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Energy Identities for the Wave Equation

Cathleen S. Morawetz

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ABSTRACT

In [1] and [2] identities for the wave equation similar to the energy identity were derived by showing that if u is sufficiently smooth, there exist linear first order operators Nu such that $Nu \Box u$, with $\Box = \partial^2/\partial t^2 - \Delta$, can be written as the divergence of a vector. Thus if $\Box u = 0$ a certain surface integral in $(\vec{x}, t)$ space vanishes identically and this yields the identity. One of these operators is $2rtu_r + (r^2+t^2)u_t + 2tu$, another is $ru_r + tu_t + u$ with $r = |\vec{x}|$. For the familiar energy identity $Nu = u_t$.

In Part I of this report these identities will be rederived for three space variables by noting that certain transformations, in particular the Kelvin transformation, leave the wave operator invariant and hence the classical energy identity can be transformed into other identities.

In Part II, the Kelvin transformation and the resulting identity are applied to incoming and outgoing waves as defined by Lax and Phillips [3,4].

In Part III the main theorem of the first part is used to prove the following result in geometrical optics: Suppose that we are given a smooth, star-shaped perfectly reflecting, three-dimensional body that extends to infinity and that a high frequency harmonic source of light illuminates the region outside the body in such a way that no shadow is cast. The field is given by a solution of a boundary value problem for the reduced wave equation. There is also an approximate solution given by geometrical optics. The theorem states that these two are asymptotically equal in the limit of infinite frequency for the harmonic source.

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ERRATA SHEET

PAGE	LINE	CHANGE
2	$\ell 9$	$((u_t^2 + (\nabla u\|^2)$ should read $((u_t^2 + (\|\nabla u\|^2)$
2	$\ell 4b$	$- u_t\, u_n$ should read $- 2u_t\, u_n$.
12	$\ell 8b$	$\|X\| < k/2$ should read $r' < k/2$
12	$\ell 7b$	$\|X\| \geq k/2$ should read $r' \geq k/2$
12	$\ell 6b$	$t = 0$ should read $t' = 0$
14	$\ell 8$	$i\omega^{-1}(\rho-t)\delta(\rho-\tau)\rho^{-1}$ should read $i\omega^{-1}(\rho-t)\delta(\rho-t)\rho^{-1}$
14	$\ell 9$	$\tau = 0$ should read $t = 0$
14	$\ell 11$	$\rho = \tau$ should read $\rho = t$
15	$\ell 5$	$= \infty$ should be omitted
15	$\ell 3b$	$\omega \int_\tau^\infty u^t e^{i\omega t} dt$ should read $\omega \int_\tau^\infty u_t e^{i\omega t} dt$
19	$\ell 5$	S should read δ
19	$\ell 8$	R should read $\mathcal{R}$
19	$\ell 11$	R should read $\mathcal{R}$
19	$\ell 5b$	$\int_R$ should read $\int_{\mathcal{R}}$
19	$\ell 1b$	$= \infty.\ K(\phi) < \infty$ should read $= K(\phi) < \infty$

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In [1] and [2] identities for the wave equation similar to the energy identity were derived by showing that if u is sufficiently smooth, there exist linear first order operators Nu such that $Nu\,\Box u$, with $\Box = \partial^2/\partial t^2 - \Delta$, can be written as the divergence of a vector. Thus if $\Box u = 0$ a certain surface integral in $(\vec{x},t)$ space vanishes identically and this yields the identity. One of these operators is $2rtu_r + (r^2+t^2)u_t + 2tu$, another is $ru_r + tu_t + u$ with $r = |\vec{x}|$. For the familiar energy identity $Nu = u_t$.

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In Part III the main theorem of the first part is used to prove the following result in geometrical optics: Suppose that we are given a smooth, star-shaped perfectly reflecting, three-dimensional body that extends to infinity and that a high frequency harmonic source of light illuminates the region outside the body in such a way that no shadow is cast. The field is given by a solution of a boundary value problem for the reduced wave equation. There is also an approximate solution given by geometrical optics. The theorem states that these two are asymptotically equal in the limit of infinite frequency for the harmonic source.

