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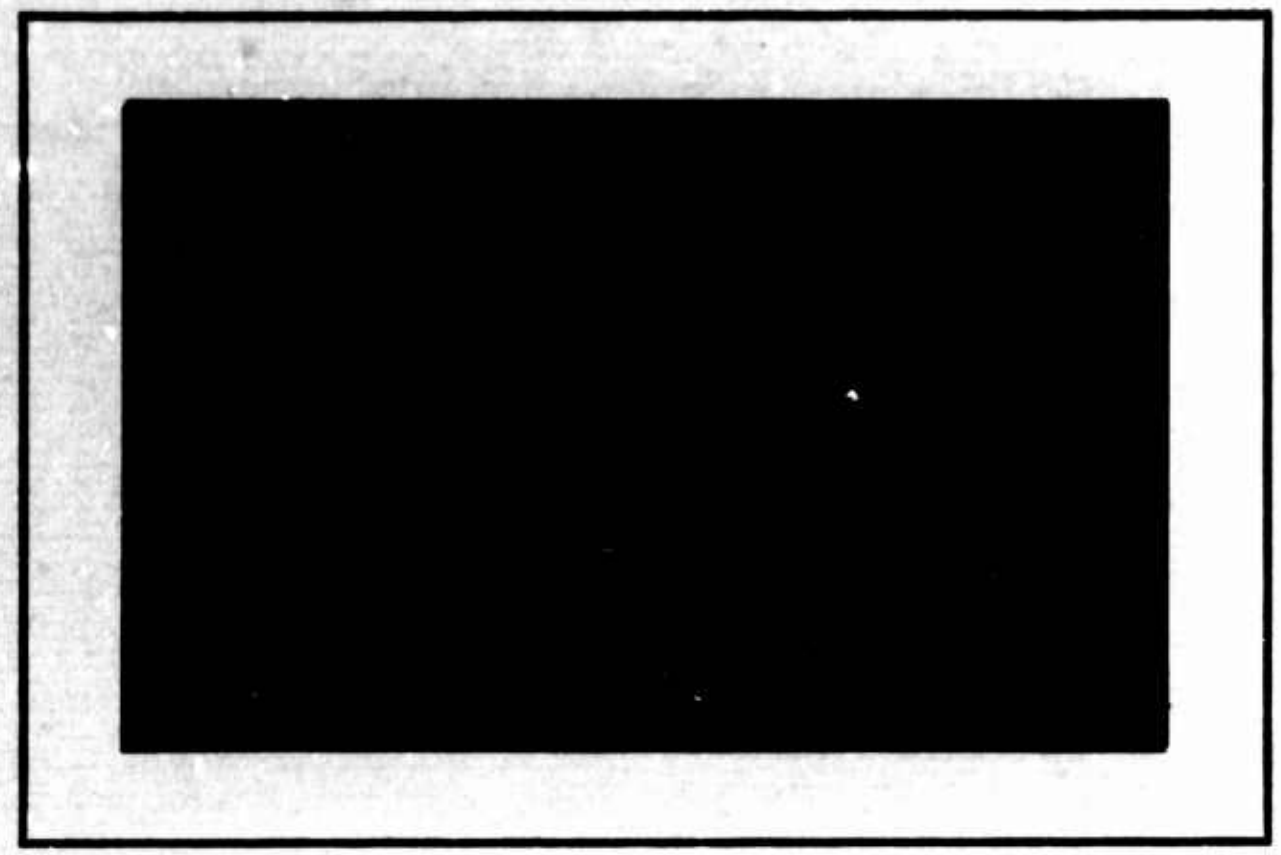

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Management Sciences Research Report No. 62

STRUCTURAL SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING AND AN EXACT PRODUCT FORM LEFT INVERSE

by

A. Charnes, Northwestern University
and
W.W. Cooper, Carnegie Institute of Technology

December 15, 1965

MANAGEMENT SCIENCES RESEARCH GROUP
GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION
CARNEGIE INSTITUTE OF TECHNOLOGY
PITTSBURGH, PENNSYLVANIA 15213

This report was prepared as part of the activities of the Management Sciences Research Group, Carnegie Institute of Technology, (under Contract NONR 760(24), NR 047-048 with the U. S. Office of Naval Research) and as part of the activities of the Systems Research Group, Northwestern University (under Contract NONR 1228(10), NR 047-021 with the U. S. Office of Naval Research). Distribution of this document is unlimited. Reproduction of this paper in whole or in part is permitted for any purpose of the United States Government.

1. Introduction:

Formulae for sensitivity analyses in linear programming problems are generally confined to parametric variations of the stipulations vector (right-hand side) and the functional coefficients. There is also no great trouble in effecting studies for possible variations in the elements of vectors that are not in the basis. This is not to say that there has been no effort or interest in effecting such studies when variations on one or more basis vectors is to be considered. See [2] and [8].[1/] There remains much to be accomplished, however, in devising more straightforward and effective approaches which do not depend on special assumptions with respect to the basis matrices being considered.

