AD631749

AD ______________

CONTRACT AF 61(052)-303
TR

March 1962

TECHNICAL REPORT

GENERATION OF TURBULENCE IN COUETTE FLOW BETWEEN EXCENTRIC CYLINDERS

BY

F. Schultz-Grunow

MARCH 1962

Institut für Mechanik
Technical University
A a c h e n, Germany

The research reported in this document has been sponsored by the AERONAUTICAL RESEARCH LABORATORY, OAR, United States Air Force, through its European Office.

9.) Evaluation of inertia terms.

It was found that the calculated Reynolds number for separation is 16 times to small compared with the experimental evidence. This discrepancy was attributed to the linearization of the inertia terms in Navier-Stokes equations Therefore an estimation of the inertia terms is of interest.

In (24) the first term represents the linearized inertia forces, the second and third one on the left side the second order inertia forces. For an estimation the surface integral of (24) over the cross section of the gap will be considered

$$\int U \Delta \psi' \, dF + \int (\psi_\varphi' \Delta \psi_r' - \psi_r' \Delta \psi_\varphi') \, dF = \nu \int r \Delta\Delta \psi' \, dF$$

Regarding (33) one sees that the integrals of the first order inertia term and of the viscosity term are zero. The surface integral of the second order inertia forces will be transformed by Greens theorem into a boundary integral:

$$\int (\psi_\varphi' \Delta \psi_r' - \psi_r' \Delta \psi_\varphi') \, dF = \oint \left(\psi_\varphi' \frac{\partial \psi_r'}{\partial n} - \psi_r' \frac{\partial \psi_\varphi'}{\partial n} \right) ds$$

Only the outer boundary with radius r_1 contributes as at the inner boundary $\psi_\varphi' = 0$, $\psi_r' = 0$. Introducing (33) and the direction of n equal to that of r one obtains when regarding the boundary conditions (44,47)

$$\int_0^{2\pi} \left(\psi_\varphi' \frac{\partial \psi_r'}{\partial n} - \psi_r' \frac{\partial \psi_\varphi'}{\partial n} \right) r_1 \, d\varphi = - \frac{r_1}{r_0^2} e^2 U^3 g' \pi$$

The following values were computed for the two numerical examples

$$K = 3 \cdot 10^3 , \qquad g^* = 196,60595\ U^*$$

$$K = 10^4 , \qquad g^* = 1972,6036\ U^*$$

Introducing the experimental values $r_1 = 25$ mm, $r_o = 21$ mm, $e = 1$ mm one obtains for $K = 3 \cdot 10^3$ as integral value $- 35\ U^{*2}$. The second order inertia terms show to be negative so that the linearized inertia term seems to be too large. If this is so a too small Reynolds number is calculated. This can be shown by introducing the dimensionless radius η and the corresponding stream function $\bar{\psi}$. Then from (22)

$$\bar{\psi}_\varphi \, \Delta \bar{\psi}_\eta - \bar{\psi}_\eta \, \Delta \bar{\psi}_\varphi = \frac{\nu}{r_1 U^*} \, \eta \, \Delta\Delta \bar{\psi}$$

is obtained. An average value of the Reynolds number

$$Re = \frac{r_1 U^*}{\nu}$$

is given by

$$Re = \frac{\int \eta \, \Delta\Delta \bar{\psi} \, dF}{\oint (\bar{\psi}_\varphi \bar{\psi}_{\eta\eta} - \bar{\psi}_\eta \bar{\psi}_{\varphi\eta}) \, ds}$$

It is indeed too small if the average value of the dimensionless inertia forces is too large.

An other procedure for the control of inertia forces is to introduce in advance smaller inertia forces than in the preceeding calculation. This is done by dividing the gap in two annulli. In the inner one inertia forces are neglected as they are anyhow zero at the inner boundary. In the outer annullus the linearized inertia forces are considered. The width of the two annulli was assumed as equal. The radius of the interface will be denoted with r' and the corresponding variable y with y'.

The solution for the inner annullus was already presented in chapter (5). It will now be written

$$\psi_i = e[c_1 r + c_2 r^{-1} + c_3 r^3 + c_4 r \ln r] \cos\varphi$$

$$+ e[b_1 r + b_2 r^{-1} + b_3 r^3 + b_4 r \ln r] \sin\varphi \;; \qquad (54$$

index i refers to the inner annulus, index a to the outer annulus. With r is denoted the dimensionless radius r/r_o. The boundary conditions are

$$\text{at } r' = 1, \quad \psi_r = 0 \;, \quad \psi_\varphi = 0 \qquad (55$$

$$\text{at } r = r',\ y = y', \quad \psi_{ri} = \psi_{ra} \;, \quad \psi_{\varphi i} = \psi_{\varphi a} \qquad (56$$

For the outer annulus one has the solution (33) and the condition (46) for the outer boundary.

