

University of Washington
College of Engineering
Department of Electrical Engineering

ON THE GENERAL SOLUTIONS OF
COUPLED-MODE EQUATIONS WITH
VARYING COEFFICIENTS

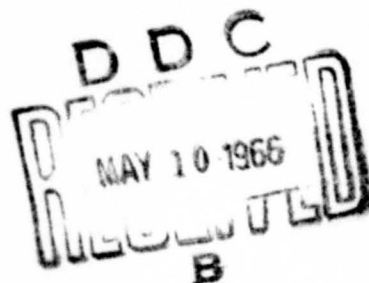
By

YUNG-SEN CHEN
AKIRA ISHIMARU

Contract No. AF19 (628)-2763

Project No. 5635

Task No. 563502



TECHNICAL REPORT NO. 103

FEBRUARY 1966

Prepared for
Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Mass.

CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION	
Hardcopy	Microfiche
2.00	0.50 38 PP. as
ARCHIVE COPY	

Code 1



ON THE GENERAL SOLUTIONS OF COUPLED-MODE EQUATIONS
WITH VARYING COEFFICIENTS

by

Yung-Sen Chen

Akira Ishimaru

University of Washington
College of Engineering
Department of Electrical Engineering

Contract No. AF19(628)-2763

Project No. 5635

Task No. 563502

Technical Report No. 103

February 1966

The research reported in this document has been sponsored by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, United States Air Force, Bedford, Massachusetts. The publication of this report does not necessarily constitute approval by the Air Force of the findings or conclusions contained herein.

Distribution of this report is unlimited

Acknowledgment

The authors would like to thank Dr. R. A. Sigelmann for helpful discussions and Dr. J. N. Lahti for comments on the manuscript. They are also grateful to Mrs. Bernice Freeman for the typing.

Table of Contents

Introduction	1
I. General Solutions of Coupled-Mode Equations	2
II. The Generalized Riccati Equation	7
III. Uniform Coupling of Two Lossless Modes of Propagation	9
IV. Coupled-Mode Description of Guided Wave Propagation through a Ferrite Rod	10
V. Case (i): $A(z) = D(z)$ $B(z) = C(z)$	12
VI. Case (ii): $A(z) = j[\beta(z) + \mu_1 \eta(z)]$ $D(z) = j[\beta(z) - \mu_2 \eta(z)]$ $C(z) = B(z) = j\mu_3 \eta(z)$	13
VII. Case (iii): $A(z) = j[\beta(z) + \mu_1 \eta(z)]$ $D(z) = j[\beta(z) - \mu_2 \eta(z)]$ $C(z) = -B(z) = j\mu_3 \eta(z)$	14
VIII. Case (iv): $A(z) = D(z)$ $B(z) = \pm C(z)$	15
IX. Concluding Remarks	17
Appendix A: Euler's Theorem Concerning the Generalized Riccati Equation	18
Appendix B: Particular Solutions of the Generalized Riccati Equation	19
References	31

Abstract

In this report a systematic mathematical method is introduced for the solution of problems involving two coupled modes in a coupled system with varying parameters. These problems involve systems of linear differential equations with varying coefficients.

By the use of a linear transformation of the dependent variables and a double diagonalization process, the coupled mode equations are reduced to two decoupled Riccati equations. The final form of the general solution is obtained in terms of four varying coupling coefficients and a transform parameter.

To illustrate some applications of the method, solutions of two special cases which have been solved previously by other workers are obtained. The solutions for a number of special cases, in which the varying coefficients are specified or interrelated, are also obtained. Further possible applications are indicated.

Introduction

During the past decade, workers in different branches of Physics and Engineering have published a considerable amount of material on the application of coupled mode theory to various kinds of coupled systems.¹⁻⁸ As far as the mathematical theory is concerned, most researchers have considered only special cases of coupled systems having constant parameters or slowly varying parameters.^{5,7} In the latter case approximate solutions have been obtained. Recently, in their study of Faraday rotation in a ferrite rod, Huang and Fan considered the case in which two self-coupling coefficients are equal and one of the mutual coupling coefficients is equal to the negative-conjugate of the other.³ This report extends the treatment further to more general case of varying parameters, and a systematic mathematical method is introduced to solve the problem of two coupled modes in a coupled system.

In coupled mode theory, the behavior of a coupled system is described in terms of the normal modes of the uncoupled system. The equations that characterize the behavior of two coupled modes takes the form:

$$\frac{da_1}{dz} = A(z)a_1 + B(z)a_2 \quad (1-a)$$

$$\frac{da_2}{dz} = C(z)a_1 + D(z)a_2 \quad (1-b)$$

In these expressions, a_1 and a_2 are mode amplitudes of the coupled system, and $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are mode coupling coefficients which may be any arbitrary functions of z . By the use of a linear transformation of the dependent variables and a double diagonalization process, we are able to obtain general solutions of the coupled mode equations in terms of the four varying parameters A , B , C , D , and a transform parameter h_0 , which is a particular solution of a generalized Riccati equation. Since the particular solutions of the generalized Riccati equation have been widely studied, and numerous results

are available in the literature of non-linear differential equations,⁹⁻¹⁵ general solutions of equations (1) can be obtained for a large number of special cases. The solutions that we have studied can be applied to many different types of systems. What is of interest here is that the results are of such generality that they allow the description and classification of a wide class of devices in a significant manner.

Further applications of our solutions to the structures of log-periodic antennas, traveling wave antennas, and directional couplers are indicated in the concluding remarks.

I. General Solutions of Coupled-Mode Equations

This section is devoted to solving the two dimensional coupled-mode equations by linear transformations of the dependent variables. In order to simplify the writing, we express the coupled mode equations in matrix form.

Let

$$\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad [A] = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \quad (2)$$

Equations (1) in matrix form become

$$\bar{a}' = [A] \bar{a} \quad (3)$$

where the prime denotes differentiation with respect to the independent variable z . Now, we introduce the following transformation of the dependent variables in equation (3).¹¹ Let $[R]$ be a continuously differentiable, non-singular matrix for $z_1 \leq z \leq z_2$. Under the linear change of variables $\bar{a} \rightarrow \bar{g}$, where

$$\bar{a} = [R] \bar{g} \quad (4)$$

(3) is transformed into

$$\bar{g}' = [R]^{-1} \{ [A] [R] - [R]' \} \bar{g} \quad (5)$$

For our purpose, we choose $[R]$ specifically to be

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} e^{\int A dz} & 0 \\ 0 & e^{\int D dz} \end{bmatrix} \quad (6)$$

It is easy to show by matrix algebra that

$$\bar{g}' = [S] \bar{g} \quad (7)$$

where

$$[S] = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 0 & B e^{-\int (A-D) dz} \\ C e^{\int (A-D) dz} & 0 \end{bmatrix} \quad (8)$$

For a second transformation, let us again assume that $[T]$ be a continuously differentiable non-singular matrix in the same range of z as considered in the first transformation: i.e. $z_1 \leq z \leq z_2$. Under the linear change of dependent variables $\bar{g} \rightarrow \bar{y}$, where

$$\bar{g} = [T] \bar{y} \quad (9)$$

(7) is transformed into

$$\bar{y}' = [T]^{-1} \{ [S] [T] - [T]' \} \bar{y} \quad (10)$$

Let us choose $[T]$ specifically to be

$$[T] = \begin{bmatrix} 1 & t_{12} \\ t_{21} & 1 \end{bmatrix} \quad (11)$$

Then (10) becomes after some mathematical manipulations

$$\bar{y}' = [L] \bar{y} \quad (12)$$

where

$$[L] = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = \begin{bmatrix} s_{12} t_{21} & 0 \\ 0 & s_{21} t_{12} \end{bmatrix} \quad (13)$$

t_{12} and t_{21} are transform parameters which are particular solutions of the following differential equations:

$$t'_{12} - s_{12} + s_{21}t_{12}^2 = 0 \quad (14-a)$$

$$t'_{21} - s_{21} + s_{12}t_{21}^2 = 0 \quad (14-b)$$

and s_{12} , s_{21} are the matrix elements given specifically in (8).

