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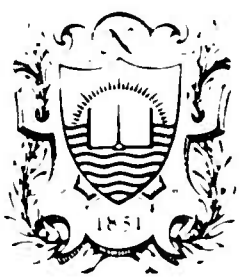

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SOME SPECIAL P-MODELS IN CHANCE-CONSTRAINED PROGRAMMING

by

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SYSTEMS RESEARCH GROUP

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Introduction

In this paper we will establish sufficient conditions for decision rules to be optimal for two n-period P-models of chance-constrained programming. [1/] Proofs of theorems will be based on results contained in our earlier papers [7, 8] on n-period E-models of chance-constrained programming.

The basic similarities and differences between the P- and E-models of chance-constrained programming can be seen by considering the following two models:

$$\max \quad E(c^T X)$$

(1) $$\text{subject to} \quad P(AX \leq b) \geq \alpha$$

$$P(X \geq 0) \geq \beta$$

and

$$\max \quad P(c^T X \geq c_o^T X_o)$$

(2) $$\text{subject to} \quad P(AX \leq b) \geq \alpha$$

$$P(X \geq 0) \geq \beta$$

In (1) and (2) E stands for the expectation operator and P for the probability operator. In both cases we compute the expectation and probability using the joint distribution of all the random variables involved in the problem, i.e., the joint distribution of all the random variables contained in the A matrix, and the b, c vectors.

The i^{th} constraint of (1) and (2) says that any set of feasible decision rules $x_j \equiv \phi_j(A, b, c)$, $j = 1, \ldots, n$, where ϕ_j is some function of the elements of A, b, c, must be such that the inequality

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

1/ See Charnes and Cooper [3] where the "P," "E," and "V" models were first explicitly introduced.

is satisfied with at least probability α_i. The second set of constraints in (1) and (2) indicate that ϕ_j, $j=1,\ldots,n$ must also have the property that the inequality

$$\phi_j(A, b, c) \geq 0$$

is satisfied with at least probability β_j.

In either problem (1) or (2) we may have additional constraints which restrict the type of functions $\phi_j(A, b, c)$ that are admissible. In the problem discussed here, as in our earlier papers on n-period E-models [7,8], these additional constraints will be dictated by our interpretation of the problem. Thus we will not impose restrictions of functional form on the admissible rules, such as by requiring them to be linear decision rules as is done in [3, 5, 10, 11]. Instead our admissible class of decision rules will be the most general class which is consistent with our interpretation of n-period models. 1/

In (1) our object is to find the feasible set of decision rules $x_1^*,\ldots,x_n^*$ which give the linear function $\sum_{j=1}^{n} c_j x_j$ as large an expected value as possible. The concept of maximizing the expected values of a linear function is common in the literature of many subjects. In economics, for example, the notion of planning so as to maximize expected profits (or minimize expected costs) under constant returns to scale is often used. Thus the formulation and analysis of chance-constrained programming problems with the objective function maximize $E(c^T X)$ is natural, particularly when we interpret the problem as one of planning over an n-period horizon.

On the other hand, the objective function of (2) perhaps requires some elaboration. In (2) the components of $c_o^T X_o$ are specified relative to some set of values which an individual or business firm regards as satisfactory whenever they are achieved. The problem is one of finding feasible decision rules which will maximize the probability of attaining the given level $c_o^T X_o$. In [3] Charnes and Cooper have discussed the relationship of the P-model approach to what H. A. Simon [19, 20] calls "satisficing" as opposed to "optimizing" behavior. In [18] a P-model objective function is used in the analysis of risky investment decisions.

1/ See sections 4 and 6 of [7] and section 2 below for further discussion of this point.

Moreover it is quite possible that employing the E-model rather than the P-model would yield a very high expected value for the $c^T X$ with a very small probability of attaining this value, while the correcponsing P-model might give a smaller expected value but with a very high probability of achieving it. Individuals or firms often prefer a gamble with lower expected value and higher probability of success, to a gamble with high expected value and a low probability of winning. It is to aid in analyzing the implications of this kind of human or business behavior that the P-model was formulated.

As mentioned above we are going to discuss two different n-period models. These models will differ chiefly in the following way. In the triangular n-period model we will assume that each period generates two new constraints, one of which requires that $P(X_j \geq 0) \geq \beta_j$ and the other couples X_j, the decision rule of the j^{th} period, to the decision rules of all preceding periods. In both these constraints we will compute the probability using the joint distribution of all the random variables involved in the problem. For this reason we will refer to these constraints as total chance constraints.