I. Energy Identities

1. The classical energy identity.

Lemma 1. If u has second derivatives in L^2 in D then

$$2\iint_D u_t \;\Box u|dx| = \int_{\dot D} ((u_t{}^2+(\nabla u|^2)t_n - 2u_t\, u_n\, x_n)d\sigma$$

where t_n, x_n are the components of the unit normal in time and space and u_n is the derivative in the direction of the outward space normal to $\dot D$.

For future reference we note:

a) On that part of $\dot D$ for which t is constant the integrand is $u_t^2 + |\nabla u|^2$.

b) On that part of $\dot D$ which is independent of time the integrand is $- u_t\, u_n$.

c) On that part of $\dot D$ which is characteristic, $x_n = \pm t_n$ the integrand is $\frac{1}{\sqrt{2}}(u_s^2 + (|\nabla u|^2 - u_n^2))$ where u_s is the derivative along the bicharacteristic in $\dot D$, u_n is the outward space normal derivative. Thus the integrand involves only derivatives in the surface.

It is convenient to have the three-dimensional wave operator in polar coordinates (r,θ,ϕ) and to introduce $w = ru$ as dependent variable. Setting $\Lambda = \sin^{-2}\theta\partial^2/\partial\phi^2 + \sin^{-1}\theta\partial(\sin\theta\partial/\partial\theta)/\partial\theta$, we find

$$\Box u = r^{-1} Lw = r^{-1}(w_{tt} - w_{rr} - r^{-2}\Lambda w)$$

By substitution in the energy identity one can find an identity for w. A more convenient expression involving only derivatives of w is obtained by adding to $u_t \Box u = r^{-2}w_t Lw$ a divergence expression which vanishes identically. In this case we add $(r^{-1}w^2)_{rt} - (r^{-1}w^2)_{tr}$. We then obtain

<u>Lemma 2</u>. If u has second derivatives in L^2 then $w = ru$ satisfies

$$2\int_{\mathcal{P}} r^2 u_t \Box u \, dv = 2\int_{\mathcal{P}} w_t Lw \, dv = \int_{\dot{\mathcal{P}}} \left\{ (w_t^2 + |\nabla w|^2)t_n - 2w_t w_n x_n \right\} d\sigma$$

Here $\mathcal{P}$ is an arbitrary domain in t,r,θ,ϕ space, dv, $d\sigma$ are volume and surface elements, w_n is the normal derivative of w in the space direction. Also

a) On that part of $\dot{\mathcal{P}}$ with t constant, the integrand is $w_t^2 + |\nabla w|^2$.

b) On that part of $\dot{\mathcal{P}}$ invariant in time the integrand is $-2w_t w_n$.

c) On that part of $\dot{\mathcal{P}}$ that is characteristic the integrand is $\frac{1}{\sqrt{2}}(w_s^2 + (|\nabla w|^2 - w_n^2))$, where w_s is the derivative along the bicharacteristic.

d) Where $\dot{\mathcal{P}}$ is space-like the integrand is positive definite in u if t_n is positive.

2. <u>The Kelvin Transformation.</u>

We consider the Kelvin transformation

$$(1) \qquad \vec{x}' = \vec{x}/(r^2 - t^2),\ t' = t/(r^2 - t^2) \text{ or } r' = r/r^2 - t^2$$

leaving θ, ϕ unchanged. It maps the exterior of the cone $r^2 = t^2$ into itself taking the origin into infinity and the point $(\infty, 0)$ into the origin. The cone $r^2 = t^2$ is mapped into a "cone at infinity" since $r^2 - t^2 = (r'^2 - t'^2)^{-1}$. The cones $r \pm t = k$ are mapped into the cones $r' \mp t' = 1/k$. Thus the "cone at infinity" orthogonal to $r - t = k$ is mapped into $r' - t' = 0$. To find how the operator L transforms we note that Λ is invariant and $\partial/\partial t \pm \partial/\partial r = (r' \mp t')^2$

