ARPA ORDER NO. 189-61

MEMORANDUM
RM-4290-ARPA
MARCH 1966

A PRELIMINARY INVESTIGATION OF TRANSPORT-BRANCHING PROCESSES

Bernard Shiffman

PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is supported by the Advanced Research Projects Agency under Contract No. SD-79. Any views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of ARPA.

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PREFACE

Many studies of one-dimensional transport processes can be found in the literature of mathematical physics. However, the literature mostly ignores the complicated problem of the determination of probability distributions involved in neutron transport processes. In this Memorandum, the author begins a probabilistic investigation of transport processes and arrives at some functional equations for generating functions involved in the process. This study evolved from research in neutron transport theory sponsored by the Advanced Research Projects Agency.

SUMMARY

A probabilistic discussion of one-dimensional multi-type neutron transport processes is given, and the invariant imbedding technique is applied to the generating functions of the emitted neutrons. A system of simultaneous partial differential equations for these generating functions is then derived. The well-known Ricatti equation for the expected reflected flux follows from these differential equations.

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A PRELIMINARY INVESTIGATION OF TRANSPORT-BRANCHING PROCESSES

1. INTRODUCTION

The purpose of this Memorandum is to begin a probabilistic study of neutron transport processes, in which neutron multiplication plays a role. We consider a model of the one-dimensional neutron transport process that has been extensively studied from a nonprobabilistic viewpoint by Bellman, Kalaba, and Wing [2, 3], and others. The essential step of this nonprobabilistic viewpoint is to define a vector representing the expected value of the neutron flux and to regard this expectation vector as being propagated through the medium. This description of the neutron transport process then bears a strong resemblance to the physical model of radiative transfer, first studied by Stokes in the nineteenth century. A brief history of the study of radiative transfer and neutron transport processes, as well as an extensive list of references, can be found in Redheffer [6]; references concerning neutron transport are given in Bellman and Kalaba [1]. In our probabilistic investigation of neutron transport processes, we also utilize the concept of a branching process [4]. We therefore refer to our mathematical model of a one-dimensional neutron transport process as a <u>transport-branching process</u>.

In the model of a one-dimensional neutron transport process that we shall use throughout this Memorandum, our basic assumption is that neutrons are point particles constrained to move along a finite line segment. This finite line segment, which we shall refer to as the rod, represents the interacting medium through which the neutrons travel. The neutrons travel through the rod, moving either to the right or to the left, and interact with the medium, occasionally producing additional neutrons by nuclear fissions. We assume that the neutrons do not interact with each other, so that the probability of a given neutron entering into a nuclear fission or otherwise interacting with the medium is independent of the presence of other neutrons. We further assume that these probabilities are independent of time. In general, energy dependence may be involved, but we shall restrict the allowable energies of the neutrons to a finite number of values, called energy states. A neutron is then completely described by its position in the rod, the direction it is traveling ("to the left" or "to the right"), and its energy state. A neutron transport process is initiated by neutrons entering the rod at the left or the right, and when a neutron leaves the rod (at the left or the right), it can no longer interact. The models of one-dimensional neutron transport processes given in [1, 2, 3] are similar to our model and differ by varying degrees of generalization and specialization. The description of the physical parameters needed to describe the process is given in Sec. 5 of this Memorandum.

We are interested in the problem of determining the probability distributions of the neutrons leaving the rod in neutron transport processes initiated by given beams of neutrons. One could consider the neutron transport process as a continuous-time stochastic processes, but such an approach does not yield explicit solutions to the particular problem that we are considering—determining the probabilities that given numbers of neutrons leave the medium (rod) throughout the process. The general approach that we use is to consider the probability distributions of the neutrons leaving the medium as functions of the initial and end points of the rod (which we consider as imbedded in the real line). This technique, for which Bellman coined the term <u>invariant imbedding</u> [1, 2, 3], has been used by Bellman and others in the study of neutron transport processes as well as in a variety of other problems. We first consider the problem of determining joint probability distributions of the neutrons "reflected" and "transmitted" through a rod composed of two adjoining rods, each with known probabilities for reflection and transmission. The approach that we use is similar to the method employed by Mycielski and Paszkowski [5] in discussing a simple particle transport process, which is a special case of the process we consider. A direct application of the Mycielski-Paszkowski method to our problem would result in immense combinatorial difficulties unless a proper choice of random variables is made. We use a set of random variables that describe a multitype (Galton-Watson) branching process [4], and we are thus able to obtain simple functional equations for the generating functions describing the reflected and transmitted neutrons. Finally, by using a limiting process, we obtain partial differential equations for these generating functions.

2. BRANCHING PROCESSES

We begin by summarizing the terminology and notation of branching processes [4] that we use throughout this Memorandum. A branching process consists of a collection of objects—e.g., neutrons in a chain reaction—each of which can produce similar additional objects. If we consider an initial collection of objects as belonging to "generation 0," then the collection of all objects produced by these objects are considered as objects of "generation 1." In general, the objects of generation n produce objects of generation n + 1. Since the probability distribution of the objects of a given generation depends only on the distribution of objects of the immediately preceding generation, we thus have a Markov chain

$$\underset{\sim}{Z}^0,\ \underset{\sim}{Z}^1,\ \underset{\sim}{Z}^2,\ \ldots,\ \underset{\sim}{Z}^n,\ \ldots\ ,$$

where $\underset{\sim}{Z}^n$ describes the objects of the n-th generation, for nonnegative n. Furthermore, the distribution of the descendants of a given object of generation n is independent of the distribution of decendants of the other objects of generation n.

We shall consider branching processes involving a finite number of types of objects, and we number the types from 1 to m. The state of the process in generation n, represented by $\underset{\sim}{Z}^n$ in the Markov chain, is described by an m-dimensional vector with nonnegative integer components, $(z_1^n, \ldots, z_m^n)$, where z_i^n is the number of objects of type i in generation n.

