Quarterly Technical Report

on

DIRECT ENERGY CONVERSION SYSTEMS

Part I

ELECTRICAL CONDUCTIVITY OF PARTIALLY IONIZED GASES

Prepared for

Advanced Research Projects Agency

Submitted by

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Mechanical Engineering Department
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ELECTRICAL CONDUCTIVITY OF PARTIALLY IONIZED GASES*

ABSTRACT

A simplification of the Chapman-Enskog method for the calculation of the electrical conductivity of a multi-component partially ionized gas in a magnetic field is presented. The calculation requires the inversion of a matrix which is of the order of the approximation, and is independent of the number of species. The third approximation to the electrical conductivity is examined for an electron-ion-neutral plasma and the results are compared with those obtained from the mixture rules of Lin, Resler, and Kantrowitz, and of Frost. It is shown that within the uncertainties in the experimental electron-neutral cross-section values, Frost's formula offers a satisfactory method of calculation for most engineering applications.

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TABLE OF CONTENTS

ELECTRICAL CONDUCTIVITY OF PARTIALLY IONIZED GASES

I. INTRODUCTION

At the present time, the Chapman-Enskog[1] method of solution of the Boltzmann equation appears to offer the soundest theoretical basis for the calculation of the electrical conductivity of a partially ionized gas. Comparison of calculations made by this method with those based on the mixture rules proposed either by Lin, Resler, and Kantrowitz[2] or by Frost[3], should provide a basis for judging the accuracy of these mixture rules.

The Chapman-Enskog method as extended by Hirschfelder, Curtiss and Bird[4] for a multi-component mixture gives rise to a set of simultaneous linear equations requiring the inversion of a matrix of the order of the number of species multiplied by the order of the approximation. In the limit of a fully ionized gas it is well known from the work of Landshoff[5], Spitzer[6], and Marshall[7] that the first and second Chapman-Enskog approximations yield values for the electrical conductivity that are 51 per cent and 98 per cent, respectively, of the Spitzer value. Therefore at least the second order approximation is required for a reasonable level of accuracy.

In the Lorentzian limit of a weakly ionized gas the rate of convergence of successive approximations is very sensitive to the speed dependence of the electron-neutral elastic scattering cross-section. For the case where the electron-neutral collision frequency ν_{en} depends on relative speed g in accordance with the relation $\nu_{en} \propto g^{2m}$, the rate of convergence of successive Chapman-Enskog approximations is illustrated in Fig. 1. (See Appendix B.) For an electron-neutral collision cross-section that is linear with electron speed, the calculation must be carried to sixth order to obtain 90 per cent of the final value. For some atoms, the electron-neutral elastic scattering cross-section is an increasing function of relative speed in the vicinity of thermal speeds of interest, in which case high orders of approximation are necessary for acceptable accuracy. The rate of convergence for a weakly-ionized gas calculated in the Lorentzian limit using experimental data for the electron-argon momentum transfer cross-section[13], is shown in Fig. 2. For a gas consisting of seeded combustion products where there may be of the order of seven (or more) species present in significant amounts, extensive calculations of the electrical conductivity to higher orders of approximation very soon become impractical for most engineering applications.

A simplification of the Chapman-Enskog method for a multi-component partially ionized gas in applied electric and magnetic fields is presented in which the order of the resulting matrix is the same as the order of the approximation and is independent of the number of species. This simplification results from being able to decouple approximately the Boltzmann equation for the electron velocity distribution function, from the Boltzmann equations for the remaining heavy particles in the mixture.

The first three approximations of the parallel electrical conductivity (zero applied magnetic field) as a function of degree of ionization have been calculated for a three component gas, taking the electron-neutral collision frequency to be proportional to a power of the relative electron speed.* These calculations are compared with several proposed mixture rules. It is concluded that the Lin, Resler, Kantrowitz rule under some circumstances may overestimate the electrical conductivity by as much as 70 per cent. On the other hand, the relatively simple mixture rule proposed by Frost agrees with the more accurate calculation to within 15 per cent over the complete range of degree of ionization and for a wide variety of collision frequency models. Calculations with actual cross-sections corresponding to cesium-seeded argon verify the order of magnitude discrepancy between these two mixture rules.

In order to evaluate Frost's proposed mixture rule for engineering applications we have examined the spread in the calculated electrical conductivity resulting from uncertainties in the electron-neutral collision cross-sections. Relying on published data, reasonable upper and lower bounds for the electron-argon and electron-cesium elastic scattering cross-section have been constructed and these have been used to estimate the resulting uncertainty in calculated electrical conductivity (using Frost's formula). At temperatures applicable to

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*For this special model, our results agree with those of Shkarofsky[8].

magnetohydrodynamic power generators, this spread in the calculated conductivity for cesium-seeded argon exceeds significantly any difference between the simplified Chapman-Enskog third approximation and the Frost formula. Based on the accuracy of present electron-neutral cross-section measurements, the magnitude of the uncertainties we have used appear to be typical[9,10]. We may therefore conclude that the mixture rule proposed by Frost provides a convenient and satisfactory method for calculating the electrical conductivity of partially ionized gases.

II. SIMPLIFICATION OF THE CHAPMAN-ENSKOG APPROXIMATIONS FOR CALCULATING ELECTRICAL CONDUCTIVITY

For purposes of brevity, we shall use similar notation and refer freely to the book by Chapman and Cowling[9]. The subscripts one and two will be used to denote electrons and ions respectively, and further subscripts 3, 4, ..., ν will denote neutrals. In accordance with the Chapman-Enskog approximation scheme, one seeks a solution of the ν Boltzmann equations for the velocity distribution functions of each species in the form $f_j = f_j^{(o)}[1 + \phi_j + \ldots]$ for $j = 1, 2, \ldots \nu$ where $f_j^{(o)}$ is a Maxwellian function. Following the development in Section (18.4) of Chapman and Cowling, the generalization of Eq. (18.4-10) for the electron function $\phi_1(\underline{c}_1)$ to the case of a multi-component partially ionized gas (as distinct from a binary mixture) is

$$\begin{aligned} f_1^{(0)}\Big\{2\underline{\mathscr{C}}_1^{\circ}\underline{\mathscr{C}}_1 : \frac{\partial}{\partial \underline{r}}\underline{c}_0 + (\mathscr{C}_1^2 - 5/2)\underline{C}_1\cdot\frac{\partial}{\partial \underline{r}}\ln T + \frac{n}{n_1}\underline{C}_1\cdot\underline{d}_1\Big\} = f_1^{(0)}\Big\{\frac{m_1}{\rho kT}\underline{C}_1\cdot(\underline{j}_1\times\underline{B}) + \\ \frac{e_1}{m_1}(\underline{C}_1\times\underline{B})\cdot\frac{\partial\phi_1^{(1)}}{\partial\underline{C}_1}\Big\} + \sum_{j=1}^{\nu}\iint f_1^{(0)}(\underline{c}_1)\, f_j^{(0)}(\underline{c})\left[\phi_1(\underline{c}_1') + \phi_j(\underline{c}') - \phi_1(\underline{c}_1) - \phi_j(\underline{c})\right] g I_{1j}\, d\Omega\, d\underline{c}\ , \end{aligned} \tag{1}$$

where in the present problem

$$\underline{d}_1 = \frac{\partial}{\partial \underline{r}}\left(\frac{n_1}{n}\right) + \left(\frac{n_1}{n} - \frac{m_1 n_1}{\rho}\right)\frac{\partial}{\partial \underline{r}}\ln p - \frac{m_1 n_1}{p}\left(\frac{e_1}{m_1} - \frac{\rho_c}{\rho}\right)\underline{E}'\ . \tag{2}$$

We have used the symbol $\underline{B}$ for the magnetic induction (M.K.S. units) and have denoted the electric field in a frame of reference having the mean mass velocity of the mixture $\underline{c}_o$ by $\underline{E}' = \underline{E} + \underline{c}_o \times \underline{B}$. The quantity $I_{1j}(\Omega, g)$ denotes the differential cross-section for the scattering of an electron having speed $g = |\underline{c}_1 - \underline{c}|$ relative to a j-type particle into the solid angle $d\Omega$ about the unit vector $\hat{\Omega}$. The charge density is denoted by ρ_c and $e_1 = -e$ is the charge on an electron. The electron velocity is $\underline{c}_1$, and $\underline{C}_1 = \underline{c}_1 - \underline{c}_o = (2kT/m_1)^{1/2}\, \underline{\mathcal{C}}_1$.

Since Eq. (1) involves the unknown functions $\phi_2, \phi_3, \ldots \phi_\nu$ in addition to ϕ_1 , it is necessary in general, to consider the simultaneous solution of a set of ν integral equations of which Eq. (1) is a proto-type. However, because the electron mass is much less than the mass of any other particle in the mixture, we may take as an approximation that the velocity of a heavy particle is unaltered by electron collisions, i.e., $\phi_j(\underline{c}') = \phi_j(\underline{c})$ for $j = 2, 3, \ldots \nu$. With the small electron mass approximation we may also drop the first term on the right hand side of Eq. (1),and re-write the equation for ϕ_1 as

$$f_1^{(0)}\left\{2\overset{\circ}{\underline{\mathcal{C}}_1\underline{\mathcal{C}}_1}:\frac{\partial}{\partial \underline{r}}\underline{c}_o+(\mathcal{C}_1^2-5/2)\underline{C}_1\cdot\frac{\partial}{\partial \underline{r}}\ln T+\frac{n}{n_1}\underline{C}_1\cdot\underline{d}_1\right\}=$$

$$-f_1^{(0)}\frac{e_1}{m_1}(\underline{C}_1\times\underline{B})\cdot\frac{\partial}{\partial \underline{C}_1}\phi_1+\iint f_1^{(0)}(\underline{c}_1)f_1^{(0)}(\underline{c})\left[\phi_1(\underline{c}_1')+\phi_1(\underline{c}')-\right. \qquad (3)$$

$$\left.\phi_1(\underline{c}_1)-\phi_1(\underline{c})\right]g\,I_{11}\,d\Omega\,d\underline{c}\;+$$

$$\sum_{j=2}^{\nu}\iint f_1^{(0)}(\underline{c}_1)f_j^{(0)}(\underline{c})\left[\phi_1(\underline{c}_1')-\phi_1(\underline{c}_1)\right]g\,I_{1j}\,d\Omega\,d\underline{c}\;.$$

By this means, we are able to de-couple the equation for the electron velocity distribution function from the remaining equations. In the small electron mass approximation the ion current may be neglected relative to the electron current so that the current density

$$\underline{j}^{(1)} \approx e_1 \int f_1^{(0)} \phi_1 \underline{C}_1 d\underline{c}_1 \tag{4}$$

depends only on ϕ_1 . Therefore to determine the electrical conductivity we need concern ourselves only with the solution of Eq. (3).

In Appendix A, it is shown that in the absence of thermal diffusion,

$$\underline{j}^{(1)} = \sigma \underline{\mathcal{E}}^{\parallel} + \sigma_1 \underline{\mathcal{E}}^{\perp} + \sigma_2 \underline{\hat{B}} \times \underline{\mathcal{E}} \quad , \tag{5}$$

where $\underline{\mathcal{E}}$ is a "generalized" electric field, $\underline{B} = B\,\underline{\hat{B}}$, $\underline{\mathcal{E}}^{\parallel} = \underline{\mathcal{E}} \cdot \underline{\hat{B}}$ and $\underline{\mathcal{E}}^{\perp} = \underline{\mathcal{E}} - \underline{\mathcal{E}}^{\parallel}$. In the ξth Chapman-Enskog approximation, the transverse electrical conductivities for an electrically neutral plasma have the values

$$\sigma_1 + i\sigma_2 = \frac{n_1^2 e_1^2}{m_1} d^{(0)}(\xi) \quad , \tag{6}$$

where $d^{(o)}(\xi)$ is determined as a solution of the ξ linear equations

$$\sum_{m'=0}^{\xi-1} Q_{mm'} d^{(m')} = \frac{3}{2} \delta_{mo} \quad . \tag{7}$$

The coefficients $Q_{mm'}$ are expressed in Appendix A in terms of certain "bracket expressions" which depend on collision integrals $\Omega_{ij}^{\ell s}$, defined by Eq. (A23). These collision integrals, in turn, are determined by the elastic differential scattering cross-sections between electrons and other species in the mixture. The parallel conductivity $\sigma = \lim_{B \to 0} \sigma_1$, and is therefore the same as the electrical conductivity in the absence of a magnetic field.