In the other model we will consider, the block-triangular model, we allow each period to generate a finite, but otherwise arbitrary, number of chance constraints. However, in contrast to the triangular model, we will regard these constraints as conditional chance constraints. That is, we will interpret the P operator in the constraints of the i^{th} period as meaning that we compute the probability using the condition distribution of the random variables of the i^{th} period given the observed values of the random variables of periods one to $i - 1$.

In sections 3 and 5 below we establish theorems which give sufficient conditions for decision rules to be optimal for the total and conditional chance constraint cases. However, in spite of the similar nature of these theorems, the methods of proof are quite different.

2. The Triangular Problem: Total Chance Constraints

The problem we are going to consider in this and the following section is

$$
\begin{aligned}
&\text{maximize} && P\Big(\sum_{j=1}^{n} c_j X_j \geq k\Big) \\
&\text{subject to} && P(a_{11} X_1 + d_1 b_1 + \omega_1 \leq 0) \geq \alpha_1 \ , \\
& && P(a_{21} X_1 + a_{22} X_2 + d_2 b_2 + \omega_2(b_1) \leq 0) \geq \alpha_2 \ , \\
(3) \qquad & && P(a_{31} X_1 + a_{32} X_2 + a_{33} X_3 + d_3 b_3 + \omega_3(b_1, b_2) \leq 0) \geq \alpha_3 \ , \\
& && \text{---} \\
& && P\Big(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \leq 0\Big) \geq \alpha_i \ , \\
& && \text{---} \\
& && P\Big(\sum_{j=1}^{n} a_{nj} X_j + d_n b_n + \omega_n(b_1, \ldots, b_{n-1}) \leq 0\Big) \geq a_n \ , \\
& && P(X_j \geq 0) \geq \beta_j \ , \quad j = 1, \ldots, n \ ,
\end{aligned}
$$

where P represents the total probability operator. Thus we compute the probability using the joint distribution of all the random variables involved in the problem.

In (3) we make the following assumptions:

(a) a_{ij}, $i \geq j$, $i, j = 1, \ldots, n$, d_i, c_j, $i, j = 1, \ldots, n$, ω_1 and k are given constants, and $a_{ii} \neq 0$, $d_i \neq 0$ for all i.

(b) $\omega_i(b_1, \ldots, b_{i-1})$, $i = 2, \ldots, n$, is a piecewise continuous function with at most a countable number of discontinuities.

(c) α_i, β_j, $i, j = 1, \ldots, n$, are known probabilities. Thus $0 \leq \alpha_i, \beta_j \leq 1$ for all i and j.

(d) b_i, $i = 1, \ldots, n$, are continuous random variables whose joint frequency function $f_n(b_1, \ldots, b_n)$ is known.

(e) X_j, $j = 1, \ldots, n$, is a function of the random variables $b_1, \ldots, b_{j-1}$ but it is not a function of $b_j, \ldots, b_n$. X_j is piecewise continuous with at most a countable number of discontinuities.

(f) For each j, $j = 1, \ldots, n$, there exists a set of $j-1$ dimensional rectangles, $\{A_k^{j-1}, k \in K^{j-1}\}$ where K^{j-1} is some indexing set, such that

(i) $\bigcup_{k \in K^{j-1}} A_k^{j-1} = E^{j-1}$, where E^{j-1} is $j-1$ dimensional Euclidean space,

(ii) $F_{j-1}(A_k^{j-1} \cap A_r^{j-1}) = 0$, for all $k, r \in K^{j-1}$ and $k \neq r$, where $F_{j-1}(G) \equiv \int \cdots \int_G f_{j-1}(b_1, \ldots, b_{j-1})\, db_1, \ldots, db_{j-1}$ for any set $G \in E^{j-1}$, and f_{j-1} is the joint frequency function of $b_1, \ldots, b_{j-1}$.

(iii) X_j^*, an optimal X_j, is of constant sign in A_k^{j-1} for all $k \in K^{j-1}$.

These six assumptions are the same as those made in [8].

The meaning of assumptions (a) through (d) is clear. Assumption (e) is dictated by our interpretation of the problem. We are going to treat (3) as an n-period, or n-stage, problem in which X_j, the decision rule of the j^{th} period, is selected after we have made decisions $X_1, \ldots, X_{j-1}$ and after we have observed the values of the random variables of periods 1 to j-1, but before b_j and all random variables and decisions of periods j+1 to n have been observed.

