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STATISTICAL ANALYSIS OF SPACECRAFT REPLENISHMENT

G. E. Modesit

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The RAND Corporation
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PREFACE

This Memorandum discusses the number of launches required to establish and maintain a spacecraft system. The mean and dispersion in number of launches are derived for a system consisting of an arbitrary number of exponentially decaying spacecraft and with a constant probability for successful replenishment launches. The report shows the dependence of these results on the spacecraft mean lifetime, the launch success probability, and the number of spacecraft per launch. The results are useful in estimating the probable costs of various spacecraft systems and should be of assistance in comparing such systems.

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SUMMARY

The mean m_R and dispersion σ_R^2 in number of launches required to maintain with a constant launch capability a system of N_o exponentially decaying spacecraft for a time t large compared with the mean system failure time are

$$m_R = \frac{1}{p}\left(\frac{t}{m_1} + \frac{\sigma_1^2}{2m_1^2} + \frac{1}{2} - \frac{m_0}{m_1}\right)$$

and

$$\sigma_R^2 = \frac{2t}{p^2 m_1}\left(1 + \frac{\sigma_1^2}{m_1^2} - \frac{m_0}{m_1} - \frac{p}{2}\right)$$

where p is the probability for a successful launch and

$$m_j = \lambda \sum_{k=N_o}^{n_j} \frac{1}{k} \qquad j = 0,1$$

and

$$\sigma_j^2 = \lambda^2 \sum_{k=N_o}^{n_j} \frac{1}{k^2} \qquad j = 0,1$$

with λ the mean lifetime of the spacecraft, and with n_0 and n_1 the number of spacecraft on orbit immediately following initial establishment and first reestablishment, respectively.

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LIST OF SYMBOLS

$A^{(k)}(s)$	pgf for $a_i^{(k)}$, $i = 0,1,2,\ldots$
$a_i^{(k)}$	probability that i launches are required to replenish system after k^{th} failure
$b_j(t)$	probability per unit time that j^{th} system failure occurs at time t
$b_j^*(\omega)$	L.T. of $b_j(t)$
$f_j(t)$	probability that j system failures occur in time t
$f_j^*(\omega)$	L.T. of $f_j(t)$
$g_k(t)$	probability per unit time that system fails at time t during interval τ_k
$g_k^*(\omega)$	L.T. of $g_k(t)$
$H_j(s)$	pgf for $h_{n,j}$, $n = 0,1,2,\ldots$
$h_{n,j}$	probability that n replenishment launches are required if j system failures occur
L.T.	Laplace transform
L_o	minimum number of successful launches required to establish system
MTBF	mean time before failure
m	m_0 or m_1
m_a	mean number of launches required to reestablish a system after a failure
m_E	mean number of establishment launches required
m_f	mean number of system failures
$m_N(t)$	mean of total number of launches required in time t
$m_R(t)$	mean number of replenishment launches required in time t
m_0	MTBF of system after establishment
m_1	MTBF of system after first failure

N_L	number of S/C per launch
N_o	number of S/C required on orbit
n_0	number of S/C on orbit immediately following establishment
n_1	number of S/C on orbit immediately following first replenishment
$P_n(t)$	probability that n replenishment launches are required in time t
$P^*(\omega,s)$	L.T. of pgf for $P_n(t)$
p	probability of successful launch
pdf	probability density function
pgf	probability generating function
q	1 - p
S/C	spacecraft
S_j	number of replenishment launches required to maintain a system through j failures
T	maintenance period
t_j	time of j^{th} failure
λ	MTBF of S/C
ν_k	number of launches required to replenish system after k^{th} failure
σ^2	$\sigma_0{}^2$ or $\sigma_1{}^2$
σ_a^2	dispersion in number of launches required to reestablish a system after a failure
σ_E^2	dispersion in number of establishment launches
σ_f^2	dispersion in number of system failures
$\sigma_N^2(t)$	dispersion in total number of launches required in time t
$\sigma_R^2(t)$	dispersion in number of replenishment launches in time t
σ_0^2	dispersion in system life after establishment
σ_1^2	dispersion in system life after first reestablishment