III. THE THIRD CHAPMAN-ENSKOG APPROXIMATION ($\underline{B} = 0$)

Beginning with this section, we shall restrict our discussion to the first three Chapman-Enskog approximations and to the case of no magnetic field; (the case of $\underline{B} \neq 0$ will be discussed in a forthcoming publication). For the charged particle interactions, we shall employ the Rutherford differential scattering cross-section

$$I_{12}(\chi, g) = \left(\frac{e^2}{8\pi\epsilon_0 m_1 g^2}\right)^2 \frac{1}{\sin^4(\chi/2)} = I_{11}/4 \tag{8}$$

with the usual Debye length cut-off. The validity of this approach is examined in Appendix C. Using Eq. (8), the collision integral for the charged particle interactions satisfy the relations

$$\left.\begin{aligned} \Omega_{11}^{1s} &= 2^{3/2}\,\Omega_{12}^{1s} \\ \Omega_{12}^{1s} &= \Gamma(s)\,\Omega_{12}^{11} = \frac{\Gamma(s)\, e^4 \ln\Delta}{32\pi(2\pi)^{1/2}\,\epsilon_0^2\,(m_1)^{1/2}(kT)^{3/2}} \,. \end{aligned}\right\} \tag{9}$$

Using the values of the bracket expressions in terms of the collision integrals provided in Refs. 1 and 4, the third Chapman-Enskog approximation for the electrical conductivity may be written as

$$\sigma(3) = \frac{3}{2}\,\frac{n_1^2 e^2}{m_1}\,\frac{\Delta_1}{\Delta} \,, \tag{10}$$

where

$$\Delta \equiv \begin{vmatrix} 8n_1\Omega_{12}^{11}+q^{00} & 12n_1\Omega_{12}^{11}+q^{01} & 15n_1\Omega_{12}^{11}+q^{02} \\ 12n_1\Omega_{12}^{11}+q^{01} & (8\sqrt{2}+26)n_1\Omega_{12}^{11}+q^{11} & (6\sqrt{2}+69/2)n_1\Omega_{12}^{11}+q^{12} \\ 15n_1\Omega_{12}^{11}+q^{02} & (6\sqrt{2}+69/2)n_1\Omega_{12}^{11}+q^{12} & (\frac{45\sqrt{2}}{2}+\frac{433}{8})n_1\Omega_{12}^{11}+q^{22} \end{vmatrix} \quad (11)$$

and where Δ_1 denotes the determinant formed from Δ by deleting the first row and column. The $q^{mm'}$ quantities are defined in terms of the electron-neutral collision integrals as follows:

$$\left.\begin{aligned} q^{00} &= 8\sum_{j=3}^{\nu} n_j\Omega_{1j}^{11}, \\ q^{01} &= 8\sum_{j=3}^{\nu} n_j\left(\tfrac{5}{2}\Omega_{1j}^{11}-\Omega_{1j}^{12}\right), \\ q^{11} &= 8\sum_{j=3}^{\nu} n_j\left(\tfrac{25}{4}\Omega_{1j}^{11}-5\Omega_{1j}^{12}+\Omega_{1j}^{13}\right), \\ q^{02} &= 8\sum_{j=3}^{\nu} n_j\left(\tfrac{35}{4}\Omega_{1j}^{11}-7\Omega_{1j}^{12}+\Omega_{1j}^{13}\right), \\ q^{12} &= 8\sum_{j=3}^{\nu} n_j\left(\tfrac{175}{16}\Omega_{1j}^{11}-\tfrac{105}{8}\Omega_{1j}^{12}+\tfrac{19}{4}\Omega_{1j}^{13}-\tfrac{1}{2}\Omega_{1j}^{14}\right), \\ q^{22} &= 8\sum_{j=3}^{\nu} n_j\left(\tfrac{1225}{64}\Omega_{1j}^{11}-\tfrac{245}{8}\Omega_{1j}^{12}+\tfrac{133}{8}\Omega_{1j}^{13}-\tfrac{7}{2}\Omega_{1j}^{14}+\tfrac{1}{4}\Omega_{1j}^{15}\right). \end{aligned}\right\} \quad (12)$$

Equations (9), (10), (11) and (12) determine the parallel electrical conductivity for arbitrary electron-neutral cross-sections. To obtain the second Chapman-Enskog approximation, one deletes the last row and column from Δ , and similarly for the first approximation.

IV. RESULTS FOR A THREE COMPONENT PLASMA

For the sake of simplicity, we shall consider a three component plasma, and represent the effects of electron-neutral collisions by the model isotropic differential cross-section

$$I_{13}(g) = A_m g^{2m-1} , \tag{13}$$

where m is a parameter.

Using Eq. (13) one obtains the relation

$$\Omega_{13}^{1s} = \frac{\Gamma(s+m+3/2)}{\Gamma(m+5/2)}\Omega_{13}^{11} = (\pi)^{1/2} A_m \left(\frac{2kT}{m_1}\right)^m \Gamma(s+m+3/2) , \tag{14}$$

and therefore

$$\left.\begin{aligned} q^{00} &= 8n_3\Omega_{13}^{11} , \\ q^{01} &= -8n_3 m\Omega_{13}^{11} , \\ q^{11} &= 4n_3(2m^2+2m+5)\Omega_{13}^{11} , \\ q^{02} &= 4n_3 m(m-1)\Omega_{13}^{11} , \\ q^{12} &= -4n_3(m^3+m^2+5m)\Omega_{13}^{11} , \\ q^{22} &= 2n_3(m^4+2m^3+13m^2+12m+35/2)\Omega_{13}^{11} . \end{aligned}\right\} \tag{15}$$

To examine the dependence of $\sigma(3)$ on degree of ionization (n_1/n_3), it is convenient and instructive to normalize $\sigma(3)$ with respect to σ_{add}, where

$$\frac{1}{\sigma_{add}} = \frac{1}{\sigma_{en}} + \frac{1}{\sigma_{ei}} . \tag{16}$$

Equation (16) is a mixture rule for calculating the electrical conductivity of a partially ionized gas proposed in Ref. (2), in which

$$\sigma_{en} = \frac{4\pi e^2}{3kT} \int_0^\infty \frac{g^4 f_1^{(0)}(g)}{\nu_{en}(g)}\, dg \qquad (17)$$

is the conductivity for a Lorentzian gas[1] and

$$\sigma_{ei} = \frac{(0.582)\,64\,(2\pi)^{1/2} \epsilon_0^2 \,(kT)^{3/2}}{(m_1)^{1/2} e^2 \ln \Lambda} \qquad (18)$$

is the Spitzer-Härm[6] conductivity for a fully-ionized gas. Here, k is Boltzmann's constant, T is the absolute temperature, $\nu_{en}(g)$ is the electron-neutral collision frequency, e the electronic charge, ϵ_o the dielectric constant of free space, and Λ is the ratio of the Debye length to the impact parameter for a 90° collision. The physical basis underlying Eq. (16) is that the resistivity is proportional to electron collision frequency, and therefore to the sum of electron-neutral and of electron-ion collision frequencies. Eq. (16) has the property of yielding the correct result in the limiting cases of either zero or infinite degrees of ionization.

For a three component plasma, $\nu_{en}(g) = g\, n_3\, Q_{13}^{(1)}$, and using Eq. (13), we obtain

$$\sigma_{en} = \frac{n_1 e^2}{m_1} \frac{\Gamma(5/2 - m)}{3(\pi)^{1/2} A_m n_3} \left(\frac{m_1}{2kT}\right)^m . \qquad (19)$$

As a convenient measure of the degree of ionization, we may introduce the dimensionless variable

$$X \equiv \frac{\sigma_{en}}{\sigma_{ei}} = \gamma_m \frac{n_1}{n_3} \frac{\Omega_{12}^{11}}{\Omega_{13}^{11}} \quad , \tag{20}$$

where

$$\gamma_m = \frac{\Gamma(m+5/2)\Gamma(m-5/2)}{6(0.582)} \quad . \tag{21}$$

Employing Eqs. (14), (15), (16), (18), (19) and (20), we may write Eq. (10) in the form

$$\frac{\sigma(3)}{\sigma_{odd}} = \frac{3\pi(1+X)}{32(0.582)} \frac{\begin{vmatrix} \beta_m + 2.072X & \delta_m + 3.582X \\ \delta_m + 3.582X & \lambda_m + 10.792X \end{vmatrix}}{\begin{vmatrix} \gamma_m + X & \alpha_m + X & \epsilon_m + 1.875 \\ \alpha_m + X & \beta_m + 2.072X & \delta_m + 3.582X \\ \epsilon_m + 1.875X & \delta_m + 3.582X & \lambda_m + 10.792X \end{vmatrix}} \quad , \tag{22}$$

where

$$\left.\begin{aligned} \alpha_m &= -\tfrac{2}{3} m \gamma_m \, , \\ \beta_m &= \tfrac{2}{9}(2m^2 + 2m + 5)\gamma_m \, , \\ \delta_m &= -\tfrac{1}{3}(m^3 + m^2 + 5m)\gamma_m \, , \\ \lambda_m &= \tfrac{1}{4}(m^4 + 2m^3 + 13m^2 + 12m + 35/2)\gamma_m \, , \\ \epsilon_m &= \tfrac{1}{4}(m^2 - m)\gamma_m \, . \end{aligned}\right\} \tag{23}$$

Figures 3 to 6 show the dependence on degree of ionization of the first 3 approximations to the electrical conductivity, given by Eq. (20), for several values of the parameter m. These curves indicate the rate of convergence of successive approximations. For larger values of the degree of ionization, charged particle interactions dominate. The conductivity is insensitive to the parameter m and convergence is rapid. The behavior at lower levels of ionization is different however. For $m = -1$ and for $m = 0$ (Maxwellian molecules) convergence is excellent and the third approximation seems to be sufficient for all levels of ionization. For $m = 1/2$ (hard spheres) and $m = 1$, convergence at low ionization is less rapid. While the third approximation for $m = 1$ is satisfactory at high levels of ionization, it is not very good at low ionization levels. This behavior, of course, is merely a reflection of the results illustrated for a Lorentzian gas in Fig. 1. Finally, it appears that the mixture rule proposed by Lin, Resler, and Kantrowitz may in some circumstances over-estimate the electrical conductivity by about 70 per cent.

V. EXAMINATION OF FROST'S MIXTURE RULE

In 1961, Frost[3] proposed calculating the electrical conductivity of a partially ionized gas using Eq. (17), valid for a Lorentzian gas, but replacing the electron-neutral collision frequency with the sum of the electron-neutral and electron-ion collision frequencies. To take into account electron-electron interactions, Frost introduced a modified expression for the electron-ion collision frequency having the form

$$\nu_{ei}(g) = K g^{-2} , \tag{24}$$

in which

$$K = .476 \, \frac{2\pi n_1 e_1^2 \ln \Delta}{(4\pi\epsilon_0)^2} \left(\frac{2e_1}{m_1}\right)^{3/2} \left(\frac{e_1}{kT}\right)^{1/2} ,$$

is determined so as to yield the Spitzer-Härm value of the electrical conductivity in the fully-ionized limit.

In order to examine the validity of Frost's procedure, we have included in Figs. 3 to 6 the results obtained using Frost's formula. At very low values of ionization Frost's results seem to be more satisfactory, as expected, since his formulation is rigorous in the limit of zero electron concentration. For high degrees of ionization, Frost's results agree well with the third approximation, and in the fully-ionized limit his results, as they must be, are slightly better than the third approximation (by less than 2%). Over the complete range of ionization and for all the cross-section models examined, Frost's results and the third approximation agree to better than 15 per cent. Considering the uncertainties in actual cross-section data (see Section V) this agreement appears quite satisfactory.

In the range of degree of ionization corresponding to $.01 \leq \frac{\sigma_{en}}{\sigma_{ei}} \leq 1$ it is not clear to what extent the difference between Frost and the third Chapman-Enskog approximation is attributable to Frost's representation of the charged particle interactions, or to a slow rate of convergence. To illustrate this point we have plotted in Figs. 7, 8 and 9 for three values of m , conductivities using Frost's formula, but with

$$\nu_{ei}(g) = K_1 g^{-3} , \tag{25}$$

and with

$$\nu_{ei}(g) = K_2 g^{-1.36} . \tag{26}$$

We have also replotted the curves obtained with Eq. (24) and the third Chapman-Enskog approximation. The constants K_1 and K_2 have been adjusted in the same manner as K . An additional comparison is provided by using Shkarofsky's[8] fourth approximation which is based on a solution of the Fokker-Planck equation resulting in the same matrix as derived from Eq. (7).*

*Shkarofsky's results are not as readily applicable for use with experimental e-n cross-sections, as are the Chapman-Enskog results.

The velocity dependence of Eq. (25) follows directly from the Coulomb cross-section of Eq. (8). For very low degrees of ionization where the electron distribution function is unaffected by electron-electron encounters, we would expect Frost's formula to apply using Eq. (25). The collision frequency of Eq. (26) has been suggested by Sodha and Varshni[11], to which Frost has given a single power fit, and is based on the Spitzer-Härm numerical solution for a fully-ionized plasma. One would expect the use of Eq. (26) with Frost's formula to supply a close approximation at high degrees of ionization.

Figures 7, 8 and 9 show that Frost's formula with Eq. (24) lies between the curves obtained with Eqs. (25) and (26) for all degrees of ionization. This results suggests that the velocity power dependence g^{-2} may be viewed as a compromise choice which achieves a reasonable approximation to the conductivity for both low and high degrees of ionization. The Frost mixture rule appears to over-estimate the conductivity at low degrees of ionization, and to under-estimate slightly at high degrees of ionization. The conclusion for low degrees of ionization is supported by additional calculations in which Eq. (17) is used with Eq. (25), but with K_1 determined by the Rutherford collision cross-section, rather than by adjustment to agree with the Spitzer-Härm conductivity.

VI. RESULTS WITH EXPERIMENTAL CROSS-SECTIONS

Figure 10 shows Frost's normalized conductivity of cesium seeded argon as a function of temperature, in the range of interest for magnetogasdynamic generators. Curve I makes use of the e-Ar. cross-section given by Frost and Phelps[12] and curve II is based on Brode's[13] cross-section for e-Ar. . Both curves use Brode's[16] cross-section for e-Cs . The same dipping in the actual conductivity curve is observed, substantiating some of the conclusions drawn from the use of an electron-neutral collision model.

We have attempted to estimate the possible spread in conductivity calculations resulting from the uncertainty in experimental values of the collision cross-sections. Figure 11 shows the momentum collision cross-section for e-Ar. obtained by different experimenters. Based on O'Malley's[14] extrapolation for low energy and the various experimental results as well as their own work, Frost and Phelps[12] have suggested an effective electron-argon cross-section which is shown on Fig. 12. In subsequent calculations we have used this curve as a lower bound and Brode's data with extrapolation at low energy as an upper bound.