$(\partial/\partial t' \mp \partial/\partial r')$. Hence $\partial^2/\partial t^2 - \partial^2/\partial r^2 = (r'^2 - t'^2)^2$ $(\partial^2/\partial t'^2 - \partial^2/\partial r'^2)$. Thus $r\Box u = Lw = (r'^2 - t'^2)L'w$ in obvious notation. Thus, setting $u' = w/r'$ we see that if $u = w/r$ satisfies $\Box u = 0$ then u' satisifes* $\Box' u' = 0$.

Futhermore, Lemma 1 or 2, the energy identity can be applied to u' in the primed space in the image of $r^2 > t^2$, i.e. in any subdomain of $r'^2 > t'^2$, and then transformed back into the unprimed space to yield an identity for $u = \frac{r'}{r} u'$ in the case of Lemma 1 or directly for w in the case of Lemma 2. However it is awkward to transform surface integrals from the primed to the unprimed variables. We note instead the transformation of the volume integral

$$I = \int_{D'} u_{t'} \Box u' dv' = \int_{\mathcal{P}'} w_{t'} Lw' \frac{dv'}{r'^2} = \int_{\mathcal{P}} (2rtw_r + (r^2+t^2)w_t) Lw \, r^{-2} dv$$

$$= 2\int (2t\vec{x}\cdot\nabla u + (r^2+t^2)u_t + 2tu) \Box u dv$$

Furthermore

$$2tx\cdot\nabla u \Box u - (2rtu_t(\vec{x}\cdot\nabla u))_t + \text{div}\,(tu_t^2\vec{x}) + 2\,\text{div}\,(t(\vec{x}\cdot\nabla u)\nabla u)$$

$$- \text{div}\,(t|\nabla u|^2\vec{x})$$

*See [5].

is a quadratic form in ∇u, u_t as is

$$(r^2+t^2)u_t \Box u - \frac{1}{2}((r^2+t^2)(u_t^2+|\nabla u|^2))_t + \text{div}((r^2+t^2)u_t\nabla u)$$

and also

$$2tu\Box u - 2(tuu_t)_t - \text{div}(2tu\nabla u) + (u^2)_t.$$

In general a quadratic form in $\nabla u, u_t$ cannot be a divergence expression. It follows therefore that when the three above expressions are added the sum must vanish since we know, from the primed space, that the sum must be a divergence. Thus we obtain

<u>Lemma 3</u>. If u has second derivatives in L^2 then

$$\int_D (2t\vec{x}\cdot\nabla u + (r^2+t^2)u_t+2tu)\Box u\, dv = \int_{\dot D} \Big\{\Big\{ 2tu_t\vec{x}\cdot\nabla u + \frac{1}{2}$$

$$((r^2+t^2)u_t^2+(\nabla u)^2) + 2uu_t - u^2 \Big\} t_n - \Big\{ t(u_t^2 - (\nabla u)^2)\vec{x}\cdot\vec{n}$$

$$+ [2t(\vec{x}\cdot\nabla u) + (|x|^2+t^2)u_t+2tu]u_n \Big\} x_n \Big\} ds$$

where $\vec{n}$ is the unit space normal out of the surface.

It is also clear that the restriction in the derivation of this result to the domain $r^2 \geq t^2$ may be dropped.