Since $[L]$ in (13) is a diagonal matrix, solutions for the y 's can easily be found by solving the decoupled differential equations of (12).

They are

$$y_1 = c'_1 e^{\int s_{12} t_{21} dz} \quad (15-a)$$

$$y_2 = c'_2 e^{\int s_{21} t_{12} dz} \quad (15-b)$$

where c'_1 and c'_2 are constants of integration.

According to transformation (9), we have

$$g_1 = y_1 + t_{12}y_2 = c'_1 e^{\int s_{12} t_{21} dz} + c'_2 t_{12} e^{\int s_{21} t_{12} dz} \quad (16-a)$$

$$g_2 = t_{21}y_1 + y_2 = c'_1 t_{21} e^{\int s_{12} t_{21} dz} + c'_2 e^{\int s_{21} t_{12} dz} \quad (16-b)$$

Also, from the transformation (4), and the expressions for the g 's in (16),

we obtain

$$a_1 = c'_1 r_{11} e^{\int s_{12} t_{21} dz} + c'_2 r_{11} t_{12} e^{\int s_{21} t_{12} dz} \quad (17-a)$$

$$a_2 = c'_1 r_{22} t_{21} e^{\int s_{12} t_{21} dz} + c'_2 r_{22} e^{\int s_{21} t_{12} dz} \quad (17-b)$$

Using the following transformation of the dependent variables in (14),

$$\begin{bmatrix} t_{12} \\ t_{21} \end{bmatrix} = [U] \begin{bmatrix} h_{12} \\ h_{21} \end{bmatrix} \quad (18)$$

where $[U]$ is given as

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} e^{-\int (A-D) dz} & 0 \\ 0 & e^{\int (A-D) dz} \end{bmatrix} \quad (19)$$

We then obtain from (14)

$$h'_{12} - (A-D) h_{12} + Ch_{12}^2 = B \quad (20-a)$$

$$h'_{21} + (A-D) h_{21} + Bh_{21}^2 = C \quad (20-b)$$

Also, (17) becomes under the transformation of (19)

$$a_1 = c'_1 r_{11} e^{\int s_{12} u_{11} h_{12} dz} + c'_2 r_{11} t_{12} e^{\int s_{21} u_{12} h_{21} dz} \quad (21-a)$$

$$a_2 = c'_1 r_{22} t_{21} e^{\int s_{12} u_{22} h_{21} dz} + c'_2 r_{22} e^{\int s_{21} u_{11} h_{12} dz} \quad (21-b)$$

Substituting all the elements of $[R]$, $[S]$, and $[U]$ specified in (6), (8), and (19), we finally obtain the general solutions of the coupled mode equations (1):

$$a_1 = c'_1 e^{\int (A+Bh_{21}) dz} + c'_2 h_{12} e^{\int (D+Ch_{12}) dz} \quad (22-a)$$

$$a_2 = c'_1 h_{21} e^{\int (A+Bh_{21}) dz} + c'_2 e^{\int (D+Ch_{12}) dz} \quad (22-b)$$

where h_{12} and h_{21} are transform parameters which are particular solutions of the generalized Riccati equations (20-a) and (20-b), respectively. It must be noted that in (22) h_{12} and h_{21} should be subject to the condition that $h_{12}h_{21} - 1 \neq 0$, which follows from the non-singular requirement on transform matrix $[T]$.

Realizing that parameters h_{12} and h_{21} are related through the differential equations (20), we can further eliminate one of these parameters. It can be shown that if h_0 is a particular solution of (20-b), $1/h_0$ will be a particular solution of (20-a). In order to satisfy the condition that $h_{12}h_{21} - 1 \neq 0$, $1/h_0$ and h_0 cannot be used as transform parameters for h_{12} and h_{21} , respectively, at the same time.

Let $1/h_0$ be a particular solution of (20-a). Then by Euler's theorem concerning the generalized Riccati equations (see appendix A), we find that

$$h_0 + \frac{\exp \int (D-A-2Bh_0)dz}{\int B \exp \{ \int (D-A-2Bh_0)dz \} \cdot dz}$$

is also a particular solution of (20-b). Since the quantity

$$h_{12}h_{21} - 1 \equiv \frac{\exp \int (D-A-2Bh_0)dz}{h_0 \int B \exp \{ \int (D-A-2Bh_0)dz \} \cdot dz}$$

does not vanish in the interval $z_1 \leq z \leq z_2$, in general, $1/h_0$ and the above expression are appropriate transform parameters for the system (22). The result of substituting the above particular solutions for the h 's in (22-a) is

$$\begin{aligned} a_1 &= c'_1 \exp \int \left[A + B \left(h_0 + \frac{\exp \int (D-A-2Bh_0)dz}{\int B \exp \{ \int (D-A-2Bh_0)dz \} \cdot dz} \right) \right] \cdot dz \\ &\quad + c'_2 \frac{1}{h_0} \exp \int \left(D + \frac{C}{h_0} \right) dz \\ &= c'_1 \left[\int B \exp \{ \int (D-A-2Bh_0)dz \} \cdot dz \right] \cdot \exp \int (A + Bh_0)dz \\ &\quad + c'_2 \frac{1}{h_0} \exp \int \left(D + \frac{C}{h_0} \right) dz \end{aligned} \quad (23)$$

From (20-a), it is found that

$$\frac{1}{h_0} \exp \int \left(D + \frac{C}{h_0} \right) dz = k \exp \int (A+Bh_0)dz \quad (24)$$

where k is an arbitrary constant. Then (23) is reduced to

$$a_1 = \left[c_1 \int B \exp \{ \int (D-A-2Bh_0)dz \} \cdot dz + c_2 \right] \cdot \exp \int (A+Bh_0)dz \quad (25-a)$$

Similarly, we have

$$\begin{aligned} a_2 &= \left\{ c_1 h_0 \int B \exp \{ \int (D-A-2Bh_0)dz \} \cdot dz + c_1 \exp \{ \int (D-A-2Bh_0)dz \} \right. \\ &\quad \left. + c_2 h_0 \right\} \exp \int (A+Bh_0)dz \end{aligned} \quad (25-b)$$

where h_0 is a transform parameter which is a particular solution of the well-known generalized Riccati equation (20-b), and c_1, c_2 are constants of integration.

