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NOTES ON THE TAXONOMY OF PROBLEMS CONCERNING PUBLIC GOODS

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Notes on the Taxonomy of Problems

Concerning Public Goods

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Martin Shubik **

> "A person may make use of his own property or ... conduct his own affairs at the expense of some harm to his neighbours. He may operate a factory whose noise and smoke cause some discomfort to others, so long as he keeps within a reasonable bound. It is only when his conduct is unreasonable, in the light of its utility and the harm which results, that it becomes a nuisance. ... As it was said in an ancient case in regard to candlemaking in a town, "Le utility del chose excusera le noisomeness del stink."
>
> Prosser on Torts [1]

1. General Considerations

This paper deals with those situations involving an intermix of political, economic and social conditions arising from the needs and desires of a society to control the use of goods or services which cannot be allocated efficiently by a market mechanism alone.

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**I am indebted to Lloyd Shapley and to Herbert Scarf for the many conversations we have held on this and closely allied topics.

The items included are those which cause nuisance or harm to others, those which must be jointly constructed by taxation of privately appropriated resources; those which are unique natural heritages such as beauty spots; items held in common (res communes) such as air and light; and a host of others, many of which are noted later.

An attempt is made to set the problems arising from the features of communal goods into a game theoretic framework. In this framework, the competitive market and voting systems may appear as special cases of a general set of games whose most important properties depend upon the nature of threats between the individual and his society and the existence or the non-existence of methods for dividing the wealth of a society so that there are some divisions against which no subset can claim that it can improve its lot by failing to cooperate. This latter condition is known as the existence of a core to the game (See Appendix A).

Leaving aside problems such as incomplete information, irrational behavior and so forth, there are two major difficulties to be dealt with in the study of market and voting mechanisms in a society. They are characterized by the roles played by numbers of participants and by the structure of preferences and technology and institutions.

Given all the conditions concerning preferences and structure necessary for the existence of a competitive equilibrium, if we make certain assumptions concerning the behavior of individuals, then not only may the equilibrium exist, we may even be able to describe a method for attaining it.

It is usually the case that for the behavioral assumptions to appear to be reasonable it is necessary to postulate that many competitors participate in all markets and that the holdings of no one are disproportionally large in comparison with the others. If these conditions exist, it is a remarkable fact that several highly different theories concerning economic behavior of individuals en masse predict the same outcome. The competitive equilibrium points turn out to be the solution to not only the competitive closed economy with all individuals acting as price-takers but to situations involving the interaction of the countervailing powers of groups as characterized by the core to an n-person game; (See Appendix A) to processes of fair division as characterized by the value (See Appendix A) of an n-person game; and to the type of economic struggle characterized by non-cooperative behavior. (See Appendix A)

Our primary concern here is however not with competitive equilibrium and the conditions necessary for it. In particular this discussion is addressed to the problem of public goods in a society, and the intermix of political voting and economic marketing mechanisms. In order to do this, government, taxing ability, law and voting processes must be explicitly blended with the market mechanism.

In order to separate difficulties, in this paper several factors which either cause conceptual problems not specific to public goods or which only cause extra complications are ruled out. In particular, it is assumed that the markets for private goods are purely competitive. In other words, we do not consider possible oligopolistic influences in these markets. Other factors to be ruled out will be noted below.

The discussion that follows is sometimes restricted to the considering of one public good for which there is a production process involving the private goods (some or all of them) and which is subject to constant or decreasing returns to scale. The inclusion of production processes for the private goods does not apparently add anything of particular conceptual value to our investigation; hence, except for the public good we may regard the situation as characterized by a trading market where the individuals all enter with initial endowments.

We consider an economy with n persons, k private goods and one public good. The i^{th} person has an initial endowment of $(A_1^i, A_2^i, \ldots, A_k^i, 0)$ A_j^i $j = 1, \ldots k$ of the private goods and none of the public good. There is a production function $y = h(x_1, x_2, \ldots x_k)$.

The i^{th} person has preferences described by

$$U_i(x_1^i, x_2^i, \ldots x_k^i, y^i)$$

where the x_j represent his holdings of the private goods and the $y^i \leq y$ gives his usage level of the public good. The $y^i = y$ reflects the condition that a public good can enter the preferences of many people without necessarily being diminished. For example, it may be argued that national defense and justice are supplied to all at the same level and that (to a good approximation) the addition of a citizen to whom national defense and justice will apply will not diminish the amount supplied to others.

Suppose that y units of the public good are produced (what physical units one should use to measure the amount of a public good gives rise to several difficult conceptual problems which have not yet been solved). The possibility for $y^i < y$ may come about in two ways. The public good may have a limited capacity such as a bridge or private park. In which case

crowding by some may reduce the ability of others to use it. Another reason for the possibility that $y^i < y$ may be that society is in a position to exclude certain members from using a public good (because they have not paid their taxes or for other reasons).

It should be noted that there are some public goods which are not produced from private goods*, but which a society has inherited as a unity. In these instances the problem becomes one of the arrangements for individual use of the communal good. The high seas, air and outer space fall in these categories.

We shall see that the prime factors in the construction, enjoyment and use of public goods can be characterized in terms of the powers, rights and threats of individuals and groups. These can be portrayed in game theoretic terms by the characteristic function [2] (and its equivalents: See Appendix A) which provides a method for reflecting the intermix of political and economic powers of groups.

2. The Meaning of Solution to the Problem of Public Goods

It is well known that the price system in general cannot be used to assign a Pareto optimal allocation of resources where public goods are involved.

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*It should further be noted that there are many shades of meaning to private property, ownership and private good, however we avoid entering into detail on this at this time.

Various taxation schemes have, on occasion, been discussed and principles suggested; however, without explicitly introducing the political mechanism and at least some type of model of the institution of government it is not possible to analyze in detail the politico-economic problems posed by the presence of public goods.

One important preliminary set of questions can be asked in the context of an extended general equilibrium model without having to consider voting in any detail or having to describe the power of coalitions. They concern the existence and the nature of the Pareto optimal surface. Foley has shown [3] that if all individuals are confronted with information concerning the amount of the public goods to be produced, the tax rates, their initial resources and the prices for the private goods, there will exist tax schemes such that all individuals when asked to choose between a particular tax scheme and set of competitive prices belonging to a certain set and any other scheme will unanimously select the first over the second. This formally establishes the existence of taxation schemes which give rise to Pareto optimality in the following sense: Given that a government body has proposed a satisfactory scheme, each individual viewing the economy as a modified competitive market may optimize by buying and selling freely and by paying his taxes in return for being able to utilize the level of public goods specified in the government proposition.

Voting as it is usually considered is not in the above model. Only the preferences of individuals separately or en masse are considered. Nevertheless even at this level several interesting general questions need to be asked. Do Pareto optimal outcomes always exist for income taxes, poll taxes and usage taxes? Do they exist if conditions of equality or equity, or progressiveness or regressiveness are imposed on the tax bills?

In general, we are interested in more than the mere existence of Pareto optimal imputations. Groups have power and exert their power; voting mechanisms are used and blend with market mechanisms. In many societies including the United States most of the assets are held by governmental or corporate bodies which are at least pro forma controlled by voting mechanisms.

Given that we can describe the power of coalitions and the nature and roles of the voting mechanism and competitive market system, we may then consider solutions more specific than just the Pareto optimal surface as a whole. Some of the solutions considered by Shapley and Shubik are the core and value. Much of the analysis is related and similar to that appearing in the study of ownership and the production function. [4/]

The analysis of the game theoretic models formulated will not be carried out in this paper. They each need to be examined for Pareto optimality and for many other solutions.

The next section of the article is devoted to presenting a taxonomy of the various public and joint goods and some of the different institutional forms designed to deal with them.

3. A Taxonomy of Public Goods and Institutional Forms.

3.1 Some Problem Areas.

The expenditures of the federal, state and local governments amount to around 20 per cent of gross national product. Corporations, trusts and other institutions swell the amount of economic activities performed by collectivities and fiduciaries. The types of goods and services that are dealt with by groups within society, or society as a whole are varied and present different basic properties which in turn pose different problems of control. It is for this reason that we must attempt to construct a taxonomy of these properties.

A partial list of goods and services involving natural monopolies, or economies or diseconomies external to the individual is given, from which examples are selected to illustrate the different features which force us to construct different models.

Among these aspects of society which may fall in the domain of public goods are:

Museums,
sponsorship of the arts,
bridges and roads,
the armed forces,
police,
government,
parks, harbors,
dams, monuments,
weather control, plague
control and drug administration

A group of goods and services whose communal properties are growing more and more important are related to communication and information processing. They are:

the post office, knowledge,
the educational system,
radio, television ,
local transportation,
libraries, copyrights
and patents.

Among the more important factors involving diseconomies are:

Smoke,

noise,

sewage and general pollution;

crowding, billboards, blocking

of light and other invasions

of privacy.

All of these have the property that either the gain or harm in production or use goes beyond the single individual. Because of this, in general, in all societies elaborate political, social and legal systems have defined and delimited the status of property and the ownership rules for inherently social or antisocial goods.

The law, institutions of society, customs and the laws of physics define the powers of individuals and of groups. From this information a characteristic function can be constructed and the possibility of different social solutions examined.

Possibly the most central game theoretic concept necessary for the formalization of the power of groups is that of threat. By threat, in this context, we mean possible actions (as contrasted with mere verbal statements) which can be taken by a group and which influence (usually negatively) the welfare of others.

It is a remarkable fact that the conditions postulated for the existence of a competitive equilibrium are such that almost any of the many alternative models for describing conflict lead to the same threat characterization. This is not so when public goods are present.

3.2 Factors Which Control the Game

Nine factors are presented which serve to describe the threat structure of individuals and groups in a society with a government, laws and a political and economic structure. They are not meant to be exhaustive, but they cover many of the features needed to delineate the mechanism of social division of public goods.

(1) The voting mechanism

Is the decision dependent upon a vote? Is the vote direct as it is in a referendum for, say, a bond issue; or is it indirect as in the case of the military budget where people who have been elected vote as trustees of the electorate?

What constitutes a winning vote? In some situations such as an act of union it may be unanimous; an impeachment may call for 75 per cent; a veto for 2/3 majority and ordinary business a simple majority. When more than one item is being voted on such as in the election of directors to a corporate board votes may be cumulative. There are several different

cumulative vote mechanisms.

Are votes for sale? For corporations votes may be purchased legally in the open market. In most political bodies, in theory, votes are not for sale; however, political payoffs and logrolling are well-known phenomena.

(2) Tax Power

Can a winning coalition legally and successfully impose its taxation scheme on others? Various individuals may oppose military budget expenditures but they are nevertheless forced to pay their taxes. It is conceivable that for some items such as the erection of a public monument a vote is needed to obtain permission to build; however, the actual erection can take place only through the private financing of the interested group.

We may distinguish the three cases where the group has the power to tax all, to tax its members or to tax none. This leaves out the many other possible restrictions on degree of taxing power.

(3) Ability to Build or Practise

Can a losing coalition build its own public good for its own use? People who do not like their school district may subscribe to the construction of a private school. In the United States it is illegal to construct

a private army or diplomatic corps. In Spain it was illegal to open a place of worship other than one of the Catholic religion. In England people often subscribe to private parks. Even though a group may believe in drug healing or rites involving the use of narcotics they may be forbidden from growing their own poppies, brewing their own beer or running their own stills.

Professions which require licenses may in some cases not be practised, even though both sides are willing; thus, although an amateur barber may cut his friends' hair with little danger of litigation, an amateur surgeon may not remove an appendix with the same lack of risk.

