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THE PERPLEXING BEHAVIOR OF THIN CIRCULAR CYLINDRICAL SHELLS IN AXIAL COMPRESSION

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THE PERPLEXING BEHAVIOR OF THIN CIRCULAR CYLINDRICAL SHELLS IN AXIAL COMPRESSION

by

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SUMMARY

The development of our knowledge of the buckling of thin-walled circular cylindrical shells subjected to axial compression is outlined from the beginning of the century until the present, with particular emphasis on advances made in the last twenty-five years. It is shown that practical shells generally buckle under stresses much smaller than the classical critical value derived by Lorenz, Timoshenko, Southwell and Flügge. A first explanation of the reasons for the discrepancy was given by Donnell and the problem was explored in detail by von Kármán, Tsien and their collaborators. More recently, Yoshimura discovered the existence of an inextensional displacement pattern which the wall of the shell can suddenly assume, and Koiter found an explanation of the sensitivity of the buckling stress to small initial deviations from the exact circular cylindrical shape.

In the last few years further interesting discoveries were made in Japan and in California regarding the effects of details of the boundary conditions, and many additional numerical results were obtained with the aid of high-speed electronic digital computers. Improvements in experimental techniques have also contributed significantly to a clarification of the problem and to an establishment of the unavoidable deviations from the exact shape as the major causes of the large differences between theory and experiment.

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INTRODUCTION

Theodore von Kármán, in whose memory this lecture is being presented, is best known for his work in aerodynamics. Nevertheless it is appropriate to speak about structures in a von Kármán Memorial Lecture because few, if any, research men have contributed as much as did von Kármán to the solution of problems of structural stability. His doctoral thesis (von Kármán 1908, 1910a) is the foundation of the theory of buckling at stresses exceeding the elastic limit of the material. It shows that the buckling load of a short column can be calculated from a modified Euler formula in which Young's modulus of elasticity is replaced by a combination of the tangent modulus and Young's modulus, generally known as the von Kármán modulus. In spite of some recent attempts to replace the von Kármán buckling load with the tangent-modulus load, it is now perfectly clear (Hoff 1965d) that the von Kármán load is, and will always be, the correct and the only solution of the buckling problem of short columns in the classical, or Eulerian, sense.

An equally important, and even more original, contribution of the man whom we honor today is the explanation of the snap-through type of buckling of thin shells subjected to compression. In a series of papers (von Kármán et al. 1939, 1940, 1941) he an his collaborators proved the existence of equilibrium states corresponding to large deformations of the compressed shell. These states are responsible for the low experimental values obtained and for the perplexing behavior of the shell when it buckles. These topics will be discussed at some length in the body of the paper.

But it would be foolish to try to assess the importance of von Kármán's work on the basis of his published contributions to aerodynamics and structural stability. The 111 original papers reprinted on the 1838 pages of his Collected Works (von Kármán 1956) deal also with stress analysis, plasticity, hydrodynamics, flight mechanics, jet and rocket propulsion, combustion, physics, mathematics, and airplane, missile and machine design. They were written in four languages, English, German, French and Hungarian. The mother tongues of Dr. von Kármán's friends, collaborators and students are even more numerous, and the influence of von Kármán on mechanics, aeronautics and astronautics will always be felt in all the countries of the world.

THE DEVELOPMENT OF THE CLASSICAL BUCKLING FORMULA

Axisymmetric Buckling

The classical buckling formula is usually written in the form

$$\sigma_{cl} = \left[3(1-\nu^2)\right]^{-1/2} E(h/a) \simeq 0.6E(h/a) \qquad (1)$$

where σ_{cl} is the critical value of the uniform axial stress, E is Young's modulus of elasticity, ν Poisson's ratio, h the thickness of the wall of the circular cylindrical shell, and $\underline{a}$ the radius of the middle surface of the shell. Essentially this expression was derived by Rudolf Lorenz, a civil engineer in Dortmund, Germany, in 1908. Starting out from the expressions in volume 5 of Föppl's Technische Mechanik (Föppl 1898), Lorenz developed the equations governing the axisymmetric deformations of initially slightly inaccurate thin-walled circular cylindrical shells subjected to uniform axial compression. For simply supported circular edges he represented both the initial deviations from the exact shape and the additional displacements due to the load by Fourier sine series and obtained the critical value of the stress as the one at which the denominator of one of the particular solutions vanished, thus implying an increase of the deformations beyond all bounds. He also found a good approximation to the correct expression for the wave length λ :

$$\lambda = \pi\left[12(1-\nu^2)\right]^{-1/4} (ah)^{1/2} \simeq 1.72(ah)^{1/2} \qquad (2)$$

Since this wave length is small for thin-walled shells, and since a small change in the relatively large number of waves in the axial direction changes but slightly the buckling stress, Eqs.(1) and (2) can be assumed to represent with engineering accuracy the buckling stress and the wave length of moderately long shells. The expressions become rigorously correct in the limit when the length L^* increases indefinitely, but, of course, very long shells can buckle as Euler columns, without any bulges or waves developing in the shell wall.

Lorenz' formulas can be obtained from the two equations given here by setting Poisson's ratio equal to zero. Thus his buckling stress was too low by 5 percent, and his wave length was too short by $2\frac{1}{2}$ percent. The correct formulas were first given in a Western European language by Timoshenko in 1910, although they were probably published earlier in Russian. In his Zeitschrift für Mathematik und Physik article the buckling stress is derived once by the energy method, and a second time through solution of an eigenvalue problem defined by a fourth-order ordinary differential equation and suitable boundary conditions, and identical results are obtained by the two approaches.

Chessboard Type of Buckling

The general case of buckling, without the restriction to axially symmetric deformations, was first tackled by Rudolf Lorenz in 1911. He started out from Love's so-called first approximation shell theory and assumed that the displacements in the axial direction, designated as the u^* displacements (see Fig. 1), were negligibly small. He arrived at a sixth-order partial differential equation which he solved in the presence of the following boundary conditions:

$$v^* = w^* = M_x = 0 \qquad \text{when} \quad x^* = 0, L^* \tag{3}$$

Here v^* and w^* are the displacements in the circumferential and radial directions, M_x is the axial bending moment resultant, x^* the axial coordinate, and L^* the length of the shell. He also established that the inclusion of the u^* displacement in the theory would yield an eighth-order differential equation whose solution would not differ appreciably from that of the sixth-order equation.

The numerical results of the calculations were presented in graphical form. They are in good agreement with values computed from Eq.(1).

A rather complete treatment of the problem of elastic stability was given by Southwell in 1914. Starting from Love's theory of thin shells, he derived and solved the equations defining the neutral equilibrium of a thin circular cylindrical shell subjected to simultaneous axial compression and lateral pressure. The same problem was treated again by Flügge in 1932 without reference to Love's theory. It follows from both treatises that Eqs.(1) and (2) define the buckling stress of the axially compressed shell of moderate length in good approximation, and that of the infinitely long shell accurately, provided that buckling takes place symmetrically to the axis (see Fig. 2). When buckling is of the chessboard type (Fig. 3), Eq.(1) is still correct with the limitations given, but Eq.(2) is replaced by a condition connecting the axial wave length with the circumferential wave length. The wave lengths are thus indeterminate; all that can be said is that knowledge of one wave length allows the calculation of the other.

These conclusions, and a comparison of the theory with test results, can be found in Timoshenko's Theory of Elastic Stability (Timoshenko 1936). The validity of Eq.(1) in the case of the chessboard type of buckling was probably first established by Timoshenko in 1914.

CRITICISM OF THE CLASSICAL RESULTS

Buckling Patterns

Figures 2 and 3 represent the two classical buckling patterns derived from the linear theory. The former can be observed easily if tests are made with tubes of comparatively small a/h ratio, say 30, and if the material of the tube is capable of large plastic deformations, which is the case, for instance, with mild steel. The deformations shown in Fig. 2 are permanent; the pattern cannot be observed when the deformations are entirely elastic. But the theory of buckling presented assumes perfect elasticity of the material which constitutes an inconsistency not easy to resolve.

The situation is even worse with the chessboard pattern shown in Fig. 3. It is the reproduction of a drawing and not a photograph of an actual test specimen, as is Fig. 2. The reason for not showing a photograph is that the pattern, although clearly defined by the solution of the equations governing buckling, has never been observed in actual experiment. The shape of a thin-walled perfectly elastic specimen after buckling can be seen in the photograph of Fig. 4. As this differs considerably from the former two patterns, the conclusion must be reached that the agreement between buckled shapes predicted by theory and those observed in experiment is very poor.

The Work of Flügge

In his classical Habilitationsschrift, that is formal lecture presented when he was appointed Privatdozent* in Göttingen, Flügge (1932)

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* Adjunct Professor

compared the results of his analysis with data on experiments described in the technical literature. As he was unable to find test results on axially compressed shells, he manufactured and tested a number of rubber and celluloid cylinders in the Institute for Applied Mechanics in the University of Göttingen. The length-to-radius ratio of the specimens varied between 1.76 and 5, and their radius-to-wall thickness ratio from 90 to 138. All the specimens buckled elastically, and the ratio of the experimental buckling stress to the buckling stress according to Eq. (1) ranged from 0.52 to 0.65 for the celluloid cylinders.

To explore the possible causes of this discrepancy, Flügge studied carefully the boundary conditions and the effects of small initial deviations from the exact shape. In the classical approach the stability of the equilibrium is examined when a perfectly cylindrical shell is under the effect of a uniformly distributed axial compressive stress σ. But if the simple support boundary conditions exist at the moment of buckling, they must have been in existence during the entire loading process during which the average axial stress was brought from zero to its value σ. Now it is well known that the shortening of the shell $L^*\sigma/E$ is accompanied by an increase in the radius amounting to $\nu a\sigma/E$, and evidently this increase is prevented by the simple support provided in the two end sections of the shell. Under an axial compressive stress σ the shell must therefore have the shape shown in Fig. 5 before it buckles.

In his investigation, Flügge solved the sixth-order ordinary differential equation defining the axially symmetric deformations of a shell whose end sections are prevented from expanding during the loading process.

He found that at the beginning of the loading the major portion of the shell is cylindrical, and the meridian is slightly curved only near the supports. As the intensity of loading approaches the critical value, the curved region extends toward the middle of the cylinder, several waves appear along the meridian, and the amplitudes of the waves increase until they become infinitely large at the classical critical value of the compressive stress. Considerably below this value, however, the stresses at the crests of the waves can be large enough to cause permanent deformations even though the average stress σ_{cr} is below the elastic limit of the material.

Similarly, Flügge showed that a small initial deviation from the exact shape in the form of a sine function of the axial coordinate multiplied by a sine function of the circumferential coordinate is increased during loading and tends to infinitely large values as the classical critical stress is approached. From this fact again large stresses and inelastic deformations follow below the critical load.

Since Flügge's experimental buckling loads differed relatively little from the theoretical critical loads, the two studies just sketched appeared to suffice for an explanation of the discrepancy. But in experiments carried out in connection with the rapid development of thin-walled aluminum alloy airplane structures in the early nineteen thirties shells having radius-to-thickness ratios up to 1500 were tested (Lundquist 1933, Donnell 1934). As these specimens often failed at stresses as low as 15 percent of the classical theoretical value, Flügge's explanation of the causes of the disagreement was no longer sufficient.

NEW SOLUTIONS OF THE CLASSICAL EQUATIONS

The Semi-Infinite Shell with a Free Edge

In 1959 the present author read a paper by Nachbar in which the effect of pressurization upon the influence coefficients of rotationally symmetric shells was evaluated. When applied to the spherical shell of not too small solid angle φ_e at the edge, the result was that the influence coefficients k_{ij}, that is the generalized edge displacements caused by unit edge stress resultants, could be expressed as

$$k_{ij} = k_{ijo} \frac{\sqrt{x+1}}{2x+1} \qquad \text{if} \quad i = j \tag{4a}$$

$$k_{ij} = k_{ijo} \frac{1}{2x+1} \qquad \text{if} \quad i \neq j \tag{4b}$$

In these equations k_{ij} is an influence coefficient in the presence of pressurization, and k_{ijo} the same influence coefficient when the shell is not pressurized. The symbol x was defined in the Nomenclature as the pressurization parameter and was given in terms of other symbols defined elsewhere.

Since the derivations contained no step that would invalidate the results if the internal pressure acting upon the shell were exchanged for an external pressure, it was reasonable to assume that $x = -1/2$ would correspond to the classical buckling stress given in Eq.(1). This equation is known to be valid for spherical shells under external pressure, and the edge displacements can be expected to increase without bounds when unit stress resultants are applied to the edge in the presence of the critical stress in the shell. With $x = -1/2$ naturally all the k_{ij} of Eqs. (4) tend to infinity.

When this conjecture was checked, it turned out that $x = -1$ rather than $x = -1/2$ corresponded to the classical critical value of the external pressure. This implied, of course, that the free edge of a spherical cap would become unstable at one-half the critical pressure of the complete spherical shell.

Because of the nature of the governing equations, the same result could be anticipated to hold also in the case of the thin-walled circular cylindrical shell. A simple derivation (Hoff 1961) presented at a symposium honoring Dr. von Kármán on his 80th anniversary proved indeed that the buckling stress of a semi-infinite circular cylindrical shell whose near edge is perfectly free to deform is one-half the classical value. A unit length of the free edge is assumed to be subjected to a stress resultant $\sigma_{cr}h$ whose magnitude and direction remain unchanged during the buckling process (see Fig. 6) in agreement with the Eulerian concept of buckling.

The eigenvalue problem was stated in the following form:

$$w^{IV} + (\sigma_x/E)4K^4 w'' + 4K^4 w = 0 \qquad (5)$$

where the differentiations indicated by superscripts must be carried out with respect to a non-dimensionalized axial coordinate x. This, and the non-dimensionalized radial displacement w are defined by

$$x = x^*/a \qquad\qquad w = w^*/a \qquad (6)$$

The following constants were introduced in the analysis:

$$4K^4 = 12(1-\nu^2)(a/h)^2 \doteq 4(E/\sigma_{cl})^2 \qquad D = Eh^3/12(1-\nu^2) \qquad (7)$$

The critical value σ_{cr} of the uniform axial stress σ_x is the one that satisfies the homogeneous boundary conditions

$$M_x = 0 = w'' \qquad \text{when} \quad x = 0$$
$$V = -(D/a^2)w''' = \sigma_{cr}hw' \qquad \text{when} \quad x = 0 \qquad (8)$$

and yields bounded values for all displacements and stresses when $x \to \infty$ in the solution of Eq.(5). This value was found to be

$$\sigma_{cr}/\sigma_{cl} = 1/2 \qquad (9)$$

for the semi-infinite shell that buckles axisymmetrically. It is perhaps worth noting that the second of Eqs.(8) expresses the fact that a small transverse shear force $\sigma_{cr}h \sin \tan^{-1}w' \approx \sigma_{cr}hw'$ must act on the edge of the shell if the direction of the applied axial compressive load remains unchanged while the generator at the edge of the shell rotates during buckling.

The restriction of axisymmetry in the deformations was removed in a follow-up paper by Nachbar and Hoff (Nachbar et al. 1961, 1962) and the critical stress of the semi-infinite shell was found to be

$$\sigma_{cr}/\sigma_{cl} = 0.38 \qquad (10)$$

Ōhira's Solution

In the discussion of a lecture the author gave to the Japan Society of Aeronautics at Meiji University in Tokyo in April 1963, Hiroichi Ōhira, a young associate professor at Kyūshū University, disclosed that he had obtained approximately 1/2 for ρ for semi-infinite shells with the near edge simply supported (Ōhira 1961). He had discovered that changes in the classical boundary conditions relating to displacements and stresses in the tangent plane to the middle surface can lead to such a reduction in the buckling stress. Ōhira used a relatively complicated differential equation and obtained solutions with the aid of a digital computer.

The Donnell Equations

It occurred to the author of the present paper that a closed-form solution of the equations defining the buckling of thin-walled circular cylindrical shells should be possible, and that it should yield the same low buckling stress as that obtained by Ōhira, if the latter's solution was correct. To check this conjecture, he and his graduate student Rehfield set up the problem (Hoff et al. 1964a, 1965c) with the aid of Donnell's small displacement equations. These equations were originally published by Donnell in 1933, when he was a research associate in the Guggenheim Aeronautic Laboratory of the California Institute of Technology whose director was von Kármán. The equations became popular after Batdorf had used them extensively in the solution of a number of stability problems, and checked their accuracy against other available theoretical solutions and test data (Batdorf 1947). Another check of their accuracy was presented by the author when he compared the roots of the characteristic equation corresponding to Donnell's equation with those obtainable from other shell equations available in the literature (Hoff 1955).