Figure 11 shows the momentum collision cross-section for e-Cs . The data spread at low energy is quite large. To determine the corresponding uncertainty in calculated conductivity we have chosen a reasonable (low energy) upper bound guided by recent theoretical calculations made by Stone and Reitz[15]

[0 to .75 $(ev)^{1/2}$] which has been merged with Brode's[16] data [1.6 to 10 $(ev)^{1/2}$] , and a lower bound as shown in Fig. 12.

Typical results of the calculations for upper and lower bound conductivities of cesium seeded argon, using Frost's formula and the third Chapman-Enskog approximation are shown in Fig. 13. As expected from the study of the electron-neutral collision model at low ionization, Frost's conductivity is higher than the conductivity calculated via the third approximation. As the degree of ionization increases (temperature increase), the difference becomes less pronounced.

The maximum difference between the Frost and Chapman-Enskog conductivities is about 25% and occurs at 2000°K. Over the entire temperature range, the difference between the Frost and the Chapman-Enskog third approximation conductivities is of the same order or less than the difference in calculated conductivity resulting from uncertainty in experimental values of the cross-sections. For comparison some recent conductivity measurements made by Harris[22] are included.

VII. CONCLUSIONS

Figures 3, 4, 5, and 6 show that the conductivities calculated by Frost's method and by the third approximation to the simplified Chapman-Enskog method differ by less than 15 per cent for a wide range of collision cross-section models. Figure 13 indicates that the uncertainties in calculated conductivity resulting from experimental cross-section uncertainties exceed, or are of the same order of magnitude as the differences resulting from these two methods of calculation. Examination of experimental and theoretical cross-section data available for elastic collision of electrons with a variety of neutral species indicates that the magnitude of the uncertainties we have used appears to be typical.[9,10] We may therefore conclude that Frost's mixture rule, at present, provides a satisfactory approximation for calculating the electrical conductivity in engineering applications.

VIII. ACKNOWLEDGMENTS

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APPENDICES

A. DERIVATION OF SIMPLIFIED CHAPMAN-ENSKOG APPROXIMATION FORMULAE

Because Eq. (3) is linear in ϕ_1 , the solution will have the structure

$$\phi_1 = -\underline{\underline{B}} : \frac{\partial}{\partial \underline{r}} \underline{c}_0 - \underline{A} \cdot \frac{\partial}{\partial \underline{r}} \ln T - n\underline{D} \cdot \underline{d}_1 \quad , \tag{A1}$$

where the unknown vector function $\underline{D}$ is determined by inserting Eq. (A1) into Eq. (3) and equating coefficients of similar gradients. The resulting integral equation for $\underline{D}$ is:

$$-\frac{f_1^{(0)}}{n_1} \underline{C}_1 = -f_1^{(0)} \frac{e_1}{m_1} (\underline{C}_1 \times \underline{B}) \cdot \frac{\partial}{\partial \underline{C}_1} \underline{D} + \sum_{j=1}^{\nu} \iint f_1^{(0)}(c_1) f_j^{(0)}(c) \left[\underline{D}(c_1') + \delta_{1j} \underline{D}(\underline{c}') - \underline{D}(c_1) - \delta_{1j} \underline{D}(\underline{c}) \right] g I_{1j} \, d\Omega \, d\underline{c} \, . \tag{A2}$$

The quantities $\underline{\underline{B}}$ and $\underline{A}$ are determined by similar equations, but we shall not consider these since the dependence of $\underline{j}^{(1)}$ on $\underline{E}'$ (and thereby the electrical conductivity) is provided through $\underline{D}$.

For the reasons discussed in Ref. (1) and (7),

$$\underline{D} = \underline{C}_1 D_1 + (\underline{C}_1 \times \underline{B}) D_2 + \underline{B} (\underline{C}_1 \cdot \underline{B}) D_3 \quad . \tag{A3}$$

The quantities D_1 , D_2 , and D_3 are scalar functions of the magnitudes C_1 and B , and are determined by the three simultaneous equations

$$-\frac{f_1^{(0)}}{n_1}\underline{C}_1 = f_1^{(0)}\frac{e_1}{m_1}B^2 D_2 \underline{C}_1 + \sum_{j=1}^{\nu}\iint f_1^{(0)}(c_1) f_j^{(0)}(c)\,[D_1(C_1')\underline{C}_1' + \delta_{1j} D_1(C')\underline{C}' - D_1(C_1)\underline{C}_1 - \delta_{1j} D_1(C)\underline{C}]\,g I_{1j}\, d\Omega\, dc, \tag{A4}$$

$$0 = -f_1^{(0)}\frac{e_1}{m_1} D_1 \underline{C}_1 + \sum_{j=1}^{\nu}\iint f_1^{(0)}(c_1) f_j^{(0)}(c)\,[D_2(C_1')\underline{C}_1' + \delta_{1j} D_2(C')\underline{C}' - D_2(C_1)\underline{C}_1 - \delta_{1j} D_2(C)\underline{C}]\,g I_{1j}\, d\Omega\, dc, \tag{A5}$$

$$0 = -f_1^{(0)}\frac{e_1}{m_1} D_2 \underline{C}_1 + \sum_{j=1}^{\nu}\iint f_1^{(0)}(c_1) f_j^{(0)}(c)\,[D_3(C_1')\underline{C}_1' + \delta_{1j} D_3(C')\underline{C}' - D_3(C_1)\underline{C}_1 - \delta_{1j} D_3(C)\underline{C}]\,g I_{1j}\, d\Omega\, dc\,. \tag{A6}$$

Multiplying Eq. (A6) by B^2 and adding to Eq. (A4) yields the equation

$$-\frac{f_1^{(0)}}{n_1}\underline{C}_1 = \sum_{j=1}^{\nu} f_1^{(0)}(c_1) f_j^{(0)}(c)\Big\{[D_1(C_1') + B^2 D_3(C_1')]\underline{C}_1' + \delta_{1j}[D_1(C') + B^2 D_3(C')]\underline{C}' - [D_1(C_1) + B^2 D_3(C_1)]\underline{C}_1 - \delta_{1j}[D_1(C) + B^2 D_3(C)]\underline{C}\Big\}\,g I_{1j}\, d\Omega\, dc \tag{A7}$$

for $(D_1 + B^2 D_3)$. Multiplying Eq. (A5) by iB and adding to Eq. (A4) yields a single complex equation

$$-\frac{f_1^{(0)}}{n_1}\underline{C}_1 = i\omega f_1^{(0)}\Upsilon\underline{C}_1 + \sum_{j=1}^{\nu}\iint f_1^{(0)}(c_1) f_j^{(0)}(c)\,[\Upsilon(C_1')\underline{C}_1' + \delta_{1j}\Upsilon(C')\underline{C}' - \Upsilon(C_1)\underline{C}_1 - \delta_{1j}\Upsilon(C)\underline{C}]\,g I_{1j}\, d\Omega\, dc\,, \tag{A8}$$

for $\zeta = D_1 + iB\,D_2$. The electron cyclotron frequency is denoted by

$$\omega = -\frac{e_1 B}{m_1} \, . \tag{A9}$$

According to Eq. (18.41-1) of Ref. (1), (7) and Eq. (4), the contribution of $\underline{D}$ to the current density is

$$\underline{j}_1^{(1)} = -\frac{n e_1}{3} \int f_1^{(0)} C_1^2 \left[D_1 \underline{d}_1 + D_2 (\underline{B} \times \underline{d}_1) + D_3 (\underline{B} \cdot \underline{d}_1) \underline{B} \right] d\underline{c}_1 \, . \tag{A10}$$

If $\underline{d}_1$ is decomposed into components parallel and perpendicular to the magnetic field, $\underline{d}_1 = \underline{d}_1^{\parallel} + \underline{d}_1^{\perp}$, then

$$\begin{aligned} \underline{j}_1^{(1)} &= -\frac{n e_1}{3} \int f_1^{(0)} C_1^2 \left[(D_1 + B^2 D_3) \underline{d}_1^{\parallel} + D_1 \underline{d}_1^{\perp} + D_2 \underline{B} \times \underline{d}_1^{\perp} \right] d\underline{c}_1 \\ &= \sigma \underline{\mathcal{E}}^{\parallel} + \sigma_1 \underline{\mathcal{E}}^{\perp} + \sigma_2 \underline{\hat{B}} \times \underline{\mathcal{E}} \, , \end{aligned} \tag{A11}$$

where

$$\underline{\mathcal{E}} = \underline{E}' - \frac{p\rho}{n_1 (e_1 - m_1 \rho_c)} \left[\frac{\partial}{\partial \underline{r}} \left(\frac{n_1}{n} \right) + \left(\frac{n_1}{n} - \frac{m_1 n_1}{\rho} \right) \frac{\partial}{\partial \underline{r}} \ln p \right] \tag{A12}$$

is a "generalized" electric field, and where $\underline{B} = B\,\underline{\hat{B}}$. The electrical conductivity parallel to the magnetic field σ , is determined by $(D_1 + B^2 D_3)$ and from Eq. (A7), σ is independent of B . The transverse electrical conductivities are determined in terms of Υ as

$$\sigma_1 + i\sigma_2 = \frac{n_1 e^2}{3kT}\left(1 - \frac{m_1 \rho_c}{\rho e_1}\right)\int f_1^{(0)} C_1^2 \Upsilon \, d\varsigma_1 \ . \tag{A13}$$

Since σ_1 reduces to σ when $B = 0$, we need be concerned only with the solution to Eq. (A8). Let us look for a solution of the form of a sum of Sonine polynomials,

$$\Upsilon(\zeta_1) = \sum_{m'=0}^{\xi-1} d^{(m')} S_{3/2}^{m'}(\zeta_1^2) \ , \tag{A14}$$

where ξ is an integer that defines the order of approximation. In terms of the coefficients $d^{(m)}$, the transverse conductivities are given by Eq. (A13) as

$$\begin{aligned} \sigma_1 + i\sigma_2 &= \frac{n_1 e^2}{3kT}\left(1 - \frac{m_1 \rho_c}{\rho e_1}\right)\sum_{m'=0}^{\xi-1} d^{(m')}\int f_1^{(0)} C_1^2 S_{3/2}^{m'}(\zeta_1^2)\, d\varsigma_1 \\ &= \frac{n_1^2 e_1^2}{m_1}\left(1 - \frac{m_1 \rho_c}{\rho e_1}\right) d^{(0)}(\xi) \ , \end{aligned} \tag{A15}$$

where we have used the orthogonality property

$$\int_0^\infty y^{2n+1} e^{-y^2} S_n^m(y^2)\, S_n^{m'}(y^2)\, dy = \frac{1}{2}\frac{(n+m)!}{m!}\delta_{mm'} \ . \tag{A16}$$

Substituting (A14) into Eq. (A8), we obtain

$$-\frac{f_1^{(0)}}{n_1}\underline{C}_1 = \sum_{m'=0}^{\xi-1} d^{(m')}\Big\{ i\omega f_1^{(0)} \underline{C}_1 S_{3/2}^{m'}(\mathcal{C}_1^2) + \sum_{j=1}^{\nu} \iint f_1^{(0)}(\underline{c}_1) f_j^{(0)}(\underline{c}) [\underline{C}_1' S_{3/2}^{m'}(\mathcal{C}_1'^2) + \delta_{1j} \underline{C}' S_{3/2}^{m'}(\mathcal{C}'^2) - \underline{C}_1 S_{3/2}^{m'}(\mathcal{C}_1^2) - \delta_{1j} \underline{C} S_{3/2}^{m'}(\mathcal{C}^2)] g I_{1j} \, d\Omega \, d\underline{c} \Big\} . \tag{A17}$$

If we now take the scalar product of this equation with $S_{3/2}^{m}(\mathcal{C}_1^2)\underline{\mathcal{C}}_1$ and integrate with respect to $d\underline{c}_1$, we obtain the set of ξ equations

$$\sum_{m'=0}^{\xi-1} Q_{mm'} d^{(m')} = \frac{3}{2}\delta_{m0} \quad , \qquad (m = 0, 1, 2, \ldots \xi-1) , \tag{A18}$$

where

$$Q_{mm'} = n_1^2 [\underline{\mathcal{C}}_1 S_{3/2}^{m}(\mathcal{C}_1^2) ; \underline{\mathcal{C}}_1 S_{3/2}^{m'}(\mathcal{C}_1^2)]_1 + \sum_{j=2}^{\nu} n_1 n_j [\underline{\mathcal{C}}_1 S_{3/2}^{m}(\mathcal{C}_1^2) ; \underline{\mathcal{C}}_1 S_{3/2}^{m'}(\mathcal{C}_1^2)]_{1j} - i\omega n_1 \frac{2}{(\pi)^{1/2}} \frac{(m+3/2)!}{m!} \delta_{mm'} , \tag{A19}$$

and where

$$[\underline{F} ; \underline{G}]_1 \equiv -\frac{1}{n_1^2} \iiint f_1^{(0)}(\underline{c}_1) f_1^{(0)}(\underline{c}) \underline{F}_1(\underline{c}_1) \cdot [\underline{G}_1(\underline{c}_1') + \underline{G}_1(\underline{c}') - \underline{G}_1(\underline{c}_1) - \underline{G}_1(\underline{c})] g I_{11} \, d\Omega \, d\underline{c} \, d\underline{c}_1 = [\underline{F}_1 ; \underline{G}_2]_{11} + [\underline{F}_1 ; \underline{G}_1]_{11} , \tag{A20}$$

$$[\underline{F}_1 ; \underline{G}_1]_{1j} \equiv \frac{-1}{n_1 n_j} \iiint f_1^{(0)}(\underline{c}_1) f_j^{(0)}(\underline{c}) \underline{F}_1(\underline{c}_1) \cdot [\underline{G}_1(\underline{c}_1') - \underline{G}_1(\underline{c}_1)] g I_{1j} \, d\Omega \, d\underline{c} \, d\underline{c}_1 , \tag{A21}$$

$$[\underline{F}_1;\underline{G}_2]_{1j} \cong \frac{-1}{n_1 n_j}\iiint f_1^{(0)}(\underline{c}_1) f_j^{(0)}(\underline{c}) \underline{F}_1(\underline{c}_1)\cdot[\underline{G}_2(\underline{c}') - \underline{G}_2(\underline{c})] g I_{1j} d\Omega d\underline{c} d\underline{c}_1 . \tag{A22}$$

These definitions correspond to the definitions of Section (4.4) in Ref. (1). Values for the bracket expressions in terms of the integrals

$$\Omega_{ij}^{\ell,s} = \left(\frac{kT}{2\pi\mu_{ij}}\right)^{1/2}\int_0^\infty e^{-\gamma^2}\gamma^{2s+3} Q_{ij}^{(\ell)}(g)\, d\gamma \quad ; \quad \gamma = \left(\frac{\mu_{ij}}{2kT}\right)^{1/2} g \, , \tag{A23}$$

where

$$Q_{ij}^{(\ell)}(g) = \int (1-\cos^{\ell}\chi)\, I_{ij}(\hat{\Omega}, g)\, d\Omega \, , \tag{A24}$$

are provided in Section (9.6) of Ref. (1) and in Ref. (4). The quantity μ_{ij} denotes the reduced mass; $\mu_{11} = m_1/2$ and $\mu_{12} \cong \mu_{13} \cong \ldots \cong m_1$.