(4) Restrictions on the Use of a Public Good

Is it possible for one group of individuals to force or to prevent another group from using a public good? The atheist may not be able to stop the chaplain from blessing the regiment he is in; the pacifist obtains the benefit of national defense whether he wants it or not. The tax dodger may be jailed but he cannot be prevented from obtaining the benefits of national government. It may not be legal to jam the radio reception of non-subscribers to a radio station. It is legal to bar an individual from a public park if he fails to pay an entrance fee or fails to observe the rules.

We find that owing to the intermix of the laws of Nature and of various societies there exist public goods or services in all four categories where the individual can or cannot be forced to or prevented from utilizing the good or service profferred.

(5) Type of Taxation Permitted

Six common types of taxation can be described as (i) income tax, (ii) property tax, (iii) poll tax, (iv) sales tax, (v) estate tax and (vi) usage tax or toll. In common parlance the distinctions between these seem to be clear enough; however, they can present analytical difficulties. Various states in the United States use either income or property taxes as a major source of revenue.

Failure to pay may lead to legal action, prison terms and the impounding of property.

A poll tax takes the form of a single equal levy on all. Those who do not pay may be barred from enjoying a right such as voting. Sales taxes are another important revenue device where a surcharge is collected on the purchase of the taxed commodity. Usage taxes or tolls are related to sales taxes in the sense that they vary directly with the amount utilized by the individual. They are often used for special highways, bridges,

ports, national parks and beaches. The individual usually has the choice of not paying and not obtaining the services regardless of whether he may already be subsidizing them.

Estate taxes fall into an especially interesting class inasmuch as they raise several fundamental questions concerning our basis of views of economic and social man and attitudes towards risk and death.

There is a seventh form of taxation much used by central governments; that is taxation by the printing of extra currency. This will be referred to again in point (9). There is also negative taxation in the form of subsidies.

(6) Conditions on Taxation

Conditions based upon considerations of equity or power may limit the way in which taxing is applied. Questions concerning equity involve progressive, constant or regressive income tax and the relationship between tax size and property or estate size. In different countries and at various times all possibilities have existed. Sometimes considerations of equity and power are intermixed. For example, the price of an admission ticket to a public park is the same for all; (except school-children and a few other classes may obtain a different rate). However, it is likely

that the proposition that each individual in society should pay a different price for his use of a park would be violently opposed both for reasons of equity and because of the difficulty in trying to administer such a usage tax scheme.

Equity in some countries has called for "one man, one vote." A poll tax or property requirement may be regarded as an extension of this to "equal vote, equal financial stake."

Does a society wish to consider taxes on special groups? In the Arab conquest of North Africa very different tax rates were levied against the unbelievers. Throughout history, Jews, merchants or other minority groups have had special tax burdens levied against them.

In summary, we must stress that it is necessary to decide if "conditions on taxation" are to be considered as part of the "rules of the game" or as part of the goals. Do we take it as axiomatic that groups distinguished from others by other than economic measures must not be specially taxed even if without doing so a condition such as Pareto optimality cannot be achieved? We may partially avoid answering a question such as this by observing that we are describing conditions which can be used to set up and study politico-economic processes; hence this social question can be treated in either form as a limit imposed or not imposed upon the rules of the game.

(7) The Role of Diseconomies

It appears to be important to distinguish three states of interaction between individuals which may exist as the result of the behavior of any individual. They can be characterized as positive, zero and negative interactions. The zero interaction is at the very foundation of the theory of the competitive market and individual property. As has already been noted it can be formally characterized by situations in which the only threat that an individual has over his fellows is to fail to cooperate. Once he has decided not to cooperate his fate and theirs are no longer interlinked.

With the usual public goods the fates of non-cooperators are often still positively interlinked in the sense that a better police force and more beautiful parks may benefit even those who do not pay for them. It must be noted, however, that the forced consumption of a public good by some who would otherwise welcome it may be regarded as a diseconomy in the sense that its free distribution whether you want it or not may be regarded as an invasion of privacy. Some examples are drug control, fluorination and "comes the revolution, everyone will eat strawberries and cream whether he likes it or not!"

The negative interaction between people, or the presence of external diseconomies grows as formerly free goods such as clean air is polluted and smoke, noise, crowding, billboards and the other effluvia of the metropolis and high density populations make themselves felt.

Basically in the law and our ethical values there appears to be an important difference in attitude towards the rules governing producers of external economies or diseconomies. The man who fills his front garden with highly odiferous roses can expect a different legal treatment than had he planted only skunk cabbage.

(8) The Interaction of the Competitive Market, Voting and Government

In order to specify the politico-economic mechanism for the implementation of policy concerning public goods or the control of diseconomies, it is necessary to introduce explicitly a governmental body and a voting mechanism. In our search for the Philosopher's Stone we would like to keep our models as "institution-free" as possible and to avoid becoming cluttered up with details. Unfortunately, even a model such as that for the competitive economy, in spite of its aesthetic appeal and apparent freedom from institutional features needs many implicit assumptions which make it far less general and institution-free than it may appear to be.

We begin by trying to restrict ourselves to a most rudimentary form of institution which we shall call "the government" and to a description of the voting process and its interaction with the market process.

In many instances, as we have already noted in (1), votes on expenditures may be indirect such as when an elected representative votes on a bill authorizing expenditure. Although eventually, it will be necessary to analyze this case, to begin with we restrict ourselves to a simpler situation. We consider some type of voting system to be specified. A government exists which performs the following tasks: (i) It administers the vote, (ii) It collects taxes and disburses subsidies, (iii) It arranges for the production, supervision and distribution of public goods in coordination with the competitive sector of the economy.

Each one of these points brings up a host of problems. Even though the administration of the vote appears to be a simple point we must immediately ask how does the item to be voted upon come into being. There are thousands of alternative proposals concerning what super highway or national park should be built. How is a proposal selected for the vote? Voting is often time consuming and expensive; how many different alternatives can be considered by means of a voting mechanism?

To begin, we could make some heroic assumptions concerning zero costs of data processing and zero expense to voting. In this instance, we could then consider that some board of experts prepares the motions and our interests will be in the properties of the motions which would be selected by the system, and possibly the relationship between the acceptable motions under one system and those under other related systems.

Technically, the problem of the generation of alternatives upon which to vote causes no problem when we consider solution concepts such as the Pareto optimal set, the core, the value or other concepts of equity. It does cause trouble if one wishes to give an extensive form specification of a political process.

The second role for the government is to collect taxes and disburse subsidies. We discuss below in (9) the importance of distinguishing between money and commodity taxes and subsidies. All instances have been and still are encountered in human affairs. Taxes have been gathered in money, salt, hours of labor and so forth and subsidies even in the United States of today may be paid in food.

As has already been noted, tax collection and concepts of equity cannot be easily treated as separate items. "Unpopular taxes" may be

hard and costly to collect. The law of administrative inconvenience says that if a tax is easy to collect and the group against whom it is levied is not in a position to cause too much trouble, one can more or less forget abstract problems of equity and continue the tax.

Following the policy of abstracting whenever possible from extra difficulties we assume that at this level of the discussion there are no problems concerning the levying or collection of taxes provided that the individuals have the amounts demanded from them.

The third point, that the government acts as the trustee and agent for the purchase and/or construction of public goods, and in many instances, for their supervision, maintenance and distribution, must be considered in terms of several separate problems. They are: (i) Does the government as an institution have an independent set of preferences (or utility function), (ii) Does a winning vote give the successful coalition a chance to bias the functioning of the mechanism of government? and (iii) What is the relationship between private and public purchasing of goods from the competitive market? What are the problems of timing, information leakages and conflict of interest which can arise?

A possible way in which the government may be considered as a motivated participant in the market is to assume that it is given a

task to achieve by a bill that is passed and that it attempts to minimize the cost of carrying it out. There are several alternative formulations which all have representations in politico-economic life. The method and size of taxation and the market purchases may be considered simultaneously; or the tax bill may be considered given; or the expenditure given and the taxation possibilities may be assumed to be variable. In the latter case further problems are faced in defining the utility function of the government. In all of these cases, there is an important difference between taxation in money or in kind. This will be discussed in (9).

If individuals know that a certain motion will be passed, after which the government will enter the open market as a purchaser, it may pay them to arrange their assets to take advantage of this eventuality. Thus the trading rules will make a difference to the game. This observation which arises from a consideration of the strategic possibilities in a formal model has its counterpart in many well-known phenomena such as when groups buy up the farm lands prior to the vote on a new zoning law.

Neither in economic theory nor in game theory to date has the concept of an institution been adequately reflected. A coalition is not necessarily an ongoing organization. Yet in practical politics there are organizations and party faithfuls who "can deliver the vote." The "city machine" could exert a control which removed much of the guesswork from

estimating the type of civic expenditures to be approved and who would be the contractors and suppliers.

Closely interlinked with the direct power that comes about in the control and influence in voting systems and administrative control over government procurement is the effect of "insider's knowledge." The leak to the son-in-law of information as to where the new state highway will go; the discrete use of information by members of a board of directors to their families and friends about a change in the price of sulphur or a new uranium strike, provide examples.

In order to reflect the insider's information advantage the extensive form of the process must show the appropriate time lags such as when a vote is passed and when various groups are told about the results of the vote. This, of course, is very different from the other important problem caused by uncertainty. That is the estimation of a vote whose outcome is not a foregone conclusion.

It must be emphasized that although market processes may often be naturally modeled as mechanisms or non-cooperative games; voting systems are naturally better considered as cooperative games. Hence in a world without lack of knowledge or major random elements, in general, the outcome of the vote should be known after the private bargaining has taken place.

(9) Means of Taxation and Payment

The importance of the difference between taxation and purchase of resources by the government by actual commodities or by an institutionally accepted money only arises when the distribution of public goods and public control are looked at from a strategic viewpoint.

The basic difference between asking simple general questions concerning the existence of distributions satisfying conditions of optimality and equity and viewing processes as a strategic game is that certain strategies which are legitimate may call for the violation of Walras' law. Bankruptcy conditions and failure to comply, penalities must be specified. A government may find that it is unable to construct the park it promised with the funds voted for it.

There is an important difference between a mandate to spend $10,000,000 on a public good and a mandate to build a public good of a certain specification spending no more than $10,000,000. There is also an important difference among situations in which eminent domain or the printing of money are weapons or are not.

If there is a chance that the government is not able to carry out a mandate within a stated budget, then a simple utility function such as "minimize the cost of a project" is not sufficient to describe its goals. We need to know how it evaluates a weapons system or an opera house 90% completed.

3.3 Games of Fair Division and Information

There is an important factor in the design of distribution mechanisms, welfare, fair division or social choice games, which has not been discussed. That is the possibility that when all are not completely informed about each others' preferences or position, it may be strategically valuable to lie. Problems with means tests, individuals violating the spirit of unemployment payment laws and so forth provide examples. There are many interesting problems in theory posed by considerations of how to lie optimally and how to design an optimal inspection system 5/. This problem is not considered further at this time. Parenthetically, it is my belief that in the actual application of welfare schemes, the effects of lying are probably considerably overestimated by those who call for complex policing mechanisms.

4. Towards a General Theory of Welfare and Public Goods

> "Everyone agrees to symmetry,
> but it's the definition that
> counts."

The enumeration of items to be taken into account in the construction of a Welfare Game was large. In Appendix B it is shown that several million different cases can easily be constructed. Among these cases the purely competitive market is merely one very special instance; as is the simple majority vote. Both of these special cases have important distinguishing

properties as is shown in Appendix C; however, they still are only two among many possibilities. At this point, the economic or political theorist may react by saying that it's all very well to claim that the politico-economic world consists of millions of different cases. The institutionalist has always taken refuge in the thickets of special instances; but where is the theory which brings order to and the understanding of these myriads of situations?