Furthermore there must exist a steady connection of the velocities on both sides of the interface. This gives the additional boundary conditions at r = r', y = y' resp.

$$r = r',\ y = y'; \qquad \psi_{rra} = \psi_{rri} \;, \quad \psi_{\varphi ra} = \psi_{\varphi ri} \qquad (57$$

Each boundary condition gives two equations one for the sine and one for the cosine term of (54). Therefore the boundary conditions lead to 16 equations by which the 8 constants c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4, of (54) and the 8 constants a_o, a_1, a_2, a_3, m_o, m_1, m_2, m_3 of (34) should be determined. However two of these equations show to be linearily dependent. These equations are $\psi_{\varphi ra} = \psi_{\varphi ri}$ at r = r', y = y' resp. They are linear combinations of (57).

As the purpose of this calculation is merely to analyse the influence of too large inertia terms on the Reynolds number for separation a further boundary condition may be introduced which also reduces the inertia forces. This is the condition that at the interface the second radial derivatives of the tangential velocity components on each side of the interface coincide. This gives two more equations, one for the sine and one for the cosine terms.

$$\psi_{rrra} = \psi_{rrri} \tag{58}$$

With this additional condition which has no physical meaning the linear dependency is removed. It was already pointed out that a lack of physical meaning is here of no importance as in this calculation only the effect of smaller inertia terms than in the previous calculation should be investigated.

Introducing (54) and (33) into the conditions (55, 56, 57, 58, 47) one obtains the following boundary conditions

1.) $r = 1$

$$\begin{aligned} c_1 - c_2 + 3c_3 + c_4 &= 0\,; \quad & b_1 - b_2 + 3b_3 + b_4 &= 0 \\ c_1 + c_2 + c_3 &= 0\,; \quad & b_1 + b_2 + b_3 &= 0 \end{aligned} \tag{59}$$

2.) $r = r'$, $y = y'$

$$\begin{aligned}
c_1 - c_2 r'^{-2} + 3c_3 r'^2 + c_4(\ln r' + 1) &= f'(y') \\
b_1 - b_2 r'^{-2} + 3b_3 r'^2 + b_4(\ln r' + 1) &= g'(y') \\
c_1 + c_2 r'^{-2} + c_3 r'^2 + c_4 \ln r' &= \frac{f(y')}{y'+1} \\
b_1 + b_2 r'^{-2} + b_3 r'^2 + b_4 \ln r' &= \frac{g(y')}{y'+1} \\
2c_2 r'^{-3} + 6c_3 r' + c_4 r'^{-1} &= f''(y') \\
2b_2 r'^{-3} + 6b_3 r' + b_4 r'^{-1} &= g''(y') \\
-6c_2 r'^{-4} + 6c_3 - c_4 r'^{-2} &= f'''(y') \\
-6b_2 r'^{-4} + 6b_3 - b_4 r'^{-2} &= g'''(y')
\end{aligned} \tag{60}$$

3.) $y = \delta_0$

see (47)

The condition for separation

$$\frac{\partial}{\partial r}(U + u) = 0$$

at r = 1 leads with (51a, 54) to the expression

$$2\frac{\delta + 1}{\delta(\delta + 2)} U^* - \varepsilon\{[2c_2 + 6c_3 + c_4]\cos\varphi + [2b_2 + 6b_3 + b_4]\sin\varphi\} = 0 \qquad (61$$

Numerical calculations were carried through for

$$K = 3 \cdot 10^3, \quad \delta_0 = 0,2, \quad y' = 0,1, \quad r' = 1,1$$

With these quantities the 16 constants were evaluated from (59, 60, 47). The following results were obtained.

$a_0 = 0,93709$ V ; $c_1 = -5,21248$ V ;

$a_1 = -2,43007$ V ; $c_2 = 20,20620$ V ;

$a_2 = 41,86091$ V ; $c_3 = -14,99372$ V ;

$a_3 = -201,64799$ V ; $c_4 = 70,39985$ V ;

$m_0 = -0,52414$ V ; $b_1 = -5,10597$ V ;

$m_1 = 0,62465$ V ; $b_2 = -8,08644$ V ;

$m_2 = -1,21061$ V ; $b_3 = 13,14616$ V ;

$m_3 = 118,39102$ V ; $b_4 = -42,46519$ V .

