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BEARING ANGLE ESTIMATION OF ATMOSPHERIC SONIC PLANE WAVES USING GROUND ARRAYS

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PREFACE

This Memorandum presents a method of estimating the bearing angle of an incoming plane wave using an arbitrary ground array of sensors. It was prepared for the Advanced Research Projects Agency's VELA Analysis study. The project is a broad and continuing system-oriented study of the detection of nuclear bursts above the ground.

The Memorandum should be useful to those concerned with acoustics and seismology, as well as those interested in data processing.

SUMMARY

This study is concerned with developing data processing techniques to obtain bearing angle estimates of plane sonic waves using arbitrary ground arrays of microphones. The evaluation of the accuracy obtainable as measured by the rms bearing angle error is computed in detail for a 16-station square array. A novel feature of the method is that the ground trace velocity of sound need not be known a priori or measured independently, but can be derived from the same measurements as the bearing angle.

ACKNOWLEDGMENTS

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LIST OF SYMBOLS

A	matrix of unknown coefficients A_j
$\hat{A}$	minimum variance estimate of A
A^*	least-squares estimate of A
$\hat{B}$	covariance matrix of estimate $\hat{A}$
B^*	covariance matrix of estimate A^*
B_2	upper 2 x 2 minor of matrix $B = \{B_{ij}\}$, $i,j = 1,2$; $\sigma^2 \{b_{ij}\}$, $i,j = 1,2$
c	local velocity of sound, $c_o + \frac{dc}{dz} \cdot z$
c_g	effective ground trace velocity, in m/sec
c_o	nominal velocity of sound, 344 m/sec at 20°C
d	distance scale factor, distance between adjacent stations in both the x and y direction in the square array of 16 stations, in meters
$\frac{dc}{dz}$	$\approx$ - 4.4 m/sec/km for $z \leq 10$ km
$E(\ )$	expected value of ( )
I	identity matrix
K_1	wind velocity gradient, in m/sec/km
$L(\theta)_{L.S.}$	normalized rms bearing angle error in rad when $\rho = I$
$L(\theta)_{min}$	normalized rms bearing angle error in rad when $\rho \neq I$
$M(\theta)_{L.S.}$	normalized error when $\rho = I$
$M(\theta)_{min}$	normalized error when $\rho \neq I$
$N + 1$	number of ground stations processed including the reference station; the number of transit times measured is N
n	index of refraction
p	perpendicular distance from station (x, y) to the phase plane of the sound wave incident at the reference station

R	$L(\theta)_{L.S.}/L(\theta)_{min} \geqq 1$, measure of information gain by using the matched estimator $\hat{A}$ instead of the mismatched least-squares estimator A^*
R'	$M(\theta)_{L.S.}/M(\theta)_{min} \geqq 1$
r_{ij}	distance between normalized coordinates of the i^{th} and j^{th} station
T	column matrix of observations
T_i	observed transit time after processing signals from station i and the reference station; $i = 1, 2, ..., N$
v_n	projection of the wind velocity vector onto the direction of propagation of the sound wave, in m/sec
X_i, Y_i	$\begin{Bmatrix} (x_o - x_i)/d \\ (y_o - y_i)/d \end{Bmatrix}$ normalized coordinates of the i^{th} station in the array
(x_o, y_o)	(x, y) rectangular coordinates of the reference station in the ground array, in meters
Z	$N \times 5$ or $N \times 2$ matrix, depending on the coordinates (X_i, Y_i)
z	altitude above the ground plane
(α, β)	coefficients in the representation of τ $$\tau = \alpha p + \beta^2 p^2$$
$\Delta\tau_i$	random error in estimate of transit time τ_i
θ	bearing angle of plane wave, in deg (see Fig. 1)
$\pi/2 - \phi$	elevation angle of plane wave, in deg (see Fig. 1)
ρ	positive definite symmetric matrix with elements ρ_{ij} used to weight the residual errors in deriving the parametric estimates $\hat{A}$ or A^*
σ^2	mean square transit time errors, $E(\Delta\tau_i^2)$
$\sigma_{\Delta\theta}$	$\left(\frac{\sigma}{\tau_o}\right) L(\theta)$, rms bearing angle error, in radians
$\sigma_{\Delta\tau_o}$	$\sigma\, M(\theta)$, rms ground transit time error, in sec

τ true transit time of the plane wave from the reference station to the station with coordinates (x, y)

$$\tau = A_1 X + A_2 Y + A_3 X Y + A_4 X^2 + A_5 Y^2$$

τ_o d/c_g, ground transit time for plane wave over scale distance d, in sec

ψ normalized covariance matrix of $(\Delta\tau_i)$ with elements ψ_{ij}

ψ_{ij} $\frac{E(\Delta\tau_i \, \Delta\tau_j)}{\sigma^2}$, $\exp\left\{-r_{ij}/k\right\}$

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I. INTRODUCTION

STATEMENT OF THE PROBLEM

Given an array of nondirectional microphones which measure sound pressure, it is desired to measure the bearing angle of an arriving plane acoustic wave in the infrasonic regions, that is, in the frequency range of from .1 to 1 cps. The array may be of arbitrary geometry in the ground plane. A novel aspect of the problem is that the local velocity of sound propagation is not presumed known except for a nominal value of $c_o = 344$ m/sec. The actual velocity may deviate by 5 to 10 percent. The local ground trace of sound propagation is also obtainable from the measurements as described; however, estimates of the elevation angle of the plane wave are not.

DESCRIPTION OF THE MODEL

The plane-acoustic wave is presumed to be generated a large distance from the array. As the sound wave is propagated through the atmosphere, the wave undergoes changes in both orientation of the phase plane and amplitude. The amplitude decreases slightly due to atmospheric absorption, but primarily due to the dilution of the sound energy over a greater volume. Superimposed on these systematic effects there are also random changes in phase at each point on the phase plane caused by turbulence in the atmosphere. Thus, the wave which arrives at the array is not strictly a plane wave. The surfaces of constant phase are taken to consist of a plane plus random deviations from the plane. An excellent discussion of the propagation properties of infrasonic sound waves through the atmosphere is given in Ref. 1.

The acoustic plane wave energy (noted as the signal) is presumed to be small compared to the atmospheric turbulence pressure (noted as noise) in the same frequency range. It is assumed that the signal has been detected by other means and that the gross direction (within, say, one quadrant) of the wave has been determined. This Memorandum is therefore not concerned with the detection problem but with the improvement of the estimate of the local bearing angle of the sound wave. The data at each array point are the result of processing the received data through noise-reducing line microphones to improve the signal-to-noise ratio. This Memorandum does not, however, attempt to evaluate the nature of the background noise or the effects of various data processing operations on the statistical properties of the signal and noise. These problems will be considered in future studies. A class of bearing estimation methods are developed and the effect of two specific methods is evaluated for certain standardized error models. The measure of merit used is a normalized standard deviation of bearing angle error, noted as $L(\theta)$. A set of computations of $L(\theta)$ is performed for a square array consisting of 16 equally spaced array points.

II. DISCUSSION OF THE ESTIMATION METHOD

The basic data required are the transit time of the wave from a fixed station or array point with coordinates (x_o, y_o) to each of the other stations with coordinates (x_i, y_i). Let this time be noted as τ_i. Then the estimation process involves the following: If each of the values of τ_i is plotted in the (X, Y) plane

$$X = \frac{x_o - x}{d}, \qquad Y = \frac{y_o - y}{d}$$

where d is a normalizing scale factor, at the value (X_i, Y_i), it will be shown that the transit time for a plane wave can be represented as

$$\tau = A_1X + A_2Y + A_3XY + A_4X^2 + A_5Y^2 \tag{1}$$

The bearing angle θ and the ground velocity c_g are estimated from the coefficients A_1 and A_2. The process then involves estimating the coefficients A_j by curve fitting of Eq. (1) to the data set $\{\tau_i\}$, i = 1, 2, ..., N for an (N + 1) station array. Let the measured value of τ_i be given by

$$T_i = \tau_i + \Delta\tau_i \tag{2}$$

where τ_i is the "true" transit time given by Eq. (1) and $\Delta\tau_i$ is the random error in transit time due to such causes as initial phase errors or deviation from the plane phase surface, and errors in estimating τ_i from the processing of signals from the array microphones. As an example, an obvious method of estimating τ_i is by cross correlation. That is

$$\tau_i = \max_{\tau} \rho(\tau) = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{+t} s_o(u)\, s_i(u + \tau)\, du \qquad (3)$$

where s_i is the observed signal from the i^{th} array point and s_o is the observed signal from the reference array point. The observed T_i and the statistics of $\Delta\tau_i$ will be determined by such factors as the actual interval of time over which the cross correlation is performed; whether the computation of the cross correlation is for sampled data or for continuous data, and the sampling rates; the time and space correlation properties of both the signal and noise components of the observed signal; and the difference in the initial timing errors of each observed signal due to initial phase errors of the plane wave.

The above effects, as well as alternate methods for generating the τ_i, will be considered in subsequent studies. For the purpose of this study, the random variable $\Delta\tau_i$ is assumed to have the following properties

$$\begin{aligned} E(\Delta\tau_i) &= 0 \\ E(\Delta\tau_i\ \Delta\tau_j) &= \sigma^2\ \delta_{ij} \qquad \text{Case A} \\ E(\Delta\tau_i\ \Delta\tau_j) &= \sigma^2\ \psi_{ij} \qquad \text{Case B} \end{aligned} \qquad (4)$$

where E( ) signifies the expected value and $\delta_{ij} = 1,\ i = j;\ = 0,\ i \neq j$. The quantity ψ_{ij} is a normalized correlation coefficient and is assumed to have the form

$$\psi_{ij} = \exp\left\{-r_{ij}/k\right\} \qquad (5)$$

where r_{ij} is the normalized distance between the i^{th} and j^{th} station

$$r_{ij} = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2}$$

and k is a constant ≥ 0.

The coefficients A_j of Eq. (2) are obtained by generalized least-squares procedures as follows: Let N + 1 be the number of stations so that the number of transit times τ_i measured from the reference station is N. Define an N × N positive, definite, symmetric matrix ρ with elements $\{\rho_{u,v}\}$. Then let

$$Q = \sum_{i=1}^{N} \sum_{j=1}^{N} (T_i - \tau_i)\, \rho_{ij} (T_j - \tau_j) \tag{6}$$

The values of A_j are selected which minimize Q.

When the following conditions hold, the solutions are as indicated:

ρ = I (identity matrix) = least-squares solution (7a)

$\rho = \text{diag}\,\{\rho_{ii}\}$ = weighted least-squares solution (7b)

$\rho = \{\rho_{u,v}\}$ = generalized weighted least-squares solution (7c)

$\rho\psi$ = I = minimum variance solution (7d)

The general formulation for Eq. (7c) is shown, from which Eqs. (7a), (7b), and (7d) are given as special cases. Computations of $L(\theta)$ are performed for the square array consisting of 16 equally spaced arrays separated by distance d between x and y coordinates of adjacent stations. Similar computations are performed for the linear case ($A_3 = A_4 = A_5 = 0$) and for certain subsets of stations to measure the improvement rate in $L(\theta)$ as more stations are processed.

CORRELATION MATCH VERSUS MISMATCH

The effects of mismatching the weighting matrix ρ and the $\Delta\tau_i$

correlation matrix ψ are computed as follows

Case I: $\rho = I$; ψ, given by Eq. (4), Case B; $k > 0$

Case II: $\rho = \psi^{-1}$; ψ, given by Eq. (4), Case B; $k > 0$

That is, the $\Delta\tau_i$ data correlation is actually as given by Eq. (5), but a least-squares solution Eq. (7a) is used. Note that as $k \to 0$, $\min(r_{ij})$ fixed, $i \neq j$, $\psi \to I$, so that the solution for the A_j approaches the matched condition given by Eq. (7d), i.e., Case II. The matched condition is optimum in the following sense. The estimates $\hat{A}_i$ obtained are random variables with zero mean and covariance matrix $B = \{B_{u,v}\}$ $u,v = 1,2,\ldots 5$. B is positive definite (in the quadratic case, A_3, A_4, $A_5 \neq 0$) and $\leq$ the covariance matrix of any other linear unbiased estimator of $A' = (A_1, A_2, \ldots A_5)$.

Thus, a comparison of the values of $L(\theta)$ for the matched and mismatched case shows how much is gained by using a minimum variance estimator as opposed to a least-squares estimator. Comparison of the subsets $N = 3, 7, 15$ (linear) and $N = 7, 15$ (quadratic) shows how much is gained by using the additional stations. Finally, a comparison of $L(\theta)$ for the quadratic curve fit and the linear curve fit shows how much additional root mean square error is caused in assuring an unbiased estimate of the bearing angle θ. It may be desirable to accept a linear model for Eq. (1) and a small bias in θ with smaller rms.

ADVANTAGES OF GENERALITY OF METHOD

The technique does not depend on the specific geometry of the array. Thus, the method lends itself to field data measurement procedures since dropping bad data does not upset the computations. Further unreliable data can be weighted to have less effect.

In Appendix A the solution for the coefficients A_i is given in terms of the observations T_i. The equations for the covariance matrix B required to evaluate the variances of θ and c_g are also derived.