The argument sketched here, presented more abstractly in Appendix C and developed in detail by Shapley and Shubik elsewhere [6], is that there is an important natural taxonomy of social affairs in terms of games exhibiting different basic properties. These concern the characterization of the strategic possibilities of the players and their interactions. They are manifested in the definition, description and measurement of properties, such as the level of strategic interlinkage between players; the degree of symmetry in the game; the inherent degree of structural competitivity (in the sense of how close is it to a game of pure opposition or of pure correlation of interests) and in the structure of threats.

Even though there may be a natural scheme for constructing and classifying myriads of different politico-economic mechanisms according to certain basic properties, this by itself does not constitute any general

theory. The remainder of the argument is that there are a series of fundamental properties that might be required of a politico-economic mechanism or of a society. They can be described in terms of words such as equality, equity, efficiency, freedom of choice, decentralization, power of groups, power of the individual and so forth. Each one of these may be given a more precise meaning in the form of a solution concept to an n-person non-constant sum game. There may be many competing concepts of equity; but at least it has been possible to formalize and axiomatize some of them. As more are formalized, our understanding of the basic variables and the fine structure of the axioms (such as the various ways in which symmetry is introduced) is improved.

The general theory is attained by adopting a pluralistic approach. This involves drawing up a check list of "solution concepts" or properties that we may deem to be desirable from a normative viewpoint. We may then wish to ask if we can simultaneously satisfy some or all of the properties that we deem to be desirable. Can one have decentralization and freedom of individual choice simultaneously with an equitable distribution of goods and services under a system of social choice involving mixed voting and market mechanisms?

Arrow has shown the impossibility of obtaining a social welfare function based upon voting mechanisms [7] if the usual assumptions are made about individual preferences. This does not imply, however, that certain sets of desiderata cannot be achieved simultaneously if the appropriate conditions on tastes, technology, and administrative structure prevail. For example, in certain classes of games the value (an equity concept) will be inside of the core (a concept reflecting the countervailing power of groups) if the game is sufficiently symmetric; otherwise it will lie outside.

Let us consider the above remarks in terms of a specific model. We make the following assumptions:

(1) All goods have an individual owner and are freely transferable (say, fungible commodities).

(2) All individuals have preferences depending only upon their own possessions.

(3) Production processes are homogeneous of order 1, i.e., there are constant returns to scale. Technology is available to anyone who has the physical inputs required by any process.

(4) Initially resources are distributed so that there are very many individuals holding each commodity.

(5) Individuals are rational, well informed, honest and need no mechanism to enforce contract or supervise their actions.

These are the conditions required for the "Garden of Eden." There is no need for the state; it has withered away. There is no need for taxes as there are no public goods.

Those who hold that decentralization and efficiency are values of importance to society know that if the conditions above hold true there will exist a system or systems of prices attainable by trading in an open competitive market which satisfy these conditions. They are the competitive equilibrium points of the market.

If we view human socio-economic behavior in terms of the powers of coalitions then (as has been briefly noted in Section 1), the core solution (which may and usually does have an existence independent of the competitive equilibrium) is the same as the competitive equilibria in this case.

If equity is the criterion to be satisfied, then several game solutions reflecting the properties of symmetry and fair division have been formulated which under the conditions noted above converge to the competitive equilibrium points.

If curbs on noncooperative behavior are being sought, so that even if the individual were permitted to behave in a strategically noncooperative manner towards his fellow man, (i.e., each individual is left with his freedom of choice, even though the strategic interlinkage of his fate and the fates of others is recognized) the resulting distribution would be optimal; then, there are indications that in this model the noncooperative equilibrium points converge to the competitive equilibrium points (See Appendix A).

Thus it can be seen that, although the model for the purely competitive economy is only one among millions of related models reflecting major aspects of the politico-economic and socio-economic world, it is a very special one.

The purely competitive economy model is a limiting case which has deep significance both from the viewpoint of the basic structure of the model and the coincidence of diverse solution concepts. It is a fixed threat game with a core; which means that almost any type of view on noncooperation or aggressiveness leads to the same game model. The existence of the core means that there will always be some outcome which cannot be ruled out by the noncooperative behavior of groups.

If the conditions described prevail, then it is possible for a society to not only give a broad freedom of choice to its citizens but to meet conditions of efficiency, decentralization, equity and power distribution simultaneously.

Unfortunately even though the competitive market model has many desirable properties it is unlikely that there has ever existed a socio- or politico-economic system for which it is a good approximate model. In all societies, the state has existed and goods common to groups or public goods have played an important role.

The competitive model serves as an ideal situation; a limit point of many apparently totally unconnected processes described by diverse theories arising from different concepts. But it is not a description of the world we live in. For reasonable approximations to parts of our and other economies many others of the host of models already noted must be considered.

Confronted with other models we may ask the general question: Can we design tax laws, voting schemes, and/or other legislation such that the checklist of desirable properties which we know to hold true for the conditions noted above, hold true for other situations?

Although the general answer to this question is not known at this time, the question and allied questions appear to be sufficiently clear to be precisely formulated. The basic properties of the models (as was indicated in 3.2) can be adequately categorized to describe them; and most of the social and/or ethical desiderata may be characterized by different game theory concepts of solution.

4.1. Some Basic Concepts of Welfare

There are four important features concerning previous discussions of welfare theory that are brought up here. They are:

(1) The role of Pareto optimality,

(2) The Arrow paradox,

(3) The relationship between the market and voting

and (4) The Social Welfare Function.

In almost all theories and discussions of welfare, it is usually stated as a desideratum that the outcome should be Pareto optimal. There is no indication, however, either about the shape of the Pareto optimal surface or of any metric which can tell you when a point is not on the surface, how close it is to the surface.

Although the assumptions of ordinality, independence and non-comparability of preferences are from some points of view attractive, it is equally (and in some cases more) attractive to assume that statements such as: "I inconvenienced myself slightly to do him a service which was of considerable importance to him," have meaning. Reasonably socially adjusted individuals may easily make more or less quantifiable comparisons. Thus, when someone at a slight inconvenience to himself directs an anxious stranger to a train, he does not apply ordinal non-comparable measures. If he did, his actions would be the same if he were in no rush himself or if he were dashing to catch his own train.

The import of the above remarks is that the condition of Pareto optimality is by no means the most natural or dominant desideratum. Equity, for example, may be a more desirable condition. If it is not possible to satisfy both simultaneously, then possibly Pareto optimality should be given up. In giving up Pareto optimality it may be possible to show that it need only fail by a "little bit"; but this calls for a measure.

Closely related to the shape of the Pareto optimal surface and the measure of the distance of distribution of wealth from it, is the

problem of devising a measure of the innate degree of conflict or coincidence of interests built into a game. In the three games illustrated in Figure 1, it can

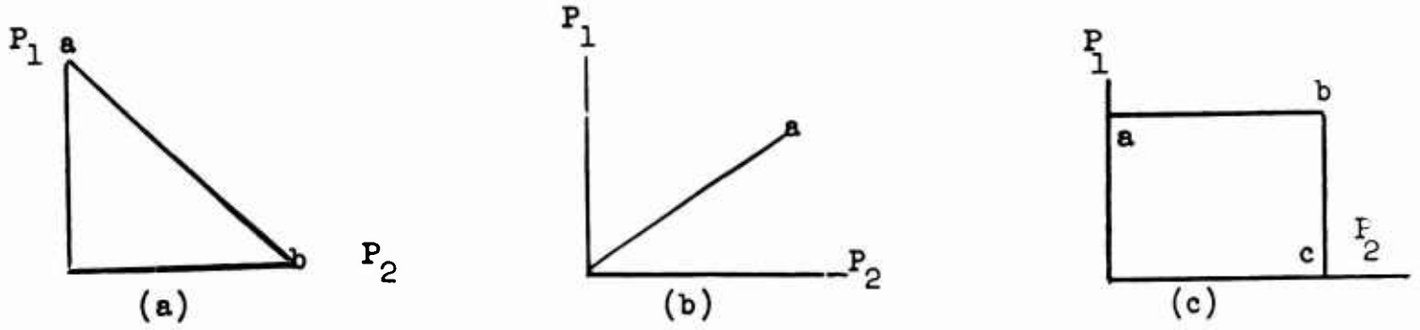

Figure 1

be seen that in (a) the domain ab of the Pareto optimal surface leaves much room for competition and conflict of interest between the players. In (b) the Pareto optimal surface consists of the single point a. In (c) the surface consists of abc, but with compelling reasons to select b.

These remarks are in part due to conversations with R. Axelrod who has been attempting to devise a measure for the structural conflict in two-person bargains.

Arrow has shown [8] that if we assume each individual has an ordered set of preferences for the various outcomes of society then, in general, it is not possible to construct a completely ordered social welfare function based upon using any voting scheme to aggregate their preferences. Inada has shown that, given the conditions for the existence of a linear transferable utility [9] it will be possible to construct a completely ordered welfare function.

There are two important points to be noted. We should not confuse the construction of a social welfare function by the means of a voting procedure with the design of a politico-economic game as a method for defining the choice mechanisms in society. Nor should we assume that of necessity we must build up a social welfare function from the aggregation of individual independent single person preferences.

It is possible to consider a society with several institutions such as the church, political parties and so forth and to formalize a model in which individuals belong to several organizations which have their own value systems and resources and to which the individual as a matter of his beliefs delegates or surrenders his decision-making powers.

The voting system has often been discussed as an alternative for or supplement to a market mechanism. Under what circumstances do we have the alternative of constructing a market or a voting game to serve to transmit the decisions in a system?

Given that for some situations it is possible to consider either; how radically do they differ? When a voting mechanism is used, it is possible that a non-Pareto optimal distribution results?

4.2. Some Basic Concepts in Game Theory

In many ways the advent of the theory of games represents direct extension of the rational utilitarian approach to the problems of economic and political man. We leave the comfort and aesthetically pleasing simplicity of maximizing man. Cross-purposes maximization is considered. No simple resolution of what people do or should do is found once we venture into the realms of non-constant sum payoffs, where cooperation and competition work together and against each other simultaneously.

"Minimaxing man" appears to be a logical extension of individually rational utilitarian man for situations involving pure conflict. But these are few and far between; and it is unreasonable to expect that where mutual gain is to be had individuals will act as mortal enemies.

Left out of consideration in all of this discussion are all aspects of dynamics. Many of the processes of society are continuous; hence the evaluation of threats is complex and dependent upon the future. Persuasion plays its role in changing perceived desires; but along with the effects of incomplete information these factors are excluded from consideration at this point.

In spite of the severe limitations of the game theoretic approach, or of any purely economic or political approach assuming fixed preferences, there are several important features of the choice structure of a society which can be examined.

The concepts of game theory when applied to the politico-economics of welfare and public goods may be used to show the importance of:

(1) Fair division games,

(2) The extensive form,

(3) The strategy sets of the players,

(4) The structure of threats,

(5) The approach to welfare via a multiplicity of solutions.

We have already noted, but stress here once more, that the game theory approach to welfare problems is via fair division games. The problem is not one of the description of individual or of aggregate preferences, but of what happens when preferences are given and a game or

decision mechanism is described through which individuals and/or groups act.

The cake problem where two individuals divide a cake by having one cut and the other choose is the prototype example of a fair division game. More sophisticated versions involve voting and taxation.

