DISTRIBUTION OF THIS
DOCUMENT IS UNLIMITED

Best Available Copy

THE TURBULENT BOUNDARY LAYER IN A COMPRESSIBLE FLUID

Donald E. Coles*
Consultant, The RAND Corporation

P-2417

August 22, 1961

*Associate Professor of Aeronautics
California Institute of Technology

CONTENTS

THE TURBULENT BOUNDARY LAYER IN A COMPRESSIBLE FLUID

APPENDIX A

A MANUAL OF EXPERIMENTAL BOUNDARY-LAYER PRACTICE

APPENDIX E

THE RELATED ANALYTICAL LITERATURE

THE TURBULENT BOUNDARY LAYER IN A COMPRESSIBLE FLUID

Donald Coles

SUMMARY

The first object of this paper is to develop a transformation which reduces the boundary-layer equations for compressible two-dimensional mean turbulent motion to incompressible form. The second object is to apply this transformation to the special case of the adiabatic turbulent boundary layer on a smooth wall. An important difference between the present transformation and others which have been previously proposed is that the present transformation represents at every stage a genuine kinematic and dynamic correspondence between two real flows, both of which are capable of being observed experimentally. Since the mean pressure and mean velocity can then be measured in either flow, the mean acceleration of the fluid can in principle be determined, and the shearing stress can be adequately and accurately defined as the stress which is necessary to account for this acceleration. This formulation leads to a general transformation valid for laminar or turbulent flow in wakes and boundary layers, without regard to the state or energy equations or the viscosity law for the compressible fluid, and without regard to the boundary conditions on surface pressure or temperature in the event that a surface is involved.

Given a boundary-layer flow of a Newtonian fluid past a smooth wall, but with no other restrictions, it is shown that the combination $(\rho_\infty\mu_\infty/\rho_w\mu_w)C_f R_\theta$ (where $C_f = 2\tau_w/\rho_\infty u_\infty^2$ = local friction coefficient, $R_\theta = \rho_\infty u_\infty \theta/\mu_\infty$ = local momentum-thickness Reynolds number) is an invariant of the transformation. This result, which is both new and useful, is referred to as the law of corresponding stations.

The special properties of the transformation for the case of the turbulent boundary layer at constant pressure are next derived. It is found that one further assumption has to be made in order to apply the transformation to this problem. After a survey of experimental data on local surface friction in compressible flow, the assumption chosen in this paper is that the sublayer Reynolds number is unaffected by compressibility or heat transfer provided that the density and viscosity are evaluated at a mean sublayer temperature defined by the transformation. The remaining empirical quantity, the numerical magnitude of the sublayer Reynolds number, is evaluated experimentally and is found to be of the order of the least Reynolds number for which turbulent flow can be observed.

Finally, explicit formulas are obtained for the effect of Mach number on surface friction when the fluid is a perfect gas and the stagnation temperature is constant or linear in the velocity. Three novel implications of the analysis are that streamlines are not transformed into streamlines by the transformation in the case of turbulent flow, that laminar and turbulent shearing stresses are transformed by different rules, and that the ratio τ_w/p (or the product $M_\infty^2 C_f$) remains finite at large Mach numbers. An appendix contains an exhaustive critical survey of the experimental literature of low-speed turbulent flow at constant pressure, and a second appendix contains a brief critical discussion of the mean-temperature hypothesis, the laminar-film hypothesis, and other analytical ideas which are related to the present treatment of compressible flow.

I. INTRODUCTION

OUTLINE AND PROSPECTUS

In this paper I will be concerned primarily with the effect of compressibility on turbulent skin friction in adiabatic flow at constant pressure. It has become a tradition in the literature of this problem to present the final result of both analytical and experimental work in the manner of Fig. 1, where the ratio of compressible to incompressible friction coefficient is plotted against Mach number. The present study, however, will follow perhaps inadvertently another and stronger tradition. This is that research in turbulence is most respectable when it is aimed at the reduction of a given set of empirical data to a straight line—and hence to one or two empirical constants—in suitably chosen coordinates. That the problem at hand can be treated in this way is illustrated schematically by the evolution of Fig. 1 into Fig. 2 and then into Fig. 3. In fact, a description of the coordinate changes involved in going from one figure to the next will serve both as an outline and as a summary of the analysis to be given in this paper.

The real starting point of the analysis is not so much Fig. 1 as a problem which arises in the preparation of this figure from empirical data. This problem is to specify some condition connecting the local friction coefficients C_f and $\bar{C}_f$. Most authors have agreed that these coefficients should be evaluated at the same Reynolds number, without agreeing as to the particular variables, especially the length, which should make up this Reynolds number. If any exit from this difficulty exists, it is probably through the use of a transformation representing a genuine kinematic and dynamic correspondence between the two flows. In Sections I-III, therefore, such a transformation is carefully derived from first principles. Among the immediate consequences of this transformation is the desired condition, in the form of an invariant which I call the law of corresponding stations. In essence, this law states that the product $C_f R_\theta$ is the same at corresponding points in any two flows related by the transformation, provided only that the friction in both cases is Newtonian at the wall. No other important restrictions have to be put on the fluid properties or on the nature of the flows in question.

Fig. 1 having been established on relatively solid ground, the transformation next suggests a minor modification in the abscissa and a major modification in the ordinate of the figure. The change in abscissa is one which has been proposed independently by Matting, et al. (1960) as a means of reconciling their measurements in air and helium; it is from M_∞ to $(\gamma-1) M_\infty^2/2$ (or for practical purposes from M_∞ to T_w/T_∞ or to q_∞/p). This change not only removes the effect of variations in γ for fixed Prandtl number, but also makes use of the fact that the curve of T_∞/T_w against $(\gamma-1) M_\infty^2/2$ is the only analytical curve which can be plotted in Fig. 1 without a detailed knowledge of the boundary-layer flow. Finally, the change in ordinate is a simple normalization of $C_f/\bar{C}_f$ with respect to this same thermodynamic parameter T_∞/T_w. These two changes of coordinate taken together lead to Fig. 2, in which the emphasis is on the scatter of the data due to variations in Reynolds number from one experiment to another.

At this stage the inferences which can be drawn solely from the transformation and from measurements of gross properties of the boundary layer, such as the local surface friction, are exhausted. More information is needed, and I assume in Section IV that this information can be derived from the similarity law known as the law of the wall. The physical interpretation given to this law is that the sublayer flow in a turbulent boundary layer is characterized by a constant Reynolds number. An argument based on the transformation and on the data of Fig. 2 then suggests that this Reynolds number might be unaffected by compressibility if the density and viscosity are evaluated at a certain mean temperature defined by the transformation. Given this assumption, which I call the substructure hypothesis, it is found that the ordinate in Fig. 2 does not depend explicitly on conditions in the free stream, but only on conditions at or very near the wall. In order for the abscissa to have the same property, it suffices to change the independent variable from $(\gamma-1)\,M_\infty^2/2$ to $(\gamma-1)\,M_\infty^2\,C_f/2$ (or for practical purposes from q_∞/p to τ_w/p). The final result, as shown in Fig. 3, is that the dependence of the experimental data in Fig. 2 on Reynolds number is very nearly accounted for.

For the sake of brevity the material of this summary has been highly abridged, and several essential arguments have been oversimplified or omitted altogether. For example, I have implied that the experimental data in Fig. 3 define a straight line. Strictly speaking, this statement is correct only if the fluid is a perfect gas, if the viscosity is proportional to temperature, and if the stagnation temperature of the flow is constant. The analysis in the text, however, is not always limited to this special case, or even to the case of flow at constant pressure. On the contrary, the central element of the analysis at every point is a transformation which is remarkably free of restrictive conditions. Many of the relationships of this paper, therefore, may eventually be applicable to problems which have been less thoroughly studied experimentally than the classical problem of adiabatic flow at constant pressure past a smooth solid surface.

STATEMENT OF THE PROBLEM

My first objective in this paper will be to find conditions under which the boundary-layer equations for compressible flow can be reduced to incompressible form by a suitable transformation. My second objective will be to apply the transformation in question to the turbulent boundary layer. Neither problem is new, and almost all of the results to be derived for laminar flow have been obtained previously by Stewartson (1949) and others. More recently, several writers have attempted to extend the concept of a transformation to turbulent flow as well. However, little attention has so far been paid to several important issues raised by the latter application. Among these issues are the question of the complete identification of corresponding points and the question of the use of dimensionless similarity laws to describe the profiles of mean temperature and mean velocity. In order to avoid compromising these and other issues in advance, I propose to develop the transformation from first principles by means of arguments which are as rigorous and as general as I can make them.

The first problem stated in brief is this. The boundary-layer equations of mean motion for a compressible fluid of variable density $\rho(x,y)$ are given in the form

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1.1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \tag{1.2}$$

It is desired to find a formal transformation from $(x, y, u, v, p, \rho, \tau)$ to $(\bar{x}, \bar{y}, \bar{u}, \bar{v}, \bar{p}, \bar{\rho}, \bar{\tau})$ such that the original equations (1.1)-(1.2) are reduced to the form

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1.3}$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{d\bar{p}}{d\bar{x}} + \frac{\partial \bar{\tau}}{\partial \bar{y}} \tag{1.4}$$

with the parameter $\bar{\rho}$ independent of position in the barred flow.

The validity of these equations will not be examined here.* They evidently incorporate the usual boundary-layer approximation, in that the stresses in the fluid have been taken to be adequately represented by the static pressure p (or $\bar{p}$) and by a single additional scalar component τ (or $\bar{\tau}$). However, the relationship of the latter quantity to the other variables of the problem has deliberately been left unspecified. Furthermore, no state or energy equation has been included for the compressible flow. This formulation of the problem, while incomplete, has the advantage of being relatively free of assumptions which are either controversial or unduly restrictive. The present partial formulation of the problem also recognizes the fact that the transformations for laminar and turbulent flow can be expected to have certain properties in common. These properties, to the extent that they can be inferred from the basic equations (1.1)-(1.4) without invoking the concepts of turbulent shearing stress and turbulent heat transfer, are therefore likely to have a relatively permanent significance.

---

*There is always the possibility that the usual boundary-layer approximation for the stress tensor may fail under certain conditions. In addition, there is room for argument about the importance in high-speed turbulent flows of various correlation terms (involving fluctuations in density, temperature, or viscosity) which are neglected here in order to treat the mean value of a product such as ρu or ρT (but not necessarily ρv) as a product of mean values.

PHYSICAL CONSIDERATIONS

In what follows I will make frequent use of a physical principle whose importance does not seem to have been fully recognized in previous work on the transformation. A brief discussion will show both the need for such a principle and the way in which it is to be applied.

Suppose that a general transformation between Eqs. (1.1)-(1.2) and (1.3)-(1.4) has been established. Suppose further that the original compressible flow is characterized in some region by essentially constant density. This might be the case near the stagnation point of a blunt body, in the upstream part of a nozzle flow, or far downstream in a submerged jet. In such a region the compressible equations can be reduced directly (i.e., without reference to the transformation) to incompressible form by the usual limiting process of retaining only the leading terms in expansions of appropriate dimensionless variables in powers of M^2. The resulting equations will be identical in form with (1.3)-(1.4), provided that the density in the latter is evaluated at the appropriate stagnation condition in the original flow. A question then arises: under these conditions, is it necessary for the transformation itself to reduce to the identity transformation?

In most of the previous work on the problem it has been tacitly assumed, e.g. by taking the reservoir conditions for the two flows to be the same, that the answer to this question is yes. However, it then becomes necessary to suppose that the same fluid can behave entirely differently in the two flows—like a perfect gas (say) in one case, but like an incompressible liquid in the other. Whatever else may be said about this situation, it is unlikely that both flows can be observed experimentally. At the same time, it is certain that an experimental description of both flows will be an essential part of any useful comparison in the case of turbulent flow.

The way out of this dilemma is to avoid the presumption that the transformation should reduce to the identity transformation in a region where both flows are incompressible—or anywhere else, for that matter. On the contrary, it is more reasonable to suppose that the transformation should reduce instead to an affine transformation of the kind commonly used to investigate the question of similarity. This being so, it must be possible to choose the density $\bar{\rho}$ and the viscosity $\bar{\mu}$ for the barred flow quite arbitrarily. To this end I will adopt as fundamental the proposition that the two flows represented by Eqs. (1.1)–(1.2) and by (1.3)–(1.4) must be treated in every respect as real physical flows, capable of being observed experimentally. It is a natural inference that corresponding variables should have the same physical dimensions and that the external and boundary conditions should be at least qualitatively the same in both problems. This principle will have several useful consequences, of which the first will be to allow the introduction of the parameter $\bar{\rho}$ on purely dimensional grounds.

Finally, the point has sometimes been raised in private discussions that an application of the transformation to turbulent flow may involve unwarranted assumptions about the effect of compressibility on the mechanism of turbulence. That this need not be the case can be seen from an argument based in part on the physical principle just presented, but also in part on the fact that only one component of the stress tensor besides the pressure is present in the equations of motion considered here.

In any boundary-layer flow described by either of the momentum equations

(1.3) or (1.4), the accelerations and the pressure forces are well defined quantities which can be evaluated experimentally. The unbalance between these quantities may simply be taken to define an apparent shearing stress, and numerical values for the latter may then be inferred from measurements of velocity and pressure. It is clearly irrelevant, for example, that this apparent stress may have a turbulent part which can be independently defined in terms of certain velocity fluctuations capable of being measured directly. For if a transformation can be found to transform the acceleration and pressure terms in the equations, and if this transformation is required to be physically realistic, then it must follow without reference to any special definition of the shearing stresses that these stresses can also be transformed by the transformation. This argument may seem to be academic, but I believe that it is the only one which can be relied on to provide an explicit transformation for the undefined shearing-stress terms in Eqs. (1.1)-(1.4), and it will be used for this purpose in the next section.

II. THE FORMAL TRANSFORMATION

THE CONTINUITY EQUATION

I will begin the development of the transformation by considering two stream functions ψ and $\bar{\psi}$, each of which will be interpreted physically as an integral of mass rate rather than volume rate of flow. The two continuity equations (1.1) and (1.3) will be satisfied by introducing these stream functions in the usual way,

$$\rho u = \frac{\partial \psi}{\partial y} \quad , \quad \rho v = -\frac{\partial \psi}{\partial x} \tag{2.1}$$

$$\bar{\rho}\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}} \quad , \quad \bar{\rho}\bar{v} = -\frac{\partial \bar{\psi}}{\partial \bar{x}} \tag{2.2}$$

where the physical principle just discussed requires an unspecified constant density $\bar{\rho}$ to be included in the second pair of equations in order to provide the same physical dimensions for $\bar{\psi}$ as for ψ.

The two stream functions are evidently constant on streamlines of their respective flows. In previous work on the transformation it has usually been assumed that these functions are the same at corresponding points and hence that streamlines in one flow are transformed into streamlines in the other. For reasons which will become apparent (see the remarks following Eq. 3.17), I do not want to make this assumption. Instead, I will leave open the relationship between ψ and $\bar{\psi}$ by writing

$$\bar{\psi}(\bar{x},\bar{y}) = \sigma(x,y)\ \psi(x,y) \tag{2.3}$$

where σ is a completely unspecified function of x and y.

THE TRANSPORT TERMS

The formal rule for the transformation of a derivative from coordinates $(\bar{x}, \bar{y})$ to coordinates (x, y) may now be applied to the stream function $\bar{\psi}$ and the result expressed in terms of velocities with the aid of Eqs. (2.1) and (2.2). It is found that at corresponding points in the two flows

$$\rho u = \frac{1}{\sigma}\left(\bar{\rho}\bar{u}\,\frac{\partial \bar{y}}{\partial y} - \bar{\rho}\bar{v}\,\frac{\partial \bar{x}}{\partial y}\right) - \frac{\psi}{\sigma}\,\frac{\partial \sigma}{\partial y} \qquad (2.4)$$

$$\rho v = \frac{1}{\sigma}\left(\bar{\rho}\bar{v}\,\frac{\partial \bar{x}}{\partial x} - \bar{\rho}\bar{u}\,\frac{\partial \bar{y}}{\partial x}\right) + \frac{\psi}{\sigma}\,\frac{\partial \sigma}{\partial x} \qquad (2.5)$$

Further differentiation and substitution in the transport terms on the left-hand side of Eq. (1.2) yields, after some extremely tedious algebra, an identity which is given in full for the record:

$$
\begin{aligned}
\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = {} & \left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right) \frac{\bar{\rho}^{2}}{\sigma^{2}} J(\bar{x}, \bar{y}) \frac{1}{\rho} \frac{\partial \bar{y}}{\partial y} \\
& - \left(\bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \frac{\bar{\rho}^{2}}{\sigma^{2}} J(\bar{x}, \bar{y}) \frac{1}{\rho} \frac{\partial \bar{x}}{\partial y} \\
& + \bar{u}^{2} \frac{\bar{\rho}^{2}}{\sigma} J\left(\frac{1}{\sigma\rho} \frac{\partial \bar{y}}{\partial y}, \bar{y} \right) \\
& - \bar{u}\bar{v} \frac{\bar{\rho}^{2}}{\sigma} \left[J\left(\frac{1}{\sigma\rho} \frac{\partial \bar{x}}{\partial y}, \bar{y} \right) + J\left(\frac{1}{\sigma\rho} \frac{\partial \bar{y}}{\partial y}, \bar{x} \right) \right] \\
& + \bar{v}^{2} \frac{\bar{\rho}^{2}}{\sigma} J\left(\frac{1}{\sigma\rho} \frac{\partial \bar{x}}{\partial y}, \bar{x} \right) \\
& + \bar{v} \frac{\partial \bar{u}}{\partial \bar{x}} \frac{\bar{\rho}}{\sigma^{3}} J(\sigma, \bar{x}) \frac{1}{\rho} \frac{\partial \bar{y}}{\partial y} \\
& + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \frac{\bar{\rho}}{\sigma^{3}} J(\sigma, \bar{y}) \frac{1}{\rho} \frac{\partial \bar{y}}{\partial y} \\
& - \bar{v} \frac{\partial \bar{v}}{\partial \bar{x}} \frac{\bar{\rho}}{\sigma^{3}} J(\sigma, \bar{x}) \frac{1}{\rho} \frac{\partial \bar{x}}{\partial y} \\
& - \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \frac{\bar{\rho}}{\sigma^{3}} J(\sigma, \bar{y}) \frac{1}{\rho} \frac{\partial \bar{x}}{\partial y} \\
& - \bar{v}\bar{u} \frac{\bar{\rho}}{\sigma^{3}} \left[J\left(\frac{1}{\rho} \frac{\partial \bar{y}}{\partial y}, \sigma \right) + \sigma J\left(\frac{1}{\sigma\rho} \frac{\partial \sigma}{\partial y}, \bar{y} \right) \right] \\
& + \bar{v}\bar{v} \frac{\bar{\rho}}{\sigma^{3}} \left[J\left(\frac{1}{\rho} \frac{\partial \bar{x}}{\partial y}, \sigma \right) + \sigma J\left(\frac{1}{\sigma\rho} \frac{\partial \sigma}{\partial y}, \bar{x} \right) \right] \\
& + \bar{v}^{2} \frac{1}{\sigma^{2}} J\left(\frac{1}{\sigma^{2}\rho} \frac{\partial \sigma}{\partial y}, \sigma \right)
\end{aligned}
\tag{2.5}
$$

where $J(A,B)$ denotes the Jacobian $\partial(A,B)/\partial(x,y)$. In vector notation $\vec{k}\, J(A,B) = \text{grad}\, A \times \text{grad}\, B$, with $\vec{k}$ the unit vector normal to the xy plane. Hence $J(A,B)$ vanishes if A or B is a constant or more generally if A is a function of B alone.

The terms on the right-hand side of Eq. (2.6) have been arranged in descending order with respect to derivatives of $\bar{v}$. At least the first term must appear in the transformed momentum equation (1.4). The other terms are for the most part expendable, and the next step is to reduce Eq. (2.6) to manageable proportions by suitably restricting the dependence of the functions σ, $\bar{x}$, and $\bar{y}$ on x and y.

For this purpose it is instructive to look at the behavior of the various terms in (2.6) outside a shear layer in the barred coordinates. Suppose first that there is a pressure gradient in the orginal flow, so that the left-hand side of (2.6) is non-zero but bounded for large values of y. If the regions outside the two shear flows are to correspond both physically and formally, there should also be a pressure gradient in the barred flow, and the right-hand side of (2.6) should be non-zero but bounded for large values of $\bar{y}$. Now as far as the factors $\bar{v}$, $\bar{u}$, $\partial\bar{u}/\partial\bar{y}$, etc., are concerned, it appears that the term in $\bar{v}\, \partial\bar{u}/\partial\bar{y}$ will ordinarily vanish for large values of $\bar{y}$ and that the terms in $D\bar{u}/D\bar{t}$ and $\bar{u}^2$ will become at most functions of $\bar{x}$ only, while the remaining terms will behave either like $\bar{y}$ or like $\bar{y}^2$. The simplest condition which will assure the proper behavior of the right-hand side of (2.6) is the requirement that each of the terms behaving like $\bar{y}$ or like $\bar{y}^2$ should vanish identically. This condition in turn is easily shown to be equivalent* to requiring each of the quantities

---

*An identical result is obtained if the flow at infinity is assumed to be a uniform stream, but the condition on $\bar{y}$ seems to be unnecessary if the

ρ, $\bar{x}$, and $\partial\bar{y}/\rho\partial y$ to be independent of y.

THE MOMENTUM EQUATION

At this point the transformation, except possibly for the jet, can be represented in the form

$$\frac{\bar{\Psi}}{\Psi} = \sigma(x) \tag{2.7}$$

$$\frac{d\bar{x}}{dx} = \xi(x) \tag{2.8}$$

$$\frac{\bar{\rho}}{\rho}\frac{\partial\bar{y}}{\partial y} = \eta(x) \tag{2.9}$$

where ξ and η, like σ, are dimensionless functions of x yet to be determined. Note that a second constant reference density is needed in (2.9) to give the proper physical dimension to the transformed variable $\bar{y}$, but that this parameter can be identified with $\bar{\rho}$ in Eq. (2.2) and any difference absorbed in $\eta(x)$.

The dependent variables $\bar{u}$ and $\bar{v}$ as derived from Eqs. (2.4) and (2.5) are now

$$\bar{u} = \frac{\sigma}{\eta} u \tag{2.10}$$

---

flow is bounded at infinity by fluid at rest. A closer examination of this case, i.e., of the jet entering a stagnant fluid, will not be undertaken here. Even for the other cases mentioned the transformation in question may not be the most general or the most useful one which can be found, firstly because the sufficient conditions used to bound the right-hand side of Eq. (2.6) for large $\bar{y}$ have not been shown to be necessary, and secondly because these conditions have been applied not only for large $\bar{y}$ but throughout the barred flow. Furthermore, it is not obvious that the two flows involved in the transformation must always have the same character; i.e., that if one flow is a wake, so is the other; if one flow is a boundary layer with pressure gradient and mass transfer, so is the other, and so on. On the other hand, the transformation considered here is more general than any which has been considered previously, and has several properties which will prove to be useful in any study of turbulent flow. The possibility of a more general transformation will therefore not be investigated further.

$$\bar{v} = \frac{\rho\sigma}{\bar{\rho}\xi} v + \frac{\sigma}{\xi\eta} u \frac{\partial \bar{y}}{\partial x} - \frac{\psi}{\bar{\rho}\xi} \frac{d\sigma}{dx} \tag{2.11}$$

and Eq. (2.6) can be written

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right)\frac{\rho}{\bar{\rho}}\frac{\xi\eta^2}{\sigma^2}$$

$$+ \frac{\rho u^2}{\eta/\sigma}\frac{d\eta/\sigma}{dx} + \frac{\psi}{\sigma}\frac{\partial u}{\partial y}\frac{d\sigma}{dx} \tag{2.12}$$

$$= -\frac{dp}{dx} + \frac{\partial \tau}{\partial y}$$

Of the three terms in (2.12) generated by the transformation of the transport terms in the original momentum equation (1.2) subject to the conditions (2.7)-(2.9), the first provides the transport terms in the transformed momentum equation (1.4). The other two terms, those in ρu^2 and $\psi\, \partial u/\partial y$, will therefore have to be suitably combined with the terms dp/dx and $\partial\tau/\partial y$.