Following the same procedures, we may obtain the solutions in a slightly different form. They are

$$a_1 = \left\{ c_1 h_0 + c_2 \left[h_0 \int C \exp\left\{\int (A-D-2Ch_0)dz\right\} \cdot dz + \exp\left\{\int (A-D-2Ch_0)dz\right\} \right] \right\} \cdot \exp\int (D+Ch_0)dz \quad (26-a)$$

$$a_2 = [c_1 + c_2 C \exp\left\{\int (A-D-2Ch_0)dz\right\} \cdot dz] \exp\int (D+Ch_0)dz \quad (26-b)$$

where h_0 is a particular solution of the generalized Riccati equation (20-a), and c_1, c_2 are constants of integration.

Our general solutions of the coupled-mode equations have thus been obtained by three successive linear transformations of dependent variables and a double diagonalization process. There is a transform parameter involved in our solutions, which is a particular solution of a generalized Riccati equation. The problem of finding general solutions of the coupled-mode equations is now reduced to that of finding a particular solution of a generalized Riccati equation.

The next section will be concerned with a discussion of the generalized Riccati differential equation. A method of obtaining particular solutions of the generalized Riccati equation is given in Appendix B.

II. The Generalized Riccati Equation

It is customary to give the name "generalized Riccati equation" to any equation of the form

$$\frac{dy}{dz} + Py + Qy^2 = R \quad (27)$$

where P, Q , and R are given functions of z .

This equation has considerable theoretical importance, since its solutions are free from movable branch points and can have only movable poles.¹⁰ It is a special case of the Abel equation. It is supposed that neither R nor Q is identically zero. If $Q = 0$, the equation is linear; if $R = 0$, the equation is reducible to the linear form by taking $1/y$ as a new variable.

It has been shown by Euler that, if a particular solution of the generalized Riccati equation is known, the general solution can be obtained by two quadratures; if two particular solutions are known, the general solution is obtainable by a single quadrature. And it follows from theorems by Weyr and Picard that, if three particular solutions are known, the general solution can be effected without a quadrature.

The equation (27) is easily reduced to a linear equation of the second order by taking a new dependent variable u defined by the equation

$$y = \frac{1}{Q} \frac{d \log u}{dz} \quad (28)$$

The equation then becomes

$$\frac{d^2 u}{dz^2} + \left(P - \frac{1}{Q} \frac{dQ}{dz} \right) \frac{du}{dz} - Q R u = 0 \quad (29)$$

Conversely, if in the general linear equation of the second order

$$P_0 \frac{d^2 u}{dz^2} + P_1 \frac{du}{dz} + P_2 u = 0 \quad (30)$$

(where P_0, P_1, P_2 are given functions of z), one may write

$$u = e^{\int Q y dz} \quad (31)$$

From which the equation defining y becomes

$$\frac{dy}{dz} + \left(\frac{P_1}{P_0} + \frac{1}{Q} \frac{dQ}{dz} \right) y + Q y^2 = - \frac{P_2}{P_0 Q} \quad (32)$$

which is of the same type as (27). The complete equivalence of the generalized Riccati equation with linear equation of the second order is consequently established.

In the following sections, we will apply our method to a number of special cases for which solutions have been previously established by other workers.

III. Uniform Coupling of Two Lossless Modes of Propagation

Consider two waves with time dependence $e^{j\omega t}$ which are weakly coupled.

It can be shown that the coupled-mode equations for this situation are

$$\frac{da_1}{dz} = -j\beta_1 a_1 + c_{12} a_2 \quad (33-a)$$

$$\frac{da_2}{dz} = c_{21} a_1 - j\beta_2 a_2 \quad (33-b)$$

The c_{12} and c_{21} are the mutual coupling coefficients per unit length. The coupling is assumed uniform over the length of the coupler, so that c_{12} and c_{21} are independent of z . The modes are assumed lossless. For this case, the matrix elements of $[A]$ are constants. Let

$$[A] = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} -j\beta_1 & c_{12} \\ c_{21} & -j\beta_2 \end{bmatrix} \quad (34)$$

where β_1 , β_2 , c_{12} , and c_{21} are independent of z .

From (34), (20-b) becomes

$$h' - j(\beta_1 - \beta_2) h + c_{12} h^2 = c_{21} \quad (35)$$

A particular solution of (35) can easily be found to be

$$h_0 = -\frac{\beta_b}{c_{12}} \tan \beta_b z + j \frac{\beta_d}{c_{12}} \quad (36)$$

where

$$\beta_d = \frac{\beta_1 - \beta_2}{2} \quad (37)$$

and

$$\beta_b = \sqrt{\beta_d^2 + c_{12} c_{21}} \quad (38)$$

Substituting (36) in (25) leads to

$$a_1 = \left[c_1 \left(\frac{-c_{12}}{\beta_b} \right) \sin \beta_b z + c_2 \cos \beta_b z \right] e^{-j\beta_a z} \quad (39-a)$$

$$a_2 = \left[\left(c_1 + c_2 \frac{j\beta_d}{c_{12}} \right) \sin \beta_b z - \left(c_1 \frac{j\beta_d}{\beta_b} + c_2 \frac{\beta_b}{c_{12}} \right) \cos \beta_b z \right] e^{-j\beta_a z} \quad (39-b)$$

where

$$\beta_a = \frac{\beta_1 + \beta_2}{2} \quad (40)$$

Equations (39) can be rewritten in a different form

$$a_1 = A_1 e^{-j\gamma_1 z} + A_2 e^{-j\gamma_2 z} \quad (41-a)$$

$$a_2 = \frac{1}{jc_{12}} \left[(\gamma_1 + \beta_1) A_1 e^{-j\gamma_1 z} + (\gamma_2 + \beta_1) A_2 e^{-j\gamma_2 z} \right] \quad (41-b)$$

where

$$\gamma_1 = \beta_a + \beta_b \quad (42-a)$$

and

$$\gamma_2 = \beta_a - \beta_b \quad (42-b)$$

Here γ_1 and γ_2 are normal mode propagation constants and A_1, A_2 are arbitrary constants.

These solutions agree with those obtained by other methods.

IV. Coupled-Mode Description of Guided Wave Propagation through a Ferrite Rod

A general set of coupled mode equations for wave propagation in a waveguide partially filled with a gyromagnetic medium has been derived by Huang and Fan. They are

$$\frac{da_e}{dz} = -K_{ee} a_e - K_{eo} a_o \quad (43-a)$$

$$\frac{da_o}{dz} = -K_{oe} a_e - K_{oo} a_o \quad (43-b)$$

Also

$$K_{ee}(z) = K_{oo}(z) \quad (44-a)$$

$$K_{eo}(z) = -K_{oe}(z) \quad (44-b)$$

where the K's are varying mode coupling coefficients.