B. LORENTZIAN GAS RESULTS

In thi. appendix, we primarily summarize some results for a Lorentzian gas as discussed by Chapman and Cowling[1]. The formula for the electrical conductivity of a Lorentzian gas, Eq. (17) follows from Chapman and Cowling's Eqs. (10.5-7), (9.33-2), (18.11-5) and is shown to correspond to the <u>infinite</u> Chapman-Enskog approximation $\sigma_{en}(\infty)$. (This result has also been derived by Allis[21] using an expansion in spherical harmonics.) The <u>first</u> Chapman-Enskog approximation to the electrical conductivity of a Lorentzian gas may be shown from Eqs. (9.81-1), (9.8-8), (18.11-5) to be

$$\sigma_{en}(1) = \frac{3}{16} \frac{n_1 e_1^2}{n_3 m_1} \frac{1}{\Omega_{13}^{1,1}} , \tag{B1}$$

where $\Omega_{13}^{1,1}$ is defined by our Eq. (A.23).

For the special case of an interparticle interaction force proportional to (interparticle separation)$^{-\nu}$, Chapman and Cowling in Eq. (10.53-10) show that the successive Chapman-Enskog approximations may be written

$$\frac{\sigma_{en}(\xi)}{\sigma_{en}(1)} = 1 + \frac{p^2}{q \cdot 1} + \frac{p^2(p+1)^2}{q(q+1) \cdot 2!} + \ldots \text{ to } \xi \text{ terms,} \tag{B2}$$

where $p = (\nu - 5)/2(\nu - 1)$, $q = 3 - 2/(\nu - 1)$. From Chapman and Cowling's Eq. (10.3-8), the parameter ν may be related to our parameter m introduced in Eq. (13) according to

$$\frac{\nu - 5}{\nu - 1} = 2m \quad . \tag{B3}$$

Combining Eqs. (14), (19) and (B1), we obtain for this case

$$\frac{\sigma_{en}(\infty)}{\sigma_{en}(1)} = \frac{16}{9\pi}\,\Gamma(\tfrac{5}{2} - m)\;\Gamma(\tfrac{5}{2} + m) \tag{B4}$$

subject to the restrictions $-\frac{5}{2} < m < \frac{5}{2}$. The results plotted in Fig. 1 are based on Eqs. (B.2) and (B.4).

C. EFFECT OF SMALL $\ln\Lambda$

The kinetic theory description of collisions between charged particles has, until recently[20], required some sort of ad hoc cut-off procedure in order to prevent the occurrence of divergent integrals. The source of this difficulty stems from the long-range nature of the Coulomb interaction and the treatment of all collisions as two-body encounters. In actuality, the interaction potential is essentially shielded by the collective behavior of the particles at sufficiently large distances. A convergent theory is obtained by introducing some device for ignoring collisions with large impact parameters in excess of the Debye length. The resulting theory is valid to order $(\ln\Lambda)^{-1}$, where $\Lambda = 1.24 \times 10^7 \, (T^3/n)^{1/2}$ is the ratio of the Debye length to the impact parameter for a 90° deflection. (The quantities T and n denote the temperature in °K and the number density per m^3 respectively.) In particular, the widely-quoted value of Spitzer and Härm for the electrical conductivity of a fully-ionized plasma is correct to this order.

In Table (1), we have calculated the values of Λ and $\ln\Lambda$ for a range of temperatures and number densities which encompass conditions expected in magnetohydrodynamic generators. Typical values of $\ln\Lambda$ are seen to range between 4 and 5 which implies an uncertainty of about 25 per cent in the Spitzer-Härm conductivity.

TABLE 1

Values of Λ and $(\ln\Lambda)$

T_e(°K)	$n_e(m^{-3})$				
	10^{18}	10^{19}	10^{20}	10^{21}	10^{22}
1000	392 (5.97)	124 (4.82)	39.2 (3.67)	12.4 (2.52)	3.92 (1.37)
1500	721 (6.58)	224 (5.41)	72.1 (4.28)	22.4 (3.11)	7.21 (1.98)
2000	1110 (7.01)	351 (5.86)	111 (4.71)	35.1 (3.56)	11.1 (2.40)
2500	1550 (7.35)	491 (6.20)	155 (5.04)	49.1 (3.89)	15.5 (2.74)
3000	2040 (7.62)	645 (6.47)	204 (5.32)	64.5 (4.17)	20.4 (3.02)

The region to the left of the heavy line in this table indicates roughly conditions for which the uncertainty in this theory is less than 20 per cent.

In about 1960, Lenard, Balescu, Rostoker and Rosenbluth[17] as well as others, obtained a kinetic equation which automatically took into account the collective behavior of a plasma, and converged for large impact parameters. However, this so-called "fluid-approximation" diverged for small impact parameters, and so it was necessary to introduce an ad hoc small cut-off limit.

A more recent advance in the kinetic-theory of charged particles was made in 1963 by Kihara and Aono[20]. These authors proposed a "unified theory" for combining both the close and distant encounter contributions to the kinetic descriptions of collisions, in such a way as to yield a divergenceless theory without the need of ad hoc assumptions. The validity of this new theory is of order Λ^{-1}, and thus provides ample accuracy for magnetohydrodynamic generator applications. The region to the left of the double-line in the preceding table indicates conditions for which this new theory has an uncertainty of 10 per cent or less.

The theory of Kihara and Aono has been applied by Ikitawa[18] to the calculation of the electrical conductivity of a fully-ionized plasma. The method of calculation is similar to that of Chapman and Enskog, and involves an expansion in terms of Sonine polynomials. The results of Ikitawa may be incorporated into the framework of the standard Chapman-Enskog theory with the following identificiation of the bracket expressions:

$$\left[\underline{\mathcal{C}}_1 S^{m}_{3/2}(\mathcal{C}_1^2);\underline{\mathcal{C}}_1 S^{m'}_{3/2}(\mathcal{C}_1^2)\right]_1 = \frac{(2)^{1/2} e^4}{4\pi (2\pi)^{1/2} \epsilon_0^2 (m_1)^{1/2} (kT)^{3/2}} \left\{\left[\ln\left(\frac{4\Delta}{3\gamma^2}\right)+\frac{1}{2}\right] C_{mm'} - \langle D\rangle_{mm'} - \tilde{D}_{mm'}\right\}, \tag{C1}$$

$$\left[\underline{\mathcal{C}}_1 S^{m}_{3/2}(\mathcal{C}_1^2);\underline{\mathcal{C}}_1 S^{m'}_{3/2}(\mathcal{C}_1^2)\right]_{12} = \frac{e^4}{4\pi (2\pi)^{1/2} \epsilon_0^2 (m_1)^{1/2} (kT)^{3/2}} \left\{\left[\ln\left(\frac{4\Delta}{3\gamma^2}\right)\right] A_{mm'} - \langle B\rangle_{mm'}\right\} \tag{C2}$$

where

$$C_{mm'} = \begin{pmatrix} 0 & 0 & 0 \cdots \\ 0 & 1 & 3/4 \cdots \\ 0 & 3/4 & 45/16 \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} ; \quad \langle D \rangle_{mm'} = \begin{pmatrix} 0 & 0 & 0 \cdots \\ 0 & 0 & 1/2 \cdots \\ 0 & 1/2 & 1/2 \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} ; \tag{C3}$$

$$A_{mm'} = \begin{pmatrix} 1 & 3/2 & 15/8 \cdots \\ 3/2 & 13/4 & 69/16 \cdots \\ 15/8 & 69/16 & 433/64 \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} ; \quad \langle B \rangle_{mm'} = \begin{pmatrix} 0 & 1 & 2 \cdots \\ 1 & 2 & 35/8 \cdots \\ 2 & 35/8 & 121/16 \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} ; \tag{C4}$$

$$\tilde{D}_{mm'} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0.08075 & 0.42062 & \cdots \\ 0 & 0.42062 & -0.07688 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} ; \qquad \ln \gamma = 0.57722 \; . \tag{C5}$$

In Fig. 14 we have plotted the ratio of Ikitawa's second approximation to the Spitzer-Härm value of the electrical conductivity, as a function of $\ln\Lambda$. For values of $\ln\Lambda \approx 4$, the more accurate value of the electrical conductivity is about 30 per cent higher than the Spitzer-Härm value.

In 1964, Rynn[19] reported a measurement of the electrical conductivity in a fully-ionized plasma which was 20 per cent higher than the Spitzer-Härm theoretical value. The probable error of Rynn's measurement was stated to be $\pm$ 10 per cent. At the value of $\ln\Lambda = 7$ for which Rynn's experiment was performed, Ikitawa's value of conductivity is about 10 per cent larger than the Spitzer-Härm value, which places Ikitawa's theoretical value within the probable error of the experiment. This comparison with experiment appears to provide some support for the new theory.

Under magnetohydrodynamic generator conditions, the plasma is only partially ionized, and so the magnitude of the correction discussed above will be proportionately smaller. When the charged particle encounters are equally important to electron-neutral collisions, we may expect a correction of the order of 15 per cent.

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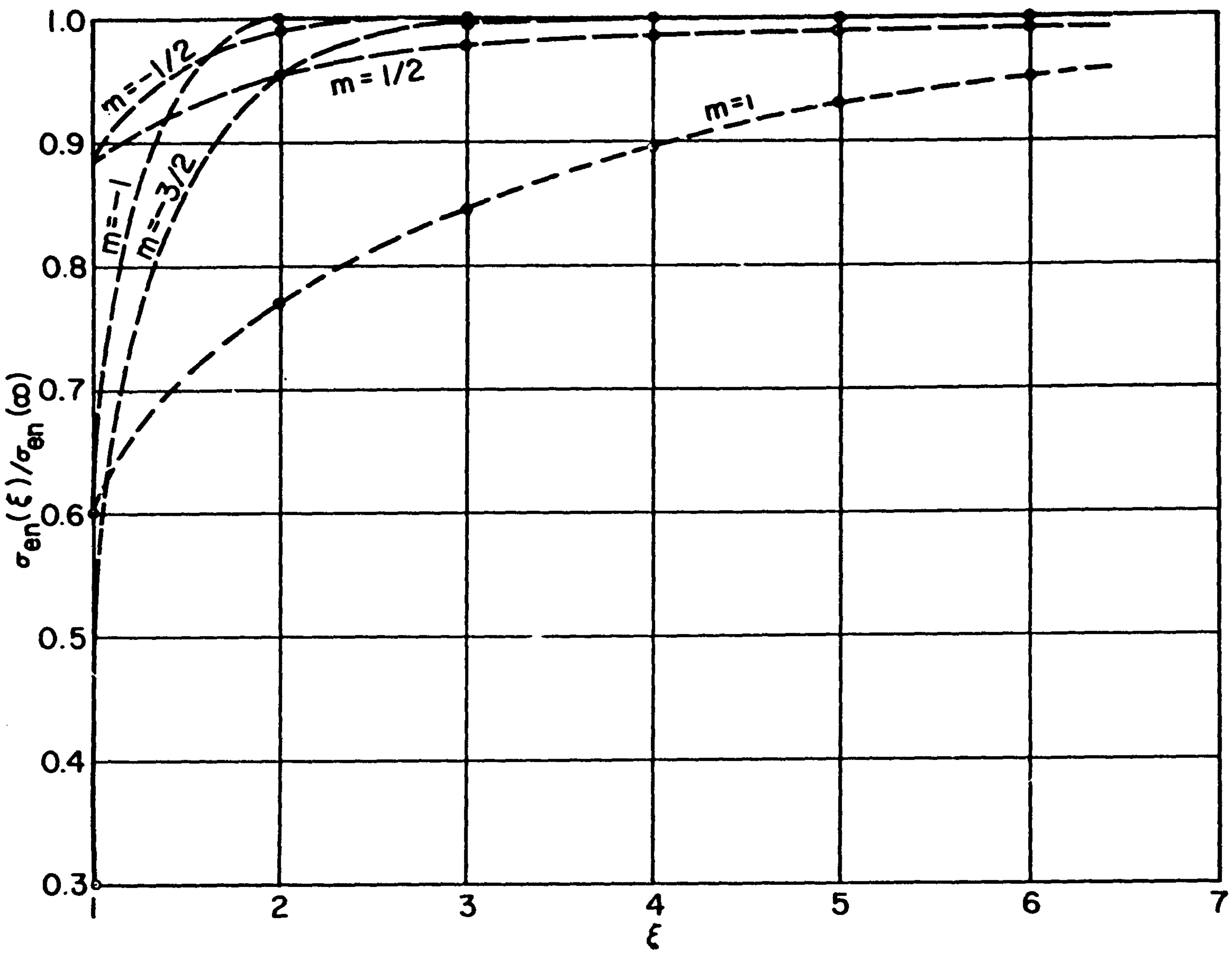