Consider now a companion equation to (2.12), obtained from the latter by solving for the transport terms in barred coordinates and using (1.4);

$$\bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \frac{\bar{\rho}}{\rho}\frac{\sigma^2}{\xi\eta^2}\left(-\frac{dp}{dx} + \frac{\partial\tau}{\partial y} - \frac{\rho u^2}{\eta/\sigma}\frac{d\eta/\sigma}{dx} - \frac{\psi}{\sigma}\frac{\partial u}{\partial y}\frac{d\sigma}{dx}\right)$$

$$= -\frac{d\bar{p}}{d\bar{x}} + \frac{\partial\bar{\tau}}{\partial\bar{y}} \tag{2.13}$$

The second equality in Eq. (2.13) is a relationship involving primarily pressures and shearing stresses. The right-hand side of this equality consists of two terms, $d\bar{p}/d\bar{x}$ and $\partial\bar{\tau}/\partial\bar{y}$. According to the physical principle underlying the present transformation, the first of these terms must depend only on $\bar{x}$ (or x), and the second must vanish for large $\bar{y}$ (or y). It is therefore necessary to write the left-hand side in the same fashion, and it follows immediately that the transformed quantities $\bar{p}$ and $\bar{\tau}$ have to be defined by

$$\frac{d\bar{p}}{d\bar{x}} = \frac{\bar{\rho}\sigma^2}{\xi\eta^2}\left(\frac{1}{\rho_\infty}\frac{dp}{dx} + \frac{u_\infty^2}{\eta/\sigma}\frac{d\eta/\sigma}{dx}\right) \tag{2.14}$$

and by

$$\frac{\partial\bar{\tau}}{\partial\bar{y}} = \frac{\bar{\rho}\sigma^2}{\xi\,\eta^2}\left[\frac{1}{\rho}\left(\frac{\partial\tau}{\partial y} - \frac{\xi}{\sigma}\frac{\partial u}{\partial y}\frac{d\sigma}{dx}\right) + \frac{dp}{dx}\left(\frac{1}{\rho_\infty} - \frac{1}{\rho}\right) + \frac{1}{\eta/\sigma}\frac{d\eta/\sigma}{dx}\left(u_\infty^2 - u^2\right)\right] \qquad (2.15)$$

respectively.

The slightly awkward form of the last two relationships is justified by their generality. Neither an equation of state nor an energy equation for the compressible flow has yet been cited. Furthermore, it has not been necessary to define the shearing stresses τ and $\bar{\tau}$ explicitly, and no conditions have been imposed which require the presence of a solid boundary. The fact that the form of the transformation can be established under such general conditions is due largely to the emphasis which is put here on physical considerations. In particular, the argument relating to Eqs. (2.12) - (2.15) can be recognized as a repetition of the argument outlined at the end of Section I in connection with the irrelevance of the shearing-stress mechanism.

However, the transformation is not in a useful form at this juncture, inasmuch as nothing is known about the properties of the three scaling functions $\sigma(x)$, $\xi(x)$, and $\eta(x)$ which appear in Eqs. (2.7) - (2.9). In the next section I will show that some information about these functions can be obtained at a quite reasonable cost measured in terms of generality. First, however, I want to demonstrate one important and useful property of the transformation for the special case of a flow bounded by a smooth wall.

THE LAW OF CORRESPONDING STATIONS

So far there has been neither need nor opportunity to introduce a viscosity $\bar{\mu}$ for the barred flow. An obvious and natural means for introducing this parameter is the use of the condition $\bar{\tau} = \bar{\mu}\,\partial\bar{u}/\partial\bar{y}$ for Newtonian friction, provided of course that this condition is appropriate. In the case of a free turbulent flow such as a jet or wake, or in the case of a boundary layer or a rough wall, the condition is evidently not appropriate. The viscosity is in a certain sense an artificial parameter, and the Reynolds number plays at most a secondary role in the description of the mean motion. When these cases are excluded there remains the case of a boundary layer on a smooth wall, and here the viscosity clearly plays a primary role.

Consider such a flow, with the wall taken for convenience to be at $y = \bar{y} = 0$. Whether the flow is laminar or turbulent, and whether or not there is mass transfer through the surface, the conditions $\bar{\tau}_w = \bar{\mu}(\partial\bar{u}/\partial\bar{y})_w$ and $\tau_w = (\mu\partial u/\partial y)_w$ can be expected to hold at the wall. The transformations (2.9) and (2.10) for u and y then require $\bar{\tau}_w$ and τ_w to be related by

$$\bar{\tau}_w = \frac{\bar{\rho}\bar{\mu}}{\rho_w\mu_w}\frac{\sigma}{\eta^2}\tau_w \tag{2.16}$$

Local friction coefficients C_f and $\bar{C}_f$, defined in the usual way, are then found to satisfy

$$C_f = \frac{2\tau_w}{\rho_\infty u_\infty^2} = \text{[illegible]} = \frac{\sigma\mu_w}{\bar{\mu}}\frac{\rho_w}{\rho_\infty}\bar{C}_f \tag{2.17}$$

where the subscript ∞ refers to the external flow. Upon introducing a momentum thickness

$$\theta = \int_0^\infty \frac{\rho u}{\rho_\infty u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy \tag{2.18}$$

and a corresponding thickness $\bar{\theta}$, it appears that

$$\bar{\theta} = \int_0^\infty \frac{\bar{u}}{\bar{u}_\infty}\left(1 - \frac{\bar{u}}{\bar{u}_\infty}\right)d\bar{y} = \frac{\rho_\infty}{\bar{\rho}}\eta\theta \tag{2.19}$$

Finally, Reynolds numbers based on θ and $\bar{\theta}$ may be defined in the usual way and connected by

$$R_\theta = \frac{\rho_\infty u_\infty\theta}{\mu_\infty} = \frac{\bar{\mu}}{\sigma\mu_\infty}\frac{\bar{\rho}\bar{u}_\infty\bar{\theta}}{\bar{\mu}} = \frac{\bar{\mu}}{\sigma\mu_\infty}\bar{R}_{\bar{\theta}} \tag{2.20}$$

where $\bar{\mu}$ in the last equation is again to be interpreted as the viscosity of the barred fluid. When σ is eliminated between (2.17) and (2.20), it is seen that

$$\bar{C}_f \bar{R}_{\bar{\delta}} = \frac{\rho_\infty \mu_\infty}{\rho_w \mu_w} C_f R_\delta \qquad (2.21)$$

and this is the desired result. The relationship (2.21), which seems to have been overlooked in previous work on the transformation, will be called the law of corresponding stations for the boundary layer on a smooth wall. A special case of this law is well known in the theory of similar solutions for the laminar boundary layer. However, I want to emphasize that Eq. (2.21) has been derived here by considering the structure of the incomplete equations of motion rather than the behavior of certain special solutions of these equations. This law will obviously be of considerable value in any use of experimental data to test the validity of the present transformation for the case of turbulent flow.

III. THE THREE SCALING FUNCTIONS σ, ξ, AND η

LAMINAR FLOW

The remaining step in completing the transformation is the specification of the three scaling functions σ, ξ, and η which determine the transformation of the stream function ψ and the coordinates x and y. These functions will be studied first for the laminar case, with the object of providing at least a qualitative model for the later discussion of turbulent flow.

If the flow is laminar the shearing stresses τ and $\bar{\tau}$ are no longer unspecified, being given by $\mu \partial u/\partial y$ and by $\bar{\mu}\partial\bar{u}/\partial\bar{y}$ throughout the two flows related by the transformation. It follows that Eq. (2.15) is valid without the subscript w,

$$\bar{\tau} = \frac{\bar{\rho}\bar{\mu}}{\rho\mu}\frac{\sigma}{\eta^2}\tau \tag{3.1}$$

and therefore that

$$\frac{\partial\bar{\tau}}{\partial\bar{y}} = \frac{\bar{\rho}^2\bar{\mu}\sigma}{\rho^2\mu\eta^3}\left(\frac{\partial\tau}{\partial y} - \frac{\tau}{\rho\mu}\frac{\partial\rho\mu}{\partial y}\right) \tag{3.2}$$

This indirect transformation (3.2) for $\partial\bar{\tau}/\partial\bar{y}$, which is a consequence of the separate transformations for u and y, must be consistent with the direct and more general transformation (2.15), which does not assume Newtonian friction. The simplest way to assure consistency, although perhaps not the only one, is to require the two expressions for $\partial\bar{\tau}/\partial\bar{y}$ to be identical. This will certainly be the case if

$$\frac{d\sigma}{dx} = 0 \tag{3.3}$$

$$\frac{\partial \rho\mu}{\partial y} = 0 \tag{3.4}$$

and

$$\frac{dp}{dx}\left(\frac{1}{\rho_\infty} - \frac{1}{\rho}\right) + \frac{1}{\eta/\sigma}\frac{d\eta/\sigma}{dx}\left(u_\infty^2 - u^2\right) = 0 \tag{3.5}$$

in which case it also follows that

$$\frac{\ell}{\sigma_1} = \frac{\rho\mu}{\bar{\rho}\bar{\mu}} = \frac{\rho_w\mu_w}{\bar{\rho}\bar{\mu}} = \frac{\rho_\infty\mu_\infty}{\bar{\rho}\bar{\mu}} \tag{3.6}$$

The condition (3.4) will evidently be satisfied for a perfect gas having the usual equation of state

$$p = \rho RT \tag{3.7}$$

with $R = c_p - c_v$ = constant, if in addition the viscosity is proportional to the temperature. The condition (3.5) will be automatically satisfied if there is no pressure gradient in either flow, since Eqs. (2.14) and/or (2.10) then require the ratio η/σ to be independent of x. More generally, however, condition (3.5) can be rewritten after a separation of variables as

$$\frac{2}{\eta/\sigma}\frac{d\eta/\sigma}{dp} = \frac{\frac{1}{\rho} - \frac{1}{\rho_\infty}}{\frac{u_\infty^2}{2} - \frac{u^2}{2}} = c(x) \tag{3.8}$$

(say). If the compressible fluid is a perfect gas, the second equality immediately suggests that $c(x)$ should be taken as $R/c_p p$. Eq. (3.8) then yields the two conditions

$$\frac{u_\infty^2}{2} + c_p T_\infty = c_p T_{o_\infty} = \frac{u^2}{2} + c_p T = c_p T_o = \text{constant} \tag{3.9}$$

and

$$\frac{\eta^2}{\sigma^2} = \frac{T_\infty}{T} \tag{3.10}$$

where $\bar{T}$ in the second equation denotes a constant of integration with the dimensions of temperature. Both of the last two expressions make indirect use of the condition $p\rho_{\infty}^{-\gamma}$ = constant for an isentropic external flow, with $\gamma = c_p/c_v$ = constant.

As might be expected, these results for laminar flow are simply a restatement of the well-known conditions (perfect gas, μ/T constant, p or T_o constant) under which the momentum and continuity equations can be uncoupled from the energy equation and integrated separately in barred coordinates.* The problem of the laminar boundary layer was first approached from this direction by Dorodnitsyn (1942 a,b). In Dorodnitsyn's formulation, however, the omission of the scaling function which I have called $\eta(x)$ led to the appearance of a factor T/T_{∞} multiplying the transformed pressure-gradient term, so that the reduction to incompressible form was achieved only for the case η = constant or dp/dx = 0. Cope and Hartree (1948), in the course of some research in a different direction, independently proposed a partial transformation in the spirit of (2.9) for the normal coordinate y. Howarth (1948), again independently, also introduced a partial transformation equivalent to (2.9), but with $\eta(x)$ proportional to $1/\sqrt{p(x)}$ rather than to

---

*The transformation of the present paper has been designed specifically for turbulent flow, and strict attention has been paid from the beginning to the physical principle that it must be possible to observe experimentally any flow whose properties are supposed to be known. This principle, which makes it essential to keep in close touch with the physical variables and with the conservation laws connecting them, is most appropriate for problems which cannot be treated by analytical means. For problems which can be so treated, on the other hand, almost any transformation or change of variable is acceptable if it leads to equations which are mathematically more tractable than the equations of motion in their original form. The principle involved in such cases is a mathematical principle, however, not a physical one, and may be as much out of place in discussions of turbulence as is the physical principle in most discussions of laminar flow.

An example of the use (and misuse) of the alternative mathematical principle is supplied by the transformation first proposed by Stewartson

$\sqrt{T_\infty(x)}$. Howarth also elected to preserve the streamwise velocity component u rather than the stream function ψ at corresponding points, and the pressure-gradient term again emerged in quite awkward form in the final equation for the transformed stream function. Stewartson (1949), in continuing Howarth's work, chose to preserve the stream function instead and to use a suitable scaling function $\sigma(x)$, and thus arrived at a laminar transformation equivalent to the transformation considered here. Minor differences include Stewartson's use of boundary-layer variables $y/\sqrt{\nu}$ and $v/\sqrt{\nu}$ (but not $\psi/\sqrt{\nu}$), and his early consolidation of the various scaling functions. The general conditions under which the compressible equations of motion can be reduced to incompressible form were also discovered simultaneously by Illingworth (1949), working in von Mises coordinates x, ψ rather than in physical coordinates x, y.

TURBULENT FLOW AT CONSTANT PRESSURE

If the flow is turbulent, the arguments leading to Eqs. (3.3) and (3.6) have to be discarded and a fresh beginning made. For the present I will consider only flow at constant pressure, for which the general

---

(1949) and later cited for turbulent flow by some writers. The transformation in question may be obtained by simply dropping the subscript ∞ in Eqs. (2.14) and (2.15) of Section II. For a perfect gas the quantity in parenthesis in Eq. (2.14) can then be written as $RT_o\,dp/pdx$, where T_o is the local stagnation temperature. In this formulation the principle of physical existence for the barred flow seems to require T_o = constant if $dp/dx \neq 0$ as before, whether the flow is laminar or turbulent. However, this requirement is entirely without force for the turbulent case, because the physical principle is being applied one stage too late in the argument. Even for laminar flow the requirement in question is largely irrelevant, and Stewartson's transformation has been used (e.g., by Cohen and Reshotko 1955, Coles 1957) to obtain physically useful similarity solutions for laminar flow with heat transfer and pressure gradient, in spite of the fact that the transformed equations do not correspond to any flow which can be observed experimentally.

transformation (2.15) for $\partial\tau/\partial y$ can be integrated in the relatively simple form

$$\frac{\eta}{\sigma^2}\,\bar{\tau} = \tau + \frac{1}{\sigma}\frac{d\sigma}{dx}\int_y^\infty \psi\,\frac{\partial u}{\partial y}\,dy \tag{3.11}$$

At the wall this expression becomes

$$\frac{\eta}{\sigma^2}\,\bar{\tau}_w = \tau_w + \frac{1}{\sigma}\frac{d\sigma}{dx}\left(\rho_\infty u_\infty^2\,\theta + \psi_w u_\infty\right) \tag{3.12}$$

where the notation makes use of an identity

$$\int_0^\infty \psi\,\frac{\partial u}{\partial y}\,dy = \rho_\infty u_\infty^2\,\theta + \psi_w u_\infty \tag{3.13}$$

which can be obtained from the definition (2.18) for θ by putting $\partial\psi/\partial y$ for ρu and integrating by parts. These relationships are independent of the state and energy equations and the viscosity law, and can be applied to rough as well as to smooth surfaces as long as τ_w and $\bar{\tau}_w$ are interpreted as tangential force per unit area.

For the particular case of a smooth wall, the shearing stresses at $y = 0$ must also be connected by Eq. (2.16),

$$\frac{\rho_w\mu_w}{\bar{\rho}\bar{\mu}}\,\frac{\eta^2}{\sigma}\,\bar{\tau}_w = \tau_w \tag{3.14}$$

Moreover, if there is no mass transfer the combination $\psi_w\,d\sigma/dx$ in (3.12) may be taken as zero (cf. Eq. 2.11 evaluated at $y = 0$). Eqs. (3.12) and (3.14), together with the momentum-integral equation $\tau_w = \rho_\infty u_\infty^2\,d\theta/dx$, then imply

$$\frac{\xi}{\sigma\eta}\,\frac{\bar{\rho}\bar{\mu}}{\rho_w\mu_w} - 1 = \frac{\rho_\infty u_\infty^2\,\theta}{\tau_w\,\sigma}\,\frac{d\sigma}{dx} = \frac{\theta}{\sigma}\,\frac{d\sigma}{d\theta} \tag{3.15}$$

This expression provides almost the last point of contact between the analyses for laminar and for turbulent flow. For the laminar case the two sides of Eq. (3.15) vanish identically, by virtue of the condition

$d\sigma/dx = 0$ or σ = constant which was originally imposed by requiring Newtonian friction away from the wall. For the turbulent case, on the other hand, the experimental evidence to be cited shortly implies that $d\sigma/d\theta$ is not zero, so that Eq. (3.15) is no longer a trivial identity but a genuine condition connecting the three scaling functions with each other and with the ostensible data of the problem. This single condition for turbulent flow, moreover, differs from the corresponding two conditions (3.3) and (3.6) for laminar flow in one important respect, since Eq. (3.15) involves not only quantities like u_w and u_∞ which are usually regarded as part of the given data for a particular problem, but also quantities like $\tau_w(x)$ and $\theta(x)$ which are usually regarded as part of the solution.

Finally, if σ is in fact a function of x for a turbulent boundary layer at constant pressure, it follows from an argument inverse to the one employed for laminar flow that τ does not transform like $\mu \partial u/\partial y$ except at the wall. Consequently there seems to be no reason in the case of turbulent flow to require the compressible fluid to be a perfect gas or to require the viscosity to be proportional to temperature. Neither is there any reason to suppose that the energy equation plays any significant part in the direct transformation, at least when the pressure gradient is zero.

THE FUNCTION $\sigma(x)$ FOR TURBULENT FLOW

I now want to consider briefly some experimental investigations which have recently been carried out in adiabatic turbulent boundary layers in supersonic flow at constant pressure. The discussion will be limited to experiments which include measurements of local surface friction as well as measurements of local mean-velocity distribution. That the number of such experiments is by this time usefully large is evident from Fig. 4, in which are collected all of the local friction data so far obtained in air.

In what follows I will overlook any slight discrepancies between the data of various observers in Fig. 4, and will assume that there is a unique relationship among the three parameters C_f, R_θ, and M_∞, at least for air at typical wind-tunnel temperatures. The main objective of this paper is to determine the relationship in question. The immediate problem, however, is to determine empirically the value of the combination $\bar{\mu}/\sigma\mu_w$, which is expressed in terms of readily measured quantities by Eq. (2.17),

$$\frac{\bar{\mu}}{\sigma\mu_w} = \frac{\rho_w}{\rho_\infty}\frac{\bar{C}_f}{C_f} \tag{3.16}$$

The local friction coefficients C_f and $\bar{C}_f$ in Eq. (3.16) are supposedly connected by the law of corresponding stations,

$$\bar{C}_f R_{\bar{\theta}} = \frac{\rho_\infty \mu_\infty}{\rho_w \mu_w} C_f R_\theta \tag{3.17}$$

and the coordinates of Fig. 4 have been chosen accordingly.* In order to use the expression (3.17) to determine $\bar{C}_f$ when M_∞, C_f, R_θ, and the fluid properties are known, it is necessary to know the dependence of $\bar{C}_f$ on $R_{\bar{\theta}}$ (or on $\bar{C}_f R_{\bar{\theta}}$) in incompressible flow, especially at the lower Reynolds numbers. This dependence is also shown in Fig. 4. Although the curve drawn to represent the incompressible data can be described analytically with the aid of certain similarity laws, this curve should be viewed for the present as a strictly empirical result, and the calculations involving Eqs. (3.16) and (3.17) should be viewed as nothing more complicated that a comparison of one set of experimental data with another.

---

*Although Fig. 4 is equivalent to Fig. 1 of the introduction, the earlier figure emphasizes the dependence of skin friction on Mach number, whereas Fig. 4 emphasizes the dependence on Reynolds number. In a different sense Fig. 5 is equivalent to Fig. 2, since the coordinates in one figure are essentially reciprocals of the coordinates in the other. In this case

When the parameter $\bar{\mu}/\sigma\mu_w$ determined with the aid of Eq. (3.16) is plotted (say) against the temperature ratio T_∞/T_w, as shown in Fig. 5, it is found that the data at a given Mach number show a weak but definite dependence on Reynolds number, quite apart from unavoidable experimental scatter. The necessary conclusion is that the combination $\bar{\mu}/\sigma\mu_w$, which is here equivalent to the ratio $C_f/\bar{C}_f$, is not only a function of Mach number for these experiments, but is also a function of Reynolds number. Because μ_w and $\bar{\mu}$ are constants for these data, it follows that σ must be treated as a function of Reynolds number and hence of x for the flows in question.

---

the earlier treatment of the data represents a normalization of $C_f/\bar{C}_f$ with respect to temperature ratio, whereas Fig. 5 represents an inquiry into the properties of the scaling function $\sigma(x)$ of the transformation.

In the construction of Figs. 4 and 5 the ratio T_w/T_∞ has been referred to a recovery factor of 0.885 in all cases where no experimental value is given in the original reference. Values of viscosity have been taken from recent NBS-NACA tables. The usual practice has been followed of assuming constant stagnation temperature T_o for the purpose of computing u(y) and $\rho(y)$ and thus $\Theta(x)$ when only M(y) can be inferred from the measurements. The floating-element data have been corrected for gap effect by using the combined area of element and gap rather than the area of the element alone to relate force and shearing stress. Finally, the values of $\bar{C}_f$ for low-speed flow have been estimated by fitting measured mean-velocity profiles to the law of the wall, using methods which are discussed at some length in Appendix A.

IV. THE NOTION OF A SUBSTRUCTURE

RECAPITULATION

Up to this point the search for a transformation capable of reducing the compressible turbulent boundary-layer equations to incompressible form has been carried out in three steps. The physical basis of the transformation was laid down in Section I; the transformation itself was formally constructed in Section II; and a preliminary study was made in Section III of the three scaling functions whose properties must be known before the transformation can be applied to a given problem.

For turbulent flow, even for the special case of turbulent flow at constant pressure, only two independent relationships have so far been found for the three scaling functions σ, ξ, and η. One of these relationships,

$$\frac{\sigma}{\eta} = \text{constant} = \frac{\bar{u}_\infty}{u_\infty} \text{ (say)} \tag{4.1}$$

follows from the condition of constant pressure; the other,

$$\frac{\xi}{\eta} = \frac{\rho_w \mu_w}{\bar{\rho}\bar{\mu}} \frac{d\sigma\theta}{d\theta} \tag{4.2}$$

then follows from the additional condition of Newtonian friction at the wall. For flow in a wake, the condition which replaces (4.2) is evidently η = constant = $\bar{\rho}\bar{\theta}/\rho_\infty\theta$ (say), in view of Eq. (2.19). None of these conditions implies any restriction on the state or energy equations or on the viscosity law.

Before the transformation can be applied to turbulent flow, it will be necessary to find a third condition corresponding to (4.1) and (4.2). The form of this third condition can be expected to vary with the circumstances of the problem. For example, the condition σ = constant, which has already been shown to be generally valid for laminar flow or for a turbulent wake, must be rejected on experimental grounds if the flow is an adiabatic turbulent boundary layer at constant pressure. For the latter case, according to Fig. 5, the quantity $\sigma\mu_w/\bar{\mu}$ depends both on Mach number and on Reynolds number. The analytical nature of this dependence is so far unknown, and will presumably have to be established with the aid of some physical argument beyond those already cited. Whatever the nature of this argument, it should be clear that the problem that now has to be solved is a different problem from the one originally formulated in Section I. The issue is no longer the transformation of the equations of mean motion to incompressible form. Instead, the issue is now the search for a hypothesis which will allow this transformation to be applied to the turbulent boundary layer. There are a great variety of physical considerations which might be taken as characteristic of turbulent boundary-layer flow, and thus a great variety of arguments which might serve to complete the transformation* by determining the scaling function $\sigma(x)$. It goes without saying that different writers may prefer different arguments for this purpose, and therefore that the particular argument which follows can and should be judged by less critical standards than those used to judge the material of Sections I-III.

*It is important to note that the end to be served is the completion of the transformation, not the completion of the equations of motion through the introduction of an explicit relationship connecting the shearing stresses τ and $\bar{\tau}$ with the other dependent or independent variables as in the case of laminar flow. On the contrary, the use of any such relationship is quite likely to prejudice the concepts of turbulent shearing stress and turbulent heat transfer in much the same way that these concepts are prejudiced by the mixing analogies of the older literature.

THE LAW OF THE WALL

The physical considerations which I propose to take as characteristic of turbulent flow near a smooth wall are contained in an empirical similarity relationship known as the law of the wall. For incompressible flow this law has the form

$$\frac{\bar{u}}{\sqrt{\bar{\tau}_w/\bar{\rho}}} = f\left(\frac{\bar{\rho}\,\bar{y}\sqrt{\bar{\tau}_w/\bar{\rho}}}{\bar{\mu}}\right) = f(z) \qquad (4.3)$$

When translated into compressible variables with the aid of the transformation formulas for u and y, this statement becomes

$$\sqrt{\frac{\sigma\mu_w}{\bar{\mu}}}\;\frac{u}{\sqrt{\tau_w/\rho_w}} = f\left(\sqrt{\frac{\sigma\mu_w}{\bar{\mu}}}\;\frac{\sqrt{\tau_w/\rho_w}}{\mu_w}\int_0^y \rho\,dy\right) \qquad (4.4)$$

If the flow is incompressible, two alternative physical interpretations are available for the law of the wall, Eq. (4.3). One is that the edge of the viscous sublayer, defined by specifying a suitable numerical value for the function f or its argument z, is a streamline of the mean flow. The other is that the sublayer Reynolds number, defined by specifying a suitable numerical value for the product fz (say), is a constant. Both of these interpretations are expressed in terms which allow their immediate extension to compressible flow. However, the first interpretation has now to be disqualified on the ground that it requires the scaling function σ to be a constant.* The second interpretation, which appears to be free of this

---

*I have previously attempted an extension of the first interpretation to compressible flow (1955), using only part of the full transformation given here. This earlier paper, in which the quantity now called $\sigma\mu_w/\mu$ was denoted by ρ_τ/ρ_w, includes a demonstration that this quantity (and hence σ) must be independent of x if $u/\sqrt{\tau_w/\rho_\tau} = f$ is constant on mean streamlines of the compressible flow. Except for the one negative conclusion just quoted, the discussion of compressibility in my 1955 paper may now be suppressed.

restriction, implies for compressible flow

$$R_s = f_s z_s = \frac{\bar{\rho}\,\bar{u}_s \bar{y}_s}{\bar{\mu}} = \frac{\sigma u_s}{\bar{\mu}} \int_0^{y_s} \rho \, dy = \text{constant} \qquad (4.5)$$

THE SUBLAYER HYPOTHESIS

The last equation will now be manipulated in such a way as to suggest a possible connection between the ratio $\bar{\mu}/\sigma$ and the properties of the flow in the sublayer. First, define a mean density ρ_s for the sublayer as

$$\rho_s = \frac{1}{y_s} \int_0^{y_s} \rho \, dy \qquad (4.6)$$

whereupon Eq. (4.5) can be written

$$R_s = \frac{\rho_s u_s y_s}{\bar{\mu}/\sigma} = \text{constant} \qquad (4.7)$$

Further, suppose that an equation of state is given in the form $T = T(\rho, p) = T(\rho)$ at constant p, so that a definite mean temperature $T_s = T(\rho_s)$ can be associated with the mean density defined by (4.6). Suppose also that a viscosity law is given in the form $\mu = \mu(T)$, so that a corresponding mean viscosity $\mu_s = \mu(T_s)$ can be associated in turn with this mean temperature.