τ_k	time interval between k^{th} and $(k+1)^{st}$ system failure
$\varphi(t)$	probability per unit time that a S/C in operation up to time t, fails at t
$\Psi_j(t)$	probability of continued successful system operation to time t following j^{th} reestablishment
$\psi(t)$	probability of continued successful operation of single S/C

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I. INTRODUCTION

In the course of some recent studies of costs of various spacecraft (S/C) systems for detection of nuclear explosions, a number of useful results applicable to the statistical analysis of S/C replenishment have been derived. In general it is desired to determine the number of launches required to maintain a system of, say, N_o S/C for a time T with a launch capability of N_L S/C per launch. The model assumes system failure when the number of operating S/C falls below N_o and assumes that replenishment launches can be effected immediately after failure. It is usually assumed that the probability for failure of a S/C known to be in successful operation is independent of the S/C age; hence, the asic parameters of the analysis are λ, the S/C expected lifetime (or MTBF), and p, the probability of a successful launch.* The most general solution is the probability P_n that n launches are required to maintain the system for the time T, although cost estimates are usually based on the mean and variance of the distribution. In order to consider possible extensions in the model, the treatment below is occasionally somewhat more general than that required for the simple model.

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*In typical detection systems, N_o may lie in the range 1 to 10, $N_L \sim 1$ to 5, $T \sim 5$ yr, $p \sim 0.5$ to 0.9, and $\lambda \sim 1/2$ to 3 yr.

II. MAINTENANCE COMPOUND DISTRIBUTION

It is convenient to write the probability P_n as a compound probability distribution

$$P_n(t) = \sum_{j=0}^{\infty} f_j(t)\, h_{n,j} \qquad (1)$$

where $f_j(t)$ is the probability that j system failures occur during the time t, and $h_{n,j}$ is the probability that n launches are required when j failures occur. This approach is useful since the events determined by the probability distribution f are independent of those determined by the probability distribution h.

III. DISTRIBUTION OF NUMBER OF SYSTEM FAILURES (f_j)

The distribution $f_j(t)$ for the number of system failures in time t will be derived in terms of the probability $g(t)dt$ that the system fails in a small interval dt at time t. That is, $g(t)$ is assumed known,* and $f_j(t)$ is to be determined.

It is convenient to introduce the probability $b_j(t)dt$ giving the probability that the j^{th} failure occurs in the interval dt at time t. Then, since the change $df_j(t)$ in f_j at time t during the interval dt is the difference between the probability for entering j (the probability that the j^{th} system failure occurs) during the interval and the probability for leaving j (the probability that the $(j + 1)^{st}$ failure occurs) during the interval, it may be expressed as

$$df_j(t) = b_j(t)dt - b_{j+1}(t)dt$$

or

$$\frac{df_j(t)}{dt} = b_j(t) - b_{j+1}(t) \tag{2}$$

With the notation that the Laplace transform (L.T.) of a function of time such as $g(t)$ is represented by $g^*(\omega)$, that is, if

$$g^*(\omega) = \int_0^\infty dt\ e^{-\omega t}\ g(t) \tag{3}$$

the L.T. of Eq. (2) is

$$\omega f_j^*(\omega) = b_j^*(\omega) - b_{j+1}^*(\omega) \tag{4}$$

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*In Section VII, the probability $g(t)$ for a simple system is derived from the failure probabilities for the individual S/C.

It is useful to observe that the L.T. of a probability density function (pdf) is $< e^{-\omega t} >$, the expectation value of $e^{-\omega t}$ over the distribution.*

The system is assumed in operation at time $t = 0$ with the pdf for failure given by $g_0(t)$. The interval to the first failure is represented by the random variable τ_0. At time $t = \tau_0$ the system suffers its first failure and is immediately reestablished, after which the failure pdf is $g_1(t)$. The process continues, with intervals between failures represented by the independent random variables $\tau_0, \tau_1, \ldots$, with failures at times $t_1, t_2, \ldots$ given by

$$t_j = \sum_{k=0}^{j-1} \tau_k$$

and with $g_k(t)$ giving the failure pdf appropriate to the interval following the k^{th} reestablishment. It is convenient to include the $j = 0$ term with the assumption $t_0 = 0$. Then, since

$$b_j^*(\omega) = \left\langle e^{-\omega t_j} \right\rangle$$

it follows that

$$b_0^*(\omega) = 1$$

and, for $j \geq 1$

$$b_j^*(\omega) = \left\langle e^{-\omega \sum_{k=0}^{j-1} \tau_k} \right\rangle$$

$$= \left\langle \prod_{k=0}^{j-1} e^{-\omega \tau_k} \right\rangle$$

$$= \prod_{k=0}^{j-1} \left\langle e^{-\omega \tau_k} \right\rangle$$

*The functions $g(t)$ and $b_j(t)$ give probabilities per unit time and are density functions, whereas $f_j(t)$ and $P_n(t)$ give probabilities in j and n and are not density functions.

