b_1] + c} \sum_{j=0}^{\infty} (q_A q_B^{\,c})^j$$

$$= \frac{p_A q_B^{[\tau_A/b_1] + c}}{1-q_A q_B^{\,c}} \quad . \tag{26}$$

In like manner,

$$P(AB) = p_A \sum_{j=1}^{\infty} q_A^{j-1} \left(q_B^{\langle jc-\tau_B/b_1 \rangle} - q_B^{[jc+\tau_A/a_1]} \right) .$$

The second series in the expression above is simply P(A). The first must be broken up into two series, one for $j \leq \left[\frac{[\tau_B/b_1]+1}{c}\right] = R$, and one for $j > R$. Thus,

$$P(AB) = p_A \sum_{j=1}^{R} q_A^{j-1} q_B^0 + p_A \sum_{j=R+1}^{\infty} q_A^{j-1} q_B^{\langle jc-\tau_B/b_1 \rangle} - P(A)$$

$$= 1 - q_A^R + p_A \sum_{j=R+1}^{\infty} q_A^{j-1} q_B^{jc-[\tau_B/b_1]-1} - P(A)$$

$$= 1 - q_A^{[\frac{\alpha}{c}]} \left\{ 1 - \frac{p_A q_B^{c-\alpha+[\frac{\alpha}{c}]c}}{1 - q_A q_B^c} - \frac{p_A q_B^{[\tau_A/b_1] + c}}{1 - q_A q_B^c} \right\} , \tag{27}$$

where $\alpha = [\tau_B/b_1] + 1$.

IV. THE DUEL WITH DELAY

Only random firing-time will be considered in this section. The preceding discussions show that the general solutions for this case have the same form as for the no-delay case and are given by equations (15) and (16). The only difference is that the distributions of times-to-kill are somewhat different. These distributions have been developed under the marksman section and their characteristic functions are (see Equation (8)),

$$\Phi_A(u) = \frac{p_A\phi_A(u)}{1-q_A\phi_A(u)\psi_A(u)} , \tag{28a}$$

and

$$\Phi_B(u) = \frac{p_B\phi_B(u)}{1-q_B\phi_B(u)\psi_B(u)} . \tag{28b}$$

Example 7

Again using exponential density functions with r_A and r_B as rates of fire and τ_A and τ_B as mean-times-of-flight, we have from equations (15c) and (28),

$$P(A) = \frac{1}{2\pi i} \int_L \frac{\frac{1}{1+i\tau_A u}\left(p_A r_A p_B r_B\right)}{\left(r_A+iu - \frac{q_A r_A}{1+i\tau_A u}\right)\left(r_B-iu - \frac{q_B r_B}{1-i\tau_B u}\right)} \frac{du}{u} .$$

The integrand has two poles in the lower half-plane at

$$u = -i \frac{(1+\tau_B r_B) \pm\sqrt{(1-\tau_B r_B)^2 + 4\tau_B r_B q_B}}{2\tau_B} .$$

After much manipulation, using the residue calculation,

$$P(A) = \frac{p_A r_A [p_A r_A \tau_B^2 - p_B r_B \tau_A \tau_B - (\tau_B + r_A \tau_A \tau_B + \tau_A + r_B \tau_A \tau_B)(p_B r_B \tau_B - 1 - r_B \tau_B)]}{(p_A r_A \tau_B - p_B r_B \tau_A)^2 + (\tau_B + r_A \tau_A \tau_B + \tau_A + r_B \tau_A \tau_B)[p_A r_A (1 + r_B \tau_B) + p_B r_B (1 + r_A \tau_A)]} . \quad (29)$$

We shall not continue this example since the calculations, although straightforward, are lengthy. The case in which both contestants have fixed time-delay again runs into the problem of essential singularities in the denominator and will not be attempted.

The fixed firing-time case will not be developed. However, it is easy to see that all the results in Section IIIB hold if one simply replaces a_1 by $a_1+\tau_A$, b_1 by $b_1+\tau_B$, and let a/b be the reduced ratio of $\frac{a_1+\tau_A}{b_1+\tau_B}$ if numerator and denominator contain a common factor.

It is instructive to look at some cases of mixed procedures.

V. MIXED PROCEDURES

Clearly both sides in the duel need not use the same procedure in respect to delay or no-delay. Also the weapons may have such different characteristics that although time-of-flight may be significant for one, it may be entirely negligible for the other. There are many such possible combinations and we shall look at only two here. The is best done by example.