In Appendix B the justification for Eq. (1) and the interpretation of the coefficients in terms of the geometry of the plane wave and local meteorological conditions are shown. The condition for accepting a linear model is derived; that is, setting $A_3 = A_4 = A_5 = 0$ in Eq. (1).

In Appendix C the normalized rms bearing angle error $L(\theta)$ is derived in terms of the covariance matrix B of the parameter estimates $\hat{A}$. The ground trace transit time to travel the distance d given by the scale factor in defining X and Y is defined as τ_o. The normalized rms error in τ_o, $M(\theta)$, is also derived in terms of the same variables. The results are presented in tables following Appendix D. Table 1 presents $L(\theta)$ for the Case I, (ρ = I) versus selected values of k for the linear case $\theta = 0^o, 15^o, 30^o$ and 45^o and N = 3, 7, 15. The value of the ratio

$$R = L(\theta)_{L.S.} / L(\theta)_{min}$$

is also shown in the table where $L(\theta)_{min}$ is the matched processing case $\rho\,\psi = I$. The value of R, which is ≥ 1, shows the gain obtained by using the matched processing. The same information is presented for the quadratic case for N = 7, 15 in Table 2. In Table 3, the same information is presented for $M(\theta)$ for $\theta = 45^o$. As shown in

Appendix C, $M(\theta) = L(\theta)$ for $\theta = 0^o$ and 90^o and the maximum $|M^2(\theta) - L^2(\theta)|$ occurs at $\theta = 45^o$.

In Appendix D the computations of $M(\theta)$ and $L(\theta)$ for a specific square array of sensors is described. The Fortran program for the computations is given. The results are presented in figures following Appendix D. Figures 3 to 8 are plots of $L(\theta)$ versus θ for the parameters as plotted for fixed values of k = .125, 4 and 256. In Fig. 3, k = .125 is taken as indicating independent timing errors so that, since $\psi \cong I$, the least-squares solution is a matched solution. For Fig. 4, k = 4 is taken as a moderately mismatched least-squares solution. In Fig. 5, k = 256 is taken as a heavily mismatched solution. For Figs. 6, 7 and 8, the matched solution is presented for the corresponding cases of k of Figs. 3, 4 and 5. Figures 9 and 10 show $L(\theta)$ versus k for fixed θ, for the linear case N = 3, 7 and 15 and the quadratic case N = 7, 15. Figure 9 is for $\theta = 0^o$ and Fig. 10 is for $\theta = 45^o$. Both are for Case I, $\rho = I$. The same data are presented in Figs. 11 and 12 for Case II, the matched case for $\rho\,\psi = I$. Other angles are obtainable from Tables 1 and 2. Note that for Case I, $L(\theta)$ is labeled $L(\theta)_{L.S.}$ and for Case II, $L(\theta)_{min}$. The value R of Tables 1 and 2 is given by

$$R = \frac{L(\theta)_{L.S.}}{L(\theta)_{min}} \geqq 1$$

where corresponding values of each of the parameters are used in the ratio.

By inspection of the tables and graphs conclusions can be made as to the accuracy in bearing angle obtainable as a function of bearing angle θ, increasing station numbers, using linear versus quadratic curve fitting, the degree of mismatch for the least-squares estimate, and the accuracy gain using a minimum variance estimate.

For example, in Fig. 4 for linear curve fitting there is apparently little to be gained at any angle θ by processing more than $N = 3$. However, in the quadratic case there is a substantial gain by going from $N = 7$ to $N = 15$. This gain is dependent on θ and increases monotonically from $\theta = 0^{\circ}$ to $\theta = 45^{\circ}$.

III. CONCLUSIONS

A method of estimating the bearing angle of a plane sonic wave using an arbitrary* ground array of sensors has been developed. The method does not require knowledge of the propagation velocity of sound. In fact, the ground trace velocity of sound can be derived from the data processing.

Equations for evaluating the rms bearing angle error and the rms ground trace timing error were developed.

Computations of $L(\theta)$ and $M(\theta)$, the normalized rms errors, were performed for a specific square array consisting of 16 equally spaced microphones. For this array, the computations demonstrate the accuracy obtainable in terms of the rms timing errors and provide a basis for determining how to efficiently process the field data.

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*Subject to certain mild restrictions, e.g., the stations shall not all be colinear and $N \geq 2$ (linear case) and $N \geq 5$ (quadratic case).

Appendix A

DERIVATION OF PARAMETER ESTIMATE EQUATIONS

QUADRATIC MODEL

It will be convenient to relabel the variables of Eq. (1) as follows: Let

$$Z_i^{(1)} = X_i, \quad Z_i^{(2)} = Y_i, \quad Z_i^{(3)} = X_i Y_i, \quad Z_i^{(4)} = X_i^2, \quad Z_i^{(5)} = Y_i^2$$

Then Eq. (1) can be written in matrix form as

$$\tau = Z\,A \qquad \text{(A-1)}$$

$$\begin{aligned} \tau &= \{\tau_i\} = N \times 1 \text{ (column matrix)},\ N \geq 5 \\ A &= \{A_i\} = 5 \times 1 \text{ (column matrix), of unknown parameters } A_i \\ Z &= [Z^{(1)}, Z^{(2)}, \ldots, Z^{(5)}] = N \times 5 \end{aligned} \qquad \text{(A-2)}$$

and $Z^{(u)}$ is an N x 1 column matrix, u = 1, 2, ..., 5.

In particular, values of A, noted as A^*, are sought which minimize the quadratic form

$$Q = (Z\,A - T)'\rho(Z\,A - T) \qquad \text{(A-3)}$$

where T is the N × 1 column matrix of observations of T_i and the prime indicates the transpose. Upon setting the gradient Q = 0 one obtains the well known result* (1)

$$A^* = (Z'\,\rho\,Z)^{-1}\,Z'\,\rho\,T \qquad \text{(A-4)}$$

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*$(\ )^{-1}$ indicates the inverse of the matrix ( ), and ′ the transpose.

where it is assumed that the columns of Z are linearly independent so that $(Z' \rho Z)^{-1}$ exists.

It is easily demonstrated that A^* is unbiased; that is

$$E(A^*) = A \tag{A-5}$$

The covariance matrix of A^* is given by (see Eq. (4))

$$\begin{aligned} B^* = E\left[[A^* - A][A^* - A]'\right] &= \sigma^2 (Z' \rho Z)^{-1} Z' \rho \psi \rho Z (Z' \rho Z)^{-1} \quad \text{Case B} \\ &= \sigma^2 (Z' \rho Z)^{-1} Z' \rho^2 Z (Z' \rho Z)^{-1} \quad \text{Case A} \end{aligned} \tag{A-6}$$

If $\rho = I$, the matched least-square case gives $(\psi = I)$

$$B^* = \sigma^2 (Z' Z)^{-1} \tag{A-7}$$

It is well known that case 7d, the minimum variance estimator, is given by[1]

$$\hat{A} = (Z' \psi^{-1} Z)^{-1} Z' \psi^{-1} T \tag{A-8}$$

and the corresponding smallest covariance matrix for the matched correlated case, corresponding to $\rho\psi = I$, is

$$\hat{B} = \sigma^2 (Z' \psi^{-1} Z)^{-1} \tag{A-9}$$

LINEAR MODEL

The derivation for the linear case is the same as the quadratic case except that since $A_3 = A_4 = A_5 = 0$, the definition of Z in Eq. (A-2) is changed to

$$Z = [Z^{(1)}, Z^{(2)}]$$

and $N \geq 2$ is required. All equations (A-1) to (A-9) then hold with the above changes. For example, B^* and $\hat{B}$ are 2×2 matrices instead of 5×5 and $\hat{A}$ is a 2×1 instead of a 5×1.

Appendix B

DERIVATION OF CURVE FITTING EQUATIONS

It is assumed that a plane acoustic wave is incident at the array at bearing angle θ and elevation angle $\left(\frac{\pi}{2} - \phi\right)$, as defined in Fig. 1. Since the quadrant is assumed known, there is no loss in generality by assuming the wave as incident in the first quadrant such that

$$0 \le \phi \le \frac{\pi}{2}, \qquad 0 \le \theta \le \frac{\pi}{2}$$

The equation of the phase plane is

$$(\sin\phi \cos\theta)x + (\sin\phi \sin\theta)y + (\cos\phi)z - P = 0 \tag{B-1}$$

Consider the position of the phase plane when the plane is incident at the reference station with coordinates $(x_o, y_o, 0)$; then P is given by

$$P = \sin\phi \left[(\cos\theta)x_o + (\sin\theta)y_o\right]$$

and the equation of the phase plane is

$$\sin\phi \left[(\cos\theta)(x - x_o) + (\sin\theta)(y - y_o)\right] + (\cos\phi)\, z = 0 \tag{B-2}$$

It is required to compute the transit time of the phase plane from its position when incident at station (x_o, y_o) to the time when the phase plane is incident at (x, y). Note first that the distance of the point (x, y) from the phase plane through station (x_o, y_o) as given by Eq. (B-2) is

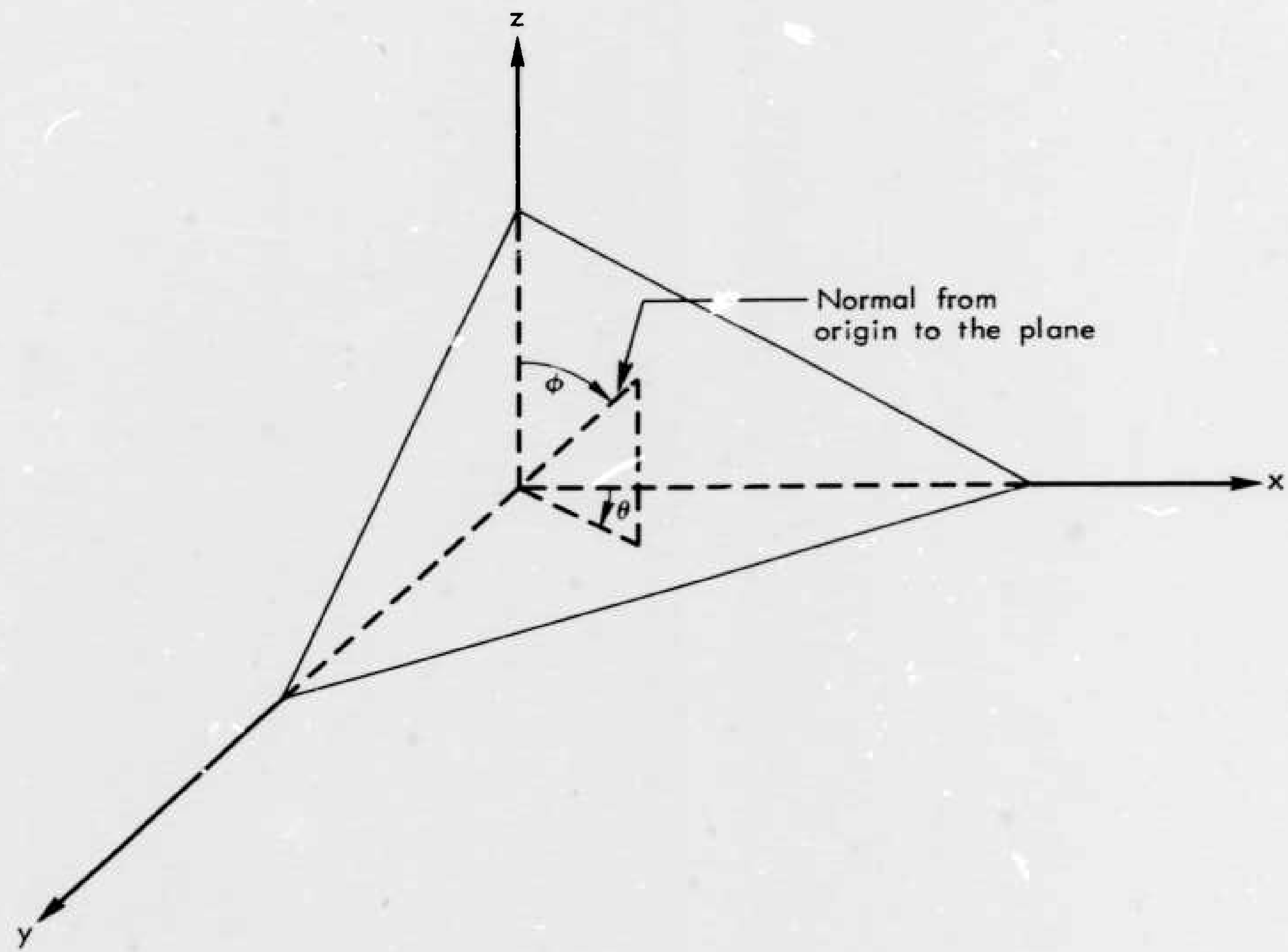

Fig. 1—Geometry of incident plane sonic wavefront

$$p = -\sin \phi \left[(\cos \theta)(x - x_o) + (\sin \theta)(y - y_o)\right] \qquad \text{(B-3)}$$

where x_o and y_o are selected such that $-(x_i - x_o) \geq 0$, $-(y_i - y_o) \geq 0$ for each of the station coordinates. The transit time is given by ray theory as[2,3]

$$\tau = \frac{1}{c_o} \int_o^p n(r)\, dr \qquad \text{(B-4)}$$

where

$$n(r) = \frac{c_o}{c + \vec{v} \cdot \vec{n}} = \frac{c_o}{c + v_n} \qquad \text{(B-5)}$$

is the index of refraction at a distance r along the ray from the station at (x, y) to the plane, given by Eq. (B-2), formed by a line perpendicular to the plane and

c_o = nominal velocity of sound = 344 m/sec at 20°C

$\vec{n}$ = unit vector in the direction of wave propagation, or perpendicular to the phase plane