The Donnell equations differ from the many other equations defining the deformations of thin circular cylindrical shells in so far as they are much simpler and more symmetric in their structure than most of the others. At the same time they are sufficiently ac urate in problems of buckling if the number N of waves around the circumference is large enough. It is advantageous to write the equations in a non-dimensional form, as was done by the author (Hoff et al. 1954) in the absence of prestresses in the median plane. A further simplification was proposed by Nachbar in 1962 who normalized the coordinates and displacements for the case of uniform axial prestress. For the present purposes the most convenient form of the equations is

$$\nabla^4 w = F_{,xx} - 2\rho w_{,xx} \tag{11}$$

$$\nabla^4 F = - w_{,xx} \tag{12}$$

where the normalized coordinates are defined as

$$x = (x^*/a)(2E/\sigma_{cl})^{1/2} \qquad \varphi = \varphi^*(2E/\sigma_{cl})^{1/2} \tag{13}$$

and the displacements are given by

$$u = (u^*/a)(2E/\sigma_{cl})^{1/2} \qquad v = (v^*/a)(2E/\sigma_{cl})^{1/2} \qquad w = (w^*/a) \tag{14}$$

In these expressions σ_{cl} has the value given in Eq.(1) and ρ is the ratio of the critical stress σ_{cr} of the present theory to the classical value

$$\rho = \sigma_{cr}/\sigma_{cl} \tag{15}$$

The stress function F is implicitly defined by the following expressions for the membrane stresses acccompanying buckling (that is, not including the prestress $\sigma_x = \sigma_{cr}$):

$$\sigma_x/E = F_{,\varphi\varphi} \qquad \sigma_\varphi/E = F_{,xx} \qquad \tau_{x\varphi}/E = -F_{,x\varphi} \tag{16}$$

Finally it is noted that subscripts following a comma indicate differentiation and the biharmonic operator ∇^4 is defined as

$$\nabla^4 z = \left[(\partial^4/\partial x^4) + 2(\partial^4/\partial x^2 \partial\varphi^2) + (\partial^4/\partial\varphi^4)\right] z \tag{17}$$

Derivation of the Classical Solution

It is easy to show that the solution of Eqs.(11) and (12) is the classical buckling stress of Eq.(1) if the classical displacement pattern is assumed. Indeed, with

$$w = \sin mx \sin n\varphi \tag{18}$$

Equation (12) yields

$$F = \frac{m^2}{(m^2+n^2)^2} \sin mx \sin n\varphi \tag{19}$$

Substitution in Eq.(11) leads to the condition

$$Z + Z^{-1} = 2\rho \tag{20}$$

where

$$Z = (m^2+n^2)^2/m^2 \tag{21}$$

If Z is considered a continuous variable, the condition of a minimum of ρ is that the derivative of the left-hand member of Eq.(20) with respect to Z vanishes. Thus

$$1 - Z^{-2} = 0 \tag{22}$$

whose solution is the condition imposed on the two reduced wave numbers

$$(m^2+n^2)^2/m^2 = 1 \tag{23}$$

Finally substitution in Eq. (20) gives

$$\rho = 1 \tag{24}$$

which is the classical solution.

Of course, there must be an integral number of waves around the circumference, and an integral number of half-waves along the length of the shell. The first condition can be written as

$$N = n(2E/\sigma_{cl})^{1/2} = \text{integer} \tag{25}$$

If N is assumed, n computed, and m determined from Eq.(23), in general, the number of half-waves in the axial direction

$$M^* = ML^*/\pi a \tag{26}$$

where

$$M = m(2E/\sigma_{cl})^{1/2} \tag{27}$$

will not be an integer. However, the value will be close enough to an integer in an engineering approximation whenever the shell is very long. This is the reason why Eq.(1) has been accepted generally as the buckling formula.

In a recent paper Mann-Nachbar and Nachbar have shown that a statistical and probabilistic analysis of the solution for prescribed nominal dimensions and prescribed manufacturing tolerances in the dimensions leads to most probable wave numbers and buckling stresses that are in reasonable agreement with experimental data (Mann-Nachbar et al. 1965).

When the number of half-waves in the axial direction is an integer, obviously

$$w = w_{,xx} = 0 \qquad \text{at} \quad x^* = 0,L^* \tag{28}$$

But from Eqs.(16) and (19) it follows that

$$\sigma_x = \sigma_\varphi = 0 \qquad \text{and} \qquad \tau_{x\varphi} \neq 0 \qquad \text{at} \quad x^* = 0,L^* \tag{29}$$

This implies that $w \equiv 0$ and $\sigma_\varphi \equiv 0$ at $x^* = 0,L^*$ and thus

$$v \equiv 0 \qquad \text{at} \quad x^* = 0,L^* \tag{30}$$

It can be concluded therefore that the classical solution implies that

$$w = w_{,xx} = \sigma_x = v = 0 \qquad \text{at} \quad x^* = 0,L^* \tag{31}$$

Generalization of the Simple-Support Conditions

This is obviously not the only possible generalization to a cylindrical shell of what is usually assumed to represent simple supports ($w = w_{,xx} = 0$) for an ordinary beam. The four fundamental sets of boundary conditions proposed by the author (Hoff et al. 1964a, 1965c) for the circular cylindrical shell are:

$$
\begin{array}{lllll}
SS1 & w = 0 & w_{,xx} = 0 & \sigma_x = 0 & \tau_{x\varphi} = 0 \\
SS2 & w = 0 & w_{,xx} = 0 & u = 0 & \tau_{x\varphi} = 0 \\
SS3 & w = 0 & w_{,xx} = 0 & \sigma_x = 0 & v = 0 \\
SS4 & w = 0 & w_{,xx} = 0 & u = 0 & v = 0
\end{array}
\tag{32}
$$

Evidently, the SS3 condition corresponds to the classical solution. Experiments performed in the ordinary tension-compression testing machine should be represented by cases SS2 or SS4, depending upon the friction, or its absence, between testing machine and test specimen.

The study of the effects of the boundary conditions on the buckling stress began with the analysis of the semi-infinite shell. At the near end of the shell, where the axial compressive load was applied, one of the four sets of boundary conditions given in Eqs.(32) was prescribed. At the far end, $x \to \infty$, the displacements and stresses were required to remain bounded.

The solution was assumed in the form

$$w = e^{px} \sin n\varphi \tag{33}$$

Substitution in Eqs.(11) and (12) yields eight values for p ; these roots were first published by Nachbar in 1962. Of the eight, four roots have positive real parts; these must be ruled out for the semi-infinite shell because of the conditions at infinity. The remaining four provide us with four solutions of the type shown in Eq.(33), each multiplied by an unknown integration constant. The constants must then be determined from that one of the four sets of boundary conditions given in Eqs.(32) which is prescribed at $x = 0$. Since the conditions are homogeneous, a non-trivial solution of the four simultaneous linear equations exists only if the determinant formed of the coefficients of the unknowns vanishes. Solution of this buckling condition yields for cases SS1 and SS2

$$\rho = 0.5 + \left[12(1-\nu^2)\right]^{-1/2} N^2(h/a) + \ldots \tag{34}$$

The dots indicate terms that are of the second and of higher powers of (h/a). They must be discarded because in the derivation of the differential equations similar terms were disregarded.

The lowest critical stress corresponds to $N = 2$ as the case $N = 1$ must be ruled out since it represents a rigid-body displacement rather than an elastic deformation of the cross-section of the shell. With $N = 2$, Eq.(34) becomes

$$\rho = 0.5 + 1.21(h/a) + \ldots \tag{35}$$

Since $h/a \ll 1$, the critical stress ratio is one-half in a first approximation.

Unfortunately the Donnell equations are reliable only when $N^2 \gg 1$, as has already been mentioned. It would appear therefore that Eqs.(34) and (35) cannot be trusted. But this is not the case, because the actual value of N is not important for very thin-walled shells. For instance, when $a/h = 1000$, the assumption of $N = 2$ yields $\rho = 0.50121$. The Donnell equations are already satisfactory when $N = 4$. But with this value Eq.(34) gives $\rho = 0.50484$ which is the same result for all practical purposes. The fact that the buckling stress is insensitive to N indicates that Eq.(35) is acceptable even though it was derived from Donnell's equations for $N = 2$.

This conclusion was confirmed when the author and his doctoral student T. C. Soong recalculated the critical stress values on the basis of Sanders' differential equations. As Fig. 7 shows, the difference between the results based on Donnell's and Sanders' equations is hardly noticeable (Hoff et al. 1964b, 1965b). On the other hand, the Sanders equations are considered to be the best first-approximation theory today, and they are certainly valid when $N = 2$ (Sanders 1959). The drawback of Sanders' equations is that their characteristic roots cannot be given in closed form. Thus Soong had to evaluate the critical stresses with the aid of a digital computer.

Shells of Finite Length

On the other hand, the author has succeeded (Hoff 1964a, 1965c) in obtaining closed-form solutions for the buckling stresses of shells of finite length when both circular edges are simply supported in accordance with the conditions SS1. In this case all the eight solutions of the type given in Eq.(33) must be retained, but the eight-by-eight buckling determinant breaks up into two four-by-four determinants, one of them defining symmetric buckling, and the other antisymmetric buckling. Of the two types of buckling, the one whose deformations are symmetric with respect to the plane perpendicular to the axis and situated half-way between the two end planes of the shell yields the lower buckling stress. For engineering purposes the value of ρ can be taken as $1/2$ for all values of the length L^* of the shell (see Fig. 7).

On the basis of the closed-form solutions (Hoff et al. 1964a, 1965c) and (Hoff 1964a, 1965c) as well as the digital computer solutions (Hoff et al. 1964b, 1965b) it can be stated that shells of all lengths whose edges are free to rotate ($w_{,x}$ not restricted) and are free to displace in the circumferential direction (v not restricted) buckle at one-half the classical critical value of the axial compressive stress, except that the critical stress is even lower when the edges are entirely free. In the latter case $\rho = 0.5$ for axisymmetric buckling, and 0.38 for multilobed buckling for the semi-infinite shell. For short shells ρ can be much smaller, as can be seen from the report by Hoff and Soong (Hoff et al. 1964b, 1965b). In all other cases of boundary conditions the minimal value of the critical stress ratio ρ is unity or greater. A summary of this work was presented to the Eleventh International Congress of Applied Mechanics (Hoff 1964b).

A further digital computer solution of the classical equations was published recently by Thielemann and Esslinger (Thielemann et al. 1964). For shells of finite length they obtained buckling stresses equal to the one given by the classical formula of Eq.(1) when the length of the shell was equal to or greater than the natural wave length of the shell and the boundary conditions were those designated in this paper by the code symbols SS3 and SS4. The same results were obtained for the cases RF2, RF3 and RF4, where the symbols RF indicate rigid end fixation and the numerals have the same meaning as in Eqs.(32). When the length of the shell was decreased further, the buckling stress increased. The same effect of the length was also found to exist for shells reinforced longitudinally or circumferentially.

Recent Work by Ōhira

The results quoted are in complete agreement with those obtained by Ōhira. In a recent private communication to the author, Professor Ōhira states: "I got the original idea of attempting local buckling theory when I crushed a number of Coca Cola paper cups in a do-it-yourself laundry during the time I was staying at Purdue in 1958". His efforts to develop a theory were successful and in 1961 he presented his first paper on the low buckling stresses of semi-infinite shells with modified simple-support conditions (Ōhira 1961). The following year this work was enlarged (Ōhira 1962), and in 1963 a detailed paper on the subject was presented to the Fifth International Symposium on Space Technology and Science. The solutions were calculated with the aid of a digital computer and detailed diagrams were presented for the SS3 and SS4 cases as well as for two cases of a free edge; in one of these the u displacement was kept constant, as in the ordinary testing machine, while in the second all displacements were free. The former resulted in $\rho = 0.5$, and the latter in $\rho = 0.38$. It is noted that unpublished calculations carried out at Stanford University confirm the value 0.5 for the first of the two free edge conditions. In all the numerical work ν was taken as 0.3 and a/h as 300.

As yet unpublished reports by Ōhira, whose material has already been presented at open meetings (Ōhira 1964, 1965), deal with the buckling of shells of finite length. In particular, in the paper of 1965 the edge conditions are assumed to be more severe at one end than at the other. The result is a satisfactory decrease in the number of waves at buckling and in the buckling stress as the length of the shell is increased.

The Question of Priority

The question of priority in the discovery of scientific information is always an interesting one, although perhaps not an important one. It appears from the preceding paragraphs that Lorenz was the first to calculate the critical stresses of axially compressed circular cylindrical shells for both the axisymmetric and the general cases of buckling. Of course, his results were not quite accurate, and were later improved upon by Timoshenko in the case of axisymmetric buckling, and by Southwell and Timoshenko in the general case. The classical formula of Eq.(1) was apparently first derived by Timoshenko.

The discovery of the existence of other solutions of the classical equations yielding buckling stresses smaller than the classical one was made independently by Ōhira and by the author and his collaborators. The first oral presentation of the discovery was made by Hoff at a symposium honoring Dr. von Kármán in Washington on May 11, 1961. The text of the talk was published and distributed to a limited number of recipients (about 200) as a Stanford University report in August 1961; the volume honoring Dr. von Kármán, on the other hand, appeared only at the end of 1962. In a similar manner, Ōhira's first oral presentation was made to the Eleventh Japan National Congress of Applied Mechanics sometime in 1961 but the proceedings of the congress appeared in print only in 1962 or 1963. There were significant differences between the two solutions. Ōhira dealt with the general type of buckling of semi-infinite shells with a simply supported edge while the solution by the present author was valid for the axisymmetric buckling of semi-infinite shells with a perfectly free edge. Moreover, Hoff's solution was in closed form, while Ōhira had obtained his results with the aid of a digital computer. Hoff's result was generalized in 1962 by Nachbar and Hoff to hold for the general case of buckling.

Up to this point, Ōhira and Hoff had worked independently, without knowledge of each other's work. Subsequently Hoff acknowledged that his closed-form solutions for the simply supported edge were undertaken after he had received information of Ōhira's digital computer solution. Similarly, in his paper presented to the Fifth International Symposium on Space Technology and Science in 1963, Ōhira acknowledged that his latest results dealing with free edges were obtained after he had received information on Hoff's and Nachbar's work.

For a clarification of these problems the dates of first presentation and/or publication of new results have been collected in Table 1. It is well to remember, however, that proceedings of conferences generally appear a year or more after the date of the conference.

Significance of New Solutions

It is equally incorrect to overestimate or to underestimate the significance of the new solutions of the small-displacement equations. From the theoretical standpoint they are most interesting as they show that solutions other than the classical one exist for the problem of the buckling of axially compressed thin-walled circular cylindrical shells even though the problem is defined by linear equations. As a matter of fact the only distinction of the classical solution is that it is the easiest one to obtain.

From a practical standpoint the new solutions show that a thin shell placed between the platens of a testing machine can buckle anywhere between $\rho = 0.5$ and 1.0, depending on the friction between specimen and testing machine. This fact certainly helps to explain the large scatter observed in the test results.

If the design engineer maintains that he already has enough empirical information on the practical buckling stresses of thin shells to make him uninterested in refinements of the theory, one can counter his criticism by saying that the empirical information is only on shells already built, and not on new types of shells to be constructed in the future. Improvements to be brought to shell design in the future can be evaluated in advance only if a complete and reliable theory has been established, and the new solutions of the classical linear shell equations make a contribution to this goal.

Of course, most shells in engineering have their edges attached to other shells, or to reinforcing rings. It is almost obvious that such rings can provide an almost perfect restriction of the circumferential displacements. This was shown to be the case in a recent publication by Almroth (1965b). On the other hand, as yet unpublished calculations carried out at Stanford University indicate that a relaxation of the boundary restraint over the distance of a single half wave can reduce the buckling stress significantly. Such a relaxation can be the consequence of a broken rivet or a poor bond.