Example 8

Let us assume that A has a fixed time-of-flight, τ_A, and uses the delay procedure, while B has essentially a zero time-of-flight. Then one form of the general solution is

$$P(A) = \frac{1}{2\pi i} \int_L \exp[-i\tau_A u]\Phi_A(-u)\Phi_B(u)\,\frac{du}{u} \tag{30a}$$

$$P(B) = 1 + \frac{1}{2\pi i} \int_U \Phi_B(-u)\Phi_A(u)\,\frac{du}{u} \tag{30b}$$

$$P(AB) = 1 - P(A) - P(B) \quad , \tag{30c}$$

where

$$\Phi_A(u) = \frac{p_A\phi_A(u)}{1-q_A\phi_A(u)\exp[i\tau_A u]} \tag{31a}$$

$$\Phi_B(u) = \frac{p_B\phi_B(u)}{1-q_B\phi_B(u)} \quad . \tag{31b}$$

Again, for exponential density functions,

$$P(A) = \frac{p_A r_A \exp[-p_B r_B \tau_A]}{r_A(1-q_A\exp[-p_B r_B \tau_A]) + p_B r_B} \tag{32a}$$

and

$$P(AB) = \frac{p_A r_A (1-\exp[-p_B r_B \tau_A])}{r_A(1-q_A\exp[-p_B r_B \tau_A]) + p_B r_B} \quad . \tag{32b}$$

In this case we see that, compared to the fundamental duel, P(A) is reduced by even more than the factor $\exp[-p_B r_B \tau_A]$.

Example 9

For this example let us suppose both combatants have fixed times-of-flight but that A uses the delay procedure while B uses the no-delay procedure. Then P(A) is precisely the same as in the preceding example and the exponential assumption gives equation (32a). However, P(B) and P(AB) again contain essential singularities and will not be attempted.

VI. DUELS WHERE TIME-OF-FLIGHT VARIES LINEARLY WITH TIME

As a final problem let us consider a situation in which both A and B use the no-delay procedure, both have continuous random firing-times, and times-of-flight vary in some linear fashion. This latter feature might be approximated if either one or both contestants are moving at a fairly constant velocity. Consider Figure 8.

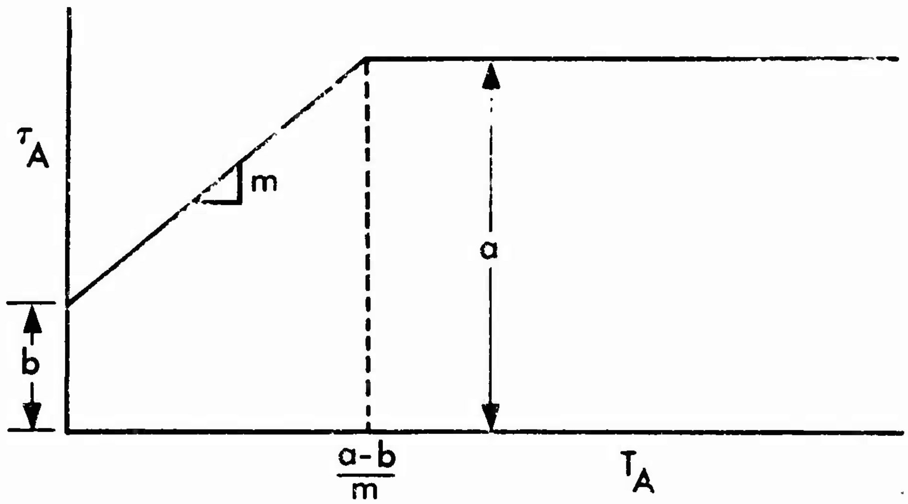


Figure 8.
Linearly Increasing Time-of-Flight

A's time-of-flight, τ_A, increases linearly with elapsed time from a value b to another value a and then remains at a. This corresponds to a situation in which A and B are separating at a uniform rate until some maximum separation occurs, which then remains constant. B's time-of-flight varies similarly. Other situations are included in this formulation as we shall see later. The probability that A wins is

$$
\begin{aligned}
P(A) &= P\,\{T_A + \tau_A \leq T_B\} \\
&= P\,\{T_A + b + m\,T_A \leq T_B\} \qquad 0 \leq T_A \leq \frac{a-b}{m} \\
&+ P\,\{T_A + a \leq T_B\} \qquad T_A \geq \frac{a-b}{m}
\end{aligned}
$$

where, as before, T_A, T_B are A's and B's times-to-a-kill. The parameter m is directly related to the rate at which separation is occurring. P(A) may now be written in terms of our usual density functions as

$$
P(A) = \int_0^{\frac{a-b}{m}} h_A(t_A)dt_A \int_{(1+m)t_A+b}^{\infty} h_B(t_B)dt_B + \int_{\frac{a-b}{m}}^{\infty} h_A(t_A)dt_A \int_{t_A+a}^{\infty} h_B(t_B)dt_B \;. \tag{33}
$$