In this paper we present another possible approach. This is done by generalizing a theorem[2/] which provides an exact inverse in the form of a product modification to a given inverse when the matrix has been altered additively by a matrix of a certain class. Since basis matrices consist of linearly independent column vectors, they always have a left inverse.[3/] In what follows we shall utilize this property and thereby provide exact formulae for such sensitivity analyses in the general case of linear programming matrices, including such cases as networks and distribution (or transportation) model types.

2. Mathematical Theory:

We first introduce the following

1/ See also [3] where the problem of optimal alteration of structural and other coefficients is considered.

2/ See [1] and [7].

3/ Vide, e.g., [4] for further discussion of left inverses.

Theorem 1: (α) Let the $m\times n$ matrix B have a left inverse $B^{\#}$ so that $B^{\#}B = I$, where I is the $n\times n$ identity matrix. (β) Let D be an $m\times n$ matrix such that $B^{\#}DB^{\#}D = pB^{\#}D$ for some real scalar p.

Then a left inverse of $B+\sigma D$, $-p^{-1} \neq \sigma$, a scalar, is $B^{\#}(I+\tau DB^{\#}) = (I+\tau B^{\#}D)B^{\#}$, where

$$\tau = -\frac{\sigma}{1+p\sigma} = -\sigma(1+p\sigma)^{-1} \tag{1}$$

Proof: We wish to show, by suitable choice of τ, that

$$I = B^{\#}(I+\tau DB^{\#})(B+\sigma D)\,. \tag{2}$$

Because of the symmetry of the product of the first two terms this may be rewritten

$$I = (I+\tau B^{\#}D)B^{\#}(B+\sigma D) = (I+\tau B^{\#}D)(I+\sigma B^{\#}D) = I+(\tau+\sigma)B^{\#}D+\tau\sigma B^{\#}DB^{\#}D\,. \tag{3}$$

By assumption

$$(B^{\#}D)^2 = pB^{\#}D \tag{4}$$

and so

$$I = I+(\tau+\sigma+\tau\sigma p)B^{\#}D. \tag{5}$$

Thus, for $B^{\#}(I+\tau DB^{\#})$ to be a left inverse it suffices to set

$$\tau+\sigma+\tau\sigma p = 0 \tag{6}$$

or

$$\tau = -\frac{\sigma}{1+p\sigma} = -\sigma(1+p\sigma)^{-1}. \tag{7}$$

Q.E.D.

In order to have a usable form for D which has the assumed property (4) regardless of the entries in B we can take D to be a matrix with only one non-zero column. Evidently multiplication of any matrix, A, on the right by D results in a matrix with at most one non-zero column--which appears in the same position as the non-zero column for D.[1/] As may then be verified, (4) holds. It should be noted, however, that the value of p may vary with $B^{\#}$.

For such D we have

$$(8)\quad \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} 0 & \cdots & d_1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & d_n & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \sum_j a_{1j}d_j & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \sum_j a_{ij}d_j & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \sum_j a_{nj}d_j & \cdots & 0 \end{bmatrix}$$

In particular, if D has only one non-zero element d_r, say, in its r^{th} row and s^{th} column, then the right-hand side of (8) reduces to

$$(9)\quad d_r \begin{bmatrix} 0 & \cdots & a_{1r} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{mr} & \cdots & 0 \end{bmatrix} .$$

It might also be further noted that this product form for the new inverse--i.e., $(I+\tau B^{\#}D)B^{\#}$--provides an alternative to the iteration procedure in the modified simplex method.[2/] This may be evident on taking $\sigma = 1$ and the non-zero column in D to be the entering column vector minus

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1/ Similar formulae can, of course, be developed for right inverses and the usable forms for D would then consist of matrices with at most one non-zero row.

2/ And hence also to the revised simplex method. See [2] and [5].

the departing column vector so that then $\tau = -(1+p)^{-1}$.

3. Applications to Linear Programming:

We now proceed to determine new simplex tableau elements as required when moving from B to $B + \sigma D$. Distinguishing between old and new quantities by inserting carets[1/] over the latter,

$$\hat{Y}_j = (B + \sigma D)^{\#} P_j = (I + \tau B^{\#} D)\, B^{\#} P_j = (I + \tau B^{\#} D)\, Y_j \tag{10}$$

or, when D is of the form (8),

$$\hat{Y}_j = Y_j + \tau y_{sj}\, B^{\#} d \tag{11}$$

where $d^T = (d_1, \ldots, d_m)$ is the non-zero column, s, in D, and $Y_j^T = (y_{1j}, \ldots, y_{sj}, \ldots, y_{mj})$. From (11) one may further note that if d has only one non-zero element, say d_r, then $B^{\#} d$ is d_r times the r^{th} column vector of $B^{\#}$.