In order to solve this set of equations, we write

$$[A] = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} -K_{ee}(z) & K_{eo}(z) \\ K_{eo}(z) & -K_{ee}(z) \end{bmatrix} \quad (45)$$

Then (20-b) becomes

$$h' - K_{eo}(h^2 + 1) = 0 \quad (46)$$

It is easy to show that a particular solution of (46) is

$$h_0 = \tan \int K_{eo} dz \quad (47)$$

Substituting h_0 in (25), we obtain the general solutions of (43):

$$\begin{aligned} a_1 &= \left[c_1 \int (-K_{eo}) \exp\left\{ \int 2K_{eo} \tan\left(\int K_{eo} dz\right) dz \right\} \cdot dz + c_2 \right] \\ &\quad \cdot \exp \left[- \int \{K_{ee} - K_{eo} \tan(\int K_{eo} dz)\} \cdot dz \right] \\ &= \left[-c_1 \int K_{eo} \sec^2\left(\int K_{eo} dz\right) dz + c_2 \right] \cdot \cos\left(\int K_{eo} dz\right) \cdot \exp\left(-\int K_{ee} dz\right) \\ &= \left[-c_1 \tan\left(\int K_{eo} dz\right) + c_2 \right] \cdot \cos\left(\int K_{eo} dz\right) \cdot \exp\left(-\int K_{ee} dz\right) \\ &= \left[-c_1 \sin\left(\int K_{eo} dz\right) + c_2 \cos\left(\int K_{eo} dz\right) \right] \cdot \exp\left(-\int K_{ee} dz\right) \\ &= \frac{d_1}{\sqrt{2}} e^{-\int (K_{ee} + jK_{eo}) dz} + \frac{d_2}{\sqrt{2}} e^{-\int (K_{ee} - jK_{eo}) dz} \end{aligned} \quad (48-a)$$

$$\begin{aligned}
a_2 &= \left[-c_1 \tan^2\left(\int K_{eo} dz\right) + c_2 \sec^2\left(\int K_{eo} dz\right) \right. \\
&\quad \left. + c_2 \tan\left(\int K_{eo} dz\right) \right] \cdot \cos\left(\int K_{eo} dz\right) \cdot \exp\left(-\int K_{ee} dz\right) \\
&= \left[c_1 \cos\left(\int K_{eo} dz\right) + c_2 \sin\left(\int K_{eo} dz\right) \right] \cdot \exp\left(-\int K_{ee} dz\right) \\
&= j \left[\frac{d_1}{\sqrt{2}} e^{-\int (K_{ee} + jK_{eo}) dz} - \frac{d_2}{\sqrt{2}} e^{-\int (K_{ee} - jK_{eo}) dz} \right] \quad (48-b)
\end{aligned}$$

where d_1 and d_2 are arbitrary constants. These expressions agree with the solutions given in Reference 3.

V. Case (i)

Here we consider the situation in which

$$A(z) = D(z) \quad (49-a)$$

and

$$B(z) = C(z). \quad (49-b)$$

For this special case, the generalized Riccati equation (20-b) becomes

$$h' + B(h^2 - 1) = 0 \quad (50)$$

A particular solution of (50) is

$$h_0 = \coth \int B dz \quad (51)$$

The solutions which follow are then

$$\begin{aligned}
a_1 &= \left[c_1 \int B \exp\left\{ \int -2B \coth\left(\int B dz\right) dz \right\} \cdot dz + c_2 \right] \\
&\quad \cdot \exp \int \{A + B \coth(\int B dz)\} \cdot dz \\
&= \left[c_1 \int B \operatorname{csch}^2\left(\int B dz\right) dz + c_2 \right] \cdot (\sinh \int B dz) \cdot \exp\left(\int A dz\right)
\end{aligned}$$

$$\begin{aligned}
&= (-c_1 \cosh \int B dz + c_2 \sinh \int B dz) \exp(\int A dz) \\
&= d_1 e^{\int (A+B) dz} + d_2 e^{\int (A-B) dz}
\end{aligned} \tag{52-a}$$

$$\begin{aligned}
a_2 &= \{c_1 \coth^2(\int B dz) + c_1 \operatorname{sech}^2(\int B dz) + c_2 \coth(\int B dz)\} \\
&\quad \cdot (\sinh \int B dz) \cdot \exp(\int A dz) \\
&= -d_2 e^{\int (A+B) dz} + d_2 e^{\int (A-B) dz}
\end{aligned} \tag{52-b}$$

If $B(z)$ and $C(z)$ are purely imaginary, then $B(z) = -C^*(z)$. This situation represents a passive coupling mechanism in the coupled-mode of propagation.

VI. Case (ii)

Case (ii) deals with the situation in which

$$A(z) = j [\beta(z) + \mu_1 \eta(z)] \tag{53-a}$$

$$D(z) = j [\beta(z) - \mu_2 \eta(z)] \tag{53-b}$$

And

$$B(z) = C(z) = j\mu_3 \eta(z) \tag{53-c}$$

where $\beta(z)$ and $\eta(z)$ are any arbitrary function of z and μ_1, μ_2, μ_3 are arbitrary constants. Substitution of (53) in (20-b) yields

$$h' + j(\mu_1 + \mu_2) \eta(z) h + j\mu_3 \eta(z) h^2 = j\mu_3 \eta(z) \tag{54}$$

A particular solution of (54) is

$$h_0 = j \sqrt{1 + \left(\frac{\nu}{2}\right)^2} \tan\left(\mu \sqrt{1 + \left(\frac{\nu}{2}\right)^2} \int \eta(z) dz\right) - \frac{\nu}{2} \tag{55}$$

The results obtained by substituting of (55) into (25) are

$$a_1 = \left[c_1 \cos \lambda \int \eta(z) dz + c_2 \left\{ j \sin \lambda \int \eta(z) dz + \frac{\nu}{2} \cos \lambda \int \eta(z) dz \right\} \right] \cdot \exp j \int \left\{ \beta(z) + \frac{\mu_1 - \mu_2}{2} \eta(z) \right\} \cdot dz \quad (56-a)$$

$$a_2 = \left[c_1 j \left\{ \sin \lambda \int \eta(z) dz - \frac{\nu}{2} \cos \lambda \int \eta(z) dz \right\} + c_2 \cos \lambda \int \eta(z) dz \right] \cdot \exp j \int \left\{ \beta(z) + \frac{\mu_1 - \mu_2}{2} \beta(z) \right\} \cdot dz \quad (56-b)$$

where

$$\lambda = \sqrt{\mu_3^2 + \left(\frac{\mu_1 + \mu_2}{2} \right)^2} \quad (57)$$

When μ_3 is real, we have the condition $B(z) = -C^*(z)$. This is the passive coupling case in which the group velocities of the waves are in the same direction.