We can obtain an identity for w corresponding to Lemma 2 by adding the appropriate divergence expression which in this case is $\int_{D'} \{-\frac{1}{2}(w^2r^{-1}(r^2+t^2))_{rt} + \frac{1}{2}(w^2r^{-1}(r^2+t^2))_{tr}\}$ $d\theta d\phi dr dt$ and we obtain

Lemma 4. If w has second derivatives in L^2 then

$$\int (2rtw_r+(r^2+t^2)w_t)Lw\,\frac{dv}{r^2} = \int\Big\{\Big\{\frac{1}{4}(r+t)^2(w_r+w_t)^2 + \frac{1}{4}(r-t)^2$$

$$(w_r-w_t)^2 + \frac{1}{2}(r^2+t^2)(|\nabla w|^2 - w_r^2)\Big\}t_n - \Big\{t(w_t^2 - |\nabla w|^2)$$

$$\vec{x}\cdot\vec{n} + (2trw_r + (r^2+t^2)w_t)w_n\Big\}x_n\Big\}\frac{dS}{r^2}$$

This identity was derived in [2].

The surface integrand is positive definite for space-like surfaces with $t_n > 0$ and involves only surface derivatives on characteristic surfaces.

The last statement can be derived by noting that the integrands of the surface integrals in both Lemmas 2 and 4 involve only derivatives of w. Now $\int w_{t'}L'w\,\frac{dv'}{r'^2} =$ $\int (2rtw_r + (r^2+t^2)w_t)Lw\,\frac{dv}{r^2}$. Hence the difference between the two corresponding surface integrals vanishes identically. If the integral over S' is transformed directly into an integral over S we see that we have then two equal integrals over S whose integrands are quadratic forms in ∇w and w_t. Hence we would have an identically vanishing integral over S whose integrand is of the form $\vec{Q}\cdot\vec{n}x_n + Pt_n$ where $\vec{Q}$, P are quadratic forms in ∇w, w_t. Thus $\int (\mathrm{div}\,\vec{Q} + P_t)dv$ vanishes identically. But since $\vec{Q}$ and P are quadratic forms in ∇w, w_t it follows that $\vec{Q} \equiv 0$ and $P \equiv 0$. Hence the integrands of the two surface integrals are the same. Hence the surface integral of Lemma 4 is what we would have obtained by transforming directly on the surface integral of Lemma 2.

If the surface in the unprimed space is characteristic the image in the primed space is also characteristic and the integrand of Lemma 2 involves only characteristic derivatives in the primed space and hence after transformation only characteristic derivatives in the unprimed space as we wanted to show.

4. Other identities.

It might now appear that we could obtain still another identity by applying the Helmholtz transformation T to the primed space. However T^2 is the identity transformation and hence no new information is obtained.

Every invariant transformation leads to an identity but no other new identity can be derived from transforming the classical energy identity. For, the remaining invariant transformations are a translation, rotation or stretching of the coordinates. From the identity in Lemma 1 it is clear that a translation produces the same identity and from Lemma 2 a rotation does the same. Stretching the variables $\vec{x}' = k\vec{x}$, $t' = kt$ also leaves the identity of Lemma 1 invariant.

However when a translation or rotation is applied to the coordinates one obtains from Lemma 4 new identities. The most interesting comes by taking $t' = t+c$ in the identity of Lemma 4 and equating coefficients of c. Thus one finds

$$\iint w(rw_r+tw_t)Lw\, r^{-2}dv = \int \{ (\tfrac{1}{2}(r+t)(w_r+w_t)^2 + \tfrac{1}{2}(r-t)(w_r-w_t)^2$$

$$+ t(|\nabla w|^2 - w_r^2))t_n - ((w_t^2 - |\nabla w|^2)\, \vec{x}\cdot\vec{n}$$

$$+ 2(rw_r + tw_t))w_n)xn \} dS.$$

This identity was used in [1].

A shift of the origin in space or a rotation of the axes also lead to new identities. However stretching the variables plainly leaves these identities invariant.

5. Hyperbolic systems.

A somewhat similar principle can be applied to a vector solution u of the symmetric hyperbolic system

$$u_t = A^x u_x$$

where $x = (x_1, \ldots x_n)$ and $A^x = (A^{x_1}, A^{x_2} \ldots)$ are constant matrices.