FIGURE 1. Rate of Convergence of Successive Chapman-Enskog Approximations for the Conductivity of a Lorentzian Gas. $\sigma_{en}(\xi)$ denotes the ξth approximation to the electrical conductivity. The electron-neutral collision frequency is assumed to depend on the relative speed

2m

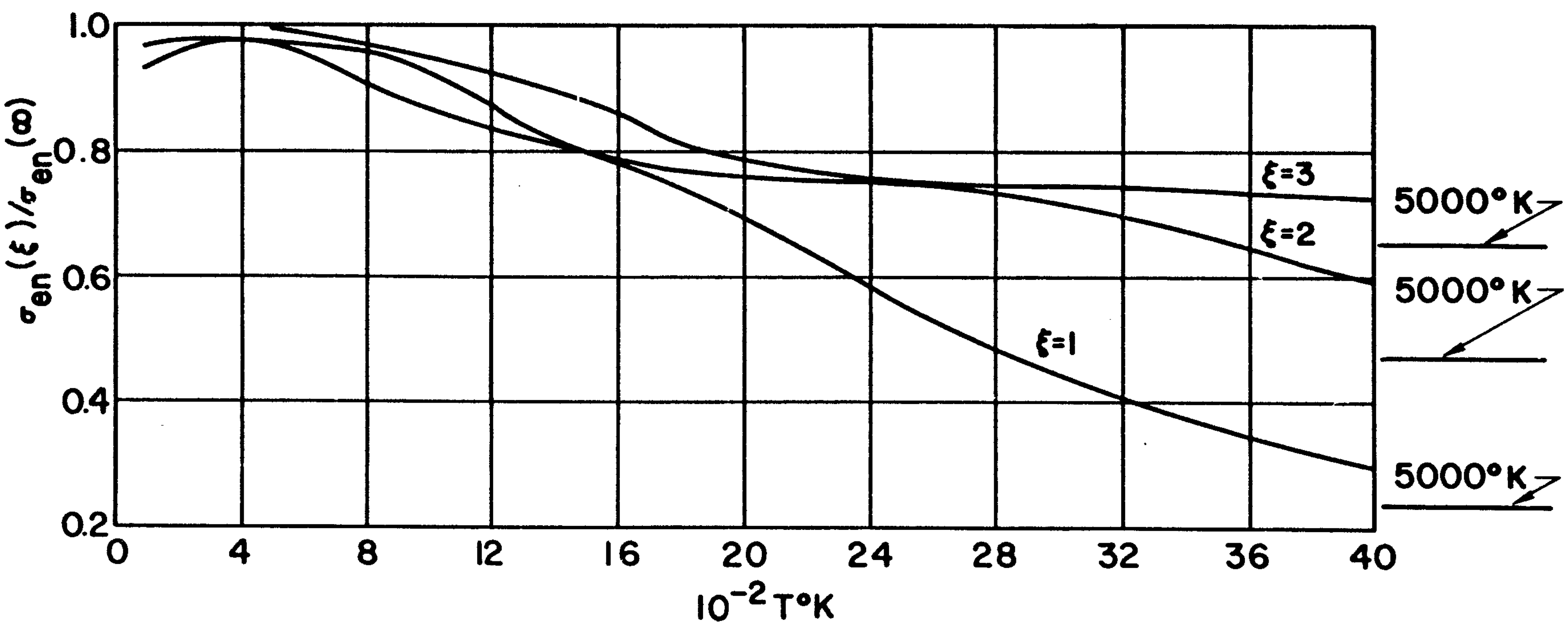

FIGURE 2. Rate of Convergence of Successive Chapman-Enskog Approximations for Conductivity as a Function of Temperature, T , for Weakly Ionized Argon. Calculations are based on electron-argon elastic momentum transfer cross-section as measured by Brode[13].

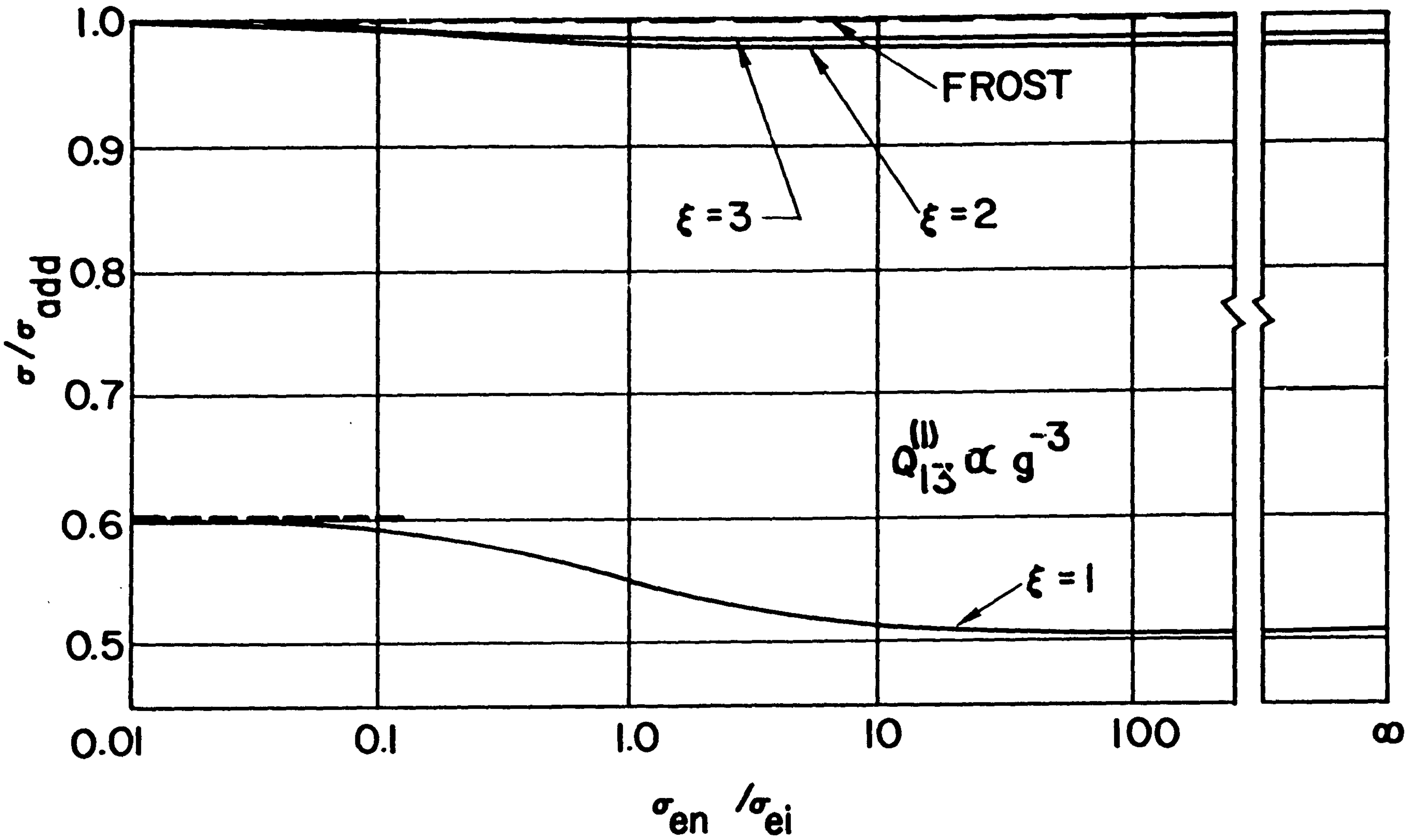

FIGURE 3. Normalized Conductivity of First Three Chapman-Enskog Approximations and of Frost as a Function of Degree of Ionization for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g^{-3} (m = -1)$. The normalizing conductivity, σ_{add}, is defined by Eq. (16) and σ_{en}/σ_{ei} is proportional to the degree of ionization as defined by Eq. (20). The quantity g denotes the relative electron speed.

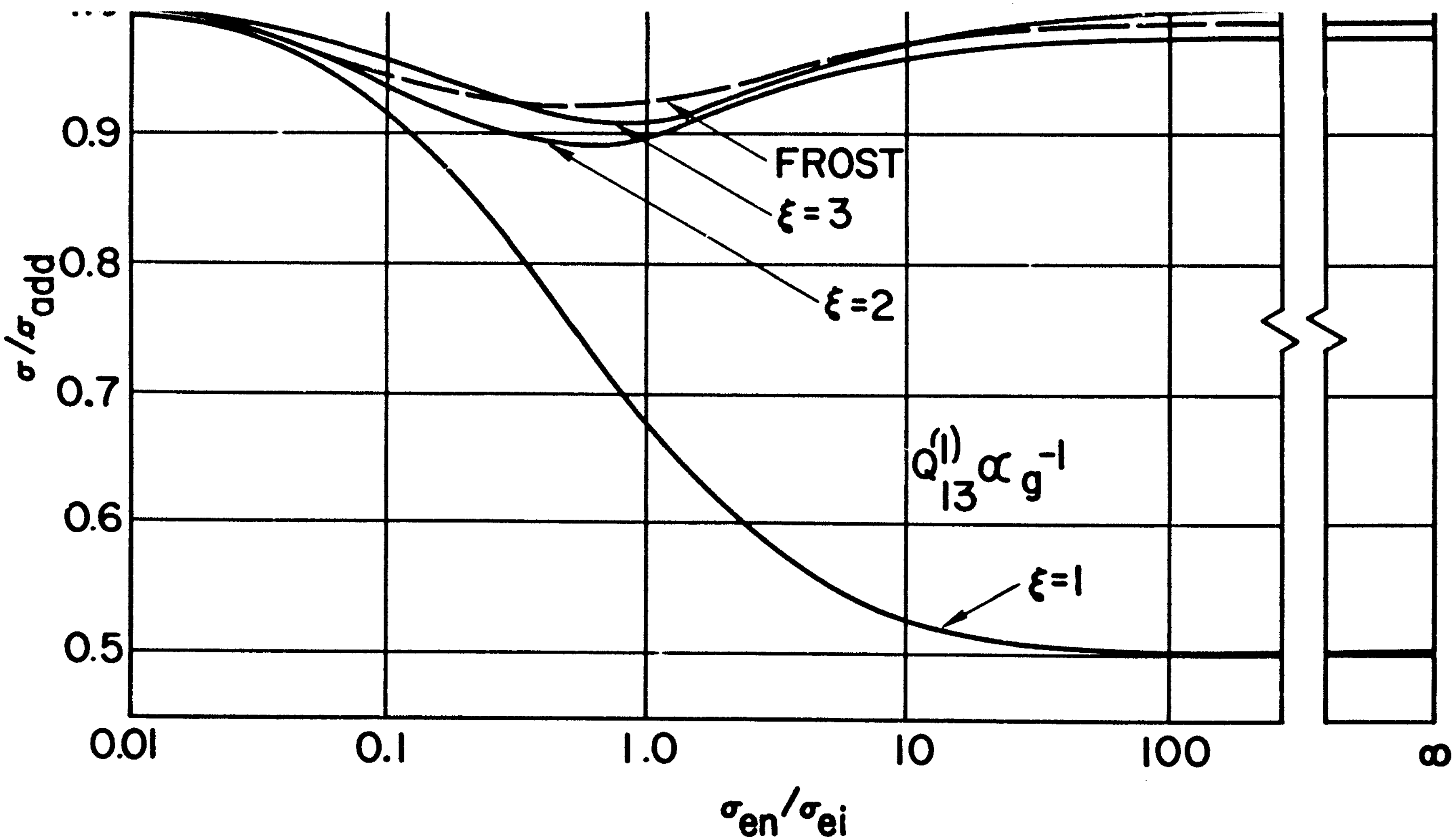

FIGURE 4. Normalized Conductivity of First Three Chapman-Enskog Approximations and of Frost as a Function of Degree of Ionization for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g^{-1}$ ([illegible], Maxwellian molecules). The normalizing conductivity, σ_{add}, is defined by Eq. ([illegible]) and σ_{en}/σ_{ei} is proportional to the degree of ionization as defined by Eq. ([illegible]). The quantity g denotes the relative electron speed.