THE DISCOVERIES OF VON KÁRMÁN AND HIS COLLABORATORS

Between 1939 and 1941, Dr. Theodore von Kármán and the very capable and enthusiastic group of students and research men gathered around him in the Guggenheim Aeronautic Laboratory of the California Institute of Technology laid the foundations of the large-displacement theory of the buckling of thin shells (von Kármán et al. 1939, 1940, 1941; Tsien 1942a,b). Kármán was puzzled on the one hand by the large difference between the complete predictability of the buckling stress of rods and thin plates, and on the other hand by the very substantial difference between theoretical and experimental values of the stress at which thin shells collapsed. He also observed that the development of buckles and bulges was gradual with elastic rods and plates while it was sudden, and even explosive, with shells. After buckling, columns and plates continued to carry the buckling load, and were even capable of supporting further increased loads, but with shells the load supported after buckling always dropped to a fraction of the buckling load. Yet the equations defining the equilibrium and the stability of rods, plates and shells were all based on the same well-proven hypotheses of the theory of elasticity.

Sudden buckling with a drop in the load carried had been observed by von Kármán much earlier (von Kármán 1910a) in the case of short columns. The phenomenon had been explained by him completely as an interaction between the inelastic behavior of the short column and the elasticity of the testing machine (see also Hoff 1961). But von Kármán was reluctant to accept a similar explanation for the puzzling behavior of the shell, because very thin-walled shells appeared to recover completely after removal of the load and thus they showed the same phenomenon without plastic deformations.

The clue to the puzzle seemed to lie in the form of the solution of the classical equations, as given in Eq.(18). Evidently the expression forthe radial displacement w remains valid if the right-hand member of the equation is multiplied by -1 ; it represents displacements of equal magnitude in the inward and outward directions. But real shells tested in the laboratory always show a preference to buckle inward, and their displacements in the outward direction are much smaller than those oriented inward. It appears therefore that the classical equations fail to represent properly the difference between the inward and outward directions in the case of shells while they are perfectly satisfactory in the case of rods and plates. This difference must be a consequence of the curvature of the shell because in all other respects the basic hypotheses upon which stability theory is built are equally valid for rods and plates the one hand, and shells on the other.

The only manner in which this shortcoming of shell theory could be remedied was to add terms to the equations that were non-linear in the displacements. But relatively simple equations containing such terms had already been developed by a former collaborator of von Kármán, Lloyd H. Donnell, who had worked in the Guggenheim Aeronautic Laboratory of the California Institute of Technology between 1930 and 1933. To the linear terms of his small displacement equations (Donnell 1933) which were to become famous later, Donnell added the non-linear terms contained in the von Kármán large-deflection plate equations (von Kármán 1910b)* to obtain what are generally known today as the von Kármán-Donnell large-displacement shell equations (Donnell 1934). The trouble with these equations is of course that they cannot be solved rigorously because they are not only non-linear, but also of a very high order (the eighth).

In his paper of 1934 Donnell showed an unusual amount of ingenuity and engineering insight when he introduced for the radial displacements at buckling expressions equivalent to those that are considered today the most important ones in the representation of the shape of the buckles:

$$w^* = A_{11} \cos (\pi x^*/L_x^*) \cos (\pi y^*/L_y^*) + A_{20} \cos (2\pi x^*/L_x^*) \tag{36}$$

where $y^* = a\varphi^*$ is the circumferential coordinate, L_x^* and L_y^* are the half-wave lengths in the axial and circumferential directions, and A_{11} and A_{20} are constants. After substitution of this expression in his compatibility equation, he obtained a rigorous solution for the stress function. He used the principle of virtual displacements to calculate the load necessary to maintain equilibrium at any value of the amplitude of the buckles.

Donnell also assumed that the shell was inaccurately manufactured. For the small initial deviations from the exact shape he introduced an expression of the type represented by Eq.(36) with the values of the constants somewhat complicated functions of the nominal geometry of the shell. The shell was assumed to fail when the stress at the crest of

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* Apparently Donnell was not aware of the existence of these equations when he derived his large-displacement theory.

the buckles reached the yield stress of the material of which the shell was constructed. Donnell ended his paper by proposing a semi-empirical design formula, based on his analysis and test results, in which three material constants appeared, namely Young's modulus, Poisson's ratio, and the yield stress.

The first innovation in the analysis by von Kármán and Tsien (von Kármán et al. 1941) was the introduction of a buckle pattern based on visual observation of the shells after failure. The pattern was defined by the following expression for the radial displacement:

$$w^* = A_{00} + A_{11} \cos(mx^*/a) \cos(ny^*/a) + A_{20} \cos(2mx^*/a) + A_{02} \cos(2ny^*/a) \tag{37}$$

The first term in the right-hand member represents the classical small-displacement solution. The sum of the second and third terms with $A_{02} = A_{20}$ defines deformations quite similar to those of the middle portion of the test specimen of Fig. 4, but, of course, the mathematical expressions imply a continuous set of buckles covering the entire surface of the shell. Such a pattern, denoted the diamond pattern, can be realized only if a close-fitting mandrel is placed inside the shell (Horton et al. 1965) (see Fig. 8).

The compatibility equation

$$(1/E)\nabla^4 F = -(1/a)w^*_{,x^*x^*} + (w^*_{,x^*y^*})^2 - w^*_{,x^*x^*}w^*_{,y^*y^*} \tag{38}$$

was solved rigorously after the expression of Eq. (37) was substituted for w^* in the right-hand member. The second of the von Kármán-Donnell equation is*

$$(D/h)\nabla^4 w^* = -\sigma w^*_{,x^*x^*} + F_{,y^*y^*}w^*_{,x^*x^*} - 2F_{,x^*y^*}w^*_{,x^*y^*}$$
$$+ F_{,x^*x^*}w^*_{,y^*y^*} + (1/a)F_{,x^*x^*} \qquad (39)$$

where $-\sigma$ is the initial axial compressive stress causing buckling and F is a stress function from which the additional membrane stresses accompanying buckling can be calculated. This equation was not solved directly, but instead the direct approach of the variational calculus was used when the total potential energy of the system was minimized with respect to the three independent displacement parameters A_{00}, A_{11} and $A_{20} = A_{02}$.

From the three conditions of a minimum of the total potential energy the three constants could be calculated, and thus the value of the initial compressive stress $-\sigma$ that corresponds to equilibrium could be determined for prescribed values of the parameters μ and η. Of these μ was defined as the ratio of the wave lengths

$$\mu = m/n \qquad (40)$$

and η was proportional to the square of the number of waves around the circumference

$$\eta = n^2(h/a) \qquad (41)$$

In one set of curves representing the results of the calculations μ was arbitrarily taken as one because the wave length ratio was found to be close to unity in specimens after they had buckled in the testing machine. The parameter of the family of curves was η, and the minimal

* This is in essence the form in which the equation was given by Kempner.

absolute value of the stress was

$$-\sigma_{min} = 0.194E(h/a) \tag{42}$$

The relationship between the stress and the shortening ϵL^* of the distance between the two ends of the cylindrical shell was also calculated and plotted. The curve obtained differed little from the one labeled "Case 1" in Fig. 9, which was obtained by Kempner in 1954.

In Fig. 9, the stress σ is the axial stress, and it is considered positive when compressive. The abscissa is the average compressive strain ϵ multiplied by the ratio a/h. The gap between the straight-line and the curved portions of the diagram was not filled in because computation of the intervening unstable states of equilibrium involves great difficulties. In their 1941 paper, von Kármán and Tsien also calculated the equilibrium curve for $\mu = 1/2$ and obtained a minimum for σ which was negative indicating tension. They attributed this unlikely result to the inaccuracies of their analysis. As a matter of fact, they were very modest about their contribution to science and stated that their rough first approximation to the true solution of the problem would have to be replaced by a much more rigorous solution.

Yet von Kármán and Tsien accomplished a great deal. They discovered the existence of three states of equilibrium corresponding either to a prescribed displacement of the loading head of the testing machine (loading in a rigid testing machine), or to a prescribed value of the load (so-called dead-weight loading). They conjectured that in the first part of the loading process the states of stress and shortening would follow the straight-line portion of the diagram (Fig. 9) which is stable in the presence of infinitesimal disturbances; this would happen, however, only if the shell were perfect geometrically and if it had been built of a completely homogeneous and isotropic linearly elastic material.

In the presence of initial deviations from uniformity the maximum value of the compressive stress would be smaller than that indicated by the letter C in Fig. 9, and the difference between the numerical value 0.605 for $(\sigma/E)(a/h)$ and the experimental value would increase with increasing values of the initial deviation. However, the random nature of the initial deviations makes it very difficult to evaluate the practical maximal value of the buckling stress. Hence von Kármán and Tsien suggested that for design purposes the engineer use the minimal value of the equilibrium stress, that is $0.194E(h/a)$; from an unbuckled state corresponding to a somewhat higher value of the stress the cylindrical shell would jump into a state of large displacements, and the minimal stress just quoted could well serve as a lower limit to the stress at which the jump could take place.

Von Kármán and Tsien also concluded that the elasticity of the testing machine would have a significant effect on the stress at which the jump takes place and that disturbances of the test, such as vibrations of the foundation of the testing machine, would be an important contributing factor to the early failure of the specimen. These two conclusions were to be proved incorrect by later investigators.

The same can be said of the ingenious Tsien criterion proposed in 1942 (Tsien 1942b). On the basis of a detailed and completely rigorous analysis of a non-linear model of a shell, namely a column supported laterally by an arbitrary number of nonlinear springs (Tsien 1942a), Tsien suggested that the minimal equilibrium stress of Fig. 9 be replaced as the lower bound of the practical buckling stress by that particular stress value at which the strain energy before buckling is equal to the strain energy after buckling in a test in a perfectly rigid testing machine. In a so-called dead-weight test, the total potential energy takes the place of the strain energy. Tsien realized that he was replacing a lower bound by another lower bound. However, Tsien's lower bound was higher, and thus closer to the empirical buckling stress, than the earlier one, and the scanty experimental data against which the Tsien criterion was checked indicated satisfactory agreement between theory and experiment. Incidentally, a slightly simpler non-linear model than the one studied by Tsien had been analyzed earlier by H. L. Cox in 1940.

Systems for which the Tsien criterion is a poor approximation were mentioned by Fung and Sechler in a rather complete survey dealing with the instability of shells (Fung et al. 1960) and presented at the First Symposium on Naval Structural Mechanics held at Stanford University in 1958. Much recent experimental evidence, to be discussed later, also shows that the proper answer to the question of the practical buckling stress of thin shells is not furnished by the Tsien criterion.

THE YOSHIMURA BUCKLING PATTERN

Although the investigations of von Kármán and his collaborators have resulted in the discovery of the physical and mathematical reasons for the perplexing behavior of the axially compressed cylindrical shell, it was left to Yoshimaru Yoshimura, an imaginative professor in the Aeronautical Research Institute of Tōkyō University, who, unfortunately, died relatively young, to find the geometric reason for this physical behavior. Yoshimura proved in a Japanese paper published in 1951 that the middle surface of the circular cylindrical shell is developable into a polyhedral surface consisting of identical plane triangles; such a surface is shown in Fig. 10. His work was republished in English by the National Advisory Committee for Aeronautics in 1955.

The shell can therefore be transformed into such a polyhedral surface without stretching its middle surface, that is without causing any membrane stresses to develop. Small bending stresses are required, of course, to eliminate the initial curvature of what are the plane triangles after buckling, and the curvature becomes infinite along the edges of the triangles which form the ridges of the polyhedron. It is not obvious whether in the limit as h/a approaches zero the work necessary to produce the infinite curvature along the ridges of a perfectly elastic shell is finite or infinite, but for an ideally elastic-plastic material certainly a finite amount of work suffices to develop the ridges. But the bending stiffness of the thin wall of the shell is proportional to h^3 while its extensional stiffness is proportional to h. Hence a practical shell is likely to have a tendency to avoid extensional deformations more and more as its thickness is decreased. For this reason very thin shells can be expected to buckle in accordance with the Yoshimura pattern while thicker ones should have more ample curvature along the ridges.

This conclusion is borne out by experiment except for one important modification: the diamond-shaped buckles of Yoshimura appear only in one or two rows rather than cover the entire surface of the shell as can be seen from Fig. 4. This difference must be a consequence of the conditions at the boundaries because the pure Yoshimura pattern is incompatible with the circular edge of the cylindrical shell.

The Yoshimura pattern was discovered independently by Kirste in 1954. It was also enthusiastically adopted and studied by Ponsford in the Guggenheim Aeronautic Laboratory of the California Institute of Technology (Ponsford 1953).

FURTHER DEVELOPMENT OF THE LARGE-DISPLACEMENT THEORY

During the quarter of a century that has passed since the publication of the von Kármán-Tsien paper of 1941, most of the advance in our understanding of the buckling of circular cylindrical shells subjected to uniform axial compression has been achieved through investigations using the von Kármán-Donnell equations and the techniques developed in that paper A number of corrections and improvements were made by Leggett and Jones in 1942 (but due to war conditions their paper was distributed widely only in 1947), by Michielsen in 1948, and by Kempner a doctoral student of the author, in 1954. In particular, the total potential energy was minimized with respect to the parameters μ and η defined in Eqs.(40) and (41). This minimization showed that the buckled state found by von Kármán and Tsien for tension (for $\mu = 1/2$) was not a state of equilibrium and thus eliminated an inconsistency from the theory. The results of Kempner's analysis are shown in Fig. 9 as the curve labeled "Case 1".

The analysis was extended to orthotropic shells in a paper presented by Thielemann at the Durand Centennial Conference (Thielemann 1960) and the results of the calculations were compared with experiments carried out by Thielemann with extreme care. This work was continued by Thielemann in a report to a NASA conference held at Langley Field in 1962 in which he objected to the minimization of the total potential energy with respect to the wave lengths because evidently only integral numbers of waves can occur in the shell. It is not clear, however, from the very concise paper what procedure he introduced to replace this minimization.

Thielemann used electronic digital computers to solve, in an approximate manner, the von Kármán-Donnell equations. The use of the digital computer was exploited even more completely by Almroth who investigated many combinations of the various terms in the series

$$w^* = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_{jk} \cos \left(j\pi x^*/L_x^* \right) \cos \left(k\pi a\varphi^*/L_y^* \right) \tag{43}$$

in order to obtain the minimum of the total potential energy. The curves labeled "Case 2" and "Case 3" in Fig. 9 represent Almroth's results; in the calculations of the former only the coefficients A_{00}, A_{11}, A_{22}, A_{20} and A_{40} were assumed to be different from zero while in the latter the coefficients A_{33} and A_{60} were also included. In order to make the computational work tolerable, all the coefficients not listed were assumed to be zero.

It can be seen that for a fixed value of $\epsilon a/h$ the equilibrium stress decreases as the number of terms considered is increased. The important question to ask is therefore where the limiting curve is situated when the number of terms considered is further increased and made to approach infinity. Almroth felt that his nine-term* approximation (not shown in the figure) approached closely enough the limiting curve; he was unable to obtain significant changes by selecting different coefficients, or considering additional ones. Moreover, his results were in excellent agreement with those obtained both theoretically and experimentally by Thielemann (1962).

The author of the present paper was not convinced that Almroth's nine-term* approximation was a sufficiently good approximation to the limiting curve but he realized that it would be extremely difficult to continue the process by adding more and more terms to the displacement expression. In particular, the replacement of the products and powers of trigonometric terms by trigonometric terms of multiple angles was such a lengthy job in the analysis of the problem that it was almost impossible to avoid errors. For this reason the author suggested that this work should be programmed for the computer; the method developed for this purpose has been described in two reports (Madsen et al. 1965b; Bushnell et al. 1965).

With the aid of this computer program the author and his collaborators (Hoff et al. 1965a) succeeded in calculating equilibrium curves on the basis of up to 14-term approximations. The manner in which the equilibrium stress decreases with the number of terms considered for a fixed value of $\epsilon a/h$ can best be seen from the entries under cases 4, 5 and 6 in Table 2. The normalized stress values are not minimal values in this instance, but they correspond to $\epsilon a/h = 3.4$. They are 0.0856, 0.0706 and 0.0528 when the number of terms retained is 8, 10 and 12. When two more terms were added to the series the value of $(\sigma/E)(a/h)$ became 0.0427, which is the minimum of the curve labeled "Case 7" in Fig. 9.

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* In the discussion that follows the A_{00} term is not counted when the total number of terms is indicated. The value of A_{00} is obtained from considerations of the continuity of deformations, and not from a minimization.