Energy conservation yields, over any domain D,

$$0 = \int_D 2u(u_t - A^x u_x)dv = \int_D (u^2 t_n - u(A^x \cdot x_n)u)d\sigma$$

where t_n is the time component of the normal and x_n is the space component.

This system is invariant if length and time are stretched. Hence if $u(x,t)$ is a solution, so is $u(kx,kt)$ and hence

$\left.\frac{\partial u}{\partial k}\right|_{k=1} = \vec{x} \cdot \nabla u + t u_t$ is also a solution.

Applying the identity one obtains:

$$0 = \int \left\{ (\vec{x} \cdot \nabla u + t u_t)^2 t_n - (\vec{x} \cdot \nabla u + t u_t)(A^x x_n)(\vec{x} \cdot \nabla u + t u_t) \right\} d\sigma .$$

From this one may, for example, conclude that if u has initially compact support then in any finite region the $\int u_t^2 |dx|$ decays like $\frac{1}{t^2}$ since $\int_{t=T} (\vec{x} \cdot \nabla u + t u_t)^2 d\sigma$ is bounded.

This method can in fact be used on the wave equation and the identity of Lemma 4 obtained after some further integration by parts.

II. Application of the Kelvin transform to incoming and outgoing waves.

In [3,4] an outgoing solution u of the wave equation in free space is defined as one which vanishes identically for $r \leq t+k$, $t \geq 0$. By the Kelvin transformation, $r = r'/r'^2-t'^2$, $t = t'/r'^2-t'^2$ and thus the corresponding solution $u' = \frac{ru}{r'}$ to an outgoing solution vanishes for $r' \geq -t'+k^{-1}$, $t' \geq 0$.

We introduce the Hilbert space of Cauchy data $X = (\psi,\phi)$ found by the closure of data of compact support in the unprimed space. The norm is the energy norm $\|X\| = \int (\phi^2 + |\nabla\psi|^2)|dx|$. The corresponding Cauchy data for a solution w of $Lw = 0$ will be $(r\psi, r\phi)$ and the identity of Lemma 2 suggest the norm $\|X\|_w = \int ((r\phi)^2+|\nabla(r\psi)|^2) \frac{dx}{r^2}$ which we call the w-norm. Since the space of data is found by closing the space of data of compact support these two norms are the same.

On the other hand the Cauchy data (ϕ,ψ) are carried by the Helmholtz transformation into the data

(ψ', ϕ') with $\psi' = \frac{1}{r^1} \psi \left(\frac{x'}{r'^2}, \frac{t'}{r'^2}\right)$, $\phi = \frac{1}{r'^4} \phi \left(\frac{x'}{r'^2}, \frac{t'}{r'^2}\right)$

but these data do not necessarily form an element of the H'-space, i.e. Cauchy data of finite energy norm in the primed space.

However there do exist subclasses of data in the space H'.

It is also clear that any set of data can be split into its outgoing part and orthogonal complement by making use of the geometry of the primed space. In the primed space the outgoing wave vanishes on the cone $r'+t' = u$ and the orthogonal complement vanishes in $r' < t'$. The two components of the initial data are found by the following algorithm. First solve in the primed space the Cauchy problem for the corresponding data up to the time $t' = k/2$. Call the data at $t' = k/2$, X_1. Then solve two Cauchy problems backwards in time where the first problem has the same data as X_1 for $|X| < k/2$ and zero outside and the second problem has the same data as X_2 for $|X| \geq k/2$ and zero inside. The two sets of corresponding data at $t = 0$ are the outgoing part and the orthogonal complement. The first set yield the data of the outgoing component.

To make the argument rigorous we note the following. We apply the identity of Lemma 4 in the primed space to the slab $0 \leq t' \leq k/2$ and replace w by aw_1+bw_2. The coefficient of ab in this identity gives us a "scalar product" identity in the primed space. Now if w_1 and w_2 are the two solutions to the two Cauchy problems described above this identity reduces because of the choice of data on $t' = k/2$ to

$$0 = \int_{t'=0} r'^2(\nabla' w_1 \cdot \nabla' w_2 + w_{1t'} w_{2t'})dv'$$

If this is transformed to the unprimed space we have

$$0 = \int_{t=0} (\nabla w_1 \cdot \nabla w_2 + w_{1t}\, w_{2t})dv$$

or the two sets of data are orthogonal as required.