First, we shall make two differenc transformations in (33). For the first double integral, let $t_A = \frac{\xi-b}{1+m}$; for the second, let $t_A = \xi-a$. This gives

$$
P(A) = \int_b^{\frac{a-b}{m}+a} h_A\left(\frac{\xi-b}{1+m}\right) \frac{d\xi}{1+m} \int_{\xi}^{\infty} h_B(t_B)dt_B + \int_{\frac{a-b}{m}+a}^{\infty} h_A(\xi-a)d\xi \int_{\xi}^{\infty} h_B(t_B)dt_B \;. \tag{34}
$$

We have previously shown [1] that the inner integrals may be replaced by $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-iu\xi}[\Phi_B(u)-1]\,du}{iu}$. When this substitution is made and the order of integration reversed,

$$
\begin{aligned}
P(A) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{[\Phi_B(u)-1]du}{u} \int_b^{\frac{a-b}{m}+a} e^{-iu\xi} h_A\left(\frac{\xi-b}{1+m}\right) \frac{d\xi}{(1+m)} \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{[\Phi_B(u)-1]du}{u} \int_{\frac{a-b}{m}+a}^{\infty} e^{-iu\xi} h_A(\xi-a)d\xi \;.
\end{aligned} \tag{35}
$$

Again we shall transform the inner integrals by $\eta = \frac{\xi - b}{1+m}$ for the first and $\eta = \xi - a$ for the second. Thus

$$P(A) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} [\phi_B(u)-1]e^{-ibu}\, du/u \int_0^{\frac{a-b}{m}} e^{-i(1+m)\eta u} h_A(\eta)\,d\eta$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} [\phi_B(u)-1]e^{-iau}\, du/u \int_{\frac{a-b}{m}}^{\infty} e^{-iu\eta} h_A(\eta)\,d\eta \quad . \tag{36}$$

It can be shown that for density functions of our type,

$$\int_0^a e^{iut} f(t)\,dt = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\phi(w+u)\,[1-e^{-iwa}]}{w}\,dw \; .$$

Using this result on the inner integrals we have, after some simplification

$$P(A) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{-iau}\phi_A(-u)[\phi_B(u)-1]\,\frac{du}{u}$$

$$+ \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{\{\exp[-i\,(\frac{a-b}{m})\,u]-1\}}{u}\left(\int_{-\infty}^{+\infty} \frac{[\phi_B(w)-1]}{w}\left\{e^{-ibw}\phi_A[u-(1+m)w]-e^{-iaw}\phi_A(u-w)\right\}dw\right)du \tag{37}$$

This result also applies for the situation depicted in Figure 9 if we replace m with -m in equation (37). Of course, a cannot be negative, so if the sloping line goes to the x-axis simply let a = 0. This represents the situation in which the opponents are closing on one another. The expression for P(B) may be written down at once by interchanging A and B in (37) and replacing the parameters a, b and m by the corresponding ones for B, say c, d and n. As usual, P(AB) = 1- P(A) - P(B), which completes the solution.

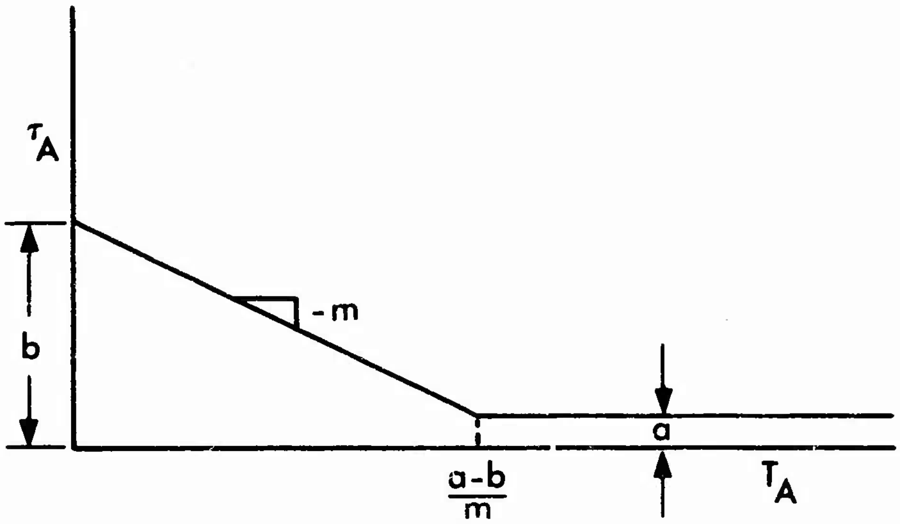


Figure 9.
Linearly Decreasing Time-of-Flight

Example 10

Again for exponential firing times,

$$P(A) = \frac{p_A r_A}{2\pi}\int_{-\infty}^{+\infty}\frac{ie^{-iau}}{(p_A r_A+iu)}\left(\frac{1}{p_B r_B-iu}\right)du$$

$$+\frac{p_A r_A}{4\pi^2}\int_{-\infty}^{+\infty}\left\{\frac{\exp\left[-i\left(\frac{a-b}{m}\right)u\right]-1}{u}\right\}\int_{-\infty}^{+\infty}\frac{i}{p_B r_B-iw}\left[\frac{e^{-ibw}}{p_A r_A-i[u-(1+m)w]}-\frac{e^{-iaw}}{p_A r_A-i(u-w)}\right]dw\left.\vphantom{\int}\right)du\ .$$