where the last step follows from the independence of the intervals τ_k. Thus

$$b_j^*(\omega) = \begin{cases} 1 & j = 0 \\ \prod_{k=0}^{j-1} g_k^*(\omega) & j \geq 1 \end{cases}$$

and, from Eq. (4)

$$f_j^*(\omega) = \left[\frac{1 - g_j^*(\omega)}{\omega g_j^*(\omega)}\right] \prod_{k=0}^{j} g_k^*(\omega) \tag{5}$$

Equation (5) gives the L.T. of the failure distribution f_j in terms of the failure pdf's $g_k(t)$.

IV. DISTRIBUTION OF NUMBER OF LAUNCHES FOR FIXED NUMBER OF FAILURES ($h_{n,j}$)

If the system fails exactly j times, and if the random variable ν_k represents the number of launches required to reestablish the system after the k^{th} failure, then $h_{n,j}$ is the probability that $S_j = n$, where S_j is given by the sum

$$S_j = \sum_{k=1}^{j} \nu_k \tag{6}$$

Consider first the sum of two non-negative integral random variables ν_1 and ν_2. If $a_i^{(1)}$ and $a_j^{(2)}$ represent respectively the probabilities that $\nu_1 = i$ and $\nu_2 = j$, then the probability that the sum $(\nu_1 + \nu_2)$ equals n is given by

$$h_n = \sum_{k=0}^{n} a_k^{(1)} \; a_{n-k}^{(2)} \tag{7}$$

Let $A^{(k)}(s)$ be the probability generating function (pgf) for $a_i^{(k)}$, defined by

$$A^{(k)}(s) = \sum_{i=0}^{\infty} a_i^{(k)} s^i \tag{8}$$

Then, from Eq. (7), the generating function H(s) for h_n is given by

$$H(s) = A^{(1)}(s) \; A^{(2)}(s)$$

The generalization* of this result to higher-order sums is obvious: if $a_i^{(k)}$ is the probability that i launches are required to reestablish

*See Ref. 1, p. 251.

the system after the k^{th} failure, the generating function for the distribution in total number of launches required to maintain the system through j failures is given by the product

$$H_j(s) = \prod_{k=0}^{j} A^{(k)}(s) \qquad (9)$$

For completeness, the term for j = 0 is included; after the "zeroth" failure, zero launches are required to reestablish, that is, $a_i^{(0)} = 0$ for all i except i = 0, whereas $a_0^{(0)} = A^{(0)}(s) = H_0(s) = 1$.

V. GENERAL SOLUTION FOR DISTRIBUTION OF TOTAL NUMBER OF REPLENISHMENT LAUNCHES

In Section III, the L.T. of $f_j(t)$, the number of system failures in time t, was determined in terms of the L.T. of the system failure pdf's $g_k(t)$; and in Section IV, the pgf for the probabilities $h_{n,j}$, the distribution in number of launches for a fixed number of system failures, was determined in terms of the pgf for the probabilities $a_i^{(k)}$, the distribution in number of launches required to reestablish after a single system failure. From Eq. (1), the L.T. of the pgf for $P_n(t)$, the total number of replenishment launches required in time t, is given by

$$P^{*}(\omega,s) = \sum_{j=0}^{\infty} f_j^{*}(\omega)\, H_j(s)$$

Hence, from Eqs. (5) and (9)

$$P^{*}(\omega,s) = \sum_{j=0}^{\infty} \left[\frac{1 - g_j^{*}(\omega)}{\omega g_j^{*}(\omega)}\right] \prod_{k=0}^{j} g_k^{*}(\omega) \prod_{i=0}^{j} A^{(i)}(s) \tag{10}$$

Since the functions $g_j^{*}(\omega)$ and $A^{(k)}(s)$ are given in Eqs. (3) and (8), in terms of known functions $g_j(t)$ and $a_i^{(k)}$, respectively, Eq. (10) respresents the general solution for the distribution P_n.