Finally, consider once more the experimentally established properties of the ratio $\bar{\mu}/\sigma$ appearing in the denominator of Eq. (4.7). According to Fig. 5, $\bar{\mu}/\sigma$ is a quantity having the dimensions of viscosity and having moreover a quite definite value at a given station in an adiabatic compressible turbulent boundary layer at constant pressure. This value appears to lie between μ_∞ and μ_w and to approach the latter as the Reynolds number increases for fixed Mach number. Now the mean viscosity μ_s corresponding to the mean sublayer temperature T_s ought to have very nearly these same properties, not only for flow at constant pressure but for adiabatic flow in general. It is therefore natural to ask, especially after consulting the numerator of Eq. (4.7), whether or not the quantities $\bar{\mu}/\sigma$ and μ_s might in fact be identical. If so, then

$$\frac{\bar{\mu}}{\sigma\mu_w} = \frac{\mu_s}{\mu_w} = \frac{\mu(T_s)}{\mu(T_w)} = \frac{\rho_w}{\rho_\omega}\,\frac{\bar{C}_f}{C_f} \qquad (4.8)$$

The first equality in this expression, the equality $\sigma = \bar{\mu}/\mu_s$, will be referred to as the sublayer hypothesis. This relationship evidently provides the third condition needed to define the three scaling functions of the transformation, and is the main contribution of the present paper to the empirical description of the turbulent boundary layer in compressible flow. If the sublayer hypothesis is adopted, Eq. (4.7) becomes finally

$$R_s = \frac{\rho_s u_s y_s}{\mu_s} = \text{constant} \qquad (4.9)$$

The essence of this argument is the proposal* that the sublayer Reynolds number R_s may be independent of compressibility when the density and viscosity

*Unfortunately this proposal, like most inventions, is not unique. For example, the sublayer Reynolds number might equally well have been defined (say) as the definite integral of fdz rather than as the product fz. The mean density ρ_s would then be replaced by a mean mass flow, and the transition to a mean temperature and a mean viscosity would become more difficult. Even the latter transition is ambiguous unless it is assumed, as in the text, that mean thermodynamic variables for the sublayer are related by the same laws which apply for corresponding local variables. This assumption, however, requires the mean temperature to be treated as a derived rather than a fundamental quantity. In this respect, as well as in the application to the sublayer and the formation of the mean in terms of an integral determined by the structure of the incomplete equations of motion, the present definition of mean temperature differs from similar but more empirical definitions in the heat-transfer literature.

are evaluated at a suitably defined mean temperature T_s. The constant B_s in Eq. (4.9), being determined by the properties of the law of the wall for incompressible flow, is conceptually independent of viscosity law, state equation, Mach number, Prandtl number, specific heat ratio, surface temperature distribution, and heat transfer rate in the compressible flow, and may also be taken as independent of Reynolds number and pressure gradient to the same extent as the law of the wall itself. On the other hand, this constant will certainly be affected by surface roughness and by mass transfer, and may possibly be affected by external turbulence level and by changes in geometry such as the introduction of lateral curvature.

FLOW OF A PERFECT GAS

Given the sublayer hypothesis (4.8), the immediate problem becomes the calculation of the sublayer mean temperature T_s. For this purpose it is convenient to take the compressible fluid to be a perfect gas having the state equation $p = \rho RT$, and to write from (4.6)

$$c_pT + \frac{u^2}{2} = c_pT_o = c_pT_w - \frac{q_w}{\tau_w} u \tag{4.14}$$

which is known to be valid for laminar flow with Pr = 1 if either the pressure gradient or the heat transfer is zero. On using Eq. (4.14) to evaluate the definite integral in (4.12), there is obtained

$$\frac{T_s}{T_w} = \frac{\rho_w}{\rho_s} = 1 - \langle f \rangle \sqrt{\frac{\mu_s}{\mu_w}} \frac{q_w}{\rho_w c_p T_w \sqrt{\tau_w/\rho_w}} - \langle f^2 \rangle \frac{\gamma-1}{2\gamma} \frac{\mu_s}{\mu_w} \frac{\tau_w}{p} \tag{4.15}$$

where by definition $q_w = -(k\,\partial T/\partial y)_w$ and

$$\langle f \rangle (z_s) = \frac{1}{z_s} \int_0^{z_s} f dz = f_s \frac{\int_0^{y_s} \rho u \, dy}{u_s \int_0^{y_s} \rho \, dy} \tag{4.16}$$

An interesting alternative form of Eq. (4.15) follows when τ_w/p is replaced by $\gamma M_\infty^2 c_f/2$ and q_w/τ_w is replaced by $c_p(T_w - T_{o_\infty})/u_\infty$, the latter from Eq. (4.14). If M_∞ is eliminated from the resulting expression using $T_{o_\infty}/T_\infty = 1 + (\gamma - 1) M_\infty^2/2$, and if c_f is eliminated using (4.6), the result is the simple expression

$$\frac{T_s}{T_w} = 1 + \langle f \rangle \left(\frac{T_{o_\infty}}{T_w} - 1\right) \sqrt{\frac{\bar{c}_f}{2}} - \langle f^2 \rangle \left(\frac{T_{o_\infty}}{T_w} - \frac{T_\infty}{T_w}\right) \frac{\bar{c}_f}{2} \tag{4.17}$$

Given the Mach number and the wall temperature ratio for the compressible flow, therefore, the two formulas (4.17) and

$$c_f = \frac{T_\infty}{T_w} \frac{\mu_w}{\mu_s} \bar{c}_f \tag{4.18}$$

$$\frac{1}{T_s} = \frac{R\rho_s}{p} = \frac{R}{p y_s}\int_0^{y_s} \rho \, dy = \frac{1}{y_s}\int_0^{y_s} \frac{dy}{T} \tag{4.10}$$

This equation may first be put in the alternative form

$$T_s = \frac{1}{\bar{y}_s}\int_0^{\bar{y}_s} T \, d\bar{y} = \frac{1}{z_s}\int_0^{z_s} T \, dz \tag{4.11}$$

where z is the independent variable used to represent the mean-velocity profile by way of the law of the wall, Eq. (4.3). The last result may be rewritten once more in terms of the local stagnation temperature $T_0 = T + u^2/2c_p$ and partially evaluated with the aid of (4.4) to obtain

$$\frac{T_s}{T_w} = 1 - \left\langle f^2 \right\rangle \frac{\gamma - 1}{\gamma} \frac{\mu_s}{\mu_w} \frac{\tau_w}{p} + \frac{1}{z_s}\int_0^{z_s} \left(\frac{T_0}{T_w} - 1\right) dz \tag{4.12}$$

where by definition

$$\left\langle f^2 \right\rangle (z_s) = \frac{1}{z_s}\int_0^{z_s} f^2 \, dz = f_s^{\,2} \, \frac{\int_0^{y_s} \rho u^2 \, dy}{u_s^{\,2} \int_0^{y_s} \rho \, dy} \tag{4.13}$$

is the mean value of f^2 in the sublayer.

THE PUTATIVE CASE Pr = 1

The form of the last term in Eq. (4.12) calls attention to the importance of the special case $T_0 = T_w$ = constant, and suggests an examination of the consequences of the full Crocco energy integral together provide a parametric relationship — with the viscosity law for $\mu_s = \mu(T_s)$ as parametric function — expressing the local friction coefficient C_f in terms of $\bar{C}_f$ alone. Once C_f is known, R_θ may be found from the law of corresponding stations (2.21), and the heat transfer may be computed from the usual formula

$$C_h = \frac{q_w}{\rho_\infty u_\infty c_p (T_w - T_{o_\infty})} = \frac{C_f}{2} \tag{4.19}$$

These results, of course, apply only for flows characterized by the Crocco energy integral (4.14).* Thus the Prandtl number must be unity, the wall temperature must be constant, and in all probability either the heat transfer or the pressure gradient must be zero, although there is so far no restriction on the viscosity law. Within these limitations, and in particular

*For turbulent flow the validity of the Crocco energy integral (4.14)— or of any other energy integral in analytical form, for that matter—is a matter for conjecture. The most that can be argued from the linear behavior of T and u near y = 0 is that the integral (4.14), if it is valid at all for turbulent flow with heat transfer, can only be valid if the laminar Prandtl number is unity. It follows that calculations based on this integral, which is to say calculations based on Eqs. (4.17) and (4.19), cannot be quantitatively correct for gases such as air, although they may be useful as a means of estimating the effects of compressibility and heat transfer on skin friction.

With regard to Eq. (4.17), it should be remembered that the quantity T_s is actually defined, e.g., by Eq. (4.15) or more generally by Eq. (4.10), without reference to conditions in the free stream. On the other hand, (4.17) has the advantage of being independent of the form of the viscosity law. Furthermore, Eq. (4.17) is readily applied to various special cases of practical interest, including the case of low-speed heat transfer ($M_\infty = 0$ or $T_{o_\infty} = T_\infty$); the case of adiabatic flow ($q_w = 0$ or $T_w = T_{o_\infty}$); and the case of a very cold wall ($T_w = 0$, say). Finally, Eq. (4.17) expresses the sublayer mean temperature T_s as a linear combination of the three characteristic temperatures T_∞, T_{o_∞}, and T_w, and thus bears a strong resemblance to certain empirical formulas which are discussed at more length in Appendix B.

for the case $dp/dx = 0$, the only new empirical information needed is the numerical value of the sublayer Reynolds number R_s (or of any one of the equivalent quantities f_s, z_s, $\langle f \rangle$, or $\langle f^2 \rangle$).

THE FURTHER RESTRICTION μ/T = Constant

It will be recalled from Section III that the equations of motion for laminar compressible flow could be reduced to incompressible form only for a perfect gas with viscosity proportional to temperature. Although the latter restriction is unlikely to be necessary for turbulent flow, it does lead to a particularly simple relationship between the local friction coefficients C_f and $\bar{C}_f$. For the case μ/T = constant (more generally, if μ is any explicit function of T), the sublayer mean temperature T_s can be eliminated from the parametric representation given by Eqs. (4.17) and (4.18) to obtain

$$C_f = \frac{\bar{C}_f}{\frac{T_w}{T_\infty} - \langle f \rangle \left(\frac{T_w - T_{o\infty}}{T_\infty}\right)\sqrt{\frac{\bar{C}_f}{2}} - \langle f^2 \rangle \left(\frac{T_{o\infty} - T_\infty}{T_\infty}\right)\frac{\bar{C}_f}{2}} \tag{4.20}$$

where corresponding stations are now connected by (2.21) in the form

$$C_f R_\theta = \bar{C}_f R_{\bar{\theta}} \tag{4.21}$$

Incidentally, for flow at constant pressure the transformation for the streamwise coordinate x can be found from Eq. (4.2), which has so far not been used. In combination with (4.1) and (2.19), this equation can be written

$$\frac{1}{\xi} = \frac{dx}{d\bar{x}} = \frac{1}{\eta}\,\frac{\bar{\rho}\bar{\mu}}{\rho_w \mu_w}\,\frac{d\bar{\theta}/\sigma}{d\theta} \tag{4.22}$$

But σ is equal to $\bar{\mu}/\mu_s$ by the sublayer hypothesis, and μ_s is an implicit function of θ through the dependence of both $\bar{R}_{\bar{\theta}}$ and μ_w/μ_s on $\bar{C}_f$, a dependence given by Fig. 4 in the one case and by the viscosity law and Eq. (4.17)(say) in the other. Again for the special case T_o = constant and μ/T = constant, Eq. (4.22) and the definitions

$$dR_x = 2\,\frac{dR_\theta}{C_f}\ ,\quad d\bar{R}_{\bar{x}} = 2\,\frac{d\bar{R}_{\bar{\theta}}}{\bar{C}_f} \tag{4.23}$$

together imply

$$dR_x = \left(\frac{T_s}{T_\infty}\right)^2 \left[1 - \left(\frac{T_w}{T_s} - 1\right)\frac{d\,\ln \bar{C}_f}{d\,\ln \bar{R}_{\bar{\theta}}}\right] d\,\bar{R}_{\bar{x}} \tag{4.24}$$

where $\bar{R}_{\bar{x}}$ is known empirically as a function of $\bar{R}_{\bar{\theta}}$ from a numerical integration of the second of Eqs. (4.23). The spirit of this calculation, incidentally, is typical of most boundary-layer problems, in that x and $\bar{x}$ have to be treated as dependent variables whose evaluation is possible only after the dependence of $\bar{C}_f$ on $\bar{R}_\theta$ has been specified in the definition (4.23).

THE NUMERICAL VALUE OF R_s

It remains to determine the numerical magnitude of the sublayer Reynolds number R_s. For this purpose the special formulas just obtained are of academic interest only, as these formulas refer for the most part to the flow of a fictitious perfect gas with Pr = 1 and μ/T = constant. Ideally, the best procedure is a direct evaluation of R_s from the definition (4.9), using measured values for T_w, u(y), and T(y). Such measurements would first determine the ratio $\mu_s/\mu_w = \bar{\mu}/\sigma\mu_w$, either by the methods already employed in Fig. 5 or more generally by a fit to the law of the wall (4.4). Given μ_s and a viscosity law, the implied value of T_s together with the static temperature profile $T(y)$ would then determine ρ_s and y_s, the former from the state equation and the latter as the upper limit of integration in Eq. (4.10). With ρ_s, μ_s, y_s, and $u_s = u(y_s)$ known, the value of $R_s = \rho_s u_s y_s / \mu_s$ follows immediately. Moreover, a comparison of values of R_s obtained for various flows by this method would provide a direct and decisive test of the sublayer hypothesis. Unfortunately, the necessary measurements have never been made, so that less direct and hence less decisive methods must be found.

Consider therefore the general equation (4.12) rewritten in the form

$$\frac{T_w}{T_s}(1 + \epsilon) = 1 + \left\langle f^2 \right\rangle \frac{\mu_s}{\mu_w} \frac{T_w}{T_s} \frac{\gamma - 1}{2} M_\infty^2 \frac{C_f}{2} \qquad (4.25)$$

Experimentally speaking, the only unknown quantities in this equation are the fundamental sublayer parameter $\left\langle f^2 \right\rangle$, defined by Eq. (4.13), and the auxiliary parameter ϵ, defined by

$$\epsilon = \left\langle \frac{T_o}{T_w} - 1 \right\rangle = \frac{1}{z_s} \int_0^{z_s} \left(\frac{T_o}{T_w} - 1 \right) dz \qquad (4.26)$$

In particular, the effect of variations in stagnation temperature, whatever their nature or origin, is now concentrated in the parameter ϵ. This parameter will ordinarily vanish only for adiabatic flow of a fluid with a Prandtl number of unity. If the Prandtl number is not too far from unity, however, the definition (4.26) indicates for adiabatic flow that ϵ will be of order $1 - r$, where r is the recovery factor. For the purpose of determining $\left\langle f^2 \right\rangle$ (and hence R_s) from Eq. (4.25), therefore, a relatively rough estimate of ϵ should be sufficient. This estimate in turn depends on a knowledge of an energy integral; i.e., a relationship between temperature and velocity such that $T_o(y)$ in Eq. (4.26) can be expressed in terms of u(y) and so in terms of f(z). The most accurate measurements of $T_o(y)$ in adiabatic supersonic flow at constant pressure are probably those by Lobb, Winkler, and Persh (1955)... by Kistler (1956), both of whom measured T_o using an unheated hot wire as a resistance thermometer. The measurements in question are shown in Fig. 6; they suggest that the observed relationship between velocity and temperature might be satisfactorily approximated—in the region which is important for the evaluation of ϵ, and only for the purpose of this evaluation--by the straight line*

$$\frac{T_o}{T_w} - 1 = \left(\frac{T_{o\infty}}{T_w} - 1\right) \frac{u}{u_\infty} \qquad (4.27)$$

But this empirical expression for adiabatic flow with Pr ≠ 1 is formally identical with the Crocco energy integral (4.14) for flow with heat transfer but with Pr = 1. It follows that ϵ can be computed in either case from the formula

$$\epsilon = \left\langle f \right\rangle \left(\frac{T_{o\infty}}{T_w} - 1\right) \sqrt{\frac{\overline{C}_f}{2}} \qquad (4.28)$$

where $\langle f \rangle$ is defined by Eq. (4.16). It also follows that Eq. (4.17) for the mean sublayer temperature T_s, although originally derived from the Crocco integral, should also be reasonably accurate for air, at least when the pressure is constant and the flow is nearly adiabatic.

---

*The accuracy of this approximation for both low-speed and high-speed flows with heat transfer will be discussed in Appendices A and B. I want to emphasize that my object here is to estimate the effect of Prandtl number on skin friction, not to determine the heat transfer. Neither the derivative $(\partial T/\partial y)_w$, which is zero for adiabatic flow, nor the integral of $\rho u (T_o - T_{o\infty})$ through the boundary layer, which is constant, can be correctly obtained from the empirical formula (4.27).

Given Eqs. (4.25) and (4.28), a simple iteration starting with $\epsilon = 0$ may be used to find $\langle f \rangle$ and $\langle f^2 \rangle$ for the data previously collected in Figs. 1-5. The final result of this iteration is shown in Fig. 7. The straight heavy line in the figure* corresponds to the values $\langle f \rangle = 17.2$, $\langle f^2 \rangle = 305$; and the lighter lines illustrate the effect of Reynolds number on T_w/T_s (or on $\bar{\mu}/\sigma\mu_w$) at constant Mach number, an effect which was originally cited as the major cause of the scatter in Figs. 2 and 5.

Whether or not the argument for a straight line in Fig. 7 is convincing, one last inference to be drawn from this argument is that any such line must terminate at a finite point which I will call the hypersonic limit. The reason is that C_f varies like $1/M_\infty^2$ for large M_∞, so that T_w/T_s is a bounded function of both Mach number and Reynolds number. From Eq. (4.17) it then follows, for $\langle f^2 \rangle = 305$ and $\bar{C}_{f_{max}} = 0.005$ (say), that T_w/T_s has a maximum value of about 5 in adiabatic flow. Furthermore, the product $M_\infty^2 C_f$ is equivalent to τ_w/p, so that for fixed $\bar{C}_f$ the ratio of τ_w to p (or to $\rho c_p T$) remains finite in the limit of large Mach number, despite the fact that τ_w/q_∞ and p/q_∞ separately approach zero. Although these conclusions are strictly correct only for the case of a perfect gas with μ/T = constant and T_0 = constant, similar conclusions may be expected to hold for real turbulent flows under appropriate conditions.

Finally, the over-all accuracy of the transformation and of the experiments may be tested by reducing the available supersonic data to equivalent

---

*Fig. 7 is the same as Fig. 3 of the introduction, except that the ordinate in the earlier figure is T_w/T_s rather than $(1 - \epsilon)T_w/T_s$. Here and elsewhere in this section the sublayer parameters have been computed from the definitions, e.g., Eqs. (4.5), (4.13), or (4.16), using the particular function f(z) tabulated in another paper (Coles 1955), but with f reduced by two per cent for reasons set out in Appendix A of the present paper.

incompressible form. Values of $\bar{C}_f$ computed from Eqs. (4.8) and (4.17), using experimental values for C_f, M_∞, T_w/T_{o_∞}, the viscosity law, and the sublayer constants, are superimposed in Fig. 8 on the low-speed friction curve of Fig. 4. In view of the discrepancies which are already present in the raw data as a result of unavoidable irregularities in tunnel conditions, residual effects of tripping devices, and other causes, I feel that this final correlation is entirely satisfactory.

THE NOTION OF A SUBSTRUCTURE

The sublayer constants of Section III have now acquired the values

$$\langle f^2 \rangle = 305$$

$$\langle \zeta \rangle = 17.2$$

$$f_s = 19.8$$

$$z_s = 430$$

$$R_s = 8500$$

At first glance the figure quoted for R_s is unprepossessing, because it is very much larger than the Reynolds number of perhaps 150 which is usually taken to be characteristic of the viscous sublayer in turbulent flow. Equally unprepossessing is the fact that the parameter R_s, although it is formally defined entirely in terms of the properties of incompressible flows, seems to play no explicit part in such flows. In other words, R_s seems to be important only insofar as the effects of compressibility are concerned, and can in fact be determined only by a study of these effects.

I will try to counter both of these objections by proposing a new interpretation for the characteristic Reynolds number R_s. This interpretation emerges from the study in Appendix A of the available data for low-speed flow when attention is focussed on the inverse transition from turbulent to laminar flow as the Reynolds number decreases under strongly disturbed conditions. Not only is there a fairly definite critical Reynolds number for this inverse transition, but the limiting turbulent profile near transition is adequately described by the law of the wall, Eq. (4.3), terminated not far from the point $(f_s, z_s) = (19.8, 430)$ which now defines the outer edge of the sublayer. The analogy with the critical Reynolds number for flow in a circular pipe is clear, especially since this critical Reynolds number for pipe flow is known to describe not a region marked primarily by large direct dissipation of energy or by strong damping of velocity fluctuations, but a region which has all the essential properties of turbulent shear flow, including the ability to extract energy from the environment -- through the action of large eddies having a scale distinct from the dissipation scale -- at a rate sufficient to prevent the decay of the turbulence. If this tentative interpretation is adopted for the boundary layer as well, the region in question should probably be given a special name to distinguish it from the ordinary sublayer. I therefore propose the term "turbulent substructure" as one which conveys something of the connection with transition and at the same time indicates the subordi[illegible] role played by this region in flow at large Reynolds numbers.

Experimental evidence can be found both for and against the proposition that the substructure parameter R_s can be identified with the limiting Reynolds number for turbulent boundary-layer flow. According to the data plotted in Fig. 10 of Appendix A, turbulent flow may have been observed at Reynolds numbers $R_{\theta} = \bar{u}_{\infty}\theta/\nu$ as small as 290, whereas the value of R_{θ} which is characteristic of the substructure flow (with $\bar{u}_{\infty}\delta/\nu = 8500$) is 750.

However, most of the profiles surveyed in the appendix have been classified as turbulent on the basis of indirect evidence, and there is no assurance in some cases that the profiles nearest transition are fully developed in the sense (say) that the intermittency factor is unity near the surface. The main difficulty in the case of boundary-layer flow is that the inherent increase in Reynolds number with distance may act to convert a temporary and abnormal response to a strong distribance into a permanent one. No such difficulty is encountered in the case of pipe flow, for which the critical Reynolds number can be determined with good precision, regardless of initial disturbances, simply by observing the flow far downstream. Preston (1958) has used this property of pipe flow in estimating a minimum value $\bar{R}_\theta = 320$ for the boundary layer, but his arguments also rely in part on a defect law which is now known to fail at small Reynolds numbers. In fact, the disappearance of the wake component in both pipe and boundary-layer flow as the Reynolds number decreases almost certainly corresponds to the disappearance of the large eddies which are primarily responsible for the transfer of energy from the mean flow to the turbulence. I have made this same point in Appendix A in connection with a study of flows recovering from the effects of tripping or thickening devices; and perhaps it is also relevant here to mention the value $\bar{R}_\delta = 680$ observed by Dutton (1955b) for the asymptotic turbulent boundary layer with suction, a flow in which the wake component is again absent and the survival of the turbulence is again marginal.

Another and less ambiguous approach to the problem of a limiting flow can be made in terms of the maximum value of the local friction coefficient, inasmuch as the properties of any limiting flow should be suitable grist for the mill of the transformation. In several of the experiments with floating elements, cited earlier, transition was observed to move over the element in response to changes in tunnel pressure or velocity or in response to changes in the strength of a variable tripping device. The measured values of $C_{f_{max}}$ are plotted against M_∞ in Fig. 9 without regard to the corresponding values of R_θ, which were usually not measured and might in any case be difficult to guarantee. Also shown are some curves of constant $\bar{C}_f$ computed for air and helium at wind-tunnel conditions with the aid of the general transformation, using for the sublayer parameters the numerical values just quoted. The figure indicates that values of $\bar{C}_f$ larger than 0.0043 have not in fact been observed* in any of these floating-element experiments, although strenuous efforts were made in most cases to stimulate early transition to turbulent flow. Because these direct measurements are entirely consistent with the plausible upper bound on $\bar{C}_f$ supplied by the value $2/f_s^2 = 0.0051$ for the hypothetical substructure flow with $\bar{R}_{\bar{\theta}} = 780$,

---

*The correction for gap effect (the inclusion of the gap area as part of the element area) has not been made for Dhawan's data, as it amounts to a decrease in C_f of some 15 percent. Fig. 9 also confirms a conclusion which I reached and reported at the time of my own experiments (Coles 1953); this conclusion is that the fence tripping device sometimes had the peculiar effect of inhibiting rather than stimulating transition.