We can thus consider $\underset{\sim}{Z}^n$ as a vector-valued random variable, which we shall refer to as a random vector. The generating function $g_{(n)}$ of the random vector $\underset{\sim}{Z}^n$ is defined by

$$g_{(n)}(s_1, \ldots, s_n) = \sum_{k_1, \ldots, k_m = 0}^{\infty} b^{(n)}_{k_1 \ldots k_m} s_1^{k_1} \ldots s_m^{k_m} , \tag{2.1}$$

where

$$b^{(n)}_{k_1 \ldots k_m} = P(\underset{\sim}{Z}^n = (k_1, k_2, \ldots, k_m))$$

for $n = 0, 1, 2, \ldots$. We consider the domain of $g_{(n)}$ to be D^m, where D is the closed unit disk in the complex plane, and we note that $g_{(n)}: D^m \to D$ and that $g_{(n)}(1, \ldots, 1) = 1$.

We quote now the basic result of Kolmogorov and Dmitriev (see Harris [4], p. 36) for branching processes. Let $a^i_{k_1 \ldots k_m}$ (for $i = 1, \ldots, m$; $k_1, \ldots, k_m = 0, 1, 2, \ldots$) represent the probability that a particle of type i will produce exactly k_j particles of type j for $j = 1, \ldots, m$. We define the function $\underset{\sim}{f}: D^m \to D^m$ by

$$\underset{\sim}{f}(\underset{\sim}{s}) = (f^1(\underset{\sim}{s}), \ldots, f^m(\underset{\sim}{s})) ,$$

where

$$\begin{aligned} f^i(\underset{\sim}{s}) &= \sum_{k_1, \ldots, k_m = 0}^{\infty} a^i_{k_1 \ldots k_m} s_1^{k_1} \ldots s_m^{k_m} , \\ \underset{\sim}{s} &= (s_1, \ldots, s_m) . \end{aligned} \tag{2.2}$$

The resulting relation describing the generating functions $\{g_{(n)}\}$ is given by

$$(2.3) \qquad g_{(n)}(\underset{\sim}{s}) = g_{(0)}(\underset{\sim}{f}_{(n)}(\underset{\sim}{s})) ,$$

where $\underset{\sim}{f}_{(n)}$ is the n-th iterate of $\underset{\sim}{f}$ (defined recursively by $\underset{\sim}{f}_{(n+1)}(\underset{\sim}{s}) = \underset{\sim}{f}(\underset{\sim}{f}_{(n)}(\underset{\sim}{s}))$, $\underset{\sim}{f}_{(0)}(\underset{\sim}{s}) = \underset{\sim}{s}$). We call $\underset{\sim}{f}$ the <u>branching generating function</u> of the branching process $\{\underset{\sim}{Z}^n\}$,

Some notation that we find convenient to use throughout this Memorandum is summarized below. We denote vector quantities by an underscored "~" and write $\underset{\sim}{s} = (s_1, \ldots, s_m)$, $\underset{\sim}{s}^1 = (s_1^1, \ldots, s_m^1)$, etc. The symbol $\underset{\sim}{1}$ represents the vector (1, ..., 1). We also use this convention for subscripts, and the symbol $\sum_{j}$ means the sum over all nonnegative values of $j_1, j_2, \ldots,$ and j_m (where $\underset{\sim}{j} = (j_1, \ldots, j_m)$). For example, Eq. (2.1) becomes

$$g_{(n)}(\underset{\sim}{s}) = \sum_{k} b^{(n)}_{\underset{\sim}{k}} s_1^{k_1} \cdots s_m^{k_m} .$$

We also utilize the concept of the conditional generating function of a vector-valued random variable. If $\underset{\sim}{W}$ is an m-dimensional random vector, and $\mathcal{A}$ is an event, then the conditional generating function φ of $(\underset{\sim}{W}|\mathcal{A})$ is defined by

$$(2.4) \qquad \varphi(s) = \sum_{k} P(\underset{\sim}{W} = \underset{\sim}{k}|\mathcal{A}) s_1^{k_1} \cdots s_m^{k_m} .$$

3. THE TRANSPORT GENERATING FUNCTIONS

We now consider the one-dimensional neutron transport process described in Sec. 1. Let the process take place on a rod extending from 0 to x on the real line and assume that there are m distinct energy states numbered from 1 to m. In order to allow a simple mathematical solution of this one-dimensional transport-branching process, we shall consider the following variables:

X_i = the number of neutrons of energy state i leaving the reactor at the left,

Y_i = the number of neutrons of energy state i leaving the reactor at the right,

$i = 1, \ldots, m.$

We also define the quantities:

U_i = the number of neutrons of energy state i incident at the left (traveling toward the right),

V_i = the number of neutrons of energy state i incident at the right (traveling toward the left),

$i = 1, \ldots, m.$

(See Fig. 1.)

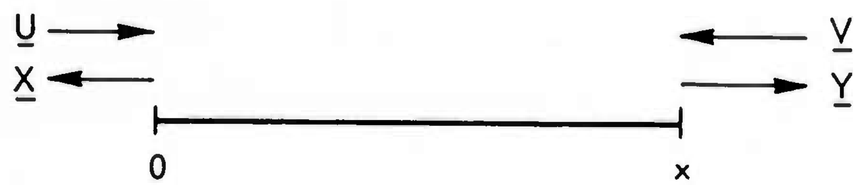


Fig. 1

We then define the random vectors

$$\underset{\sim}{X} = (X_1, \ldots, X_m), \quad \underset{\sim}{Y} = (Y_1, \ldots, Y_m),$$
$$\underset{\sim}{U} = (U_1, \ldots, U_m), \quad \underset{\sim}{V} = (V_1, \ldots, V_m),$$

and

$$\underset{\sim}{Z} = (\underset{\sim}{X}, \underset{\sim}{Y}) = (X_1, \ldots, X_m, Y_1, \ldots, Y_m) .$$

We also let $\underset{\sim}{E}^i$ represent the "i-th unit vector" with the i-th coordinate equal to one, and other coordinates zero.