Now from (61) the excentricity e with which separation occurs can be determined. Introducing the two angles $\varphi = 0^\circ, 90^\circ$ one obtains

$$\varphi = 0^\circ: \quad 5,45455 - \varepsilon \cdot 20,84991 = 0$$

$$\varepsilon = 0,26161, \quad e = 5,45021 \text{ mm}$$

$$\varphi = 90^\circ: \quad 5,45455 - \varepsilon \cdot 20,23889 = 0$$

$$\varepsilon = 0,26951, \quad e = 5.61477 \text{ mm}$$

In the preceeding calculation with linearized inertia terms covering the whole gap it was found e = 1 mm for the same values of δ_0 and $K = 3 \cdot 10^3$ or $Re = 1,3 \cdot 10^3$ resp. The experiments had shown that the Reynolds number for separation was calculated 16 times too small. Now with reduced inertia terms that means with the assumption that only in the outer half of the gap inertia forces are acting e = 5,61477 mm is found, which result indeed corresponds more closely to the experiments. A direct comparison is not possible as experiments were performed only with a maximum excentricity e = 3,5 mm corresponding to $\varepsilon = 0,167$. For separation the experiments gave $Re = 2,1 \cdot 10^4$ or $K = 4,8 \cdot 10^4$ resp. This quantity relating to e = 3,5 mm must indeed be larger than for e = 5,6 mm. This is in agreement with the calculation which gave e = 5,6 mm for $K = 3 \cdot 10^3$ or $Re = 1,3 \cdot 10^3$ resp. One sees that the order of magnitude of experimental and calculated Reynolds numbers now agrees.

Table II

Estimation of inertia terms.

First calculation

$K = 3.10^3$; Re $= 1,3.10^3$, $e = 1,01392$ mm; $\varepsilon = 0,048668$

Second calculation

$K = 3.10^3$; Re $= 1,3.10^3$, $e = 5,45021$ mm; $\varepsilon = 0,26161$

Experiment

$K = 5.10^4$; Re $= 2,1.10^4$; $e = 1$ mm; $\varepsilon = 0,048$

This shows clearly that the linearization causes the discrepancy of calculated and measured Reynolds numbers for separation. By omitting partly the inertia terms in the second calculation indeed a satisfactory agreement with experiments is obtained.

10.) Stability proof of Couette flow with regard to perturbation waves.

In the preceeding theoretical investigation it was found that separation of the Couette flow may occur when the flow is bounded by excentric cylinders the outer rotating the inner at rest. The close agreement of the calculated Reynolds number for separation and the experimentally determined Reynolds number for the first occurance of perturbations confirms that with excentric cylinders the generation of turbulence is a separation effect. It may be stated that the same effect may be due to vibrations, as separation is affected by inertia forces independently of the means by which they are created i.e. by stationary or nonstationary motion. Those circumstances may explain the generation of turbulence in earlier experiments [1,2,3,4]. In fact the review of the earlier experimental work does not exclude the conjecture that excentricities and vibrations could have been present. This could mean that in the absence of excentricities and vibrations a Couette flow with rotating outer cylinder and the inner cylinder at rest should be stable. But this conclusion would not be complete if there could not be given a direct proof of the stability of such a "pure" Couette flow. By eliminating excentricities and vibrations as far as possible stability was found to the highest speeds of revolutions experimentally obtainable [5]. The theoretical treatment applying the method of small perturbation waves on the complete Navier Stokes equations leads to Bessel functions of complex order as eigenfunctions [5]. As the zeros of these functions are not known the eigenvalues cannot be determined in this way. Therefore the solution was derived as a series expansion. This expansion contains few numerical errors which fortunately showed to be of subordinate influence on the numerical results. In eq.2o of [5] the coefficients - 16.67; 33.33 have to be replaced by 5.56; 22.22 and in eq. 21 of [5] the δ^2 terms are positive. The corrected equations will be given with the following notations. The fraction of the width d of the gap and the radius r of the outer cylinder is denoted by δ,

$$\delta = \frac{d}{r} ,$$

the wave length of the perturbation by λ and the number of waves along the circumference by k

$$k = \frac{2\pi}{\lambda} r ,$$

the ratio of the propagation velocity c of the perturbation and the velocity V of the outer cylinder by ξ,

$$\xi = \xi_r + i\,\xi_i = \frac{c}{V}$$

where ξ_r, ξ_i are the real and imaginary part of ξ, the Reynolds number by R,

$$R = \frac{Vr}{\nu}$$

Finally the abbreviations

$$q = k\ R\ \delta^2 \xi_i$$
$$r = k\ R\ (1 - \xi_r)$$
$$s = k\ R\left[\xi_r - \frac{1}{\delta(2-\delta)}\right]$$

are introduced.