By the term "wind-tunnel conditions" in Fig. 9 is meant that the stream stagnation temperature for air has been taken as 550° R for $M_\infty < 4.7$, and the stream static temperature as 100° R for $M_\infty > 4.7$. The increase in $C_f/\bar{C}_f$ with increasing Mach number when $\bar{C}_f$ is large corresponds qualitatively to the change in sign of the factor multiplying M_∞^2 in the denominator of Eq. (B-18). For the special conditions represented by the latter formula it then follows, whenever $\bar{C}_f$ exceeds the value $2/\langle f^2 \rangle = 0.0066$, that T_w/T_∞ is less than unity. Since this condition cannot be met in an adiabatic flow in which T decreases monotonically from T_w at the wall to T_∞ in the free stream, a definite upper bound in the neighborhood of $\bar{C}_f = 0.0066$ is seen to be inherent in the formalism of the substructure concept itself.

I am not inclined to attach too great a significance to the few cases encountered in Appendix A for which larger values of $\bar{C}_f$ seem to be implied by the law of the wall.

Appendix A

A MANUAL OF EXPERIMENTAL BOUNDARY-LAYER PRACTICE

I. LOW-SPEED FLOW

INTRODUCTION

The analysis reported in the main body of this paper assumes an empirical knowledge of certain properties of turbulent boundary layers in low-speed flow at constant pressure. The most important of these properties is the value of the local friction coefficient, especially for low Reynolds numbers. I have therefore felt obliged to undertake yet another survey of the experimental literature, the survey this time being as complete as I can make it from both the clerical and the critical point of view. Inasmuch as I have found this survey highly instructive on the subject of proper experimental technique, I want to record some of my impressions here in the hope that they may be relevant for future as well as for past research.

I have listed in the adjacent Table I all of the papers which I know to contain experimental information about the mean-velocity distribution in turbulent boundary layers on smooth surfaces in low-speed flow at nominally constant pressure. Some of these papers are listed primarily for the sake of completeness, and will not be represented in the later figures. The reason is usually that the data are incompletely reported or that they contain certain anomalies which in my opinion are not compensated for by some definite contribution, either positive or negative, to experimental knowledge of the problem. Thus the measurements by Baines (1950) are omitted because they were carried out in a flow marked by high turbulence level and nonuniform pressure, and because they are superseded by some later experiments by Landweber and Siao (1958) in the same tunnel under much improved

EXPERIMENTAL INVESTIGATIONS OF TURBULENT BO

Author	Date	Tunnel Configuration*	Stream Turbulence Level+	Model	
Albertson	1948	cj, or (d)	Moderate?	Partially shielded plate	Tap
Allan and Cutland	1953	Tank	---	Partially submerged plank	Wir
Ashkenas et al.	1952	cj,cr	0.0004	Tunnel wall	Non
Ashkenas and Riddell	1955	cj,cr	0.0004	Full-span plate	Dist
Ashkenas	1956	cj,cr	0.0004	Full-span plate	Dis
Baines	1950	cj,cr	0.012	Partially shielded plate	Blu rou
Barrow	1958	oj,or	High?	Unshielded plate	Blu wir
Cermak and Lin	1955	cj,or(d)	0.007	Full-span plate	Tap
Dhawan	1951	cj,or(d)	0.0003	Full-span plate	Wir
Dryden	1936	cj,or(d)	0.005, 0.030	Full-span plate	Non
Dutton	1955	cj,cr	Low?	Full-span plate	Blu or
Ede and Saunders	1958	Water channel	---	Partial-span plate	Non
Edwards and Furber	1956	cj,or(d)	Variable	Full-span plate	Non
Elias	1929	oj,cr	High?	Unshielded plate	Non
Favre et al.	1955	cj,cr	0.0004	Full-span plate	Dis
Furber	1954	cj,cr	High?	Partially shielded plate	Blu
Grant	1957	cj,or(d)	Low?	Tunnel floor	Fen

* oj = open jet
  cj = closed jet
  or = open return
  cr = closed return
  (u) = fan upstream
  (d) = fan downstream

+ Low = less than 0.001
  Moderate = 0.001 to 0.005
  High = more than 0.005

Table I.

INVESTIGATIONS OF TURBULENT BOUNDARY LAYERS IN LOW-SPEED FLOW AT CONSTANT PRESSURE

Model	Tripping Device	Probe Dimensions	Source of Data
tially shielded plate	Tape ahead of model	Tungsten hot wire	Fig. 8
tially submerged plank	Wire; pins	1.6 mm dia. (rake)	Private comm.
nel wall	None	0.7 x 2.5 mm	Figs. 12, 19-23
l-span plate	Distributed roughness	0.3 x 2 mm	Fig. 17a
l-span plate	Distributed roughness	0.3 x 2 mm	Fig. 4
tially shielded plate	Blunt leading edge; roughness	0.7 mm dia.	Fig. 5
hielded plate	Blunt leading edge; wire	1 mm dia.	Figs. 2, 7-9; private comm.
l-span plate	Tape ahead of model	Tungsten hot wire	Table 29
l-span plate	Wire	0.1 x 0.6 mm	Fig. 12
l-span plate	None	Platinum hot wire	----
l-span plate	Blunt leading edge; wire or roughness	0.13 x 1.5 mm	Private comm.
tial-span plate	None	1.8 mm dia.	----
l-span plate	None	1 mm dia.	Fig. 7
hielded plate	None	0.3 mm dia.	Figs. 18, 32, 40
l-span plate	Distributed roughness	1 mm dia.	Figs. 8, 9
tially shielded plate	Blunt leading edge	1.8 mm dia.	----
nel floor	Fence	0.5 mm high	Fig. 20

ERS IN LOW-SPEED FLOW AT CONSTANT PRESSURE

Device	Probe Dimensions	Source of Data	Number of Profiles Studied	Other Data of Interest
model	Tungsten hot wire	Fig. 8	--	Mass transfer (evaporation, water)
	1.6 mm dia. (rake)	Private comm.	20	Edge effect
	0.7 x 2.5 mm	Figs. 12, 19-23	6	Reynolds stresses
oughness	0.3 x 2 mm	Fig. 17a	11	Sweep
oughness	0.3 x 2 mm	Fig. 4	3	Reynolds stresses; sweep
edge;	0.7 mm dia.	Fig. 5	--	Intensity of u-fluctuations
edge;	1 mm dia.	Figs. 2, 7-9; private comm.	22	Mass transfer (blowing at slot, air)
model	Tungsten hot wire	Table 29	--	Mass transfer (evaporation, water)
	0.1 x 0.6 mm	Fig. 12	1	Floating element
	Platinum hot wire	----	--	Transition; intensity of u-fluctuations
edge; wire	0.13 x 1.5 mm	Private comm.	19	Preston tube; mass transfer (distributed suction, air)
	1.8 mm dia.	----	--	Heat transfer
	1 mm dia.	Fig. 7	6	Heat transfer
	0.3 mm dia.	Figs. 18, 32, 40	4	Heat transfer
ughness	1 mm dia.	Figs. 8, 9	12	Time-space correlations (velocity)
edge	1.8 mm dia.	----	--	Heat transfer; mass transfer (condensation, water)
	0.5 mm high	Fig. 20	1	Space correlations

C

Author	Date	Tunnel Configuration*	Stream Turbulence Level+	Model
Hama	1947	oj,cr	Moderate?	Shielded plate
Hansen	1928	oj,cr	High?	Unshielded plate
van der Hegge Zijnen	1924	cj,or(d)	High?	Unshielded plate
Johnson	1955	cj,or(u)	0.0007	Tunnel floor
Jürges	1924	oj,or	High?	Unshielded plate
Karlsson	1958	cj,or(u)	Variable	Tunnel floor
Kay	1953	cj,cr	Moderate?	Full-span plate
Klebanoff and Diehl	1951	cj,cr	0.0003	Full-span plate
Klebanoff	1954	cj,cr	0.0003	Full-span plate
Kline et al.	1960	cj,or(d)	Variable	Full-span plate
Landweber and Siao	1958	cj,cr	0.002	Full-span plate
Ludovici	1926	oj,or	High?	Unshielded plate
McCullough and Gambucci	1952	cj,or(u)	Moderate?	Tunnel wall
Michel	1950	cj,or(d)	High?	Tunnel floor
Mickley and Davis	1957	cj,or(u)	Moderate?	Tunnel ceiling
Parmelee and Huebscher	1947	cj,or(d)	0.013	Partial-span plate
Peters	1938	cj,or(u)	0.002	Full-span plate

* oj = open jet
cj = closed jet
or = open return
cr = closed return
(u) = fan upstream
(d) = fan downstream

+Low = less than 0.001
Moderate = 0.001 to 0.005
High = more than 0.005.

A

Table I. -- continued

Model	Tripping Device	Probe Dimensions	Source of Data
Shielded plate	Blunt leading edge; wire	0.25 mm high	Fig. 1; private com
Unshielded plate	Blunt leading edge	0.3 mm dia.	Fig. 17
Unshielded plate	None	Pt.-ir. hot wire	Tables I-VII
Tunnel floor	Large wire	0.7 x 1.9 mm	Fig. 10
Unshielded plate	Blunt leading edge	0.8 mm dia.	Figs. 23, 24
Tunnel floor	Pins	Platinum hot wire	----
Full-span plate	None	0.6 mm high	----
Full-span plate	Various	0.35 mm high	Figs. 5,7,8,12-15
Full-span plate	Distributed roughness	0.35 mm high; platinum hot wire	Fig. 3
Full-span plate	Blunt leading edge; wire	0.25 mm high	Table 3; Figs. 12,1
Full-span plate	Blunt leading edge	1 mm dia.	Tables 2,3
Unshielded plate	None	Platinum hot wire	----
Tunnel wall	None	0.25 mm high	----
Tunnel floor	None; leading edge step	0.05, 0.1 mm high	Tables 1, 3; Plate
Tunnel ceiling	Intrinsic roughness	0.2 mm high	Tables I, II
Partial-span plate	Blunt leading edge	0.9 mm dia.	Fig. 2
Full-span plate	None	0.4 mm dia.	Fig. 4

005

B

Device	Probe Dimensions	Source of Data	Number of Profiles Studied	Other Data of Interest
ing edge; wire	0.25 mm high	Fig. 1; private comm.	11	----
ing edge	0.3 mm dia.	Fig. 17	13	Roughness
	Pt.-ir. hot wire	Tables I-VII	19	----
	0.7 x 1.9 mm	Fig. 10	6	Heat transfer; structure of turbulence
ing edge	0.8 mm dia.	Figs. 23, 24	--	Heat transfer
	Platinum hot wire	----	--	Oscillating free stream
	0.6 mm high	----	--	Mass transfer (distributed suction, air)
	0.35 mm high	Figs. 5,7,8,12-15	51	Spectrum of u-fluctuations
d roughness	0.35 mm high; platinum hot wire	Fig. 3	1	Structure of turbulence
ing edge; wire	0.25 mm high	Table 3; Figs. 12,17	9	Intensity of u-fluctuations
ing edge	1 mm dia.	Tables 2,3	10	Preston tube
	Platinum hot wire	----	--	Heat transfer
	0.25 mm high	----	--	Mass transfer (distributed suction, air)
ing edge step	0.05, 0.1 mm high	Tables 1, 3; Plates 2,4	19	Pressure gradient
roughness	0.2 mm high	Tables I, II	18	Mass transfer (distributed blowing, air)
ing edge	0.9 mm dia.	Fig. 2	3	Heat transfer
	0.4 mm dia.	Fig. 4	1	Transition; intensity of u-fluctuations

C

Table I

Author	Date	Tunnel Configuration*	Stream Turbulence Level+	Model	T
Preston and Sweeting	1944	cj,or(d)	High?	Tunnel floor	None
Preston	1954	cj,or(d)	Moderate?	Tunnel wall	None
Reynolds	1957	oj,cr	0.025	Partially shielded plate	Blun surf
Scesa	1954	cj,or(d)	Moderate?	Tunnel ceiling	None
Schubauer and Klebanoff	1955	cj,cr	0.0003	Full-span plate	None
Schultz-Grunow	1940	cj,or(u)	Moderate?	Full-span plate	Blun
Smith and Walker	1958	cj,cr	0.0002	Shielded plate	Blun air
Sogin and Goldstein	1960	cj,or(u)	0.012	Full-span plate	Wire
[illegible]ngos	1956	cj,cr	0.010	Tunnel floor	Distr
Sugawara et al.	1953	cj,or(?)	Variable	Full-span plate	Blun
Tillmann	1945	cj,or(u)	0.0025	Full-span plate	Blunt
Townsend	1951	cj,cr	0.0006	Tunnel floor	Ledge
Tulin and Wright	1949	cj,cr	High?	Tunnel wall	None
Wieghardt	1943	cj,or(u)	0.0025	Full-span plate	Blunt
Wieghardt	1944	cj,or(u)	High	Full-span plate	Blunt
Willmarth	1959	cj,cr	Low?	Pipe wall	Air

* oj = open jet
cj = closed jet
or = open return
cr = closed return
(u) = fan upstream
(d) = fan downstream

+Low = less than 0.001
Moderate = 0.001 to 0.005
High = more than 0.005

A

Table I. -- continued

Str[illegible]am Turbu[illegible]ence Lev[illegible]l+	Model	Tripping Device	Probe Dimensions	Source of I[illegible]
[illegible]igh?	Tunnel floor	None	1 mm dia.	Fig. 2
[illegible]der[illegible]e?	Tunnel wall	None	0.15 mm high	Fig. 7
[illegible].025	Partially shielded plate	Blunt leading edge; surface tape	0.5 mm dia.	Table I
[illegible]der[illegible]e?	Tunnel ceiling	None	0.1 mm high	Fig. 5
[illegible].000	Full-span plate	None	0.35 mm high?	Fig. 3
[illegible]der[illegible]e?	Full-span plate	Blunt leading edge	1 mm dia.?	Figs. 11, 1[illegible]
[illegible]000	Shielded plate	Blunt leading edge; air jets	0.2 x 2 mm	Tables I-II[illegible]
[illegible]012	Full-span plate	Wire	0.28 mm high	Fig. 2
[illegible]010	Tunnel floor	Distributed roughness	Platinum hot wire	Figs. 7, 10[illegible]
[illegible]ria[illegible]e	Full-span plate	Blunt leading edge	Hot wire	----
[illegible]002	Full-span plate	Blunt leading edge; ledge	1 mm dia. (rake)	Private com[illegible]
[illegible]000	Tunnel floor	Ledge	1 mm dia.	Fig. 2
[illegible]gh?	Tunnel wall	None	1.2 mm dia.	Fig. 2
[illegible]002[illegible]	Full-span plate	Blunt leading edge	1 mm dia. (rake)	Private com[illegible]
[illegible]gh	Full-span plate	Blunt leading edge	1 mm dia. (rake)	Private com[illegible]
[illegible]r?	Pipe wall	Air jets	0.16 mm high	Private com[illegible]

[illegible]an [illegible].001
[illegible].00[illegible] to 0.005
[illegible]han 0.005

B

ed

e	Probe Dimensions	Source of Data	Number of Profiles Studied	Other Data of Interest
	1 mm dia.	Fig. 2	3	-----
	0.15 mm high	Fig. 7	--	Preston tube
e;	0.5 mm dia.	Table I	6	Heat transfer (variable wall temperature)
	0.1 mm high	Fig. 5	--	Heat transfer; film cooling
	0.35 mm high?	Fig. 3	2	Mechanics of transition
e	1 mm dia.?	Figs. 11, 12	7	Floating element
e;	0.2 x 2 mm	Tables I-III	70	Floating element; Preston tube
	0.28 mm high	Fig. 2	3	Mass transfer (sublimation, napthalene)
ness	Platinum hot wire	Figs. 7, 10	12	Heat transfer, Reynolds stress
e	Hot wire	----	--	Heat transfer; intensity of u-fluctuations
e; ledge	1 mm dia. (rake)	Private comm.	20	Roughness; pressure gradient
	1 mm dia.	Fig. 2	3	Structure of turbulence
	1.2 mm dia.	Fig. 2	--	Shock-wave interaction
e	1 mm dia. (rake)	Private comm.	43	Roughness; pressure gradient
e	1 mm dia. (rake)	Private comm.	14	Intensity of u-fluctuations
	0.16 mm high	Private comm.	7	Time-space correlations (pressure).

C

conditions. The highly competent measurements by Ludwieg and Tillmann (1949) are not specifically included because they are in some respects a duplication of earlier work by Wieghardt (1943, 1944a) and by Tillmann (1945). The two related series of experiments by Albertson (1948) and by Cermak and Lin (1955) are omitted because the tungsten hot wires which were used were apparently subject to large and erratic changes in calibration*, and I have not even been able to study the influence of the unorthodox tripping device which was employed in these experiments. A paper by Zysina-Molozhen (1956) is not included in the survey because no copy is available to me. A turbulent boundary-layer paper by Nikuradse (1942) is not included because I think that this paper should not be taken seriously, for reasons which will be apparent to anyone who consults the original. And so on.

METHOD OF ANALYSIS

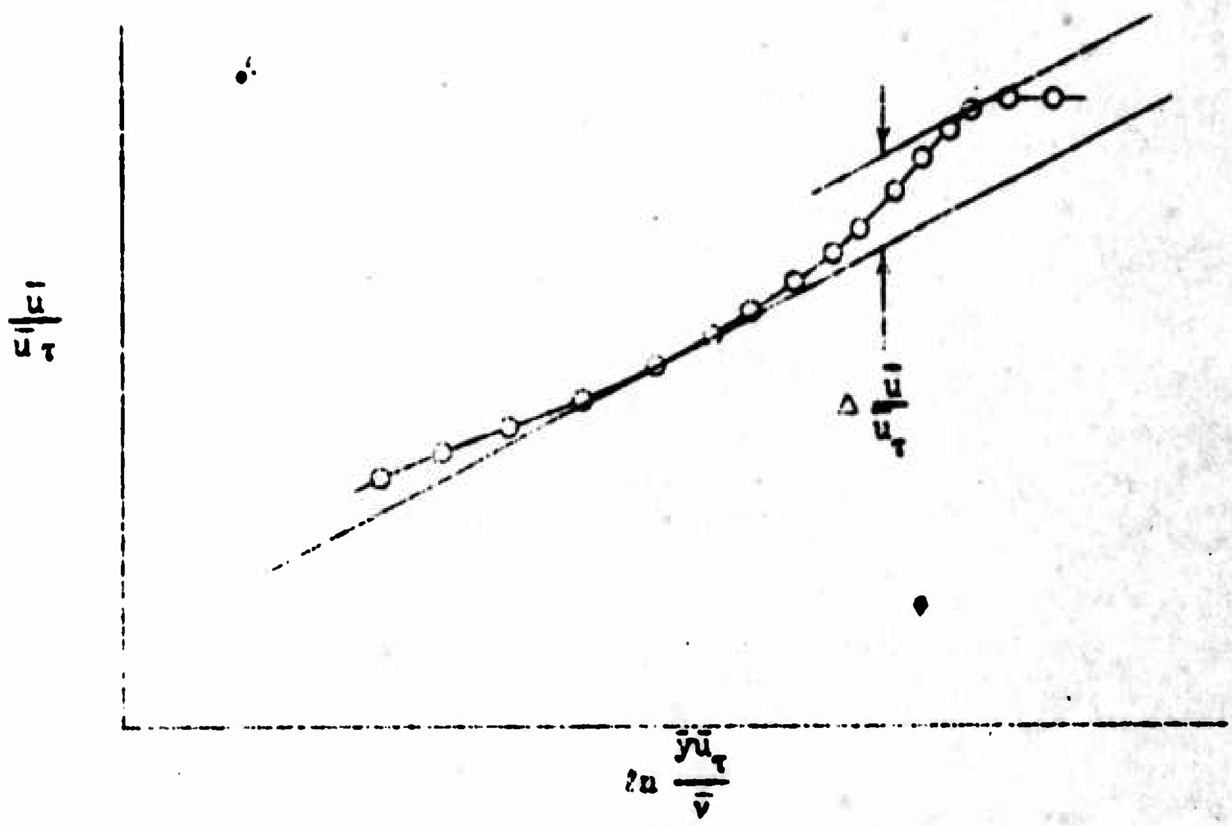
It is no secret that most writers on the problem of the turbulent boundary layer tend to take an intensely personal view of their subject. I should therefore state at the outset that I am definitely prejudiced in favor of the similarity laws known as the law of the wall and the law of the wake (or the defect law), primarily because I feel that to abandon these concepts is to revert to the most primitive kind of empiricism in any description of turbulent boundary-layer flow. In particular, these similarity laws are useful as a more rational criterion than that of simple majority rule in the classification of the nearly five hundred profiles represented by the table. My first step in this process of classification has therefore been to recover the original mean-velocity distributions in the form

*Similar difficulties with tungsten wires are mentioned by Kline et al. (1960).

$\bar{u}(\bar{y})$, removing where possible any corrections for probe displacement effect. My next step has been to determine a parameter $\bar{u}_\tau$ for each mean-velocity distribution by fitting the central part of the profile to a logarithmic formula of the form

$$\frac{\bar{u}}{\bar{u}_\tau} = \frac{1}{K} \ln \frac{\bar{y}\,\bar{u}_\tau}{\bar{\nu}} + c \qquad \text{(A-1)}$$

When this fitting operation is carried out for a typical profile, as shown in the adjacent sketch, the data near the free stream are found to deviate noticeably from the logarithmic formula. The maximum deviation will



be denoted by $\Delta\bar{u}/\bar{u}_\tau$ and will also be referred to as the strength of the wake component for the boundary-layer flow. The accuracy with which this quantity can be determined is probably no better than five or ten per cent, given a typical uncertainty of perhaps 0.01 in $\bar{u}/\bar{u}_\infty$. Within this limitation the parameter $\Delta\bar{u}/\bar{u}_\tau$ is found to be distinguished by an almost exquisite sensitivity to the history and environment of a particular flow, and this property in turn makes possible not only a precise classification of boundary-layer flows, but a refinement and rationalization of the similarity laws which are implicit in the fitting operation itself. For the moment, however, my objective is to compare various profiles one with another, and for this purpose no significance need be attached to the parameter $\bar{u}_\tau$ or to the values of the numerical constants κ (eventually taken as 0.41) and c (eventually taken as 5.0) in the logarithmic formula. In particular, another choice for these constants would not noticeably affect the outcome, at least as far as the process of classification is concerned.

THE EQUILIBRIUM BOUNDARY LAYER

One immediate result of the present survey is the tentative identification of a normal (equilibrium, ideal, fully-developed, asymptotic) state for the turbulent boundary layer at constant pressure. For the experiments which I believe to represent this state, the quantity $\Delta\bar{u}/\bar{u}_\tau$ is plotted against $\bar{R}_\theta$ in Fig. 10. It goes without saying that there should exist a well defined relationship between the two quantities. However, it is perhaps surprising to find that $\Delta\bar{u}/\bar{u}_\tau$ decreases, and hence that the traditional momentum-defect law fails, for values of $\bar{R}_\theta$ less than about 6000. In fact, the wake component seems to have disappeared entirely -- and rather abruptly -- by the time $\bar{R}_\theta$ has reached a value in the neighborhood of about 500.

The validity of this conclusion depends on the validity of the logarithmic formula, in which it is commonly understood that the parameter $\bar{u}_\tau$ can be identified with $\sqrt{\bar{\tau}_w/\bar{\rho}}$. Inasmuch as $\bar{\tau}_w$ was not measured in any of the experiments classified as normal in Fig. 10, the equivalence of $\bar{\tau}_w$ and $\bar{\rho}\,\bar{u}_\tau^2$ can only be tested indirectly, in terms of the approximate equivalence of $\bar{\tau}_w$ and $\bar{\rho}\,\bar{u}_\infty^2\,d\bar{\theta}/d\bar{x}$ for flow at constant pressure. For the data of Fig. 10, therefore, the ratio of $d\bar{\theta}/d\bar{x}$ to $\bar{u}_\tau^2/\bar{u}_\infty^2$ has been plotted against $\bar{R}_\delta$ in Fig. 11. In this and later figures only the general level of the various curves is significant, since not only have effects of pressure gradients and three-dimensionality in the mean flow been ignored, but no account has been taken of the deleterious effects of turbulence, probe interference, and other factors in the determination of $\bar{u}_\tau$ and $\bar{\theta}$. There is also a large inherent uncertainty in the calculation of $d\bar{\theta}/d\bar{x}$ by graphical differentiation. The real issue in Fig. 11, therefore, is not the validity of the relationship $\bar{\tau}_w = \bar{\rho}\,\bar{u}_\tau^2$ over the whole range of Reynolds numbers involved, but rather the classification of the flows in question as normal. To emphasize this point, I want next to consider some flows which cannot be so classified.