In discussions of bargaining and competitive equilibrium, dynamic or semi-dynamic arguments are often presented in the form of tatonnement schemes describing the actual bargaining steps in a market. In some game theoretic models this detailed description is important and can be modelled easily. When mixed market and voting schemes are to be considered this is the case. For example, there is a large strategic difference between schemes in which individuals are permitted to trade private resources, vote on taxation and government purchases and then trade private resources once more, and the situation in which voting and taxation takes place before trading.

Possibly the most important characterization of a strategic situation be it social, political or economic is in terms of the strategic freedom and interlinkage of the individuals' control. This is closely related to the extensive form or move-by-move description of the game, and enables us to gain an appreciation of the degree in which the fates of individuals are intertwined. Three simple games in which each player has only two

strategies are given below to illustrate varying degrees of inter-relationship.

	1	2
1	1,2	1,5
2	3,2	3,5

	1	2
1	3,6	-16
2	3,-2	-1,-2

	1	2
1	4,4	0,0
2	0,0	1,-1

The first number in each cell is the payoff to the first player, while the second number is the payoff to the second player. For example, if each player employs his second strategy in the first game the payoff to the first is 3 and to the second is 5.

The first example is that of an inessential non-constant sum game. The strategies of the players are not interlinked. The payoff to each player i is a function of only his own strategy and could be written as $P_i(s_i)$.

The second example is constructed so that the payoff to the first individual depends only upon the strategy of the second and vice-versa. The payoffs could be written as $P_1(s_2)$ and $P_2(s_1)$.

Joint control and correlation of interest is illustrated by the third example. Each payoff depends on the strategies of both. The values

of $P_1(s_1, s_2)$ and $P_2(s_1, s_2)$ are however highly correlated.

In each of these games the structure influences the threat possibilities for one individual against the other if they fail to cooperate.

In the first as they are not interlinked any behavior by one has no effect on the other. In the second each controls the amount of damage he can do to the other but has no control over his own fate. In the third the price of damaging the other individual is to damage oneself at the same time.

The last point to be stressed has been noted earlier, but merits re-emphasis. We are willing to take as given any structure of individual or group preferences. Given this as a basis, we wish to ask how will these preferences be realized for different choice mechanisms or "games" such as systems involving combinations of trading and voting?

A pluralistic and axiomatic view is taken of the various solution concepts. The various axioms may be regarded in terms of concepts such as equity, power and freedom of choice.

5. Some Examples

Tax me not in mournful numbers
Come and make a total haul
For the residue that slumbers
Is no good to me at all.

Samuel Hoffenstein

In this section a few simple examples are given to help to illustrate the discussion in sections 3 and 4 . They are:

(1) The trading game for fungible chattels;

(2) The simple majority vote;

(3) Corporate voting in a world with jointly held goods;

(4) Private and public goods, the direct vote, free market and taxation.

5.1. The Trading Game for Fungible Chattels

In most textbooks describing bilateral monopoly or discussing consumer behavior with an illustration of a two commodity indifference map, the implication is that the commodities are in the form of essentially indistinguishable units, such as the same grade of apple and are most probably easily transportable items to which no particularly special set of laws or restrictions applies. Most foods and other consumer items fit this description fairly well. Although in certain senses firearms, liquor, automobiles or grand pianos become different goods as they are moved into different jurisdictions where the laws pretaining to them change.

We have already noted that various game theoretic solutions coincide with the competitive equilibrium points. This model is reintroduced at this point in order to stress the special features of the assumptions needed for it to exhibit the powerful properties it possesses. The three broad categories in which assumptions must be made are:

(1) The individual: his preferences and intents

(2) The nature of the economic goods

(3) The role of society: political and social institutions.

5.2. The Simple Majority Vote

In one sense politics may be regarded as the economics of jointly owned goods and services. Ownership may have many different shades of meaning depending upon the conditions (physical, legal and social) which define an individual's control over resources. It is an institutional manifestation of the more basic concept of strategy space. As such it is easy to conceive of items which are jointly owned in one society but privately held in another.

Consider a society in which ab initio all resources are publically owned. An example would be a religious or utopian group moving into empty lands such as parts of the West and Canada. If all goods are public then it does not matter if the physical properties of some are such that they could be appropriated easily and traded privately. For our example consider n individuals with $k + 1$ goods where the utility function for each individual is of the form $U_i(x_1, x_2, \ldots x_{ki} y)$ and there is a production function $y = h(z_1, z_2 \ldots z_k)$ which specifies the output of the $k + 1$st product as a function of inputs of the first k products. Let us assume that the initial resources of the society are $(A_1, A_2 \ldots A_k, A_{k+1})$

We assume that economic wealth is distributed by vote and that a simple majority vote is used. Free exchange and price system trading are expressly forbidden by law.

It is not enough to state that a voting mechanism is used as this does not specify how a proposal is selected. In particular unless the method for selecting alternatives is stated, the society if presented with a vote to accept or reject a distribution of wealth that is nonoptimal might easily vote to accept it.

By definition there will be a nonempty set of Pareto optimal distributions. We wish to examine the nature of different solutions given the simple majority vote.

1. Optimality: this is not a property inherent in the voting system. A method for selecting Pareto optimal distributions upon which to vote must be specified.

2. Decentralization: by fiat this is ruled out. Neither the competitive equilibrium nor the noncooperative equilibrium solutions are defined.

3. The Power of Groups:

We consider four cases here for illustrative purposes. They are the symmetric and nonsymmetric games, and the games where we assume that utilities are comparable (and transferable) and where they are not.

The games with comparable utilities are considered first. The characteristic function $v(S)$ is defined on all sets S of N, where) . N is the set of all players. In order to make the threat structure specific we assume that the ultimate threat of an individual or group of individuals

against the rest is to vote on the other side of a proposal. They cannot riot or force others to vote differently or leave the society.

Where there is comparability and transferability, optimality conditions become very simple. There will be (except for degenerate cases) a unique joint maximum. The proposals before the society will not be on how to operate it (as that is uniquely deferred) but on how to divide the proceeds.

If the democratic form is observed, then the minority must go along with the decision of the majority. The characteristic function will then be:

$$v(s) = 0 \qquad \text{for } s \leq n/2 \quad \begin{array}{l}\text{(where } s \text{ and } n \text{ are}\\ \text{the number of voters}\\ \text{in } S \text{ and } N \text{ respectively)}\end{array}$$

$$= \max \sum_{i=1}^{n} v\,(r_1^i, r_2^i, \ldots r_k^i, y) \quad \text{for } s > n/2$$

where $\Sigma r_j^i = A_j - z_j$ for $j = 1 \ldots k$.

This form reflects a particularly stark version of democratic process where, at least in the formal structure of the game (as contrasted with the type of solution considered) there is no real protection of minority rights. A coalition of less than $n/2$ could conceivably by heavily exploited. In studying game theoretic or strategic models of political or economic processes it is desirable to specify the nature of minority rights.

Although the utility functions of the individuals are different, the players all enter strategically into the game in a completely symmetric manner. Hence the symmetric form of the characteristic function. This is a simple game [10] which is equivalent to

$$V(s) = 0 \qquad s \leq n/2$$
$$= 1 \qquad s > n/2 \ .$$

As the role of all individuals is the same we may write s instead of S in the characteristic function because all coalitions of the same size are equivalent hence the number of members is sufficient to identify the coalition. The characteristic function in this case can be shown diagrammatically as is indicated below. It has a single step at $s = n/2$.

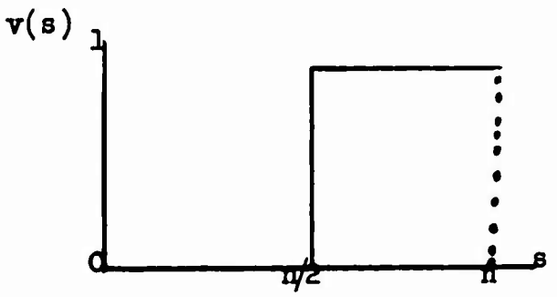

Figure 2

The players may divide the proceeds in an imputation of wealth $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ where $\sum_{i=1}^{n} \alpha_i = 1$. The condition that $\alpha_i \geq 0$ implies that there is some legal protection of the rights of the individual.

It is immediately evident that this situation has no core. If there were a core, then the symmetric imputation $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ would be in it. However any subset S where $\frac{n}{2} < s < n$ can obtain $\frac{1}{s}$ for each of its members and as $\frac{1}{s} > \frac{1}{n}$ this implies that the imputation $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ can be blocked or dominated by it.

4. Equity: As the game is symmetric and all equity concepts employ a basic symmetry axiom, they all lead to $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ in this case.

We may equally well investigate the voting game without assuming that preferences can be compared. When this is the case a simple numerical characteristic function cannot be defined as the summation of utilities has no meaning. We can use a notation which is an extension of the characteristic function. The values of $w(s)$ are given by the set of imputations which are Pareto optimal among the members of the set S. The game under consideration here is not quite symmetric hence all 2^n values of the modified characteristic function need to be specified.

$$w(s) = \left\{ U_i(0, 0, \ldots, 0), U_j(0, 0, \ldots, 0), \ldots \right\}$$

for all $i, j, \ldots \in S$ for $|S| = s \leq n/2$

$$= \left\{ U_i(r_1^i, r_2^i, \ldots r_k^i, y), U_j(r_1^j, r_2^j, \ldots r_k^j, y), \ldots \right\}$$

for all $i, j, \ldots \in S$ for $|S| = s > n/2$

for all values of $r_1^i, r_2^i, \ldots, r_1^j, r_2^j, \ldots$ such that the resultant set of imputations are Pareto optimal among members of the set S. A similar argument to that for the comparable utility case establishes that there is no core for this game.

The equity solutions will be different for the game above and the totally symmetric case where we assume that $U_i(x_1, x_2, \ldots, x_k, y)$ $= U(x_1, x_2, \ldots, x_k, y)$ for all $i \in N$. In the latter case the value is the symmetric point $\left(U(r_1, r_2, \ldots, y), U(r_1, r_2, \ldots, y), \ldots\right)$ where $nr_i = A_1 - z_1$; $nr_2 = A_2 - z_2$, ... and $y = h(z_1, z_2, \ldots, z_k)$.

In summary we note that this model with a simple majority voting scheme and public ownership of all means of production implies among other things that the individual's personal skills and abilities are at the disposal of the majority to decide how to employ them.

The rules of the game rule out decentralization, limit individual power, but leave sufficient power to large coalitions that there is an inherent weakness to any social stability, evinced by the absence of the core.

5.3. Corporate Voting in a World with Jointly Held Goods

> Corporations cannot commit treason,
> nor be outlawed nor excommunicated,
> for they have no souls.
>
> Sir Edward Coke 1612

The corporate form in modern United States society is probably a better model of the political process than is the simple majority vote. It reflects both the possibility for trading votes and the formalization of rules concerning the protection of minorities.

Individuals may command more than one vote. A simple majority is required for control of the decision-making (for some purposes more than

a simple majority may be needed). Any payout of resources however, must be made on a pro rata basis to all stockholders. This may be regarded as an equity concept built in as part of the rules of the game rather than applied as a solution criterion.

Unlike the simple majority game where one individual is restricted to one vote, here nonsymmetry can occur in two ways. The first is if the preferences of the individuals are different and the second is if the number of votes they control are different. We will look primarily at the first.

As in the previous example in 5.2. decentralization and the power of the individual are both formally ruled out by the rules. Furthermore there is nothing inherent in the system which guarantees the selection of Pareto optimal choices.

