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CHANCE-CONSTRAINED GENERALIZED NETWORKS

by

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Introduction

In this paper we consider extensions of the current theory of generalized network problems[1/] (from a linear programming viewpoint) to cover situations in which the nonzero entries of the generalized incidence matrix may be random variables. Our extension involves an interpretation of the constraints as chance constraints and thereby the extended problem becomes a chance-constrained programming problem. We solve this problem for the optimal zero-order rule and, in doing so, we obtain a chance-constrained problem which is the dual of our original problem. This dual chance-constrained problem is obtained through the use of dual "deterministic equivalents." Thus, we extend to this class of models, the chance-constrained duality theorem of [2] which allowed random variables only in the stipulations vector. Further extensions of the chance-constrained duality theorem to other general models will be forthcoming elsewhere.

We discuss the interpretation of this dual problem and show how it can be used to help solve the given problem. We also show that our results hold regardless of the distributions of the random variables involved in the problem. Finally, we indicate how similar techniques can be used to handle the case in which the primal objective function is also stochastic in nature.

Investigation of this problem was motivated by consideration of the problem of optimal design of wastewater treatment plants discussed in [3]. In that problem, the nonzero elements ϵ_j in the generalized incidence matrix represent "process factors" associated with the jth process. In particular, ϵ_j is the factor by which the amount of flow out of process j differs from the flow into process j as a result of the flow undergoing the jth process. From the nature of the problem it is clear that ϵ_j is a random variable, since the efficiency with which the process operates depends on such stochastic quantities as the density or composition of the flow, the temperature of the treatment chambers and ingredients, etc. Hence ϵ_j is not constant but rather fluctuates in some way over a range of possible values. Thus, it is not possible in general to specify a flow pattern, in advance, which

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1/ See [1], Volume II, page 628 ff.

will be optimal (or even physically feasible) for all possible values of the ϵ_j. However, we can find a flow pattern which is both feasible and optimal within certain preassigned probability limits. This type of interpretation leads to constraints which are conveniently expressed in the form of chance-constraints. Thus our problem becomes one of finding the optimal flow pattern within the probability limits specified by our chance constraints. Interestingly enough, since this is not true generally for dual deterministic linear programming problems, for our particular problem we are able to give the dual chance-constrained problem a meaningful physical interpretation.

As we indicated above we will solve our chance-constrained problem for the optimal zero order rule. Zero order rules were used in [4] in a discussion of PERT-type scheduling problems in which the dual stipulations vector of project completion times was assumed to be random. However, in contrast to the work in [4] in which the object was to interpret the PERT problem as a chance-constrained programming problem and also to study the distribution of total project completion time we are concerned here with the computational aspects of our model as well as with the theoretical results already mentioned. Specifically, the deterministic equivalent for our problem turns out to be a deterministic generalized network, and thus can be solved using existing algorithms for such problems (see [5], [6], [7]). In particular, we show that for a certain class of problems the deterministic equivalent can be solved using the one pass algorithms presented in [6]. Extensions of these one pass algorithms to include problems which involve decision rules more general than the zero order rule are under investigation.

Generalized Network Problems: Deterministic Case

A (pure) network is an oriented connected graph with the following additional features: associated with each link (or arc) is not only a direction but a unit cost of flow and associated with each node (or vertex) is a quantity representing an influx or efflux. Flow is regarded as taking place along the links from nodes at which influx is present to nodes at which efflux is to occur; flow on any link incurs a per unit cost in an amount given by the cost associated with that link. Capacitated networks, meaning networks in which there is an upper bound to the flow on each link, are not considered here.

Such a network, having m nodes and n links, can be described by its incidence matrix A, an $m \times n$ matrix in which the jth column (corresponding to link j) contains -1 in row k, $+1$ in row q, and zeros elsewhere when link j leads <u>from</u> node k <u>to</u> node q. An m-vector, b, contains in its ith position the influx (with a - sign) or efflux (with a + sign) associated with node i; the n-vector, c, contains in its jth position the unit cost associated with link j.

If it is desired to minimize total cost while satisfying the influent and effluent restrictions, the optimal flow pattern x is the solution to the linear-programming problem:

$$\text{Minimize} \quad c^T x$$

$$\text{(1)} \qquad \text{Subject to:} \quad Ax = b$$

$$x \geq 0,$$

where $x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$, x_j is the flow on link j; and the ith constraint is a statement of the Kirchoff conservation condition at the ith node. There exist many variations on this theme (for example, some of the equations in (1) may be replaced by inequalities) but all such variations can be converted into problems with structure of (1).