MISCELLANEOUS ANOMALIES

Some measurements are collected in Fig. 12 to show that good intentions on the part of the experimenter are not always sufficient to guarantee a normal state for a turbulent boundary layer. The classification of these particular flows as anomalous is sometimes only a matter of degree, although a check of the momentum-integral equation for these data, as shown in Fig. 13, is frequently highly unsatisfactory.

In attempting to account for the observed anomalies, I think it is a significant point of technique that practically all of the data which I have called normal in Fig. 10 were obtained in closed wind tunnels on plates having blunt leading edges fitted in some cases with a tripping device of quite modest dimensions. The one exception to this rule is the experiment of Hickley and Davis (1957), in which the original boundary layer was removed by suction through a forward compartment of the smooth screen forming the working surface. On the other hand, much of the data which I have called anomalous in Fig. 12 was obtained in open-jet tunnels using models not equipped with adequate side plates. Examples are the measurements by Hansen (1928), Reynolds (1957), and Barrow (1958). Thus I feel sure that many of the discrepanies encountered in the latter experiments can definitely be blamed on three-dimensionality in the mean flow. That not even the direction of the departure from the normal state can be safely predicted in advance is clear on comparing the data obtained by Hansen (1928) and by Elias (1929) in what is presumably the same free-jet tunnel at Aachen. The experience of Hama (1947; see Fig. 10) shows that adequate shielding of a model by side plates or by a complete enclosure can overcome these difficulties. However, it does not follow, in view of the experience for example of Ashkenas et al. (1952) and Johnson (1955), that a closed channel automatically insures a satisfactory flow.

The work of Schultz-Grunow (1940) is a major landmark in the experimental literature of turbulent boundary layers, and deserves special consideration. I have presented these data here rather than in Fig. 10 because of a slight but consistent difference in the magnitude of the wake component for Schultz-Grunow's measurements at 19.4 meters per second as compared to later measurements by Wieghardt (1943) at 17.8 and 33.0 meters per second in the same tunnel. The discrepancies involved are entirely trivial by any ordinary standard, amounting to perhaps one per cent in the mean velocity in the vicinity of the point $\bar{y} = \bar{\theta}$ or to perhaps two per cent in the local friction coefficient. When I consulted Professor Wieghardt about this difference, he called my attention to his account (1947) of some changes which were made in the Göttingen tunnel after Schultz-Grunow's experiments had been completed. These changes were intended to reduce the residual turbulence and large-scale rotation in the calming section between the fan and the test section of the tunnel. A second honeycomb and two screens were added, and the length of the calming section was increased by four meters. Thus the test-section conditions in the later experiments should have been appreciably closer to the ideal of an undisturbed free stream. In particular, since the turbulence level was definitely reduced (to 0.0025) by these changes, I believe that the difference between the two sets of data is probably associated with weak large-scale vorticity which caused the flow studied by Schultz-Grunow to be slightly three-dimensional.

One or two comments should also be made about the experiments by van der Hegge Zijnen (1924), as these experiments have never been adequately reviewed in the light of contemporary experimental practice. Not only were these measurements the first important contribution to the experimental boundary-layer literature, but they involved a use of the then new technique of hot-wire anemometry in a way which was years ahead of its time. On the other hand, there were a number of serious defects in the management of the experiments. First, the plate model did not completely span the tunnel, so that the test configuration had all the disadvantages of the open jet and simultaneously had the disadvantage of an appreciable negative pressure gradient caused by growth of the boundary layers on the test-section walls. Second, it was the free-stream velocity at each particular station, rather than the velocity at some standard reference station, which was always adjusted to a pre-determined value. Third, the model was occasionally moved upstream or downstream in an unsystematic way as various stations were surveyed, and this in a flow having a relatively high level of decaying turbulence because of the presence of a honeycomb at the tunnel entrance not far upstream of the model. Fortunately, it will be seen that these factors do not altogether depreciate the value of van der Hegge Zijnen's measurements.

FREE-STREAM TURBULENCE LEVEL

Effects of external turbulence level have so far not been considered, although large values are associated with some of the anomalous data in Fig. 12 (van der Hegge Zijnen 1924, Michel 1950, Spengos 1956, Reynolds 1957, Sogin and Goldstein 1960). To investigate this point, data for a few additional cases in which the turbulence level was deliberately raised in order to study any effects on the flow in the boundary layer are collected in Figs. 14 and 15. I have tried to indicate by the size of the symbols the measured or estimated intensity of free-stream turbulence for each profile, the necessary estimates being based on empirical formulas for the decay of turbulence behind grids. Except possibly at very low Reynolds numbers, the effect of increased stream turbulence is to decrease the strength of the wake component and thus to increase the local friction coefficient at a fixed Reynolds number. Roughly speaking, the quantity $\Delta \bar{u}/\bar{u}_\tau$ can be expected to decrease to about half of its normal value for an external turbulence level of 0.018, and to decrease to zero for a turbulence level of 0.045. The amount of the decrease may also depend on

Reynolds number and on the scale of the turbulence, although the data are far from allowing either effect to be determined quantitatively. In any case, the typical very slow approach to the normal state as the external turbulence decays is well illustrated in the measurements by Wieghardt (1944a).

TRIPPING DEVICES; THE APPROACH TO EQUILIBRIUM

Several experimental studies have recently been made of turbulent boundary layers which were artificially thickened by various means. In principle, it is not difficult to increase the momentum defect in a boundary layer by injecting fluid of low velocity at the boundary or by making use of the drag increment from isolated or distributed roughness elements. All of these devices will also tend to promote transition, and may sometimes be used primarily for this reason. In practice, however, it seems that such techniques must be handled with care, inasmuch as a turbulent boundary layer may recover very slowly from the effects of certain kinds of disturbances. This conclusion is based on the evidence of Fig. 16 and 17, which show the behavior of the wake component and the momentum balance respectively for a number of such artificially thickened boundary layers. These data are quite instructive, and will therefore be discussed at some length.

Consider first the case of a boundary layer artificially thickened by distributed roughness near the leading edge. If a flow at constant pressure is characterized by a defect law which is insensitive to roughness, then the quantity $\Delta\bar{u}/\bar{u}_\tau$ should increase sharply (inversely as the square root of the local friction coefficient, if $\Delta\bar{u}$ remains fixed) immediately on passing from a rough to a smooth surface. It follows that the large eddies, whose energy is proportional on the average to the strength of the wake component, will have too much energy compared to a normal flow at the same Reynolds number. This surplus energy will presumably have to be disposed of by the relatively slow process of transfer to progressively smaller eddies followed finally by dissipation as heat. At the same time, the boundary layer can be expected to adjust itself in such a way as to reduce the local rate of transfer of energy from the mean flow to the turbulence. But the local rate of energy transfer to the large eddies is itself measured by the part of the product $-\overline{u'v'}\,\partial\bar{u}/\partial\bar{y}$ having the boundary-layer thickness as characteristic length, and thus again by the strength of the wake component. It follows that there may be a tendency for the quantity $\Delta\,\bar{u}/\bar{u}_\tau$ to decrease temporarily below its normal value pending the transfer of any excess energy to a sufficiently remote part of the spectrum. Similar but more emphatic statements should apply for a flow which is completely separated at a trip wire or spoiler and which then becomes reattached at some point farther downstream.

These observations account at least qualitatively for the behavior of the flows depicted in Fig. 16, and especially for the fact that the strength of the wake component sometimes drops below the normal or equilibrium value following a strong disturbance. In the case of the experiments by Klebanoff and Diehl (1951) with screen or sand roughness at low free-stream velocities, the recovery is apparently complete by the time the end of the plate is reached, and the momentum balance is also normal. It can safely be assumed, therefore, that the later measurements by Klebanoff (1954) of the structure of turbulence are representative of a turbulent boundary-layer flow in equilibrium. A similar statement can be made for Willmarth's observations (1959) of wall pressure fluctuations, to be taken up in Fig. 18.

The situation for the measurements of Favre et al. (1955) is only slightly less satisfactory. However, the measurements of structure by Townsend (1951) were made downstream of a single relatively large spoiler. These data, although they probably correspond to a later stage of recovery than (say) the data of Klebanoff and Diehl with the 1/4-inch diameter rod or the data of Tillmann (1945) with the ledge, are therefore likely to be representative of a flow with an abnormal distribution of turbulent energy.

It should be noted in Fig. 16 that the data of Klebanoff and Diehl for the 0.040-inch diameter rod at a free-stream velocity of 108 feet per second do not appear to be approaching the normal state far downstream, in spite of the fact that the wake of the thickening device has almost certainly been absorbed. This anomalous behavior seems to be typical of the NBS data at the higher tunnel speed, as the flow with natural transition at 108 feet per second is also anomalous (cf. Fig. 12), while the flow with natural transition at 50 feet per second as reported by Schubauer and Klebanoff (1955) is nearly normal (but cf. Fig. 12). It is possible that the anomalies in question are caused by slight distortions of the flat-plate model by aerodynamic loads at the highest tunnel speeds, although this remark is pure speculation. In any case, similar anomalies which might be present in the data for the 1/4-inch diameter rod or for the sand roughness at 108 feet per second in Fig. 16 are necessarily masked by direct effects of the thickening device.

FLOW AT LARGE REYNOLDS NUMBERS

The data so far considered extend only to values of $\bar{R}_\delta$ in the neighborhood of 15,000. When measurements at higher Reynolds numbers are treated in the same way, as shown in Fig. 18, they are found to disagree both with each other and, except for Willmarth's data (1959), with experiments at lower values of $\bar{R}_\delta$. In particular, if the elegant and elaborate measurements by Smith and Walker (1958) are accepted as definitive, the concept of a defect law must be called into question.

I think it may be significant that a small but definite reduction in scatter in the values of $\Delta\bar{u}/\bar{u}_\tau$ for the data of Smith and Walker can be achieved by plotting these values against Reynolds number per unit length, as shown in Fig. 19, rather than against $\bar{R}_\delta$ directly. In other words, the strength of the wake component for these experiments seems to be constant along the model for a fixed tunnel condition, implying a defect law whose parameters depend on tunnel conditions rather than on distance from the leading edge of the model. This observation in turn suggests that some factor such as external turbulence level, probe interference, or three-dimensionality may be affecting these data. Stream turbulence can probably be eliminated on the ground that measurements by Boltz et al. (1960) in the same tunnel show values much too low to account for the observed effects. Three-dimensionality is a possible but somewhat improbable source of difficulty, inasmuch as the momentum balance shown in Fig. 20 is satisfactory. By contrast, the data of Allan and Cutland (1953) in Fig. 18 refer to a towed model with a free edge. Three-dimensional effects were definitely present near this edge, and were investigated in some detail. The data of Willmarth (1959), finally, are presumably free of three-dimensional effects, in view of the circular symmetry of his channel, although there may be some influence of the slight negative pressure gradient.

Smith and Walker, having made direct measurements of $\bar{\tau}_w$ for most of their mean-velocity profiles, adopted the view that the law of the wall in the form

$$\frac{\bar{u}}{\sqrt{\bar{\tau}_w/\bar{\rho}}} = \frac{1}{\kappa} \ln \frac{\bar{y}\sqrt{\bar{\tau}_w/\bar{\rho}}}{\bar{\nu}} + c \qquad \text{(A-2)}$$

could be taken to be independently defined by their data. Fig. 21 shows a typical profile at $\bar{R}_\delta$ = 26,600 in the coordinates of Eq. (A-2), with $\bar{\tau}_w$ the directly measured value corrected for gap effect.* The solid line in the figure is Eq. (A-1) with κ = 0.41 and c = 5.0, as determined by a concensus of other measurements; it is a fit to this line which defines $\bar{u}_\tau$. As shown by Fig. 22, which also includes measurements by Schultz-Grunow (1940) and Dhawan (1951), the ratio $\bar{\tau}_w/\bar{\rho}\ \bar{u}_\tau^2$ is roughly 0.98 for the data of Smith and Walker over the whole range of Reynolds numbers. The data in question therefore support the law of the wall in principle. Unfortunately, any reasonable revision of the constants κ and c in going from Eq. (A-1) to Eq. (A-2) in order to produce agreement in Fig. 21 will not remove the difficulties in Fig. 18. On the other hand, the variations in the strength of the wake component for Smith and Walker's data in Fig. 18 can be reduced by about half if the solid line in Fig. 21 and the measured value of $\bar{\tau}_w$ are both accepted as correct, but $\bar{u}$ or $\bar{y}$ is assumed to be in error because of probe interference (say; cf. the similar but more serious difficulties reported by Matting et al. (1960) at high supersonic speeds).

---

*My own experience with the floating-element technique has convinced me that some account should be taken of the finite drag of the gap around the element. Because of symmetry, half of this drag should act on the element and half on the surrounding surface. The gap drag is unlikely to be less than the drag on an equal area of unbroken surface, and can reasonably be estimated at twice this value. I have therefore recomputed τ_w as (measured force)/(element area + gap area) for all of the floating-element data considered in this paper, including my own. The correction in the case of the measurements by Smith and Walker, by Schultz-Grunow, and by Dhawan is one per cent, two per cent, and fifteen per cent respectively.

However, it is then necessary to explain why similar effects of probe interference are not present in Willmarth's data, since the tripping devices, probe dimensions, Mach numbers, and unit Reynolds numbers were quite similar in the two experiments. I am willing to leave this question open, noting only that it seems to be necessary to choose between two alternatives; either (1) some of the experimental data at large Reynolds numbers must be rejected, on grounds which are obscure at best; or (2) the similarity laws for the mean velocity profile, particularly the defect law, are not valid at the high level of precision attempted in this survey.

THE LOCAL FRICTION LAW

Starting with the local friction coefficient $\bar{C}_f(\bar{R}_\delta)$ given by the formula

$$\sqrt{\frac{2}{\bar{C}_f}} = \sqrt{\frac{\bar{\rho}\,\bar{u}_\infty^2}{\bar{\tau}_w}} = \frac{\bar{u}_\infty}{\bar{u}_\tau} = \frac{1}{\kappa}\ln\frac{\bar{\delta}\,\bar{u}_\tau}{\bar{\nu}} + c + \frac{2\Pi}{\kappa} \qquad \text{(A-3)}$$

a more convenient expression for $\bar{C}_f(\bar{R}_\delta)$ can be obtained with the aid of the definition

$$\bar{\theta} = \int_0^\infty \frac{\bar{u}}{\bar{u}_\infty}\left(1 - \frac{\bar{u}}{\bar{u}_\infty}\right) d\,\bar{y} \qquad \text{(A-4)}$$

where

$$\frac{\bar{u}}{\bar{u}_\infty} = \frac{\bar{u}_\tau}{\bar{u}_\infty}\left[f\left(\frac{\bar{y}\,\bar{u}_\tau}{\bar{\nu}}\right) + \frac{\Pi}{\kappa}\,w\left(\frac{\bar{y}}{\bar{\delta}}\right)\right] \qquad \text{(A-5)}$$

The functions $f(\bar{y}\bar{u}_\tau/\bar{\nu})$ and $w(\bar{y}/\bar{\delta})$ in Eq. (A-5) have been tabulated elsewhere (for f see Coles 1955, but note that f is now reduced by two per cent, so that the constants κ and c are to be taken as 0.41 and 5.0 respectively*;

*My most important reason for this revision can be documented with the aid of Fig. 3 of the present paper. Whatever the motivation for choosing the particular coordinates in this figure, the fact remains that these for $\bar{\nu}$ see Coles 1956). The parameter Π is given as an empirical function of $\bar{R}_\delta$ by Fig. 16 of the present paper; it increases from zero at about $\bar{R}_\delta = 500$ to an asymptotic value of 0.55 ($\Delta\bar{u}/\bar{u}_\tau = 2\Pi/\kappa = 2.70$) for $\bar{R}_\delta > 6000$.

Figs. 4, 8, and 23 and the adjacent Table II show the local friction law and related quantities as computed from Eqs. (A-3)-(A-5). For $\bar{R}_\delta > 20{,}000$ the relationship between $\bar{\theta}$ and $\bar{\delta}$ and between $\bar{\delta}^*$ and $\bar{\delta}$ can be obtained directly from the asymptotic formulas

$$\kappa \frac{\bar{u}_\infty}{\bar{u}_\tau} \frac{\bar{\delta}^*}{\bar{\delta}} = 1 + \Pi \tag{A-6}$$

$$\left(\kappa \frac{\bar{u}_\infty}{\bar{u}_\tau}\right)^2 \frac{\bar{\theta}}{\bar{\delta}} = \kappa \frac{\bar{u}_\infty}{\bar{u}_\tau} (1 + \Pi) - (2 + 3.200\Pi + 1.522\Pi^2) \tag{A-7}$$

which neglect the effect of viscosity in the sublayer, in the sense that $f(\bar{y}\bar{u}_\tau/\bar{\nu})$ in Eq. (A-5) is replaced by $(1/\kappa) \ln \bar{\delta}\,\bar{u}_\tau/\bar{\nu} + c + (1/\kappa) \ln \bar{y}/\bar{\delta}$. In the range $\bar{R}_\theta > 10{,}000$ the data in Fig. 16 indicate a slight decrease in the parameter Π, but the evidence is not really conclusive. I also have some reservations about the local friction law in the range $\bar{R}_\theta < 6000$. The behavior of the boundary-layer flow in this region is reminiscent of the

---

coordinates lead to a satisfactory correlation of the floating-element measurements in supersonic flow, with very little residual effect of Mach number and Reynolds number. If the low-speed friction law is obtained by the fitting operation described in this appendix, but with $\kappa = 0.40$ and $c = 5.1$ as recommended in my earlier papers, the best straight line in Fig. 5 seems to intersect the vertical axis slightly above the point (0,1). Inasmuch as the extrapolation in question amounts to the inference of a low-speed friction law from data obtained in compressible flow, consistent results can only be achieved if the values of $\bar{C}_f$ are increased by about four per cent, or the values of the function $f(z)$ defining the law of the wall are reduced by about two per cent (except of course near $z = 0$, where $f \rightarrow z$). Other evidence can be found to support this revision; e.g., the resulting friction law for low-speed flow in Fig. 4 is a reasonably good fit to the floating-element measurements by Smith and Walker (1958) and Schultz-Grunow (1940), at least in the range where these data overlap.

Table II

THE LOCAL FRICTION LAW FOR THE TURBULENT BOUNDARY LAYER AT CONSTANT PRESSURE

$\frac{\bar{\delta} \bar{u}_\tau}{\nu}$	Π	$\bar{C}_f$	$\bar{R}_{\bar{\theta}}$	$\frac{\bar{\delta}^*}{\bar{\theta}}$	$\bar{C}_f \bar{R}_{\bar{\theta}}$
240	0	.00590	425	1.535	2.51
300	.12	.00524	590	1.500	3.10
400	.23	.00464	855	1.470	3.97
500	.30	.00426	1150	1.445	4.88
600	.36	.00398	1450	1.425	5.73
800	.43	.00363	2050	1.405	7.41
1000	.48	.00340	2650	1.390	8.94
1500	.53	.00308	4150	1.365	12.75
2000	.55	.00290	5650	1.350	16.36
3000	.55	.00269	8600	1.325	23.2
4000	.55	.00255	11500	1.315	29.6
5000	.55	.00246	14500	1.305	35.9
6000	.55	.00238	17500	1.295	41.8
8000	.55	.00227	23500	1.290	53.6
10000	.55	.00219	29000	1.285	64.8

slow initial approach to full similarity in a free shear flow such as a wake or jet, and can be interpreted either as a development of the large-eddy structure toward equilibrium or as a change in the conditions defining equilibrium. Under these circumstances it may not be proper to refer to the flows in question as normal or as fully developed. On the other hand, this behavior is typical not only of the data plotted in Fig. 10, but also of the data plotted in Figs. 12 and 14. Thus there is substantial evidence in favor of the existence of the hypothetical substructure flow defined in Section IV as a real limiting flow, and experiments dealing specifically with this point seem to me to be worth while in spite of their difficulty.

SOME IMPRESSIONS

Perhaps the most important point to emerge from this study of experimental data is that it is not as easy to produce a normal boundary layer as is commonly supposed. Such a layer once produced can be easily recognized, for example by means of a joint test of momentum balance and of the magnitude of the wake component in the manner demonstrated here. Considerable care should be exercised in the choice of tripping or thickening devices, as the recovery of a boundary-layer flow from strong disturbances may be very slow and may involve three-dimensional effects. A blunt leading edge seems to be a satisfactory tripping device for most purposes.

In fundamental work it is desirable to limit the external turbulence level to a value well below one per cent, and to remove any large-scale rotation produced by fans or compressors located upstream of the test surface. Even so, difficulties with three-dimensional flow should be expected, especially in tunnels which are not fully enclosed, and any model mounted in an open jet should be carefully shielded. One test for two-dimensionality sometimes used, a comparison of profiles measured at several stations across the flow, has at most a negative value. Some attention should be paid to the advantages of configurations having axial symmetry, as used for example by Feindt (1956), Brevoort and coworkers (1958), and Willmarth (1959). This suggestion is of course not original here, and is most relevant for flows involving separation or reattachment.

If scatter is to be avoided in measurements extending over a considerable period of time, it is important to monitor secular changes in Reynolds number caused by changes in fluid density or viscosity. One technique worth considering is the use of multiple impact tubes or hot wires. The rake developed at Göttingen (Wieghardt 1944b), for example, was undoubtedly a significant factor in the success of the experiments in the roughness tunnel. For estimation of surface shearing stress the Preston tube (Preston 1954) provides a means so simple and reliable that its use ought to be made compulsory. Finally, more emphasis should be put on hot-wire instrumentation in measurements of mean velocity and mean temperature.

One interesting byproduct of this survey is the observation that none of the low-speed measurements using floating elements (Schultz-Grunow 1940, Dhawan 1951, Smith and Walker 1958) seem to be completely free of anomalies. A similar statement applies for several experiments aimed at the problem of low-speed turbulent heat transfer (Elias 1929, Johnson 1955, Spengos 1956, Reynolds 1957 and others), although it does not follow that the usefulness of the heat-transfer data is seriously impaired. By contrast, two recent experiments aimed at the problem of mass transfer (Dutton 1955b, Mickley and Davis 1957) recommend themselves, so that the associated measurements with v_w different from zero can be treated with some confidence. In future research in heat and mass transfer, more effort should be made to work in terms of local conditions. In the present state of the art, it is similarity laws for mean velocity, mean temperature, and mean concentration which seem to provide the best means for correlation of data from various sources. Unfortunately, the validity of the commonly accepted similarity laws is under a small cloud at the close of this survey, and further measurements of skin friction are needed at large Reynolds numbers in low-speed flow in order to clarify this question. For this purpose it is probably better to achieve a large Reynolds number by increasing scale, i.e., by using a large facility, rather than by increasing velocity or density or both.

Finally, I will make a plea for experimenters to report their results more completely, either in graphical or in tabular form, for the sake of future studies like the present one. At any given moment the last word on the subject of turbulence may be an analytical one, but the next to the last word is usually experimental.

Appendix B
THE RELATED ANALYTICAL LITERATURE

THE EFFECTIVE-TEMPERATURE HYPOTHESIS

Because the analysis presented in this paper has elements in common with several previous contributions to the literature of turbulent compressible flow, a brief review of the relevant literature is in order. This review will consider, without attempting to be exhaustive, (1) the concept of a mean or effective temperature as a basis for estimating the effect of compressibility on turbulent skin friction; (2) the concept of a laminar film, or sublayer, whose properties control the friction and especially the heat transfer; and (3) previous work on transformations designed to reduce the equations of mean motion partly or wholly to incompressible form.

The main assumptions of the effective-temperature hypothesis are that a friction law is given for incompressible flow in the form $\bar{C}_f = \bar{C}_f(\bar{R})$, and that this same law is valid for compressible flow provided only that the density and viscosity appearing in the friction law (as distinguished from the density and viscosity appearing in the definitions of C_f and R) are evaluated at a suitably chosen mean or effective temperature T_m.