VI. TOTAL REPLENISHMENT DISTRIBUTION FOR CONSTANT LAUNCH SUCCESS PROBABILITY

The model described in the Introduction assumes a constant probability for a successful launch. Hence, the replenishment launch number probabilities $a_i^{(k)}$ do not depend on k, the system failure number, and may be written as

$$a_i^{(k)} = a_i \qquad k \geq 1$$

where a_i, the probability that i launches are required to replenish the system after failure (that is, the probability that the i^{th} launch is the first successful launch following a system failure) is given by

$$a_i = \begin{cases} 0 & i = 0 \\ pq^{i-1} & i \geq 1 \end{cases} \tag{11}$$

and $q = 1 - p$. The pgf for the probabilities a_i is

$$A(s) = p \sum_{i=1}^{\infty} q^{i-1} s^i = \frac{ps}{1-qs} \tag{12}$$

Hence, from Eq. (9)

$$H_j(s) = \prod_{k=0}^{j} A^{(k)}(s) = [A(s)]^j$$

$$= \left[\frac{ps}{1-qs}\right]^j \tag{13}$$

Expansion of $H_j(s)$ in a power series in s gives

$$H_j(s) = (ps)^j \sum_{k=0}^{\infty} \frac{(j+k-1)!}{(j-1)!\ k!} (qs)^k$$

$$= \sum_{n=j}^{\infty} \left[\frac{(n-1)!}{(j-1)!\ (n-j)!} q^{n-j} p^j \right] s^n \tag{14}$$

Hence, the probability $h_{n,j}$ that n replenishment launches are required if j failures ($j \geq 1$) occur is, from Eq. (14)

$$h_{n,j} = \begin{cases} 0 & n < j \\ \frac{(n-1)!}{(j-1)!\ (n-j)!} q^{n-j} p^j & n \geq j \end{cases} \tag{15}$$

If j = 0 (no failures), no replenishment launches are required; therefore, $h_{0,0} = 1$ and $h_{n,0} = 0$ for $n \geq 1$. Substitution of this expression for $h_{n,j}$ into Eq. (1) gives

$$P_n(t) = \begin{cases} f_0(t) & n = 0 \\ \sum_{j=1}^{n} \frac{(n-1)!}{(j-1)!\ (n-j)!} q^{n-j} p^j f_j(t) & n \geq 1 \end{cases} \tag{16}$$

VII. AGELESS SPACECRAFT

Consider a system consisting of a single S/C placed in operation at time $t = 0$, and let $\psi(t)$ represent the probability that it remains in successful operation (is "alive") at age t. Assume that the S/C is observed to be alive at age t_0, i.e., $\psi(t) = 1$ for $0 \le t \le t_0$. For times beyond the last observation time t_0, the decrease $d\psi$ in $\psi(t)$ in an interval dt (the probability that the S/C fails in dt) is the product of the probability that the S/C is alive at age t and the conditional probability $\varphi(t)dt$ that the S/C, if alive at age t, fails in the interval dt

$$-d\psi = \psi(t)\ \varphi(t)dt$$

With the given boundary conditions, $\psi(t)$ becomes

$$\psi(t) = \begin{cases} 1 & 0 \le t \le t_0 \\ e^{-\int_{t_0}^{t} dt\ \varphi(t)} & t \ge t_0 \end{cases}$$

For convenience, and because of reasonable agreement with experience, it is usually assumed that the failure probability of an operating S/C is independent of its age; hence

$$\varphi(t) = \frac{1}{\lambda}$$

where λ is a constant, and

$$\psi(t) = e^{-\frac{t-t_0}{\lambda}} \qquad (17)$$

from which it is easily determined that λ is the MTBF measured from t_0, the (most recent) time of known successful operation.