This value is substantially smaller than Almroth's 0.0652. It is also smaller than the value of 0.0518 obtained by Sobey (1964b) with a 23-term approximation. The inclusion of such a large number of terms was possible only because of the availability of a superior computer program. The reason why Sobey's stress value is higher for 23 terms than the author's value for 14 terms is tha. many of the coefficients of the terms retained by Sobey have very small numerical values. Hence these terms are unimportant in the definition of the displacement pattern. Incidentally, Sobey's paper had a very limited distribution and was unknown to the author at the time he wrote his paper jointly with Madsen and Mayers.

The most interesting result of the paper by Hoff, Madsen and Mayers (1965a) is the observation that with increasing numbers of terms retained in the expression for w the coefficients of the terms of the double Fourier series approach the values characterizing the Fourier expansion of the Yoshimura buckle pattern. At the same time, μ, η and σ approach zero.

It appears therefore that the shell buckles into an exact Yoshimure pattern, with a finite wave length in the axial direction but a vanishing wave length in the circumferential direction ($\mu = 0$).* At the same time $\eta = N^2h/a$ approaches zero; since the number of waves around the circumference cannot be less than two, obviously h/a must approach zero. In other words, the limiting curve obtained when the number of terms is increased and made to approach infinity is a rigorous, but trivial, solution because it is valid only for an infinitely thin shell. Evidently the stress under which an infinitely thin shell can be in equilibrium after buckling is infinitely small.

*Another possibility is a finite wave length in the circumferential direction and an infinite wave length in the axial direction.

The solution of this puzzle was presented in a follow-up report by Madsen and Hoff (1965a). For a given cylindrical shell, that is for a prescribed value of a/h, η cannot assume a value smaller than $4a/h$. Hence minimization with respect to η means a differentiation of the total potential energy with respect to η, setting the resulting expression equal to zero, and solving for η, provided that a value equal to or greater than $4a/h$ is obtained by this procedure (and provided that the inaccuracy connected with the replacement of the integral values of N with a continuous function is considered admissible). If the value obtained for η is less than $4a/h$, it has to be replaced by $4a/h$. In this manner a lower bound exists for η and the minimal value of the postbuckling stress is greater than zero.

In another extension of the large-displacement investigations initiated by von Kármán and Tsien in 1941, the behavior of initially slightly inaccurate circular cylindrical shells was studied (see Fig.11). The importance of initial deviations from the exact shape had been recognized much earlier as it had already been studied by Flügge in 1932 and by Donnell in 1934. But in 1950, Donnell and Wan greatly altered the procedure followed by Donnell sixteen years earlier and developed a new method of calculation which was to be copied by several other investigators. Through a rather complex reasoning, and on the basis of his broad experience in engineering, Donnell came to the conclusion that the most dangerous initial deviations of the middle surface of a circular cylindrical shell from the exact shape could be represented by the equation

$$w_o^*/h = (U/\pi^2)(L_x^{*1.5}L_y^{*0.5}/h^2)\ f(x^*,y^*) = (Ua^2/\mu^{1.5}N^2h^2)\ f(x^*,y^*) \qquad (44)$$

where U is the unevenness parameter and $f(x^*,y^*)$ the function given in Eq.(37). The additional radial displacements caused by the load were represented in the same form and were multiplied by an amplification factor. The compatibility equation (38) was solved rigorously for the stress function F. The expressions for w^*_{tot} and F were then substituted in the expressions for the total potential energy and the expression so obtained was minimized with respect to the amplification factor and A_{20}, A_{02}, m and n.

This implies that the shape of the displacements caused by the loads was taken to be the same as the shape of the initial deviations. This is obviously a restriction on the generality of the solution, but in view of the difficulties inherent in any solution of the governing equations it is a justifiable one. If the minimization had been carried out only with respect to the amplification factor, the result could be accepted as a usable approximation. Unfortunately, the total potential energy was also minimized with respect to the parameters defining the shape of <u>initial</u> deviations. This means that the system whose total potential energy was minimized was not defined at all, but changed its initial shape during minimization. A correct and complete analysis should define the initial shape by means of the coefficients A^o_{11}, A^o_{20}, A^o_{02} and the additional displacements by means of the coefficients A^1_{11}, A^1_{20}, A^1_{02}. The minimization should then be carried out with respect to A^1_{11}, A^1_{20} and A^1_{02}, and not with respect to the coefficients A^o_{11}, A^o_{20} and A^o_{02}.

The same error can be found in a number of publications based on the paper by Donnell and Wan; they are the articles by Loo (1954), Lee (1962) and Sobey (1964b).

The error was avoided by Madsen and Hoff who used a two-term expression to define the shape of initial deviations and a three-term expression for the additional displacements (Madsen et al. 1965a). The minimization was carried out with respect to the three coefficients of the additional displacements and the results are shown in Fig. 11.

It is evident from this figure that small initial deviations from the exact cylindrical shape have a large effect upon the maximum load carried by the compressed shell; and it is worth noting that this maximum load is the only quantity that can be observed directly in a compression test. For instance, an initial amplitude of the nonsymmetric deviations amounting to one-tenth of the wall thickness coupled with an amplitude of the axisymmetric deviations amounting to one-fortieth of the wall thickness reduces the maximum load to 60 percent of the classical value calculated for the perfect shell.

A more complete calculation by Almroth (1965b, 1966) resulted in somewhat lower maximal values of the stress.

All the solutions of the large-displacement equations quoted assume that the shell is very long and that its surface is completely covered with uniform bulges after it has buckled. Yet Fig. 4 clearly shows that in the laboratory shells buckle only over a small area and that the remainder of their surface remains smooth. This fact was already mentioned by Yoshimura (1951). The localized buckle pattern was introduced into an energy solution of the large-displacement equations by Uemura, a former collaborator of Yoshimura, while he was working on a research project at Stanford University. His solution (Uemura 1963, 1964) indicates a tendency on the part of the shell to prefer local buckles to uniformly distributed ones, but the results are not really conclusive because of the comparatively small number of terms retained in the series representing the radial displacements.

This chapter would be incomplete without mention of the efforts made to check whether the von Kármán-Donnell equations are sufficiently accurate for an analysis of the postbuckling behavior of thin-walled circular cylindrical shells. On the one hand it is easy to show that the Donnell expressions for the membrane strain are completely inadequate to represent the inextensional deformations of the Yoshimura pattern when there are 5 to 10 triangles around the circumference of the shell (Hoff et al. 1965a), and on the other the curvature expressions become inaccurate and the stresses can exceed the yield stress of the material when the computations are carried out with the retention of more and more terms of the infinite series for a prescribed value of the a/h ratio. The latter two observations were made by Mayers and Rehfield in a report published in 1964.

Moreover the tremendous effort made by many investigators in the last 25 years has resulted only in a reduction of the value of the coefficient k in the buckling stress formula $\sigma_{cr} = kE(h/a)$, but the value of k has remained a constant, independent of the a/h ratio. Experiments show, however, that k can be as low as 0.3 when a/h is 100, and 0.06 when a/h is 3000.

It was desirable therefore to investigate the effect upon k of the use of equations more accurate than the von Kármán-Donnell equations. This was done by Mayers and Rehfield in the paper cited; they found, however, that the dependence of k on the a/h ratio is negligibly small. The same conclusion was drawn by Tsao (1965) and by Madsen and Hoff (1965). Unfortunately, the calculations of the former were shown to be unreliable by Mayers and Rehfield. In the Madsen-Hoff article perfectly rigorous membrane strain expressions and almost perfectly rigorous curvature expressions were developed for arbitrarily large displacements and for strains that are small compared to unity. The calculations involved the minimization of a total potential energy expression containing more than 12,000 terms.

The perplexing conclusion must be drawn therefore that even though displacement patterns can easily be devised for which the Donnell strain-displacement and curvature-displacement relations are grossly inadequate, and although these relations form the basis of the von Kármán-Donnell large-displacement equations, replacement of these equations by more accurate ones does not change noticeably the equilibrium states obtainable from the equations. The explanation of the paradox is probably that the procedures used to solve the equations always lead to displacement patterns involving so many waves around the circumference that the shell can be considered a shallow one.

THE EFFECT OF PREBUCKLING DEFORMATIONS

It has already been mentioned that in his fundamental paper of 1932 on the buckling of cylindrical shells, Flügge investigated the effect on the buckling load of deformations that occur during the loading of the shell before it buckles. These deformations arise because of the tendency of the compressed shell to expand uniformly, and because of the restriction of this expansion at the supports. Flügge's study indicated that because of the prebuckling deformations the yield stress of the material is reached in the shell slightly before the critical value of the stress is reached even though the critical stress is well within the elastic limit of the material.

Recently the problem was attacked again by two investigators who worked almost simultaneously and entirely independently, without knowledge of each other's work. But in this new research the full power of the electronic digital computer was used to obtain the results. In the United States the veteran investigator Manuel Stein (1962, 1964) and in Germany the young research man G. Fischer (1963) solved first the relatively simple axisymmetric problem of the prebuckling deformations. Next extensive computer programs were developed for the solution of the linearized stability problem of the deformed and still axisymmetric, but no longer cylindrical shell.

When the results were finally compared, surprisingly Fischer's buckling stress was found to be about twice as high as that obtained by Stein. The former showed buckling at about 82 percent of the classical critical stress while the latter's shells buckled at stresses amounting to 42 to 48 percent of the classical critical value. The discrepancy was explained when the new solutions of the classical linear equations obtained by Ōhira and Hoff, described at the beginning of this paper, became known. In his analysis Fischer used the boundary conditions designated by the code symbol SS3, and Stein those denoted SS2 (see Eqs.(32)).

A final comparison of the two solutions was made by Almroth (1965b) who studied eight sets of boundary conditions, namely those indicated by the present author by the symbols SS1 to SS4, and RF1 to RF4. He confirmed Stein's and Fischer's solutions and concluded that the effect upon the buckling stress of the boundary conditions was large, and that of the prebuckling deformations was small.

THE KOITER THEORY

Probably the easiest way to acquire an understanding of the fundamental idea of the Koiter theory is to work out an example in some detail, and for the example the non-linear model of a shell analyzed recently by the author (Hoff 1965a) may well be chosen. The model (Fig. 12) consists of two pin-jointed bars whose common end point is supported laterally by a non-linear spring, and whose far ends are under the action of equal and opposite forces P. In the original paper the bars were elastic and two linear springs attached to their far ends represented the elasticity of the testing machine. Since the effect of these features of the model on the buckling phenomenon is small, they are omitted from the present analysis in order to save space and effort.

The spring force S is characterized by the equation

$$S = 2Kf(\zeta) \tag{45}$$

where the non-dimensional displacement ζ is defined as

$$\zeta = \eta/h = (y/h) - (e/h) \tag{46}$$

and e and h are the eccentricity of the system and the initial vertical component of the length of each bar. In the original publication the spring characteristic was defined as

$$S = 100\eta^3 - 200\eta^2 + 105\eta \tag{47}$$

A graph of this relationship is shown in Fig. 13. In the present paper the quantities appearing in Eq. (45) are defined as

$K = 525$ lb $\qquad\qquad h = 10$ in.

$$f(\zeta) = a_3\zeta^3 + a_2\zeta^2 + \zeta \tag{48}$$

$a_3 = (100/105)100 \qquad\qquad a_2 = -(200/105)10$

It is useful now to study the equilibrium and the stability of the system with the aid of energy considerations. The strain energy W stored in the non-linear spring when it is displaced a distance ζh to the right is

$$W = 2hk\int_0^{\zeta} f(\xi)d\xi \tag{49}$$

where ξ is a dummy variable representing the instantaneous value of ζ. The potential V of the external loads being

$$V = -2Pu \tag{50}$$

where u is the axial displacement of the ends of the bars, the total potential energy U is

$$U = 2hK\int_0^{\zeta} f(\xi)d\xi - 2Pu \tag{51}$$

From the geometry of the system (Fig. 12) it follows that

$$u = h - \left[h^2+e^2-y^2\right]^{1/2} = h\left\{1 - \left[1+\delta^2-(\zeta+\delta)^2\right]^{1/2}\right\} \tag{52}$$

where

$$\delta = e/h \tag{53}$$

Substitution in Eq. (51) yields

$$U = 2hK\left\{\int_0^{\zeta} f(\xi)d\xi - \lambda\left\{1 - \left[1 + \delta^2 - (\zeta+\delta)^2\right]^{1/2}\right\}\right\} \tag{54}$$

with

$$\lambda = P/K \tag{55}$$

First the system without eccentricity, the so-called perfect system, will be examined. For such a system

$$U = 2hK\left\{\int_0^{\zeta} f(\xi)d\xi - \lambda\left[1 - (1-\zeta^2)^{1/2}\right]\right\} = 0 \tag{56}$$

since

$$e = 0 = \delta \tag{56a}$$

In agreement with the principle of virtual displacements the first variation of the total potential energy must vanish for equilibrium. The variation must be carried out with respect to the only independent displacement quantity ζ . One obtains

$$\delta U = (dU/d\zeta)\delta\zeta = 2hK\left[f(\zeta) - \lambda\zeta(1-\zeta^2)^{-1/2}\right]\delta\zeta = 0 \tag{57}$$

This equation has two kinds of solutions. Since $f(0) = 0$, evidently one solution is

$$\zeta = 0 \tag{58}$$

This means that the system is in equilibrium in its fundamental state, the initial straight-line configuration, whatever the value of the load factor λ .

In addition, equilibrium is possible for certain combinations of load and displacement characterized by

$$\lambda = (1/\zeta)(1-\zeta^2)^{1/2} f(\zeta) \quad \text{when } \zeta \neq 0 \tag{59}$$

This equation defines a buckled state adjacent to the fundamental state which will be called the adjacent state.

The stability of the fundamental state depends upon the sign of the second derivative of the total potential energy. From Eq. (57) one obtains:

$$(d^2U/d\zeta^2) = 2hK\left[f'(\zeta) - \lambda(1-\zeta^2)^{-3/2}\right] \tag{60}$$

But for the fundamental state $\zeta = 0$; hence

$$(d^2U/d\zeta^2)_{\zeta=0} = 2hK\left[f'(0) - \lambda\right] \tag{61}$$

From Eq. (48) evidently

$$f'(0) = 1 \tag{62}$$

Consequently the fundamental state is stable when $\lambda < 1$ and unstable when $\lambda > 1$. The critical point is characterized by

$$\lambda_{cr} = 1 \qquad \text{that is} \quad P_{cr} = 525 \text{ lb} \tag{63}$$

The critical point is a bifurcation point, or branching point, where equilibrium configurations adjacent to the fundamental configuration appear for the first time in the loading process. The brancning point is labeled Q in Fig. 14.

Koiter was the first to call attention to the importance of the stability of the system in the branching point itself. There the second derivative of the total potential energy is zero and thus stability depends on the derivatives of higher order. From Eq.(60) the third derivative is easily obtained:

$$(d^3U/d\zeta^3) = 2hK\left[f'''(\zeta) - 3\lambda\zeta(1-\zeta^2)^{-5/2}\right] \tag{64}$$

Since in the branching point $\zeta = 0$ and $\lambda = 1$, the expression becomes

$$(d^3U/d\zeta^3)_{\substack{\zeta=0\\ \lambda=1}} = 2hKf'''(0) \tag{65}$$

From Eq.(48) one calculates

$$f'''(0) = 2a_2 = -4000/105 \approx -38 \tag{66}$$

Thus

$$(d^3U/d\zeta^3)_{\substack{\zeta=0\\ \lambda=1}} \approx -76hK \neq 0 \tag{67}$$

When the second derivative of the total potential energy vanishes and at the same time the third derivative is not zero, the system is unstable (see, for instance, Hoff 1956). Evidently in such a case the equilibrium corresponds to a minimax, and the total potential energy decreases during a small positive excursion if it increases during a small negative excursion, and vice versa. Koiter has shown that under such conditions the system is very sensitive to small initial deviations from the perfect shape.