We next obtain, for the most general permissible $B^{\#} D$,

$$\hat{\omega}^T = c_B^T (B + \sigma D)^{\#} = c_B^T B^{\#} (I + \tau D B^{\#}) = \omega^T (I + \tau D B^{\#}) . \tag{12}$$

When D has form (8), this may be simplified to

$$\hat{\omega}^T = \omega^T + \tau\, \omega^T d (B^{\#})_s , \tag{13}$$

where $(B^{\#})_s$ is the s^{th} row of $B^{\#}$.

Similarly,

$$\hat{z}_j = \hat{\omega}^T P_j = \omega^T (P_j + \tau D Y_j) = z_j + \tau\, \omega^T D\, Y_j \tag{14}$$

1/ The notation conventions used here conform to the ones in [2].

in general, while under the assumptions for (8)

$$\hat{z}_j = z_j + \tau\, y_{sj}\, \omega^T d. \tag{15}$$

We may now obtain our desired formulae. First, the new value of the functional is

$$\hat{z}_o = z_o + \tau\, \omega^T DY_o \tag{16}$$

in general, and under (8), this reduces to

$$\hat{z}_o = z_o + \tau\, y_{so}\, \omega^T d. \tag{17}$$

For $B + \sigma D$ to be a dual feasible basis, it is necessary and sufficient that $\hat{z}_j \geq c_j$ or, using the above expression,

$$z_j - \sigma(1 + p\sigma)^{-1}\, \omega^T D\, Y_j \geq c_j, \text{ all } j. \tag{18}$$

For the form D specialized as (8), this becomes

$$z_j - \sigma(1 + p\sigma)^{-1}\, y_{sj}\, \omega^T d \geq c_j, \text{ all } j\ . \tag{19}$$

Primal feasibility on the other hand corresponds to

$$Y_o - \sigma(1 + p\sigma)^{-1}\, B^{\#} DY_o \geq 0 \tag{20}$$

and thus, under (8),

$$Y_o - \sigma(1 + p\sigma)^{-1}\, y_{so}\, B^{\#} d \geq 0. \tag{21}$$

Cf. (11).

We may note that the necessary and sufficient conditions for primal and dual feasibility reduce to a system of linear inequalities to be satisfied by the scalar σ when Y_o, $B^{\#}D$ and p are regarded as fixed. These systems thereby define the range of parametric variation for σ when these conditions are to be maintained.

Here the developments have proceeded by reference to left inverses. Analogous results may be obtained for right inverses if (β) of Theorem is replaced by the condition $DB^{\#}\ DB^{\#} = pDB^{\#}$ for some real scalar p and $BB^{\#} = I$. Convenient forms for D would consist of matrices with with only one non-zero row. The form for the perturbed inverse would again be $B^{\#}(I + \tau\ DB^{\#})$.

Further weakening of the hypotheses of Theorem 1 seem to be possible only at the expense of requiring various specializing assumptions for the matrices D and B. For although Theorem 1 holds with left inverse replaced by the "triplet property" $A^{\#}AA^{\#} = A^{\#}$, it need not hold for the von Neumann-Rao generalized inverse property.[1/] $AA^{\#}A = A$ where $B^{\#}$ is neither a left nor a right inverse. The von Neumann-Rao property is the one that is essential for explicit representation of the general solution and consistency of the general linear system $Ax = b$.

1/ See [4].

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*
Graduate School of Industrial Administration
Carnegie Institute of Technology

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP
Not applicable

3. REPORT TITLE
STRUCTURAL SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING AND AN EXACT PRODUCT FORM LEFT INVERSE

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report, December 1965

5. AUTHOR(S) *(Last name, first name, initial)*
Charnes, A., and Cooper, W. W.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 15, 1965	8	8

8a. CONTRACT OR GRANT NO.
NONR 760(24)

b. PROJECT NO.
NR 047-048

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)
Management Sciences Research Report No. 62 *

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
Systems Research Memo 141 (see item 11)

10. AVAILABILITY/LIMITATION NOTICES
Releasable without limitations on dissemination

11. SUPPLEMENTARY NOTES
Also under Contract NONR 1228(10)
Project NR 047-021 at
Northwestern University

12. SPONSORING MILITARY ACTIVITY
Logistics and Mathematical Statistics Branch
Office of Naval Research
Washington, D. C. 20360

13. ABSTRACT

This paper shows how to obtain an exact inverse in the form of a product modification to a given inverse when the matrix has been altered additively by a matrix of a certain class. Explicit formulae are derived for sensitivity analyses, e.g., in linear programming, wherein the elements of the structural matrix are to be varied.

* Joint with Northwestern University Systems Research Group. See items 9b and 11.

DD FORM 1 JAN 64 1473

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Linear Programming Sensitivity Analysis Matrix Inverse Left Inverses Generalized Inverse						

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