With this the real and the imagninary part of the series expansion are

$$83,\overline{3}\,(1+\delta) + \left[80,\overline{5} + 11,\overline{1}\,k^2 + 5,\overline{5}\,q\right]\delta^2 \qquad (54$$

$$+ \left[77,\overline{7} + 22,\overline{2}\,k^2 + 5,\overline{5}\,q\right]\delta^3$$

$$+ \left[-\,19,791\overline{6} - 43,0\overline{5}\,k^2 - 2,08\overline{3}\,q - 0,69\overline{4}\,(k^4 + k^2 q) - 0,347\overline{2}\,(q^2 - p^2)\right]\delta^4$$

$$+ \left[-\,2,430\overline{5} - 4,1\overline{6}\,k^2 + 2,\overline{7}\,q + 3,47\overline{2}\,k^4 + 2,08\overline{3}\,k^2 q + 0,115\overline{740}\,q^2 - 0,578\overline{703}\,p^2 - 0,\overline{4629}\,ps\right]\delta^5 = 0$$

and

$$5,\overline{5} + 22,\overline{2}\,\delta - (25,\overline{4629} + 2,3\overline{148}\,k^2)\,\delta^2$$

$$+ (11,34\overline{259} + 6,71\overline{295}\,k^2)\delta^3$$

$$- (1,\overline{851} + 3,00\overline{925}\,k^2)\delta^4 \qquad (55$$

$$+ \xi_r\left[-\,11,\overline{1} - 5,\overline{5}\,\delta + (9,7\overline{2} + 1,3\overline{8}\,k^2)\delta^2 - (7,63\overline{8} + 4,86\overline{110}\,k^2)\delta^3 + (2,\overline{7} + 2,08\overline{3}\,k^2)\delta^4\right]$$

$$= q\delta^2\left[1,\overline{851} - 2,08\overline{3}\,\delta + 0,69\overline{4}\,\delta^2 + \xi_r\,(-\,1,3\overline{8} + 1,15\overline{740}\,\delta - 0,23\overline{148}\,\delta^2)\right].$$

There exists stability when $\xi_i < 0$, $q<0$ resp. For different width of the gap and different wave numbers k the following stability regions for ξ_r were calculated. It has to be mentioned that $\xi_r > 1$ has no physical sense.

Table III

Regions of ξ_r in which $\xi_i < 0$

δ	K^2	$\xi_{r\,min}$	$\xi_{r\,max}$
0,01	1	0,51722	1,28571
	100	0,51586	"
	1000	0,50339	"
	10000	0,36239	"
	100000	0	"
0,1	1	0,64988	1,29412
	100	0,55012	"
	1000	0	"
	10000	0	"
0,2	1	0,76593	1,25391
	100	0,47515	"
	1000	0	"
	10000	0	"
0,4	1	0,90171	1,17310
	100	0,58857	"
	1000	0	"
	10000	0	"
0,6	1	0,97159	1,09525
	100	0,85994	"
	1000	0,81647	"
	10000	0,81026	"
	∞	0,80954	"
0,8	1	0,99654	1,03030
	100	0,97705	"
	1000	0,97399	"
	10000	0,97363	"
	∞	0,97359	"
0,9	1	0,99957	1,00866
	100	0,99362	"
	1000	0,99295	"
	10000	0,99288	"
	∞	0,99287	"

One sees that the larger the width of the gap the smaller the stability region of ξ_r which in the limiting case $\delta = 1$ seems to converge to $\xi_r = 1$. It will be shown later that this is in agreement with the exact solution, which will be derived for $\delta = 1$. It will be shown that $\xi_r = 1$ is the only possible propagation velocity. Therefore a larger or smaller region of ξ_r does not indicate the reliability of this calculation.

11.) Stability of a fluid rotating as rigid body

As the series expansion does not give a complete stability proof it is of interest to search for special cases in which an exact solution of the stability problem is possible. As will be shown this is true in the limiting case of a centric inner cylinder with zero radius. Here the fluid is rotating as rigid body.

Denoting the radius of the outer cylinder by r, the rotatory speed of this cylinder by V, the velocity of the liquid by U and introducing the radial coordinate y with the origin at the outer cylinder one has

$$U = \frac{V}{r}(r-y) \qquad (56$$

If γ denotes the circumferential angle the circumferential coordinate

$$x = r\gamma \qquad (57$$

is introduced.