Since the number of neutrons emitted from the reactor depends on the incident neutron beams, the generating function for $\underset{\sim}{Z}$ depends on the assumptions about the values of $\underset{\sim}{U}$ and $\underset{\sim}{V}$. It is useful to define

Ψ_i^1 = the conditional generating function of

$$(\underset{\sim}{Z} | \underset{\sim}{U} = \underset{\sim}{0}, \underset{\sim}{V} = \underset{\sim}{E}^i) ,$$

Ψ_i^2 = the conditional generating function of

$$(\underset{\sim}{Z} | \underset{\sim}{U} = \underset{\sim}{E}^i, \underset{\sim}{V} = \underset{\sim}{0}) ,$$

for $i = 1, \ldots, m.$

We also define the vector generating functions $\underset{\sim}{\Psi}^1$ and $\underset{\sim}{\Psi}^2$ by

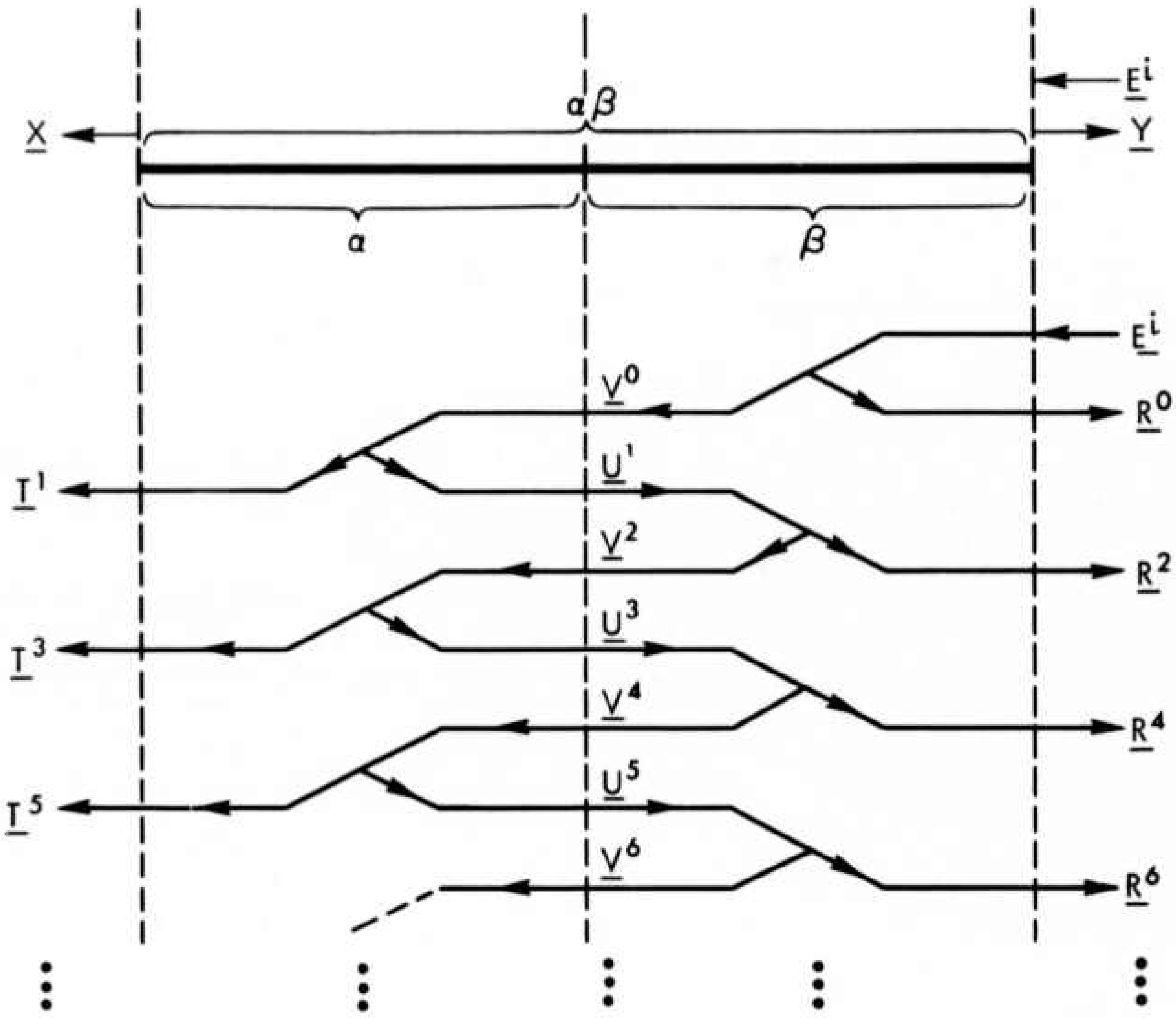
$$\underset{\sim}{\Psi}^\ell(\underset{\sim}{s}) = (\Psi_1^\ell(\underset{\sim}{s}), \ldots, \Psi_m^\ell(\underset{\sim}{s})), \quad \ell = 1, 2 .$$

Assume that $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are random vectors, and let $g: D^{2m} \to D$ be the generating function of the 2m-dimensional vector $(V_1, \ldots, V_m, U_1, \ldots, U_m)$. Since the neutrons do not interact with each other, it is easily seen that the generating function for $\underset{\sim}{Z}$ is given by

$$g(\underset{\sim}{\Psi}^1(\underset{\sim}{s}), \underset{\sim}{\Psi}^2(\underset{\sim}{s})) .$$

Our eventual goal is to determine the generating functions $\underset{\sim}{\Psi}^1$ and $\underset{\sim}{\Psi}^2$, which we call the <u>transport generating functions</u>. Following the Mycielski-Paszkowski method [5] of considering multiple "reflections" and "transmissions," we first determine the transport generating functions for a rod composed of two segments in terms of the transport generating functions of the segments.