This sensitivity can be checked if the imperfect system is investigated. The total potential energy is given by Eq.(54); its first derivative is

$$dU/d\zeta = 2hK\left\{f(\zeta) - \lambda(\zeta+\delta)\left[1 + \delta^2 - (\zeta+\delta)^2\right]^{-1/2}\right\} = 0 \tag{68}$$

With a non-vanishing eccentricity $e = h\delta$ the above expression of the principle of virtual displacements has only one solution:

$$\lambda = (\zeta+\delta)^{-1} f(\zeta)\left[1 + \delta^2 - (\zeta+\delta)^2\right]^{1/2} \tag{69}$$

The second derivative of U is

$$d^2U/d\zeta^2 = 2hK\left\{f'(\zeta) - \lambda(1+\delta^2)\left[1 + \delta^2 - (\zeta+\delta)^2\right]^{-3/2}\right\} \tag{70}$$

For sufficiently small absolute values of δ and λ this expression is certainly positive; hence the load-displacement curve defined by Eq. (69) is stable when the load is small. The stability vanishes when the second derivative becomes zero. Substitution of the expression for λ from Eq. (69) and equation to zero of the second derivative result in

$$f'(\zeta) - (1+\delta^2)(\zeta+\delta)^{-1}\left[1 + \delta^2 - (\zeta+\delta)^2\right]^{-1} f(\zeta) = 0 \tag{71}$$

When ζ and δ are sufficiently small, this simplifies to

$$f'(\zeta) - (\zeta+\delta)^{-1} f(\zeta) = 0 \tag{72}$$

In view of the graph of $S = 2Kf(\zeta)$ shown in Fig. 13 this equation has no real solution when δ and ζ are negative. The curves representing the displacements of a system whose eccentricity is negative are stable everywhere. A critical point can exist, however, when δ and ζ are positive, but this critical point is not a branching point but a limit point, that is a maximum of the load-displacement curve.

The curves shown in Fig. 14 indicate that the maximum of the load reached by a slightly imperfect system can be much lower than the classical critical load of the perfect system. This is always the case when the branching point of the perfect system (point Q in Fig. 14) is unstable.

The opposite is true when the branching point of the perfect system is stable, as will now be shown. Let us attach a second spring to the joint of the system shown in Fig. 12, but in the opposite direction. The horizontal force S' provided by the second spring will then be

$$S' = 100\eta^3 + 200\eta^2 + 105\eta \tag{73}$$

To maintain unchanged the classical critical load, the dimensions of the springs will be reduced until each provides only one-half the force it did before. The combination of the two springs will now be characterized by

$$S'' = (1/2)(S+S') = 100\eta^3 + 105\eta \tag{74}$$

and in Eqs.(48) the only change to be made is to write

$$f(\zeta) = a_3\zeta^3 + \zeta \tag{75}$$

The total potential energy expression of Eq.(56) and the expressions for the derivatives given in Eqs.(57), (60) and (64) remain unchanged. Again, the bifurcation of the equilibrium states occurs at $P = 525$ lb and the fundamental state is stable below, and unstable above this value. But the second derivative of $f(\zeta)$ is different:

$$f''(\zeta) = 6\zeta \tag{76}$$

In the fundamental state this obviously vanishes. Hence in the branching point

$$(d^3U/d\zeta^3)_{\substack{\zeta=0\\ \lambda=1}} = 0 \tag{77}$$

The stability of the system in the branching point now depends on the sign of the fourth derivative of the total potential energy. From Eq.(64) one obtains

$$d^4U/d\zeta^4 = 2hK\left[f'''(\zeta) - 3\lambda(1+4\zeta^2)(1-\zeta^2)^{-7/2}\right] \tag{78}$$

At the critical point this becomes

$$(d^4U/d\zeta^4)_{\substack{\zeta=0\\ \lambda=1}} = 2hK\left[f'''(0) - 3\right] \tag{79}$$

But from Eqs.(48) and (75)

$$f'''(\zeta) = 6a_3 = 600(100/105) \tag{80}$$

It can be concluded therefore that in the branching point the fourth derivative of the total potential energy is positive, and thus the equilibrium of the branching point is stable.

The equilibrium states were also investigated in the presence of small initial eccentricities and the curves representing the behavior of the system are shown in Fig. 15. It can be seen from the figure that imperfections have no significant effect upon the load the system can carry.

Figure 15 is representative of the behavior of a flat rectangular plate compressed in its plane with its edges simply supported. After buckling the load can be increased further and small deviations from flatness have little effect on the load-carrying capacity of the plate.

Figure 14 is characteristic of the behavior of an axially compressed thin-walled circular cylindrical shell. The maximum load it can carry is greatly affected by small deviations from the exact cylindrical shape and the load drops suddenly when the critical value of the imperfect system is reached in the testing machine.

The connection between postbuckling behavior and the stability of the system was explored in detail for such complex systems as shells, and criteria for determining the stability of the branching point were established rigorously in a doctoral dissertation written by Koiter in 1945. Unfortunately, the dissertation was published in the Dutch language and for a long time it did not receive the attention it merited. A concise presentation of the principles involved was made by Koiter at the Symposium on Non-Linear Problems in Madison, Wisconsin, in 1963 and the printing of the paper in the Proceedings of the symposium has contributed greatly to the recognition of the importance of the theory in the analysis of structural stability. A third publication by the same author (Koiter 1963[b]) contains a rigorous solution for the imperfect circular cylindrical shell.

It follows from Koiter's general theory that the ratio ρ of the maximum stress of the imperfect shell to the classical critical stress of the perfect shell is given by the equation

$$\rho\psi = \left[4/27(1-\nu^2)\right]^{1/2}(1-\rho)^2 \qquad (81)$$

if the imperfections are axially symmetric and ψ is the ratio of the amplitude of the sinusoidal initial deviations from the exact circular cylindrical shape and the wall thickness. Introduction of the new symbols

$$z = 1 - \rho$$

$$C = \left[27(1-\nu^2)/4\right]^{1/2} \psi \simeq 2.48\psi \tag{82}$$

permits the writing of the solution of Eq. (81) in the form

$$z = \frac{\sigma_{cl} - \sigma_{cr}}{\sigma_{cl}} = -(C/2) + C^{1/2}\left[1 + (C/4)\right]^{1/2} \tag{83}$$

When $C/4 \ll 1$, this is equivalent to

$$z = C^{1/2} - (1/2)C + (1/8)C^{3/2} + \ldots \tag{84}$$

In an earlier paper (Madsen et al. 1965a) it was proposed that in a first approximation the inital deviation amplitude should be assumed to be proportional to the radius of the shell. Since ψ is this amplitude divided by the wall thickness h , one can write

$$\psi = K^*(a/h) \tag{85}$$

The formulas given lead to reasonable agreement with experimental data, as was indicated by the author in his lecture at the Seventh International Aeronautics Congress in Paris (Hoff 1965b), if the value of K^* is chosen as 10^{-4} . If one wants to obtain a formula valid for less carefully manufactured specimens he may choose

$$K^* = 4 \times 10^{-4} \tag{86}$$

Substitutions yield

$$C \simeq 10^{-3}(a/h) \tag{87}$$

From Eqs. (82) and (83) the values of ρ become about 0.64, 0.47, 0.38 and 0.21 when the a/h ratio is 200, 600, 1000 and 3000.

The Koiter theory has recently been taken up by investigators at University College in London and at Harvard University in Boston and a number of interesting results were obtained in the former place by J. M. T. Thompson (Thompson 1961, 1963, 1964)* and in the latter by J. Hutchinson (Hutchinson 1965).

* Part of Thompson's work was carried out at Stanford University.

EXPERIMENTAL VERIFICATION

The development of the theory has always gone hand in hand with increasingly careful experimentation. No details will be given here of experiments conducted to determine the buckling stresses of circular cylindrical shells. It may be mentioned, however, that the derivation of the classical buckling stress formula was preceded by Lilly's experiments in 1908, and that the revival of interest in buckling theories in the thirties was paralleled by the experimental work of Robertson (1928, 1929), Flügge (1932), Wilson and Newmark (1933), Lundquist (1933), Donnell (1934) and Kanemitsu and Nojima (1939).

In the more recent past large-scale experiments were carried out with specimens of large a/h ratios by Harris, Suer, Skene and Benjamin in 1957 and by Weingarten, Morgan and Seide in 1965. Thielemann (1960, 1962) also made a large number of tests at the time when he worked out his theory of the buckling of orthotropic circular cylindrical shells.

The fact that tests in very rigid and in very elastic testing machines lead to the same buckling stress, and that consequently the Tsien criterion must be considered invalid, has been confirmed for circular cylindrical shells by Horton, Johnson and Hoff in 1961 and by Almroth, Holmes and Brush in 1964. The same proof was brough recently for complete spherical shells by Carlson, Sendelbeck and Hoff (1965).

The sensitivity of axially compressed circular cylindrical shells to small initial deviations from the exact shape was demonstrated by Babcock and Sechler (1962, 1963) when they tested a series of very accurately fabricated shell specimens with built-in and carefuly measured deviations. The manufacturing procedure used in these investigations had originally been introduced by Thompson in 1960 who had produced thin spheres by the electroplating method. With the best specimens of this kind Babcock and Sechler reached 76 percent of the classical critical stress when the a/h ratio was 890. The values of $\rho = \sigma_{cr}/\sigma_{cl}$ obtained by Almroth, Holmes and Brush ranged from 0.43 to 0.73. Even higher values, up to 0.9, were reported by Tennyson (1963, 1964) when the a/h ratio of specimens made of a photoelastic material was between 100 and 170.

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TABLE 1

CHRONOLOGY OF SOLUTIONS OF SMALL-DISPLACEMENT EQUATIONS

Author	Date	Description of Work
Lorenz	1908	Axisymmetric buckling stress (approximate)
Timoshenko	1910	Classical formula for axisymmetric buckling stress
Lorenz	1911	Buckling stress in general case (approximate)
Southwell	1914	Buckling stress in general case
Timoshenko	1914	Buckling stress in general case (classical formula?)
Flügge	1932	More general derivation of buckling stress and exploration of reasons for disagreement with experiment
Hoff	1961	Closed-form solution of axisymmetric buckling of semi-infinite shell with free edge; $\rho = 0.5$
Ōhira	1961	Digital computer solution of general buckling of semi-infinite shells with simply supported edge; $\rho \simeq 0.5$
Nachbar & Hoff	1961,1962	Closed-form solution of general buckling of semi-infinite shell with free edge; $\rho \simeq 0.38$
Ōhira	1963,1964	Digital computer solution of general buckling of semi-infinite shell with various edge conditions
Hoff & Rehfield	1964,1965	Closed-form solution of general buckling of semi-infinite shell with simply supported edge
Hoff	1964,1965	Closed-form solution of general buckling of finite shell with simply supported edges
Hoff & Soong	1964,1965	Digital computer solutions of general buckling of finite shells with various boundary conditions
Thielemann & Esslinger	1964	Digital computer solution of general buckling of finite shells with or without reinforcements
Ōhira	1964,1966(?)	Digital computer solution of general buckling of finite shell with simply supported edges
Ōhira	1965,1966(?)	Digital computer solution of general buckling of finite shell with dissimilar edge conditions
Almroth	1965,1966(?)	Digital computer solution of general buckling of finite shells with elastic edge constraint

TABLE 2

CHRONOLOGY OF MINIMAL VALUES OF STRESS OBTAINED FOR LARGE-DISPLACEMENT EQUILIBRIUM

Case	Number of Coefficients Retained	Year of Publication	Authors	$\left(\frac{\sigma}{E}\frac{a}{h}\right)_{min.}$
--	2	1941	v. Kármán, Tsien	0.194
--		1942, 1947	Leggett, Jones	0.195
--	2	1948	Michielsen	0.195
1	3	1954	Kempner	0.182
2	4	1963	Almroth	0.0949
3	6	1963	Almroth	0.0733
	9	1963	Almroth	0.0652
--	23	1964	Sobey	0.0518
4	8	1965	Hoff, Madsen, Mayers	0.0856
5	10	1965	Hoff, Madsen, Mayers	0.0706
6	12	1965	Hoff, Madsen, Mayers	0.0528
7	14	1965	Hoff, Madsen, Mayers	0.0427

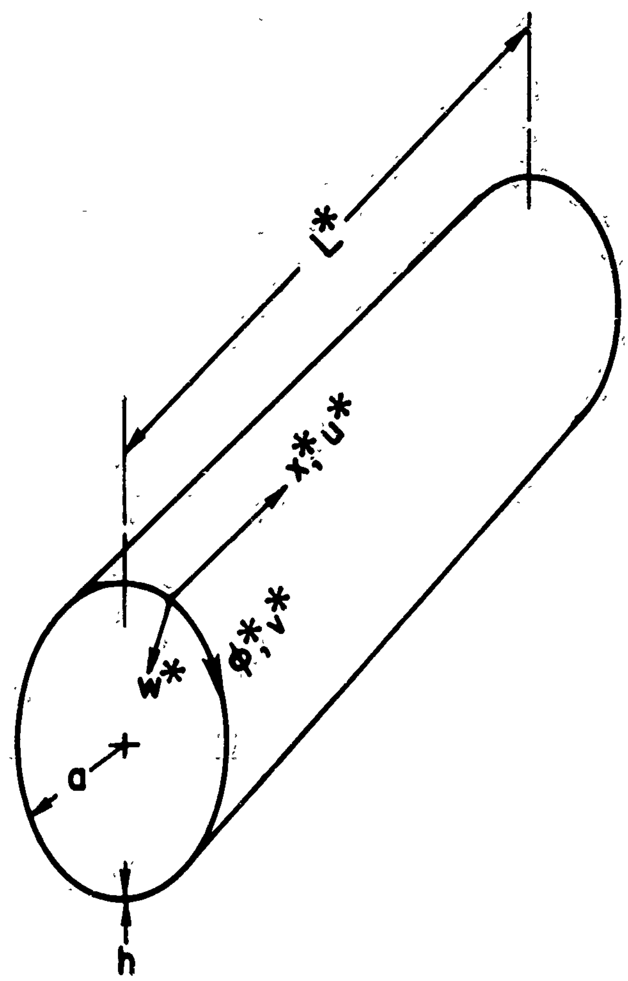

FIG. 1. Notation.

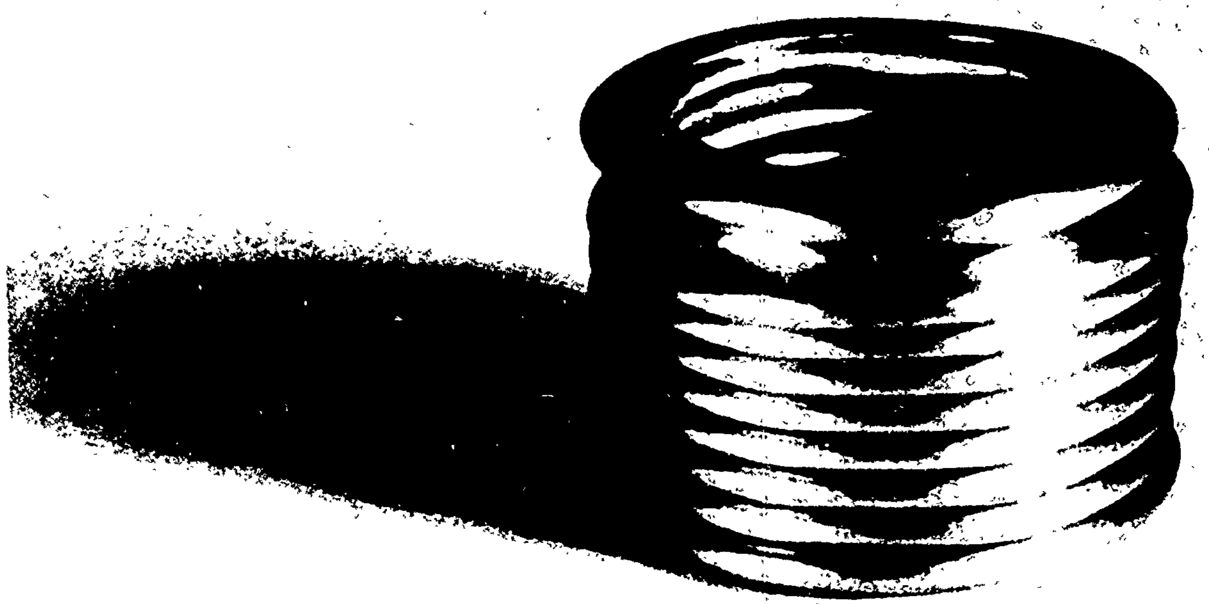

FIG. 2. Axisymmetric Buckling

(courtesy of W. H. Horton)