The symmetric game with comparable transferable utility in this instance may be described as is shown below both by the characteristic function and a representation of it in Figure 3.

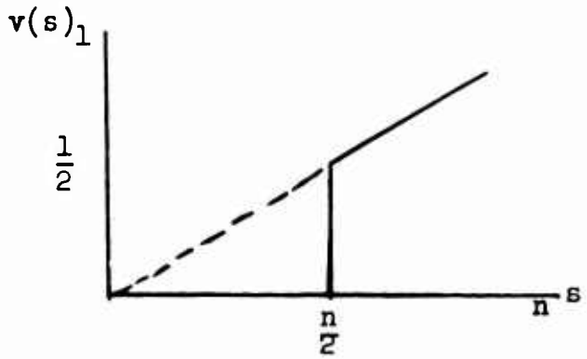

Figure 3

On the assumption that each person has only one vote; that a simple majority is needed for control and that any disbursement must be paid to all stockholders in proportion to their holdings regardless of whether they belong to the majority group the characteristic function will be:

$$v(s) = 0 \qquad \text{for} \quad s \leq n/2$$
$$= s/n \qquad \text{for} \quad s > n/2$$

It is important to note that the division of proceeds in the corporation is based on considerations of equality not equity and upon protection of the rights of all (in fact, by use of accounting practises, control of hiring, wages and so forth the group in control can appropriate more than a pro rata share).

In general when contemplating the corporation and its behavior it is natural to think in terms of money or some other one-dimensional fungible transferable commodity. There is no abstract philosophical utilitarian hidden meaning to it. A dollar is a dollar to all from the strategic viewpoint as long as it is "good for all debts public and private." The welfare question of whether it is worth more or less to J. P. Morgan or to the widow is not germaine if the game has been designed in such a manner that dollars are the chips. If a dollar has a physical meaning (as it does to the corporate treasurer) then "equal dividends for equal holdings of shares" is well defined.

Unlike the simple majority game with no protection of the minority, the corporate form with transferable utility gives rise to a game with

a core. Furthermore the core is a single point regardless of nonsymmetry in voting. It is the point at which each obtains his pro-rata share. No coalition can obtain more.

The value coincides with the core, in the symmetric game but not otherwise. A simple example is used to demonstrate this. The calculations for the example are supplied in Appendix A .

Consider a society of 4 individuals with votes of 2, 1, 1 and 1. Suppose a simple majority is needed for control. The core and the value are calculated for case 1 : A simple majority game with no protection of minorities and case 2 a simple majority corporate or stockholder game.

Case 1 core - as we have already noted, the core is empty.

value - the value is given by the imputation

$$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

Case 2 core - the core is given by the solitary imputation

$$\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$$

value - the value is given by the imputation

$$\left(\frac{9}{20}, \frac{11}{60}, \frac{11}{60}, \frac{11}{60}\right)$$

In Case 2 the higher return to the man with 2 votes in the value over the core reflects his greater relative importance for the formation of winning coalitions than the others. The value in Case 2 is somewhat less than in Case 1 for the 2 vote individual; this decrement is caused by the introduction of the protection of minority stockholders.

Although in the case of corporations it is more or less natural to consider the characteristic function and even to treat money as a commodity with the same significance to all; if we wish to consider a communal society we may not want to presume that interpersonal comparisons can be made. If this is the case, then we can define the modified characteristic function $w(s)$ as follows:

$$w(s) = \left\{ 0, 0, \ldots 0 \right\} \text{ for } |S| = s \leq n/2$$

$$= \left\{ U_i(r_1^i, r_2^i, \ldots r_k^i\, y),\ U_j(r_1^j, r_2^j, \ldots r_k^j\, y),\ \ldots \right\}$$

$$\text{for all } i, j, \ldots \epsilon S \text{ for } |S| = s > n/2$$

for all values of $r_1^i, r_2^i, \ldots, r_1^j, r_2^j, \ldots$ such that $\sum_{i \epsilon S} r_e^i = \frac{s}{n}(A_e - z_e)$ for $\ell = 1, \ldots, k$ and the resultant set of imputations are Pareto optimal among members of the set S.

As before, this game has a core. The existence of the core is guaranteed by the rules or laws which enforce <u>pro rata</u> sharing of all commodities. In general the resultant division of wealth will not be Pareto optimal unless it is understood that once each individual has been awarded his $\frac{1}{n}$th share of all commodities available all are free to make exchanges.

5.4. Private and Public Goods, the Direct Vote, Free Market and Taxation

A modern democratic society with a capitalist economy can be described by a simple majority voting game with or without the protection of minorities guaranteed; combined with open trading in a "private sector." In the previous examples in 5.2. and 5.3. voting was considered and centralization and coordination was forced on all economic activity by law. Here we consider in more detail the possibility of using the special physical properties of some items so that society centralizes only those items over which it has no physical choice while it uses a decentralized mechanism to determine the distribution of the remainder.

We are now in a position to consider in detail the special properties suggested by the utility functions for the individuals for the $k + 1$ goods. k of the goods are such that it is possible to design an open market mechanism in which they can be traded; however the $k + 1$st good is by physical nature a public good.

In order to well define the game to reflect the main aspects of this society it is necessary to specify initial conditions. Do we start with society as a whole owning everything or do we consider all items to be individually appropriated?

For specificity we assume that only the first k commodities are in supply and that the public good which is naturally a public good (say for example the administration of justice) does not exist to begin with, hence private resources must be appropriated in order to produce it.

We assume that initially all resources are privately owned. This immediately introduces an extra consideration of symmetry. Is equality of initial holdings an equitable distribution of them or should we regard whatever random distribution which happens to exist at this particular time as a just initial position?

In the first example, for ease we take the symmetric initial distribution $\left(\frac{A_1}{n}, \frac{A_2}{n}, \ldots, \frac{A_k}{n}, 0 \right)$ and the production function to be $y = \sum_{i=1}^{k} z_i$.

Before any game model is fully specified it is necessary to provide more detail. Can coalitions of any size build a public good for their own use? In this case it is assumed that they are not able to do so. Furthermore it is assumed that if a majority decides to build a public good, all will share in its use and no one can be prevented from participating regardless of how he voted.

It is assumed that a simple majority vote is required to decide upon the size of the public good to be constructed and the taxation scheme to be imposed. It is further assumed in this simple case that each individual has a single vote.

In the dymanic process of human affairs trade takes place before and after taxation and votes on appropriations. Tactically it may be important to trade before a vote has been taken or a new tax bill passed.

We assume (although it is usually not the case) that voting is direct and not via representatives. Votes are not legally directly for sale.

Trading may take place at any time.

A board of experts may be presumed to exist to perform the task of selecting a motion for the public to vote on. It might be instructed to try to pick only motions which are or will lead to a Pareto optimal outcome.

Alternatively though possibly with less institutional justification, we will assume that any winning coalition is allowed to name its own proposition.

Do we wish to consider special rules on the type of taxation? If there are no restrictions then majorities are in a position to completely exploit minorities. If taxes are to be equal for all, we have a situation analogous to the corporation where the minority is protected by the pro rata rule. Conditions such as a graduated income tax lie in between.

Do we assume that the government specifies its taxes and public works in terms of money or bundles of commodities? If it specifies taxes in money and public works in commodities (such as so many million dollars to build a specific bridge) it runs the danger being unable to meet its promises. In the first examples it is assumed that all taxes and public works promises are stated in terms of money.

As a first approximation we assume that government is an efficient costless institution with no values of its own designed to execute the will

of the majority subject to the constraints of the law. Embodied in the law may be conditions manifesting equity, protection of minorities and so forth.

A specific simple example will be explored. Let there be N/2 individuals of type 1 and N/2 of type 2. Where an individual i of type 1 has a utility function of the form

$$U_i(x_1^i, x_2^i, y) = 2\sqrt{x_1^i x_2^i} + by$$

and individual j of type 2 has:

$$U_j(x_1^j, x_2^j, y) = x_1^j + x_2^j + by$$

the production function for the public good is $y = z_1 + z_2$ where z_1 and z_2 are imputs of the private goods.

All supplies of the two private goods are held by the individuals as follows: Each of type 1 has an initial endowment of (2A, 0),and each of type 2 has (0, 2A).

Case 1

For our first case we assume not only no protection of the minority against the size of their tax burden, but furthermore the majority can award itself subsidies other than by building the public good. In other words they are in a position to legally pocket taxes.

Call the sets of the first and second type of players S_1 and S_2 respectively, and let N be the set of all players.

$$v(SuT) = 0 \quad \text{for} \quad |S| + |T| = s + t \leq \frac{n}{2} \text{ where } S \subset S_1 \; T \subset S_2$$

$$= \max_p \; [2n\,A\,(1-p) + 2bn\,Ap\,(s+t)] \text{ for } s+t > n/2 \text{ and } 0 \leq p \leq 1 .$$

This characteristic function states that any minority can be fully exploited. Any majority has at its disposal all of the resources of the society.

The example is somewhat strange, inasmuch as owing to the linear form of the payoff, depending upon whether $\frac{1}{s+t} \lessgtr b$ all resources or no resources will be used for the public good.

If $\frac{1}{n} > b$ no public good will ever be produced. Voting will still enable the majority to exploit the minority. In the range $\frac{1}{n} \lesssim b \lesssim \frac{2}{n}$ large coalitions may produce the public good.

This game has no core. In spite of the possible decentralization it is not decentralized. In spite of the lack of symmetry in preferences the value is the symmetric imputation.

If $\frac{2}{n} \lesssim b$ then the production of the public good will be supported by any majority; and the game will also have a core. The value will be in the core and will be the midpoint.

Case 2

In this case we consider that the minority is protected only to the extent that all tax monies raised must be used to construct the public good and hence cannot be used to subsidize the private expenditures of the majority.

As before: $v(S \cup T) = 0$ for $|S| + |T| = s+t \leq \frac{n}{2}$ where $S \subset S_1 \quad T \subset S_2$

$$= \max_p \quad 2(s+t)\, A\, (1-p) + 2(n-s-t)\, A\, (s+t)\, b$$

$$+ 2\, pb\, (s+t)^2\, A \text{ for } s+t > n/2$$

$$= \max_p \quad 2(s+t)\, A \left[(1-p) + b \left\{ n-(1-p)\,(s+t) \right\} \right]$$

If $\frac{1}{n} > b$ no majority will use its own resources to produce the public good. The game has no core as the members of minimal winning coalitions can achieve higher per capita gains than all other coalitions.

Noncooperative behavior has been ruled out by imposing price-taking conditions on the trading of private goods. However once taxation has been specified the economy can be viewed as decentralized if taxation and public good construction are carried out in terms of money.

The specific extensive form, making use of a monetary mechanism may be regarded as follows:

(1) A winning coalition forms and announces tax rates in monetary terms for all.

(2) A competitive market for the exchange of private commodities then takes place with the government acting to maximize the production of the public good by purchasing private goods as imputs subject to the constraint of not overspending its tax revenues. Those who have been taxed must also act so as to satisfy their extra constraints.

It is immediately obvious that even in this simple model rules must be specified concerning actions to be taken if someone is unable to fulfill his contracts or pay his taxes.

If $\frac{1}{n} < b < \frac{2}{n}$ the game will still not have a core however depending upon the size of b some winning coalitions may vote for public construction.

If $b \geq 2/n$ the game has a core and is equivalent to case 1.