In general, the probability $\Psi_j(t)$ that a system of several S/C remains in successful operation for some time t following its j^{th} reestablishment is one minus the probability that it has failed during the interval $t - t_j$, i.e.

$$\Psi_j(t) = 1 - \int_{t_j}^{t} g_j(t)\, dt \tag{18}$$

With the L.T. of this relation

$$\Psi_j^*(\omega) = \frac{1 - g_j^*(\omega)}{\omega} \tag{19}$$

f_j^* may, from Eq. (5), be expressed as

$$f_j^* = \begin{cases} \Psi_0^* & j = 0 \\ \Psi_j^* \prod_{k=0}^{j-1} g_k^* & j \geq 1 \end{cases} \tag{20}$$

Thus, $f_j(t)$, the probability that there are exactly j failures in time t, is the convolution of the probabilities for j failures g_0, $g_1, \ldots g_{j-1}$, and the probability Ψ_j for continued operation after the j^{th} failure.

The model assumes a constant launch capability N_L and a constant orbit requirement N_o. Consider, for example, a system with $N_o = 5$ and $N_L = 3$. Two successful launches are required to establish the system; hence, at t = 0, there will be six operating S/C. At the first system failure (i.e., at the second S/C failure), one successful launch is needed to replenish, immediately after which there will be seven operating S/C. Successive failures form a recurrent pattern between four and seven S/C on orbit.

In general, there will be n_0 S/C in operation after initial establishment and n_1 S/C immediately after the first and all subsequent replenishments. If, in addition, the S/C are ageless, then all system failure pdf's after the first failure are identical, i.e.

$$g_j(t) = \begin{cases} g_0(t) & j = 0 \\ g_1(t) & j \geq 1 \end{cases} \tag{21}$$

With this result, Eq. (20) becomes

$$f_j^* = \begin{cases} \Psi_0^* & j = 0 \\ \Psi_1^* \, g_0^* (g_1^*)^{j-1} & j \geq 1 \end{cases} \tag{22}$$

Following initial establishment, the number of S/C in successful operation decreases from n_0 through successive S/C failures. However, the system is considered to fail only when the number of S/C falls below N_o. Hence, the probability Ψ_0 for successful system operation is the sum of the probabilities that k S/C, where $N_o \leq k \leq n_0$, are in operation; i.e.

$$\Psi_0 = \sum_{k=N_o}^{n_0} \frac{n_0!}{(n_0-k)!k!} \psi^k (1-\psi)^{n_0-k} \tag{23}$$

and Ψ_1 is given by the same expression with n_0 replaced by n_1. In the following, the subscripts 0 and 1 will often be dropped from Ψ, g,n and associated parameters when the results apply to either subscript. The L.T. of Eq. (23) depends on the integral

$$\int_0^\infty e^{-\omega t}\left(e^{-\frac{t}{\lambda}}\right)^k \left(1 - e^{-\frac{t}{\lambda}}\right)^{n-k} dt$$

which, with the substitution $x = e^{-t/\lambda}$, may be transformed into the integral

$$\lambda \int_0^1 x^{\omega\lambda+k-1}(1-x)^{n-k}\,dx$$

This integral defines the beta function[2] and therefore may be expressed as

$$\lambda \frac{\Gamma(\omega\lambda+k)\,\Gamma(n-k+1)}{\Gamma(\omega\lambda+n+1)}$$

where Γ is the gamma function. Thus, the L.T. of Ψ is

$$\Psi^*(\omega) = \lambda \frac{n!}{\Gamma(\omega\lambda+n+1)} \sum_{k=N_o}^{n} \frac{\Gamma(\omega\lambda+k)}{k!} \tag{24}$$

Except in the case $n_0 = n_1 = N_o$, inversion of the expression Eq. (22) for f_j^* with the above value of Ψ^* is not feasible, hence, the replenishment distribution P_n cannot be determined in detail, and it is necessary to evaluate its mean and variance by other methods.

VIII. MEAN AND VARIANCE OF REPLENISHMENT DISTRIBUTION

From Eq. (1), the mean m_R of the replenishment distribution P_n is given by

$$m_R = \sum_{n=0}^{\infty} n\, P_n = \sum_{n=0}^{\infty} n \sum_{j=0}^{\infty} f_j\, h_{n,j}$$

$$= \sum_{j=0}^{\infty} f_j \left\langle S_j \right\rangle \tag{25}$$

Where $\left\langle S_j \right\rangle$, given by

$$\left\langle S_j \right\rangle = \sum_{n=0}^{\infty} n\, h_{n,j}$$

is the expected total number of launches required when j failures occur.