The stability will be investigated with the theory of small perturbation waves. As perturbation the well known expression [10] for the stream function ψ

$$\psi = \varphi(y)\, e^{i(\alpha x - \beta t)} \qquad (58$$

is introduced, t denoting time, $\alpha = 2\pi/\lambda$ with λ denoting the wave length. $\beta = \beta_r + i\beta_i$ is complex. The real part β_r is the natural frequency of the perturbation. The imaginary part β_i determines damping ($\beta_i < 0$) or axitation ($\beta_i > 0$). $\beta_i < 0$ means stability. Then the complex propagation velocity of the perturbance

$$c = \frac{\beta}{\alpha} \tag{59}$$

will be introduced. As mentioned before the number of perturbation waves around the circumference is

$$k = \alpha r \tag{60}$$

Introducing the perturbation (58) into Navier-Stokes equations one obtains [11] by linearization

$$\begin{aligned}&\left[\varphi'' - \left(\frac{r}{r-y}\right)^2\alpha^2\varphi - \frac{\varphi'}{r-y}\right]\left(U - \frac{r-y}{r}c\right) + \varphi\left(-U'' + \frac{U'}{r-y} + \frac{U}{(r-y)^2}\right)\\ &= -\frac{i\nu}{\alpha}\left[\left(\frac{r}{r-y}\right)^2\alpha^4\varphi - \frac{2r}{r-y}\alpha^2\varphi'' - \frac{2r}{(r-y)^2}\alpha^2\varphi' - \frac{4r}{(r-y)^3}\alpha^2\varphi + \frac{r-y}{r}\varphi^{IV}\right.\\ &\qquad\left. - \frac{\varphi'}{r(r-y)^2} - \frac{\varphi''}{r(r-y)} - \frac{2}{r}\varphi'''\right]\end{aligned} \tag{61}$$

This expression contains the complete frictional terms. An exact solution of eq.(61) will be deduced .

Introducing (56) one sees that the φ-term on the left hand side vanishes. Then introducing the operator [5]

$$L(y) = \varphi'' - \frac{\varphi'}{r-y} - \alpha^2\left(\frac{r}{r-y}\right)^2\varphi \tag{62}$$

one obtains

$$\left(U - \frac{r-y}{r}c\right)L = -\frac{i\nu}{\alpha}\left(\frac{r-y}{r}L'' - \frac{1}{r}L' - \alpha^2\frac{r}{r-y}L\right) \tag{63}$$

Thus the fourth order equation (61) is reduced to the two second order equations (62,63).

Introducing the dimensionless quantities

$$\eta = \frac{y}{r}\,,\quad R = \frac{Vr}{\nu}\,,\quad \xi = \frac{c}{V} = \frac{\beta}{\alpha V} = \xi_r + i\xi_i \tag{64}$$

in (62,63) one obtains

$$L(\eta) = \varphi'' - \frac{\varphi'}{1-\eta} - \kappa^2 \frac{\varphi}{(1-\eta)^2} \tag{65}$$

$$\frac{i}{\kappa R}\left[(1-\eta)^2 L'' - (1-\eta) L'\right] - \left[\frac{\kappa i}{R} + (1-\eta)^2(\xi - 1)\right] L = 0 \tag{66}$$

Substituting

$$z^2 = (1-\eta)^2 i \kappa R (\xi - 1) \tag{67}$$

as new independent variable (66) is transformed to the Bessel equation

$$z^2 L'' + z L' + (z^2 - \kappa^2) L = 0 \tag{68}$$

with the solution

$$L = c_1 J_\kappa(z) + c_2 H_\kappa^1(z) \tag{69}$$

(65) is transformed by (67) to

$$z^2 \varphi'' + z \varphi' - \kappa^2 \varphi - z^2 L = 0 \tag{70}$$

Adding (68,70) one obtains

$$z^2 (L+\varphi)'' + z (L+\varphi)' - \kappa^2 (L+\varphi) = 0$$

with the solution

$$L + \varphi = c_3 z^\kappa + c_4 z^{-\kappa}$$

Thus the complete solution of (61) is

$$\varphi = c_3 z^{\kappa} + c_4 z^{-\kappa} - c_1 J_{\kappa}(z) - c_2 H_{\kappa}^{1}(z) \tag{71}$$

with constants c_1, c_2, c_3, c_4.

The boundaries are y = 0, y = r or with (64,67)

$$z_1 = \sqrt{i\kappa R(\xi - 1)}\,, \quad z_2 = 0 \tag{72}$$

The boundary donditions are

$$\varphi = 0\,, \quad \frac{d\varphi}{dy} = 0 \tag{73}$$

As z^{-k}, H_k^{1} are infinite at z = 0 one has

$$c_4 = c_2 = 0$$

The boundary conditions at z_1 yield the homogenous equations

$$\varphi = c_3 z_1^{\kappa} - c_1 J_{\kappa}(z_1) = 0$$

$$\frac{d\varphi}{dy} = \left[c_3 \kappa z_1^{-\kappa} + c_2 J_{\kappa}'(z_1)\right] \frac{dz}{dy} = 0$$

Putting the determinant to zero one obtains

$$\left[- z^{\kappa} \frac{dJ_{\kappa}}{dz} + \kappa z^{\kappa-1} J_{\kappa}\right]_{z=z_1} = 0 \tag{74}$$