In other words, we must select X_1, our first period decision rule, before observing the value of the first period random variable b_1. Then having selected X_1 and having observed b_1 we must choose the second period decision rule, X_2, before we observe the value of b_2. This process continues with X_j depending explicitly on X_i, b_i, $i = 1, \ldots, j-1$, and only implicitly (i.e., through the coupling effect of the constraints) on b_j and X_i, b_i, $i = j+1, \ldots, n$. Our admissible class is defined in this manner because in an n period planning situation our information at any stage (aside from a knowledge of the joint distribution of b_i, $i = 1, \ldots, n$) is limited to a knowledge of the decisions and observations of the preceding stages. This explains the first part of assumption (e).

The second part of this assumption and assumption (f) are required in order that we will be able to use the differential equation methods of the isoperimetric theory of the calculus of variations. It is clear, however, that they do not restrict our admissible class of rules to any significant degree. We assume that there exists a set of feasible decision rules for (3). This assumption is similar to that made in most mathematical programming papers where it is assumed that the problem does have some feasible solution. In the model with conditional chance constraints discussed in sections 4 and 5, more is said about the question of feasible rules existing, since there it is quite likely that for some values of $b_1, \ldots, b_{i-1}$ it will not be possible to satisfy the constraints of the i^{th} period.

Note, however, that unlike most programming problems, in any P model problem we need not be concerned with whether or not the objective function is bounded, since we know a priori that its optimal value will lie in the interval $[0,1]$.

An important question in (3), and in any P model, concerns the uniqueness of the optimal decision rules. The nature of the objective function of (3) is such that we expect that several sets of optimal decision rules will exist. This is primarily due to the fact that we gain nothing by making $\sum_{j=1}^{n} c_j X_j$ strictly larger than k for any sample point if we could just as easily make it equal to k. In other words, if X_j' and X_j'', $j=1,\ldots,n$, are two sets of feasible rules for which $P(\sum_{j=1}^{n} c_j X_j' \geq k) = P(\sum_{j=1}^{n} c_j X_j'' \geq k)$, then both these rules could be optimal even though we might have $X_j' < X_j''$, $j=1,\ldots,n$, for all sample points $(b_1, \ldots, b_{j-1})$. Thus unlike most programming problems, the actual numerical value of $\sum_{j=1}^{n} c_j X_j$ is not critical as long as it is greater than k. For this reason there will, in general, exist many sets of optimal rules for (3). This is, of course, to be expected. Since, as we explained in our introduction, the P model was designed explicitly to handle the kind of situation where an individual or firm desires only to achieve a certain level of satisfaction with as large a probability as possible and any set of decision rules which do this will be considered optimal. In particular, the individual or firm is not

interested in making $\sum_{j=1}^{n} c_j X_j$ as large as possible (although one of the chance constraints may require that $\sum_{j=1}^{n} c_j X_j$ exceed a given value with at least a specified probability).

The fact that there may well exist several sets of optimal decision rules makes it very difficult to establish necessary conditions on these rules. On the other hand, by focusing on a particular subclass of feasible rules we can obtain sufficient conditions for optimality in our n-period P-model. This will be done by converting the given problem into an n+1-period E-model problem, and using our previous results on necessary conditions for optimality in the E model to derive sufficient conditions for optimality of the P model. This is done in the next section.

3. Sufficient Conditions for Optimality

In order to use the results contained in [8] we shall employ the following device. We will replace the constant k in (3) by the random variable b_{n+1} which is $N(k, \epsilon)$ where $\epsilon > 0$ is very small. This device is employed because the results in [8] hold only for continuous random variables. However, because the frequency function of b_{n+1} converges uniformly to the frequency function of k (since treating k as a random variable its frequency function is $\delta(x; k)$, the Dirac delta function at $x = k$), it is easy to show that as $\epsilon \to 0$ the optimal decision rules for the new problem converge uniformly to the optimal decision rules of (3).

Thus we are now going to consider the problem

$$
\begin{aligned}
&\text{maximize} && P\Big(\sum_{j=1}^{n} c_j X_j \geq b_{n+1}\Big) \\
(4)\qquad &\text{subject to} && P\Big(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \leq 0\Big) \geq \alpha_i \ , \quad i=1,\ldots,n \\
& && P(X_j \geq 0) \geq \beta_j \ , \quad j=1,\ldots,n
\end{aligned}
$$

where P means that we compute the probability using f_{n+1}, the known joint frequency function of $b_1, \ldots, b_{n+1}$. However, because b_{n+1} is independent of $b_1, \ldots, b_n$ and because X_j is a function of only $b_1, \ldots, b_{j-1}$, it is clear that the constraints of (4) do not depend on b_{n+1}. Thus we will get the same result

if we compute the probability in the constraints using f_n rather than f_{n+1}. Hence the constraints of (4) are identical to those of (3).