$\vec{v}(r)$ = wind velocity vector

c = local velocity of sound

$v_n = \vec{v} \cdot \vec{n}$ projection of $\vec{v}$ on $\vec{n}$

It is assumed that the medium is horizontally stratified so that both c and v are functions of height only. For a standard atmosphere one may write

$$c(z) = c_o + \frac{dc}{dz} \cdot z \qquad 0 \leq z \leq 10 \text{ km} \qquad \text{(B-6)}$$

where

$$\frac{dc}{dz} \cong -4.4 \text{ meters/sec/km}$$ [2]

v(z) versus z increases logarithmically with z for heights up to 30 to 50 meters and then at a slower rate.(3) However, for the purpose of this discussion, wind height will be considered a slowly increasing linear function of height which can be represented over the range of altitudes of interest as

$$v_n(z) = v_n(o) + K_1 z \qquad \text{(B-7)}$$

Setting $z = r \cos \phi$, Eq. (B-4) becomes

$$\tau \cong \frac{1}{c_o \cos \phi} \int_o^p \left[1 - \frac{1}{c_o} \frac{dc}{dz} z - \frac{v_n(z)}{c_o}\right] dz \qquad \text{(B-8)}$$

where p is sufficiently small so that $\left| \frac{dc}{dz}/c_o \right| << 1$, and $(v_n(z)/c_o) << 1$. Substituting Eq. (B-7) into Eq. (B-8) and integrating Eq. (B-8) gives

$$\tau = \frac{p}{c_o}\left[1 - \frac{v_n(o)}{c_o}\right] - \left(\frac{p}{c_o}\right)^2 \cdot \frac{\cos \phi}{2} \left\{\frac{dc}{dz} + K_1\right\} \qquad \text{(B-9)}$$

Equation (B-9) is a quadratic in p which can be written in the form

$$\tau = \alpha p + \beta p^2 \qquad \text{(B-10)}$$

On substituting Eq. (B-3) into Eq. (B-9) one finds

$$\tau = A_1 X + A_2 Y + A_3 X Y + A_4 X^2 + A_5 Y^2 \qquad \text{(B-11)}$$

The coefficients A_j are given by

$$A_1 = (\alpha \sin \phi \cos \theta)\, d$$

$$A_2 = (\alpha \sin \phi \sin \theta)\, d$$

$$A_3 = (2\beta \sin^2 \phi \sin \theta \cos \theta)\, d^2 \qquad \text{(B-12)}$$

$$A_4 = (\beta \sin^2 \phi \cos^2 \theta)\, d^2$$

$$A_5 = (\beta \sin^2 \phi \sin^2 \theta)\, d^2$$

$$\alpha = \frac{\left[1 - \frac{v_n(o)}{c_o}\right]}{c_o}$$

$$\text{(B-13)}$$

$$\beta = -\frac{1}{c_o^2}\, \frac{\cos \phi}{2} \left\{\frac{dc}{dz} + K_1\right\}$$

Define the effective ground trace velocity c_g by

$$c_g^{-1} = \alpha \sin \phi$$

Then estimates of both c_g and θ may be obtained as follows:

Note that

$$\tan \theta = \frac{A_2}{A_1}, \qquad \theta = \tan^{-1} \left(A_2/A_1\right) \qquad \text{(B-14)}$$

$$\tau_o \equiv \left(d\, c_g^{-1}\right) = \left[A_1^2 + A_2^2\right]^{\frac{1}{2}} \qquad \text{(B-15)}$$

Thus the estimates of A_1 and A_2 provide estimates of the bearing angle θ and the effective ground trace velocity. The quantity τ_o is the time for the wave to travel a distance d on the ground.

If $\beta = 0$, so that the linear model for τ can be used, the amount of data processing is reduced and the rms of the estimates of θ and τ_o is decreased. From Eq. (B-11)

$$\tau = \alpha\, p \left\{1 + \frac{\beta}{\alpha}\, p\right\} \tag{B-16}$$

From Eq. (B-12)

$$\alpha = \frac{\left[A_1^{\,2} + A_2^{\,2}\right]^{\frac{1}{2}}}{d \sin \phi} \tag{B-17}$$

$$\beta = \frac{A_4 + A_5}{d^2 \sin^2 \phi}$$

so that Eq. (B-16) can be written

$$\tau = \alpha\, p \left\{1 + \frac{A_4 + A_5}{\sqrt{A_1^{\,2} + A_2^{\,2}}}\ \frac{p}{d \sin \phi}\right\} \tag{B-18}$$

The value of $p/d \sin \phi$ is clearly determined from Eq. (B-3) as

$$p/d \sin \phi = X \cos \theta + Y \sin \theta$$

so that the maximum magnitude of $p/d \sin \phi$ = the maximum normalized dimension of the array. Let this characteristic value be D where

$$D = \max_{\theta,\, x_i,\, y_i} \left(\frac{p}{d \sin \theta}\right)$$

Then, if

$$\frac{A_4 + A_5}{\left[A_1^{\,2} + A_2^{\,2}\right]^{\frac{1}{2}}}\, D << 1$$

one may take $\beta = 0$, and therefore $A_3 = A_4 = A_5 = 0$, and use the linear model.

Appendix C

BEARING ANGLE ACCURACY

The relationship between the bearing angle θ and the coefficients of the curve fit A_1, A_2 is given in Eq. (B-14). This relationship is nonlinear. However, if the errors in the coefficients are small, then the errors in θ can be determined as follows

$$\tan \theta = A_2/A_1$$

$$\Delta\theta = \cos^2 \theta \, \frac{\left\{A_2 \Delta A_1 - A_1 \Delta A_2\right\}}{A_1^2} \tag{C-1}$$

where $\Delta\theta$ is the random error in θ due to random errors ΔA_1 and ΔA_2 in the parameter estimates A_1 and A_2. Since $E(\Delta\tau_i) = 0$, then $E(\Delta A_j) = 0$, $j = 1, 2, 3, \ldots, 5$, and $E(\Delta\theta) = 0$. Then

$$E(\Delta\theta^2) = \left(\frac{\cos \theta}{A_1}\right)^4 [A_2 \ - A_1] \ B_2 \ [A_2 \ - A_1]' \tag{C-2}$$

where B_2 is the 2×2 submatrix of the covariance matrix given by Eq. (A-7) or Eq. (A-9); e.g.

$$B_2 = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = \sigma^2 \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \tag{C-3}$$

where the b_{ij} are the normalized covariance $E(\Delta A_i \ \Delta A_j) = \sigma^2 \ b_{ij}$ (C-4)

$$| \ b_{ij} \ | \leq 1$$

The value of A_j used in the estimation is matched to the appropriate choice of ρ for a given ψ to determine which B matrix to use.

From Eq. (B-12)

$$A_1 = \tau_o \cos\theta, \quad A_2 = \tau_o \sin\theta, \tag{C-5}$$

so that

$$E(\Delta\theta^2) = \left(\frac{\sigma}{\tau_o}\right)^2 \left\{\cos^2\theta\, b_{11} + \sin^2\theta\, b_{22} - 2\, b_{12} \cos\theta \sin\theta\right\} \tag{C-6}$$

Define

$$L^2(\theta) = \frac{E(\Delta\theta^2)}{(\sigma/\tau_o)^2} \tag{C-7}$$

the normalized variance of θ.

Equation (C-6) shows that $E(\Delta\theta^2)$ is inversely proportional to τ_o, the ground transit time of the wave over the distance given by the scale factor d. Assuming d to be fixed (say the x, y coordinate distance between adjacent stations in a square array), then $\tau_o \to 0$ as $\phi \to 0$. (See Fig. 1.) In this case the ground trace velocity is infinite and θ becomes indeterminant, as is expected. Thus, it is required to limit ϕ so that $\phi \geq \phi_o$ before an attempt to estimate θ is considered. Define

$$\sigma_{\Delta\theta} = \sqrt{E(\Delta\theta^2)} = \frac{\sigma}{\tau_o} \cdot L(\theta) \quad \text{in radians}$$

L(θ) is shown in Tables 1 and 2 and gives the bearing angle accuracy in radians.

From Eq. (C-6) note that if $b_{11} = b_{22}$

$$E(\Delta\theta^2) = \left(\frac{\sigma}{\tau_o}\right)^2 \{b_{11} - 2\, b_{12} \cos\theta \sin\theta\} \tag{C-8}$$

so that L(θ) is symmetrical with respect to $\theta = 45^o$. When the stations are placed symmetrically with respect to the line y = x,

Table 1

NORMALIZED BEARING ANGLE rms ($L(\theta)$) AND RATIO[a] R VERSUS CORRELATION PARAMETER k FOR THE LINEAR CASE

N	θ \ k		0.125	0.25	0.5	1	2	4	8	64	256
3	0	L	0.2722	0.2722	0.2723	0.2731	0.2714	0.2620	0.2490	0.2268	0.2234
		R	1.	1.	1.	1.0001	1.0042	1.0352	1.1374	2.3130	4.4227
	15	L	0.3043	0.3043	0.3043	0.3034	0.2928	0.2658	0.2321	0.1711	0.1608
		R	1.	1.	1.	1.	1.0018	1.0166	1.0725	1.8702	3.4377
	30	L	0.3258	0.3258	0.3258	0.3238	0.3075	0.2685	0.2189	0.1143	0.0909
		R	1.	1.	1.	1.	1.0004	1.0043	1.0203	1.3232	2.0733
	45	L	0.3333	0.3333	0.3333	0.3309	0.3127	0.2695	0.2139	0.0844	0.0427
		R	1.	1.	1.	1.	1.	1.	1.	1.	1.
7	0	L	0.2076	0.2099	0.2252	0.2525	0.2682	0.2639	0.2508	0.2258	0.2220
		R	1.	1.0001	1.0025	1.0123	1.0239	1.0551	1.1533	2.3146	4.4155
	15	L	0.2393	0.2418	0.2579	0.2838	0.2899	0.2676	0.2340	0.1707	0.1599
		R	1.	1.0001	1.0027	1.0131	1.0226	1.0380	1.0909	1.8775	3.4386
	30	L	0.2601	0.2628	0.2794	0.3047	0.3048	0.2703	0.2209	0.1146	0.0906
		R	1.	1.0001	1.0028	1.0135	1.0219	1.0265	1.0405	1.3365	2.0817
	45	L	0.2673	0.2700	0.2869	0.3120	0.3101	0.2713	0.2159	0.0852	0.0431
		R	1.	1.0001	1.0028	1.0136	1.0217	1.0225	1.0209	1.0179	1.0174
15	0	L	0.1745	0.1781	0.2031	0.2479	0.2784	0.2846	0.2790	0.2642	0.2617
		R	1.	1.0001	1.0067	1.0369	1.0896	1.1916	1.3939	3.1413	6.1016
	15	L	0.2006	0.2046	0.2311	0.2747	0.2931	0.2780	0.2497	0.1963	0.1876
		R	1.	1.0001	1.0061	1.0302	1.0608	1.1113	1.2207	2.3515	4.4204
	30	L	0.2177	0.2219	0.2496	0.2927	0.3034	0.2731	0.2258	0.1252	0.1039
		R	1.	1.0001	1.0058	1.0267	1.0441	1.0579	1.0871	1.5075	2.4684
	45	L	0.2237	0.2280	0.2561	0.2990	0.3071	0.2712	0.2163	0.0854	0.0432
		R	1.	1.0001	1.0057	1.0256	1.0387	1.0393	1.0361	1.0307	1.0300

[a] R is the ratio of $L(\theta)$ for the mismatched case to the matched case
N + 1 is the number of stations
θ is the bearing angle

Table 2

NORMALIZED BEARING ANGLE rms (L(θ)) AND RATIO[a] R VERSUS CORRELATION PARAMETER k FOR THE QUADRATIC CASE

N	θ \ k		0.125	0.25	0.5	1	2	4	8	64	256
7	0	L	1.5172	1.5121	1.4631	1.3015	1.0561	0.8202	0.6381	0.3870	0.3476
		R	1.	1.	1.0004	1.0013	1.0018	1.0019	1.0016	1.0006	1.0002
	15	L	1.5065	1.5029	1.4606	1.3014	1.0471	0.7977	0.6009	0.3089	0.2561
		R	1.	1.	1.0004	1.0014	1.0020	1.0022	1.0021	1.0010	1.0004
	30	L	1.4985	1.4961	1.4588	1.3013	1.0405	0.7808	0.5721	0.2358	0.1585
		R	1.	1.	1.0004	1.0014	1.0021	1.0024	1.0024	1.0019	1.0010
	45	L	1.4956	1.4936	1.4582	1.3013	1.0381	0.7745	0.5612	0.2025	0.1015
		R	1.	1.	1.0005	1.0015	1.0022	1.0025	1.0026	1.0026	1.0026
15	0	L	0.6677	0.6755	0.7176	0.7634	0.7624	0.7312	0.6985	0.6543	0.6484
		R	1.	1.	1.0022	1.0193	1.0784	1.2217	1.4941	3.6171	7.1002
	15	L	0.7275	0.7345	0.7660	0.7756	0.7206	0.6380	0.5666	0.4738	0.4613
		R	1.	1.	1.0013	1.0111	1.0454	1.1348	1.3182	2.9044	5.6129
	30	L	0.7683	0.7748	0.7996	0.7845	0.6884	0.5600	0.4459	0.2747	0.2468
		R	1.	1.	1.0007	1.0055	1.0183	1.0486	1.1148	1.8462	3.2985
	45	L	0.7828	0.7891	0.8115	0.7877	0.6763	0.5286	0.3926	0.1447	0.0727
		R	1.	1.	1.0006	1.0036	1.0076	1.0099	1.0106	1.0104	1.0103

[a] R is the ratio of L(θ) for the mismatched case to the matched case
N + 1 is the number of stations
θ is the bearing angle

then it is obvious that one may interchange y and x and demonstrate that $b_{11} = b_{22}$ so that the symmetry conditions given by Eq. (C-8) hold.