This expression for the eigenvalues z_1 can be transformed with the differential formula for Bessel functions

$$\frac{dJ_{\kappa}}{dz} = \frac{\kappa}{z} J_{\kappa} - J_{\kappa+1} \tag{75}$$

to

$$J_{\kappa+1}(z_1) = 0 \tag{76}$$

As the Bessel function J of first kind has only real roots the eigenvalues z_1 are real. This requires according to (67)

$$\xi_r - 1 = 0 \quad , \quad \xi_i < 0 \tag{77}$$

where ξ_r, $i\xi_i$ are the real and imaginary parts of ξ according to (64). This means that a perturbation once originated can only rotate with the angular velocity of the rotating fluid. $\xi_i < 0$ means damping. Thus a flow representing a rigid body rotation is stable. This confirms the former result obtained by series expansion (s. TableIII). It showed that the larger the width of the gap the more the stability region converges to $\xi_r = 1$.

The solution $z_1 = 0$ of (76) has to be excluded as $z = 0$ represents the inner boundary according to (72).

The exact stability proof also can be given if a liquid annullus of the width d is rotating as a rigid body. The boundary z_1 is the same as in the preceeding case, s. (72) but z_2 is now not zero. Therefore the constants c_1, c_2 in (71) are now not zero.

The coordinate of the inner boundary is

$$z_2 = \sqrt{i\kappa R(\xi - 1)}\,\left(1 - \frac{d}{r}\right) \tag{78}$$

With

$$1 - \frac{d}{r} = \alpha \tag{79}$$

one has

$$z_2 = \alpha z_1 \tag{80}$$

α is real.

Introducing (71) into the two boundary conditions (73) for each boundary one obtains

$$c_3\, z_1^{\kappa} + c_4\, z_1^{-\kappa} - c_1 J_\kappa(z_1) - c_2 H_\kappa^1(z_1) = 0$$
$$c_3 \kappa z_1^{\kappa-1} - c_4 \kappa z_1^{-\kappa-1} - c_1 J_\kappa'(z_1) - c_2 H_\kappa^{1'}(z_1) = 0$$
$$c_3\, z_2^{\kappa} + c_4\, z_2^{-\kappa} - c_1 J_\kappa(z_2) - c_2 H_\kappa^1(z_2) = 0$$
$$c_3 \kappa z_2^{\kappa-1} - c_4 \kappa z_2^{-\kappa-1} - c_1 J_\kappa'(z_2) - c_2 H_\kappa^{1'}(z_2) = 0$$

The condition that the determinant must be zero yields if the Wronski determinant

$$J_\kappa(z) H_\kappa^{1'}(z) - J_\kappa'(z) H_\kappa^1(z) = \frac{2i}{\pi z} \tag{81}$$

is introduced

$$(-z_1^{\kappa} z_2^{-\kappa} + z_1^{-\kappa} z_2^{\kappa})\left[J_\kappa'(z_1) H_\kappa^{1'}(z_2) - J_\kappa'(z_2) H_\kappa^{1'}(z_1) - \frac{\kappa^2}{z_1 z_2}\left(J_\kappa(z_1) H_\kappa^1(z_2) - J_\kappa(z_2) H_\kappa^1(z_1)\right)\right]$$
$$+(z_1^{\kappa} z_2^{-\kappa} + z_1^{-\kappa} z_2^{\kappa})\left[-\frac{\kappa}{z_2}\left(J_\kappa'(z_1) H_\kappa^1(z_2) - J_\kappa(z_2) H_\kappa^{1'}(z_1)\right) + \frac{\kappa}{z_1}\left(J_\kappa(z_1) H_\kappa^{1'}(z_2) - J_\kappa'(z_2) H_\kappa^1(z_1)\right)\right]$$
$$= i\frac{8\kappa}{\pi z_1 z_2}$$

By rearranging one obtains

$$z_1^{\kappa} z_2^{-\kappa}\Big[-J_\kappa'(z_1)\left(H_\kappa^{1'}(z_2) + \frac{\kappa}{z_2} H_\kappa^1(z_2)\right) + J_\kappa'(z_2)\left(H_\kappa^{1'}(z_1) - \frac{\kappa}{z_1} H_\kappa^1(z_1)\right)$$
$$+ \frac{\kappa}{z_1} J_\kappa(z_1)\left(H_\kappa^{1'}(z_2) + \frac{\kappa}{z_2} H_\kappa^1(z_2)\right) + \frac{\kappa}{z_2} J_\kappa(z_2)\left(H_\kappa^{1'}(z_1) - \frac{\kappa}{z_1} H_\kappa^1(z_1)\right)\Big]$$
$$+ z_1^{-\kappa} z_2^{\kappa}\Big[J_\kappa'(z_1)\left(H_\kappa^{1'}(z_2) - \frac{\kappa}{z_2} H_\kappa^1(z_2)\right) - J_\kappa'(z_2)\left(H_\kappa^{1'}(z_1) + \frac{\kappa}{z_1} H_\kappa^1(z_1)\right)$$
$$+ \frac{\kappa}{z_1} J_\kappa(z_1)\left(H_\kappa^{1'}(z_2) - \frac{\kappa}{z_2} H_\kappa^1(z_2)\right) + \frac{\kappa}{z_2} J_\kappa(z_2)\left(H_\kappa^{1'}(z_1) + \frac{\kappa}{z_1} H_\kappa^1(z_1)\right)\Big]$$
$$- \frac{8i\kappa}{\pi z_1 z_2} = 0$$