VII. Case (iii)

Here we consider A, B, C, and D to be related in the following manner:

$$A(z) = j \left[\beta(z) + \mu_1 \eta(z) \right] \quad (58-a)$$

$$D(z) = j \left[\beta(z) - \mu_2 \eta(z) \right] \quad (58-b)$$

$$C(z) = -B(z) = j\mu_3 \eta(z) \quad (58-c)$$

For this situation, the particular solution of (20-b) can be found to be

$$h_0 = -j \sqrt{\left(\frac{\nu}{2}\right)^2 - 1} \tanh \mu_3 \sqrt{\left(\frac{\nu}{2}\right)^2 - 1} \int \eta(z) dz + \frac{\nu}{2} \quad (59)$$

The general solutions for a_1 and a_2 are then

$$a_1 = \left[c_1 \cosh \kappa \int \eta(z) dz + c_2 \left\{ j \sinh \kappa \int \eta(z) dz + \frac{\nu}{2} \cosh \kappa \int \eta(z) dz \right\} \cdot \exp j(\mu_1 + \mu_2) \int \eta(z) dz \right] \cdot \exp j \int \left\{ \beta(z) + \frac{\mu_1 - \mu_2}{2} \eta(z) \right\} dz \quad (60-a)$$

$$a_2 = \left[c_1 \left\{ j \sinh \kappa \int \eta(z) dz - \frac{\nu}{2} \cosh \kappa \int \eta(z) dz \right\} + c_2 \left\{ \cosh \kappa \int \eta(z) dz \right\} \cdot \exp j(\mu_1 + \mu_2) \int \eta(z) dz \right] \cdot \exp j \int \left\{ \beta(z) + \frac{\mu_1 - \mu_2}{2} \eta(z) \right\} dz \quad (60-b)$$

where

$$\kappa = \sqrt{\left(\frac{\mu_1 + \mu_2}{2}\right)^2 - \mu_3^2} \quad (61-a)$$

$$\nu = \frac{\mu_1 + \mu_2}{\mu_3} \quad (61-b)$$

If $C(z)$ and $B(z)$ are purely imaginary, $C(z) = B^*(z)$. This is the active coupling case in which the group velocities for the two waves are in opposite directions.

VIII. Case (iv)

In this case we consider the relations

$$A(z) \approx D(z) \quad (62-a)$$

$$B(z) = \pm C(z) \quad (62-b)$$

The approximation (62-a) is a fundamental assumption commonly made when treating a pair of weakly coupled modes. Under this assumption, we will neglect the second term, which is small compared with others, in the generalized Riccati equation.

Thus (20-b) becomes

$$h' + B(z) h^2 = \pm B(z) \quad (63)$$

$$h_0 = \tan \int B dz \quad (64)$$

The expression (64) is a particular solution of (63) taking the positive sign, and

$$h_0 = \coth \int B dz \quad (65)$$

is a particular solution of (63) corresponding to the negative sign. On substitution of these particular solutions in (22), we obtain approximate solutions for a_1 and a_2 .

With the upper sign in (62-b), we have

$$a_1 = \left\{ -c_1 \left[\int \{ B \exp \int (D-A) dz \} \cdot dz \right] \cosh \int B dz + c_2 \sinh \int B dz \right\} \cdot \exp \int A dz \quad (66-a)$$

$$a_2 = \left\{ c_1 \left[\int \{ B \exp \int (D-A) dz \} \cdot dz \right] \coth^2 \left(\int B dz \right) + \exp \int (A-D) dz \cdot \operatorname{sech}^2 \left(\int B dz \right) \right\} \cdot \sinh \int B dz + c_2 \cosh \int B dz \cdot \exp \int A dz \quad (66-b)$$

With the lower sign in (62-b), we have

$$a_1 = \left\{ c_1 \left[\int \{ B \exp \int (D-A) dz \} \cdot dz \right] \sin \int B dz + c_2 \cos \int B dz \right\} \cdot \exp \int A dz \quad (67-a)$$

$$a_2 = \left\{ c_1 \left[- \int \{ B \exp \int (D-A) dz \} \cdot dz \right] \tan^2 \int B dz + \exp \int (A-D) dz \cdot \sec^2 \int B dz \right\} \cdot \cos \int B dz - c_2 \sin \int B dz \cdot \exp \int A dz \quad (67-b)$$

IX. Concluding Remarks

A systematic method has been introduced to obtain solutions of the two dimensional coupled-mode equations with varying coefficients. Some special cases for which solutions are already known have been given to illustrate the application of our method. The solutions of sections V-VIII may be applied to physical problems: e.g. directional couplers, microwave antennas, etc.

The coupled-mode theory of propagation has been applied by R. A. Sigelmann and D. K. Reynolds in their study of a traveling wave antenna with broad bandwidth.⁸ Because of the systematic method of solving the coupled-mode equations developed here, it may be possible to find solutions for the log-periodic structure by the coupled mode approach.

It is also anticipated that the coupled mode theory as applied to propagation problems will have further application in the design of broadband directional couplers in which the coupling mechanism is of the log-periodic type.

Appendix A

Euler's Theorem Concerning the Generalized Riccati Equation:

It has been shown by Euler that, if a particular solution of the generalized Riccati equation is known, the general solution can be obtained by two quadratures.

To prove the result, let y_0 be a particular solution of

$$\frac{dy}{dz} + Py + Qy^2 = R \quad (A-1)$$

and write $y = y_0 + \frac{1}{v}$. The equation in v is

$$\frac{dv}{dz} - (P + 2Qy_0)v + Q = 0 \quad (A-2)$$

of which the solution is

$$v \exp\left\{-\int(P + 2Qy_0)dz\right\} - \int Q \exp\left\{-\int(P + 2Qy_0)dz\right\} \cdot dz = 0 \quad (A-3)$$

and, since $b = \frac{1}{(y-y_0)}$, the truth of the theorem is manifest.

Appendix B

Particular Solutions of the Generalized Riccati Equation

Sometimes one or more particular solutions can be found by inspection or by a lucky guess. More formal procedures apply in certain cases.

I. The form of the generalized Riccati equation is

$$y' = f(x) + g(x)y + h(x)y^2 \quad (B-1)$$

Let $y = u(x)/h(x)$ and (B-1) is converted into

$$u' = F(x) + G(x)u + u^2 \quad (B-2)$$

where $F(x) = f(x)h(x)$; $G(x) = g(x) + \frac{h'}{h}$. If both $F(x)$ and $G(x)$ are polynomials. See Sections (a), (b), (c) in turn. If $f(x)$, $g(x)$, and $h(x)$ are polynomials, it will not necessarily be true that $G(x)$ is a polynomial, hence the methods of those sections may not apply; see, however, Section (d) for a possible procedure.

(a) The equation becomes

$$u'(x) = F(x) + u^2 \quad (B-3)$$

and $F(x)$ is a polynomial with $G(x) = 0$. There are two possibilities.

- (i) The degree of $F(x)$ is odd. There is no polynomial solution of (B-3); hence, none of (B-1).
- (ii) The degree of $F(x)$ is even. Two possible polynomial solutions may exist. To find them, first note that if $P(x)$ were a polynomial of even degree $2n$, then $\sqrt{P(x)}$ could be expanded in series of the form.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots$$

Perform such an expansion on $\sqrt{-F(x)}$ but stop the calculation with the constant term. Call this result $X(x)$; it is a polynomial of degree n , if $F(x)$ is of degree $2n$. The coefficients in the

polynomial could be found by a simple modification of the square-root extraction method of elementary algebra, by the undetermined coefficients, or by expansion in a Maclaurin series.

If the differential equation has special polynomial solutions, they are given by

$$u = \pm X(x)$$

Test both of them, for both, one, or neither may satisfy the differential equation. If neither is a solution of (B-3), there are no polynomial solutions of (B-3) and hence none of (B-1).

- (b) $G(x) \neq 0$. Let $u = w(x) - \frac{G(x)}{2}$ and (B-2) becomes $w^2(x) = H(x) + w$ where $4H(x) + G^2(x) = 4F(x) + 2G'(x)$. The equation in $w(x)$ is similar to (B-3); hence, with slight modifications the procedure is very much like that in (a). Calculate

$$Q(x) = G^2 - 4F - 2G'$$

There are two cases.