In other words, $\bar{C}_f$ and $\bar{R}$ are to be replaced in the friction law by

$$\bar{C}_f = \frac{2\tau_w}{\rho_m u_\infty^2} = \frac{\rho_\infty}{\rho_m}\frac{2\tau_w}{\rho_\infty u_\infty^2} = \frac{\rho_\infty}{\rho_m} C_f \qquad \text{(B-1)}$$

and by

$$\bar{R} = \frac{\rho_m u_\infty L}{\mu_m} = \frac{\rho_m}{\rho_\infty}\frac{\mu_\infty}{\mu_m}\frac{\rho_\infty u_\infty L}{\mu_\infty} = \frac{\rho_m}{\rho_\infty}\frac{\mu_\infty}{\mu_m} R \qquad \text{(B-2)}$$

respectively. These two relationships, insofar as they provide rules for computing C_f and R when $\bar{C}_f$ and $\bar{R}$ are given, are equivalent to a transformation. The proper choice of the reference length L in Eq. (B-2), however, is obvious only for certain special cases of laminar flow. For example, if the pressure is constant and the fluid is a perfect gas with viscosity proportional to temperature, the combinations $C_f\sqrt{R_x}$ and $C_f R_\theta$ are known to be independent of Reynolds number, Mach number, Prandtl number, and heat-transfer rate. It follows that the equations $\bar{C}_f = C_f T_m/T_\infty$ and $\bar{R}_L = R_L(T_\infty/T_m)^2$ will be consistent for an arbitrary choice of T_m if $L = x$, but will be inconsistent if $L = \theta$. In the latter case it is evidently necessary to put, instead of (B-2),

$$\bar{R}_\theta = \frac{\rho_\infty u_\infty \theta}{\mu_m} = \frac{\mu_\infty}{\mu_m}\,\frac{\rho_\infty u_\infty \theta}{\mu_\infty} = \frac{\mu_\infty}{\mu_m} R_\theta \qquad \text{(B-3)}$$

or perhaps

$$\bar{R}_\theta = \frac{\rho_m u_\infty \theta}{\mu_\infty} = \frac{\rho_m}{\rho_\infty}\,\frac{\rho_\infty u_\infty \theta}{\mu_\infty} = \frac{\rho_m}{\rho_\infty} R_\theta \qquad \text{(B-4)}$$

Thus if $L = \theta$ the viscosity or the density, but not both, should be evaluated at the mean temperature T_m. Alternatively, Eq. (B-3) can be derived by assuming that Eq. (B-2) is correct when $L = x$ and that mean and local friction coefficients should transform according to the same rule.

The quantitative formulation of the effective-temperature concept* begins with a combination of Eqs. (B-1) and (B-2) in the form

*The real origins of this concept are to be found in early work on heat transfer in enclosed channels. For such problems the idea of a bulk or mixing-cup temperature arises quite naturally in any heat-balance calculation. In this review, however, I will restrict myself to the concept of mean temperature as it has developed in the context of the boundary-layer equations for compressible flow.

$$\bar{C}_f \sqrt{\bar{R}_x} = \sqrt{\frac{\rho_\infty \mu_\infty}{\rho_m \mu_m}} \; C_f \sqrt{R_x} \qquad \text{(B-5)}$$

which is appropriate for laminar flow with similarity when the pressure is constant. If the fluid is a perfect gas but the viscosity is not proportional to temperature, the product $C_f \sqrt{R_x}$ will depend on Prandtl number, Mach number, and heat-transfer rate. This dependence can first be determined with the aid of exact solutions of the laminar boundary-layer equations, whereupon Eq. (B-5) can be used to find an appropriate value of T_m, hopefully as a linear combination of T_∞, T_w, and T_{o_∞} (or T_{a_w}, the adiabatic wall temperature). This formula for T_m, in conjunction with the transformation (B-1)-(B-2) with L = x, then serves as an interpolation and extrapolation device for estimating skin friction and heat transfer when exact solutions are not available.

The argument given here was first outlined by Karman in his Volta Congress paper (1935). Karman also suggested a similar interpolation device for turbulent flow, and proposed as a first approximation for T_m the wall temperature T_w. The merit of this intuitive suggestion has been established by a gradual accumulation since 1935 of exact solutions of the laminar equations on the one hand and of experimental data for turbulent flow on the other. An almost direct line of descent from Karman's formulation to the one in use today can be traced through papers by Rubesin and Johnson (1949), Young and Jansson (1952), and Eckert (1954, 1960). Experimental contributions to the turbulent problem have also been made by Tucker (1951), Sommer and Short (1955), and others. Eckert, in particular, has acquired a kind of proprietary interest in the reference-temperature method by virtue of his participation in applications of this method to laminar problems marked by chemical changes (including dissociation) and by variations in surface pressure and temperature. As a result, his 1954 formula for the reference temperature T_m,

$$T_m = 0.50\,(T_w + T_\infty) + 0.22\,(T_{a_w} - T_\infty) \qquad \text{(B-6)}$$

is frequently quoted, as is an equivalent formula for reference enthalpy. At least two writers (Rott 1957, Burggraf 1961) have derived formulas similar to Eq. (B-6) by assuming that the characteristic temperature for turbulent flow is the temperature at the edge of the sublayer (but see also my comments following Eq. (B-13) and following Eqs. (B-15)-(B-17) of this appendix). In these two papers, however, the main object is to determine not skin friction but heat transfer in terms of skin friction, and the main emphasis is therefore on the question of an energy integral for turbulent flow. A formula similar to (B-6) has also been derived by Monaghan (1955) using a mean representation of the Crocco energy integral.

In most of this work the independent variable has been the Reynolds number R_x, for reasons which have little relevance for turbulent flow. One difficulty is that there is no generally accepted definition for R_x in a turbulent boundary layer. Thus Eckert's formula (B-6) requires $\bar{R}_x/R_x = (T_\infty/T_m)^2$ to be a rapidly decreasing function of Mach number (i.e., $T_m/T_\infty = 1 + 0.72\, r(\gamma-1)\, M_\infty{}^2/2$ for adiabatic flow), and there is no assurance that turbulent compressible flows cannot be observed at values of M_∞ and R_x such that $\bar{R}_x$ is small enough to be off scale from the experimental point of view. If such cases occur, extrapolation of a particular analytical formula for $\bar{C}_f(\bar{R}_x)$ to low Reynolds numbers in order to generate values for $\bar{C}_f$ and C_f is a questionable procedure at best.

It is therefore unfortunate that much of the conceptual simplicity of the effective-temperature method is lost in working with local quantities, as for example when Eq. (B-1) is combined with the awkward Eq. (B-3) to obtain

$$\bar{C}_f \bar{R}_\theta = \frac{\rho_\infty \mu_\infty}{\rho_m \mu_m} C_f R_\theta \tag{B-7}$$

This expression bears a superficial resemblance to the law of corresponding stations, Eq. (2.21) of the text. The law of corresponding stations, however, is part of a genuine transformation of physical flow variables, while the relationship (B-7) is probably best described as a plausible but somewhat arbitrary mapping of certain dimensionless parameters of the problem. The same distinction can be made between Eqs. (4.22) and (B-6) for the characteristic temperatures T_s and T_m respectively, and between Eqs. (2.17) and (B-1) for the local friction coefficients. In fact, for the case of viscosity proportional to temperature the latter two equations become formally identical when T_s is identified with T_m, and Eckert's formula (B-6) for adiabatic flow can then be plotted in Fig. 5 as the straight line $T_m/T_w = T_\infty \bar{C}_f / T_w C_f = 0.72 + 0.28\ T_\infty/T_w$. Granted that a better overall fit to the turbulent data could be achieved by altering the numerical constants in Eckert's formula, a more important point here is that this formula is incapable in any case of representing the effect of Reynolds number in the figure.

THE LAMINAR-FILM HYPOTHESIS

A second empirical approach to the problem of the turbulent boundary layer makes use of the film or sublayer hypothesis. For incompressible flow, for example, the assumption is that the mean-velocity profile can be adequately represented by a combination of a linear profile $\bar{u} = \bar{\tau}_w \bar{y}/\bar{\mu}$ near the wall and by (say) a power law $\bar{u}/\bar{u}_\infty = (\bar{y}/\delta)^{1/n}$ elsewhere in the boundary layer. The two curves intersect at the edge of the sublayer, where $\bar{u}$ and $\bar{y}$ have the values $\bar{u}_f$ and $\bar{y}_f$ respectively. Once the ratios $\bar{u}_f/\bar{u}_\infty$ and $\bar{y}_f/\delta$ have been expressed in terms of $\bar{C}_f$ and $\bar{R}_\delta$, it is clear that one further condition relating $\bar{u}_f$ to $\bar{y}_f$ will determine a local friction law in the form $\bar{C}_f = \bar{C}_f(\bar{R}_\delta)$. The condition in question may be obtained by assuming a constant sublayer Reynolds number $R_f = \bar{u}_f \bar{y}_f/\bar{\nu}$, or alternatively by assuming a constant ratio of laminar to turbulent shearing stress at the edge of the sublayer, the stresses being evaluated with the aid of a suitable mixing-length formula. For the case of a power-law profile outside a linear sublayer these two assumptions can be shown to be completely interchangeable.

This formulation of the film hypothesis was begun independently by Prandtl (1910) and Taylor (1916) in connection with the problem of turbulent heat transfer. It was later carried essentially to completion by Prandtl (1928), who derived a suitable sublayer condition from the Blasius local friction law rather than vice versa. The formulation described here is due to Donaldson* (1952), who also extended his analysis to the case of high-speed flow by assuming in effect that the film Reynolds number, redefined as $R_f = \rho_f u_f y_f/\mu_f$, should be unaffected by compressibility or heat transfer. The new variables ρ_f and μ_f at the edge of the sublayer appear both in R_f and in the formula $\tau_w = \mu_f u_f/y_f$, and can be determined when a viscosity law and an energy integral have been specified.

*I believe that Donaldson's paper deserves a prominent place in any review of the analytical literature of the compressible turbulent boundary layer. The reason is that this paper has stood almost alone, pending the first application of coordinate transformations to turbulent flow, in recognizing the increase in the relative thickness of the sublayer at large Mach numbers as the dominant effect of compressibility.

Among the relationships derived explicitly or implicitly in Donaldson's original paper are

$$C_f R_\theta = 2 \frac{\theta}{\delta} \frac{T_f}{T_\infty} \left(\frac{u_f}{u_\infty} \right)^{1-n} \qquad \text{(B-8)}$$

$$C_f = \frac{2}{R_f} \frac{T_\infty}{T_f} \left(\frac{u_f}{u_\infty} \right)^{2} \qquad \text{(B-9)}$$

$$\frac{T_f}{T_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^2 \left[1 - \left(\frac{u_f}{u_\infty} \right)^2 \right] \qquad \text{(B-10)}$$

where it has been assumed for the sake of simplicity that T_o is constant and that μ is proportional to T. The first of these equations follows on eliminating y_f between the linear and power-law profiles at their intersection*; the second on eliminating y_f in favor of R_f in the linear profile; and the third on putting $u = u_f$ and $T = T_f$ in the energy integral $C_pT + u^2/2 = C_pT_o$ = constant (say).

If now u_f/u_∞ is eliminated between (B-8) and (B-9) on the one hand and between (B-9) and (B-10) on the other, a local friction law for $C_f(R_\theta)$ is obtained in parametric form as

$$C_f = 2 R_f^{\frac{1-n}{1+n}} \left[\left(\frac{T_\infty}{T_f} \right)^{\frac{n-3}{n+1}} \frac{\delta}{\theta} R_\theta \right]^{-\frac{2}{n+1}} \qquad \text{(B-11)}$$

$$\frac{T_w}{T_f} = 1 + \frac{R_f}{2} \frac{\gamma-1}{2} M_\infty^2 C_f \qquad \text{(B-12)}$$

For $M_\infty = 0$ and $n = 7$, Eq. (B-11) becomes the Blasius law $C_f = 0.045 R_\delta^{-1/4}$,

---

*Tables of the ratio θ/δ, which depends on M_∞, on T_w/T_∞, and on the exponent n in the power-law profile, have been published by several authors (e.g., Tucker 1951; Persh and Lee 1956) for the more general density-velocity relationship given by the Crocco energy integral.

in which the numerical coefficient 0.045 implies the value $R_f = 158$ noted by Donaldson.

It is worth noting that Eq. (B-11), which is the same as Eq. (17) of Donaldson's paper (TN 2692), can be rewritten for a general viscosity law in the terminology of the effective-temperature concept; thus

$$\frac{2\,\tau_w}{\rho_f u_\infty^2} = 2\,R_f^{\frac{1-n}{1+n}} \left(\frac{\rho_f u_\infty \delta}{\mu_f}\right)^{-\frac{2}{n+1}} \qquad \text{(B-13)}$$

Within the limitations of the power-law approach, therefore, the local friction law in the form $C_f = C_f(R_\delta)$ is found to be unaffected by compressibility or heat transfer if the density and viscosity are evaluated at a film temperature T_f which is defined for the case of constant stagnation temperature by Eq. (B-12). Consequently, one effect of the special assumptions of the film hypothesis, as compared to the much less specific assumptions of the effective-temperature hypothesis, is to provide an explicit formula (showing a weak dependence on Reynolds number) for an effective temperature T_f. The film analysis, however, does not appear to be equivalent to a transformation, as no rules are provided for the identification of corresponding stations.

Finally, the close resemblance of the formula (B-12) to my equation (4.33), which can be written

$$\frac{T_w}{T_s} = 1 + \frac{\langle f^2 \rangle}{2}\,\frac{\gamma-1}{2}\,M_\infty^2\,C_f \qquad \text{(B-14)}$$

also deserves a comment. The linear sublayer profile is a rudimentary form of the law of the wall, and the condition R_f = constant is a general consequence of the latter law, at least for the case of incompressible flow.

Donaldson's intuitive and indirect extension of this condition to the compressible case (in terms of the ratio of shearing stresses rather than the Reynolds number R_f, and without benefit of any real similarity law for the mean-velocity profile) sets a precedent which I am pleased to follow. However, my own arguments at the beginning of Section IV proceed by way of the full law of the wall, the transformation, and the presentation of experimental data in Fig. 5, and the most important element common to the two approaches is the emphasis on conditions within and near the edge of the sublayer. Moreover, Donaldson did not notice, perhaps because he did not actually write out Eq. (B-12) or perhaps because his analysis neither suggested nor required such a result, that the effective temperature ratio T_f/T_w in this equation is in fact a function only of γ and τ_w/p and is thus formally independent of conditions in the outer flow, including the exponent n in the power-law representation of the mean-velocity profile.

An analysis not unlike that of Donaldson has also been described by Spence (1959), who combined a power-law outer profile with the law of the wall near the surface, the two curves intersecting (more properly, osculating) at a point in the logarithmic region of the latter law. On matching the velocity gradients as well as the velocities at this point, he obtained a power law for $\bar{C}_f(\bar{R}_\delta)$. Spence's treatment of the compressible case included the substitution of ρ_m for $\bar{\rho}$ and μ_m for $\bar{\mu}$, with ρ_m and μ_m evaluated at Eckert's mean temperature. He also introduced a partial transformation by substituting $\eta = \int_0^y (\rho/\rho_m)\, dy$ for $\bar{y}$. The resulting local friction law differs only slightly from Donaldson's formula (B-13). Spence also examined a limited amount of experimental data in transformed coordinates. His most significant contribution to the problem of the turbulent boundary layer, however, is probably his ingenious derivation of a power-law shearing-stress profile for use in the integration of the energy equation.

TRANSFORMATIONS

The prior art in transformation of the turbulent boundary-layer equations begins with papers by Dorodnitsyn (1942b) and Van Le (1953), both of whom considered only the momentum-integral equation of von Karman. The first attempt to transform the differential equations of mean motion for a turbulent boundary layer was made by Mager (1957), in an analysis which is restricted to the case of adiabatic flow of a perfect gas with Pr = 1 and μ/T = constant. Mager's three main assumptions are (1) that streamlines should be transformed into streamlines by the transformation; (2) that laminar and turbulent shearing stresses should be transformed by the same rules (in Mager's paper this assumption is expressed by a remarkable rationalization which has sometimes been repeated by later writers, e.g., Vaglio-Laurin 1958); and (3) that the transformation should reduce to the identity transformation when both flows are incompressible*.

---

*The limitations inherent in these assumptions have controlled the evolution of my own work on the transformation. My original point of departure was a previous generalization of the law of the wall to compressible flow (Coles 1959). This generalization resembled Eq. (4.4) of the present paper, except that $\sigma\mu_w/\bar{\mu}$ was written as ρ_τ/ρ_w, with $\tau_w = \rho_\tau u_\tau^2$ by definition. The sublayer hypothesis was first formulated essentially as in Eq. (4.9), except that ρ_s was written as ρ_τ and μ_s was written as $\rho_w\mu_w/\rho_\tau$. For the special case T_e = constant and μ/T = constant, a formula for μ_s/μ_w equivalent to Eq. (4.12) was then obtained. This formula, however, required μ_s to vary with τ_w and hence with x in flow at constant pressure, whereas the original generalization of the law of the wall required $\mu_s = \rho_w\mu_w/\rho_\tau$ to be independent of x.

In the hope of resolving this inconsistency, I next took up the question of a full transformation of the differential equations of mean motion. My first essay differed from the one reported in this paper in two important respects. Firstly, the parameter σ was omitted throughout, inasmuch as I assumed (following Stewartson, Mager, and others) that streamlines should be transformed into streamlines. Secondly, physical considerations played a part only in the reduction of the equation corresponding to

The third of these three assumption accounts for the fact that the concept of effective temperature is frequently invoked in connection with the transformation. Most writers on the subject have considered one or more of the relationships (here written in my notation, with viscosity taken as proportional to temperature)

$$\frac{C_f}{\bar{C}_f} = \frac{\bar{\rho}}{\rho_\infty} = \frac{\mu_\infty}{\bar{\mu}} \tag{B-15}$$

$$\frac{R_\theta}{\bar{R}_{\bar{\theta}}} = \frac{\rho_\infty}{\bar{\rho}} = \frac{\bar{\mu}}{\mu_\infty} \tag{B-16}$$

$$\frac{R_x}{\bar{R}_{\bar{x}}} = \frac{\rho_\infty}{\bar{\rho}} \frac{\bar{\mu}}{\mu_\infty} = \left(\frac{\rho_\infty}{\bar{\rho}}\right)^2 = \left(\frac{\bar{\mu}}{\mu_\infty}\right)^2 \tag{B-17}$$

The only distinction ordinarily made between these equations and the corresponding equations (B-1)-(B-3) of the effective-temperature analysis is that the reference quantities $\bar{\rho}$ and $\bar{\mu}$ represent fluid properties in an incompressible flow which is related to the compressible one by the transformation, while the reference quantities ρ_m and μ_m refer to fluid properties

---

(2.8). The resulting analysis, although it did yield the law of corresponding stations, did not succeed in removing the inconsistency at issue. The ordinate of Fig. 5, for example, appeared as $\bar{\mu}/\mu_w$; consequently, this ratio was fixed by the transformation, and the parameter $\bar{\mu}$ could not be chosen arbitrarily. Furthermore, the effect of Reynolds number in this same figure could only be accounted for by allowing $\bar{\mu}$ to depend on x.

There followed a considerable pause, during which the logical premises of the analysis were reexamined. It finally became clear that the fundamental assumption $\bar{\psi} = \psi$ for the transformation was in fact an unnecessary and unjustifiable extrapolation of experience with laminar flow. The function σ was then introduced by writing $\bar{\psi} = \sigma\psi$, as in Eq. (2.3), and the transformation was worked out substantially as given here except that $\bar{p}$ and $\bar{\tau}$ were still defined (again following Stewartson and Mager) in the conventional way, without regard to the principle of observability. As a result the at some intrinsic reference state for the compressible flow itself. If the transformation is required to reduce to the identity transformation when both flows are incompressible, this becomes a distinction without a difference, and the choice of a reference state for the transformation is equivalent to an application of the effective-temperature method.

Thus Mager took $\bar{\rho}$ and $\bar{\mu}$ to be the isentropic stagnation values for the compressible flow, and used Eqs. (B-15) and (B-17) and a suitable low-speed friction law to compute $C_f/\bar{C}_f = \mu_\infty/\mu_{o_\infty}$ for fixed $C_f\sqrt{R_x}$. In this calculation, which is very like that of Karman (1935), Mager assumed a power law for $\mu(T)$ in place of the relationship μ/T = constant appropriate to his transformation. Burggraf (1961) later modified Mager's result by evaluating $\bar{\rho}$ and $\bar{\mu}$ at the edge of the sublayer; i.e., at the intersection of the linear and logarithmic representations for the mean-velocity profile, without commenting on the fact that the fluid properties in the incompressible flow must then depend on x. Culick and Hill (1958), making a point of Eq. (B-16) but not of the law of corresponding stations, determined $\bar{\mu}/\mu_\infty$ from Eq. (B-15) by using experimental values of $C_f/\bar{C}_f$ for fixed $C_f R_\theta$. They also concluded, independently of Mager, that $\bar{\rho}$ and $\bar{\mu}$ should be evaluated at the isentropic stagnation condition (but note the contradictory

---

validity of the transformation seemed to be limited by physical considerations to the cases p = constant or T_o = constant (cf. the footnote following Eq. 3.10). The law of corresponding stations and the substructure hypothesis, on the other hand, were presumably not limited to these cases. After another pause, I was able to reformulate the principle of observability as in the introduction to this paper. At the same time several other issues were clarified, including the reduction to an identity transformation, the recovery of the condition σ = constant for laminar flow, and the role of the state and energy equations and the viscosity law in the direct and inverse transformations.

evidence in my Fig. 5). And so on through a number of other analyses*, most of which are applications of the transformation technique to various practical problems.

I will close by commenting briefly on what might be called the unfinished business of the transformation. For the purposes of this discussion it is instructive to view the transformation as a kind of bilingual dictionary used to translate statements about compressible flows into corresponding statements about associated incompressible flows, and vice versa. The construction of the dictionary should insure that the sense of certain fundamental statements, particularly physical laws for the conservation of mass and momentum together with supplementary ideas about boundedness and observability of physical quantities, will be correctly translated. For more specialized subjects, however, the vocabulary of the transformation may have to be enlarged. By definition, any material used for this purpose must itself be capable of translation, although it may be empirical and heuristic in nature. Typical examples of such special vocabularies are those provided by (1) the assumption of Newtonian friction for laminar flow and (2) the substructure hypothesis for the turbulent boundary layer on a smooth wall.

A number of problems not yet touched upon may be classified in terms of this metaphor. The turbulent wake and the turbulent boundary layer on a rough surface, for example, will also require special vocabularies. In

---

*In the present analysis the equations corresponding to (B-15)-(B-17) are (2. 17), (2.20), and (4.24); the solution proposed for the problem of a reference temperature is the substructure hypothesis; and the x-dependence of the reference properties is assigned to the parameter $\sigma(x) = \bar{\mu}/\mu_s(x)$ rather than to the parameter $\bar{\mu}$ alone.

both cases the viscosity is an artificial parameter, and some of the relationships derived here -- beginning with the law of corresponding stations in the case of the boundary layer -- will have to be reconsidered. Inasmuch as an adequate description of the mean properties of low-speed flow is available in both cases, and inasmuch as the effects of compressibility on drag for rough surfaces are also known, neither of these problems seems to be really formidable.

Certain other problems, however, will require some preliminary work in composition before the question of translation can even be raised. Prominent among these problems are the effects of mass transfer and of pressure gradient in compressible turbulent flow. In the one case the methods of this paper seem to require first a satisfactory generalization of the law of the wall for low-speed flow with mass transfer, and then a corresponding generalization of the substructure hypothesis. In the other case, that of flow with pressure gradient, there is no real difficulty with the substructure hypothesis, or with formulas like

$$\frac{c_f}{\bar{c}_f} = \frac{1}{1 + \left(1 - \langle r^2 \rangle \frac{\bar{c}_f}{2}\right) \frac{\gamma - 1}{2} M_\infty^{\ 2}} \qquad \text{(B-18)}$$

which follow from this hypothesis for adiabatic flow with T_c = constant and μ/T = constant. Instead, the difficulty is with the condition connecting the two variables M_∞ and $\bar{c}_f$ on the right-hand side of this formula. For laminar flow it was shown in Section III that the condition $dp/dx = 0$ implies $\eta/\sigma = u_\infty/\bar{u}_\infty$ = constant, while a special argument based on the assumption that τ should transform like $\mu \partial u/\partial y$ throughout the flow was necessary in order to extend this result -- in the form $\eta^2/\sigma^2 = T_\infty/\bar{T}$, or

$M_\infty = \bar{u}_\infty / \sqrt{\gamma RT}$, with the auxiliary conditions T_o = constant and μ/T = constant -- to the case $dp/dx \neq 0$. For turbulent flow the condition $dp/dx = 0$ still implies η/σ = constant, but the argument used for the case $dp/dx \neq 0$ no longer holds. Consequently the transformation cannot yet be applied with confidence to turbulent flow in a pressure gradient, regardless of the methods used to relate the incompressible variables $\bar{u}_\infty$ and $\bar{C}_f$ in the associated low-speed flow. Moreover, the status of the conditions T_o = constant and μ/T = constant for flows with pressure gradient is at best uncertain.

Finally, the problem of heat transfer is typical of several problems which must always lie outside the scope of any transformation incorporating the idea of observability. The reason is that equations of state, energy, or concentration involve concepts existing only in one language and not in the other, and so are inherently incapable of being translated. At the same time, such statements will enter into the description of physical quantities in the compressible flow, and will frequently complicate both the direct and the inverse transformations. As long as the common practice of honoring the energy equation in the breach is continued, therefore, the most elegant results of any transformation are likely to be the least realistic ones, and no new contributions to the subject of heat transfer in turbulent flow can be expected. My own feeling is that more attention should be paid, both experimentally and analytically, to the development of similarity laws for static and stagnation temperature. I have emphasized in this paper that the turbulent shearing stress can be readily inferred from the corresponding similarity laws for velocity; in fact, a vehicle for this calculation in the case of compressible flow has been provided in Eq. (3.11). At least in principle, the turbulent heat transfer can be inferred in the same way from an empirical knowledge of the velocity and temperature fields, and the relationship between shearing stress and heat transfer in turbulent flow can thus be made experimentally accessible without recourse to difficult hot-wire techniques for the direct measurement of q and τ.