With the differential formulas (75) and

$$\frac{dJ_K}{dZ} = -\frac{K}{Z} J_K + J_{K-1} \tag{82}$$

which hold for all kinds of Bessel functions this expression can be transformed to

$$\begin{aligned}
&Z_1^{K} Z_2^{-K}\left[-H'_{K-1}(Z_2)\left(J'_K(Z_1) - \frac{K}{Z_1} J_K(Z_1)\right) - H'_{K+1}(Z_1)\left(J'_K(Z_2) + \frac{K}{Z_2} J_K(Z_2)\right)\right] \\
&+Z_1^{-K} Z_2^{K}\left[-H'_{K+1}(Z_2)\left(J'_K(Z_1) + \frac{K}{Z_1} J_K(Z_1)\right) - H'_{K-1}(Z_1)\left(J'_K(Z_2) - \frac{K}{Z_2} J_K(Z_2)\right)\right] \\
&\qquad - \frac{8iK}{\pi Z_1 Z_2} = 0
\end{aligned}$$

Here again introducing (75,82) one obtains

$$\begin{aligned}
&Z_1^{K} Z_2^{-K}\left[H'_{K-1}(Z_2) J_{K+1}(Z_1) - H'_{K+1}(Z_1) J_{K-1}(Z_2)\right] \\
&+Z_1^{-K} Z_2^{K}\left[-H'_{K+1}(Z_2) J_{K-1}(Z_1) + H'_{K-1}(Z_1) J_{K+1}(Z_2)\right] - \frac{8iK}{\pi Z_1 Z_2} = 0
\end{aligned}$$

Expressing the Hankel function by J and the Neumann function N one has

$$\begin{aligned}
&Z_1^{K} Z_2^{-K}\left[N_{K-1}(Z_2) J_{K+1}(Z_1) - N_{K+1}(Z_1) J_{K-1}(Z_2)\right] \\
&+Z_1^{-K} Z_2^{K}\left[-N_{K+1}(Z_2) J_{K-1}(Z_1) + N_{K-1}(Z_1) J_{K+1}(Z_2)\right] = \frac{8K}{\pi Z_1 Z_2}
\end{aligned}$$

Introducing (80), multiplying the whole expression by α and differentiating with respect to α one has

$$\begin{aligned}
&\alpha^{-K}\left[-(K-1)\left(N_{K-1}(Z_2) J_{K+1}(Z_1) - N_{K+1}(Z_1) J_{K-1}(Z_2)\right) + Z_2\left(N'_{K-1}(Z_2) J_{K+1}(Z_1) - N_{K+1}(Z_1) J'_{K-1}(Z_2)\right)\right] \\
&+\alpha^{K}\left[(K+1)\left(-N_{K+1}(Z_2) J_{K-1}(Z_1) + N_{K-1}(Z_1) J_{K+1}(Z_2)\right) - Z_2\left(N'_{K+1}(Z_2) J_{K-1}(Z_1) + N_{K-1}(Z_1) J'_{K+1}(Z_2)\right)\right] \\
&\qquad = 0
\end{aligned}$$

By rearranging and introducing (75,82) one obtains

$$z_2 N_K(z_2)\left[-\alpha^{-K} J_{K+1}(z_1) - \alpha^{K} J_{K-1}(z_1)\right] + z_2 J_K(z_2)\left[\alpha^{-K} N_{K+1}(z_1) + \alpha^{K} N_{K-1}(z_1)\right] = 0$$

As $z_2 \neq 0$ one has the condition

$$\frac{N_K(z_2) J_{K+1}(z_1) - J_K(z_2) N_{K+1}(z_1)}{J_K(z_2) N_{K-1}(z_1) - N_K(z_2) J_{K-1}(z_1)} = \alpha^{2K} \qquad (83)$$

As α is real this condition means that the ratios of the real and the imaginary parts of the nominator and denominator must be equal. This is not possible as the nominator and the denominator are of different degree in the lowest power of z. Therefore the condition only can be satisfied with a real independent variable z. This again means stability. It can be seen easily that (83) reduces to (76) if α tends to zero. Then the J_K terms vanish and $N_K(z_2)$ can be cancelled. What is left is (76).