- (i) The degree of $Q(x)$ is odd. There is no polynomial solution of either (b-2) or (B-1).
- (ii) The degree of $Q(x)$ is even. Expand $\sqrt{Q(x)}$ as in (a) again, stopping with constant term and call the resulting polynomial $X(x)$.

There are two possible polynomial solutions of (B-2)

$$u = - \frac{(G \pm X)}{2}$$

Test both, for neither may satisfy the differential equation, see (c).

- (c) $Q(x)$ is a constant. This is the necessary and sufficient condition that both solutions of (b) satisfy the differential equation. There are two cases.

(i) $X(x) = k \neq 0$. Introduce a new variable with the relation

$$u = -\frac{(G+k)}{2} + \frac{1}{v}$$

so that (B-2) becomes

$$v'(x) = 1 + kv$$

It is separable with solution $v = ce^{kx} + \frac{1}{k}$ and the corresponding solution of (B-2) is

$$u = -\frac{(G+k)}{2} + (ce^{kx} + \frac{1}{k})^{-1}$$

Two special polynomial solutions of it result with $c = 0, \infty$.

(ii) $X(x) = k = 0$. The equation in $v(x)$ becomes $v'(x) + 1 = 0$, which is separable, with solution $v + x = c$. The solution of (B-2) is

$$u = -\frac{G}{2} - \frac{1}{c-x}$$

(d) Polynomial coefficients:

Suppose that (B-1) has the form

$$\phi(x)y' = f(x) + g(x)y + h(x)y^2 \quad (B-4)$$

where all of the coefficients are polynomials. Assume it to have a special polynomial solution $y_1 = R(x)$ and use a new variable $y = u + y_1$ so that it becomes

$$\phi(x)u' = F(x)u + h(x)u^2$$

where $F(x) = g(x) + 2h(x)R(x)$. If there is a polynomial solution of (B-4), having the form $u = (x-a)^m$, $m \geq 1$ then $\phi(x)$ will also contain a factor $(x-a)$. This property restricts the possible types of polynomial solutions. Unlike the previous cases, where only two possible polynomials may exist, the more general solution can have a larger number of such solutions. See (e) for further discussion of this case.

(e) Several polynomial solutions

Given (B-4), any specified number of polynomial solutions can be constructed. The general solution can be taken as

$$y = \frac{cf_1(x) + f_2(x)}{cf_3(x) + f_4(x)}.$$

If $f_1f_4 - f_2f_3 \neq 0$, then (B-1) can be written in the form of (B-4) with $\phi(x) = f_1f_4 - f_2f_3$; $f(x) = f_1f_2' - f_1'f_2$; $g(x) = f_2f_3' - f_2'f_3 + f_4f_1' - f_1f_4'$; $h(x) = f_3f_4' - f_4f_3'$.

Now suppose that the $f_i(x)$ are polynomials. Select $f_1(x) = 0$; choose any polynomials desired as $f_3(x)$ and $f_4(x)$. Assign n special values $c_1, c_2, \dots, c_n$ to the arbitrary constant and require that

$$f_2(x) = (c_1f_3 + f_4)(c_2f_3 + f_4) \cdots (c_nf_3 + f_4)$$

The differential equation (B-4) will have $(n+1)$ polynomial solutions,

$$y = 0 \quad \text{and} \quad y_i = \frac{f_2}{c_1f_3 + f_4} \quad i = 1, 2, \dots, n$$

The special solution $y = 0$ can be avoided and $(n+1)$ special solutions retained if the variables are transformed with the relation $u(x) = g(x) + P(x)$ where $P(x)$ is any polynomial.

The considerations of this section could be very helpful in constructing an equation with a predetermined number of polynomial solutions.

II. The following properties of the generalized Riccati equation sometimes apply in attempting to solve it.

(a) Removal of the linear term

The result, which is similar to (B-3) can be achieved in three different ways. It might then be treated as in I, but see also (b) for a special case.

- (i) Let $y = u(x)e^{\phi(x)}$, $\phi(x) = \int g(x)dx$ and (B-1) becomes

$$u'(x) = F(x) + G(x)u^2 \quad (B-5)$$

where $F(x) = f(x)e^{-\phi(x)}$, $G(x) = h(x)e^{\phi(x)}$.

- (ii) Let $y = u - v(x)$, $v(x) = \frac{g(x)}{2h(x)}$. If g, h are differentiable and $h(x) \neq 0$, the result is again like (B-5) but now with

$$F(x) = f + v' - \frac{g^2}{4h}; \quad G(x) = h(x).$$

- (iii) Let $y = u(z) e^{\phi(x)}$; $\phi(x) = \int g(x)dx$; $z = - \int h e^{\phi} dx$. In this case (B-1) becomes

$$u'(z) = F(z) - u^2(z) \quad (B-6)$$

with $F(z)h(z) = - f(z) e^{-2\phi}$

- (b) Relation between the coefficients when certain relations exist between the coefficient of (B-1) its solution may be easy to obtain.

- (i) Look for two constants a, b with $|a| + |b| > 0$ so that

$$a^2f + abg + b^2h = 0$$

If $a = 0$, then $f(x) = 0$ and the equation is linear. If $a \neq 0$, then $y_1 = b/a$ is a particular solution of (B-1). A simple case arises if $f + g + h = 0$, for then $a = b = 1$ and $y_1 = 1$.

- (ii) Use (a) to remove the linear term in (B-1). Then if $F(x)$ is proportional to $G(x)$ in (B-5), the result is separable. In the original variable and functions the requirement is $f(x) = A^2 h(x) \int \exp(2 g(x)dx)$ where A is a constant for proportionality. The solution of (B-1) is then

$$y = \sqrt{\frac{f}{h}} \tan\left(\int \sqrt{fh} dx + C\right); \quad fh > 0$$

if $fh < 0$, replace $\tan$ by $\tanh$ and insert a minus sign under both radicals.

(iii) Assume that a special solution of (B-1) exists so that

$$2hy_1 = X(x) - g(x)$$

where $X(x)$ is determined from the relation

$$f(x) = hy_1^2 - X(x)y_1 + y_1'$$

The last equation imposes a severe restriction on the form of $f(x)$ but if it is satisfied, y_1 can be found. A few cases that might be tested are

$$X = 0, \quad 4f = g^2/h - 2(g/h)';$$

$$X = -h'/h, \quad 4f = 2\left(\frac{1}{h}\right)'' - 2(g/h)' - h(1/h)'^2 + g^2/h;$$

$$X = g - 2\sqrt{fh}, \quad 2g = 4\sqrt{fh} + f'/f - f'/h.$$

III. If all the tests in I and II fail, transform the equation into one of second order. Change the dependent variable by the transformation

$$yh(x) u(x) + u'(x) = 0; \quad \ln U(x) + \int yh(x)dx = 0$$

The result is

$$u''(x) + P(x)u' + Q(x)u = 0$$

where $p(x) = -(g + h'/h)$, $Q(x) = f(x)h(x)$. It is linear and of second order. Such equations have been studied extensively and this may be the most suitable procedure unless the given first-order equation has some special property.