Consider the composite rod $\alpha\beta$ made up of the two rods α and β as in Fig. 2. We let $\underset{\sim}{\Psi}^{\ell}_{\alpha}$, $\underset{\sim}{\Psi}^{\ell}_{\beta}$, and $\underset{\sim}{\Psi}^{\ell}_{\alpha\beta}$ $(\ell = 1, 2)$ be the transport generating functions for the neutron transport processes on the rod α, the rod β, and the rod $\alpha\beta$, respectively. Let i be a fixed integer between 1 and m, and consider the transport-branching process initiated by one neutron of energy state i incident upon the rod $\alpha\beta$ at the right (and no neutrons incident at the left). Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be the random vectors representing the neutrons emerging from the rod $\alpha\beta$, as defined above. The generating function for $\underset{\sim}{Z} = (\underset{\sim}{X}, \underset{\sim}{Y})$ is thus given by the i-th component of $\underset{\sim}{\Psi}^1_{\alpha\beta}$. This process is to be viewed as consisting of multiple "reflections" and "transmissions" through the segments α and β (see Fig. 2). We let $(\underset{\sim}{V}^o, \underset{\sim}{R}^o)$ represent



$\underline{E}^i$
$\alpha\beta$
$\underline{X}$
$\underline{Y}$
α
β
$\underline{E}^i$
$\underline{V}^0$
$\underline{R}^0$
$\underline{T}^1$
$\underline{U}^1$
$\underline{V}^2$
$\underline{R}^2$
$\underline{T}^3$
$\underline{U}^3$
$\underline{V}^4$
$\underline{R}^4$
$\underline{T}^5$
$\underline{U}^5$
$\underline{V}^6$
$\underline{R}^6$

Fig. 2

the neutrons emerging directly from β with the single neutron of energy state i incident at the right. Then we let $(\underset{\sim}{T}^1, \underset{\sim}{U}^1)$ represent the neutrons emerging directly from α with incident neutron beam represented by $\underset{\sim}{V}^o$ at the right. In general, $(\underset{\sim}{V}^n, \underset{\sim}{R}^n)$ represents the neutrons emerging directly from β with incident neutron beam $\underset{\sim}{U}^{n-1}$ at the left; $(\underset{\sim}{T}^{n+1}, \underset{\sim}{U}^{n+1})$ represents the neutrons emerging directly from α with incident neutron beam $\underset{\sim}{V}^n$ at the right. ($\underset{\sim}{V}^n$ and $\underset{\sim}{R}^n$ are considered as $\underset{\sim}{0}$ for n odd; $\underset{\sim}{T}^n$ and $\underset{\sim}{U}^n$ are considered as $\underset{\sim}{0}$ for n even.) We define

$$\underset{\sim}{X}^n = \sum_{k=0}^{n} \underset{\sim}{T}^k ,$$

(3.1)

$$\underset{\sim}{Y}^n = \sum_{k=0}^{n} \underset{\sim}{R}^k .$$

This complicated description can be greatly simplified by considering the branching process involving 4m objects, where the object of type i represents

$$\begin{cases} \text{a neutron of the energy state } i \text{ emerging from } \alpha\beta \text{ to the left, for } 1 \le i \le m; \\ \text{a neutron of energy state } i - m \text{ emerging from } \alpha\beta \text{ to the right, for } m+1 \le i \le 2m; \\ \text{a neutron of energy state } i - 2m \text{ emerging from } \beta \text{ to the left, for } 2m+1 \le i \le 3m; \\ \text{a neutron of energy state } i - 3m \text{ emerging from } \alpha \text{ to the right, for } 3m+1 \le i \le 4m. \end{cases}$$

This branching process is described by the Markov chain

$$\underset{\sim}{Q}^0, \underset{\sim}{Q}^1, \ldots, \underset{\sim}{Q}^n, \ldots ,$$

where the 4m-dimensional random vector

$$(3.2) \qquad \underset{\sim}{Q}^n = (\underset{\sim}{X}^n, \underset{\sim}{Y}^n, \underset{\sim}{V}^n, \underset{\sim}{U}^n) , \qquad \text{for } n = 0, 1, 2, \ldots .$$

Let $g_{(n)}$ be the generating function of $\underset{\sim}{Q}^n$ and let $\underset{\sim}{f}: D^{4m} \to D^{4m}$ be the branching generating function for the branching process $\{\underset{\sim}{Q}^n\}$.

Writing

$$(3.3) \qquad \underset{\sim}{f} = (\underset{\sim}{f}^1, \underset{\sim}{f}^2, \underset{\sim}{f}^3, \underset{\sim}{f}^4) ,$$

where $\underset{\sim}{f}^\ell: D^{4m} \to D^m$, $\ell = 1, \ldots, 4$, the reader can easily see from the above description that

$$(3.4) \qquad \begin{aligned} \underset{\sim}{f}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{s}^3, \underset{\sim}{s}^4) &= \underset{\sim}{s}^1 , \\ \underset{\sim}{f}^2(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{s}^3, \underset{\sim}{s}^4) &= \underset{\sim}{s}^2 , \\ \underset{\sim}{f}^3(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{s}^3, \underset{\sim}{s}^4) &= \underset{\sim}{\Psi}_\alpha^1(\underset{\sim}{s}^1, \underset{\sim}{s}^4) , \\ \underset{\sim}{f}^4(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{s}^3, \underset{\sim}{s}^4) &= \underset{\sim}{\Psi}_\beta^2(\underset{\sim}{s}^3, \underset{\sim}{s}^2) . \end{aligned}$$

Writing $\underset{\sim}{f}_{(n)} = (\underset{\sim}{f}^1_{(n)}, \underset{\sim}{f}^2_{(n)}, \underset{\sim}{f}^3_{(n)}, \underset{\sim}{f}^4_{(n)})$, we have the recursive formulas:

$$
(3.5)\qquad
\begin{aligned}
\underset{\sim}{f}^1_{(n)} &= \underset{\sim}{s}^1 , \\
\underset{\sim}{f}^2_{(n)} &= \underset{\sim}{s}^2 , \\
\underset{\sim}{f}^3_{(n)} &= \underset{\sim}{\Psi}^1_{\alpha}(\underset{\sim}{s}^1, \underset{\sim}{f}^4_{(n-1)}) = \underset{\sim}{\Psi}^1_{\alpha}(\underset{\sim}{s}^1, \underset{\sim}{\Psi}^2_{\beta}(\underset{\sim}{f}^3_{(n-2)}, \underset{\sim}{s}^2)) , \\
\underset{\sim}{f}^4_{(n)} &= \underset{\sim}{\Psi}^2_{\beta}(\underset{\sim}{f}^3_{(n-1)}, \underset{\sim}{s}^2) = \underset{\sim}{\Psi}^2_{\beta}(\underset{\sim}{\Psi}^1_{\alpha}(\underset{\sim}{s}^1, \underset{\sim}{f}^4_{(n-2)}), \underset{\sim}{s}^2) .
\end{aligned}
$$