Finally, Eq. (C-1) seems to require $\theta \neq \pi/2$. However, one can define cotan $\theta = A_1/A_2$ and derive Eq. (C-6) as the end result so that Eq. (C-6) does hold for all θ.

TRANSIT TIME ERROR

From Eq. (B-15)

$$\Delta\tau_o \cong \frac{A_1 \ \Delta A_1 + A_2 \ \Delta A_2}{\tau_o} \qquad \text{(C-9)}$$

so that

$$E(\Delta\tau_o) = 0$$

$$E(\Delta\tau_o^{\ 2}) = \sigma^2 \{b_{11} \cos^2 \theta + b_{22} \sin^2 \theta + 2\cos \theta \sin b_{12}\} \qquad \text{(C-10)}$$

If $b_{11} = b_{22}$, which holds for stations symmetrically placed with respect to the line Y = X

$$\frac{E(\Delta\tau_o^{\ 2})}{\sigma^2} = \{b_{11} + 2 \cos \theta \sin \theta \ b_{12}\} = M^2(\theta) \qquad \text{(C-11)}$$

The normalized variance $M^2(\theta)$ is symmetric with respect to $\theta = 45^o$. Note that when

$$\theta = 0 \text{ or } \theta = \pi/2, \quad M(\theta) = L(\theta) \qquad \text{(C-12)}$$

for any value of θ

$$M^2(\theta) - L^2(\theta) = 4b_{12} \sin \theta \cos \theta \qquad \text{(C-13)}$$

and therefore

$$M^2(\theta) - L^2(\theta) \leq 2\, b_{12} \qquad \text{(C-14)}$$

the equality sign holding for $\theta = 45^\circ$ when the symmetry conditions $b_{11} = b_{22}$ hold and

$$M^2(\theta) + L^2(\theta) = 2\, b_{11} \qquad \text{(C-15)}$$

independent of θ.

Table 3 presents $M(\theta)$, $\theta = 45^\circ$, for the linear and quadratic case with k from .125 to 256 and values of N as indicated.

Table 3

NORMALIZED GROUND TRACE rms $M(\theta)$ AND RATIO[a] R' VERSUS CORRELATION PARAMETER k FOR FIXED BEARING ANGLE $\theta = 45^{\circ}$

Case	N \ k		0.125	0.25	0.5	1	2	4	8	64	256
LINEAR	3	M	0.1925	0.1925	0.1928	0.1992	0.2226	0.2542	0.2797	0.3094	0.3130
		R′	1.	1.	1.	1.0005	1.0127	1.0796	1.2497	2.8130	5.4686
	7	M	0.1213	0.1234	0.1382	0.1736	0.2183	0.2563	0.2814	0.3078	0.3109
		R′	1.	1.	1.0011	1.0080	1.0283	1.0955	1.2603	2.8062	5.4482
	15	M	0.1043	0.1073	0.1299	0.1831	0.2464	0.2973	0.3300	0.3637	0.3676
		R′	1.	1.0002	1.0106	1.0690	1.1860	1.3867	1.7263	4.2637	8.3925
QUADRATIC	7	M	1.5386	1.5303	1.4680	1.3017	1.0739	0.8635	0.7066	0.5085	0.4810
		R′	1.	1.	1.0003	1.0011	1.0015	1.0013	1.0010	1.0003	1.0001
	15	M	0.5281	0.5385	0.6095	0.7383	0.8398	0.8887	0.9064	0.9139	0.9140
		R′	1.	1.0001	1.0052	1.0382	1.1331	1.3362	1.6944	4.3115	8.5218

[a] R' is the ratio of $M(\theta)$ for the mismatched case to the matched case

N + 1 is the number of stations

Appendix D

COMPUTATION OF NORMALIZED BEARING ACCURACIES $L(\theta)$ and $M(\theta)$ FOR A SQUARE ARRAY

In this appendix computations of $L(\theta)$, the normalized bearing accuracy, and $M(\theta)$, the normalized ground trace timing accuracies given by Eq. (C-11), are described for a specific array configuration shown in Fig. 2.

The numbers in Fig. 2 show the normalized coordinates (X, Y) and station index number.

For the linear case, $L(\theta)$ and $M(\theta)$ are computed for the first four stations ($N = 3$), the first eight stations ($N = 7$), which includes the previous stations, and all the stations ($N = 15$).

For the quadratic case, $L(\theta)$ is computed for $N = 7$ and $N = 15$ defined over the same set of stations as in the linear case for corresponding N.

Computations are performed for $\theta = 0^o$, 15^o, 30^o and 45^o for values of the correlation parameter $k = (2)^j$, $j = -3$ to 8, in steps of 1. For small values of k (.125 or .25) the effect is essentially the same as taking $\psi = I$, so that this case will not be computed separately. For large values of k, the $\Delta\tau_i$ errors at each station are heavily correlated, and one may note the effect of using a mismatched processing such as least squares on this data versus using the matched processing, $\rho\ \psi = I$, of Eq. (7d). The Fortran program from which $L(\theta)$ and $M(\theta)$ are computed is shown on the following pages. Figures 3 to 12 present the $L(\theta)$ values graphically for possible interpolation and visual comparison. Tables 1 to 3 present numerical results of the program.

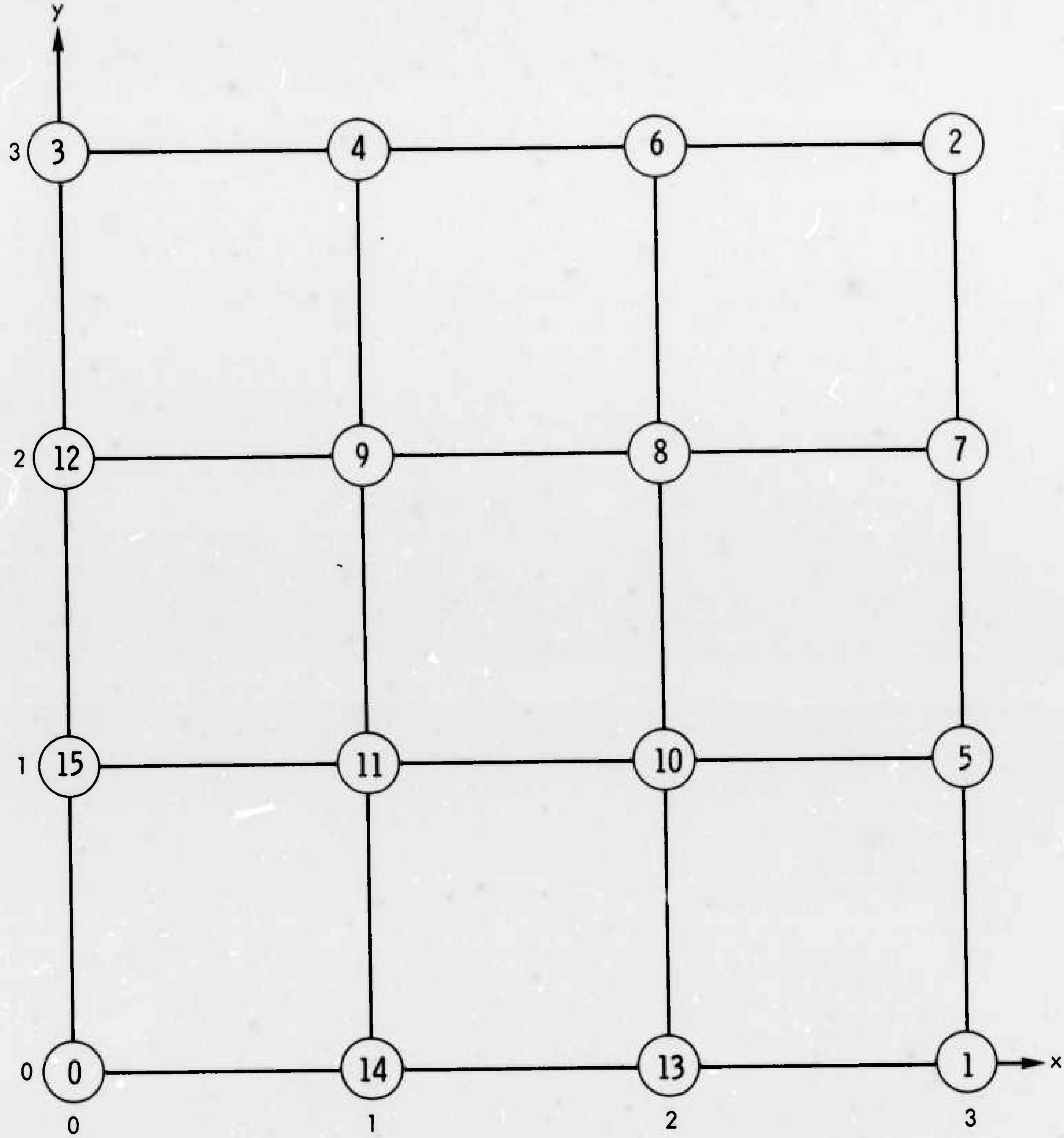

Fig. 2—Square array station layout in normalized coordinates (station identification number in circle)