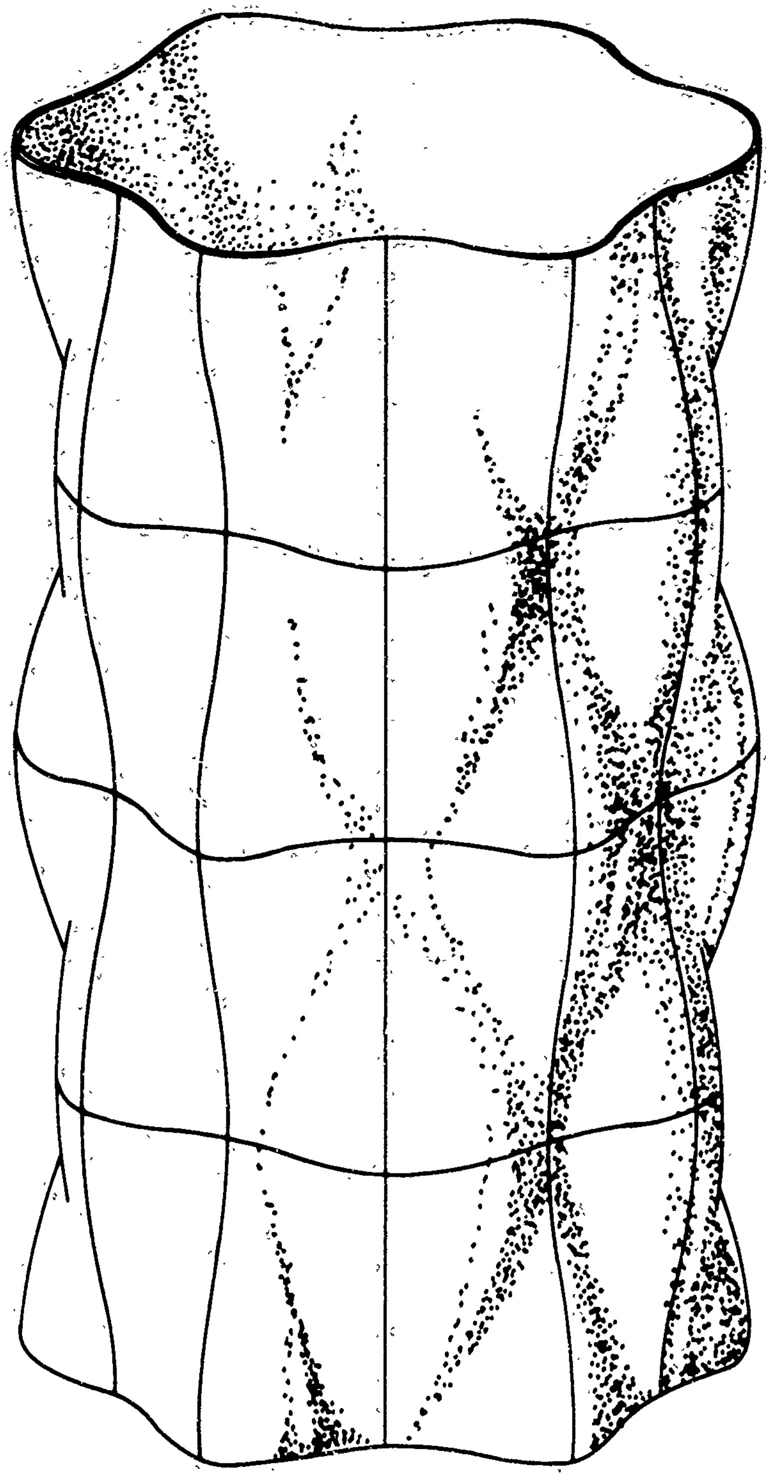

FIG. 3. Chessboard Type of Buckling.

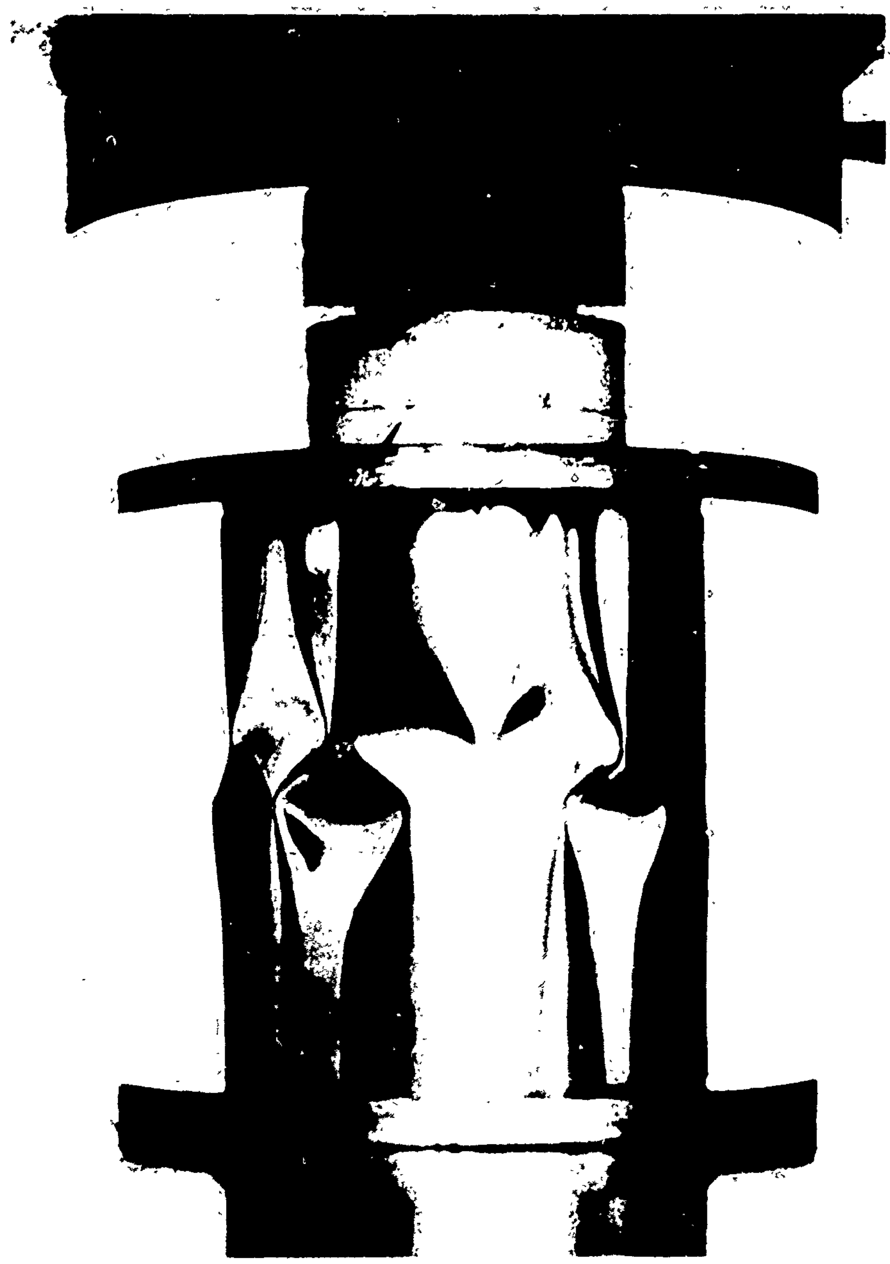

FIG. 4. Photograph of Thin-Walled Shell after Buckling.

(courtesy of W. H. Horton)

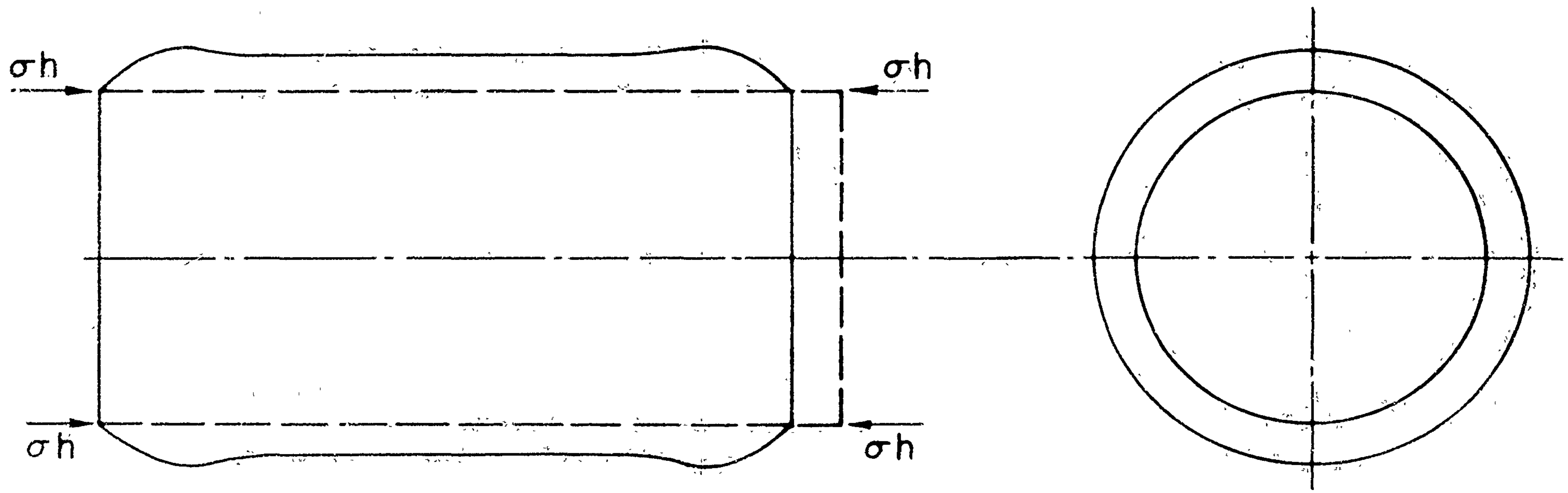

FIG. 5. Axisymmetric Deformation of Shell before Buckling (Dotted line: before load application; full line: after load application) (Displacements grossly exaggerated).

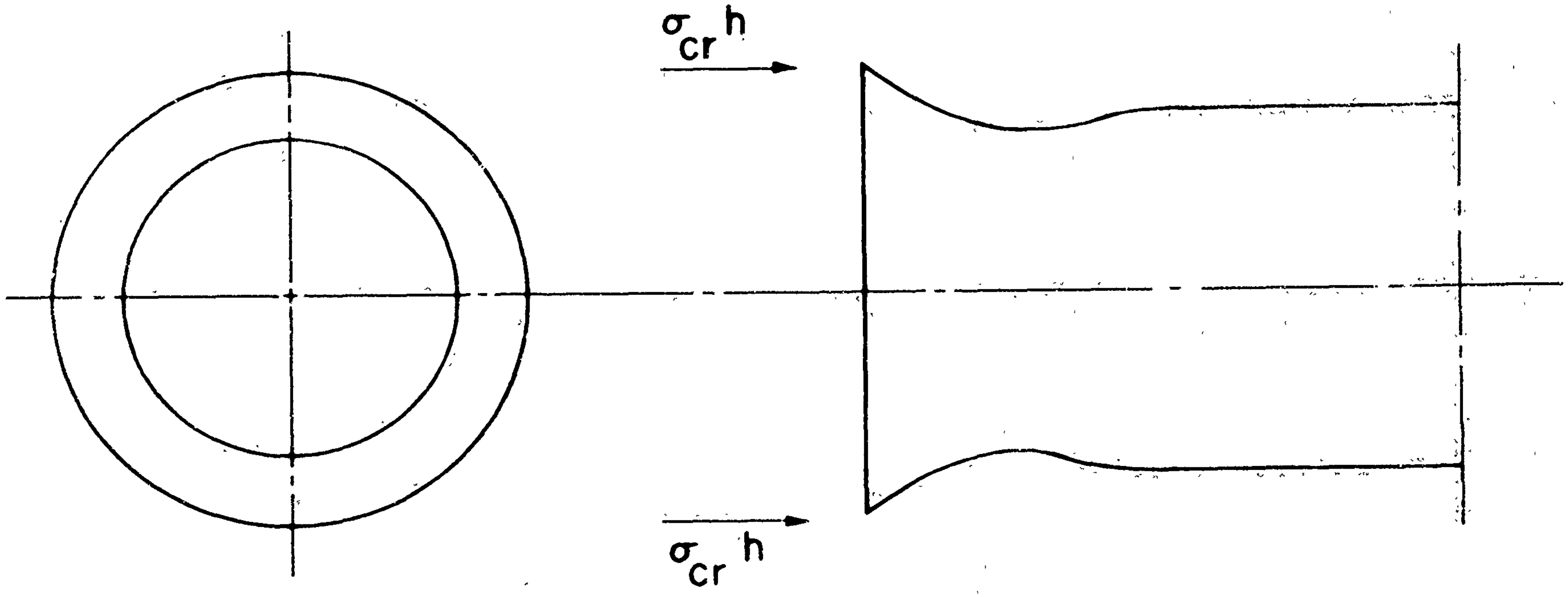

FIG. 6. Symmetric Buckling of Free Edge of Semi-Infinite Circular Cylindrical Shell Subjected to Uniform Axial Compression (Displacements grossly exaggerated).

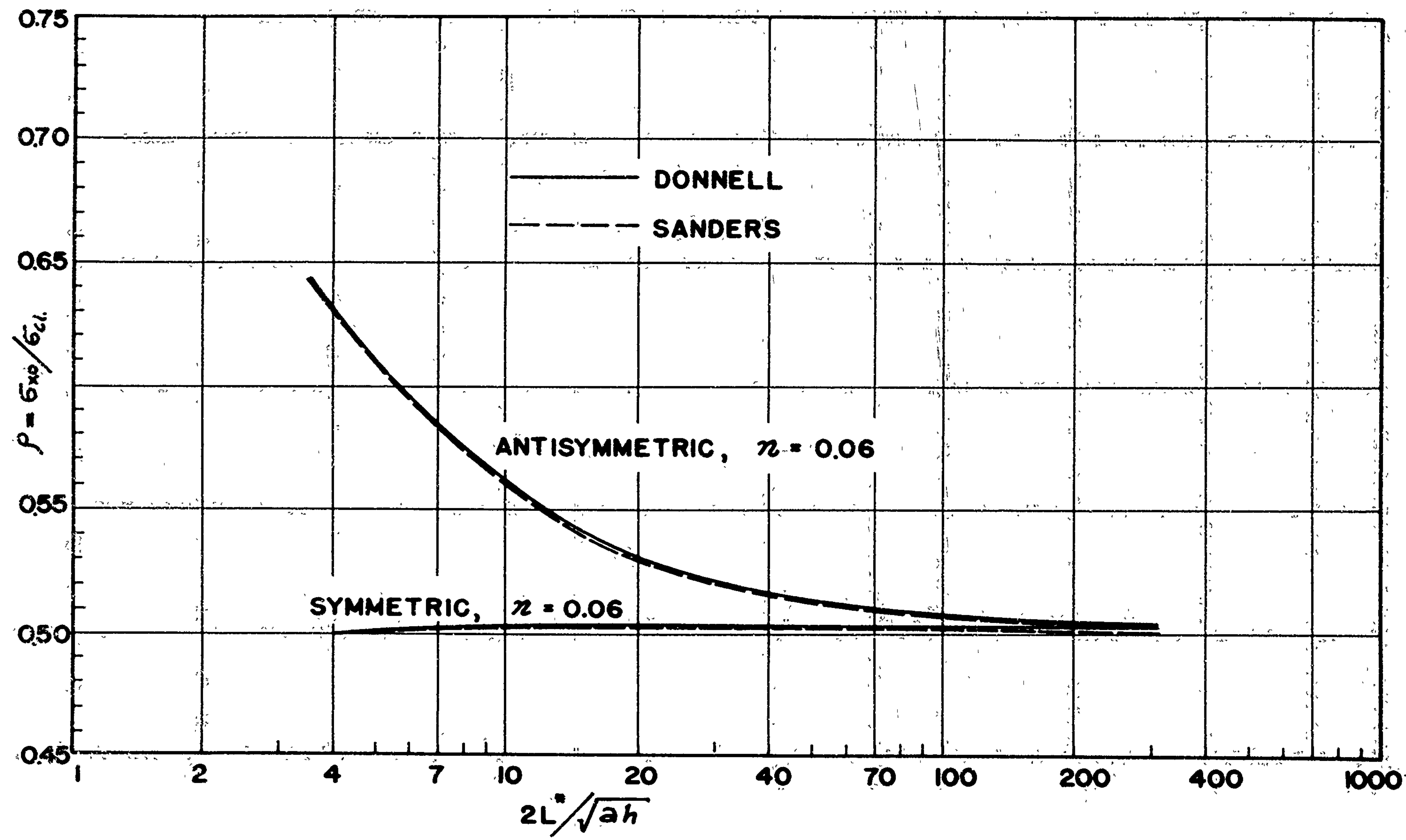

FIG. 7. Comparison of ρ Values Obtained from Donnell's and Sanders' Equations for Case SS1 (From International Journal of Mechanical Sciences)