The poles in the lower half-plane (both for the first integrand and the inner integrand of the second expression) are at $u,w = -ip_B r_B$. From which

$$P(A) = \frac{p_A r_A \exp[-ap_B r_B]}{p_A r_A+p_B r_B}$$

$$-\frac{p_A r_A}{2\pi i}\int_{-\infty}^{+\infty}\left\{\frac{\exp\left[-i\left(\frac{a-b}{m}\right)u\right]-1}{u}\right\}\left[\frac{\exp[-bp_B r_B]}{p_A r_A+(1+m)p_B r_B-iu}-\frac{\exp[-ap_B r_B]}{p_A r_A+p_B r_B-iu}\right]du\ .$$

The integrand has two poles in the lower half-plane at $u = -i\,[p_A r_A+(1+m)p_B r_B]$ and $u = -i\,[p_A r_A+p_B r_B]$. Using these,

$$P(A) = p_A r_A \exp[-bp_B r_B]\left\{\exp\left[-\left(\frac{a-b}{m}\right)\left[p_A r_A+(1+m)p_B r_B\right]\right]\left(\frac{1}{p_A r_A+p_B r_B} - \frac{1}{p_A r_A+(1+m)p_B r_B}\right) + \frac{1}{p_A r_A+(1+m)p_B r_B}\right\}. \tag{38}$$

We may immediately get the case of opponents who are closing on one another by setting $a = 0$ and replacing m by $-m$.

$$P(A) = p_A r_A \exp[-bp_B r_B]\left\{\exp\left[-\left(\frac{b}{m}\right)\left[p_A r_A+(1-m)p_B r_B\right]\right]\left(\frac{1}{p_A r_A+p_B r_B} - \frac{1}{p_A r_A+(1-m)p_B r_B}\right) + \frac{1}{p_A r_A+(1-m)p_B r_B}\right\}. \tag{39}$$

Two limiting cases are easily checked from (39). If $m\to\infty$ we should get the fundamental due with zero time-of-flight, which we do, i.e.,

$$P(A) = \frac{p_A r_A}{p_A r_A+p_B r_B} \quad . \tag{40}$$

If $m = 0$ we should get the fixed time-of-flight case or

$$P(A) = \frac{p_A r_A \exp[-bp_B r_B]}{p_A r_A+p_B r_B} \quad , \tag{41}$$

which again checks, since we interpret b to be τ_A.

If we let $a\to\infty$ in (38), the expression is for τ_A increasing without limit. Thus

$$P(A) = \frac{p_A r_A \exp[-bp_B r_B]}{p_A r_A+(1+m)p_B r_B} \tag{42}$$

This differs from the fixed time-of-flight case only by the (1+m) factor in the denominator. Since this factor may vary from 0 to ∞ it may change P(A) not at all, or reduce it to zero, or put it at any value in between.

VI. CONCLUSION

Various situations in which time-of-flight may be significant have been examined. General solutions to the problem of the marksman versus a passive target and to the problem of the duel have been given, with examples of each. In all cases, as one might expect, if time-of-flight is large relative to the other parameters (particularly with respect to time between rounds fired) it has a major influence on the outcome. Some idea of the quantitative effect may be obtained from the examples.

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Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) System Development Corporation Santa Monica, California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Stochastic Duels With Time-Of-Flight Included

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (Last name, first name, initial)

Ancker, C. J.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
19 May 1966	28	9

8a. CONTRACT OR GRANT NO. Independent Research for the Scientific Community b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) SP-1017/009/00 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

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The fundamental duel in the theory of Stochastic Duels is extended to include projectile time-of-flight. As a preliminary, the marksman firing at a passive target is studied. Two firing procedures are assumed. In one, refiring proceeds as rapidly as possible, whereas in the other the duelist delays until he observes the effect of each round. Both fixed (discrete) and continuous (random) firing-times are considered. Also included is the case where time-of-flight varies uniformly with elapsed time. General solutions and examples are given. The analysis indicates in a quantitative way how time-of-flight may radically influence the outcome.

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Stochastic Duels Projectile Time-of-flight Passive Target						

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