IV. There are fourteen transforms listed below in pairs for solving a certain class of Riccati's equation.¹⁶ The intermediate steps dealing with transforms and solutions of these equations are avoided. However, it is appropriate to cite the first case of the transforms listed below to explain the intermediate steps omitted. Take transform 1,

$$y = z/(Pz + Qz').$$

(B-7)

Substitute this into a generalized Riccati's equation,

$$y' = \alpha(x)y + \beta(x)y^2 + \gamma(x) \quad (\text{B-8})$$

where α , β and γ are undetermined variable coefficients and only α can become zero, typical of Riccati's class of equations.

Substitute (B-8) into (B-7)

$$\begin{aligned} Qz'^2 - P'z^2 - (Q'z' + Qz'')z + (Pz + Qz')\alpha z + \beta z^2 \\ = (Pz + Qz' + 2PQzz') \end{aligned} \quad (\text{B-9})$$

There are four places where z^2 appears on the other hand, there are two places where z'^2 appears. The key idea is to eliminate terms z^2 and z'^2 in (B-9), thus degenerate (B-9) into a first-order linear differential equation. Because z'^2 appears in two places, the unique choice for $\gamma(x)$ is made first,

$$\gamma(x) = \frac{1}{Q(x)} \quad (\text{B-10})$$

The coefficients for z^2 are collected to specify $\alpha(x)$ and $\beta(x)$,

$$-P' + \alpha P + \beta - \frac{P^2}{Q} = 0. \quad (\text{B-11})$$

Note that there are three ways of specifying while retaining (B-8) in Riccati's class of equations. If

$$\alpha = 0 \quad (\text{B-12})$$

then

$$\beta = P' + \frac{P^2}{Q}. \quad (\text{B-13})$$

If

$$\alpha = \frac{P}{Q} \quad (\text{B-14})$$

then

$$\beta' = P'. \quad (\text{B-15})$$

If

$$\alpha = (\log_e P)' \quad (\text{B-16})$$

then

$$\beta = \frac{p^2}{Q} \quad (B-17)$$

Judging from these developments, it is quite obvious that a transform of type

$$y = \frac{z'}{Pz + Qz'} \quad (B-18)$$

does not solve a Riccati's equation. If (B-18) is substituted in (B-8), there is only one place where z^2 appears. This forces P to zero to eliminate the term of z^2 . If P is zero in (B-18), y is not related to another variable z. Following the above steps, the transforms listed below are obtained and may be applied for exact solutions for a certain class of Riccati's non-linear differential equations. Only fourteen transforms in pairs are listed below but it is possible to find many more transforms in pairs.

Transform 1:

$$y = \frac{z}{Pz + Qz'} \quad (B-19)$$

solves

$$y' + (P' + \frac{P^2}{Q})y = \frac{1}{Q} \quad (B-20)$$

$$y' + \frac{Py}{Q} + P'y^2 = \frac{1}{Q} \quad (B-21)$$

$$y' + (\log_e P)'y + \frac{P^2 y^2}{Q} = \frac{1}{Q} \quad (B-22)$$

Transform 2:

$$y = \frac{(Pz + Qz')}{z} \quad (B-23)$$

solves

$$y' + \frac{y^2}{Q} = P' + \frac{P^2}{Q} \quad (B-24)$$

$$y' - \frac{Py}{Q} + \frac{y^2}{Q} = P' \quad (B-25)$$

$$y' - (\log_e P)'y + \frac{y^2}{Q} = \frac{P^2}{Q} \quad (B-26)$$

Transform 3:

$$y = \frac{z}{z + PQz'} \quad (B-27)$$

solves

$$y' - \left[\beta(x) - \frac{1}{PQ}\right] y + \beta(x)y^2 = \frac{1}{PQ} \quad (B-28)$$

where $\beta(x)$ is any arbitrary function of x .

Transform 4:

$$y = \frac{(z + PQz')}{z} \quad (B-29)$$

solves

$$y' + \left[\gamma(x) - \frac{1}{PQ}\right] y + \frac{y^2}{PQ} = \gamma(x) \quad (B-30)$$

where $\gamma(x)$ is any arbitrary function of x .

Transform 5:

$$y = \frac{PQz}{(Pz + Qz')} \quad (B-31)$$

solves

$$y' + \frac{(P - Q')y^2}{Q^2} = P \quad (B-32)$$

$$y' + \frac{Py}{Q} + (Q^{-1})'y^2 = P \quad (B-33)$$

$$y' - (\log_e Q)'y + \frac{Py^2}{Q^2} = P \quad (B-34)$$

Transform 6:

$$y = \frac{(Pz + Qz')}{PQz} \quad (B-35)$$

solves

$$y' + Py^2 = \frac{(P-Q')}{Q^2} \quad (B-36)$$

$$y' - \frac{Py}{Q} + Py^2 = (Q^{-1})' \quad (B-37)$$

$$y' + (\log_e Q)'y + Py^2 = \frac{P}{Q^2} \quad (B-38)$$

Transform 7:

$$y = \frac{z}{(PQz + z')} \quad (B-39)$$

solves

$$y' + [(PQ)^2 + (PQ)']y^2 = 1 \quad (B-40)$$

$$y' + PQy + (PQ)'y^2 = 1 \quad (B-41)$$

$$y' + [\log_e PQ]'y + (PQ)^2y^2 = 1 \quad (B-42)$$

Transform 8:

$$y = \frac{(PQz + z')}{z} \quad (B-43)$$

solves

$$y' + y^2 = (PQ)^2 + (PQ)' \quad (B-44)$$

$$y' - PQy + y^2 = (PQ)' \quad (B-45)$$

$$y' - [\log_e PQ]' + y^2 = (PQ)^2 \quad (B-46)$$

Transform 9:

$$y = \frac{Pz}{(Pz + Qz')} \quad (B-47)$$

solves

$$y' - [\beta(x) - \frac{P}{Q}]y + \beta(x)y^2 = \frac{P}{Q} \quad (B-48)$$

where $\beta(x)$ is any arbitrary function of x .

Transform 10:

$$y = \frac{(Pz + Qz')}{Pz} \quad (B-49)$$

solves

$$y' + [\gamma(x) - \frac{P}{Q}]y + \frac{Py^2}{Q} = \gamma(x) \quad (B-50)$$

where $\gamma(x)$ is any arbitrary function of x .

Transform 11:

$$y = \frac{Qz}{(Pz + Qz')} \quad (B-51)$$

solves

$$y' + \left\{ \left[\log_e \left(\frac{P}{Q} \right) \right]' + \frac{P}{Q} - \frac{Q \beta(x)}{P} \right\} y + \beta(x) y^2 = 1 \quad (B-52)$$

where $\beta(x)$ is any arbitrary function of x .

Transform 12:

$$y = \frac{(Pz + Qz')}{Qz} \quad (B-53)$$

solves

$$y' - \left\{ \frac{P}{Q} + \left[\log_e \frac{P}{Q} \right]' - \frac{Q \gamma(x)}{P} \right\} y + y^2 = \gamma(x) \quad (B-54)$$

where $\gamma(x)$ is any arbitrary function of x .