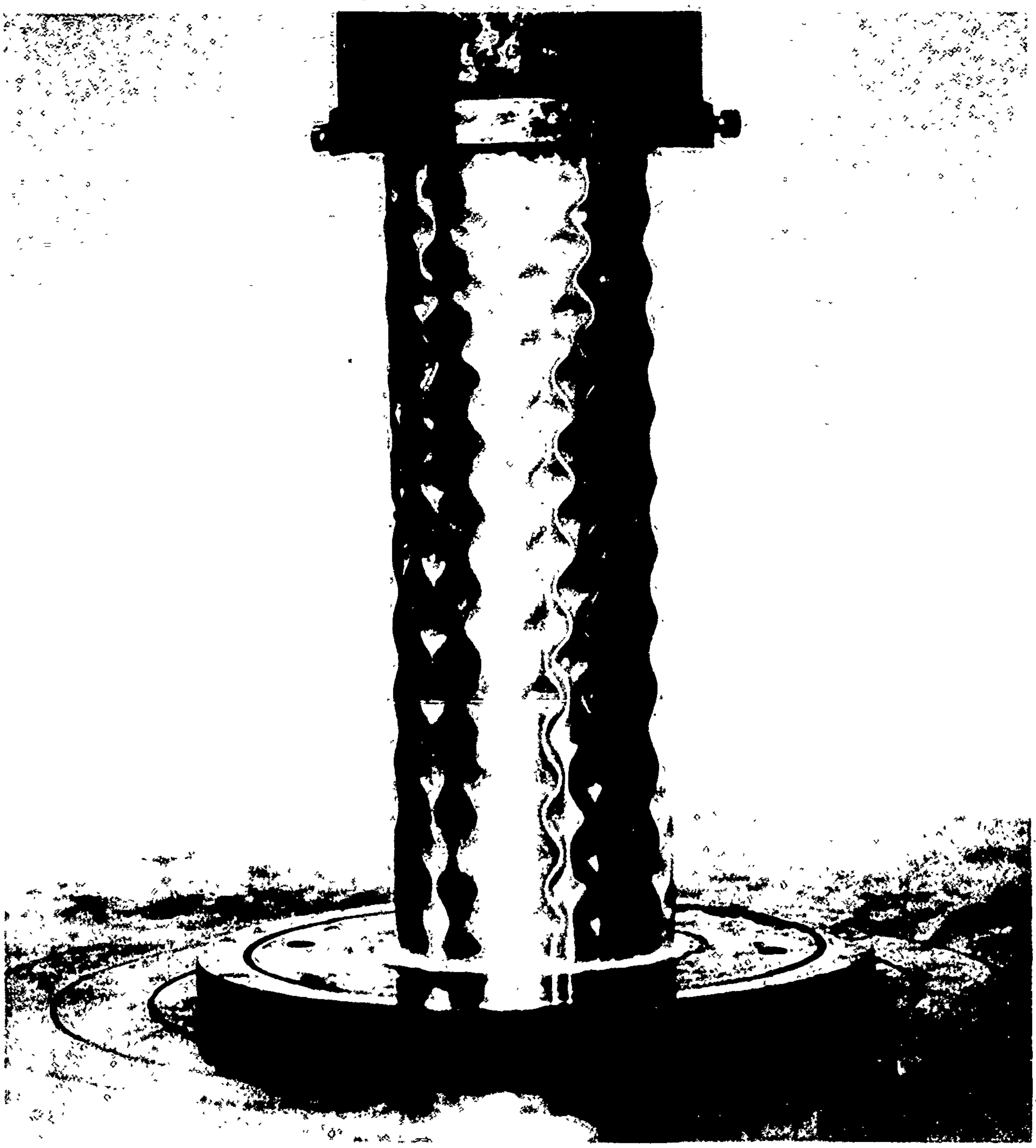

FIG. 8. Buckles Covering the Entire Surface of a Shell Provided with a Close-Fitting Mandrel (From International Journal of Solids and Structures)

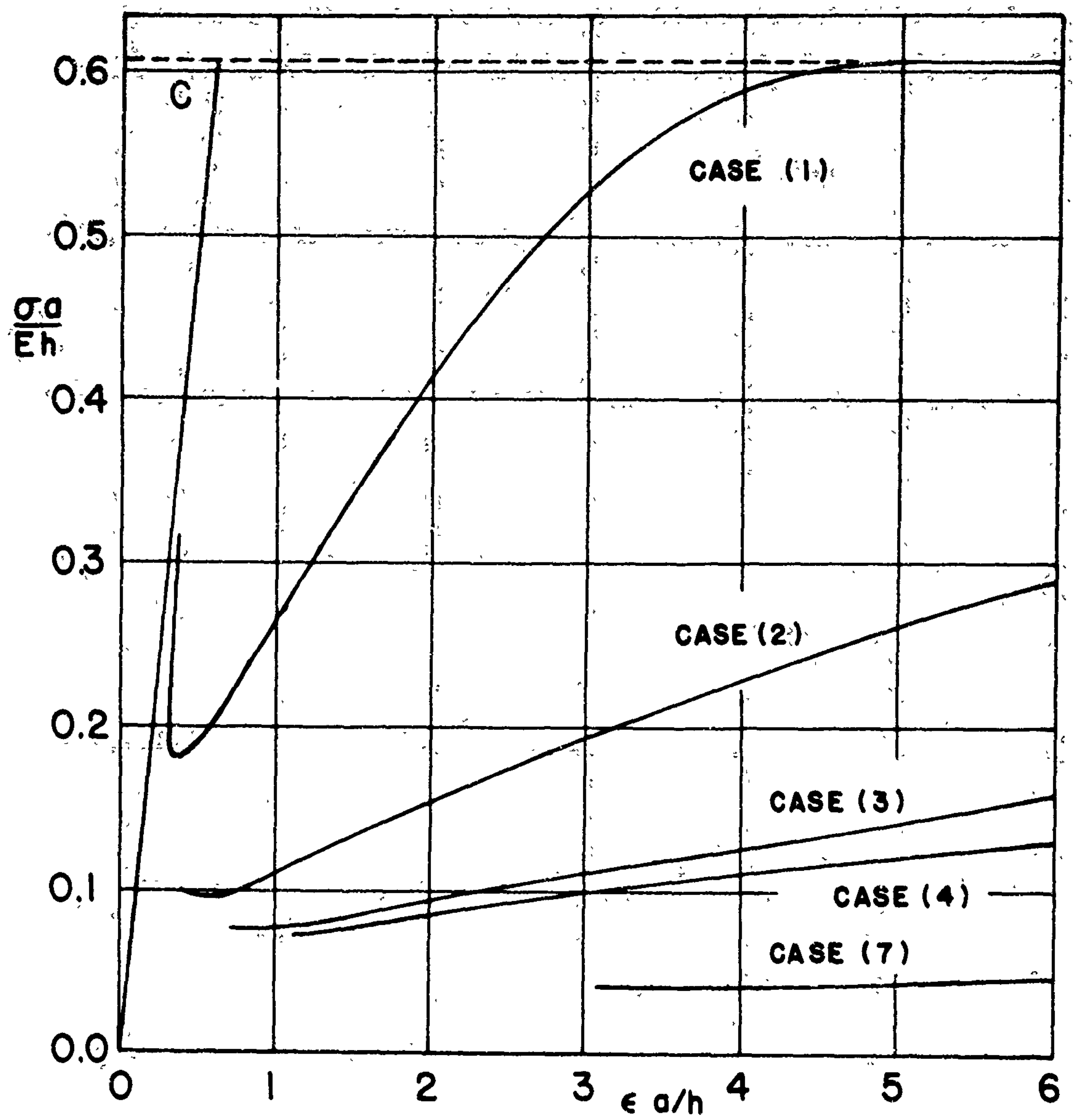

FIG. 9. Postbuckling Curves Calculated by Authors Listed in Table 2 (From SUDAER No. 221).

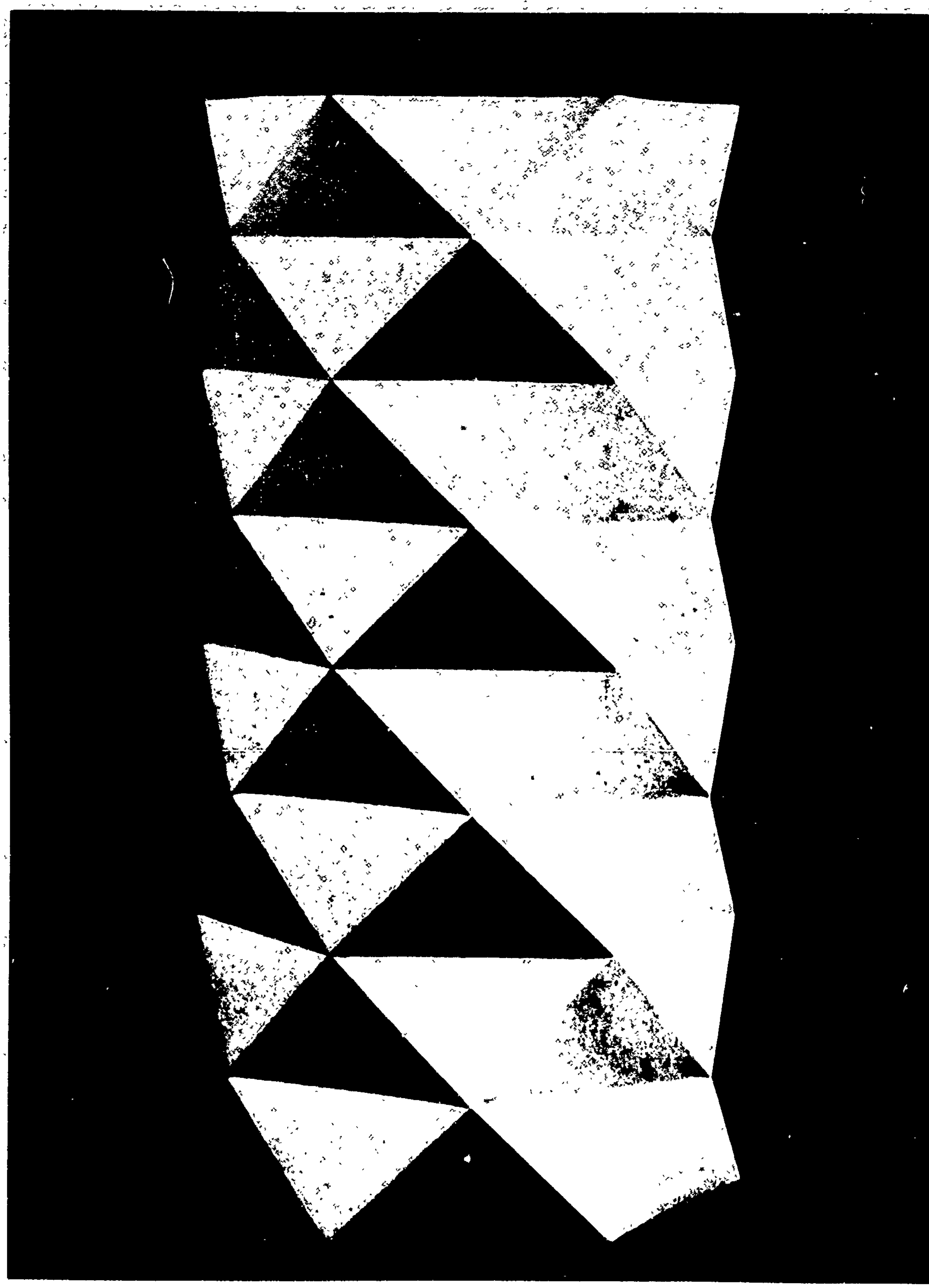

FIG. 10. Yoshimura Buckling Pattern

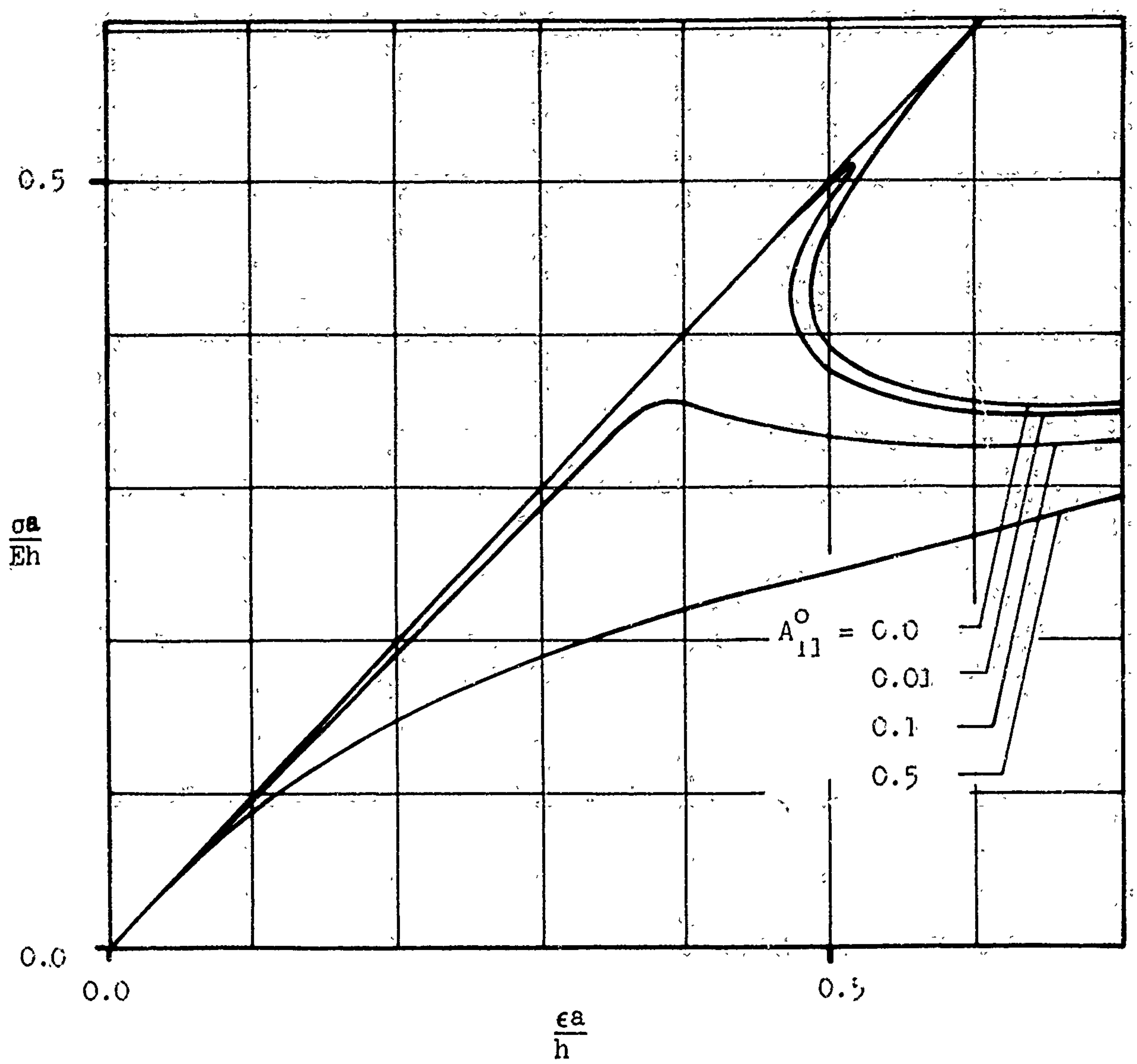

FIG. 11. Postbuckling Behavior of Initially Imperfect Shells (From SUDAER No. 227).