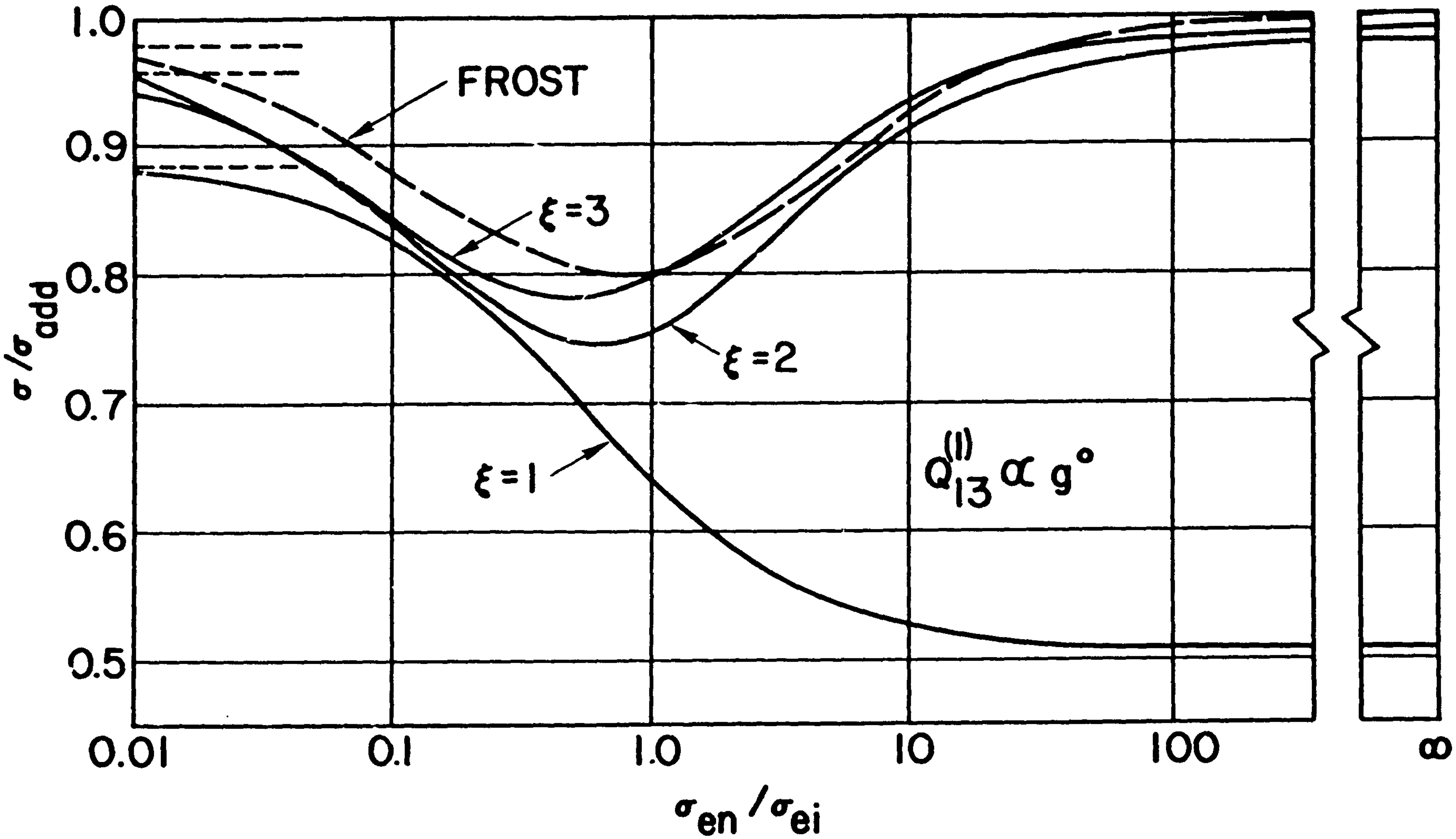

FIGURE 5. Normalized Conductivity of First Three Chapman-Enskog Approximations and of Frost as a Function of Degree of Ionization for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g^{0}$ ($m = 1/2$, Hard Spheres). The normalizing conductivity, σ_{add}, is defined by Eq. (16) and σ_{en}/σ_{ei} is proportional to the degree of ionization as defined by Eq. ([illegible]). The quantity g denotes

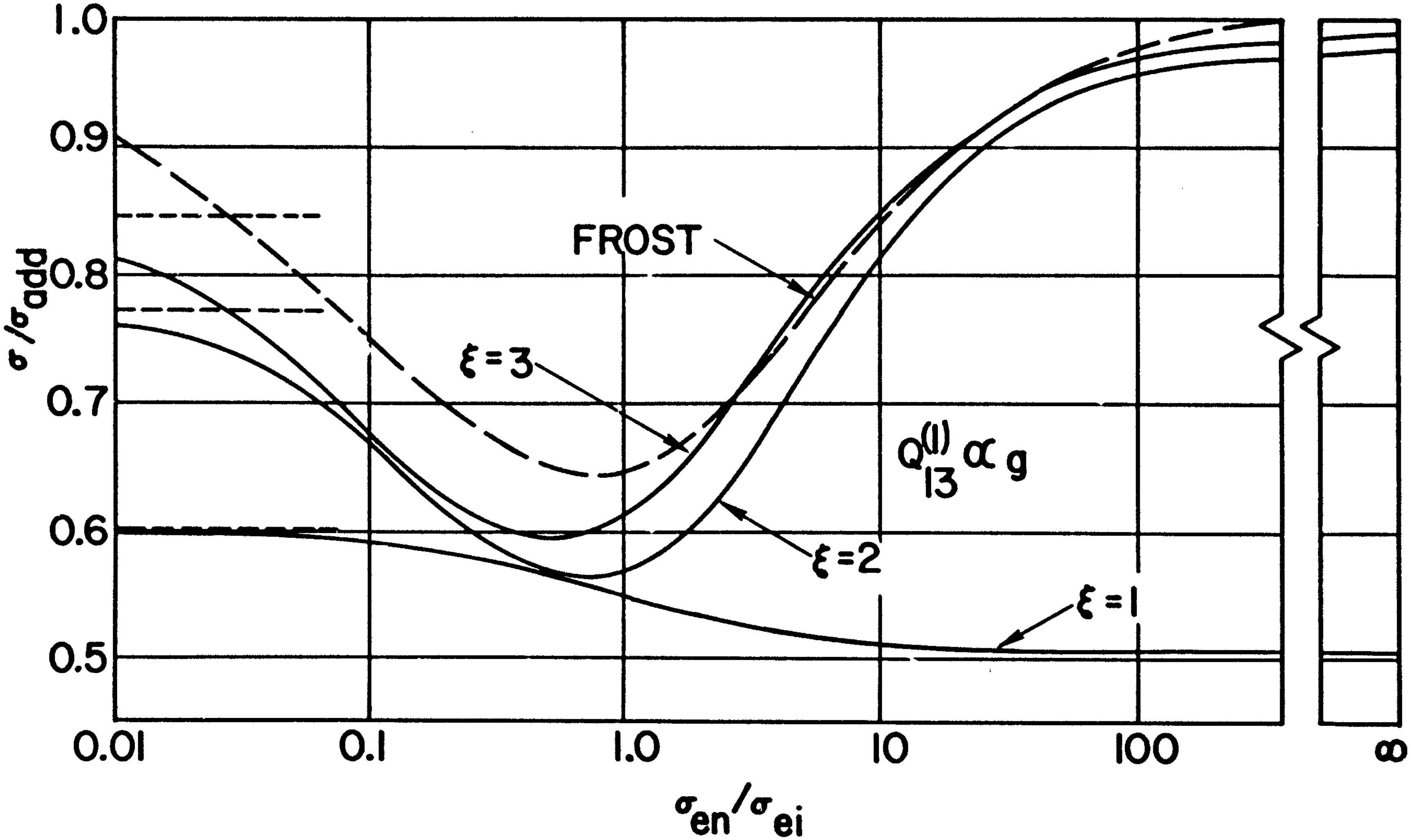

FIGURE 6. Normalized Conductivity of First Three Chapman-Enskog Approximations and of Frost as a Function of Degree of Ionization for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g$ (m = 1). The normalizing conductivity, σ_{add}, is defined by Eq. (16) and σ_{en}/σ_{ei} is proportional to the degree of ionization as defined by Eq. (20). The quantity g denotes the relative electron speed.

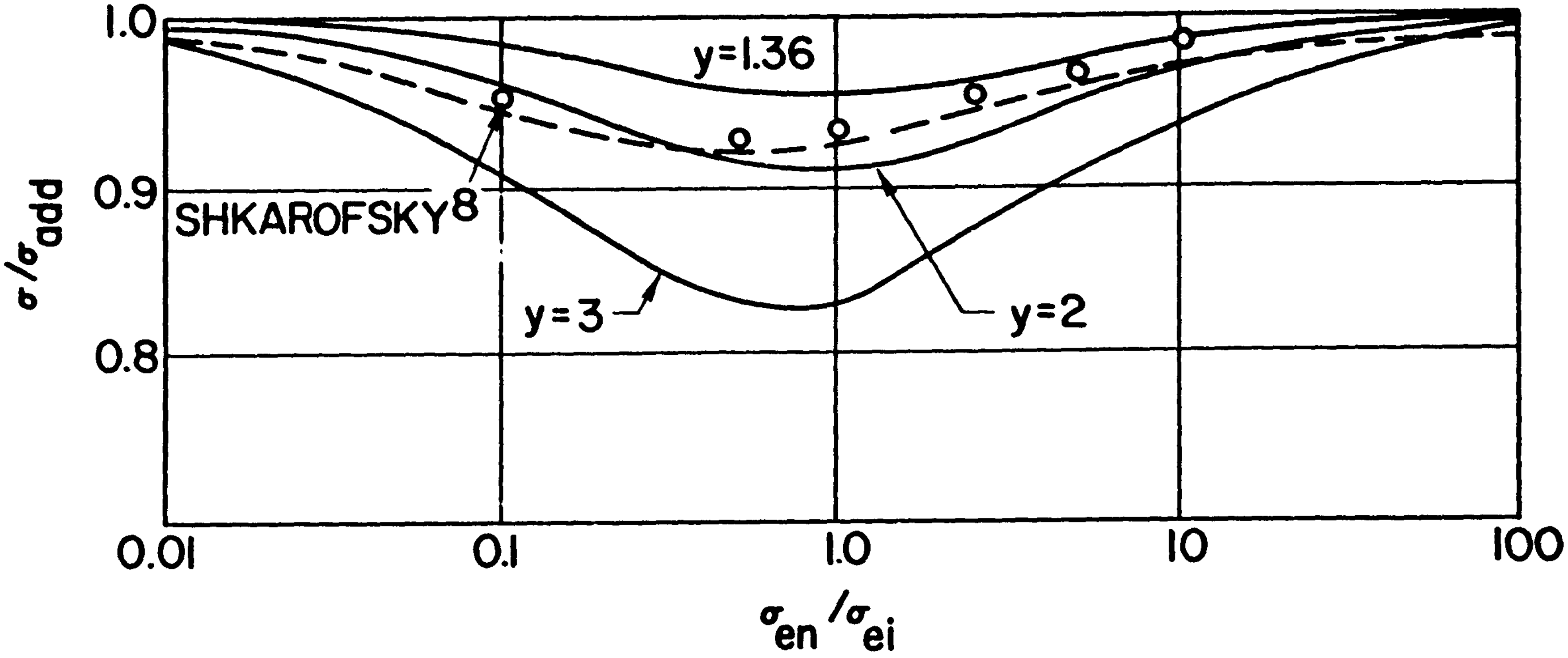

FIGURE 7. Normalized Electrical Conductivity as a Function of Degree of Ionization for Different Electron-Ion Collision Frequencies $\nu_{ei} \propto g^{-y}$, and for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g^{-1}$ ($m = 0$, Maxwellian molecules). The dashed curve represents the third Chapman-Enskog approximation.