But (4) is equivalent to the problem

$$
\begin{aligned}
&\text{maximize} && P\Big(\sum_{j=1}^{n} c_j X_j - X_{n+1} \geq b_{n+1}\Big) \\
(5)\qquad &\text{subject to} && P\Big(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \geq \alpha_i\Big), \quad i = 1, \ldots, n \\
& && P(X_j \geq 0) \geq \beta_j, \quad j = 1, \ldots, n \\
& && X_{n+1} \geq 0 .
\end{aligned}
$$

That (4) and (5) are equivalent can be seen as follows.

Let X_j', $j = 1, \ldots, n$ be feasible for (4), then X_j', $j = 1, \ldots, n$, and any $X_{n+1}' \geq 0$ are feasible for (5) and conversely. Moreover, if X_j^*, $j = 1, \ldots, n$ are optimal for (4), then X_j^*, $j = 1, \ldots, n$ and $X_{n+1}^* \equiv 0$ are optimal for (5). Since if there exist X_j'', $j = 1, \ldots, n+1$ such that X_j'' are feasible for (5) and the inequality

$$
P\Big(\sum_{j=1}^{n} c_j X_j'' - X_{n+1}'' \geq b_{n+1}\Big) > P\Big(\sum_{j=1}^{n} c_j X_j^* - X_{n+1}^* \geq b_{n+1}\Big)
$$

holds, then we also have

$$
P\Big(\sum_{j=1}^{n} c_j X_j'' \geq b_{n+1}\Big) > P\Big(\sum_{j=1}^{n} c_j X_j^* \geq b_{n+1}\Big)
$$

as X_{n+1}'', $X_{n+1}^* \geq 0$ everywhere. But X_j'', $j = 1, \ldots, n$ are feasible for (5) and hence they are also feasible for (4). Thus we have contradicted the assumed optimality of X_j^*, $j = 1, \ldots, n$, so X_j^*, $j = 1, \ldots, n$ and $X_{n+1}^* \equiv 0$ are indeed optimal for (5). Conversely if X_j^*, $j = 1, \ldots, n+1$ are an optimal set of rules for (5), then X_j^*, $j = 1, \ldots, n$ are optimal for (4). Hence (4) and (5) are equivalent problems.

Now let α_{n+1}^* be the optimal value of the objective function of (4). Then α_{n+1}^* is also the optimal value of (5). Hence <u>any</u> set of decision rules $X_1^*, \ldots, X_{n+1}^*$ which satisfy the constraints of (5) and the additional constraint

(6) $$P\left(\sum_{j=1}^{n} c_j X_j^* - X_{n+1}^* \geq b_{n+1}\right) \geq \alpha_{n+1}^*$$

will be optimal for (5) and $X_1^*, \ldots, X_n^*$ will be optimal for (4). (Of course our definition of α_{n+1}^* implies that we will only be able to satisfy the second inequality in (6) as an equality and not a strict inequality, but it is more convenient to write it as we have above). Thus our object now is to find $X_1^*, \ldots, X_{n+1}^*$ which satisfy

(7) $$\begin{cases} P\left(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \leq 0\right) \geq \alpha_i, & i=1, \ldots, n+1 \\ P(X_j \geq 0) \geq \beta_j, & j=1, \ldots, n+1 \end{cases}$$

where $\beta_{n+1} = 1$, $\alpha_{n+1} = \alpha_{n+1}^*$, $\omega_{n+1}(b_1, \ldots, b_n) = 0$, $d_{n+1} = 1$

and $$\alpha_{n+1,j} = \begin{cases} -c_j & j=1, \ldots, n \\ 1 & j=n+1 \end{cases}.$$

If we regard (7) as the constraints of a chance-constrained programming problem, (4) and (5) will be solved as soon as we have found <u>any</u> set of feasible decision rules for the problem whose constraints are (7). Thus we can use as our objective function in this problem any function we desire. It is convenient to choose as our objective maximize $E\left(\sum_{j=1}^{n+1} c_j' X_j\right)$ and so to solve the problem

$$\text{maximize} \quad E\left(\sum_{j=1}^{n+1} c_j' X_j\right)$$

(8) $$\text{subject to} \quad \text{constraints given by (7)}$$

where c_j', $j=1, \ldots, n+1$ are some set of constants But (8) is an $n+1$ period triangular E-model problem of the type considered in [8]. Moreover in corollary 2 in [8] we establish that a necessary condition that X_j^*, $j=1, \ldots, n+1$ be optimal for (8) is that X_j^* be a piecewise linear function of ω_i, $i=1, \ldots, j$ and $\{z_k^{i*}, k \in K^{i-1}\}$, $i=1, \ldots, j$, where $z_k^{i*} \equiv \max\{z_k^i : \bar{f}_i(z_k^i) = T_k^{i*}\}$, T_k^{i*} is a given constant for each A_k^{i-1} defined in assumption (f), and $\bar{f}_i(\cdot)$

is the conditional frequency function of b_i given by $b_1, \ldots, b_{i-1}$. Thus we have established