Thus, $\left\langle S_j \right\rangle$ is the expected value of the sum S_j given by Eq. (6), i.e.

$$\left\langle S_j \right\rangle = \left\langle \sum_{k=1}^{j} \nu_R \right\rangle = \sum_{k=1}^{j} \left\langle \nu_R \right\rangle = j m_a \tag{26}$$

where m_a is the mean number of launches required to reestablish a system after a single failure. The last step in Eq. (26) follows from the assumption that the distributions in number of reestablishment launches are independent of k. Thus, from Eqs. (25) and (26)

$$m_R = \sum_{j=0}^{\infty} f_j\, j m_a = m_a\, m_f \tag{27}$$

where

$$m_f = \sum_{j=0}^{\infty} j f_j$$

is the mean number of system failures. A similar calculation for the variance in total number of replenishment launches, defined by

$$\sigma_R^2 = \sum_{n=0}^{\infty} (n-m_R)^2 \, P_n$$

gives*

$$\sigma_R^2 = m_f \sigma_a^2 + m_a^2 \sigma_f^2 \tag{28}$$

where σ_a^2 is the variance in the number of launches required to reestablish after a single failure, and σ_f^2 is the variance in number of system failures. The distribution in number of launches to reestablish after a single failure is given by a_i in Eq. (11); the mean and variance of this distribution are

$$m_a = \frac{1}{p} \tag{29}$$

$$\sigma_a^2 = \frac{q}{p^2} \tag{30}$$

where p is the probability of a successful launch and q = 1-p. The expressions Eqs. (27)-(30) may be combined to give

$$m_R = \frac{m_f}{p} \tag{31}$$

and

$$\sigma_R^2 = \frac{\sigma_f^2 + q m_f}{p^2} \tag{32}$$

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*See Ref. 1, p. 276

From Eq. (22), the L.T. of the mean m_f is

$$\bar{m}_f^* = \langle j \rangle^* = \sum_{j=0}^{\infty} j f_j^*$$

$$= \Psi_1^* g_0^* \sum_{j=1}^{\infty} j (g_1^*)^{j-1}$$

$$= \Psi_1^* g_0^* \frac{d}{dg_1^*} \sum_{j=1}^{\infty} (g_1^*)^j$$

$$= \Psi_1^* g_0^* \frac{d}{dg_1^*} \left[(1-g_1^*)^{-1} - 1 \right]$$

$$= \Psi_1^* g_0^* (1-g_1^*)^{-2} = \frac{g_0^*}{\omega^2 \Psi_1^*} \tag{33}$$

The variance σ_f^2 may be obtained from the expected value of $j(j+1)$; the L.T. of this quantity is

$$\langle j(j+1) \rangle^* = \Psi_1^* g_0^* \sum_{j=1}^{\infty} j(j+1)(g_1^*)^{j-1}$$

$$= \Psi_1^* g_0^* \frac{d^2}{dg_1^{*2}} \sum_{j=1}^{\infty} (g_1^*)^{j+1}$$

$$= \Psi_1^* g_0^* \frac{d^2}{dg_1^{*2}} \left[g_1^* (1-g_1^*)^{-1} - g_1^* \right]$$

$$= 2\Psi_1^* g_0^* (1-g_1^*)^{-3} = \frac{2g_0^*}{\omega^3 (\Psi_1^*)^2} \tag{34}$$

Inversion of Eqs. (33) and (34) with the relation Eq. (24) for Ψ^* is not feasible; however, approximate expressions for the inverse function may be obtained in the most important limit, that of large times, with term by term inversion of the expansion of the L.T. in a power series in ω.