The Markov chain $\underset{\sim}{Q}^n$ will converge, providing $\underset{\sim}{V}^n$ approaches zero (or equivalently, $\underset{\sim}{U}^n \to \underset{\sim}{0}$). Consider the Markov chain

$$\underset{\sim}{V}^0, \underset{\sim}{V}^2, \underset{\sim}{V}^4, \ldots, \underset{\sim}{V}^{2n}, \ldots ,$$

which also describes a branching process with branching generating function given by

$$(3.6)\qquad \underset{\sim}{f}_V(\underset{\sim}{s}) = \underset{\sim}{\Psi}^1_{\alpha}(\underset{\sim}{1}, \underset{\sim}{\Psi}^2_{\beta}(\underset{\sim}{s}, \underset{\sim}{1})) .$$

If we assume that the Jacobian of $\underset{\sim}{f}_V$ (evaluated at $\underset{\sim}{1}$) is small enough, then $\underset{\sim}{V}^n \to \underset{\sim}{0}$ with probability one; and it follows that $\underset{\sim}{U}^n \to \underset{\sim}{0}$ with probability one, and $\underset{\sim}{Q}^n$ converges to a finite value with probability one.

We henceforth assume the convergence (with probability one) of $\underset{\sim}{Q}^n$. This means that the probability of a "chain reaction" (producing an infinite number of neutrons) occurring is zero. Then the generating functions $\underset{\sim}{f}_{(n)}$ and $g_{(n)} = g_{(0)}\underset{\sim}{f}_{(n)}$ converge, and $\lim g_{(n)}$ is the generating function of $\lim \underset{\sim}{Q}^n$. From the definition of $\underset{\sim}{Q}^n$, it is clear that

$$\lim_{n\to\infty} \underset{\sim}{Q}^n = (\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{0}, \underset{\sim}{0}) \ . \tag{3.7}$$

Let

$$\begin{aligned} &\lim_{n\to\infty} \underset{\sim}{f}^3_{(n)}(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{0}, \underset{\sim}{0}) = \underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2) \ , \\ &\lim_{n\to\infty} \underset{\sim}{f}^4_{(n)}(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{0}, \underset{\sim}{0}) = \underset{\sim}{\pi}^2(\underset{\sim}{s}^1, \underset{\sim}{s}^2) \ . \end{aligned} \tag{3.8}$$

Let φ_β and $\varphi_{\alpha\beta}$ be the i-th components of $\underset{\sim}{\Psi}^1_\beta$ and $\underset{\sim}{\Psi}^1_{\alpha\beta}$, respectively. Then

$$g_{(0)} = \varphi_\beta(\underset{\sim}{s}^3, \underset{\sim}{s}^2) \ ,$$

and therefore,

$$g_{(n)} = \varphi_\beta(\underset{\sim}{f}^3_{(n)}, \underset{\sim}{s}^2) \ .$$

The generating function for $\lim \underset{\sim}{Q}^n = (\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{0}, \underset{\sim}{0})$ is given by

$$g = \lim g_{(n)} = \varphi_\beta(\underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2), \underset{\sim}{s}^2) \ .$$

Therefore,

$$\varphi_{\alpha\beta}(\underset{\sim}{s}^1, \underset{\sim}{s}^2) = g(\underset{\sim}{s}^1, \underset{\sim}{s}^2, \underset{\sim}{0}, \underset{\sim}{0}) = \varphi_\beta(\underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2), \underset{\sim}{s}^2) \ ,$$

from which we conclude that

$$\underset{\sim}{\Psi}^1_{\alpha\beta} = \underset{\sim}{\Psi}^1_\beta(\underset{\sim}{\pi}^1, \underset{\sim}{s}^2) \ . \tag{3.9}$$

By identical reasoning, or by appealing to symmetry, we obtain our second relation

$$(3.10)\qquad \underset{\sim}{\Psi}{}^{2}_{\alpha\beta} = \underset{\sim}{\Psi}{}^{2}_{\alpha}(\underset{\sim}{s}^{1},\ \underset{\sim}{\pi}^{2})\ .$$

The functions $\underset{\sim}{\pi}^{1}$ and $\underset{\sim}{\pi}^{2}$, which are the limits of the sequences given by Eqs. (3.8), satisfy the implicit equations,

$$(3.11)\qquad \begin{aligned} \underset{\sim}{\pi}^{1} &= \underset{\sim}{\Psi}{}^{1}_{\alpha}(\underset{\sim}{s}^{1},\ \underset{\sim}{\pi}^{2}) = \underset{\sim}{\Psi}{}^{1}_{\alpha}(\underset{\sim}{s}^{1},\ \underset{\sim}{\Psi}{}^{2}_{\beta}(\underset{\sim}{\pi}^{1},\ \underset{\sim}{s}^{2})) \\ \underset{\sim}{\pi}^{2} &= \underset{\sim}{\Psi}{}^{2}_{\beta}(\underset{\sim}{\pi}^{1},\ \underset{\sim}{s}^{2}) = \underset{\sim}{\Psi}{}^{2}_{\beta}(\underset{\sim}{\Psi}{}^{1}_{\alpha}(\underset{\sim}{s}^{1},\ \underset{\sim}{\pi}^{2}),\ \underset{\sim}{s}^{2})\ . \end{aligned}$$

4. DIFFERENTIAL EQUATIONS OF INVARIANT IMBEDDING

We now use the invariant imbedding approach combined with the functional equations of Sec. 3 to obtain systems of partial differential equations describing the transport generating functions $\underset{\sim}{\Psi}^1$ and $\underset{\sim}{\Psi}^2$. One should note that these partial differential equations could also be obtained, using the techniques of this Memorandum, without recourse to the results of Sec. 3.

Let us first consider a neutron transport process taking place on a given rod [0, L]. To utilize the invariant imbedding method, we consider the family of segments [a, b] of the rod [0, L], and we let $\underset{\sim}{\Psi}^1_{[a,b]}$ and $\underset{\sim}{\Psi}^2_{[a,b]}$ be the transport generating functions defined in Sec. 3 for the transport process taking place on the (isolated) rod segment [a, b] (for $0 \leq a \leq b \leq L$).