Theorem 1 A sufficient condition that X_j^*, $j=1,\ldots,n$ be optimal for (4) is that X_j^*, $j=1,\ldots,n$ be the optimal piecewise linear function of ω_i, $i=1,\ldots,j$ and $\{z_k^{i*}, k\epsilon K^{i-1}\}$, $i=1,\ldots,j$ where z_k^{i*} is defined above.

Note that none of the quantities with subscript $n+1$ enter into the determination of X_j^*, $j=1,\ldots,n$ as given in Theorem 1. Hence we can conclude that X_j^*, $j=1,\ldots,n$ are not functions of the ϵ used in defining b_{n+1}. Thus they are, in fact, a set of optimal rules for (3) and not merely a set of rules which converge to the optimal rules of (3).

In the special case where $b_1, \ldots, b_n$ are independent random variables we have, using the results in section 6 in [8],

Theorem 2 Let $b_1, \ldots, b_n$ be independent random variables. Then a sufficient condition that X_j^*, $j=1,\ldots,n$ be the optimal piecewise linear function of ω_i, $i=1,\ldots,j$ and $\{D_k^i, k\epsilon K^{i-1}\}$, $i=1,\ldots,j$, where D_k^i is a constant for each $k\epsilon K^{i-1}$ and K^{i-1} is the indexing set defined in assumption (f).

We could proceed to establish other theorems which are similar to those contained in [8]. We will not do so, however, as our main objective in this paper is to prove the general piecewise linearity of the optimal rules which we have just done. The extensions and ramifications of this result along with examples of optimal rules in special cases will be presented in subsequent papers.

4. The Block Triangular Model: Conditional Chance Constraints

The model we treat in this and the following section is similar to that discussed in [7]. It is expressed as follows:

$$
\begin{aligned}
&\text{maximize} && P\Big(\sum_{j=1}^{n} c_j^T X_j \geq k\Big) \\
(9)\qquad &\text{subject to} && \bar{P}(A_{11} X_1 \leq b_1) \geq \bar{\alpha}_1 \\
& && \bar{P}(A_{21} X_1 + A_{22} X_2 \leq b_2) \geq \bar{\alpha}_2 \\
& && \text{----------} \\
& && \bar{P}(A_{i1} X_1 + \ldots + A_{ii} X_i \leq b_i) \geq \bar{\alpha}_i \\
& && \text{----------} \\
& && \bar{P}(A_{n1} X_1 + \ldots + A_{nn} X_n \leq b_n) \geq \bar{\alpha}_n \\
& && X_j \geq 0 \ , \quad j = 1, \ldots, n \quad .
\end{aligned}
$$

In (9) we make the following assumptions:

(a) A_{ij}, $i \geq j$, $i, j = 1, \ldots, n$ is an $m_i \times n_j$ matrix of constants,

(b) c_j^T, $j = 1, \ldots, n$ is a $1 \times n_j$ vector of constants,

(c) k is a given constant,

(d) b_i, $i = 1, \ldots, n$ is an $m_i \times 1$ vector of random variables. The joint distribution of all the random variables contained in b_i, $i = 1, \ldots, n$ is assumed to be known.

(e) $\bar{\alpha}_i$, $i = 1, \ldots, n$ is an $m_i \times 1$ vector of probabilities. $\bar{\alpha}_{ik}$, the k^{th} element of $\bar{\alpha}_i$, is a function of the random variables contained in $b_1, \ldots, b_{i-1}$, i.e., $\bar{\alpha}_{ik} \equiv \bar{\alpha}_{ik}(b_1, \ldots, b_{i-1})$.

(f) X_j, $j = 1, \ldots, n$ is the $n_j \times 1$ vector of decision rules for the j^{th} period. X_j is a function of $b_1, \ldots, b_{j-1}$ but it is not a function of $b_j, \ldots, b_n$.

With the exception of (e), these assumptions are similar to those used in the triangular problem. The chief difference is that in (9) the i^{th} period generates m_i coupling constraints, while in (3) m_i is equal to one. Also, in (9) the components of b_i, $i = 1, \ldots, n$ need not be continuous random variables as they were in (3).