Repeated application of the recursion relation for the gamma function, i.e.

$$\Gamma(1+x) = x\,\Gamma(x)$$

gives

$$\begin{aligned}\Gamma(k+\omega\lambda) &= (k-1+\omega\lambda)(k-2+\omega\lambda)\cdots(1+\omega\lambda)\,\Gamma(1+\omega\lambda)\\ &= \left[(k-1)! + \omega\lambda(k-1)!\sum_{j=1}^{k-1}\frac{1}{j} + 0(\omega^2)\right]\Gamma(1+\omega\lambda)\end{aligned}$$

Also, since

$$\Gamma(1+\omega\lambda) = \Gamma(1) + \omega\lambda\,\Gamma'(1) + 0(\omega^2)$$

it follows that

$$\frac{\Gamma(k+\omega\lambda)}{(k-1)!} = 1 + \omega(\alpha_k+c) + 0(\omega^2)$$

where c is a constant and

$$\alpha_k = \lambda\sum_{j=1}^{k-1}\frac{1}{j}$$

Then, from Eq. (24), $\Psi^*(\omega)$ is given to order ω^2 as

$$\begin{aligned}\Psi^*(\omega) &= \frac{\lambda}{1+\omega(\alpha_{n+1}+c)}\sum_{k=N_o}^{n}\frac{1+\omega(\alpha_k+c)}{k}\\ &= \lambda\sum_{k=N_o}^{n}\frac{1-\omega(\alpha_{n+1}-\alpha_k)}{k} = m - \psi\end{aligned} \qquad (35)$$

where

$$m = \lambda \sum_{k=N_o}^{n} \frac{1}{k} \tag{36}$$

and

$$\gamma = \lambda \sum_{k=N_o}^{n} \frac{\alpha_{n+1} - \alpha_k}{k} = \lambda^2 \sum_{k=N_o}^{n} \frac{1}{k} \sum_{j=k}^{n} \frac{1}{j} \tag{37}$$

The significance of the parameters m and γ (i.e., of the four parameters m_1, m_2, γ_1, and γ_2) may be seen from a comparison of the two previous expressions for g^*

$$g^* = 1 - \omega \Psi^* \tag{38}$$

and

$$g^* = \left\langle e^{-\omega t} \right\rangle$$

To order ω^2, these relations are respectively

$$g^* = 1 - \omega m + \gamma\omega^2$$

and

$$g^* = 1 - \omega \langle t \rangle + \frac{\omega^2}{2} \langle t^2 \rangle$$

Thus,

$$m = \langle t \rangle$$

is the mean system life, and

$$\gamma = \frac{\langle t^2 \rangle}{2}$$

The dispersion σ^2 in system lifetime is

$$\sigma^2 = \langle t^2 \rangle - \langle t \rangle^2 = 2\gamma - m^2$$

$$= 2\lambda^2 \sum_{k=N_0}^{n} \frac{1}{k} \sum_{j=k}^{n} \frac{1}{j} - \lambda^2 \sum_{k=N_0}^{n} \frac{1}{k} \sum_{j=N_0}^{n} \frac{1}{j}$$

$$= \lambda^2 \sum_{k=N_0}^{n} \frac{1}{k^2} \tag{39}$$

With these results, m_f^* may be expanded to order ω^{-1} as

$$m_f^* = \frac{g_0^*}{\omega^2 \Psi_1^*} = \frac{1-\omega m_0}{\omega^2 (m_1 - \gamma_1 \omega)}$$

$$= \frac{1}{\omega^2 m_1} + \frac{1}{\omega}\left(\frac{\gamma_1}{m_1^2} - \frac{m_0}{m_1}\right)$$

$$= \frac{1}{\omega^2 m_1} + \frac{1}{2\omega}\left(1 + \frac{\sigma_1^2}{m_1^2} - \frac{2m_0}{m_1}\right)$$

Insertion of this expression gives the mean number of failures in time t to order constant, that is, neglecting terms of order m_0/t and m_1/t as

$$m_f = \frac{t}{m_1} + \frac{\sigma_1^2}{2m_1^2} + \frac{1}{2} - \frac{m_0}{m_1} \tag{40}$$

Similarly, to order ω^{-2}

$$\langle j(j+1) \rangle^* = \frac{2g_0^*}{\omega^3 (\Psi_1^*)^2} = \frac{(1-\omega m_0)}{\omega^3 (m_1 - \gamma_1 \omega)^2}$$

$$= \frac{2}{\omega^3 m_1^2}\left[1 + \omega\left(\frac{2\gamma_1}{m_1} - m_0\right)\right]$$

$$= \frac{2}{\omega^3 m_1^2} + \frac{2}{\omega^2 m_1}\left(1 + \frac{\sigma_1^2}{m_1^2} - \frac{m_0}{m_1}\right)$$