Write

$$\underset{\sim}{\Psi}^{\ell}_{[0,x]}(\underset{\sim}{s}) = \underset{\sim}{\Psi}^{\ell}(x, \underset{\sim}{s}), \qquad \ell = 1, 2 ,$$

where

$$\underset{\sim}{s} = (\underset{\sim}{s}^1, \underset{\sim}{s}^2) \in D^{2m} ,$$

and consider the family of generating functions $\{\underset{\sim}{\Psi}^{\ell}(x, \underset{\sim}{s})\}$ parametrized by the ral variable x. In order to obtain differential equations for $\underset{\sim}{\Psi}^{\ell}(x, \underset{\sim}{s})$, we consider the rod [0, x + h] as composed of two segments [0, x] and [x, x + h]. (See Fig. 3.)

Fig.3

Letting

$$\frac{\partial}{\partial h} \underset{\sim}{\Psi}^{\ell}_{[x,x+h]}(\underset{\sim}{s})\Big|_{h=0} = \underset{\sim}{\omega}^{\ell}(x, \underset{\sim}{s}), \qquad \ell = 1, 2 , \tag{4.1}$$

we have

$$\underset{\sim}{\Psi}^{\ell}_{[x,x+h]}(\underset{\sim}{s}^1, \underset{\sim}{s}^2) = \underset{\sim}{s}^{\ell} + \underset{\sim}{\omega}^{\ell}(x, \underset{\sim}{s})h + \underset{\sim}{o}(h), \qquad \ell = 1, 2 . \tag{4.2}$$

(We shall assume that $\underset{\sim}{\omega}^1$ and $\underset{\sim}{\omega}^2$ are continuously differentiable.)

Applying the results of Sec. 3 with $\alpha = [0, x]$ and $\beta = [x, x + h]$ (see Fig. 3), we have

$$\begin{aligned} \underset{\sim}{\Psi}^1(x + h, \underset{\sim}{s}^1, \underset{\sim}{s}^2) &= \underset{\sim}{\Psi}^1_{[x,x+h]}(\underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2), \underset{\sim}{s}^2) \\ &= \underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2) + \underset{\sim}{\omega}^1(x, \underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2), \underset{\sim}{s}^2)h + \underset{\sim}{o}(h) \\ \underset{\sim}{\Psi}^2(x + h, \underset{\sim}{s}^1, \underset{\sim}{s}^2) &= \underset{\sim}{\Psi}^2(x, \underset{\sim}{s}^1, \underset{\sim}{\pi}^2(\underset{\sim}{s}^1, \underset{\sim}{s}^2)) . \end{aligned} \tag{4.3}$$

Applying Eqs. (3.8), we find that the functions $\underset{\sim}{\pi}^1$ and $\underset{\sim}{\pi}^2$ in the above expression are given by

$$\underset{\sim}{\pi}^1(\underset{\sim}{s}^1, \underset{\sim}{s}^2) = \underset{\sim}{\psi}^1(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2) + h \sum_j \omega_j^2(x, \underset{\sim}{\psi}^1, \underset{\sim}{s}^2) \frac{\partial \underset{\sim}{\psi}^1}{\partial s_j^2} + \underset{\sim}{o}(h),$$

(4.4)

$$\underset{\sim}{\pi}^2(\underset{\sim}{s}^1, \underset{\sim}{s}^2) = \underset{\sim}{s}^2 + \underset{\sim}{\omega}^2(x, \underset{\sim}{\psi}^1, \underset{\sim}{s}^2)h + \underset{\sim}{o}(h) .$$

We readily obtain from Eqs. (4.3) and (4.4) the <u>basic system of partial differential equations for the transport generating functions</u>,

$$\frac{\partial}{\partial x} \underset{\sim}{\psi}^1(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2)$$

$$= \sum_j \omega_j^2(x, \underset{\sim}{\psi}^1, \underset{\sim}{s}^2) \frac{\partial \underset{\sim}{\psi}^1}{\partial s_j^2} + \underset{\sim}{\omega}^1(x, \underset{\sim}{\psi}^1, \underset{\sim}{s}^2) ,$$

(4.5)

$$\frac{\partial}{\partial x} \underset{\sim}{\psi}^2(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2)$$

$$= \sum_j \omega_j^2(x, \underset{\sim}{\psi}^1, \underset{\sim}{s}^2) \frac{\partial \underset{\sim}{\psi}^2}{\partial s_j^2} .$$

The initial conditions are given by

$$\underset{\sim}{\psi}^1(0, \underset{\sim}{s}^1, \underset{\sim}{s}^2) = \underset{\sim}{s}^1 ,$$

(4.6)

$$\underset{\sim}{\psi}^2(0, \underset{\sim}{s}^1, \underset{\sim}{s}^2) = \underset{\sim}{s}^2 .$$

Note that the first equation of (4.5) does not involve $\underset{\sim}{\psi}^2$, so it can be solved independently to give the value of $\underset{\sim}{\psi}^1$. We can also regard these partial differential equations as an infinite system of ordinary differential equations for the probabilities of various combinations of particles being reflected and transmitted.

If we are interested only in the "reflection" probabilities, these basic equations can be simplified by letting

$$(4.7) \qquad \underset{\sim}{\Phi}(x, \underset{\sim}{s}) = \underset{\sim}{\Psi}^1(x, \underset{\sim}{1}, \underset{\sim}{s}) .$$

The reader should observe that $\Phi_i(x, \underset{\sim}{s})$ is the conditional generating function of $(\underset{\sim}{Y} | \underset{\sim}{U} = \underset{\sim}{0}, \underset{\sim}{V} = \underset{\sim}{E}^i)$ in the transport process taking place on the rod $[0, x]$. The first equation of (4.5) becomes

$$(4.8) \qquad \begin{aligned} &\frac{\partial}{\partial x} \underset{\sim}{\Phi}(x, \underset{\sim}{s}) = \sum_{j=1}^{m} \omega_j^2(x, \underset{\sim}{\Phi}, \underset{\sim}{s}) \frac{\partial}{\partial s_j} \underset{\sim}{\Phi}(x, \underset{\sim}{s}) + \underset{\sim}{\omega}^1(x, \underset{\sim}{\Phi}, \underset{\sim}{s}) , \\ &\underset{\sim}{\Phi}(0, \underset{\sim}{s}) = \underset{\sim}{1} . \end{aligned}$$

This last result is analogous to the matrix Ricatti equation for the expected reflected neutron flux [1]. In fact, the Ricatti equation for this reflection function follows from Eq. (4.8), as is shown at the end of the next section.