12.) Summary

The preceeding report No. 1 of June 1961 was devoted to the question how turbulence may be generated in a Couette flow between excentric cylinders when the outer one is rotating and the inner cylinder is at rest.

This question arose when recent experiments performed by the author had shown definitely stability up to considerable speeds of revolution [5] whilst the earlier experiments [1,2,3,4] clearly had shown transition. As in the new experiments excentricities and vibrations were avoided as far as possible in the former experiments disregarded excentricities and vibrations could have affected the transition to turbulence. In fact a review of the earlier experimental work does not exclude this conjecture. It therefore seemed worthwhile to investigate the generation of turbulence by excentricities and vibrations. Turbulence would occur here as consequence of separation. As separation is coupled with inertia forces independent of the means by which they are produced it is sufficient to consider excentricity. Restricting to small excentricities with regard to the width of the gap the outer boundary which is regarded as excentric can be replaced by a fictitious centric boundary representing the mean of the actual boundary. The rotating outer cylinder produces radial velocities at the fictitious boundary. Therefore nonhomogenous boundary conditions exist excluding an eigenvalue problem and necessitating the calculation of the velocity profiles. Doing this it was found that separation can occur.

However the calculated Reynolds numbers for separation are too small compared with experimental observations. The discrepancy showed to be too large as to be explained by the earlier occuring instability at the inflection point of the velocity profile. The insufficiency of a

linear approximation seemed to be more likely. In fact the mean velocity tends to zero at the inner boundary so that linearization here is not anymore justified. Obviously linearization implies to large inertia forces. Therefore in this report a new calculation was performed with inertia forces which should be smaller than the actual inertia forces. This was done by neglecting the inertia forces in the inner half of the annullus and linearizing the inertia forces in the outer half. Now a satisfactory agreement with experiments was obtained.

It may be mentioned that linearized boundary calculations along corrugated walls had shown a sensitive influence of the corrugation on transition [7] in agreement with the result obtained here that the calculated Reynolds number for separation is too small. Now as a comparison with experiments is available this sensitiveness shows to be more attributed to the insufficient linearization than to physical effects.

Having demonstrated the generation of turbulence by an excentricity and herewith as mentioned before also by vibrations a straight forward proof of the stability of the Couette flow with rotating outer cylinder on the basis of propagating perturbation waves is still of interest. The exact solution could be derived [5] . However the eigenfunctions are Bessel functions of complex order which are not yet enough explored. Thus a series expansion had to be introduced which was recalculated in this report. It shows at least stability regions. However in a special case an exact stability proof can be given. It is the rotation of a fluid as rigid body. This is also the limiting case of the centric Couette flow regarded here with vanishing radius of the inner cylinder. The eigenfunctions are Bessel functions of first kind the zeros of which are the eigenvalues. As they are real the perturbation waves are damped and rotate with the body. Damping means stability. The exact stability proof is also given for a liquid annullus rotating as a rigid body. Also in this case the eigenvalues turn out to be real so that stability exists.

List of references. *)

1.) G.I. Taylor, Stability of a viscous liquid contained between two rotating cylinders. Phil.Trans. A 223, 289 (1923); Proc.Roy. Soc. A 151, 494 (1935) and 157, 546 and 565 (1936).

2.) M. Couette Etudes sur le frottement des liquides. Ann.chim.phys. 21, 433 (1890).

3.) A. Mallock Experiments on Fluid Viscosity PhilTrans.Roy.Soc. London (A) 41,41 (1896).

4.) F. Wendt Turbulent flow between rotating coaxial cylinders Ing.Arch. 4, 577 (1933).

5.) F.Schultz-Grunow On stability of Couette flow, ZaMM 39, 101 (1959).

6.) F.Schultz-Grunow On stability of Couette flow, AGARD-Boundary Layer Meeting London April 1960.

7.) H. Görtler Influence of small wall undulation on laminar boundary layer flow ZaMM 25/27, 233 (1947); 28, 13 (1948).

8.) F.Schultz-Grunow The frictional resistance of rotating discs in casings ZaMM 15, 191 (1935).

9.) F.Schultz-Grunow and H. Hein Contribution to Couette flow Z.f.Flugwiss. 4, 28, 1956.

10.) Lord Rayleigh Scientific Papers I, 474 (1880), III, 17 (1887), IV, 197 (1913).

11.) H. Schlichting Nachr.Ges.Wiss.Göttingen Math.Phys.Kl. (1932), 160.

*) References 1 to 9 are the same as in annual summary report N° 1 of June 1961.