Note that the Fortran program is sufficiently general to handle an arbitrary set of stations and not just the square array described above, provided $N \geq 2$ in the linear case and $N \geq 5$ in the quadratic case, and not all the stations are colinear.

```
C
C PROGRAM TO COMPUTE NORMALIZED BEARING ACCURACY AND NORMALIZED GROUND
C TRACE TIMING ACCURACY FOR LEAST SQUARES AND MINIMUM VARIANCE CASES.
      REAL K
      DIMENSION Z1(35),Z2(35),Z3(35),Z4(35),Z5(35),NN(35),CAY(35),THET(3
     15),S(1,1),Z(35,35),R(35,35),SINE(35),COSINE(35),PSI(35,35),P(35,35
     2),ZT(35,35),ZTZ(35,35),Z11(35,35),Z12(35,35),Z13(35,35),B(35,35),Z
     321(35,35),C(35,35),CT(35,35),Z31(35,35),ELSQ(35,35),EL(35),Z41(35,
     435),EMSQ(35,35),EM(35),IPIVOT(35),INDEX(35,2),ELL(35),EMM(35),ELR(
     535),EMR(35)
      READ 2,N2,LB2,LC2,LI2,LA2,LF2
      READ 1,(Z1(N),N=1,N2)
      READ 1,(Z2(N),N=1,N2)
      READ 1,(Z3(N),N=1,N2)
      READ 1,(Z4(N),N=1,N2)
      READ 1,(Z5(N),N=1,N2)
      READ 2,(NN(LB),LB=1,LB2)
      READ 3,(CAY(LC),LC=1,LC2)
      READ 4,(THET(LI),LI=1,LI2)
    1 FORMAT(18F4.0)
    2 FORMAT(18I4)
    3 FORMAT(8F9.3)
    4 FORMAT(12F6.2)
      S(1,1)=1.
C
C Z MATRIX (N2 X 5) IS FORMED.
      DO 10 I=1,5
      DO 10 N=1,N2
      IF (I.EQ.1) Z(N,I)=Z1(N)
      IF (I.EQ.2) Z(N,I)=Z2(N)
      IF (I.EQ.3) Z(N,I)=Z3(N)
      IF (I.EQ.4) Z(N,I)=Z4(N)
   10 IF (I.EQ.5) Z(N,I)=Z5(N)
C
C R MATRIX (N2 X N2) IS FORMED.
      DO 20 I=1,N2
      DO 20 N=1,N2
   20 R(I,N)= SQRT((Z1(I)-Z1(N))**2+(Z2(I)-Z2(N))**2)
C
C SINE AND COSINE VALUES ARE CALCULATED HERE TO SAVE TIME.
      RAD=1.74532925E-2
      DO 25 LI=1,LI2
      THETA=THET(LI)*RAD
      SINE(LI)=SIN(THETA)
   25 COSINE(LI)=COS(THETA)
```

```
C
C PROBLEM BEGINS.
C
C LINEAR WHEN LA=1, QUADRATIC WHEN LA=2.
      DO 100 LA=1,LA2
      IF (LA.NE.1) GO TO 27
      LB1=1
      M=2
      GO TO 28
   27 LB1=2
      M=5
C
C A VALUE OF N IS PICKED.
   28 DO 100 LB=LB1,LB2
      N=NN(LB)
C
C A VALUE OF K IS PICKED, PSI MATRIX (N X N) IS FORMED.
      DO 100 LC=1,LC2
      K=CAY(LC)
      IF (LA.EQ.1) PRINT 2000,N,K
      IF (LA.EQ.2) PRINT 2001,N,K
 2000 FORMAT(1H1,2X,6HLINEAR,4X,2HN=I2,4X,2HK=F8.3///)
 2001 FORMAT(1H1,2X,9HQUADRATIC,4X,2HN=I2,4X,2HK=F8.3///)
      DO 30 LE=1,N
      DO 30 LD=1,N
      PSI(LD,LE) =EXP(-R(LD,LE)/K)
   30 P(LD,LE)=PSI(LD,LE)
C
C CASE 1.  RHO MATRIX = IDENTITY (N X N) IS FORMED.  (MISMATCHED)
C ZT MATRIX (M X N) = Z (N X M) TRANSPOSE IS FORMED.
C ZTZ MATRIX (M X M) = MATRIX PRODUCT OF ZT AND Z IS FORMED.
C ZTZ INVERSE MATRIX (M X M) IS FORMED.
C B (NORMALIZED COVARIANCE MATRIX - M X M) = MATRIX PRODUCTS OF
C ZTZ INVERSE (M X M), ZT (M X N), PSI (N X N), Z (N X M), ZTZ INVERSE
C (M X M) IS FORMED.
      DO 80 LF=1,LF2
      IF (LF.NE.1) GO TO 50
      DO 40 I=1,M
      DO 40 J=1,N
   40 ZT(I,J)=Z(J,I)
      CALL MATMUL (ZT,M,N,Z,M,ZTZ)
      CALL MATINV (ZTZ,M,S,0,IPIVOT,INDEX,ISING)
      IF (ISING.NE.0) GO TO 71
      CALL MATMUL (ZTZ,M,M,ZT,N,Z11)
      CALL MATMUL (Z11,M,N,PSI,N,Z12)
```

```
      CALL MATMUL (Z12,M,N,Z,M,Z13)
      CALL MATMUL (Z13,M,M,ZTZ,M,B)
      PRINT 1002
 1002 FORMAT(1H0,4X,19HCASE 1 (MISMATCHED))
      PRINT 1003
 1003 FORMAT(//7X,32HB (NORMALIZED COVARIANCE MATRIX)/)
      DO 43 I=1,M
   43 PRINT 1004,(B(I,J),J=1,M)
 1004 FORMAT(1H ,F17.8,4F15.8)
      GO TO 60
C
C CASE 2.  RHO MATRIX (N X N) = PSI INVERSE.  (MATCHED)
C PSI INVERSE MATRIX (N X N) IS FORMED.
C B (NORMALIZED COVARIANCE MATRIX - M X M) = MATRIX PRODUCTS OF ZT (M X
C N), PSI INVERSE (N X N), Z (N X M) INVERSE IS FORMED.
   50 CALL MATINV (P,N,S,0,IPIVOT,INDEX,ISING)
      IF (ISING.NE.0) GO TO 72
      CALL MATMUL (ZT,M,N,P,N,Z21)
      CALL MATMUL (Z21,M,N,Z,M,B)
      CALL MATINV (B,M,S,0,IPIVOT,INDEX,ISING)
      IF (ISING.NE.0) GO TO 73
C
C NORMALIZED COVARIANCE MATRIX B IS PRINTED OUT.
      PRINT 1005
 1005 FORMAT(//4X,16HCASE 2 (MATCHED))
      PRINT 1003
      DO 51 I=1,M
   51 PRINT 1004,(B(I,J),J=1,M)
C
C FOR MISMATCHED AND MATCHED CASES, NORMALIZED BEARING ACCURACY EL,
C NORMALIZED GROUND TRACE ACCURACY EM, AND RATIOS OF
C LR = EL (MISMATCHED) / EL (MATCHED) AND
C MR.= EM (MISMATCHED) / EM (MATCHED) ARE PRINTED OUT.
   60 DO 62 LI=1,LI2
      C(1,1)=COSINE(LI)
      C(2,1)=-SINE(LI)
      C(3,1)=0.
      C(4,1)=0.
      C(5,1)=0.
      CT(1,1)=COSINE(LI)
      CT(1,2)=-SINE(LI)
      CT(1,3)=0.
      CT(1,4)=0.
      CT(1,5)=0.
      CALL MATMUL (CT,1,M,B,M,Z31)
```

```
      CALL MATMUL (Z31,1,M,C,1,ELSQ)
      EL(LI)=SQRT(ELSQ(1,1))
      C(2,1)=SINE(LI)
      CT(1,2)=SINE(LI)
      CALL MATMUL (CT,1,M,B,M,Z41)
      CALL MATMUL (Z41,1,M,C,1,EMSQ)
      EM(LI)=SQRT(EMSQ(1,1))
      IF (LF.NE.1) GO TO 62
      ELL(LI)=EL(LI)
      EMM(LI)=EM(LI)
   62 PRINT 1006,THET(LI),EL(LI),THET(LI),EM(LI)
 1006 FORMAT  (/(7X,2HL(F6.2,2H)=F12.8,4X,2HM(F6.2,2H)=F12.8))
      GO TO 80
   71 PRINT 1007
 1007 FORMAT(1H0,4X,24HZTZ INVERSE IS SINGULAR.)
      GO TO 80
   72 PRINT 1008
 1008 FORMAT(1H0,4X,24HPSI INVERSE IS SINGULAR.)
      GO TO 80
   73 PRINT 1009
 1009 FORMAT(1H0,4X,30HB MATRIX (CASE 2) IS SINGULAR.)
   80 CONTINUE
      PRINT 81
   81 FORMAT(1H0)
      DO 90 LI=1,LI2
      ELR(LI)=ELL(LI)/EL(LI)
      EMR(LI)=EMM(LI)/EM(LI)
   90 PRINT 1010,THET(LI),ELR(LI),THET(LI),EMR(LI)
 1010 FORMAT(1H ,4X,3HLR(F6.2,2H)=F12.8,4X,3HMR(F6.2,2H)=F12.8)
  100 CONTINUE
      CALL EXIT
      END
```

```
C
C MATRIX MULTIPLICATION
      SUBROUTINE MATMUL (A,M,N,B,L,C)
      DIMENSION A(35,35),B(35,35),C(35,35)
      DO 20 I=1,M
      DO 20 K=1,L
      SUM=0.
      DO 10 J=1,N
   10 SUM=SUM+A(I,J)*B(J,K)
   20 C(I,K)=SUM
      RETURN
      END
```

```
C
C     MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF LINEAR EQUATIONS
      SUBROUTINE MATINV(A,N,B,M,IPIVOT,INDEX,ISING)
C
      DIMENSION A(35,35),B(1,1),IPIVOT(35),INDEX(35,2)
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM),       (T, SWAP)
C
C     INITIALIZATION
    5 ISING = 0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
C BIG LOOP ON I
   30 DO 550 I=1,N
   35 IROW = 0
   40 AMAX=0.0
C     SEARCH FOR PIVOT ELEMENT
   45 DO 105 J=1,N
   50   IF ( IPIVOT(J).EQ.1 )  GO TO 105
   60 DO 100 K=1,N
   70   IF ( IPIVOT(K).EQ.1 ) GO TO 100
   80   IF (ABS(AMAX).GE.ABS(A(J,K)) )  GO TO 100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
  107 IF (IROW.EQ.0) GO TO 750
```

```
  110 IPIVOT(ICOLUM)=1
C
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
  130   IF (IROW.EQ.ICOLUM ) GO TO 260
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205   IF (M.LE.0)  GO TO 260
  210 DO 250 L=1, M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
C
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  330 A(ICOLUM,ICOLUM)=1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/AMAX
  355   IF (M.LE.0)  GO TO 380
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/AMAX
C
C     COMPLETE THE PIVOT
C
  380 DO 550 L1=1,N
  390   IF (L1.EQ.ICOLUM) GO TO 550
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.0
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  455   IF (M.LE.0)  GO TO 550
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C
C     INTERCHANGE COLUMNS
C
  600 DO 710 I=1,N
  610 L=N+1-I
  620   IF ( INDEX(L,1).EQ.INDEX(L,2) ) GO TO 710
  630 JROW=INDEX(L,1)
```

```
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP=A(K,JROW)
  670 A(K,JROW)=A(K,JCOLUM)
  700 A(K,JCOLUM)=SWAP
  705 CONTINUE
  710 CONTINUE
  740 RETURN
C SINGULARITY FLAG
  750 ISING = 1 + N - I
  760 GO TO 740
      END
```

INPUT

Input Order	Variable Name	Explanation of Variable	Restriction*	Input Format
1	N2	Total number of stations	≤ 36	18I4
1	LB2	Number of sets of stations		18I4
1	LI2	Number of θ values	≤ 35	18I4
1	LC2	Number of k values	≤ 35	18I4
1	LA2	Linear case only when LA2=1, linear and quadratic case when LA2=2		18I4
1	LF2	Mismatched case only when LF2=1, mismatched and matched cases when LF2=2		18I4
2	Z1	x coordinate of each station	≤ 35	18F4.0
3	Z2	y coordinate of each station	≤ 35	18F4.0
4	Z3	Product of x and y coordinates of each station		18F4.0
5	Z4	x^2 of each station		18F4.0
6	Z5	y^2 of each station		18F4.0
7	NN	Number of stations minus one used in each set	NN>2 for linear NN>5 for quadratic	18I4
8	CAY	Correlation parameter	0<k<10,000	8F9.3
9	THET	Bearing angle of plane wave in degrees		12F6.2

*The program is designed to handle the parameters as indicated by the restrictions. The program however has not been checked to the limit of these restrictions. Computations for the cases presented in this report have been checked. Other cases may require additional verification.

Sample Input

```
15   3  12   4   2   2                                              N2 LB2 LC2 LI2 LA2 LF2
3.  3.  0.  1.  3.  2.  3.  2.  1.  2.  1.  0.  2.  1.  0.                  Z1
0.  3.  3.  3.  1.  3.  2.  2.  2.  1.  1.  2.  0.  0.  1.                  Z2
0.  9.  0.  3.  3.  6.  6.  4.  2.  2.  1.  0.  0.  0.  0.                  Z3
9.  9.  0.  1.  9.  4.  9.  4.  1.  4.  1.  0.  4.  1.  0.                  Z4
0.  9.  9.  9.  1.  9.  4.  4.  4.  1.  1.  4.  0.  0.  1.                  Z5
 3   7  15                                                                  N
   .125     .25      .5       1.       2.       4.       8.      16.    CAY
 32.      64.      128.      256.                                          CAY
0.   15.   30.   45.                                                       THETA
```

OUTPUT

N = Number of stations minus one

K = Correlation parameter

$L(\theta)$ = Normalized bearing angle accuracy

$M(\theta)$ = Normalized ground trace timing accuracy

$$LR(\theta) = \frac{L(\theta)_{mismatched}}{L(\theta)_{matched}}$$

$$MR(\theta) = \frac{M(\theta)_{mismatched}}{M(\theta)_{matched}}$$

Sample Output

```
LINEAR     N= 3     K=   1.000


  CASE  1  (MISMATCHED)

    B  (NORMALIZED COVARIANCE MATRIX)

     0.07459377      -0.03492071
    -0.03492071       0.07459377

    L(   0.    )=   0.27311860      M(   0.    )=   0.27311860
    L(  15.00)=   0.30340423      M(  15.00)=   0.23902597
    L(  30.00)=   0.32378387      M(  30.00)=   0.21059807
    L(  45.00)=   0.33092972      M(  45.00)=   0.19918097

  CASE  2  (MATCHED)

    B  (NORMALIZED COVARIANCE MATRIX)

     0.07457582      -0.03493867
    -0.03493867       0.07457582

    L(   0.    )=   0.27308573      M(   0.    )=   0.27308573
    L(  15.00)=   0.30338944      M(  15.00)=   0.23896962
    L(  30.00)=   0.32378016      M(  30.00)=   0.21051850
    L(  45.00)=   0.33092973      M(  45.00)=   0.19909079

  LR(   0.    )=   1.00012037      MR(   0.    )=   1.00012037
  LR(  15.00)=   1.00004874      MR(  15.00)=   1.00023580
  LR(  30.00)=   1.00001143      MR(  30.00)=   1.00037797
  LR(  45.00)=   0.99999996      MR(  45.00)=   1.00045294
```