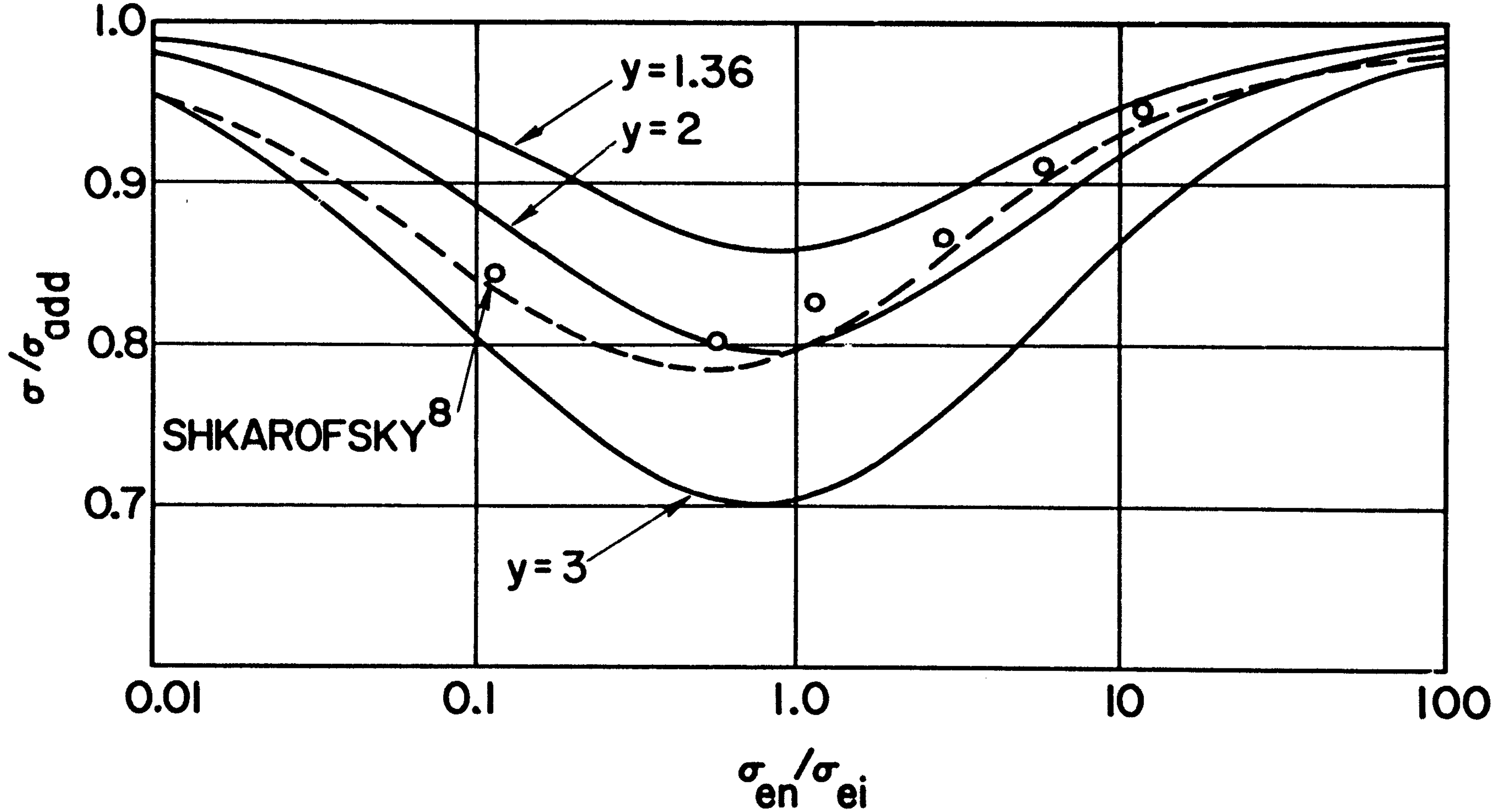

FIGURE 8. Normalized Electrical Conductivity as a Function of Degree of Ionization for Different Electron-Ion Collision Frequencies $\nu_{ei} \propto g^{-y}$, and for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g^{0}$ ($m = 1/2$, Hard Spheres). The dashed curve represents the third Chapman-Enskog approximation.

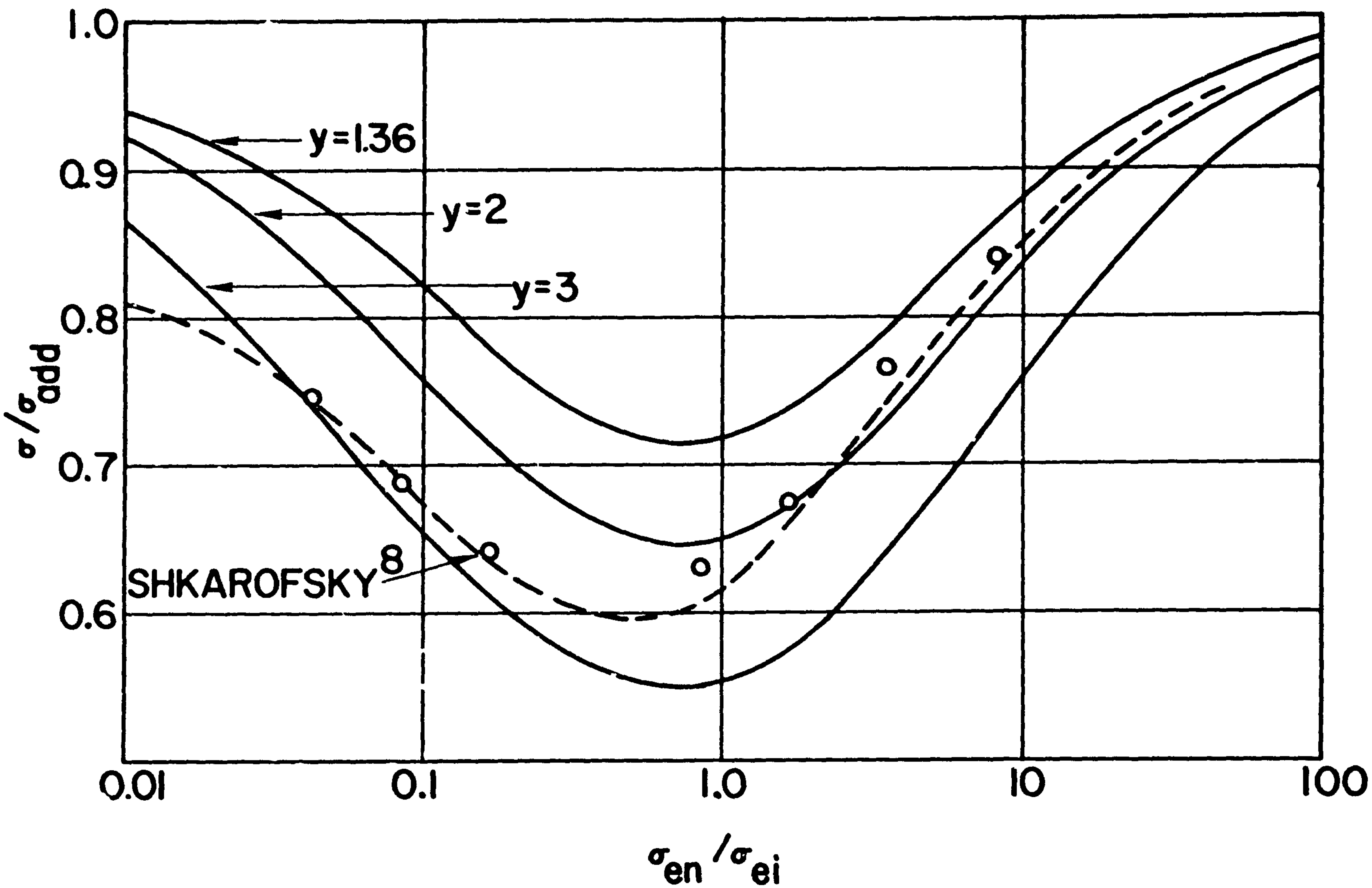

FIGURE 9. Normalized Electrical Conductivity as a Function of Degree of Ionization for Different Electron-Ion Collision Frequencies $\nu_{ei} \propto g^{-y}$, and for the Electron-Neutral Cross-Section Model $Q_{13}^{(1)} \propto g$ (m = 1). The dashed curve represents the third Chapman-Enskog approximation.

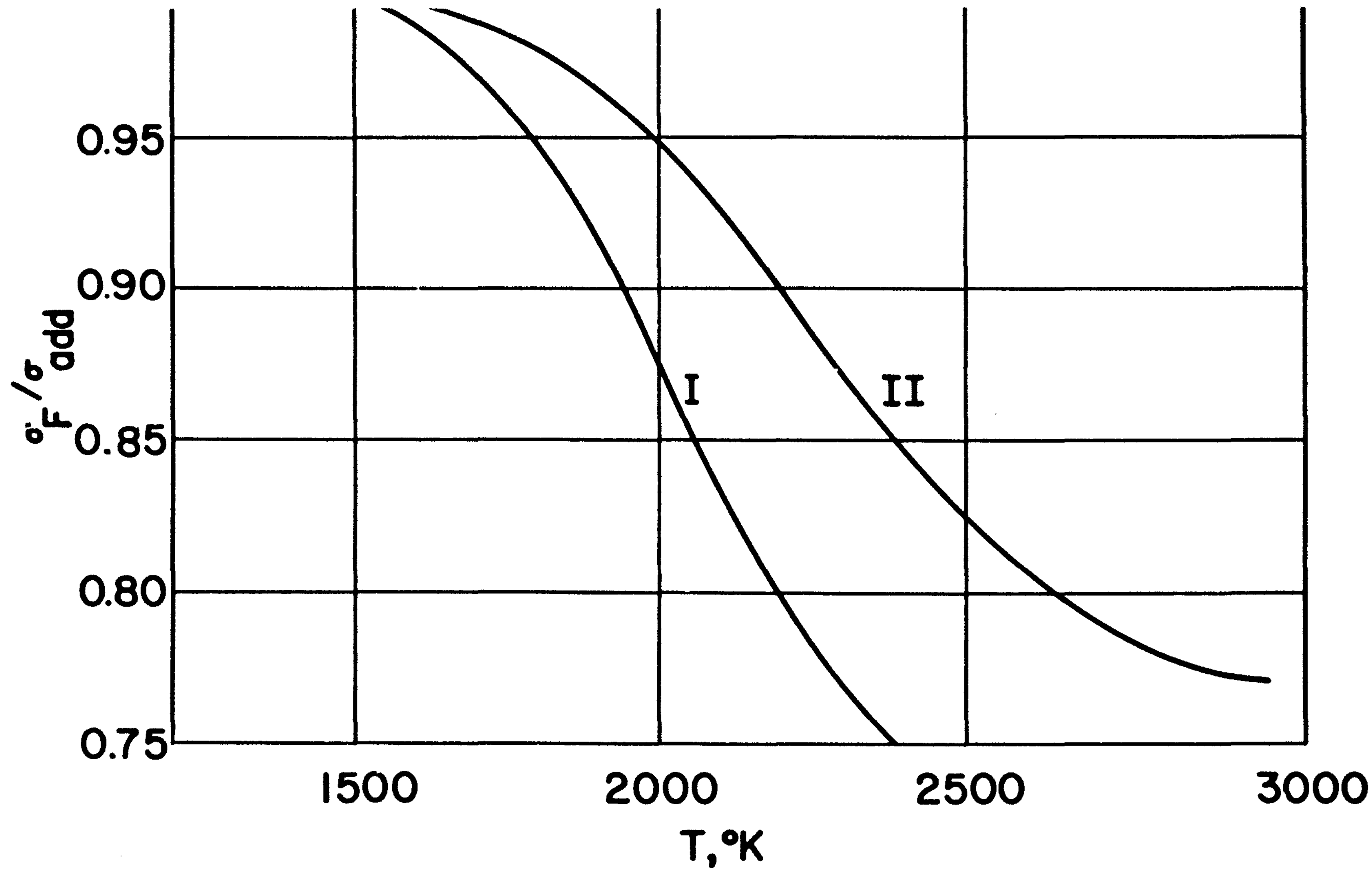

FIGURE 10. Frost's Normalized Electrical Conductivity for Cesium-Seeded Argon as a Function of Temperature. Total pressure is 1.0 atmosphere and partial pressure of Cs is [illegible] Torr. The e-Ar. cross-section of Frost and Phelps[14] and of Brode[15] have been used for curves I and II respectively. Both curves use Brode's[16] cross-section for e-Cs.

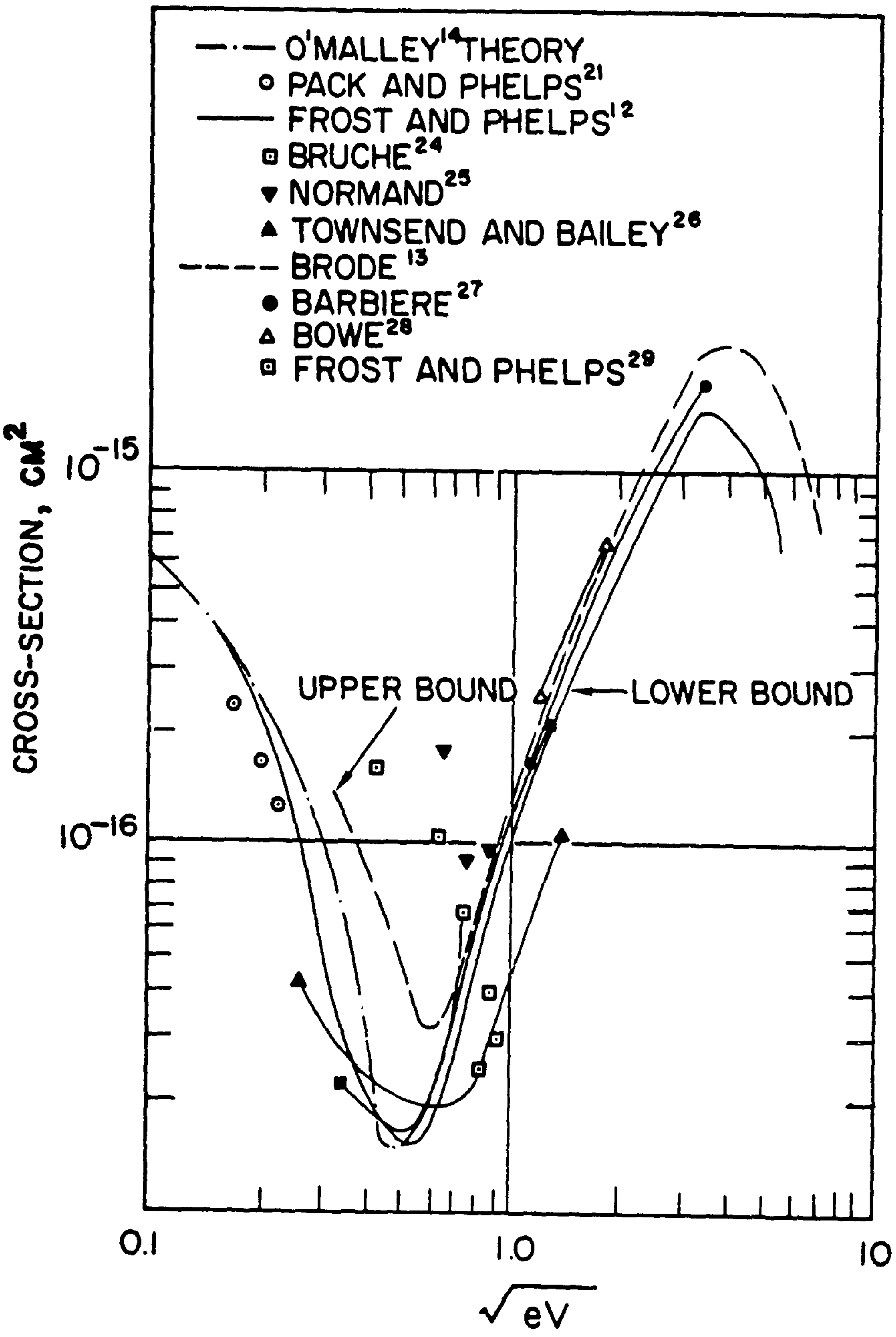

FIGURE 11. Electron-Argon Momentum Transfer Cross-Section.