$\mu = 1.0 \qquad \eta = 0.826 \qquad A^o_{20} = A^o_{11}/4$

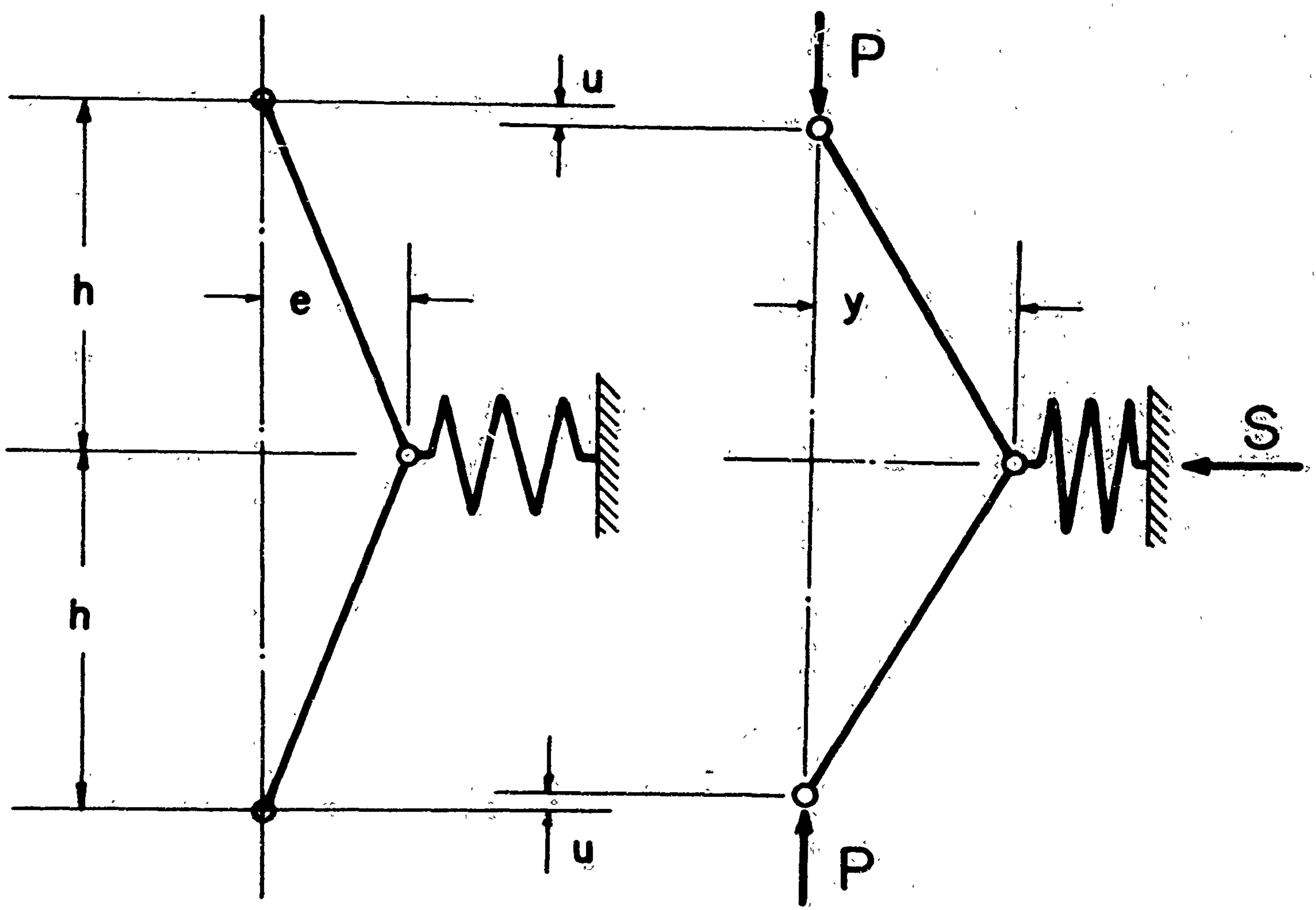

FIG. 12. Mechanical Model.

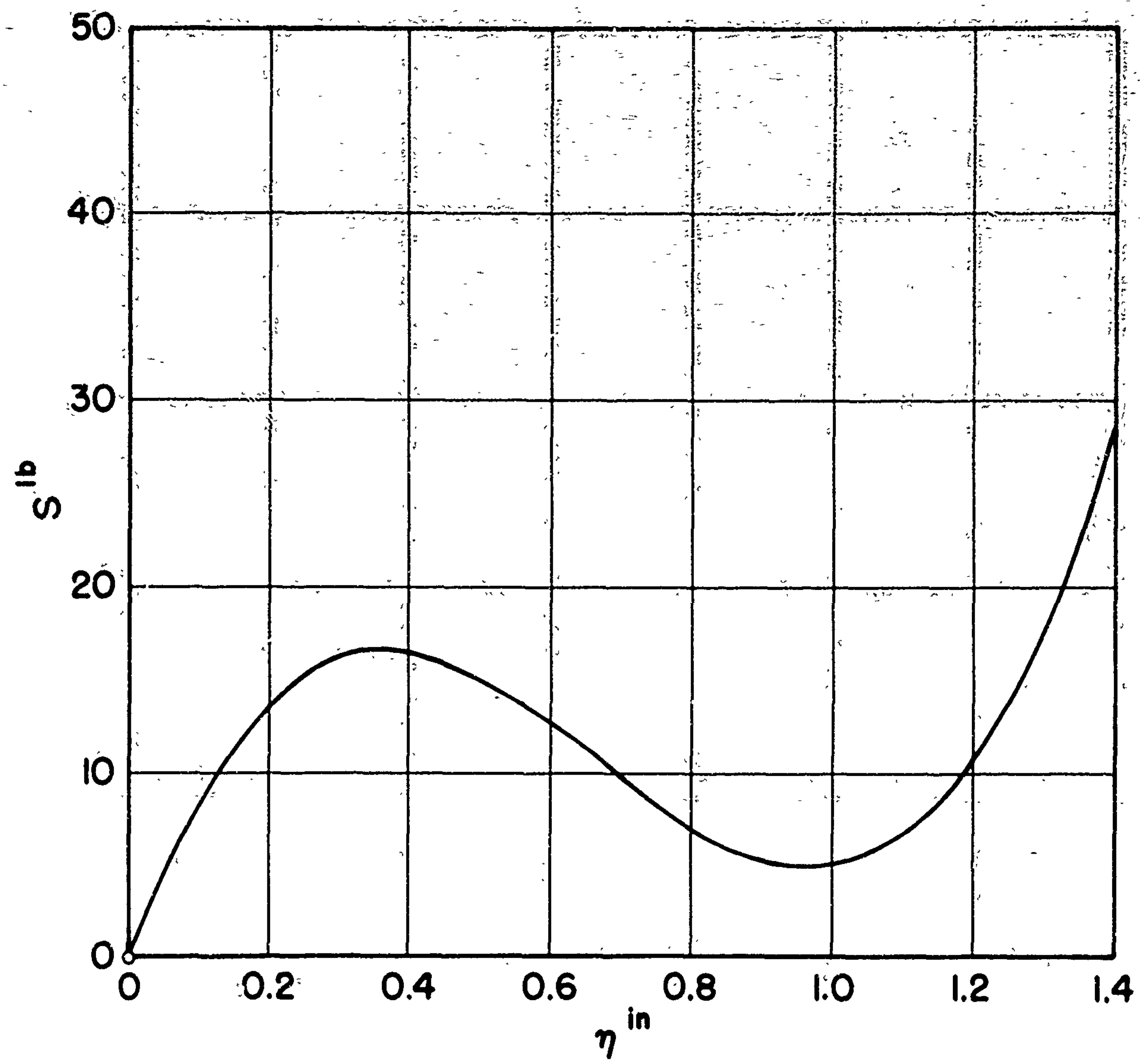

FIG. 13. Characteristic of Nonlinear Spring (From the Journal of Applied Mechanics).

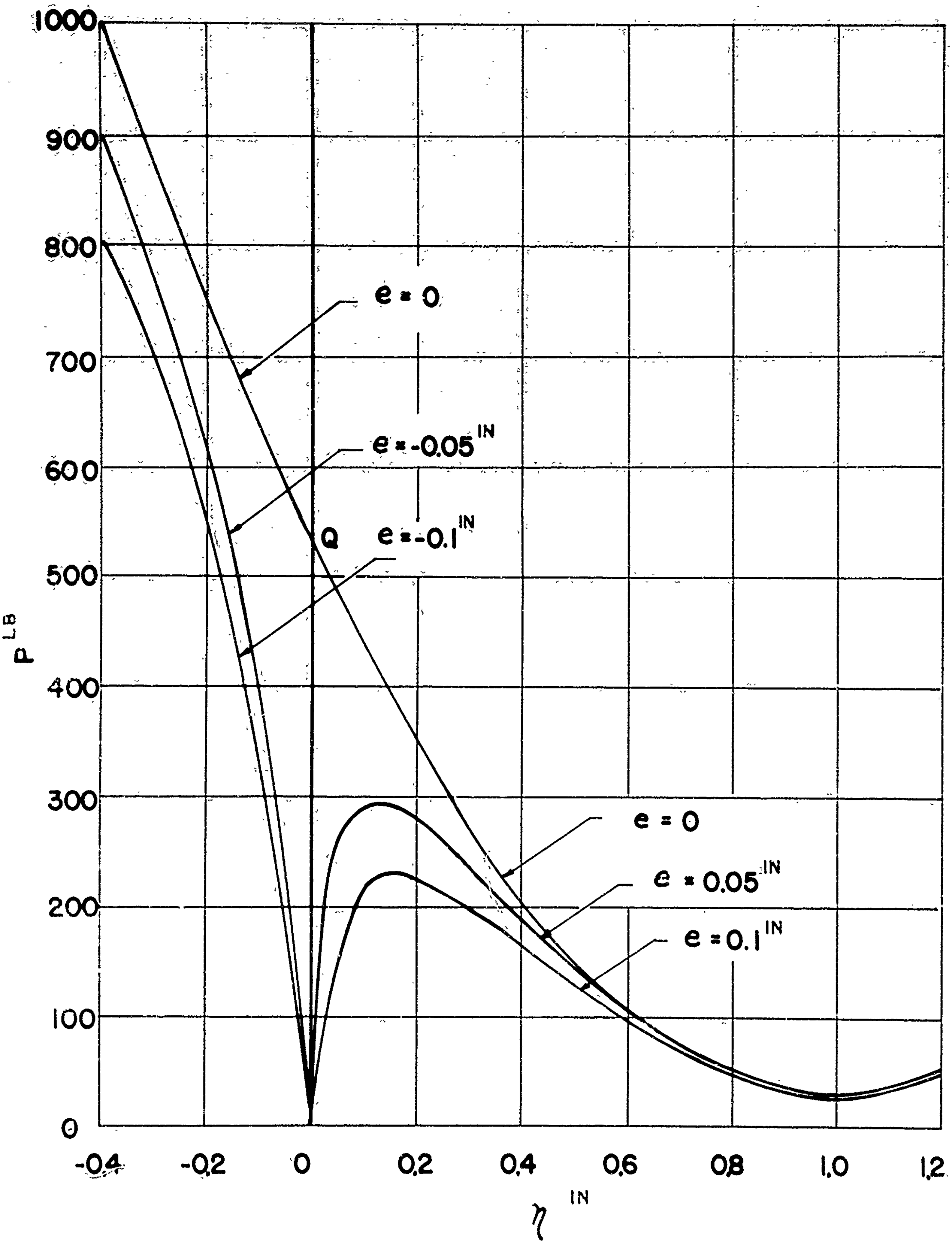

FIG. 14. Load-Displacement Diagram for Imperfection-Sensitive System (Unstable Branching Point) $S = 100\eta^3 - 200\eta^2 + 105\eta$.

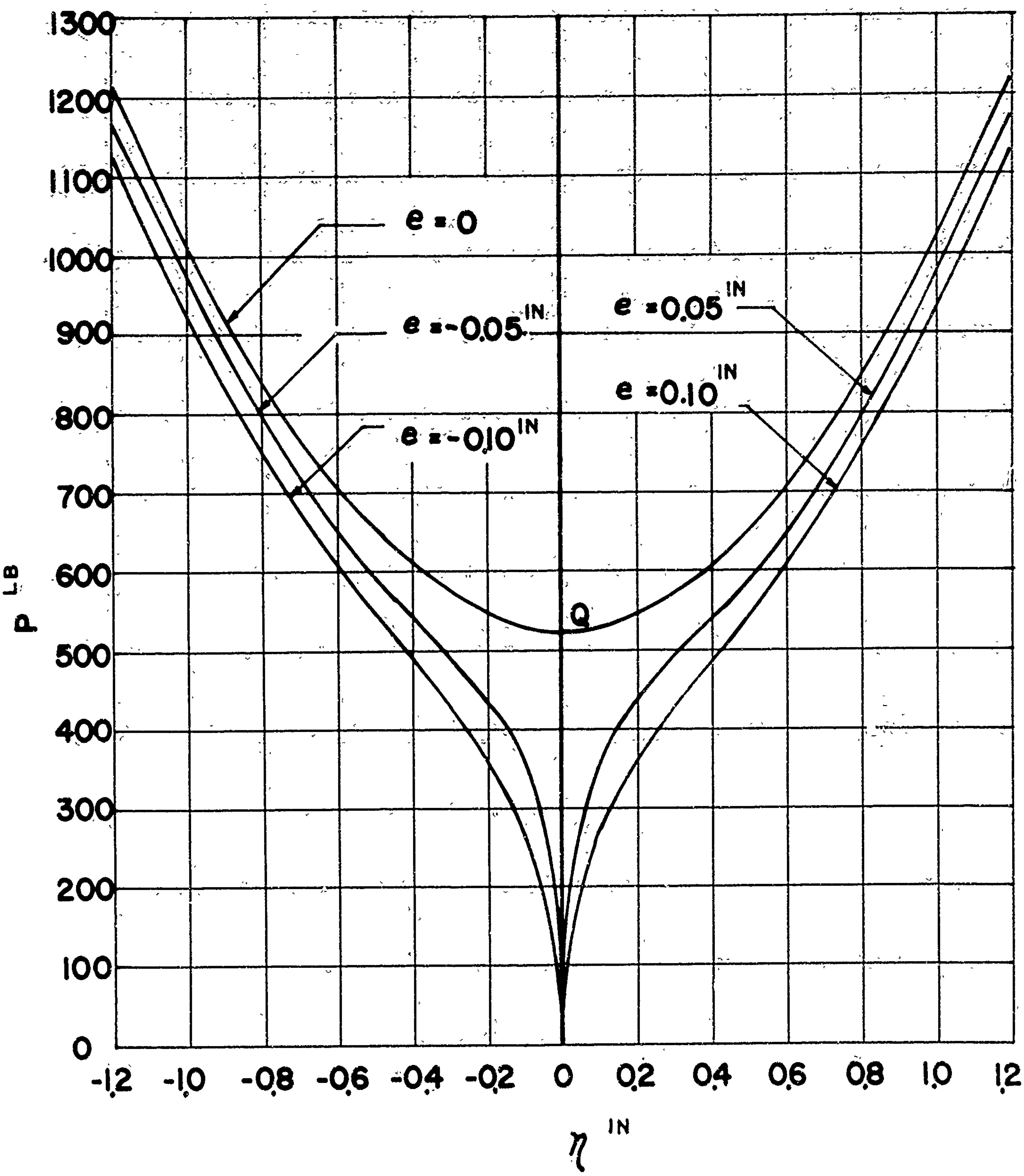

FIG. 15. Load-Displacement Diagram for Imperfection-Insensitive System (Stable Branching Point) $S = 100\eta^3 + 105\eta$.

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13. ABSTRACT

The development of our knowledge of the buckling of thin-walled circular cylindrical shells subjected to axial compression is outlined from the beginning of the century until the present, with particular emphasis on advances made in the last twenty-five years. It is shown that practical shells generally buckle under stresses much smaller than the classical critical value derived by Lorenz, Timoshenko, Southwell and Flügge. A first explanation of the reasons for the discrepancy was given by Donnell and the problem was explored in detail by von Kármán, Tsien and their collaborators. More recently, Yoshimura discovered the existence of an inextensional displacement pattern which the wall of the shell can suddenly assume, and Koiter found an explanation of the sensitivity of the buckling stress to small initial deviations from the exact circular cylindrical shape.

In the last few years further interesting discoveries were made in Japan and in California regarding the effects of details of the boundary conditions, and many additional numerical results were obtained with the aid of high-speed electronic digital computers. Improvements in experimental techniques have also contributed significantly to a clarification of the problem and to an establishment of the unavoidable deviations from the exact shape as the major causes of the large differences between theory and experiment.

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