Assumption (e) needs to be explained in more detail. Because we interpret the $\bar{P}$ operator in the constraint

$$\bar{P}\left(\sum_{j=1}^{i} A_{ij} X_j \leq b_i\right) \geq \bar{\alpha}_i$$

as meaning that we compute the probability using the conditional distribution of b_i given $b_1, \ldots, b_{i-1}$, it follows that $\bar{P}\left(\sum_{j=1}^{i} A_{ij} X_j \leq b_i\right)$ will be a function of $b_1, \ldots, b_{i-1}$. Hence we want to allow the probability with which the i^{th} period constraints must hold to be a function of the random variables $b_1, \ldots, b_{i-1}$. For this reason $\bar{\alpha}_i$ can be a function of the random variables of periods 1 to i-1.

The fact that $\bar{P}$ is a conditional probability operator explains why we have $X_j \geq 0$ in (9) rather than $\bar{P}(X_j \geq 0) \geq \bar{\beta}_j$ which would correspond more closely to (3). For in this latter case, in accordance with the above notation, we would have to treat $\bar{P}$ as meaning that we compute $\bar{P}(X_j \geq 0)$ using the conditional distribution of b_j given by $b_1, \ldots, b_{j-1}$. But given $b_1, \ldots, b_{j-1}$, X_j is deterministic by assumption (f), hence if $\bar{\beta}_j > 0$ we must have $X_j \geq 0$ if it is to be feasible. So instead of writing $\bar{P}(X_j \geq 0) \geq \beta_j$ we write the nonnegativity constraints in the simpler form $X_j \geq 0$.

In the objective function of (9) however, the P operator means that we compute the probability using the joint distribution of all the random variables involved in $b_1, \ldots, b_n$. This is similar to the interpretation used in (3).

From [7] we have the following

Lemma 1: The constraint

$$\bar{P}\left(\sum_{j=1}^{i} A_{ij} X_j \leq b_i\right) \geq \bar{\alpha}_i$$

in (9) can be replaced by the equivalent constraint

$$\sum_{j=1}^{i} A_{ij} X_j \leq \bar{F}_i^{-1}(1 - \bar{\alpha}_i),$$

where $\bar{F}_i^{-1}(1 - \bar{\alpha}_i)$ is the $m_i \times 1$ vector of $1 - \bar{\alpha}_i$ percentile (or fractile) points of the conditional distribution of b_i given $b_1, \ldots, b_{i-1}$. $\bar{F}_i^{-1}(1 - \bar{\alpha}_{ik})$, the k^{th} component of

$\bar{F}_i^{-1}(1-\bar{\alpha}_i)$, is defined by $\bar{F}_i^{-1}(1-\bar{\alpha}_{ik}) \equiv \max\{y : \bar{F}_{ik}(y) \le 1-\bar{\alpha}_{ik}\}$ where $\bar{F}_{ik}(\cdot)$ is the conditional distribution function of b_{ik}, the k^{th} component of b_i, given $b_1, \ldots, b_{i-1}$.

Using this lemma we can write (9) in the equivalent form

$$\text{maximize} \quad P\Big(\sum_{j=1}^{n} c_j^T X_j \ge k\Big)$$

$$\text{subject to} \quad \sum_{j=1}^{i} A_{ij} X_j \le \bar{F}_i(1-\bar{\alpha}_i) \quad , \quad i = 1, \ldots, n \tag{10}$$

$$X_j \ge 0 \quad , \quad j = 1, \ldots, n \quad .$$

On considering (10), an immediate question which arises is whether or not a set of feasible decision rules exists for all possible sample points $(b_1, \ldots, b_n)$. In general such a set will not exist for all sample points and so we are faced with the problem of how we compute $P(\sum_{j=1}^{n} c_j^T X_j \ge k)$ over those sample points for which a feasible set of $X_1, \ldots, X_n$ fails to exist.

One method of resolving this difficulty, and the one we shall use here, is to interpret the objective function as meaning that we want to maximize the joint probability that $\sum_{j=1}^{n} c_j^T X_j \ge k$ and feasible $X_1, \ldots, X_n$ exist, i.e., maximize $P(\sum_{j=1}^{n} c_j^T X_j \ge k \cap X_1, \ldots, X_n$ feasible). Thus for any sample point for which feasible $X_1, \ldots, X_n$ do not exist, we get no contribution to the objective function. This procedure is equivalent to defining X_{jk}, $k = 1, \ldots, n_j$, $j = 1, \ldots, n$ to be M if $c_{jk} < 0$, or to be $-M$ if $c_{jk} > 0$, at all points of inconsistency, where $M > 0$ is very large. Thus it corresponds to defining $X_{jk} = 0$ at the points of inconsistency in the E-model as was done in [7].