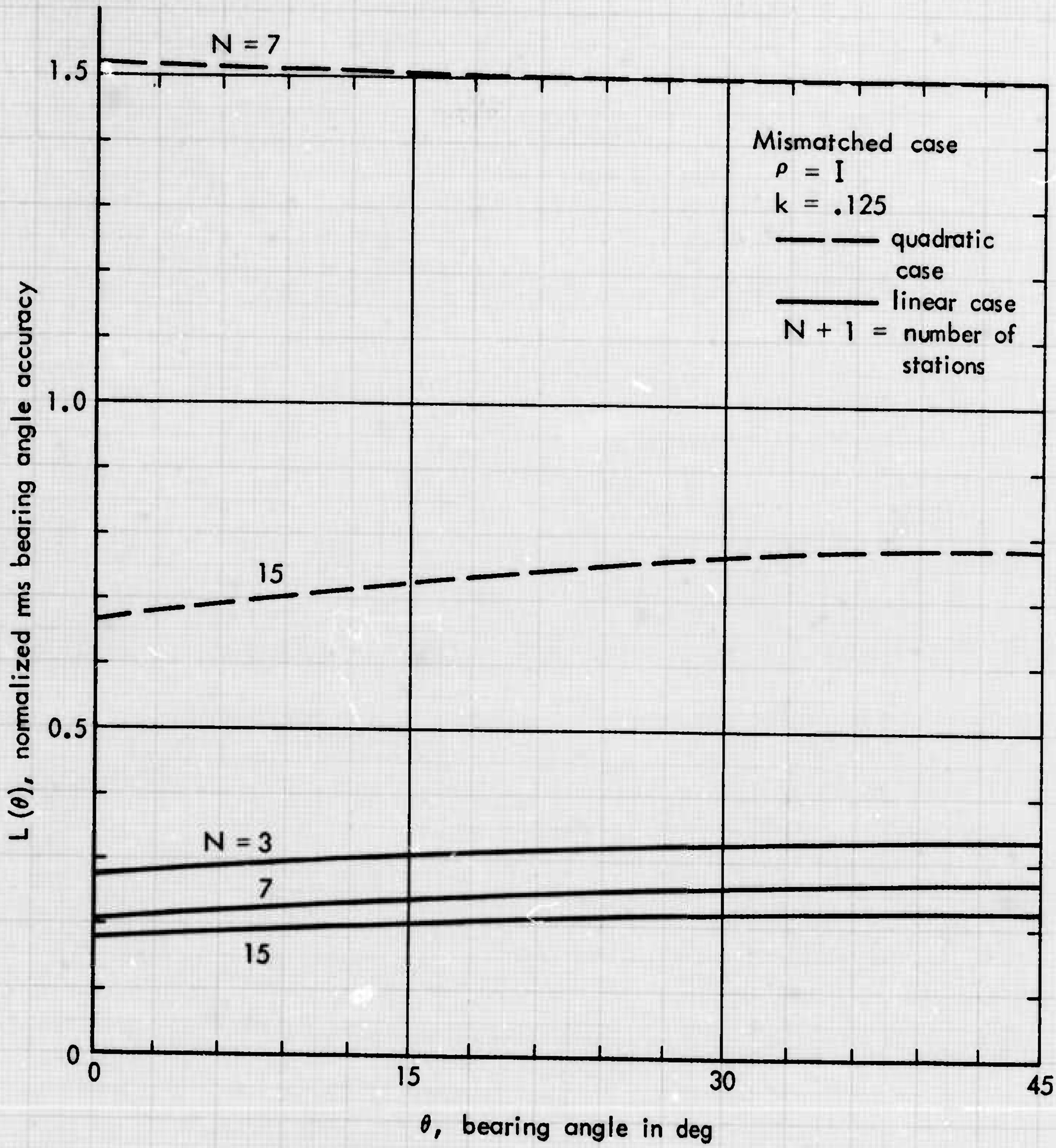

Fig.3—Normalized rms bearing angle accuracy versus bearing angle

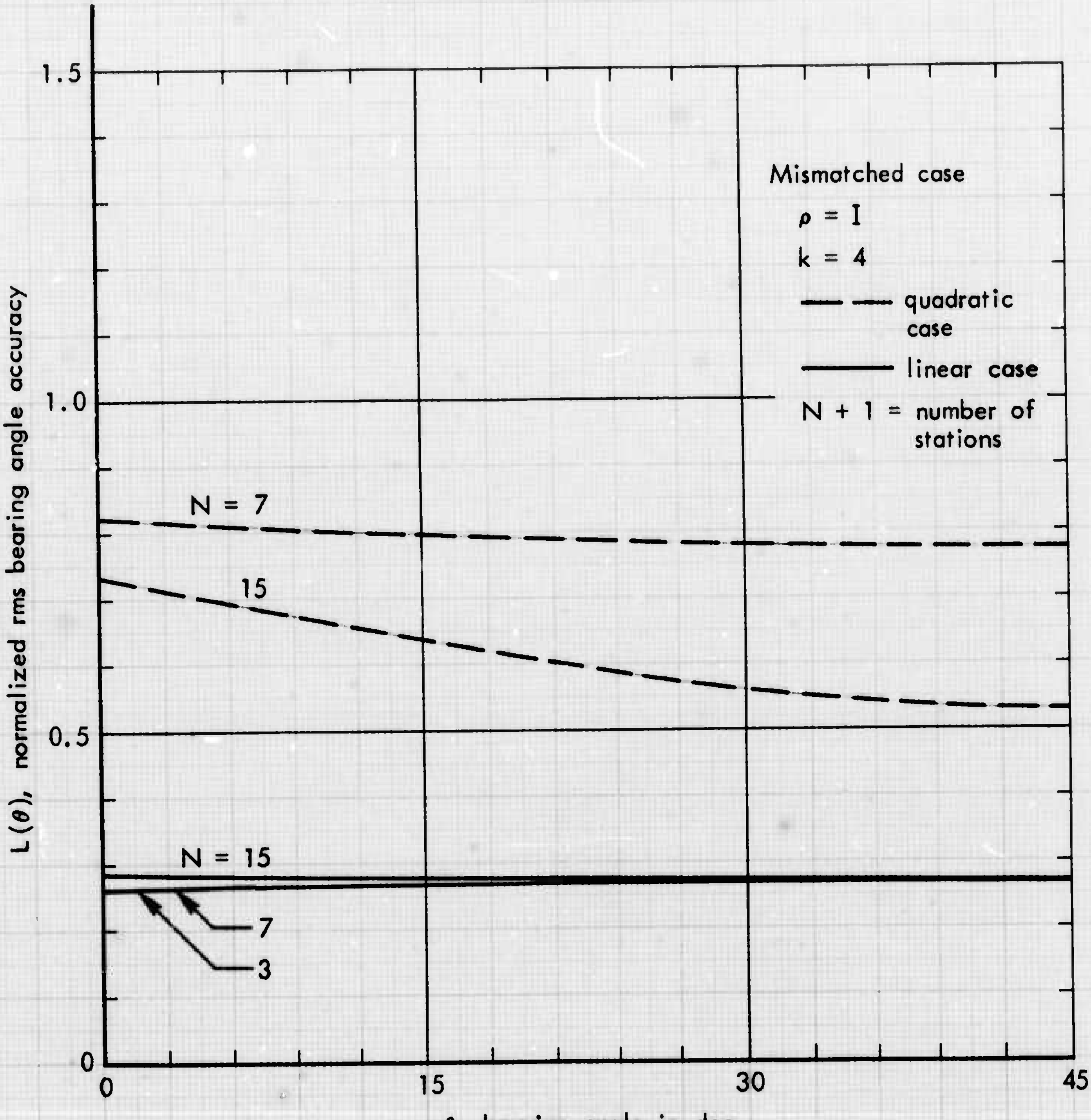

Fig. 4—Normalized rms bearing angle accuracy versus bearing angle

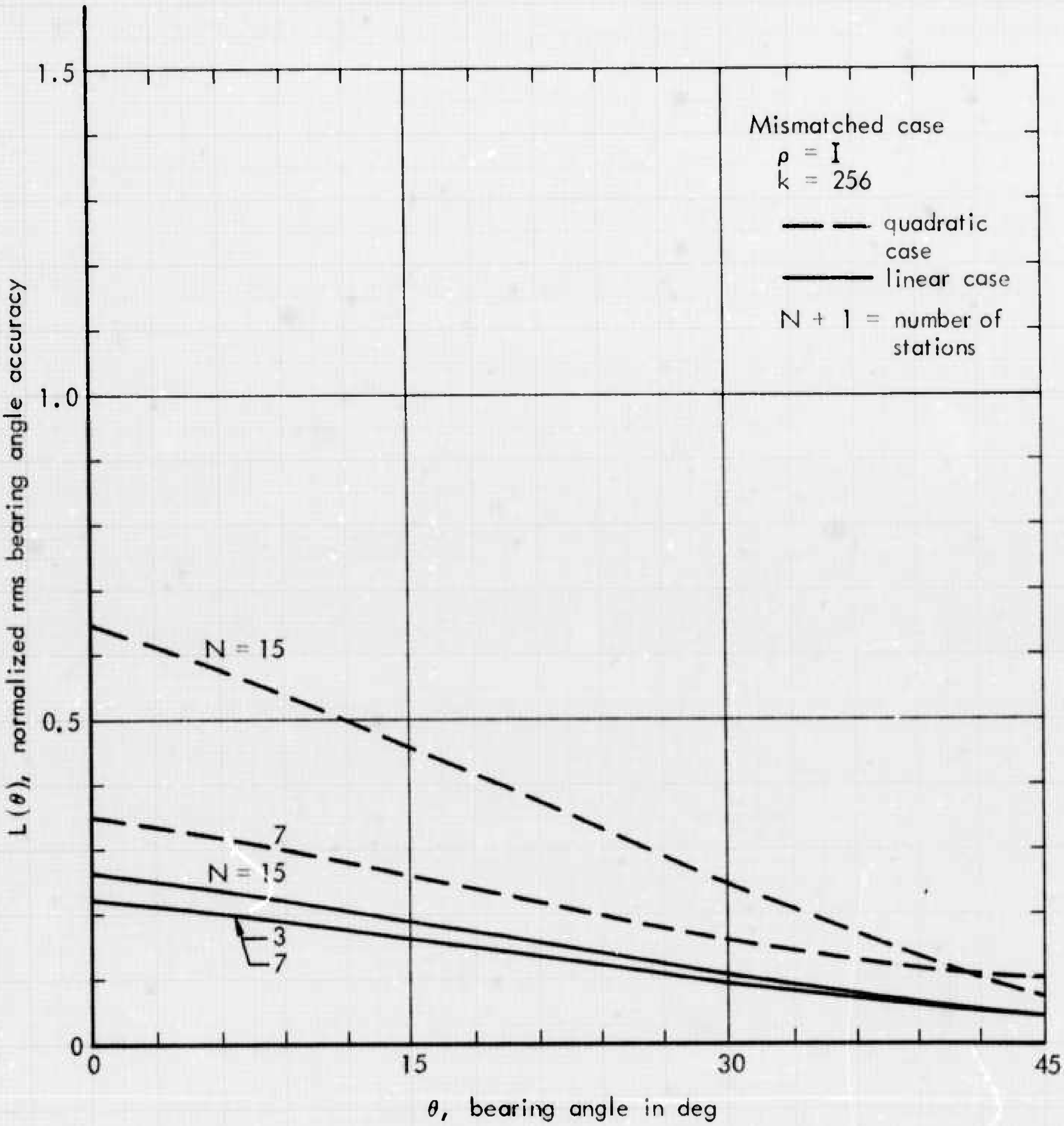

Fig. 5—Normalized rms bearing angle accuracy versus bearing angle

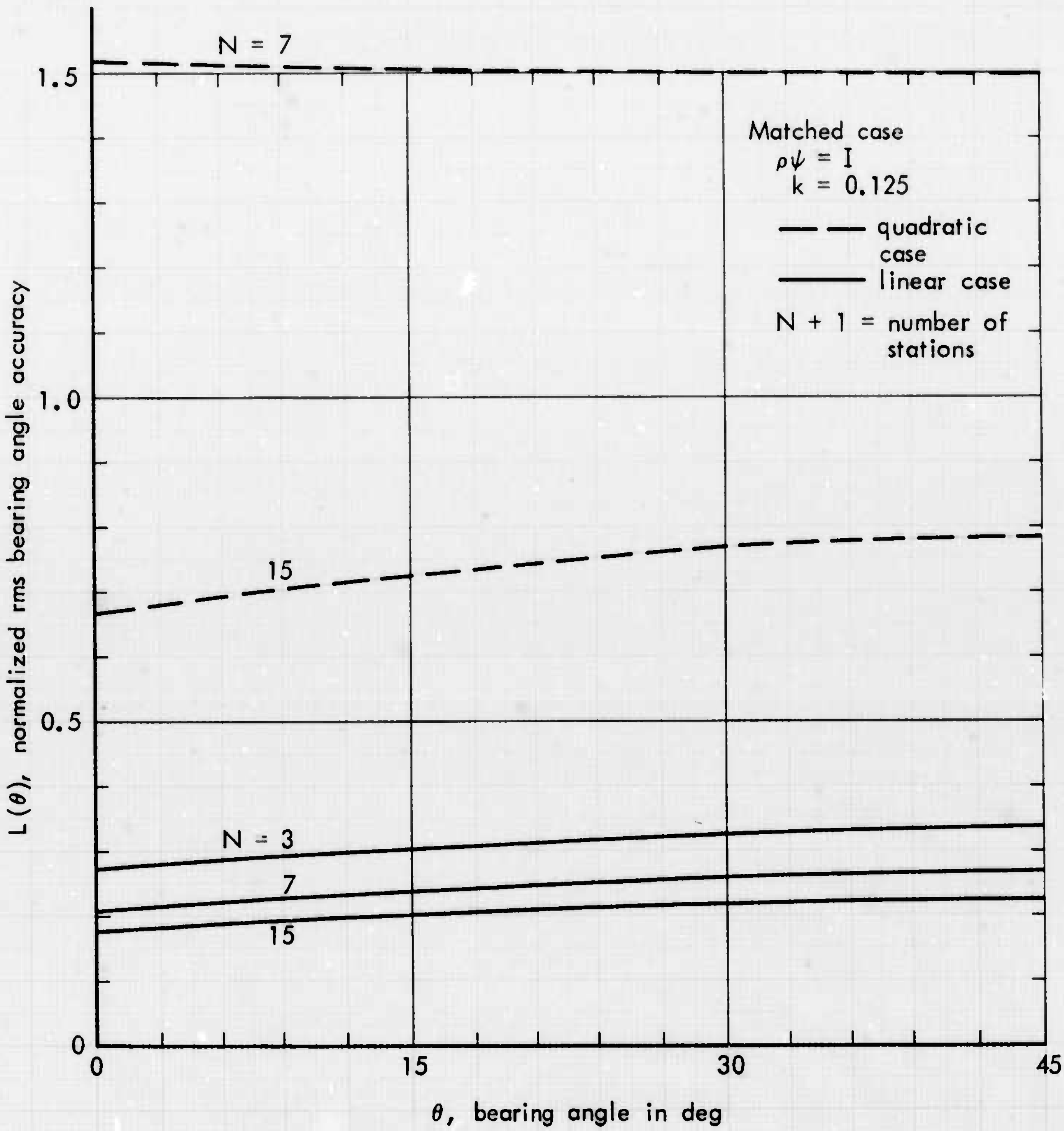

Fig. 6—Normalized rms bearing angle accuracy versus bearing angle

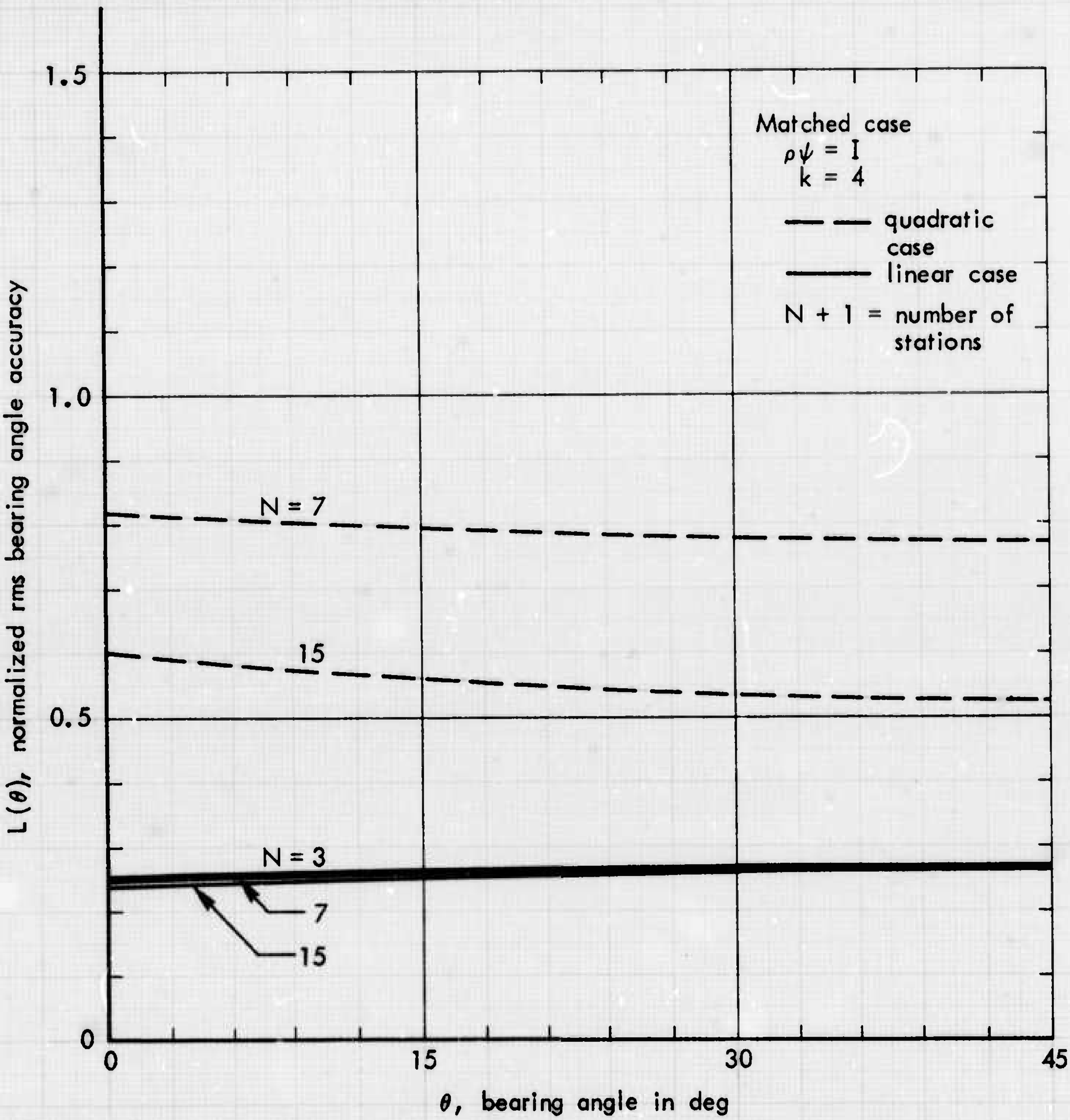

Fig. 7—Normalized rms bearing angle accuracy versus bearing angle

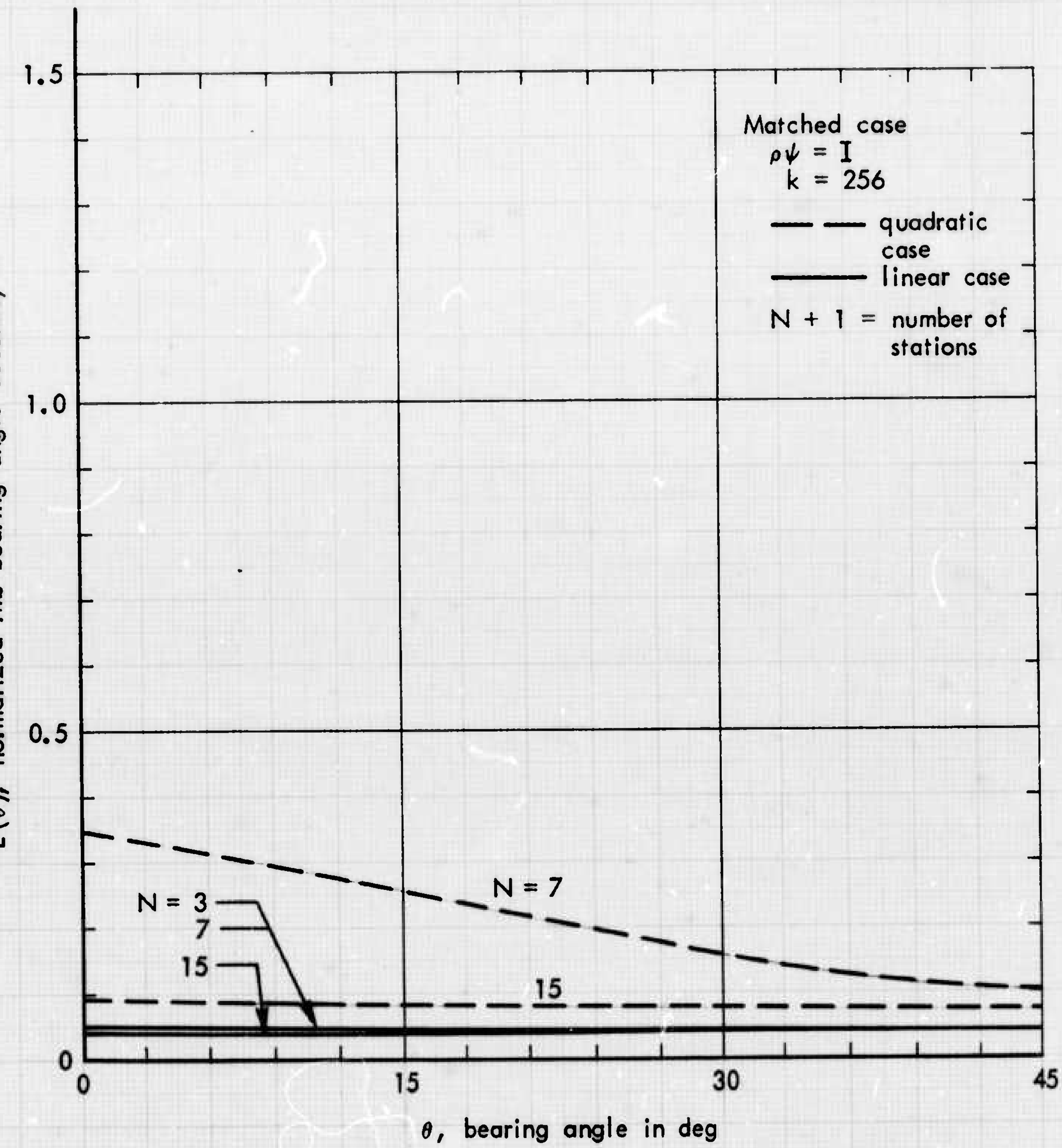

Fig. 8—Normalized rms bearing angle accuracy versus bearing angle