REFERENCES

Albertson, M. L. 1948 Ph.D. Thesis, State Univ. Iowa; see also Proc. Heat Transf. Fluid Mech. Inst., 243-254, 1951.

Allan, J. F. and Cutland, R. S. 1953 Trans. N. E. Coast Inst'n. Eng'rs. Shipbuilders 69, 245-256, D65-D78.

Ashkenas, H. I., Riddell, F. R., and Rott, N. 1952 Cornell Univ. (NACA Naw-6014).

Ashkenas, H. I. and Riddell, F. R. 1955 NACA TN 3383.

Ashkenas, H. I. 1956 Ph.D. Thesis, Cornell Univ.; NACA TN 4140, 1958.

Baines, W. D. 1950 Ph.D. Thesis, State Univ. Iowa.

Barrow, H. 1958 J. Roy. Aeron. Soc. 62, 135-138.

Boltz, F. W., Kenyon, G. C., and Allen, C. Q. 1960 NASA TN D-309.

Brevoort, M. J. and Arabian, B. D. 1958 NACA TN 4248.

Burggraf, O. R. 1961 Lockheed Rep. LMSD-395080.

Cermak, J. E. and Lin. P. N. 1955 Civil Eng. Dept., Colo. A. and M. Coll., Rep. No. 9.

Cohen, C. B. and Reshotko, E. 1955 NACA TN 3325; NACA TR 1293, 1956.

Coles, D. 1953 Ph.D. Thesis, Calif. Inst. Tech; see also JPL Rep. 20-71, 1953; J. Aeron. Sci. 21, 433-448, 1954.

Coles, D. 1955 50 Jahre Grenzschichtforschung, Braunschweig, 153-163.

Coles, D. 1956 J. Fl. Mech. 1, 191-226.

Coles, D. 1957 Proc. Heat Transf. Fluid Mech. Inst., 119-137.

Cope, W. F. and Hartree, D. R. 1948 Phil. Trans. Roy. Soc. A241, 1-69.

Culick, F. E. C. and Hill, J. A. F. 1958 J. Aeron. Sci. 25, 259-262.

Dhawan, S. 1951 Ph.D. Thesis, Calif. Inst. Tech.; NACA TN 2567, 1952; NACA TR 1121, 1953; see also Proc. 1st U.S. Nat. Cong. Appl. Mech., 869-874, 1951 (with H. W. Liepmann).

Donaldson, C. duP. 1952 NACA TN 2692; see also NACA RM L52H04, 1952; Proc. Heat Transf. Fl. Mech. Inst., 19-35 1952.

Dorodnitsyn, A. 1942a C. R. (Doklady) Acad. Sci. U.R.S.S. 34, 213-219.

Дородницын, А. 1942b ПММ 6, 449 - 486.

Dryden, H. L. 1936 NACA TR 562.

Dutton, R. A. 1955a Ph.D. Diss., Cambr. Univ.; see also ARC R & M 3058, 1957 (1956); ARC CP 453, 1959.

Dutton, R. A. 1955b Ph.D. Diss., Cambr. Univ.; see also J. Aeron. Sci. 23, 1127-1128, 1956; ARC R & M 3155, 1960 (1958).

Eckert, E. R. G. 1954 WADC Tech. Rep. 54-70; see also J. Aeron. Sci. 22, 585-587, 1955; Trans. ASME 78, 1273-1283, 1956.

Eckert, E. R. G. 1960 WADC Tech. Rep. 59-624.

Ede, A. J. and Saunders, O. A. 1958 Proc. Inst. Mech. Eng. 172, 743-756.

Edwards, A. and Furber, B. N. 1956 Proc. Inst. Mech. Eng. 170, 941-954.

Elias, F. 1929 Z. A. M. M. 9, 434-453; 10, 1-14 (translated as NACA TM 614, 1931).

Favre, A. J., Caviglio, J. J., and Dumas, R. 1955 La Rech. Aeron. No. 48, 3-14; see also J. Fl. Mech. 2, 313-342, 1957; ONERA Pub. No. 92, 1958; see also Caviglio, J., Thesis, Univ. d'Aix-Marseille, 1958.

Feindt, E. G. 1956 DFL Braunschweig, Ber. Nr. 56/10.

Furber, B. N. 1954 Proc. Inst. Mech. Eng. 168, 847-860.

Grant, H. L. 1957 Ph.D. Diss., Cambr. Univ.; see also J. Fl. Mech. 4, 149-190, 1958.

Hakkinen, R. J. 1954 Ph.D. Thesis, Calif. Inst. Tech.; NACA TN 3486, 1955.

Hama, F. R. 1947 Rep. Inst. Sci. Tech., Univ. Tokyo 1, 13-16, 49-50 (in Japanese).

Hansen, M. 1928 Z. A. M. M. 8, 185-199 (translated as NACA TM 585, 1930).

van der Hegge Zijnen, B. G. 1924 Thesis, Delft; see also Burgers, J. M., Proc. 1st Int. Cong. Appl. Mech., 113-128, 1924.

Howarth, L. 1948 Proc. Roy. Soc. A194, 16-42.

Illingworth, C. R. 1949 Proc. Roy. Soc. A199, 533-558.

Johnson, D. S. 1955 Ph.D. Thesis, Johns Hopkins Univ.; OSR Tech. Note 55-289; see also Trans. ASME 79, J. Appl. Mech. 24, 2-8, 1957; Trans. ASME 81E (J. Appl. Mech.), 325-336, 1959.

Jürges, W. 1924 Beihefte zum Gesundheits-Ing., No. 19.

Karlsson, S. F. K. 1958 Ph.D. Thesis, Johns Hopkins Univ.; see also J. Fl. Mech. 5, 622-636, 1959.

von Karman, T. 1935 Proc 5th Volta Cong., Rome, 222-276; Collected Works, Vol. III, 179-221, 1956.

Kay, J. M. 1953 ARC R & M 2628.

Kistler, A. L. 1958 BRL (Aberdeen) Rep. 1052; Phys. Fluids 2, 290-296, 1959.

Klebanoff, P. S. and Diehl, Z. W. 1951 NACA TN 2475; NACA TR 1110, 1952.

Klebanoff, P. S. 1954 NACA TN 3178; NACA TR 1247, 1955.

Kline, S. J., Lisin, A. V., and Waitman, B. A. 1960 NASA TN D-368.

Korkegi, R. H. 1954 Ph.D. Thesis, Calif. Inst. Tech.; GALCIT HWT Memo. 17 (DA-04-495-Ord-19); see also J. Aeron. Sci. 23, 97-108, 1956.

Landweber, L. and Siao. T. T. 1958 J. Ship Res., 21-33.

Ludowici, W. 1926 Z. Ver. deutscher Ing. 70, 1122-1124.

Ludwieg, H. and Tillmann, W. 1949 Ing.-Arch. 17, 288-299 (translated as NACA TM 1285, 1950).

Mager, A. 1957 Proc. Heat Transf. Fl. Mech. Inst., 85-98; J. Aeron. Sci. 25, 305-311, 1958.

Matting, F. W., Chapman, D. R., Nyholm, J. R., and Thomas, A. C. 1960 NASA TR R-82; see also Proc. Heat Transf. Fluid Mech. Inst., 90-94, 1959.

McCullough, G. B. and Gambucci, B. J. 1952 NACA RM A52D01b.

Michel, R. 1950 Thesis, Paris; Pub. Sci. Tech. Min. de l'Air, No. 258, 1951.

Mickley, H. S. and Davis, R. S. 1957 NACA TN 4017.

Monaghan, R. J. 1955 Proc. 5th Int. Aeron. Conf., Los Angeles, 277-315.

Nikuradse, J. 1942 ZWB Monograph.

Nothwang, G. J. 1956 NACA TN 3721; NACA TR 1320, 1957.

Parmelee, G. V. and Huebscher, R. C. 1947 Trans. ASHVE 53, 245-284.

Persh, J. and Lee, R. 1956 NOL Navord Rep. 4282; see also Persh, J., NOL Navord Rep. 3854, 4099, 1955.

Peters, H. 1938 Proc. 5th Int. Cong. Appl. Mech., 393-395; see also J. Aeron. Sci. 6, 104-105, 1939.

Prandtl, L. 1910 Phys. Zeitschr. 11, 1072-1078.

Prandtl, L. 1928 Phys. Zeitschr. 29, 487-489.

Preston, J. H. and Sweeting, N. E. 1944 ARC FM 671.

Preston, J. H. 1954 J. Roy. Aeron. Soc. 58, 109-121.

Preston, J. H. 1958 J. Fl. Mech. 3, 373-384.

Reynolds, W. C. 1957 Ph.D. Thesis, Stanford Univ.; see also NASA Memo. 12-1-58W, 12-2-58W, 1958 (with W. M. Kays and S. J. Kline).

Rott, N. 1957 Ramo-Wooldridge Corp., GMRD Rep. GM-TR-211.

Rubesin, M. W. and Johnson, H. A. 1949 Trans. ASME 71, 383-388; see also Trans. ASME 71, 447-456, 1949.

Scesa, S. 1954 Ph.D. Thesis, Univ. Calif.

Schubauer, G. B. and Klebanoff, P. S. 1955 NACA TN 3489; NACA TR 1289, 1956; see also NPL Symposium on Boundary-Layer Effects in Aerodynamics, 4.1-4.21, 1955.

Schultz-Grunow, F. 1940 Lufo. 17, 239-246 (translated as NCAC TM 986, 1941).

Shutts, W. H., Hartwig, W. H., and Weiler, J. E. 1955 Univ. Texas, Rep. DRL-364.

Smith, D. W. and Walker, J. H. 1958 NACA TN 4231; NASA TR R-26, 1959; see also Smith, D. W., Proc. 5th Midw. Conf. Fluid Mech., 108-121, 1957.

Sogin, H. H. and Goldstein, R. J. 1960 Brown Univ. (AFOSR TN-60-647).

Sommer, S. C. and Short, B. J. 1955 NACA TN 3391.

Spence, D. A. 1959 RAE Rep. No. Aero 2631; see also Proc. Heat. Transf. Fluid Mech. Inst., 62-76, 1960; J. Fl. Mech. 8, 368-387, 1960.

Spengos, A. C. 1956 Civil Eng. Dept., Colo. A. and M. Coll., Sci. Rep. No. 1; see also Proc. Heat Transf. Fluid Mech. Inst., 89-104, 1956 (with J. E. Cermak).

Stalmach, C. J. Jr. 1958 Univ. Texas, Rep. DRL-410.

Stewartson, K. 1949 Proc. Roy. Soc. A200, 84-100.

Sugawara, S., Sato, T., Komatsu, H., and Osaka, H. 1953 J. Japan Soc. Mech. Eng. 19, 18-25 (translated as NACA TM 1441, 1958).

Taylor, G. I. 1916 ACA R & M 272.

Tillmann, W. 1945 KWI, Göttingen, U & M 6627 (translated as MAP-VG-34, 1946; BIGS-19 (CGD 497), 1946).

Townsend, A. A. 1951 Proc. Cambr. Phil. Soc. 47, 375-395.

Tucker, M. 1951 NACA TN 2337.

Tulin, M. P. and Wright, R. H. 1949 NACA RM L9H29a.

Vaglio-Laurin, R. 1958 Polytechnic Inst. Brooklyn, WADC Tech Note 58-301; J. Aero/Space Sci. 27, 27-36, 1960; see also Proc. Heat Transf. Fl. Mech. Inst., 95-109, 1959.

Van Le, N. 1953 J. Aeron. Sci. 20, 583-584.

Wieghardt, K. 1943 KWI, Göttingen, U & M 6603; see also U & M 6612, 1944; U & M 6617, 1944 (with W. Tillmann; translated as NACA TM 1314, 1951).

Wieghardt, K. 1944a KWI, Göttingen, U & M 6612; see also Z. A. M. M., 24, 294-296, 1944.

Wieghardt, K. 1944b ZWB Technische Berichte 11, 207-211.

Wieghardt, K. 1947 AVA Mono. MAP-VG-246T, Sect. D_1 II 3.3.

Willmarth, W. W. 1959 NASA Memo 3-17-59W.

Young, G. B. W. and Janssen, E. 1952 J. Aeron. Sci. 19, 229-237.

Зысина - Моложен, Л. М. 1956 Технические Отчеты, Центральный Котлотурбинный Институт им. И. И. Ползунова, № 6807/0 - 1999.

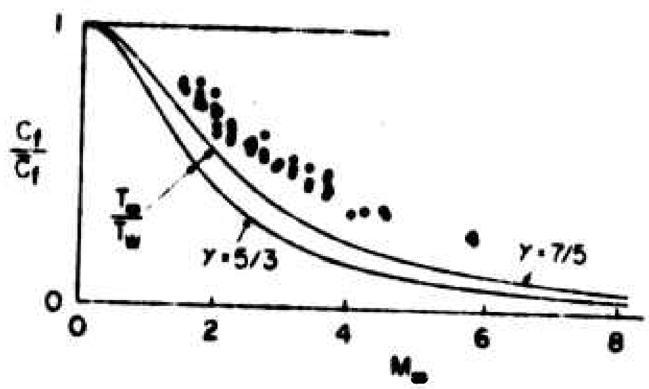


Fig. 1. Floating-element measurements of local surface friction in air.

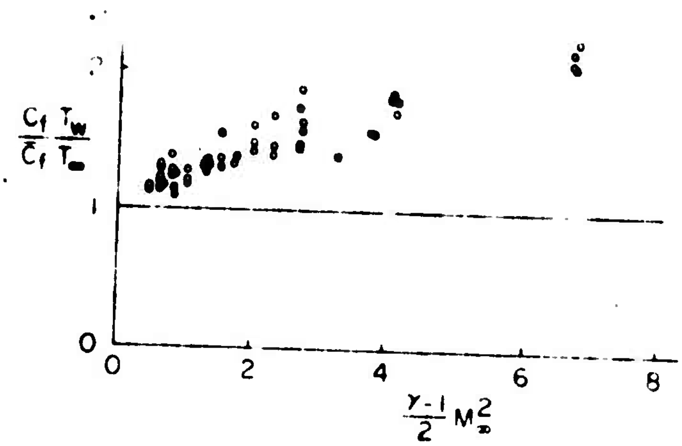


Fig. 2. Data of Fig. 1 normalized with respect to temperature ratio.

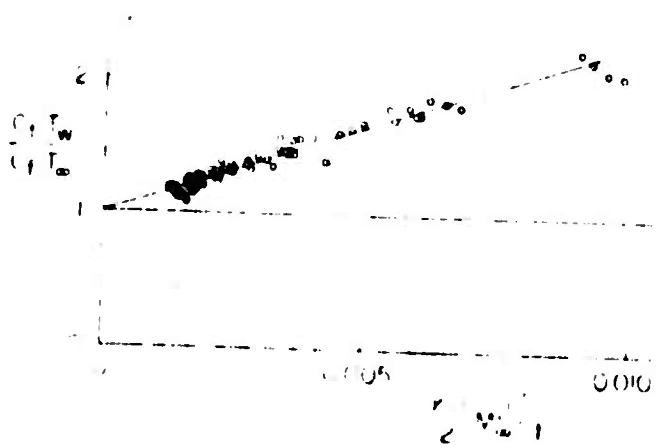


Fig. 3. Data of Fig. 2 in the coordinates of the substructure formulation.

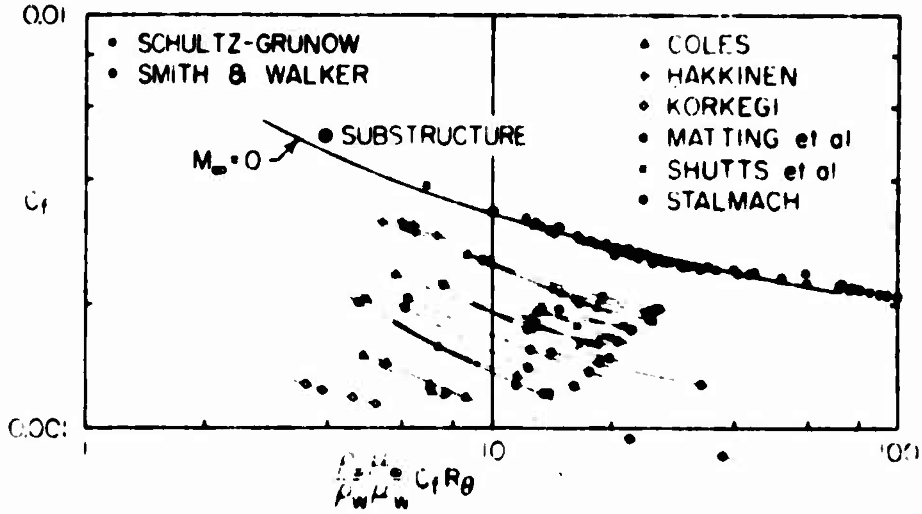


Fig. 4. Floating-element measurements of local surface friction in air.

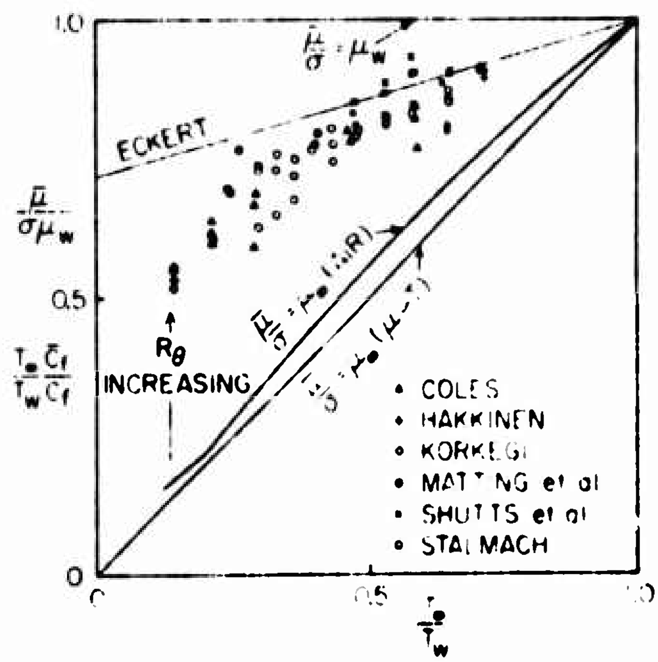


Fig. 5. Empirical evaluation of the parameter $\bar{\mu}/\sigma\mu_w$ of the transformation.

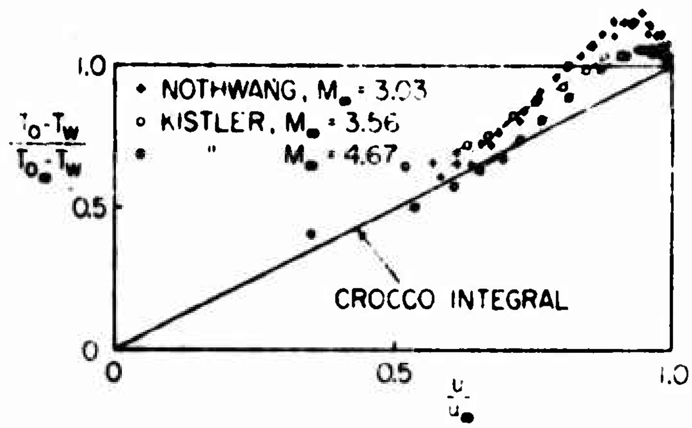


Fig. 6. Stagnation-temperature distribution in adiabatic supersonic flow.

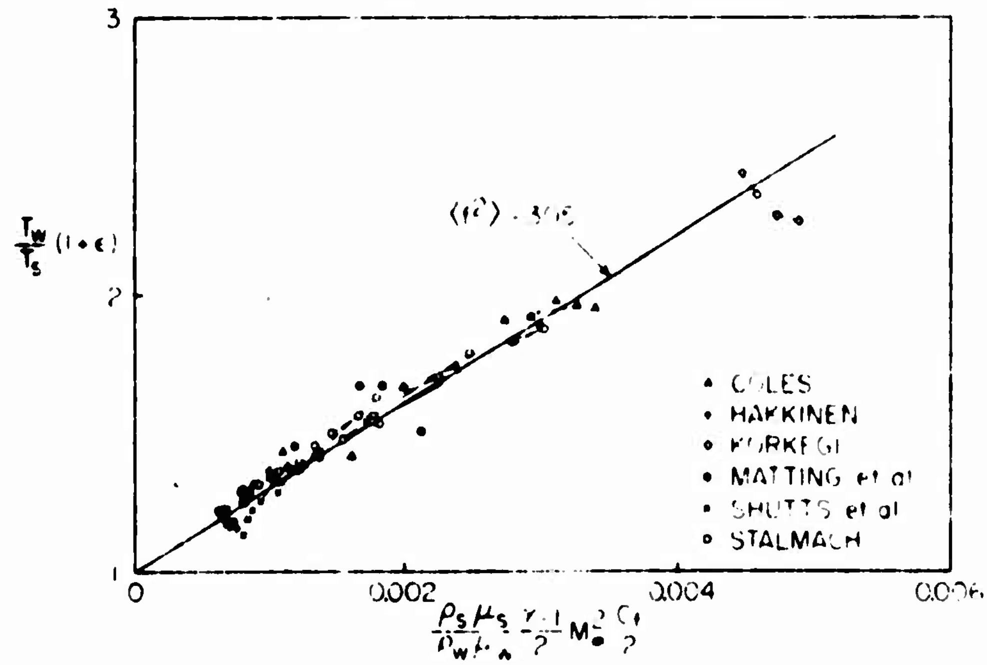


Fig. 7. Indirect evaluation of the substructure Reynolds number R_s.

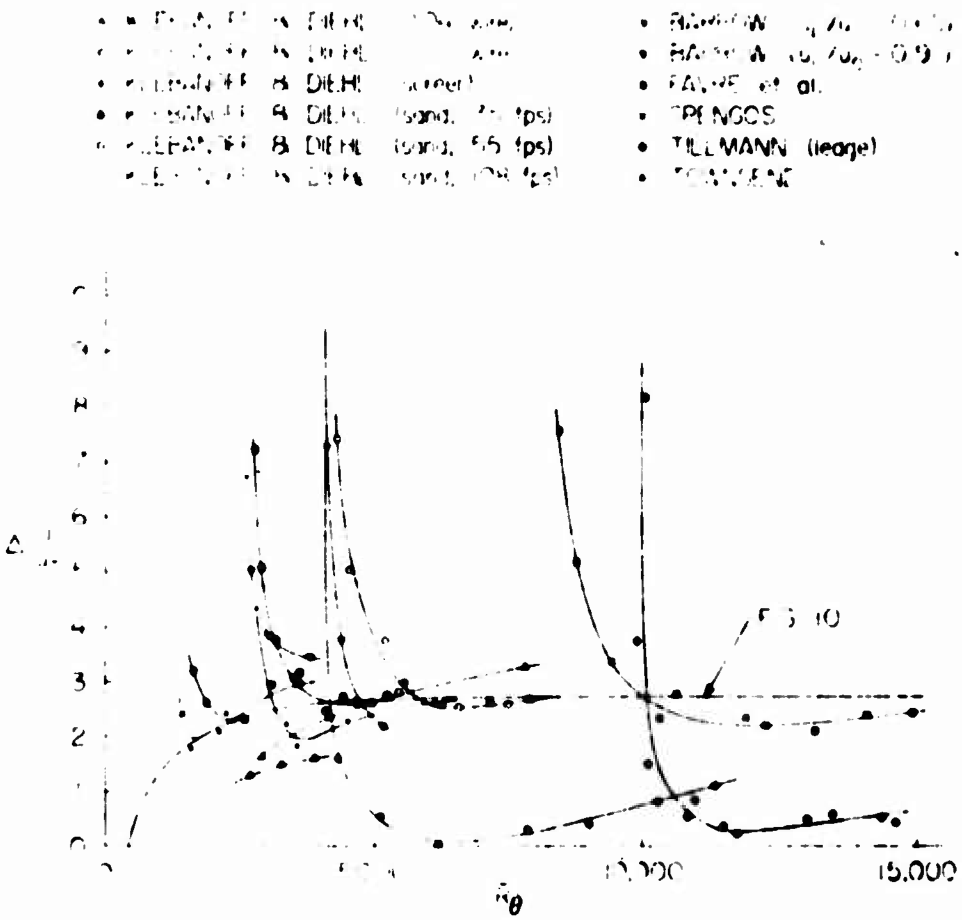


Fig. 16. Approach to equilibrium in terms of the wake component

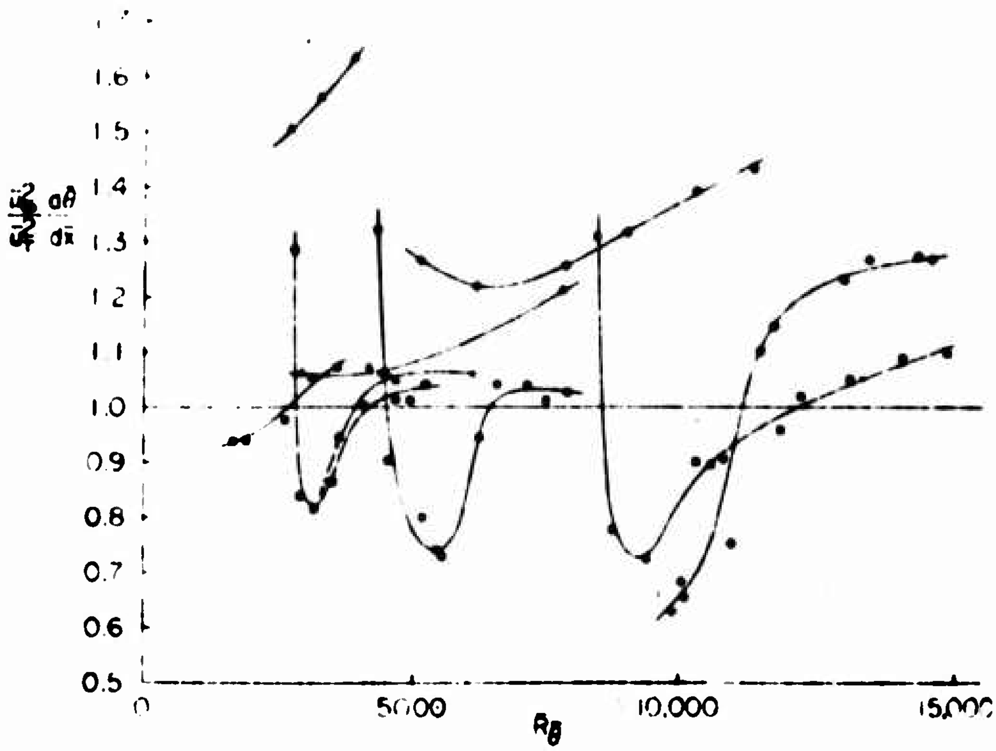


Fig. 17. Momentum balance for the data of Fig. 16.