Case 3

For the third case we consider the law that there must be an equal per capita tax and that all tax monies are to be used for building the public good.

$$v(SuT) = 0 \quad \text{for} \quad |S| + |T| = s+t \leq \frac{n}{2} \quad \text{where} \quad ScS_1 \; TcS_2$$

$$= \max_p \left[2(s+t)\, A\, (1-p) + 2n\, pb\, (s+t)\, A \right]$$

$$\text{for} \quad s+t > n/2 \qquad \text{where} \quad 0 \leq p \leq 1 .$$

p is the tax burden leveled equally on all. This game always has a core regardless of the range of b. The difference among the three cases can best be illustrated for coalitions which just have a majority, i.e., those of a size $\frac{n}{2} + 1$. If n is large the difference between $\frac{n}{2} + 1$ and $\frac{n}{2}$ is small, hence for ease assume that $n/2$ can control the vote. The payoffs to the minimal winning coalitions and the coalition of all players are illustrated for the three cases and different values of k where $b = k/n$.

	Case 1		Case 2		Case 3	
	Half of the players	All	Half of the players	All	Half of the players	All
$k \leq 1$	2n A	2n A	$\frac{3}{2}$n A	2n A	n A	2n A
$= 2$	2n A	4n A	2n A	4n A	2n A	4n A
> 2	kn A	2kn A	kn A	2kn A	kn A	2kn A

We see immediately that the rules of the game, which in our society may be interpreted as laws are progressively more effective in protecting the rights of the minority against the power of the majority vote.

As was observed in Section 3, there appear to be a plethora of conditions which must be taken into account in formulating a system which actually utilizes an intermix of voting and the competitive market. The example given here was highly special and was selected to minimize computation, yet at least illustrate directly how various quite plausible laws modify the strategic properties of the system.

Nothing in the above example appears to hinge on comparability and transferability for the calculation of the core (this is not so for the value). However in the modelling and in interpretation fiat money, or a unit of account appears to play an important role. Without the issuing of tax bills in the form of monetary levies, the administrative aspects of decentralization and the use of a market system do not appear to be feasible.

It must be further noted that all the examples have been very special inasmuch as only one public good was considered. With more than one public good it is not yet known what are the restrictions and the laws needed to guarantee the existence of a core.

6. Conclusions

In this paper an attempt has been made to suggest that for extremely fundamental reasons the problems of public finance and welfare give rise to a myriad of models, almost all of which have their counterparts in the varieties of institutional forms, political and economic systems which exist or have existed. The general equilibrium competitive model is one very special case which exhibits powerful limiting properties when various methods of solution are applied.

It is suggested that a society in its search for institutions and administrative procedures may be simultaneously attempting to satisfy different values such as the distribution of power, equity, efficiency and so forth. It may not be possible to satisfy all of them simultaneously. There does however, exist a methodology to investigate this question inasmuch as it appears that the values or goals can be formulated in terms of one of the various solution concepts of the theory of games.

For at least in the guidance of short term behavior, it appears that a society may wish to design itself as a self-policing system in which individuals behave in a manner to conform with long range values which are enforced via the rules, i.e., by the structure of the institutions, social and legal system.

This is in contrast with short term values which are reflected in the actual solution method used. In Section 5 examples were supplied which illustrate the above remark. The societal values of equity can be built into the rules by laws and can also be manifested in how individuals actually play the game with which they are confronted.

Appendix A

In order to explain the various concepts of solution, we must also define what is meant by the following terms: the characteristic function of a game, an imputation, an effective set of players, and domination of one imputation by another.*

The characteristic function specifies the worth that a coalition can achieve if they limit their trades strictly to themselves. Mathematically it is a function $v(S)$ defined on sets of players S, with the properties

$$v(\theta) = 0 ,$$

$$v(S \cup T) \geq v(S) + v(T), \text{ whenever } S \cap T = \theta .$$

The first condition merely states that the amount achievable by the null set is nothing. The second condition is the fundamental economic property of superadditivity: if two separate groups having commerce only amongst themselves are joined together, the resultant group is at least as effective as were the two independent groups. Beyond these two conditions there is nothing more than can be said a priori about a characteristic function.

If we denote the set of all players in a game by N, then $v(N)$ specifies the total amount that the whole group can obtain by cooperation. A reasonable form of "cooperative" behavior would be for the players to

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* Much of the following exposition on the characteristic function, core and von Neumann and Morgenstern stable set solution comes from a paper by Shapley and Shubik 11/. This material has been slightly rewritten and is presented in order to make this article somewhat more self-contained.

agree to maximize jointly, *# and then to decide how the proceeds are to be apportioned, or "imputed." We define an imputation α to be a division of the proceeds from the jointly optimal play of the game among all the n players:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n),$$

where

$$\alpha_i \geq v(\bar{i}) \text{ and } \sum_{i=1}^{n} \alpha_i = v(N).$$

The condition $\alpha_i \geq v(\bar{i})$ embodies the principle that no individual will ever consent to a division that yields him less than he could obtain by acting by himself. It is often convenient to normalize the individual scales so that $v(\bar{i}) = 0$.

A set of players is said to be <u>effective</u> for an imputation if by themselves they can obtain at least as much as they are assigned in that imputation. Symbolically, S is effective for α if and only if

$$v(S) \geq \sum_{i \in S} \alpha_i .$$

If ">" rather than "=" holds, we shall say that S is <u>strictly effective</u>.

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* Their utilities being transferable, this is properly represented by a single number, which denotes maximum obtainable welfare. If utilities were not transferable, $v(N)$ would instead have to represent the Pareto-optimal surface, and similarly for smaller coalitions.

In this appendix, except when otherwise stated all solution concepts are described for the transferable utility case. This is only for simplicity. Equivalent but more unwieldy definitions can be given for the nontransferable case. Both the basic concepts and almost all of the results are essentially equivalent.

An imputation α dominates an imputation β if there exist an effective set S for α such that for all members of S, $\alpha_i > \beta_i$. Following the notation of von Neumann and Morgenstern [12], we write

$$\alpha \prec \beta .$$

In other words, if a set S of players is in a position to obtain by independent action the amounts that they are offered in the imputation α, and if, when they compare the amounts offered in α to the amounts offered in β, all of them prefer the former, then α dominates β. There is a potential coalition that prefers α to β and is in a position to do something about it. Note that S is necessarily strictly effective for β, the dominated imputation.

Core and Stable Set

Finally, we may define two "solution" concepts. The core of an n-person game is the set of undominated imputations, if any. A von Neumann-Morgenstern solution or stable set on the other hand, consists of a set of imputations which do not dominate each other, but which collectively dominate all alternative imputations. There is at most one core, but there may be many solutions. All solutions contain the core, if it exists.

Some Examples

A series of simple, three-person games will illustrate these concepts. Consider first the game in which any player acting by himself obtains nothing, but any pair of players acting in concert can demand three units to share between them, while all three players in coalition are also awarded

three. The characteristic function of this game is

$$v(\theta) = 0,$$
$$v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0,$$
$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 3,$$
$$v(\overline{123}) = 3,$$

where $\overline{12}$ means "the set consisting of players 1 and 2."

We may represent the imputations in this game by triangular coordinates, as shown in Figure 1. The vertices P_1, P_2, P_3 represent the imputations (3, 0, 0), (0, 3, 0), and (0, 0, 3), respectively. The point $\omega = (1, 1, 1)$ is the center of the triangle. Consider the two imputations $\alpha = (1.9, 0, 1.1)$ and $\beta = (0, 1.5, 1.5)$. The set $\overline{23}$ is effective for β, and furthermore both 2 and 3 are better off in β than in α. Hence $\beta \curlywedge \alpha$.

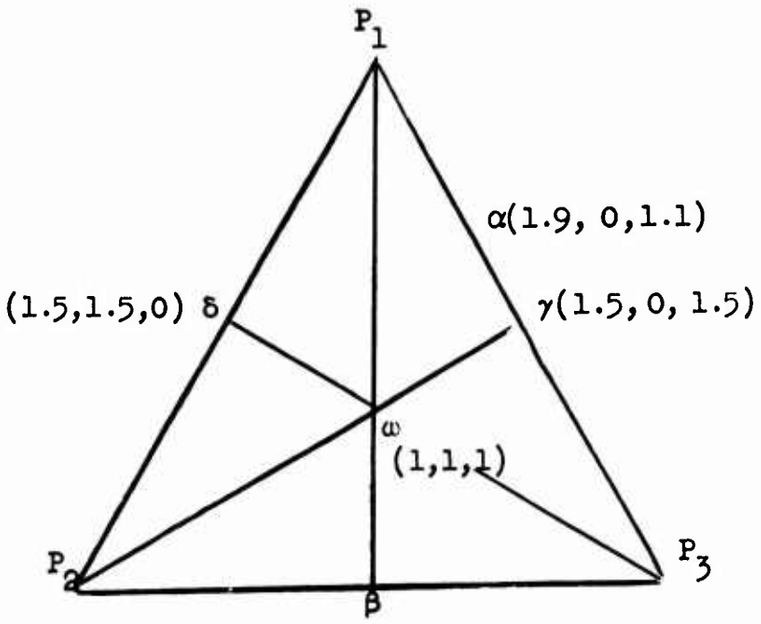

Figure 1

The trio of imputations β, γ, and δ forms a solution set to this particular game. Any other imputation gives two of the players less than 1.5 apiece, and thus is dominated by one of these three imputations, but the three do not dominate each other. (There are other solution sets, which we need not discuss.) This game has no core, since the imputations β, γ, and δ, dominating all the rest, are themselves dominated by others. For example, the imputation α, which was dominated by β, in turn dominates δ via the effective set $\overline{13}$. Note that domination is not a transitive relation: $\beta \dashv \alpha$ and $\alpha \dashv \delta$ do not entail $\beta \dashv \delta$.

We now consider three closely related games, differing from the previous one only in what the two-person coalitions obtain. In the first variant we have

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 0. *$$

In this case, all imputations are in the core. The only set of players that is effective, for most imputations, is the three-person set; however,

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* This all-or-nothing type of characteristic function, like the previous one, is associated more with political than economic processes. The previous game was a majority-take-all situation; the present one is a veto situation, since if one member wishes to be the "dog in the manger," he can prevent the others from obtaining any payoff. In economics such extremes -- called simple games -- are not typical. We shall presently consider variants in which the two-person coalitions obtain intermediate amounts, reflecting the more usual situation in which any new adherent to a coalition means added possibilities for profit.

this is useless for domination, since on examining the distribution of welfare from the viewpoint of all three players we see that if one player prefers one of two imputations, then at least one of the other players will prefer the other, the sum of the allotments being constant. In fact, it suffices to point out that no set of players is strictly effective for any imputation -- hence there is no domination. The core is therefore as large as possible, and is also the unique von Neumann-Morgenstern solution.

In our third example we assume

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 2 .$$

As shown in Figure 2, the lines which describe the amount obtainable by each coalition of two players intersect in a single point, the imputation ω with coordinates (1, 1, 1). This is the only undominated imputation of the game, and thus constitutes a single-point core. Since ω fails to dominate the three small triangles adjoining it in the diagram, however, it is not a von Neumann-Morgenstern solution by itself. To get a solution we must add some more or less arbitrary curves, as shown, traversing the three triangular regions [12].

In the final variant, we assume that the two-person coalitions are only half as profitable as in the preceding example. That is, we have

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 1.$$

The lines indicating the ranges of effectiveness of these coalitions are spread apart, as shown in Figure 3, revealing a large, hexagonal core.