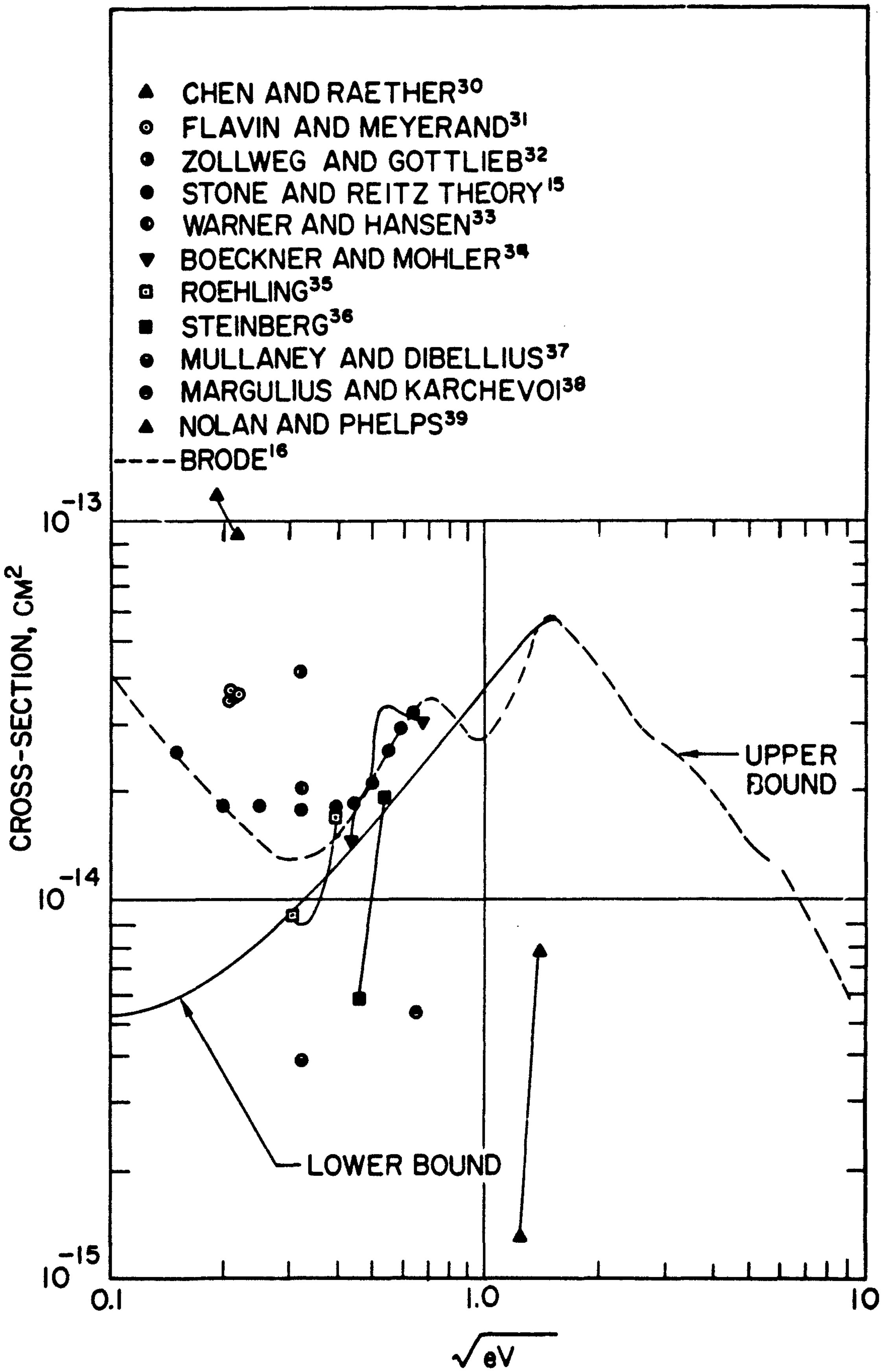

FIGURE 12. Electron-Cesium Momentum Transfer Cross-Section.

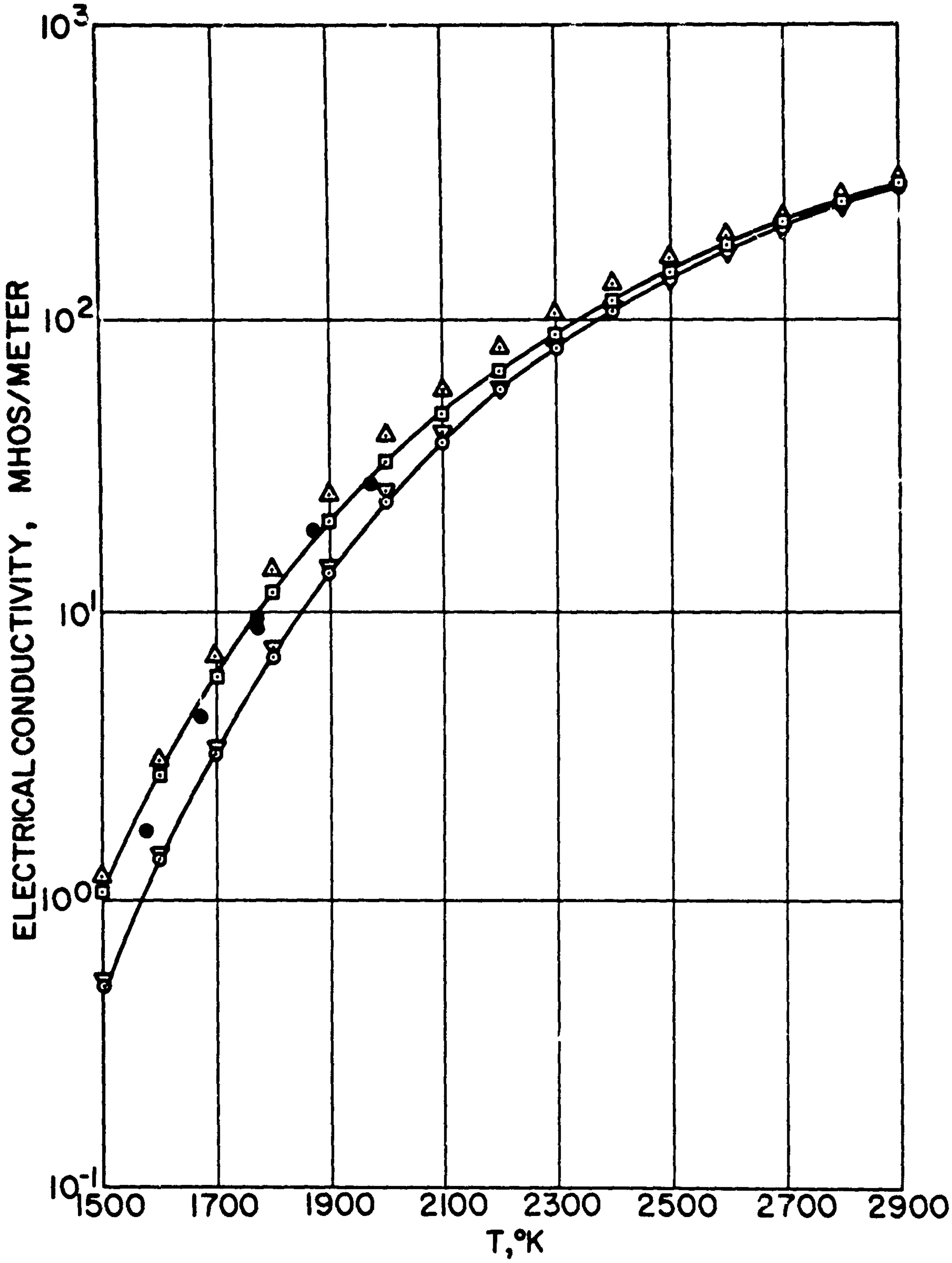

FIGURE 13. The Electrical Conductivity Calculated by Frost's Method, △, ▽, and the Third Chapman-Enskog Approximation, ⊡, ⊙, as a Function of Temperature for Cesium-Seeded Argon. Total pressure is 1.0 atmospheres and partial pressure of Cs is 1.0 Torr. The upper and lower curves are based, respectively, on the lower and upper bound cross-sections indicated in Figs. 11 and 12. The experimental data of Harris[22] is indicated by ●.

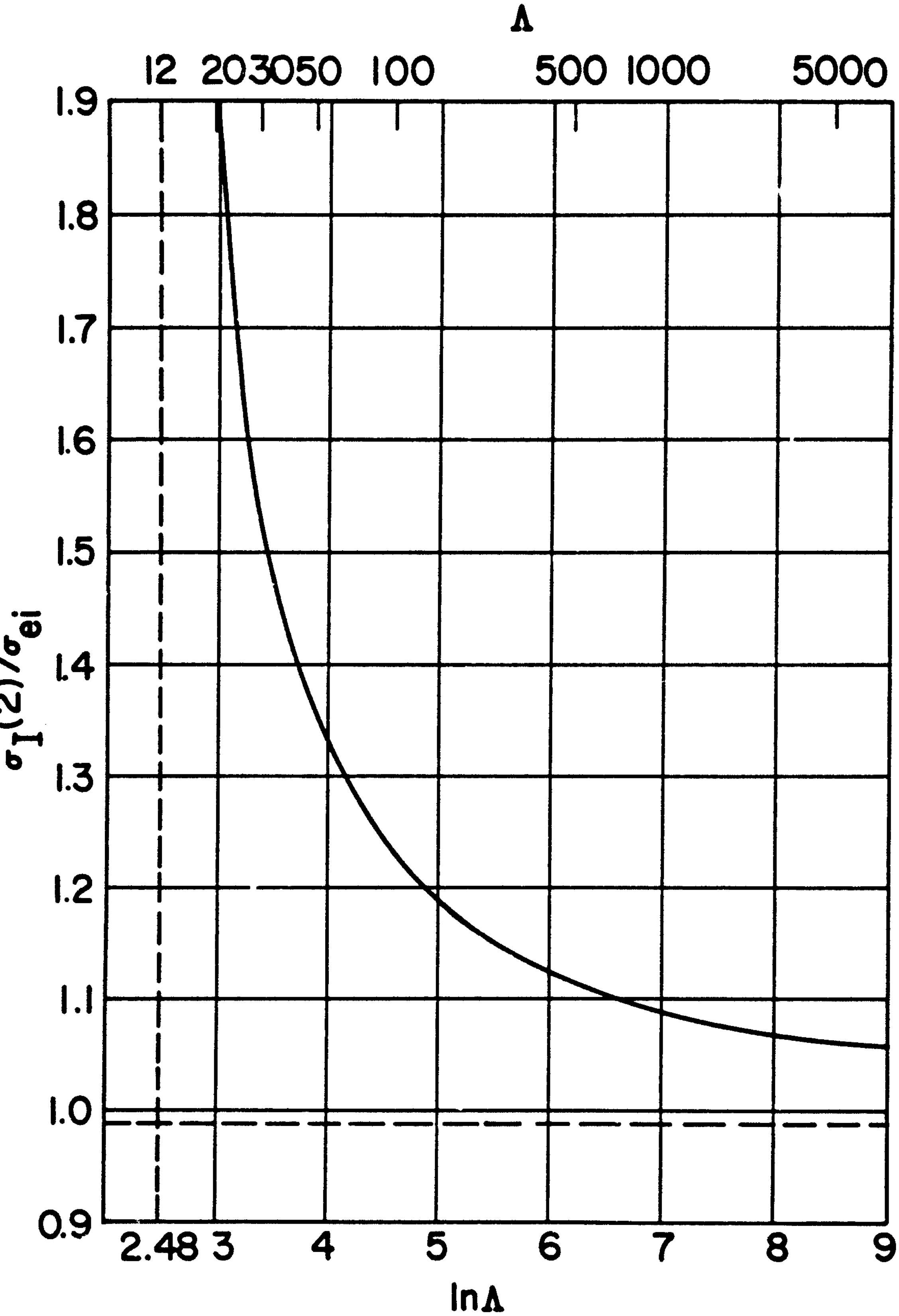

FIGURE 14. The Ratio of Itikawa's Second Approximation to the Spitzer-Härm Electrical Conductivity of a Fully-Ionized Gas as a Function of $\ln\Lambda$. The dashed lines indicate assymptotic values. For $\ln\Lambda \rightarrow \infty$, $\sigma_I(2) \rightarrow \sigma_{ei}(2)$, the second Chapman-Enskog approximation.

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13 ABSTRACT