Transform 13:

$$y = \frac{PQz}{(z + z')} \quad (B-55)$$

solves

$$y' + \left[\frac{1}{PQ} + \left(\frac{1}{PQ} \right)' \right] y^2 = PQ \quad (B-56)$$

$$y' + y + \left(\frac{1}{PQ} \right)' y^2 = PQ \quad (B-57)$$

$$y' - (\log_e PQ)' y + \frac{y^2}{PQ} = PQ \quad (B-58)$$

Transform 14:

$$y = \frac{(z + z')}{PQz} \quad (B-59)$$

solves

$$y' + PQy^2 = \left[\frac{1}{PQ} + \left(\frac{1}{PQ} \right)' \right] \quad (B-60)$$

$$y' - y + PQy^2 = \left(\frac{1}{PQ} \right)' \quad (B-61)$$

$$y' + (\log_e PQ)' y + PQy^2 = \frac{1}{PQ} \quad (B-62)$$

V. A Table of Solutions of Riccati's Equations.

Tabulated exact solution of $y'(x) + P(x)y(x) + Q(x)y(x) = R(x)$	
A particular solution	Required interrelationship
(i) $a e^{-\int P dx}$	$R = a^2 Q e^{-2\int P dx}$
(ii) $\frac{1}{\int Q dx + b}$	$R = \frac{P}{\int Q dx + b}$
(iii) $\int R dx + c$	$Q = -\frac{P}{\int R dx + c}$
(iv) $-\frac{P}{Q}$	$R = -\left(\frac{P}{Q}\right)'$
(v) $\frac{R}{P}$	$Q = \left(\frac{P}{R}\right)'$
(vi) $\sqrt{\frac{R}{Q}}$	$P = -\left(\ln \sqrt{\frac{R}{Q}}\right)'$
(vii) $\frac{e^{-\int (P-2Q) dx} - \int P e^{-\int (P-2Q) dx} dx - a}{e^{-\int (P-2Q) dx} + \int P e^{-\int (P-2Q) dx} dx + a}$	$R = Q - P$
(viii) $\frac{\int P e^{-\int (P+2Q) dx} dx + b - e^{-\int (P+2Q) dx}}{\int P e^{-\int (P+2Q) dx} dx + b + e^{-\int (P+2Q) dx}}$	$R = Q + P$
NOTE: a, b, c are arbitrary constants. These eight table entries represent a great number of specialized cases.	

References

1. W. H. Louisell, "Coupled Mode and Parametric Electronics", John Wiley & Sons, 1960.
2. C. W. Burnes, "Conservative Coupling Between Modes of Propagation - a Tabular Summary", Proceedings of the IEEE, Vol. 52, No. 1, Jan. 1964, pp 64-73.
3. H. C. Huang and D. Y. Fan, "Coupled-Mode Theory of Guided Wave Propagation through a Ferrite Rod with Variable Cross-Section", Scientia Sinica, Vol. XIV, No. 3, Mar., 1965, pp 379-395.
4. S. E. Miller, "Coupled Wave Theory and Waveguide Applications", The Bell System Technical Journal, Vol. XXXIII, No. 3, May, 1954.
5. W. H. Louisell, "Analysis of the Single Tapered Mode Coupler", The Bell System Technical Journal, Vol. XXXIV, No. 4, July, 1955.
6. J. S. Cook, "Tapered-Velocity Coupler", The Bell System Technical Journal Vol. XXXIV, No. 4, July, 1955.
7. H. C. Huang, "Generalized Theory of Coupled Local Normal Modes in Multi-Wave Guides", Scientia Sinica, Vol. IX, No. 1, Jan., 1960, pp 142-154.
8. R. A. Sigelmann and D. K. Reynolds, "Traveling Wave Antenna with Broad Bandwidth", T.R. No. 42, University of Washington, Seattle, June, 1962.
9. G. N. Watson, "Theory of Bessel Functions", (2nd Ed.) London: Cambridge University Press, 1944.
10. G. M. Murphy, "Ordinary Differential Equations and Their Solutions", D. Van Nostrand, 1960.
11. P. Hartman, "Ordinary Differential Equations", John Wiley and Sons, 1964.
12. Iwao Sugai, "A Table of Solutions of Riccati Equations", Proc. IRE (Correspondence), Vol. 50, No. 10, Oct., 1962, pp 2125-2126.
13. I. Sugai, "Riccati's Nonlinear Differential Equation", Am. Math. Monthly, Vol. 67, pp 134-139, Feb., 1960.
14. I. Sugai, "Exact Solutions for Ordinary Nonlinear Differential Equations", Electrical Communication, Vol. 37, No. 2, pp 47-55, 1961.
15. A. K. Kajagopal, "On the Generalized Riccati Equation", Am. Math. Monthly, Vol. 68, pp 777-779, Oct., 1961.
16. I. Sugai, "A Class of Solved Riccati's Equations", Electrical Communication, Vol. 37, No. 1, pp 56-60, 1961.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D		
<i>(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)</i>		
1. ORIGINATING ACTIVITY <i>(Corporate author)</i> Department of Electrical Engineering University of Washington Seattle, Washington 98105		2a. REPORT SECURITY CLASSIFICATION Unclassified
		2b. GROUP
3. REPORT TITLE On the General Solutions of Coupled-Mode Equations with Varying Coefficients		
4. DESCRIPTIVE NOTES <i>(Type of report and inclusive dates)</i> Scientific Report - Summary		
5. AUTHOR(S) <i>(Last name, first name, initial)</i> Chen, Yung-Sen Ishimaru, Akira, Dr. (P.I.)		
6. REPORT DATE February 1966	7a. TOTAL NO. OF PAGES 36	7b. NO. OF REFS 16
8a. CONTRACT OR GRANT NO. AF19(628)-2763	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 103	
b. PROJECT AND TASK NO. 5635, 02		
c. DOD ELEMENT 61445014	9b. OTHER REPORT NO(S) <i>(Any other numbers that may be assigned this report)</i>	
d. DOD SUBELEMENT 681305	AFCRL-66-163	
10. AVAILABILITY/LIMITATION NOTICES Distribution of this document is unlimited.		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Hq. AFCRL, OAR (CRD) United States Air Force L.G.Hanscom Field, Bedford, Mass.	
13. ABSTRACT In this report a systematic mathematical method is introduced for the solution of problems involving two coupled modes in a coupled system with varying parameters. These problems involve systems of linear differential equations with varying coefficients. By the use of a linear transformation of the dependent variables and a double diagonalization process, the coupled mode equations are reduced to two decoupled Riccati equations. The final form of the general solution is obtained in terms of four varying coupling coefficients and a transform parameter. To illustrate some applications of the method, solutions of two special cases which have been solved previously by other workers are obtained. The solutions for a number of special cases, in which the varying coefficients are specified or interrelated, are also obtained. Further possible applications are indicated.		

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Systematic mathematical method Coupled modes Varying parameters Linear transformation Double diagonalization process Decoupled Riccati equations Applications Special cases						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (corporate author) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (either by the originator or by the sponsor), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

- (1) "Qualified requesters may obtain copies of this report from DDC."
- (2) "Foreign announcement and dissemination of this report by DDC is not authorized."
- (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."
- (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."
- (5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (paying for) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

Unclassified

Security Classification