Inversion gives, to order t/m

$$\langle j(j+1)\rangle = \frac{t^2}{m_1^2} + \frac{2t}{m_1}\left(1 + \frac{\sigma_1^2}{m_1^2} - \frac{m_0}{m_1}\right)$$

To order t/m, the variance in number of failures is

$$\sigma_f^2 = \langle (j-m_f)^2\rangle = \langle j(j+1)\rangle - \langle j\rangle^2 - \langle j\rangle$$

$$= \frac{t}{m_1}\left[1 + 2\left(\frac{\sigma_1^2}{m_1^2} - \frac{m_0}{m_1}\right)\right], \qquad (41)$$

These results, together with the results shown in Eqs. (29) and (30), may be used in Eqs. (27) and (28) to give the mean and variance in total number of launches required in the (large) time t as

$$m_R = \frac{1}{p}\left(\frac{t}{m_1} + \frac{\sigma_1^2}{2m_1^2} + \frac{1}{2} - \frac{m_0}{m_1}\right) \qquad (42)$$

and

$$\sigma_R^2 = \frac{2t}{p^2 m_1}\left(1 + \frac{\sigma_1^2}{m_1^2} - \frac{m_0}{m_1} - \frac{p}{2}\right) \qquad (43)$$

The expression Eq. (42) is accurate to the constant term; the expression Eq. (43) is accurate to order t/m.

For the important special case $N_L = 1$, the distribution f_j may be obtained exactly. With one S/C per launch, there will be no surplus S/C in either the initial or reestablished phases, i.e., $n_0 = n_1 = N_o$. From Eqs. (36) and (37) the mean and variance in the system lifetime following establishment (and each subsequent reestablishment) are

$$m = \frac{\lambda}{N_o}$$

and

$$\sigma^2 = \frac{\lambda^2}{N_o^2} = m^2$$

From Eq. (24)

$$\Psi^*(\omega) = \frac{\lambda}{\omega\lambda + N_o} = \frac{m}{\omega m + 1}$$

and, from Eq. (22)

$$f_j^* = \Psi^*(1-\omega\Psi^*)^j = \frac{m}{(\omega m+1)^{j+1}}$$

Inversion gives the Poisson distribution

$$f_j = \frac{e^{-\frac{t}{m}}}{j!} \left(\frac{t}{m}\right)^j$$

Note that when the relations $\sigma_1^2 = m_0^2 = m_1^2$, which follow from the assumption $N_L = 1$, are used in the approximations Eqs. (40) and (41) for the mean and variance in f, the results are

$$m_f = \sigma_f^2 = \frac{t}{m}$$

in agreement with the mean and variance of the exact (Poisson) distribution.

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IX. MEAN AND VARIANCE OF ESTABLISHMENT AND TOTAL DISTRIBUTIONS

The probability distribution in number of launches required to establish a system is given essentially by $h_{n,j}$, the probability that n replenishment launches are required if j system failures occur. The establishment distribution is obtained from the expression for $h_{n,j}$ given in Eq. (15) if the random variable n represents the number of establishment launches and the integer j represents the minimum number of successful launches required to establish the system. The mean m_E and variance σ_E^2 in number of establishment launches are, according to Eq. (15), given by

$$m_E = \frac{L_o}{p} \tag{44}$$

and

$$\sigma_E^2 = \frac{L_o}{p^2} - \frac{L_o}{p} \tag{45}$$

where j has been replaced by L_o.

If N represents the total (establishment plus replenishment) number of launches required in time t, the mean m_N and dispersion σ_N^2 in N are given by

$$m_N = m_E + m_R \tag{46}$$

and

$$\sigma_N^2 = \sigma_E^2 + \sigma_R^2 \tag{47}$$

where m_R and σ_R^2 are given by Eqs. (42) and (43).

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10. ABSTRACT

A discussion of the number of launches required to establish and maintain a spacecraft system, undertaken to estimate the probable costs of various spacecraft systems and to be of assistance in comparing them.

11. KEY WORDS

Spacecraft
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Statistical methods and Procedures