This is not, however, the only way in which we could deal with the problem of inconsistency. We could, for example, explicitly introduce constraints which would result in our feasible choices of $X_1, \ldots, X_{j-1}$ being limited in such a way that we are guaranteed that a feasible X_j exists, $j = 1, \ldots, n$. The type of additional constraints we would have to include

would be of the same form as the constraints of (10) except that the right-hand side of the i^{th} block of constraints would be operated on by a projection into the range of the A_{jj} operator. This procedure is discussed in section 7 of [7].

Another alternative, and one which is much less satisfactory than the two discussed above, would be to simply assume that the constraints are consistent for all possible sample points. This procedure is adopted in the n stage linear programming under uncertainty problems discussed in [12, 13, 14, 15, 22]. Such an assumption destroys one of the major features of chance-constrained programming, namely that the decision rules which result from solving a chance-constrained problem are designed only to provide "policies" for management operation and decision. As such, the implementation of these decision rules is subject to the controls available to the manager, hence they may impute an action which, due to exceptional circumstances, cannot actually be taken. Thus our rules need not spell out in advance the actual actions that will be taken in exceptional circumstances, e.g., the circumstances under which the constraints of (10) will be inconsistent.

We now turn to the problem of finding sufficient conditions for X_j^*, $j = 1, \ldots, n$ to be optimal for (10), under the assumption that if a set of infeasible decision rules does not exist for some sample point, this point will not be included in computing $P(\sum_{j=1}^{n} c_j^T X_j^* \geq k)$.

5. Sufficient Conditions for Optimality

In a manner analogous to that used in section 3, it is easy to show that (10) is equivalent to the problem

$$\text{maximize} \quad P(X_{n+1} \geq 0)$$

$$\text{subject to} \quad \sum_{j=1}^{i} A_{ij} X_j \leq \bar{F}_i^{-1}(1 - \bar{\alpha}_i) \quad , \quad i = 1, \ldots, n \tag{11}$$

$$-\sum_{j=1}^{n} c_j^T X_j + X_{n+1} \leq -k$$

$$X_j \geq 0 \quad , \quad j = 1, \ldots, n \quad ,$$

where X_{n+1} is selected after observing $b_1, \ldots, b_n$ and in the objective function we integrate over only those values of $b_1, \ldots, b_n$ for which $X_{n+1} \geq 0$ and feasible X_j, $j = 1, \ldots, n+1$ exist.

Let b_{n+1} be a $N(0,1)$ random variable which is independent of $b_1, \ldots, b_n$.

Let $\bar{\alpha}_{n+1}$ be defined by

$$\bar{\alpha}_{n+1} \equiv 1 - \bar{F}_{n+1}(-k) , \tag{12}$$

where $\bar{F}_{n+1}(\cdot)$ is the conditional distribution function of b_{n+1} given $b_1, \ldots, b_n$. Because of the independence of b_{n+1} and b_i, $i = 1, \ldots, n$ we see that $\bar{F}_{n+1}(\cdot)$ is the distribution function of a $N(0,1)$ random variable. Thus $\bar{\alpha}_{n+1}$ defined by (12) will be constant for all b_i, $i = 1, \ldots, n$.

Let $A_{n+1,j} = \begin{cases} -c_j^T , & j = 1, \ldots, n \\ 1 , & j = n+1 \end{cases}$

so that $A_{n+1,j}$ is $1 \times n_j$, $j = 1, \ldots, n$ and $A_{n+1,n+1}$ is 1×1.

Then (11) can be written as

$$\begin{aligned} &\text{maximize} \quad P(X_{n+1} \geq 0) \\ &\text{subject to} \quad \sum_{j=1}^{i} A_{ij} X_j \leq \bar{F}_i^{-1}(1 - \bar{\alpha}_i) , \quad i = 1, \ldots, n+1 \\ &\qquad\qquad X_j \geq 0 , \quad j = 1, \ldots, n . \end{aligned} \tag{13}$$

In (13) for each sample point $(b_1, \ldots, b_{n+1})$ we want to find decision rules X_j^*, $j = 1, \ldots, n+1$ which are feasible and, if possible, also satisfy the inequality $X_{n+1}^* \geq 0$.