III. Geometrical optics with no shadow.

We want finally as an application to show that geometrical optics yields asymptotic solutions to the reduced wave equation. The problem we consider is the following. Let V be the outgoing solution of

$$\Delta V + \omega^2 V = \delta(x-a)$$

which vanishes on B where B is star-shaped with respect to a point inside it which we take as origin. We shall show that V is given by the Fourier transform $\int_0^\infty ue^{i\omega t}dt$ where u satisfies

$$\Box u = 0,\ u = 0 \text{ on } B,\ u_t = 0,\quad u = \frac{-1}{i\omega}\,\delta(\vec{x}-\vec{a}),\quad \text{for } t = 0.$$

Furthermore this transform in turn is asymptotic to the formal expansion in ω found on taking out the contributions due to the singularities of u.

Consider the solution u described above. It is given by $i\omega^{-1}(\rho-t)\delta(\rho-\tau)\rho^{-1}$, where $\rho = |\vec{x}-\vec{a}|$, in the region bounded by $\tau = 0$, the body cylinder B generated by the body in time and a characteristic surface, R, formed by the reflected rays of the cone $\rho = \tau$. Across this characteristic surface u will have a singularity of the same type; that is, if D is the interior of the reflection surface R, then in the complement of D exterior to B, the solution is given by

$$(1)\qquad u = i\omega^{-1}\left\{(\rho-t)\delta(\rho-t)\rho^{-1}+X_1^*(\xi)\delta(\xi)+X_2^*S_2(\xi)+\dots\right\}$$

Here X_1^*, X_2^* are smooth functions, ξ is the normal distance from R and S_2 satisfies $\int^{\xi}\xi'\delta(\xi')d\xi' = \delta(\xi)$ etc. For details, see [5]. In the interior of D, u is a solution of the wave equation which vanishes on B and is a smooth function $\omega^{-1}\phi(\vec{x},\vec{t})$ on R. Furthermore on R the integral

$$(2)\quad K(\phi) = \int_R \Big\{\Big\{ \tfrac{1}{4}(r+t)^2((r\phi)_r+(r\phi)_t)^2 + \tfrac{1}{4}(r-t)^2((r\phi)_r-(r\phi)_t)^2$$

$$+ \tfrac{1}{2}(r^2+t^2)(|\nabla(r\phi)|^2 - (r\phi)_r^2)\Big\} t_n$$

$$- \Big\{ t((r\phi)_t^2 - |\nabla(r\phi)|^2)\vec{x}\cdot\vec{n} + (2tr(r\phi)_r$$

$$+ (r^2+t^2)(r\phi)_t)r\phi)_n \Big\} x_n \Big\} \frac{dS}{r^2}$$

$$= \infty$$

is bounded.

Corresponding statements are true of u_t, u_{tt}, etc.

A modification of Theorem I in [2] which is given in the appendix shows that $\omega \int_\tau t^2 u dt$ converges and is bounded in terms of K. Here $\tau(x,y,z)$ is the value of t where the line $\vec{x}$ = constant cuts the reflection surface R. Hence

$$\int_{\tau(x,y,z)}^{\infty} ue^{i\omega t}dt$$

exists. Similarly $\omega \int_\tau^\infty u^t e^{i\omega t}dt$ exists as do the integrals of u_{tt}. And all these integrals are bounded independent of ω.