5. EQUATIONS FOR NEUTRON TRANSPORT

The differential equations of Sec. 4 are dependent on Eq. (4.2) (or, equivalently, on Eq. (4.1)), which defines the key parameters, $\underset{\sim}{\omega}^1$ and $\underset{\sim}{\omega}^2$, of an abstract transport-branching process. We shall now determine $\underset{\sim}{\omega}^1$ and $\underset{\sim}{\omega}^2$ in terms of some physical parameters of a neutron transport process. Let

$\sigma_i(x)$ = The reciprocal of the mean free path of a neutron of energy state i at the point x in the rod,

$q^i_{\underset{\sim}{jk}}(x)$ = the probability that a nuclear fission initiated by a neutron of state i at the point x will produce exactly j_μ neutrons of state μ traveling to the left and k_μ neutrons of state μ traveling to the right, for $\mu = 1, \ldots, m$.

(Absorption of neutrons can be taken into account by considering absorption as a "nuclear fission" in which no neutrons are produced.) We then define the generating functions

$$(5.1) \qquad Q_i(x, \underset{\sim}{s}, \underset{\sim}{t}) = \sum_{j,k} q^i_{\underset{\sim}{jk}}(x) s_1^{j_1} \cdots s_m^{j_m} \, t_1^{k_1} \cdots t_m^{k_m} ,$$

$$\text{for } i = 1, \ldots, m.$$

Consider a neutron fission process in the rod [x, x + h]. A neutron of energy state i entering the rod has the probability $\sigma_i(x)h + o(h)$ of causing a nuclear fission, and the probability that this neutron starts a chain reaction resulting in two or more fissions is o(h). Letting φ_i^ℓ represent the i-th component of $\underset{\sim}{\Psi}^\ell_{[x,x+h]}$ (ℓ = 1, 2; i = 1, ..., m), we therefore conclude that

$$\varphi_i^\ell(\underset{\sim}{s}^1, \underset{\sim}{s}^2) = \begin{cases} [1 - \sigma_i(x)h + o(h)]s_i^\ell & \text{(no fission),} \\ + [\sigma_i(x)h + o(h)]Q_i(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2) & \text{(one fission),} \\ + o(h) & \text{(two or more fissions),} \end{cases}$$

or, more simply,

$$\varphi_i^\ell(\underset{\sim}{s}^1, \underset{\sim}{s}^2) = s_i^\ell + \sigma_i(x)[Q_i(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2) - s_i^\ell]h + o(h) \quad . \tag{5.2}$$

Comparing Eq. (5.2) with Eq. (4.2), we obtain our desired result:

$$\omega_i^\ell(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2) = \sigma_i(x)[Q_i(x, \underset{\sim}{s}^1, \underset{\sim}{s}^2) - s_i^\ell] \ , \tag{5.3}$$

for ℓ = 1, 2; i = 1, ..., m.

With only one energy state (m = 1), Eqs. (4.5) become

$$\begin{aligned} \frac{1}{\sigma(x)} \frac{\partial \Psi^1}{\partial x} &= [Q(x, \Psi^1, s^2) - s^2] \frac{\partial \Psi^1}{\partial s^2} + Q(x, \Psi^1, s^2) - \Psi^1 \\ \frac{1}{\sigma(x)} \frac{\partial \Psi^2}{\partial x} &= [Q(x, \Psi^1, s^2) - s^2] \frac{\partial \Psi^2}{\partial s^2} \ . \end{aligned} \tag{5.4}$$

The neutron fission process in which each fission produces two neutrons, one traveling to the left and one traveling to the right, is often discussed in the literature [1, 2, 3]. In this case,

$$Q(x,\ s^1,\ s^2) = s^1 s^2\ ; \tag{5.5}$$

the transport generating functions are given by

$$\begin{aligned}
\frac{1}{\sigma(x)}\,\frac{\partial\psi^1}{\partial x} &= (s^2\psi^1 - s^2)\,\frac{\partial\psi^1}{\partial s^2} + s^2\psi^1 - \psi^1\ , \\
\frac{1}{\sigma(x)}\,\frac{\partial\psi^2}{\partial x} &= (s^2\psi^1 - s^2)\,\frac{\partial\psi^2}{\partial s^2}\ , \\
\psi^1(0,\ s^1,\ s^2) &= s^1,\quad \psi^2(0,\ s^1,\ s^2) = s^2\ ,
\end{aligned} \tag{5.6}$$

and the generating function for the reflected neutrons is given by

$$\begin{aligned}
\frac{1}{\sigma(x)}\,\frac{\partial\Phi}{\partial x} &= (s\Phi - s)\,\frac{\partial\Phi}{\partial s} + s\Phi - \Phi\ , \\
\Phi(0,\ s) &= 1\ .
\end{aligned} \tag{5.7}$$

This special case is discussed by Bellman, Kalaba, and Wing [2], who arrived at Eq. (5.7) using elementary methods.