A <u>generalized</u> network differs from the above in that the nonzero entries in A are not required to be ± 1, although it is still required that each column have exactly two nonzero entries which are of opposite sign. It is clear that, by appropriate scaling of the columns of A and the corresponding elements of c, an equivalent problem may be obtained in which the <u>negative</u> element in each column of A is equal to -1. The positive element in the jth column of A will be denoted k_j. The flow on link j may be regarded as incurring a cost of $c_j x_j$ and then being subjected to amplification (or attenuation) by the factor k_j. Thus the flow along link j is amplified by a factor k_j during the course of its traversal of link j. This is in contrast to the pure network case where all $k_j = 1$.

The dual to (1) is

$$\text{Maximize} \qquad w^T b$$

$$\text{(2)} \qquad \text{Subject to} \qquad w^T A \leq c^T$$

In order to understand the meaning of the dual constraints we proceed as follows. First, note that the dual variable w_i is associated with node i and that there is one dual constraint for each link (i, j) in the network. Second, recall (from linear programming theory) that a basic optimal solution to (2) will have at least m constraints satisfied as equalities; any m such equations serve to specify a basic solution to the primal problem. Satisfaction of the remaining constraints of (2) is, in fact, a criterion for optimality of the corresponding basic solution to (1).

Suppose now that we were to interpret w_i as representing the per unit net decrease in cost which would be obtained supposing that it were feasible to increase the flow out of node i along link (i, j) by modifying the flows on the other links of the network. The w_i are thus "virtual" quantities in the same sense that virtual quantities appear in other fields such as mechanics, etc.[1/] Since increasing the flow along link (i, j) by one unit involves increasing the flow into node i by one unit and increasing the flow out of node j by k_{ij} units, the per unit net decrease in cost which would be obtained by such a change is $k_{ij}w_j + (-w_i) = k_{ij}\, w_j - w_i$. Thus, under this interpretation of the dual variables, w_i, the dual constraints state that the (per unit) net decrease or "virtual decrement" in cost which would be obtained by increasing the flow along link (i, j) by one unit must be less than or equal to the per unit actual cost increase, c_{ij}, which would be incurred by such a change, i.e., $k_{ij}w_j - w_i \leq c_{ij}$ for each link (i, j). Hence a flow pattern is optimal (i.e., results in a set of w_i which satisfy the dual

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1/ See [8] and [1], pp. 646 ff.

constraints) if and only if a change in the flow along any link would cost more than the benefits which would be obtained from such a change. This then is the interpretation we will give to the dual variables and constraints.

In concluding this discussion of deterministic generalized networks one further property of such problems should be noted, namely that any generalized network is equivalent to one in which at most one link with $k_{ij} \neq 1$ is positively incident on any node (see Figure 1). Algebraically this means that, with no loss of generality, the matrix A in (1) can be assumed to have at most one positive entry per row which is not equal to 1. To see this refer to Figure 1 in which the nodes i^* and r^* are adjoined to the original network, in order to convert the original network shown in (a) to the one in (b) which has the required property. Henceforth we shall assume that all networks under discussion possess this property. Moreover, in accordance with the above diagram, all links which have attenuation factors associated with them will be denoted by link (j, j^*) and the attenuation factor will be k_{jj^*}.

Generalized Network Problems: Chance-Constrained Formulation

We now turn to consideration of the case in which the attenuation factor for link (j, j^*) is permitted to be a random variable. We place no restrictions on the distribution of this random variable, other than to assume that the joint distribution of all the k_{j,j^*} is known. Thus we allow the possibility that some of the k_{j,j^*} may be dependent, some may be discrete, others continuous, etc. Note, however, that our assumption implies that $F_{j,j^*}(\cdot)$, the marginal cumulative distribution function of k_{j,j^*}, is known.

Due to the random nature of the attenuation factors it may no longer be possible to exhibit *a priori* (i.e., before the values of the random variables have been observed) a flow pattern which satisfies the generalized Kirchoff conservation conditions at the nodes for all possible values of k_{j,j^*}. We can attempt, however, to specify a flow pattern which, *after* observations of the k_{j,j^*}, has the property that at node j^* total influx must be greater than or equal to total efflux $(100 - \alpha_{j^*})$ per cent of the time. In other words, the flows specified should not demand large amplification factors in order to be