We can now evaluate

$$\omega \int_{\tau}^{\infty} ue^{i\omega t} dt = -i \int_{\tau}^{\infty} ude^{i\omega t}$$

asymptotically by integrating by parts. The asymptotic expansion will be of the form $e^{i\omega\tau}\,(X_0^*+\omega^{-1}X_1^*\ldots)$ and the remainder will be of order ω^{-1}. On the other hand $\omega \int_0^{\tau} ue^{i\omega t} dt$ has a similar expansion for $\rho = |\vec{x}-\vec{a}| \neq 0$ by (1). Hence $\int_0^{\infty} ue^{i\omega t} dt$ is asymptotically equal to an expression of the form $e^{i\omega\tau}(X_0+\omega^{-1}X_1\ldots)$ where X_0, X_1 are functions of $\vec{x}$.

We now have

<u>Theorem</u>: If u is a weak solution of $\Box u = 0$ exterior to B and satisfies $u = 0$ on B, $u = i\omega^{-1}\delta(\vec{x}-\vec{a})$, $u_t = 0$ for $t = 0$ then $\int_0^{\infty} ue^{i\omega t} dt$ exists and is asymptotically of the form $e^{i\omega\vec{\tau}(x)}(X_0+\omega^{-1}X_1\ldots)$ where $\tau(\vec{x})$ is the value of t on the characteristic surface formed by the reflection of the cone $|\vec{x}-\vec{a}| = t$ on the body B.

We must finally show that the desired function V is in fact $\int_0^{\infty} ue^{i\omega t} dt$. We note that

$$\omega^2 \int_0^\infty ue^{-i\omega t}dt = -\int_0^\infty u_{tt}e^{-i\omega t}dt + \delta(\vec{x}-\vec{a})$$

$$= -\int_0^\infty \Delta ue^{-i\omega t}dt + \delta(\vec{x}-\vec{a})$$

By the same methods as in [2] and noting the singularity in u at $t = \tau(\vec{x})$ one can show

$$\int_0^\infty \Delta ue^{-i\omega t}dt = \Delta \int_0^\infty ue^{-i\omega t}dt.$$

Hence the integral satisfies the reduced wave equation with the non-homogeneous term $\delta(\vec{x}-\vec{a})$. The integral vanishes on the body B since $u = 0$ there. It remains to show that the solution is outgoing, i.e. that it satisfies the Sommerfeld radiation condition.* Again consider

$$\int_0^\infty ue^{-i\omega t}dt = \int_0^\tau ue^{-i\omega t}dt + \int_\tau^\infty ue^{-i\omega t}dt$$

The first integral consists of two terms $e^{i\omega\rho}/\rho$ and a term of the form $e^{i\omega\tau}X(\tau)$ which separately satisfy the Sommerfeld radiation condition since $\tau \to r$ at infinity and X becomes independent of r. The second integral is split

*A function f satisfies the Sommerfeld radiation condition if $r(f_r - i\omega f) = o(1)$ as $r \to \infty$.

by setting $u = u_I + u_B$ where u_I is the solution of $\Box u = 0$ in D which has the same data as u on R and u_B is the solution given by the retarded potential on B, i.e.

$u_B(\vec{x},t) = \int \frac{1}{\bar{R}} [\frac{\partial u}{\partial n}] d\tau$ with $\bar{R}$ the distance from $(\vec{x},t)$ to the variable point on B and [ ] means the retarded value on B . The same argument, slightly modified, as that in [2] using (vi) of the modified theorem in the appendix shows that $\int_{\tau}^{\infty} u_B e^{-i\omega t} dt$ satisfies the radiation condition. On the other hand $\int_{\tau}^{\infty} u_I e^{-i\omega t} dt$ also satisfies this condition by a similar argument involving the representation of u_I by mean values over R.

Hence V is given by the Fourier transform $\int_0^{\infty} u e^{-i\omega t} dt$ and thus has the desired asymptotic expansion. Substitution of the form in the differential equation yields the expansion explicitly. It could also be computed directly.

Appendix.

We prove here a modification of the main theorem of [2] that is necessary for the results of Part III.