Suppose for each sample point we consider the set of constraints

$$\begin{cases} \sum_{j=1}^{i} A_{ij} X_j \leq \bar{F}_i^{-1}(1 - \bar{\alpha}_i) , & i = 1, \ldots, n+1 \\ X_j \geq 0 , & j = 1, \ldots, n+1 . \end{cases} \tag{14}$$

If we find any set of decision rules which satisfies these constraints for a given sample point, then for this point such a set of rules will be optimal for (13). To see this, consider any point which has the property that feasible rules exist for (14). Then this point also yields feasible rules for (13) and gives $X_{n+1} \geq 0$. Conversely, if for some sample point we can find feasible rules for (13) which contribute to the objective function of (13), these rules will also satisfy (14). Moreover any point for which feasible X_j, $j = 1, \ldots, n$ exist in (13) but fail to give $X_{n+1} \geq 0$ can be treated as a point of inconsistency since it does not contribute to the objective function.

To obtain a set of rules which satisfies (14) let us solve the problem

$$\text{(15)} \qquad \begin{aligned} &\text{maximize} \quad E\Big(\sum_{j=1}^{n+1} c_j'^{T} X_j\Big) \\ &\text{subject to} \quad \text{constraints given by (14)} , \end{aligned}$$

where all components of $c_j'^{T}$, $j = 1, \ldots, n+1$ are strictly positive. In (15) just as we did in [7], we will set $X_j = 0$, $j = 1, \ldots, n+1$ at all points where the constraints are inconsistent. Since $c_j'^{T} > 0$, $j = 1, \ldots, n+1$ and we are maximizing, we know that any point for which $\sum_{j=1}^{n+1} c_j'^{T} X_j^* > 0$ is such that feasible rules exist for (15). Thus for such a point decision rules exist which are feasible for (10) and which contribute to the objection of (10) (i.e., they give $\sum_{j=1}^{n} c_j^{T} X_j \geq k$).

But in theorem 2 of [7] we have shown that a necessary condition that X_j^*, $j = 1, \ldots, n+1$ be optimal for the n+1 period E-model, given by (15) is that X_j^* be a piecewise linear function of $\bar{F}_j^{-1}(1 - \bar{\alpha}_j)$ and $X_1^*, \ldots, X_{j-1}^*$. Hence we have proved

Theorem 3 A sufficient condition that X_j^*, $j = 1, \ldots, n$ be optimal decision rules for (9) is that X_j^* be the optimal piecewise linear function of $\bar{F}_j^{-1}(1 - \bar{\alpha}_j)$ and X_k^*, $k = 1, \ldots, j-1$.

This theorem leads immediately to

Corollary 1 A sufficient condition that X_j^*, $j = 1, \ldots, n$ be optimal decision rules for (9) is that X_j^* be the optimal piecewise linear function of $\bar{F}_k^{-1}(1 - \bar{\alpha}_k)$, $k = 1, \ldots, j$.

6. Conclusions

Having given a general characterization of the classes of rules which contain an optimal set for (3) and (9), we will conclude with a few remarks on how these rules can be found. In general it is extremely difficult to find an optimal set of rules even though we know the rules are piecewise linear. It is not, however, impossible as is evidenced by the examples in [7, 9] and in the solution of the savings and loan problem discussed in [9]. This latter example is particularly useful since it provides us with a means of comparing the optimal piecewise linear rules with the optimal linear rules found in [11].

Moreover, even though the general problem is difficult to solve, it may be possible to generate algorithms for finding the optimal rules in special cases. For example, it is fairly easy to establish sufficient conditions that X_j^*, $j = 1, \ldots, n$ be piecewise linear in the random variables $b_1, \ldots, b_{j-1}$. Since much work had been done on finding deterministic equivalents for problems in which X_j is restricted to be a linear function of $b_1, \ldots, b_{j-1}$ (see [3, 5, 10, 11]), a perturbation technique may be able to be used to find exactly how the optimal linear rule ought to be perturbed in order to get the optimal piecewise linear rule. In the savings and loan problem referred to above an approach of this sort would have found that a rather minor perturbetion of the optimal linear rule resulted in the optimal piecewise linear rule. Thus this perturbation approach may lead to efficient means of solving certain classes of problems.

Finally, it is to be hoped that the fact that the same class of rules is optimal for the E-model and the corresponding P-model will lead to increased efforts in finding algorithms for these problems, since the resulting alforithms will then solve two of the three chance-constrained models first proposed in [3].

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Sufficient conditions are derived for decision rules to be optimal for two classes of n-period P-models of chance-constrained programming. It is shown that the optimal rule for period j is the optimal piecewise linear function of the decision rules of previous periods and certain fractile points. The optimal class of rules is shown to be the same for the n-period P-model as for the corresponding n-period E-model.

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