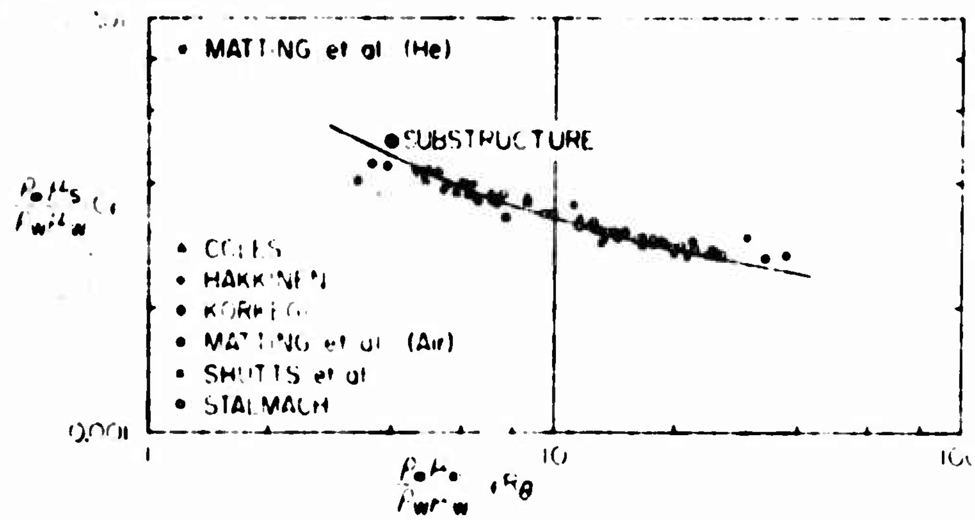


Fig. 8. Friction data in compressible flow according to the transformation.

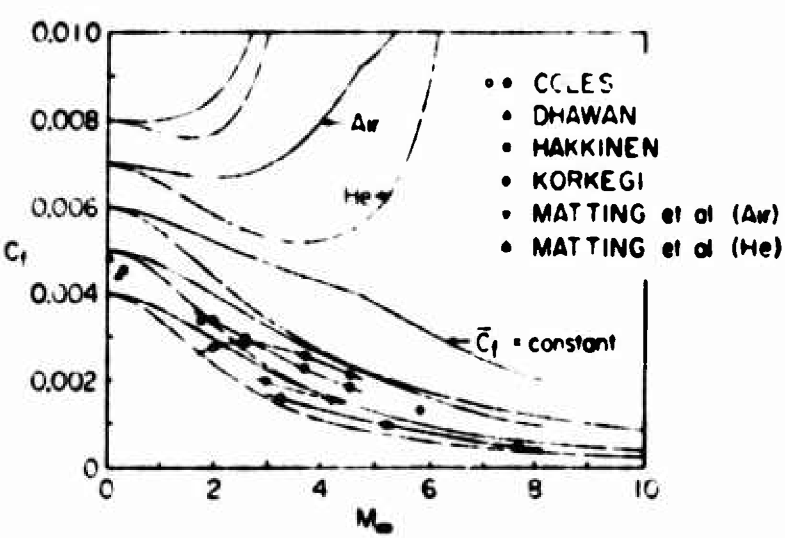


Fig. 9. Maximum local friction coefficients as observed in air and helium. (Open symbols, natural transition; solid symbols, tripped.)

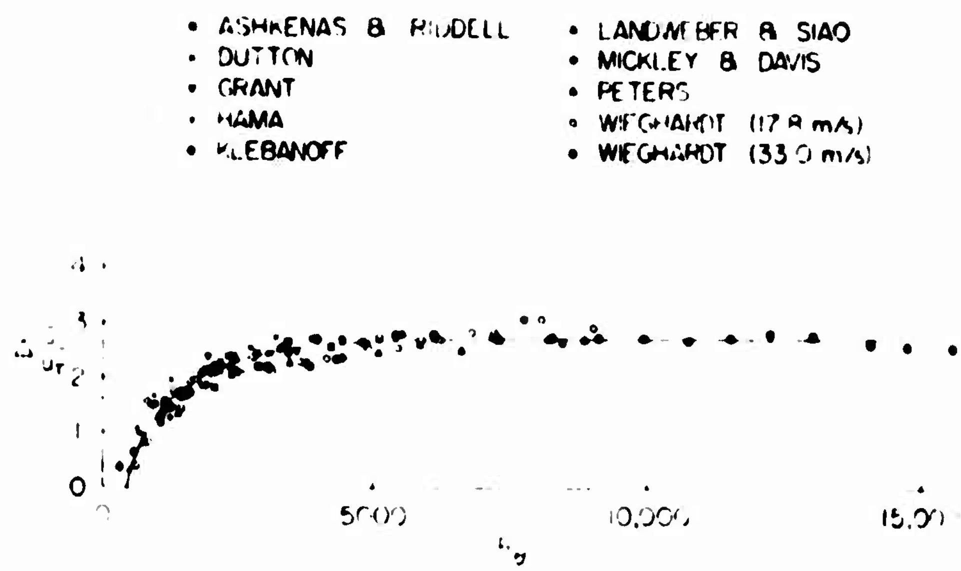


Fig. 10. Strength of the wake component in equilibrium turbulent flow.

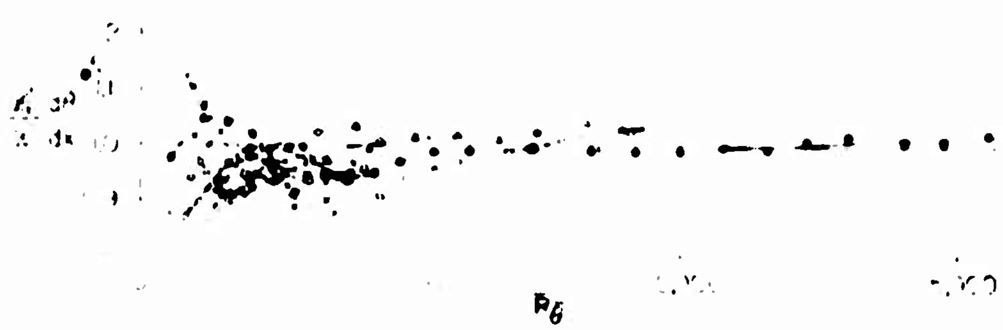


Fig. 11. Momentum balance for the data of Fig. 10.

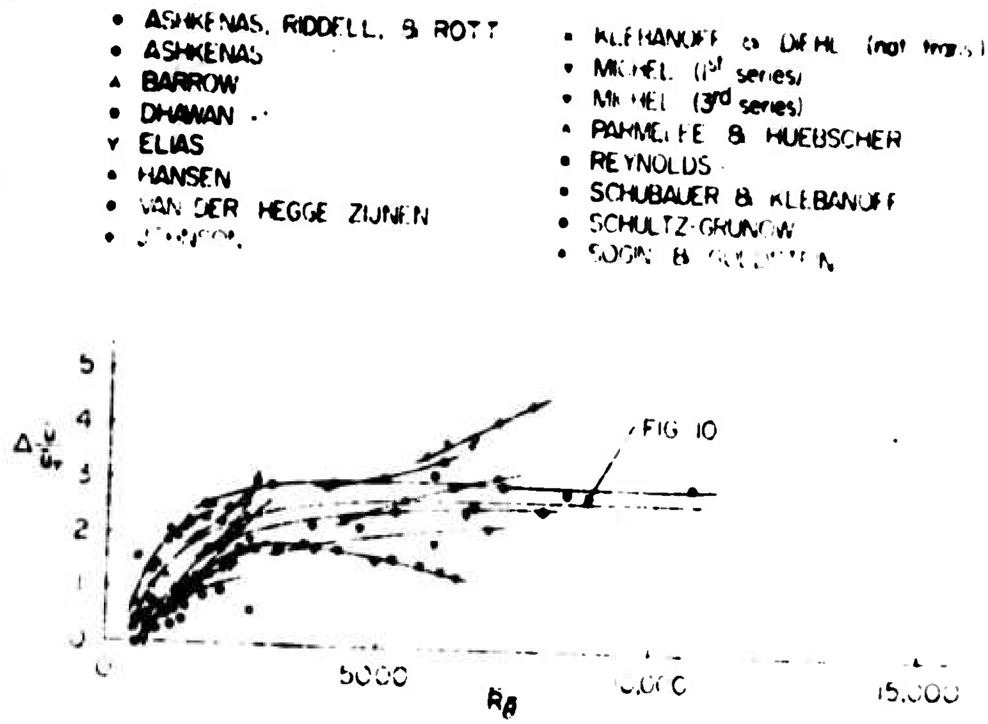


Fig. 12. Strength of the wake component in anomalous turbulent flow.

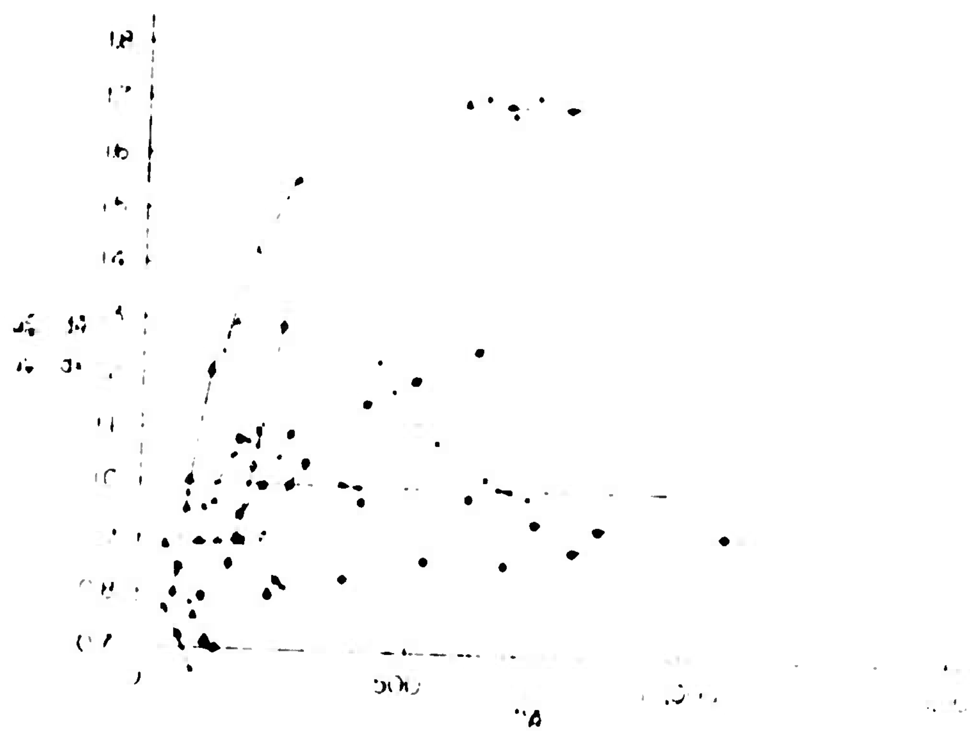


Fig. 13. Momentum balance for the data of Fig. 1.

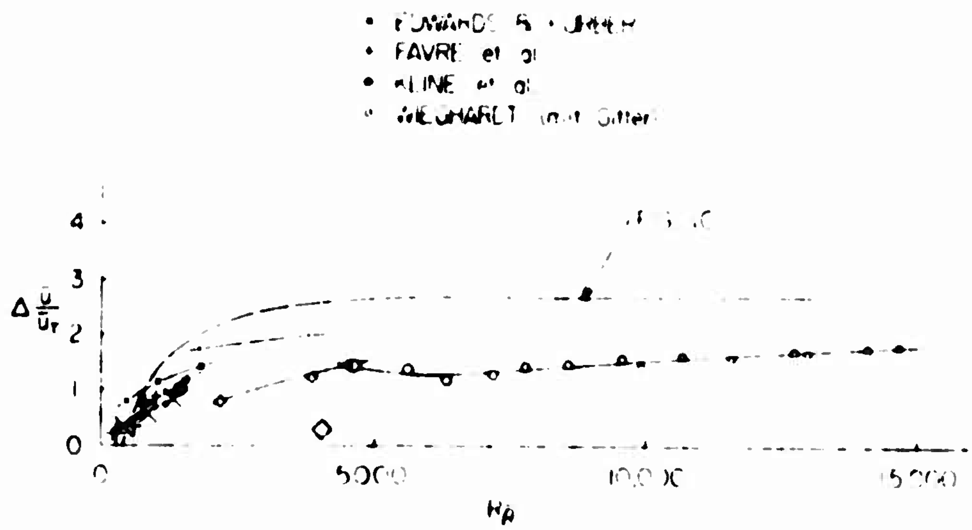


Fig. 14. Effect of free-stream turbulence on the strength of the wake component. (Symbol size indicates relative turbulence level.)

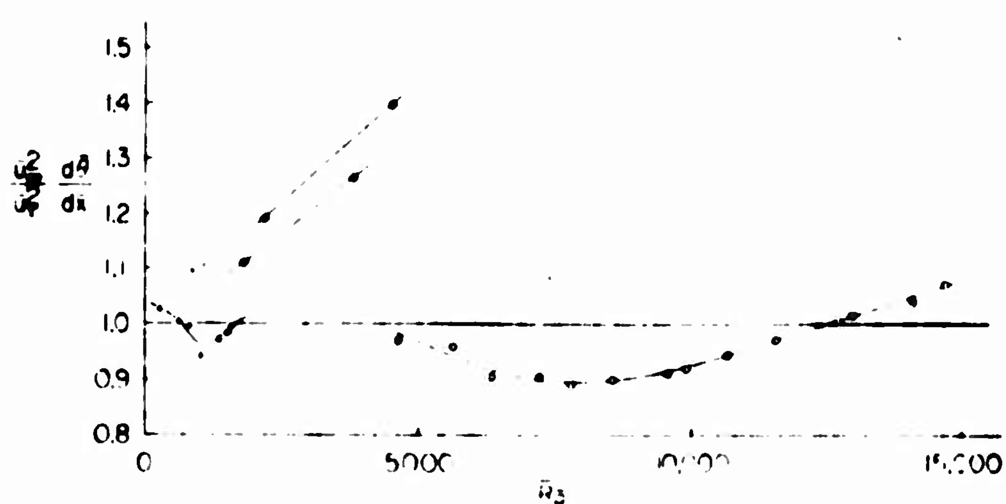


Fig. 15. Momentum balance for the data of Fig. 14.

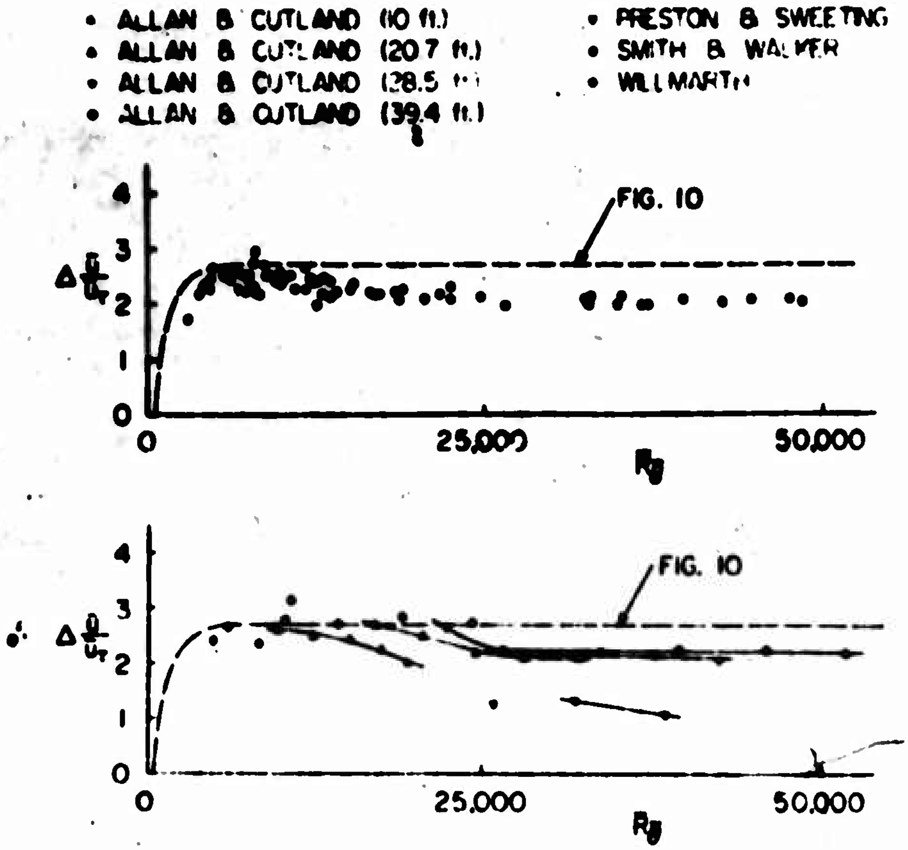


Fig. 18a, b. Strength of the wake component at large Reynolds numbers.

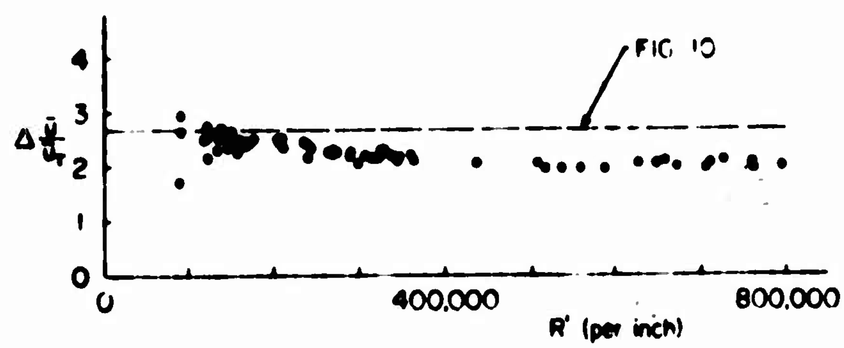


Fig. 19. Strength of the wake component for Smith and Walker's data as a function of Reynolds number per unit length. (Solid symbols are $\bar{R}_{\bar{\theta}}$ < 10,000 in Fig. 18.)

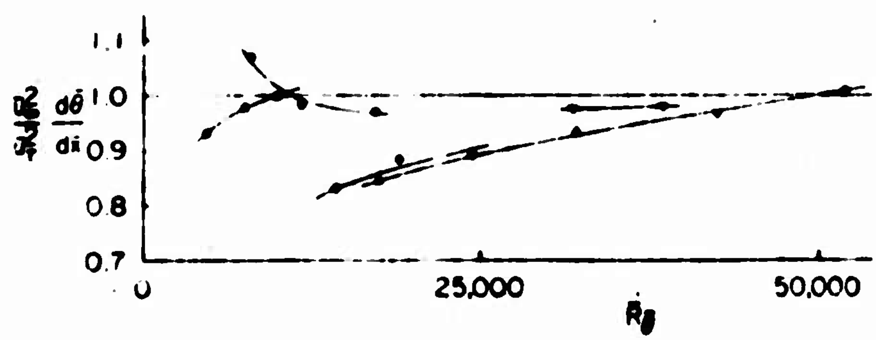


Fig. 20. Momentum balance for the data of Fig. 18.

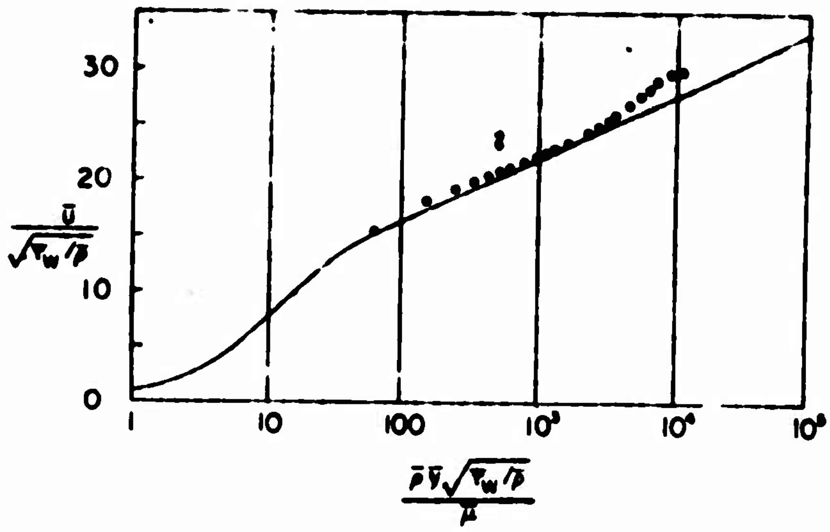


Fig. 21. Typical profile of Smith and Walker in terms of the law of the wall.

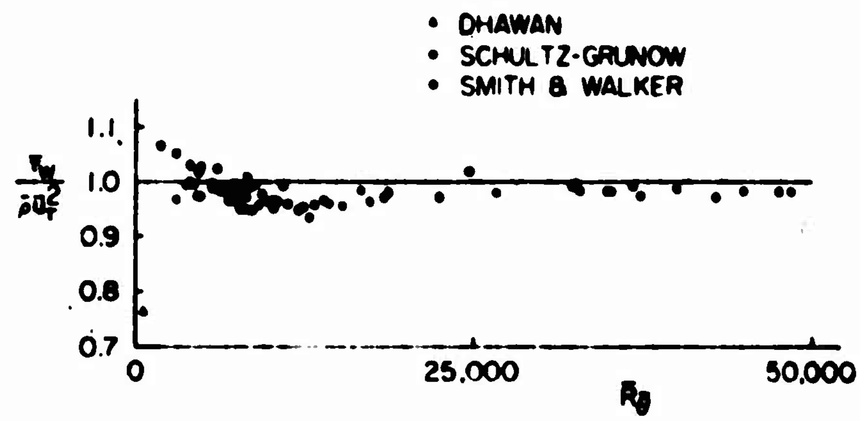


Fig. 22. Comparison of $\bar{\tau}_w$ and $\bar{\rho}\bar{u}_\tau^2$ for floating-element friction data.

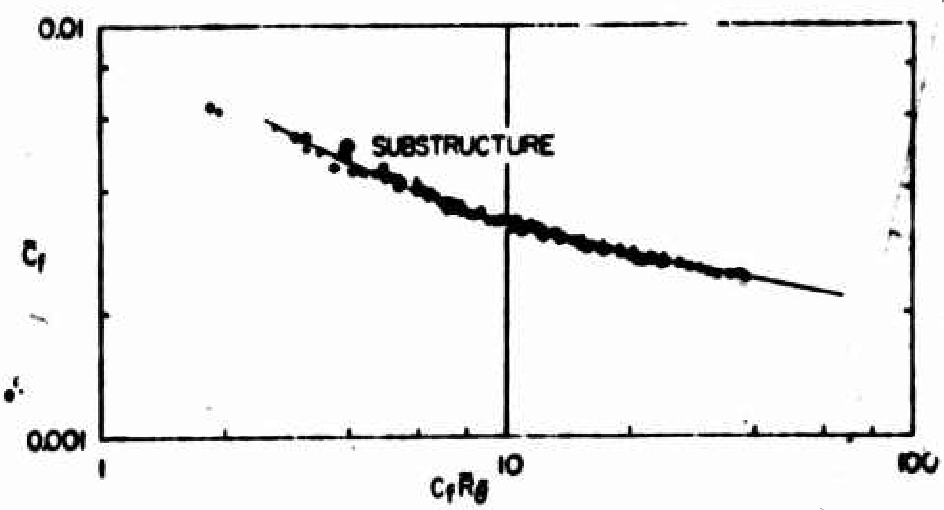


Fig. 23. Local friction law for the equilibrium boundary layer; see also Fig. 4. (Legend same as Fig. 10.)