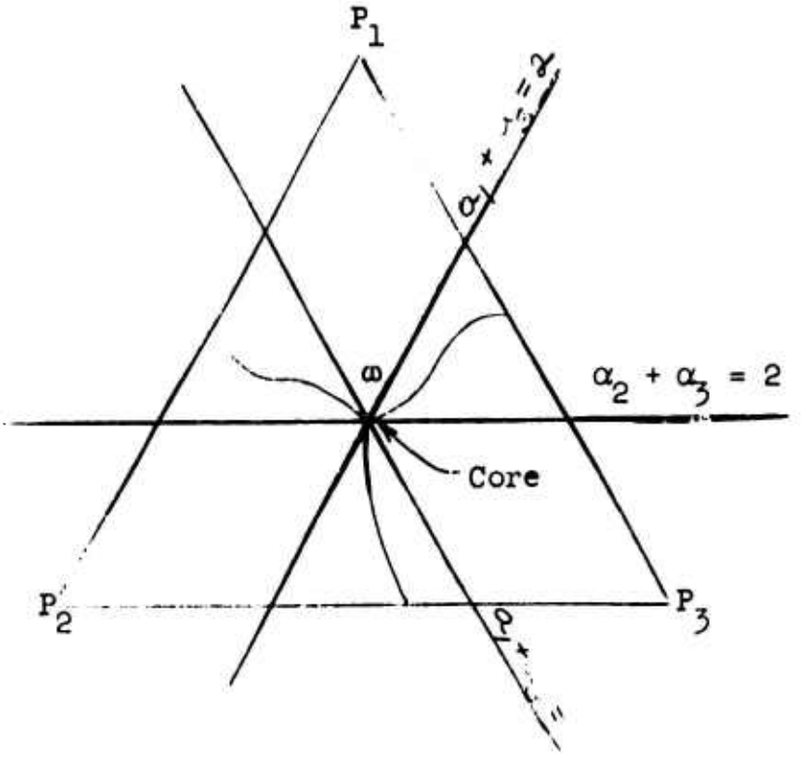

Figure 2

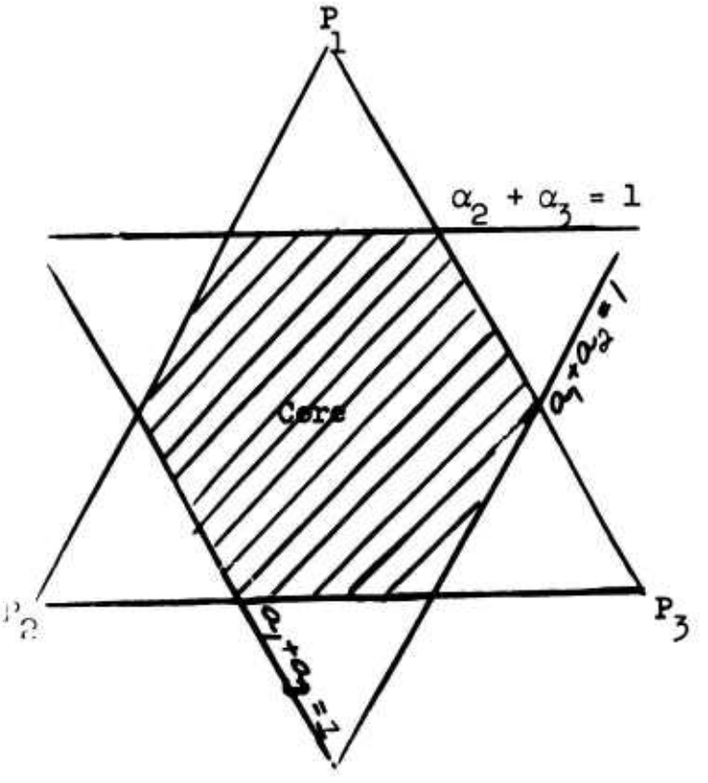

Figure 3

All imputations in that area are undominated. As in the second example, this core is the unique solution.

A superficial examination of these four examples suggests a relationship between the size of the core and the "fatness" of the coalitions in a game, i.e., how much they can promise their members per capita as compared to the per capita amount available in the whole game. In all four instances, the latter amount was $v(\overline{123})/3 = 1$. Denote $v(\overline{ij})/2$ by f_2. In the first game, f_2 was 1.5, which is greater than 1, and there was no core. In the third game, f_2 was exactly 1, and the core was a single point. In the fourth game, f_2 was 1/2, and there was a large core, while in the second game f_2 was 0, and every imputation was in the core.

Of course, in a less symmetric situation, this principle would not reveal itself in such a clean-cut manner. However, a general rule of thumb seems to persist: the more power there is in the hands of the middle-sized groups, the more narrowly circumscribed is the range of outcomes of the cooperative game.

When we do not permit the transfer of utility, it is no longer possible to regard the amount attainable by a coalition as a single number, yet cores [13] and solutions [14] can still be defined and a comparison of gains in various coalitions can still be made in a vectorial sense.

The Value

Given the description of a game by means of its characteristic function, Shapley has suggested a method to impute a unique value to

each player. One way of regarding this value is to consider that a priori all coalitions are equally likely. Furthermore the probabilities of the order in which an individual joins a coalition are the same, e.g.: in a three person coalition Player i has the same probability of being the first, second or third to join.

We consider every possible order in which every individual can enter every coalition and we credit him with his incremental contribution to the coalition. In terms of the characteristic function coalition S and Player i this is:

$$v(S) - v(S - \{i\}) \quad .$$

Adding all his contributions together we average them and award the player that amount, φ_i where:

$$\varphi_i = \sum_{\text{all } S \subset N} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v\left(S - \{i\}\right)\right] .$$

The value may be arrived at through a series of axioms which reflect basic concepts of symmetry and fair division. Various criticisms have been made to the effect that the value may not be a desirable fair division scheme. For the most part they are directed towards the difficulties which are inherent in the formulation of the characteristic function and the concept of threat. This will be referred to again after the examples.

We consider two voting games with 4 players with votes distributed 2, 1, 1 and 1. In the first game there is no protection of the minority;

any winning coalition can take all. In the second game the minority is protected by a pro rata rule. The first game is known as a simple game, all values of the characteristic function are either 0 or 1 .

Call the players a, b, c, d. We use the notation (a, b) to represent the set of a and b; and for brevity $\bar{1}$, $\bar{2}$ or $\bar{3}$ and (a, $\bar{1}$) etc., to represent a set of 1, 2 or 3 of the one vote players and (a, $\bar{1}$) the set consisting of a and 1 other. The characteristic function is:

$$v(\bar{1}) = v(\{a\}) = 0$$
$$v(\bar{2}) = 0$$
$$v(\{a, \bar{1}\}) = 1$$
$$v(\bar{3}) = 1$$
$$v(\{a, \bar{2}\}) = 1$$
$$v(\{a, \bar{3}\}) = 1 \quad .$$

The characteristic function for the second game is:

$$v(\bar{1}) = v(\{a\}) = 0$$
$$v(\bar{2}) = 0$$
$$v(\{a, \bar{1}\}) = 3/5$$
$$v(\bar{3}) = 3/5$$
$$v(\{a, \bar{2}\}) = 4/5$$
$$v(\{a, \bar{3}\}) = 1 \quad .$$

In the first game there is no core as can be seen by the inconsistency of requiring

$$\alpha_1 + \alpha_2 + \alpha_4 \geq 1$$

$$\alpha_1 + \alpha_2 + \alpha_3 \geq 1 \qquad \alpha_1 + \alpha_2 \geq 1$$

$$\alpha_1 + \alpha_3 + \alpha_4 \geq 1 \qquad \alpha_1 + \alpha_3 \geq 1$$

$$\alpha_2 + \alpha_3 + \alpha_4 \geq 1 \qquad \alpha_1 + \alpha_4 \geq 1$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

where α_1, α_2, α_3 and α_4 are the shares awarded to a, b, c and d respectively.

In the second game the conditions for the core are given by:

$$\alpha_2 + \alpha_3 + \alpha_4 \geq 3/5 \qquad \alpha_1 + \alpha_2 \geq 3/5$$

$$\alpha_1 + \alpha_3 + \alpha_4 \geq 4/5 \qquad \alpha_1 + \alpha_3 \geq 3/5$$

$$\alpha_1 + \alpha_2 + \alpha_3 \geq 4/5 \qquad \alpha_1 + \alpha_4 \geq 3/5$$

$$\alpha_1 + \alpha_2 + \alpha_4 \geq 4/5$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1.$$

These are satisfied by the imputation $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

In the first game, as an individual adds to a coalition only if he turns it from losing to winning, the following simple scheme illustrates and calculates the value.

$\dot{1}$ 2 1 1 ; for Player a 6 x 1 = 6

1 $\dot{2}$ 1 1 ; for Player a 6 x 1 = 6

1 1 $\dot{1}$ 2 ; for Player a 6 x 0 = 0

2 $\dot{1}$ 1 1 ; for Player a 6 x 0 = 0

12

Total number of cases is 24.

$$\varphi_a = \frac{12}{24} = \frac{1}{2}, \qquad \varphi_b = \varphi_c = \varphi_d = \frac{1}{6} .$$

The imputation representing the value is $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$.

In the scheme above, the dot indicates the pivotal player, i.e., the man who changes defeat to victory. Each line-up can occur 6 ways. For example, in the case 1 $\dot{2}$ 1 1 we have b $\dot{a}$ c d , c $\dot{a}$ b d , d $\dot{a}$ c b , b $\dot{a}$ d c , c $\dot{a}$ d b and d $\dot{a}$ b c . The player with two votes is pivotal in 12 out of 24 cases.

In the second example we must also take into account the fact that a player joining an already winning coalition still makes a contribution.

1 2 1 1	for	Player a	6 x 3/5
1 1 2 1	for	Player a	6 x 4/5
1 1 1 2	for	Player a	6 x 2/5
2 1 1 1	for	Player a	6 x 0
			54/5 ,

$$\varphi_a = \frac{1}{24} \times \frac{54}{5} = \frac{9}{20}, \qquad \varphi_b = \varphi_c = \varphi_d = \frac{11}{60} .$$

The imputation representing the value is $(\frac{9}{20}, \frac{11}{60}, \frac{11}{60}, \frac{11}{60})$.

The shift down in his value from that in the previous game reflects the effect of the protection of minority rights.

There is a problem introduced in the second game which did not exist in the first. In the first game, a minority could only guarantee 0 for itself and the majority could restrict the gain of the minority to 0. In the second game the minority can still only guarantee itself 0 by unilateral action however the majority cannot restrict the gain of the minority to zero owing to the pro rata rule. These differences in the threat structure of the game are not reflected in the characteristic function.

The Noncooperative Equilibrium

A completely different concept of solution is provided by Nash. The core, stable set and value all were explicitly cooperative in the sense that all players were presumed to have negotiated, planned and jointly considered their alternatives. The spirit behind the noncooperative equilibrium point is that of the exercise of individual power in an introspective manner. The individuals neither explicitly coordinate their strategies, nor do they willfully attempt to damage or threaten each other. They merely try to maximize their own welfare on the assumption that others are doing likewise. Many interpretations in terms of dymanic processes can be given; however in a static sense the existence of a noncooperative equilibrium point may be stated as follows:

Consider a game with n players. Let player i have a class of possible "strategies" S_i, and s_i denote a particular strategy belonging to the class. The payoff to the i-th player is denoted by

$P_i(s_1, s_2, s_3, \ldots, s_n)$, a function of the strategies of all the players. A strategy vector $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \ldots, \bar{s}_n)$ is said to constitute an equilibrium point if, for all i, the function

$$P_i(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{i-1}, s_i, \bar{s}_{i+1}, \ldots, \bar{s}_n)$$

is maximized by setting $s_i = \bar{s}_i$. In other words, a set of strategies one for each player, forms an equilibrium point if each player, knowing the strategies of all others, will not be motivated to change. If the equilibrium point is unique, as it proves to be in many of the classical economic models, it is termed the "noncooperative solution" to the game.