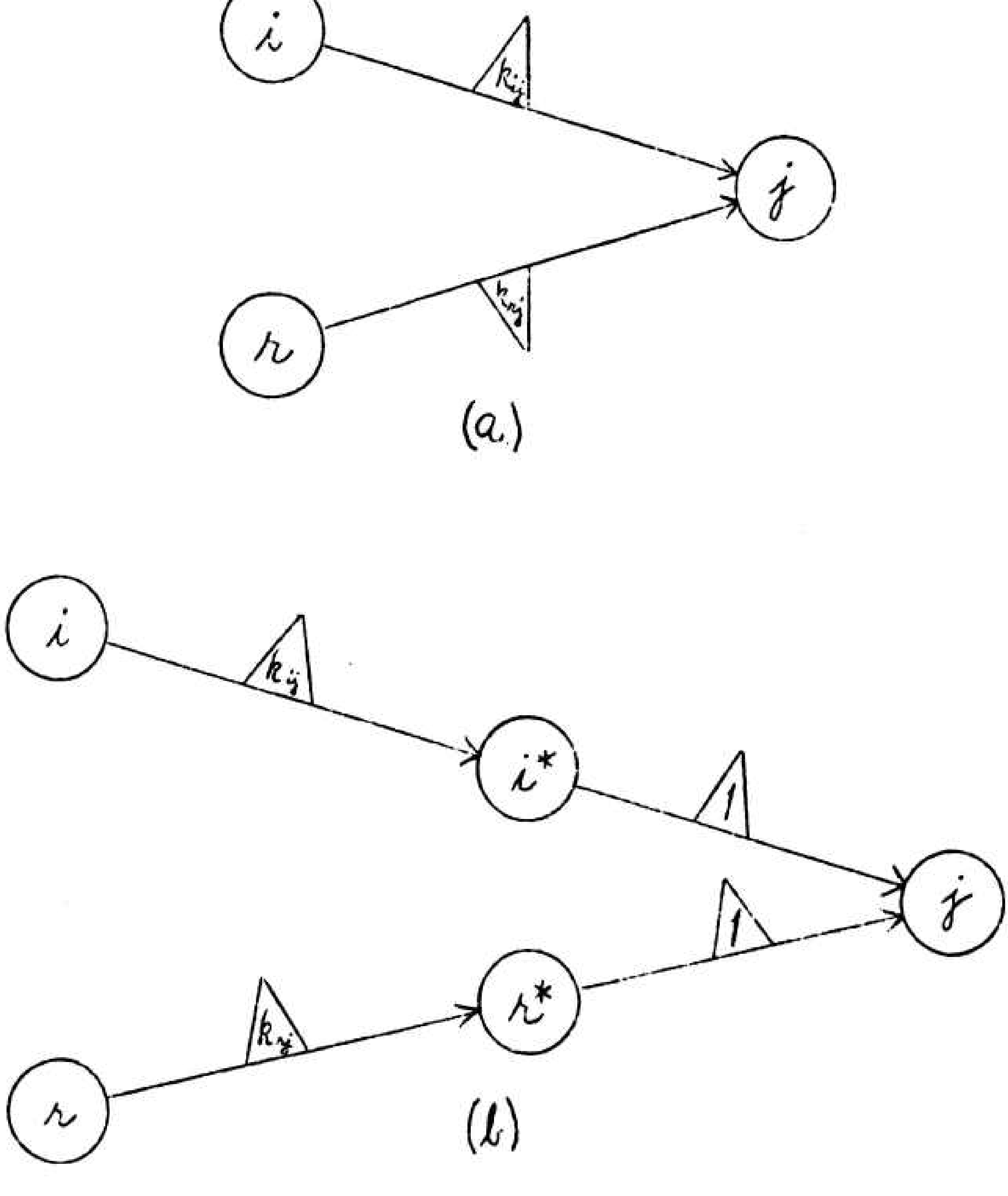

Fig. 1.

realized any more than some specified proportion (i.e., α_{j*}) of the time. Thus the generalized Kirchoff node condition which constitutes the conservation condition at the ith node becomes the chance constraint

$$(3) \qquad P\left\{k_{jj*}\ X_{jj*} \le \sum_{i\in S(j*)} X_{j*i} - \sum_{k\in R(j*)} X_{kj*} + b_{j*}\right\} = \alpha_{j*}$$

where $S(j*)$ is the set of i for which there is a link from $j*$ to i, and $R(j*)$ is the set of k for which there is a pure (i.e., unattenuated) link from k to $j*$. Any constraints for which the attenuation factors are not random should be reproduced in their deterministic form, along with the nonnegativity constraints.

Hence our problem can be written as

$$\text{Minimize} \quad \sum_{i,j} c_{ij}x_{ij}$$

(4) Subject to

$$P\left\{k_{jj*}\ X_{jj*} \le \sum_{i\in S(j*)} X_{j*i} - \sum_{k\in R(j*)} X_{kj*} + b_{j*}\right\} = \alpha_{j*}$$

for all links $(j, j*)$ with random k_{jj*},

$$k_{jj*}\ X_{jj*} + \sum_{k\in R(j*)} X_{kj*} - \sum_{i\in S(j*)} X_{j*i} = b_{j*}$$

for all links with deterministic k_{jj*},

$$x_{ij} \ge 0.$$

We must remark that the chance constraints in (4) could just as easily be written as $\quad P(\quad) \ge \alpha_{j*}$

with no change in our interpretation or subsequent results. The reason we have written the constraints in the form

$$P(\quad) = \alpha_{j*}$$

is so that the resulting deterministic equivalent constraint (see (5) below) will be similar in appearance to the other deterministic constraints in (4).