Another simple example of a transport-branching process is the continuous one-dimensional random walk. In a one-dimensional random walk, a particle enters the rod and randomly reverses its direction during its journey until it leaves the rod either to the left or to the right. Let

$$\begin{Bmatrix} \sigma_\ell(x) \\ \sigma_r(x) \end{Bmatrix} = \text{the reciprocal of the mean-free path of the particle when it is traveling to the } \begin{Bmatrix} \text{left} \\ \text{right} \end{Bmatrix} \text{ at the point } x.$$

Thus the particle has the probability $\sigma_r(x)h + o(h)$ of reversing its direction while traveling to the right from x to $x + h$, and similarly while traveling to the left. The reader can then readily see that

$$(5.8) \qquad \begin{aligned} \omega^1(x, s, t) &= \sigma_\ell(x)(t - s) , \\ \omega^2(x, s, t) &= \sigma_r(x)(s - t) . \end{aligned}$$

Equation (4.8) then becomes

$$(5.9) \qquad \begin{aligned} \frac{\partial \Phi}{\partial x} &= \sigma_r(x)(\Phi - s) \frac{\partial \Phi}{\partial s} + \sigma_\ell(x)(s - \Phi) , \\ \Phi(0, s) &= 1 . \end{aligned}$$

The solution to (5.9) is given by

$$\Phi(x, s) = 1 - p(x) + p(x)s ,$$

where $p(x)$ is the solution of the Ricatti equation

$$(5.10) \qquad \begin{aligned} p' &= \sigma_\ell - (\sigma_\ell + \sigma_r)p + \sigma_r p^2 , \\ p(0) &= 0 . \end{aligned}$$

The quantity $p(x)$ is the probability that a particle entering the rod $[0, x]$ at x will leave the rod at x, and this probability is given by the well-known equation (5.10).

The well-known matrix differential equations [1, 6] of a neutron transport process are easily derived from our general results. Let $R_{ij}(x)$ represent the expected number of neutrons of state i leaving the rod $[0, x]$ at the right, given that one neutron of state j enters the rod at the right. Then

$$(5.11) \qquad R_{ij}(x) = \frac{\partial \Phi_j}{\partial s_i}(x, \underset{\sim}{1}) .$$

If the probability that a finite number of neutrons are produced is equal to one (i.e., the probability of the occurrence of a "chain reaction" is zero), it follows that

$$\Phi_i(x, \underset{\sim}{1}) = 1 ,$$

$$\omega_i^{\ell}(x, \underset{\sim}{1}) = 0 , \qquad \ell = 1, 2 ,$$

$$i = 1, \ldots, m.$$

Differentiation of Eq. (4.8) utilizing the above identities yields the familiar matrix Ricatti equation [1] for the expected reflected neutron flux,

$$(5.12) \qquad \frac{d}{dx} R_{ij} = \frac{\partial \omega_j^1}{\partial s_i^2} + \sum_k \frac{\partial \omega_k^2}{\partial s_i^2} R_{kj} + \sum_k R_{ik} \frac{\partial \omega_j^1}{\partial s_k^1}$$

$$+ \sum_{k,\ell} R_{i\ell} \frac{\partial \omega_k^2}{\partial s_\ell^1} R_{kj} ,$$

where the partial derivatives of $\underset{\sim}{\omega}^1$ and $\underset{\sim}{\omega}^2$ are evaluated at $\underset{\sim}{s}^1 = \underset{\sim}{s}^2 = \underset{\sim}{1}$. The Jacobian matrices

$$\frac{\partial \underset{\sim}{\omega}^1}{\partial \underset{\sim}{s}^1} , \quad \frac{\partial \underset{\sim}{\omega}^1}{\partial \underset{\sim}{s}^2} , \quad \frac{\partial \underset{\sim}{\omega}^2}{\partial \underset{\sim}{s}^1} , \quad \text{and} \quad \frac{\partial \underset{\sim}{\omega}^2}{\partial \underset{\sim}{s}^2}$$

(evaluated at $\underset{\sim}{s}^1 = \underset{\sim}{s}^2 = \underset{\sim}{1}$) appearing in Eq. (5.12) have the following physical significance. Consider a neutron transport process taking place on the isolated rod segment $\beta = [x, x + h]$, in which one neutron of energy state j enters at the right (and no neutrons enter at the left). Then the expected number of neutrons of energy state i leaving β at the right is given by

$$\frac{\partial \omega_j^1}{\partial s_i^2} (x, \underset{\sim}{1}, \underset{\sim}{1}) + o(h) ,$$

and the expected number of neutrons of state i leaving at the left is given by

$$\delta_{ij} - \frac{\partial \omega_j^1}{\partial s_i^1} (x, \underset{\sim}{1}, \underset{\sim}{1}) + o(h) .$$

The matrices $\frac{\partial \underset{\sim}{w}^2}{\partial \underset{\sim}{s}^1}$ and $\frac{\partial \underset{\sim}{w}^2}{\partial \underset{\sim}{s}^2}$ are similarly interpreted.

We can similarly define $T_{ij}(x)$ to be the expected number of neutrons of state i leaving the rod [0, x] at the left, given that one neutron of state j enters the rod at the right. Then

$$(5.13) \qquad T_{ij}(x) = \frac{\partial \Psi_j^1}{\partial s_i^1}(x, \underset{\sim}{1}, \underset{\sim}{1}) ,$$

and again assuming that the probability of a "chain reaction" is zero, we similarly obtain the differential equation for the transmission matrix

$$(5.14) \qquad \frac{d}{dx} T_{ij} = \sum_{\ell} T_{i\ell}\left(\frac{\partial w_j^1}{\partial s_\ell^1} + \sum_k \frac{\partial w_k^2}{\partial s_\ell^1} R_{kj}\right) .$$

Finally, we remark that the algebraic equations for nonprobabilistic transport theory found in Redheffer [6] are also easily obtained from the results of Sec. 3 by taking expectations.

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DOCUMENT CONTROL DATA

1. ORIGINATING ACTIVITY THE RAND CORPORATION	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

A PRELIMINARY INVESTIGATION OF TRANSPORT-BRANCHING PROCESSES

4. AUTHOR(S) (Last name, first name, initial)

Shiffman, Bernard

5. REPORT DATE March 1966	6a. TOTAL NO. OF PAGES 35	6b. NO. OF REFS. 6
7. CONTRACT or GRANT NO. SD-79	8. ORIGINATOR'S REPORT NO. RM-4290-ARPA	
9a. AVAILABILITY/LIMITATION NOTICES DDC 1	9b. SPONSORING AGENCY Advanced Research Projects Agency	

10. ABSTRACT

A probabilistic discussion of one-dimensional multi-type neutron transport processes. The invariant imbedding technique is applied to the generating functions of the emitted neutrons. A system of simultaneous partial differential equations for these generating functions is then derived. The well-known Ricatti equation for the expected reflected flux follows from these differential equations.

11. KEY WORDS

Mathematics
Neutron transport
Transport theory
Radiative transfer
Differential equations
Ricatti equation