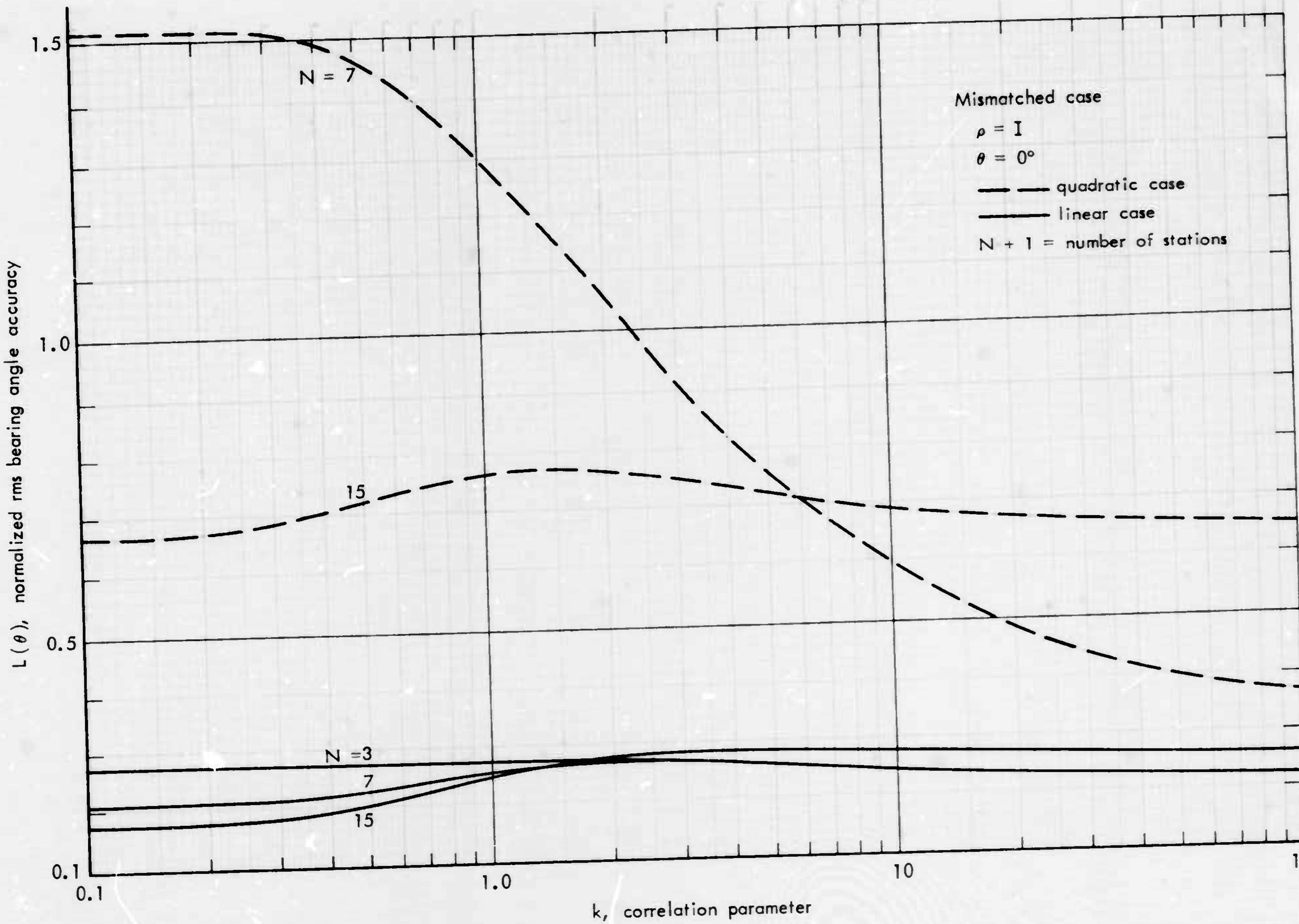

Fig. 9—Normalized rms bearing angle accuracy versus correlation parameter

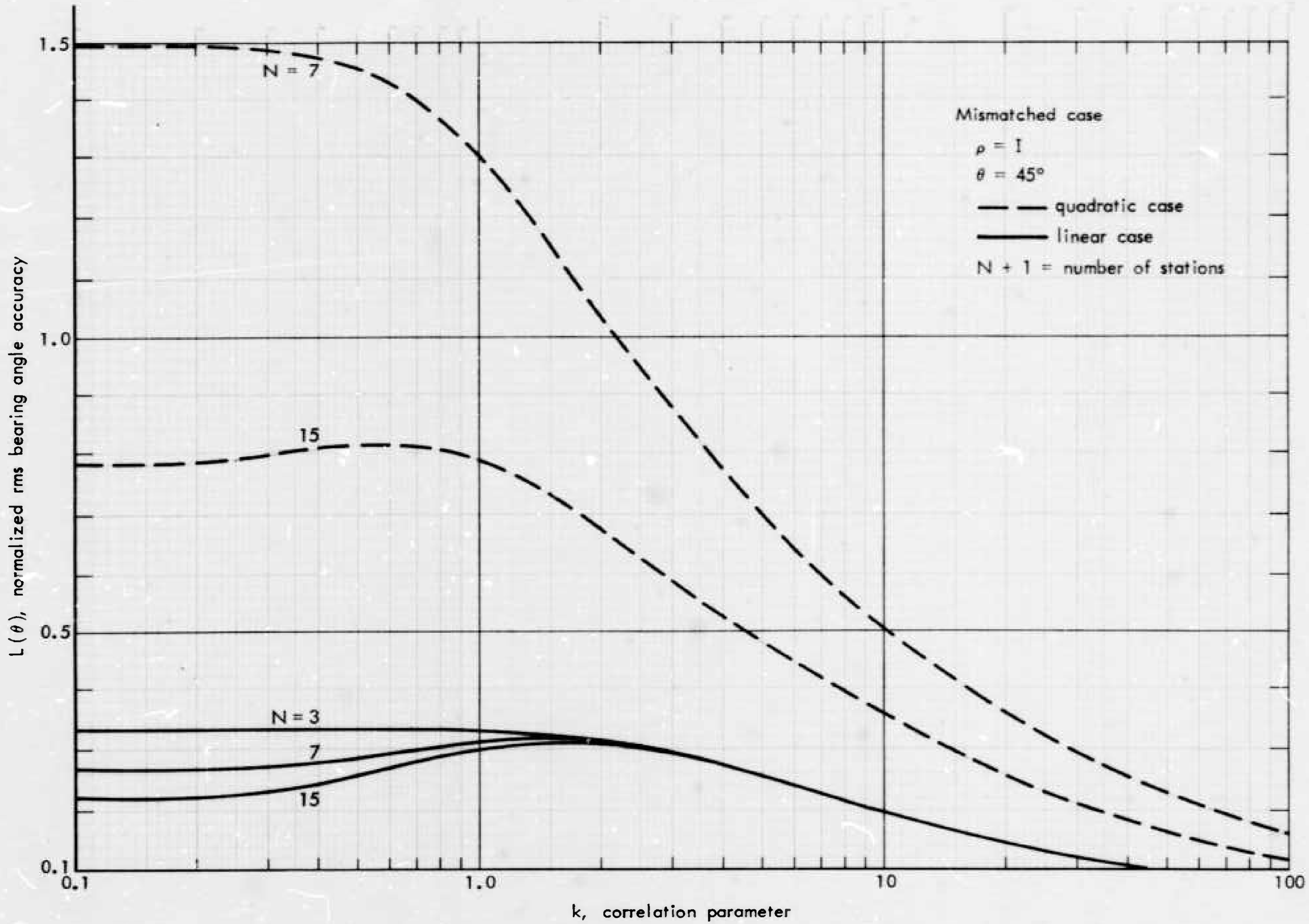

Fig. 10—Normalized rms bearing angle accuracy versus correlation parameter

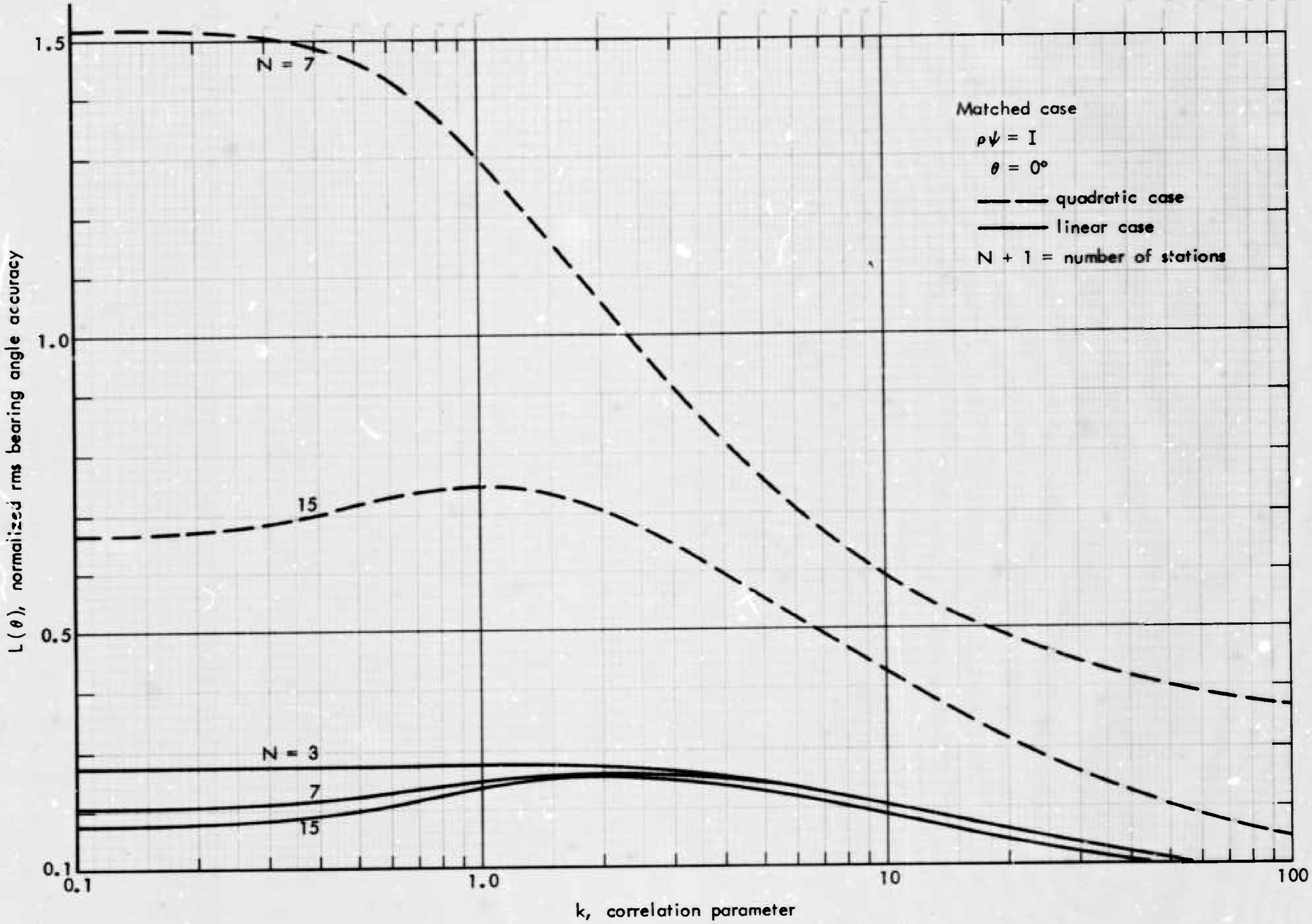

Fig. 11—Normalized rms bearing angle accuracy versus correlation parameter

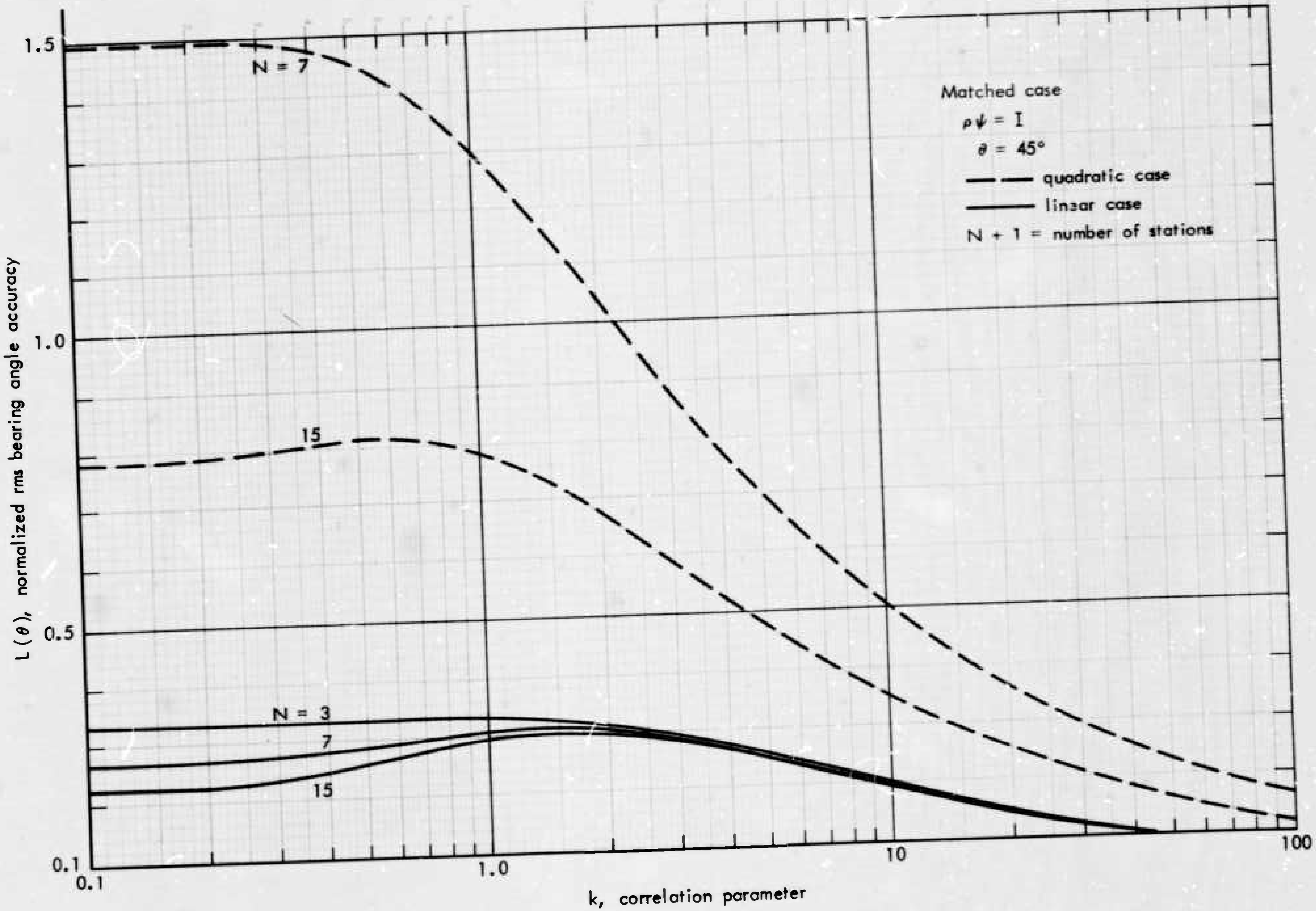

Fig. 12—Normalized rms bearing angle accuracy versus correlation parameter

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10. ABSTRACT

A presentation of data processing techniques developed to obtain bearing angle estimates of plane sonic waves using arbitrary ground arrays of microphones. The evaluation of the accuracy obtainable as measured by the rms bearing angle error is computed in detail for a proposed 16-station square array for use in the VELA detection system. A novel feature of the method is that the ground trace velocity of sound need not be already known or measured independently, but can be derived from the same measurements as the bearing angle.

11. KEY WORDS

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Atmosphere
Microphones
Acoustics
Noise

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