In accordance with procedures such as those discussed in [6] and [7], the constraints of (3) give rise to the deterministic equivalent constraints

$$(5) \qquad F_{jj*}^{-1}(\alpha_{j*})\, X_{jj*} + \sum_{k \in R(j*)} X_{kj*} - \sum_{i \in S(j*)} X_{j*i} = b_{j*}$$

Thus, the deterministic problem which is equivalent to (1) is

$$\text{Minimize} \quad \sum_{i,j} c_{ij}\, x_{ij}$$

subject to

$$F_{jj*}^{-1}(\alpha_{j*})\, X_{jj*} + \sum_{k \in R(j*)} X_{kj*} - \sum_{i \in S(j*)} X_{j*i} = b_{j*}$$

(6) for all links $(j, j*)$ with random k_{jj*}.

$$k_{jj*}\, X_{jj*} + \sum_{k \in R(j*)} X_{kj*} - \sum_{i \in S(j*)} X_{j*i} = b_{j*}$$

for all links with deterministic k_{jj*},

$$x_{ij} \geq 0.$$

Since (6) is an ordinary linear programming problem it has a dual; specifically:

$$\text{Maximize} \quad \sum_{i=1}^{m} w_i \, b_i$$

(7) Subject to $\quad k_{ij} \, w_j - w_i \leq c_{ij}$

where k_{ij} is either k_{jj*} or $F^{-1}_{jj*}(\alpha_{j*})$ in the appropriate instance.

Note that (7) is of the exact same form as (2) and can be similarly interpreted. The only difference is that whenever k_{jj*} is random we replace it by the α_{j*} fractile point of its marginal distribution.

Moreover, (6) is equivalent to the chance-constrained problem:

$$\text{Maximize} \quad \sum_{i=1}^{m} w_i \, b_i$$

Subject to $\quad P(k_{ij} \, w_j - w_i \leq c_{ij}) \geq \alpha_{ij}$ for links (i, j) for which k_{ij} is random,

$k_{ij} \, w_j - w_i \leq c_{ij}$ for links for which k_{ij} is deterministic.

Thus (8) and (4) are dual chance-constrained programming problems according to the terminology first introduced in [2], e.g.: (a), they employ the same data; (b), the functional value in (8) is less than or equal to that in (4) for any pair of feasible solutions to their constraints; and (c), the optimal values of the objective functions of (4) and (8) are equal.

Extensions and Specifications

First, we note that if the c_{ij} are also permitted to be random variables and the objective in (4) is to minimize total expected cost, then c_{ij} may be replaced by its expected value, $\bar{c}_{ij}$, and all of the preceding results remain valid.

Next, we restrict our attention to a more special situation. Let us now assume that the generalized network has the following special features:

(a) Exactly one node has nonzero net efflux.

(b) Exactly one node has nonzero net influx.

(c) There is no availability limitation at the influent node (or, alternatively, there is no restriction at the effluent node). Then the constraint at the influent node (node 1) is $\sum_j x_{1j} \geq -M$, where M may be as in the regularization techniques described in [8], an element from the(non-Archimedian) Hilbert extension field.

(d) All c_{ij} are nonnegative.

(e) All k_{jj*} are contained in the interval (0, 1) with probability one.

Under these assumptions the deterministic problem (6) can be solved using the one pass algorithm discussed in [6].

It is of interest to note that the wastewater treatment problem we mentioned in our introduction and discussed in [3] is of this special form. Thus we have solved an extension of the wastewater problem which permits process factors to be random. This result secures for us the possibility of solving the wastewater model, deterministic or chance-constrained, by one pass through the network using the techniques developed in [6], a fact which was not previously known even for the deterministic case.

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13. ABSTRACT

An extension to the theory of linear programming over generalized networks is presented which replaces the generalized Kirchoff node conditions by chance constraints. The extension is motivated by a class of problems in sanitary and chemical engineering in which the non-zero entries in the generalized incidence matrix may be random variables.

Duality relationships are established for appropriate pairs of such chance-constrained programming problems by showing that their deterministic equivalents consist of a deterministic generalized network problem and its dual. It is also shown how these duality relationships may be exploited in order to obtain actual solutions to chance-constrained generalized network problems.

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