Theorem: Let $\mathcal{B}$ be a strongly star-shaped infinite body which satisfies $\int \frac{1}{R^2}\, d\sigma < \frac{k}{\delta^{\varepsilon}}$ where R is distance from $\vec{x}$ to the point on the body $\mathcal{B}$, S is the minimum distance, ε is a constant, k is a constant, and $d\sigma$ is surface element on $\mathcal{B}$. Let D be the region bounded by the cylinder generated by $\mathcal{B}$ in time and a characteristic surface R whose generators go to infinity as $t \to \infty$: Let u be a smooth solution of $\Box u = 0$ in D which satisfies $u = 0$ on and $u = \phi$ on R where ϕ satisfies

$$\int_R \Big\{ \Big\{ \tfrac{1}{4}(r+t)^2((r\phi)_r+(r\phi)_t)^2 + \tfrac{1}{4}(r-t)^2((r\phi)_r - (r\phi)_t)^2$$

$$+ \tfrac{1}{2}(r^2+t^2)(|\nabla(r\phi)|^2 - (r\phi)_r^2) \Big\} t_n$$

$$- \Big\{ t((r\phi)_t^2 - |\nabla(r\phi)|^2)\vec{x}\cdot\vec{n}$$

$$+ (2tr(r\phi)_r+(r^2+t^2)(r\phi)_t)(r\phi)_n \Big\} x_n \Big\} \frac{dS}{r^2}$$

$$= \infty. \quad K(\phi) < \infty.$$

Then $W = ru$ satisfies the following inequalities

i) $$\int_A \frac{1}{r^2} (|\nabla w|^2 + w_t^2)dv < \frac{K}{2} t^2 \text{ for } t \text{ large enough,}$$

where A is a fixed region in space.

Furthermore

ii) $$|u(\vec{x},t)| < \frac{K}{2\delta^{\varepsilon} t} ,$$

iii) $$\int t^2 (\frac{\partial u}{\partial n})^2 dt\, d\sigma < \frac{K_1}{\alpha} ,$$

where α depends on the shape of the body, K_1 on K and δ is the minimum distance to the body, and

iv) $$\int_0^\infty t^2 q^2(x,t)dt < \frac{K_1}{\alpha\delta}\ 1+\varepsilon$$

v) $$r^{2+\varepsilon} \int_0^\infty t^2 (u_r + u_t)^2 dt < K_1$$

where q is any derivative of u.

This theorem is proved by proving modified forms of Lemmas 1-6. Lemma 1, [2], is another formulation of Lemma 4. The modification is to consider the domain D instead of a slab $0 \leq t \leq t_1$. In Lemmas 2 and 3 the estimates on the right are all replaced by suitable multiples of K. In Lemma 4, u_I is a free space solution with the same data on R instead of the free space initial value solution. In Lemma 5, A/δ^2 is replaced by K/δ^ε and in Lemma 6*, r^2 becomes $r^{2+\varepsilon}$ and A becomes k.

*In [2] the factor r^2 should read r^4. In the proofs of Lemmas 5 and 6 an obvious change in the use of Schwarz' lemma is required.

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13. ABSTRACT

Identities for the wave equation similar to the energy identity were previously derived by showing that if u is sufficiently smooth, there exist linear first order operators Nu such that $Nu\Box u$, with $\Box = \partial^2/\partial t^2 - \Delta$, can be written as the divergence of a vector. In Part I of this report these identities are re-derived for three space variables by noting that certain transformations leave the wave operator invariant and hence the classical energy identity can be transformed into other identities. In part II, the Kelvin transformation and the resulting identity are applied to incoming and outgoing waves. In Part III the main theorem of the first part is used to prove the following result in geometrical optics: Suppose that we are given a smooth, star-shaped perfectly reflecting, three-dimensional body that extends to infinity and that a high frequency harmonic source of light illuminates the region outside the body in such a way that no shadow is cast. The field is given by a solution of a boundary value problem for the reduced wave equation. There is also an approximate solution given by geometrical optics. The theorem states that these two are asymptotically equal in the limit of infinite frequency for the harmonic source.

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