Noncooperative solutions have been successfully applied to the study of oligopolistic markets; however a critical problem in model building must be faced in attempting to apply the concept of a noncooperative equilibrium point to a closed economic model.

An open economic model is one in which at least one individual or group is not considered as an individual or group with preferences and possibly strategies but as a mechanism. For example in the study of duopoly in general the customers are considered only implicitly via the demand mechanism.

In a closed economic model all relevant actors are considered explicitly. The simplest example of this type is bilateral trade. In this instance all trades and preferences can be illustrated in the well-known Edgeworth box diagram.

In general it is extremely difficult to formulate noncooperative solutions for closed economic models. A quick contemplation of the simplest of bilateral monopoly markets shows why. Suppose that a strategy by each player is to name an amount he is willing to sell and buy; in general strategies will not match. An individual will not know in advance if his plans are feasible or optimal.

In the special case of a market with only two types of players it has been possible to formulate a reasonable model for a closed economic noncooperative market without side-payments. However for the more general case it is conjectured that it may be necessary to consider utility functions of the form $U_i(x_1, x_2, \ldots x_k) + \lambda_i x_{k+1}$. The reason for this is that such preferences permit individuals to take strategic actions which may be inconsistent with each other, yet settle up the failure to earn the planned income, to sell or buy the desired commodities with a "money" side-payment. These models require an explicit formulation of how Walras' law is permitted to fail in trading and the conditions for bankruptcy and failure of contract. The special $k+1$ st commodity serves as a strategy space decoupling device. There are no implications of interpersonal comparison of utility but use is made of the lack of income effect on the holdings of the $k+1$ st commodity.

Appendix B

In general one should be careful in playing the "big numbers game" however it is my belief that the problem of the design of politico-economic mechanisms to meet criteria of welfare and social choice is a problem in mathematical institutional political economy. It requires a high level of abstraction concerning concepts such as strategy space, threat, Pareto optimality, core and value. Yet it is not institution free. The considerations noted in Section 3 are all basic and influence the possible models.

Synthesis must come from considering how to group the myriads of cases in terms of whether these "games" or administrative mechanisms enable us to satisfy the criteria of interests.

Below, is given a table of the different factors discussed in Section 3. Not all of them are mutually consistent with each other; hence the straight forward multiplication of cases somewhat overestimates the possibilities.

			Cases
1.	Election type:	Direct or indirect	2
2.	Majority needed:	The most common are 100%, 75, $66\frac{2}{3}$, 50 or dictatorship	5
3.	Veto conditions:	yes or no	2
4.	Cumulative voting:	yes or no	2
5.	Votes for sale:	yes or no	2
6.	Power to tax:	A winning coalition can tax everyone; themselves only; or no one	3

		Cases
7.	Power to construct a public good: yes or no (sometimes a minority may be forbidden from constructing certain goods).	2
8.	Enforcement of use: yes or no (sometimes a minority is forced to use a public good it does not want).	2
9.	Prevention of Use: yes or no (it may be technologically or legally not possible to prevent the use of a public good).	2
10.	Type of tax feasible: income tax, sales tax, property tax, inheritance tax, poll tax, usage tax or toll subsidy and inflation (there are many others but these are among the most important).	8
11.	Conditions on taxation: progressive, constant or regressive (there are obviously many other conditions).	3
12.	Economy or Diseconomy: Is the item a public nuisance to be controlled by society or a public benefit to be supplied?	2
13.	How is a motion or proposal selected: A controlling group may specify it; a board of experts may be asked to put forward two or more alternatives on which to vote.	2
14.	Information Conditions: are all similarly informed, or do "insiders" obtain advanced knowledge of motions, budgets, etc.?	2
15.	Trading: Does free market trading take place before during and after the vote (some markets may be closed on election days; trading in certain commodities may be suspended before and after a vote to declare war).	3
16.	Taxation in money or kind: taxation is usually in money but can be in work other services or goods.	2

		Cases
17.	Subsidies: Welfare payments may be in goods or in money.	2
18.	Bankruptcy laws: there are many different ways of handling inability to honor contract or pay taxes.	3
19.	Eminent domain: yes or no	2
20.	Governmental or Institutional value system: Does the administration as an institution have its own value system or is it a value free mechanism in its execution of public affairs?	2

There are obviously items left off this list and in each category one could argue about the number of alternatives to be considered. On the whole, the error has been in the direction of aggregation and omission of categories.

As a rough approximation the product of the alternatives in each category indicates the order of magnitude of different models which can be constructed (where for the most part each has a counterpart in known human institutions).

The number of cases $k = 2^{14} \cdot 3^4 \cdot 5 \cdot 8 \simeq 53{,}000{,}000$.

Left out of consideration are oligopolistic effects of markets with few competitors; the effects of ignorance and lack of knowledge; and an adequate modelling of the finer features of political and financial institutions which are manifested in manoeuvers such as log-rolling or the strategic use of money which converts it from a "veil" to a strategic weapon.

Appendix C

The main text has presented a pluralistic approach to the problems of public finance and welfare. Many solution concepts and many models have been suggested. In this appendix a recapitulation of the interpretation of six solutions is given together with a sketch of the problems in modeling caused by threat conditions.

Solutions

1. Pareto optimality: Although to some, Pareto optimality may be regarded as a very weak solution criterion it reflects the condition of economic efficiency.

2. The Competitive Equilibrium: The stress here appears to be on decentralization. It also can be argued that implicit in the conditions for the competitive equilibrium is the assumption that the individual has no strategic power whatsoever but acts as an isolated optimizer.

3. Noncooperative Equilibrium: In contrast with the competitive equilibrium the noncooperative solution stresses the use of individual strategic power.

4. The Core: This deals with the power of coalitions. The existence of a core implies the possibility of imputing wealth in a manner that is stable against the power of all coalitions.

5. The Value: Equity is the underlying concept behind the axioms for this solution.

6. The Stable Set: A relatively general concept of social stability is reflected in this type of solution. If the core exists it will be part of any stable set.

Threat Conditions

The characteristic function and its equivalent without side-payments can be obtained by calculating the amount that a coalition can guarantee to itself. Let the payoff to Player i be P_i, then the value for $v(S)$ is given by:

$$\max_{i \in S} \min_{j \in N-S} \sum_{i} P_i \ .$$

This way of calculating the characteristic function presents a very pessimistic view of the threat structure within a society. It may be costly for members of N-S to try to minimize the payoff to members of S. Furthermore in some situations the amount that a coalition is able to obtain may not coincide with the amount that the counter-coalition may be able to restrict it to. The corporate voting game serves as an example. Any coalition S with $S \leq n/2$ is able to obtain zero and no more, yet no counter-coalition can prevent it from obtaining a pro rata share of the proceeds.

There are some games whose strategic structure is such that no matter what type of threat behavior is used or even if noncooperative behavior is employed by a counter-coalition the amount obtainable by the

coalition S remains unchanged. This happens when:

$$\max_{S} \min_{N-S} \sum_{i \in S} P_i$$

$$\max_{S} \min_{N-S} \left\{ \sum_{i \in S} P_i - \sum_{j \in N-S} P_j \right\}$$

$$\max_{i \in S} \sum P_i$$

and

$$\max_{j \in N-S} \sum P_j$$

give the same value for $\sum_{i \in S} P_i$ as is the case for the two person matrix game:

3, 3	- 5, 7
7, -5	1, 1

In market games the first two threat conditions coincide; the third is not clearly defined.

Games and Solutions

The following table displays the current state of knowledge concerning the six solution concepts noted as applied to five games.

In the table NSP stands for no side-payments and SP for side-payments. The 2 refers to the two-person case and the ∞ to limiting behavior for when n is large.

The first game is the classical market with independent preferences and production processes.

The second is the simple majority, simple game where coalitions either win all or lose.

The third is the majority game modified by the corporate rule of pro rata payout. Without side-payments this must be modified to either proportional taxes and subsidies in money or the equivalent in a market with barter. We assume initially that the state holds all resources.

The fourth game is the majority vote with individually oriented preferences, private and public goods, equal taxation, and competitive markets for private goods.

The fifth game is the same as the fourth except we assume identical preferences for all players, i.e., a totally symmetric game.

It should be noted that there are several other solution concepts such as the "bargaining set" proposed by Aumann and Maschler; however they are not discussed here.

One interpretation of the variety of possibilities indicated in the table is that it might be desirable to construct a hierarchy of public socio-economic and socio-political problems based upon the degree of need for laws, administrative and political devices in order to obtain economic outcomes satisfying the desired sets of solution criteria.

	1. Pareto Optimality				2. Competitive Equilibrium				3. Noncooperative Equilibrium			
	S.	P.	S.	P.	S.	P.	S.	P.	S.	P.	S.	P.
	2	∞	2	∞	2	∞	2	∞	2	∞	2	∞
The Classical Market	← Exists →				Exists and is unique		Exists but not necessarily unique		Special case exists	special case converges to the C. E.	Special case exist	special case converges to the C. E.
The simple majority voting game	← yes →				← not defined →				← not defined →			
The corporate voting game	← yes →				← not defined →				← not defined →			
Voting equal taxes and competitive markets	yes		yes	? (f)	Exists subject to redefinition in order to include voting & taxation			?	exists ← (b) →			?
The symmetric voting taxation and market game	← yes →				← (a) →							

Note: C. E. (Competitive Equilibrium)

	4. Core				5. Value				6. Stable Set				Threat Condition
	S.	P.	N. S.	P.	S.	P.	N. S.	P.	S.	P.	N. S.	P.	
	2	∞	2	∞	2	∞	2	∞	2	∞	2	∞	
The Classical Market	always	converges to C.E.	always	converges to C.E.	exists	converges to C.E.	exists	exists (c)	always	^	always	^	No problem in the modeling of threat
The simple majority voting game	exi	empty	exi	empty	always	always	but not necessarily	always	exists		exists		conditions; all reasonable models are equivalent.
The corporate voting game	sts	1 point	sts	?	and is	ex		exists	and is	(d)	and is	(e)	Considerable problems in
Voting equal taxes and competitive markets	always	1 point	always	?	unique	ists	unique	? (f)	the same		the same		the modeling of threat conditions, most methods ~~give~~ different characterizations of the game. (g)
The symmetric voting taxation and market game	exists	1 point	exists	1 point	exists	midpoint	exists	midpoint	as the core	v	as the core	v	

Note: C. E. (Competitive Equilibrium)

(a) The completely symmetric game without side-payments is equivalent to the game with side-payments.

(b) There are many problems in the construction of adequate models of this case.

(c) It is conjectured that this converges to the set of competitive equilibrium points.

(d) A fair amount is known about solutions to the simple majority game; apart from that little work has been done on solutions (it is not even known if they always exist, although it is conjectured that they do). Apparently as numbers of players increase, in general stable sets do not appear to exhibit any striking limiting behavior; it is an interesting open problem to find out if they do have special properties in the case of the classical competitive market.

(e) Little is known about no-side-payment stable sets.

(f) Here we have a problem of definition concerning the value and the Pareto optimal surface. If equal taxation is a rule of the game, then it is possible that no outcome in the new game is Pareto optimal in the game without this added rule. The value however, selects an outcome which is Pareto optimal within the set of alternatives feasible according to the rules of the game.

(g) The problem of threats appears to be closely related to the law of torts and to the legal attitude to distinctions between intent and behavior.

FOOTNOTES

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